

Eric Larson 

Constructing reducible Brill–Noether curves II

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Abstract. This paper is the second of a two-part series by the author devoted to the following fundamental problem in the theory of algebraic curves in projective space: Which reducible curves arise as limits of smooth curves of general moduli? Special cases of this question studied by Sernesi (Sernesi, Edoardo (1984) On the existence of certain families of curves. *Invent Math* 75(1): 488–557), Ballico (Ballico, Edoardo (2012) Embeddings of general curves in projective spaces: the range of the quadrics. *Lith Math J* 52(2): 134–137), and others have been critical in the resolution of many problems in the theory of algebraic curves over the past half century. In this paper, we give sharp bounds on this problem for space curves, when the nodes are general points and the components are general in moduli. We also systematically study a variant in projective spaces of arbitrary dimension when the nodes are general in a hyperplane. The results given here significantly extend those cases established in the previous paper in this series (Eric Larson, Constructing reducible Brill–Noether curves, To appear in *documenta mathematica*, [arxiv:1603.02301](https://arxiv.org/abs/1603.02301)), as well as those cases established by Sernesi (Sernesi, Edoardo (1984) On the existence of certain families of curves. *Invent Math* 75(1): 488–557), Ballico (Ballico, Edoardo (2012) Embeddings of general curves in projective spaces: the range of the quadrics. *Lith Math J* 52(2): 134–137), and others. As explained in (Eric Larson, Degenerations of curves in projective space and the maximal rank conjecture, [arXiv:1809.05980](https://arxiv.org/abs/1809.05980)), the reducible curves constructed in this paper also play a critical role in the author’s proof of the maximal rank conjecture in a subsequent paper (Eric Larson Degenerations of curves in projective space and the maximal rank conjecture, [arXiv:1809.05980](https://arxiv.org/abs/1809.05980)).

1. Introduction

The goal of the present paper is continue the analysis in [9] of the following fundamental problem in the theory of algebraic curves in projective space:

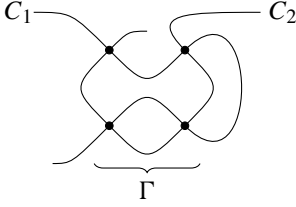
Question If $f: C_1 \cup_{\Gamma} C_2 \rightarrow \mathbb{P}^r$ is a map from a reducible curve, under what conditions can f be deformed to an immersion of a general smooth curve?

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Special cases of this question and variants have been critical in the resolution of many problems in the theory of algebraic curves over the past half century; examples include Sernesi's proof of the existence of components of the Hilbert scheme with the expected number of moduli when the Brill–Noether number is negative [13], and various cases of the maximal rank conjecture by Ballico [2] and Hirschowitz [7].

To fix notation: Write $\bar{M}_g(\mathbb{P}^r, d)$ for Kontsevich's space of stable maps $C \rightarrow \mathbb{P}^r$ of degree d , from a nodal curve C of genus g . There is a natural map $\bar{M}_g(\mathbb{P}^r, d) \rightarrow \bar{M}_g$. As in [9], we define:

Definition 1.1. We refer to a stable map $C \rightarrow \mathbb{P}^r$ as a *Brill–Noether curve (BN-curve)* if it corresponds to a point in a component of $\bar{M}_g(\mathbb{P}^r, d)$ which both dominates $\bar{M}_g$, and whose generic member is a nondegenerate map from a smooth curve, which is an immersion if $r \geq 3$, birational onto its image if $r = 2$, and finite if $r = 1$.

Additionally, we say a stable map $f: C \rightarrow \mathbb{P}^r$ is an *interior curve* if it lies in a unique component of the corresponding space of stable maps.

The Brill–Noether theorem, proven by Griffiths and Harris [5], Gieseker [4], Kleiman and Laksov [8], and others, asserts that BN-curves of degree d and genus g in $\mathbb{P}^r$ exist if and only if the *Brill–Noether number*

$$\rho(d, g, r) := (r + 1)d - rg - r(r + 1) \geq 0;$$

and that in this case, the locus of BN-curves forms an irreducible component of $\bar{M}_g(\mathbb{P}^r, d)$.

Returning to our main question: In [9], we studied this problem when the nodes were general points and the components were general in moduli. In this case the natural conjecture is:

Conjecture. Let $f: C_1 \cup_{\Gamma} C_2 \rightarrow \mathbb{P}^r$ be a stable map from a reducible curve, such that the $f|_{C_i}$ are BN-curves of degree d_i and genus g_i , and $f(\Gamma)$ is a general set of $n = \#\Gamma$ points in $\mathbb{P}^r$. Then f is a BN-curve if and only if it has nonnegative Brill–Noether number.

However, in [9] we found some counterexamples to this conjecture, all of which satisfied

$$(r + 1)d_1 - (r - 3)(g_1 - 1) = (r + 1)d_2 - (r - 3)(g_2 - 1) = (r - 1)n,$$

or equivalently such that both $f|_{C_i}$ admit no deformation passing through $f(\Gamma)$. This motivated us to introduce the following refined conjecture:

Conjecture 1.2. Let $f: C_1 \cup_{\Gamma} C_2 \rightarrow \mathbb{P}^r$ be a stable map from a reducible curve, such that the $f|_{C_i}$ are BN-curves of degree d_i and genus g_i , and $f(\Gamma)$ is a general set of $n = \#\Gamma$ points in $\mathbb{P}^r$. Then unless

$$(r+1)d_1 - (r-3)(g_1-1) = (r+1)d_2 - (r-3)(g_2-1) = (r-1)n, \quad (1)$$

we have that f is a BN-curve if and only if it has nonnegative Brill–Noether number.

In light of counterexamples for which both equalities in (1) hold, it is unsurprising that the difficulty of this conjecture is greatest when both equalities in (1) hold approximately. To date the strongest results on this conjecture were obtained by the author in [9], where we established this conjecture when one of the equalities in (1) was at least 4 away from being an equality and the curves $f|_{C_i}$ were nonspecial.

In the present paper, we establish the first known *sharp* results on this problem by proving Conjecture 1.2 for space curves:

Theorem 1.3. *Conjecture 1.2 holds for space curves ($r = 3$).*

In other words, let $f: C_1 \cup_{\Gamma} C_2 \rightarrow \mathbb{P}^3$ be a stable map from a reducible curve, such that the $f|_{C_i}$ are BN-curves of degree d_i and genus g_i , and $f(\Gamma)$ is a general set of $n = \#\Gamma$ points in $\mathbb{P}^3$. Then unless $n = 2d_1 = 2d_2$, we have that f is a BN-curve if and only if it has nonnegative Brill–Noether number.

However, in many applications — including the results of Sernesi [13], Ballico [2], and Hirschowitz [7] mentioned earlier — it is essential to consider reducible curves where $f(\Gamma)$ is not a general set of points in $\mathbb{P}^r$, but is instead a general set of points in a hyperplane $H \subset \mathbb{P}^r$. The other goal of the present paper is thus to consider a variant of this conjecture for stable maps $f: C \cup_{\Gamma} D \rightarrow \mathbb{P}^r$, where $f|_D$ factors as $\iota \circ f_D$ for $f_D: D \rightarrow H \simeq \mathbb{P}^{r-1}$ and $\iota: H \hookrightarrow \mathbb{P}^r$ the inclusion of a hyperplane $H \subset \mathbb{P}^r$, and $f|_C$ is transverse to H and of specified degree d' and genus g' , and $f(\Gamma)$ is a set of n general points in H .

We will systematically investigate when BN-curves of this form exist. Since the genus of D and degree of f_D are

$$g'' = g + 1 - g' - n \text{ and } d'' = d - d',$$

the curve $C \cup_{\Gamma} D$ must be connected, and the hyperplane section $f(C) \cap H$ contains d' points (or fewer), such curves can only exist when

$$\begin{aligned} g' &\geq 0, \\ g + 1 - g' - n &= g'' \geq 0, \\ (r+1)d' - rg' - r^2 - r &= \rho(d', g', r) \geq 0, \\ r(d-d') - (r-1)(g-g') + (r-1)n - r^2 + 1 &= \rho(d'', g'', r-1) \geq 0, \\ n-1 &\geq 0, \\ d' - n &\geq 0. \end{aligned}$$

In order to construct such reducible curves $C \cup_{\Gamma} D \rightarrow \mathbb{P}^r$, we first need to know when we can pass $f|_C$ and f_D through a set $\Gamma \subset H$ of n general points. In this

paper, we will focus on the case when results of [12] guarantee the existence of such curves $f|_C$ and f_D through Γ . Namely, Theorem 1.4 of [12] implies an hyperplane section of $f|_C$ can pass through n general points subject to the inequality

$$(2r - 3)(d' + 1) - (r - 2)^2(g' - d' + n) - 2r^2 + 3r - 9 \geq 0. \quad (2)$$

In addition, by Theorem 1.2 of [12], f_D passes through n general points provided that

$$(r - 2)n \leq rd'' - (r - 4)(g'' - 1) - 2r + 2;$$

or upon rearrangement,

$$r(d - d') - (r - 4)(g - g') - 2n - 2r + 2 \geq 0.$$

(The argument we give here is inductive but depends only on the general shape of these inequalities, and not the exact coefficients. For example, if Theorem 1.4 of [12] were known in slightly greater generality, say when the left-hand side of (2) was at least -1 , the same inductive argument would apply in this slightly more general situation.)

When all of these inequalities are satisfied, we can construct such a curve $C \cup_{\Gamma} D \rightarrow \mathbb{P}^r$; but a priori, this curve may not be a BN-curve — in fact, a priori, it may not even lie in a component of the Kontsevich space whose generic member is a map from a smooth curve. One can show, as in the proof of Corollary 4.3 of [6] *mutatis mutandis*, that when $f|_C$ and f_D are general, $C \cup_{\Gamma} D \rightarrow \mathbb{P}^r$ admits a first-order smoothing if and only if

$$2n + d + g' - d' - g - r - 1 = n + d'' - g'' - r = n - (\dim H^1(N_{f|_D}) + 1) \geq 0. \quad (3)$$

In these terms, our first theorem shows that, if there exists an n satisfying these inequalities, with (2) satisfied even when d' and n are decreased by 1 and (3) strict, then for the minimal such n , the resulting curve $C \cup_{\Gamma} D \rightarrow \mathbb{P}^r$ is in fact a BN-curve. Namely:

Theorem 1.4. *Let d, g, d', g' , and r be integers which satisfy:*

$$g' \geq 0, \quad (4)$$

$$(r + 1)d - rg - r^2 - r \geq 0, \quad (5)$$

$$(r + 1)d' - rg' - r^2 - r \geq 0. \quad (6)$$

Suppose there exists an integer n satisfying:

$$(2r - 3)d' - (r - 2)^2(g' - d' + n) - 2r^2 + 3r - 9 \geq 0, \quad (7)$$

$$g - g' - n + 1 \geq 0, \quad (8)$$

$$r(d - d') - (r - 1)(g - g') + (r - 1)n - r^2 + 1 \geq 0, \quad (9)$$

$$n - 1 \geq 0, \quad (10)$$

$$d' - n \geq 0, \quad (11)$$

$$r(d - d') - (r - 4)(g - g') - 2n - 2r + 2 \geq 0, \quad (12)$$

$$2n + d + g' - d' - g - r - 2 \geq 0; \quad (13)$$

let n be the minimal such integer. Then any curve $f: C \cup_{\Gamma} D \rightarrow \mathbb{P}^r$ of degree d and genus g , so that $f|_C$ is a general BN-curve of degree d' and genus g' ; and $f|_D$ factors as $\iota \circ f_D$, for $\iota: H \hookrightarrow \mathbb{P}^r$ the inclusion of a hyperplane $H \subset \mathbb{P}^r$, and f_D a general BN-curve; and such that $\#\Gamma = n$ and $f(\Gamma)$ is a general set of n points in H , is an interior BN-curve.

Remark 1.5. If $(2r-3)(d'+1) - (r-2)^2g' - 2r^2 + 3r - 9 \geq 0$, then Theorem 1.4 of [12] implies the hyperplane section of $f|_C$ is general. If $r \geq 4$, this implies the general such reducible curve is an immersion. So we get a curve in the boundary of the component of the Hilbert scheme corresponding to BN-curves, as opposed to just for the Kontsevich space.

In the course of proving Theorem 1.4, we also establish the following slight variant (which yields the same conclusion subject to a slightly different system of inequalities):

Theorem 1.6. Let d, g, d', g' , and r be integers which satisfy:

$$g' \geq 0, \quad (4')$$

$$(r+1)d - rg - r^2 - r \geq 0, \quad (5')$$

$$(r+1)d' - rg' - r^2 - r \geq 0. \quad (6')$$

Suppose there exists an integer n satisfying:

$$(2r-3)(d'+1) - (r-2)^2(g' - d' + n) - 2r^2 + 3r - 9 \geq 0, \quad (7')$$

$$g - g' - n \geq 0, \quad (8')$$

$$r(d - d') - (r-1)(g - g') + (r-1)n - r^2 \geq 0, \quad (9')$$

$$n - 1 \geq 0, \quad (10')$$

$$d' - n \geq 0, \quad (11')$$

$$r(d - d') - (r-4)(g - g') - 2n - 2r - 2 \geq 0, \quad (12')$$

$$2n + d + g' - d' - g - r - 2 \geq 0; \quad (13')$$

let n be the minimal such integer. Then any curve $f: C \cup_{\Gamma} D \rightarrow \mathbb{P}^r$ of degree d and genus g , so that $f|_C$ is a general BN-curve of degree d' and genus g' ; and $f|_D$ factors as $\iota \circ f_D$, for $\iota: H \hookrightarrow \mathbb{P}^r$ the inclusion of a hyperplane $H \subset \mathbb{P}^r$, and f_D a general BN-curve; and such that $\#\Gamma = n$ and $f(\Gamma)$ is a general set of n points in H , is an interior BN-curve.

Remark 1.7. The inequality (k) coincides with the inequality (k') for $k = 4, 5, 6, 10, 11, 13$, and is slightly different for $k = 7, 8, 9, 12$.

Several cases of Theorems 1.4 and 1.6 are already known: The case $n \leq r + 2$ follows from Theorem 1.9 of [9]; the cases $r = 1$ and $r = 2$ follow from classical results on the irreducibility of the Hurwitz space (c.f. [3]) and of the Severi variety (c.f. [14]). We will therefore assume for the proof of Theorems 1.4 and 1.6 that:

$$r \geq 3 \tag{14}$$

$$n \geq r + 3. \tag{15}$$

Since Γ is a general set of points, we may deform the curve f appearing in Theorems 1.4 and 1.6 to assume that (f_D, Γ) is general in the component of $M_{g'',n}(H, d'')$ corresponding to BN-curves, and that $f|_C$ is general in the component of $M_{g'}(\mathbb{P}^r, d')$ corresponding to BN-curves (hence is transverse to H). Similarly, we may deform the curves f_i appearing in Theorem 1.3 to assume that the (f_i, Γ) are both general in the component of $M_{g_i,n}(\mathbb{P}^3, d_i)$ corresponding to BN-curves. In particular, by (14), we have that f is unramified in Theorems 1.4 and 1.6, and that both f_i are unramified in Theorem 1.3.

To prove Theorems 1.3, 1.4, and 1.6, we employ the inductive strategy described in detail in the introduction to the first paper of this sequence [9], and summarized here in Section 2.

This yields a proof of Theorems 1.4 and 1.6 by simultaneous induction on n , with Theorem 1.9 of [9] (which implies both theorems when $n \leq r + 2$) serving as the base case. Namely, we show first, in Section 3, that Theorem 1.4 for any given value of n implies Theorem 1.6 for the same value of n ; then, in Section 4, we show that Theorems 1.4 and 1.6 for any given value of n imply Theorem 1.4 for $n + 1$. For Theorem 1.3, our argument will also be by induction on n .

Note: Throughout this paper, we work over an algebraically closed field of characteristic zero.

2. Preliminaries

In this section, we explain the basic inductive strategy we shall use to prove Theorems 1.4, 1.6, and 1.3, and summarize the key results from [9] and [12] that we shall need.

The basic inductive strategy is to use results of [12] to degenerate one component, say $C \rightarrow \mathbb{P}^r$, to a union $C^\circ = C' \cup C'' \rightarrow \mathbb{P}^r$, that still passes through Γ . To conclude that $C \cup D$ is a BN-curve as desired, there are three basic steps:

- 1 Using results of [9], we show that $C'' \cup D \rightarrow \mathbb{P}^r$ is a BN-curve.

2 Note that our specialization of $C \rightarrow \mathbb{P}^r$ induces a specialization of f :

$$f^\circ: C' \cup (C'' \cup D) = (C' \cup C'') \cup D \rightarrow \mathbb{P}^r. \quad (16)$$

Using our inductive hypothesis, we show that $C' \cup (C'' \cup D) \rightarrow \mathbb{P}^r$ is a BN-curve.

3 Using results of [9], we show that (16) is an interior curve.

(This implies that any deformation, including our original curve $C \cup D$, is a BN-curve as desired.)

We now summarize the key results from [9] and [12] that we shall need to complete this program. To state these results, it is convenient to make the following definition:

Definition 2.1. We say a curve is a *WBN-curve* if it is either a BN-curve, or a (possibly degenerate) nonspecial curve.

To specialize $C \rightarrow \mathbb{P}^r$ to a union $C' \cup C'' \rightarrow \mathbb{P}^r$ that still passes through Γ , we shall need the following results:

Theorem 2.2. (Theorem 1.2 of [12]) *There exists a BN-curve of degree d and genus g in $\mathbb{P}^r$ (with $\rho(d, g, r) \geq 0$), passing through n general points, if*

$$(r-1)n \leq (r+1)d - (r-3)(g-1) - 2r.$$

Theorem 2.3. (Theorem 1.4 of [12]) *The hyperplane section of a general BN-curve of degree d and genus g in $\mathbb{P}^r$ contains n general points (with $0 \leq n \leq d$) if*

$$(2r-3)(d+1) - (r-2)^2(g-d+n) - 2r^2 + 3r - 9 \geq 0.$$

These results allow us to construct specializations via the following method, which can be applied to $f|_C$, or to f_D if we replace $\mathbb{P}^r$ with H :

Method (*).

- 1 Partition $\Gamma = \Gamma' \cup \Gamma''$.
- 2 Find a WBN-curve $C' \rightarrow \mathbb{P}^r$ through Γ' , that is general in some component of the space of WBN-curves passing through Γ' .
If Γ' is not general, it spans a hyperplane; in this case, suppose $C' \rightarrow \mathbb{P}^r$ is transverse to this hyperplane.
- 3 Show that C' passes through a set Δ of general points.
- 4 Find a WBN-curve $C'' \rightarrow \mathbb{P}^r$ through $\Gamma'' \cup \Delta$, that is general.
Again, if $\Gamma'' \cup \Delta$ is not general, suppose $C'' \rightarrow \mathbb{P}^r$ is transverse to the hyperplane spanned by $\Gamma'' \cup \Delta$.
- 5 Construct the union $C' \cup_\Delta C'' \rightarrow \mathbb{P}^r$.

Next, we state the key result from [9] that we shall need to show that $C'' \cup D$ is a BN-curve (our first step above).

Theorem 2.4. (Theorem 1.6 of [9]) Let $C_i \rightarrow \mathbb{P}^r$ (for $i \in \{1, 2\}$) be WBN-curves of degree d_i and genus g_i , which pass through a set $\Gamma \subset \mathbb{P}^r$ of $n \geq 1$ general points. Suppose that, for at least one $i \in \{1, 2\}$, we have

$$(r + 1)d_i - rg_i + r \geq rn.$$

Then $C_1 \cup_{\Gamma} C_2 \rightarrow \mathbb{P}^r$ is a WBN-curve.

For the third step above, we summarize results from [9] that give conditions to guarantee that (16) is an interior curve. The key cohomological computation is:

Lemma 2.5. The twist down by Γ of the normal complex of a curve constructed via Method (*) above has vanishing first hypercohomology:

$$\mathbb{H}^1(\mathcal{N}_{C' \cup_{\Delta} C'' \rightarrow \mathbb{P}^r}(-\Gamma)) = 0.$$

Proof. This follows from combining Lemmas 3.1, 3.2, and 3.3 of [9]. $\square$

And the consequences of it that we shall need here are:

Lemma 2.6. Suppose we specialize $C_i \rightarrow \mathbb{P}^3$ in Theorem 1.3 via Method (*), while leaving the other curve general. Then f° is an interior curve.

Proof. The argument given in Lemma 3.4 of [9] reduces the desired claim to the conclusion of Lemma 2.5. $\square$

Lemma 2.7. Suppose we specialize f_D in Theorems 1.4 or 1.6 (while leaving $f|_C$ general). Then f° is an interior curve provided that

$$H^1((f_D^{\circ})^* \mathcal{O}_H(1)(\Gamma)) = 0. \quad (17)$$

Proof. The argument given in Lemma 3.5 of [9] reduces the desired claim to the conclusion of Lemma 2.5. $\square$

Lemma 2.8. Suppose we specialize $f|_C$ in Theorems 1.4 or 1.6 (while leaving f_D general), so that $f|_C$ remains transverse to H along $\Gamma' \subset \Gamma$. Then f° is an interior curve provided that

$$d'' \geq g'' + r - 1 - \#\Gamma'.$$

Proof. The argument given in Lemma 3.7 of [9] reduces the desired claim to the conclusion of Lemma 2.5. $\square$

Finally, both to verify the cohomological condition (17), and to apply our inductive hypothesis when we make such a degeneration, we shall use the following lemma from [9]:

Lemma 2.9. (Lemma 6.2 of [9]) Let $f: C \cup_{\Gamma} D \rightarrow \mathbb{P}^r$ be an unramified map from a reducible curve, such that $f|_D$ factors as a composition of a general BN-curve $f_D: D \rightarrow H$ of degree d and genus g with the inclusion of a hyperplane $H \subset \mathbb{P}^r$, while $f|_C$ is general in some component of the space of WBN-curves transverse to H along Γ . Let Δ be a set of general points on D , and $\Delta' \subset f(C) \cap H \setminus \Gamma$, such that $\Gamma \cup \Delta'$ and Δ are general sets of points in H . Write $n = \#\Gamma$ and $m = \#\Delta$.

If $d - g + n \geq \max(m, r - 1)$, then $H^1(f|_D^* \mathcal{O}_{\mathbb{P}^r}(1)(\Gamma)) = 0$, and there exists a deformation of f still passing through $\Delta \cup \Delta'$, and transverse to H along $\Delta \cup \Delta'$.

3. Theorem 1.4 implies Theorem 1.6

In this section, we show that Theorem 1.4 for a given value of n implies Theorem 1.6 for the same value of n .

From (8')xm8', we have $g'' - 1 \geq 0$; and from (9'), we have $\rho(d'' - 1, g'' - 1, r - 1) \geq 0$. We may therefore (using Theorem 2.4) specialize f_D to a map from a reducible curve $f_D^\circ: D' \cup_{\{p,q\}} \mathbb{P}^1$, with D' of genus $g'' - 1$ and $f_D^\circ|_{D'}$ of degree $d'' - 1$; and $f_D^\circ|_{\mathbb{P}^1}$ of degree 1; and $\{p, q\}$ a set of two general points on D' . By (12')xm12' and Theorem 2.2, $f_D^\circ|_{D'}$ can pass through n general points; in particular, we may specialize so that $\Gamma = \Gamma_1 \cup \Gamma_2$ is a set of n general points consisting of a set Γ_1 of 2 points on $f_D^\circ(D')$, and a set Γ_2 of $n - 2$ points on $f_D^\circ(D')$. Note that since $\Gamma_1 \cup \{p, q\}$ is general, Γ_1 and Γ_2 are independently general.

By Lemma 2.7, it suffices to show the resulting curve $f^\circ: C \cup_{\Gamma_1 \cup \Gamma_2} (D' \cup_{\{p,q\}} \mathbb{P}^1) \rightarrow \mathbb{P}^r$ is a BN-curve and $H^1((f_D^\circ)^* \mathcal{O}_{\mathbb{P}^r}(1)(\Gamma_1 + \Gamma_2)) = 0$. Moreover the exact sequence

$$\begin{aligned} 0 &\rightarrow f_D^\circ|_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^r}(1)(\Gamma_1 - p - q) \simeq \mathcal{O}_{\mathbb{P}^1}(1) \\ &\rightarrow (f_D^\circ)^* \mathcal{O}_{\mathbb{P}^r}(1)(\Gamma_1 + \Gamma_2) \rightarrow f_D^\circ|_{D'}^* \mathcal{O}_{\mathbb{P}^r}(1)(\Gamma_2) \rightarrow 0 \end{aligned}$$

reduces the vanishing of $H^1((f_D^\circ)^* \mathcal{O}_{\mathbb{P}^r}(1)(\Gamma_1 + \Gamma_2))$ to the vanishing of $H^1(f_D^\circ|_{D'}^* \mathcal{O}_{\mathbb{P}^r}(1)(\Gamma_2))$; this in turn from Lemma , together with (13')xm13' which becomes upon rearrangement (using (14)):

$$(d'' - 1) - (g'' - 1) + (n - 2) \geq \max(2, r - 1) = r - 1. \quad (18)$$

It thus remains to show f° is a BN-curve. For this, we write f° as

$$f^\circ: (C \cup_{\Gamma_1} \mathbb{P}^1) \cup_{\Gamma_2 \cup \{p,q\}} D' \rightarrow \mathbb{P}^r.$$

Note that each inequality (k') for (d, g, d', g', n) implies the corresponding inequality (k) for $(d, g, d' + 1, g' + 1, n)$. Moreover, each inequality (k) for $(d, g, d' + 1, g' + 1, n - 1)$ implies the inequality (k') for $(d, g, d', g', n - 1)$, except for $k \in \{4, 6, 11\}$ when (k') for $(d, g, d', g', n - 1)$ follows from (k') for (d, g, d', g', n) . Thus, $(d, g, d' + 1, g' + 1, n)$ satisfies the inequalities of Theorem 1.4, and n is minimal with that property.

Note that $f^\circ|_{C \cup_{\Gamma_1} \mathbb{P}^1}$ is a BN-curve by Theorem 2.4. Showing that f° is a BN-curve thus follows from Theorem 1.4 (with the same value of n), since $f^\circ|_{C \cup_{\Gamma_1} \mathbb{P}^1}$ admits a deformation still passing through $\Gamma_2 \cup \{p, q\}$ which is transverse to H along $\Gamma_2 \cup \{p, q\}$ by Lemma together with (18).

4. Proof of Theorem 1.4

In this section, we show that Theorems 1.4 and 1.6 for $n - 1$ imply Theorem 1.4 for n . Together with the inductive argument in the previous section, this will complete the proofs of both Theorems 1.4 and 1.6.

Since by assumption, n is minimal subject to the system of inequalities in Theorem 1.4, one of these inequalities must cease to hold when n is replaced by

$n - 1$. Note that all inequalities except for (9), (10), and (13) are nonincreasing in n , and that (10) continues to hold when n is replaced by $n - 1$ by (15). We must therefore be in one of two cases:

Case 1: (13) ceases to hold when n is replaced by $n - 1$: In other words, we have

$$2(n - 1) + d + g' - d' - g - r - 2 \leq -1.$$

Subtracting r times this inequality from (9), we obtain upon rearrangement

$$g - g' - n + 1 \geq r(n - 3). \quad (19)$$

In particular, combining this with (15), the genus $g'' = g - g' - n + 1$ satisfies

$$g - g' - n + 1 \geq r. \quad (20)$$

As (9') implies $\rho(d'' - r + 1, g'' - r, r - 1) \geq 0$, we may therefore (using Theorem 2.4) specialize f_D to a map from a reducible curve $f_D^\circ: D' \cup_\Delta \mathbb{P}^1$, with D' of genus $g'' - r$ and $f_D^\circ|_{D'}$ of degree $d'' - r + 1$; and $f_D^\circ|_{\mathbb{P}^1}$ of degree $r - 1$; and Δ a set of $r + 1$ points.

By Theorem 2.2, $f_D^\circ|_{D'}$ can pass through $n - 1$ general points provided that

$$(r - 2)(n - 1) \leq r(d'' - r + 1) - (r - 4)(g'' - r - 1) - 2(r - 1);$$

or substituting in $d'' = d - d'$ and $g'' = g - g' - n + 1$ and rearranging, provided that

$$r(d - d') - (r - 4)(g - g') - 2n - 4r \geq 0, \quad (21)$$

which follows by adding (9) + $3 \cdot (19)$ + $(2r + 2) \cdot (15)$ to $3r^2 - 5r + 2 \geq 0$.

Note that $f_D^\circ|_{D'}$ can always pass through $r + 1$ general points in H , since $r + 1 \leq n - 1$ by (15), and that $f_D^\circ|_{\mathbb{P}^1}$ can pass through $r + 2$ general points in H by Corollary 1.4 of [1]. We may therefore degenerate so that f_D° still passes through a set $\Gamma = \Gamma' \cup \{p\}$ of general points in H , with $\#\Gamma' = n - 1 > 0$, such that $f_D^\circ|_{D'}$ passes through Γ' and $f_D^\circ|_{\mathbb{P}^1}$ passes through p , and such that $\Delta \cup \{p\}$ is a general set of $r + 2$ points in H .

By Lemma 2.7, it suffices to show the resulting curve $f^\circ: C \cup_{\Gamma' \cup \{p\}} (D' \cup_\Delta \mathbb{P}^1) \rightarrow \mathbb{P}^r$ is a BN-curve and $H^1((f_D^\circ)^* \mathcal{O}_{\mathbb{P}^r}(1)(\Gamma' + p)) = 0$. Moreover the exact sequence

$$\begin{aligned} 0 &\rightarrow f_D^\circ|_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^r}(1)(p - \Delta) \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \\ &\rightarrow (f_D^\circ)^* \mathcal{O}_{\mathbb{P}^r}(1)(\Gamma' + p) \rightarrow f_D^\circ|_{D'}^* \mathcal{O}_{\mathbb{P}^r}(1)(\Gamma') \rightarrow 0 \end{aligned}$$

reduces the vanishing of $H^1((f_D^\circ)^* \mathcal{O}_{\mathbb{P}^r}(1)(\Gamma' + p))$ to the vanishing of $H^1(f_D^\circ|_{D'}^* \mathcal{O}_{\mathbb{P}^r}(1)(\Gamma'))$; this follows in turn from Lemma 2.9, together with (13) which becomes upon rearrangement

$$(d'' - r + 1) - (g'' - r) + (n - 1) \geq \max(r + 1, r - 1) = r + 1. \quad (22)$$

It thus remains to show f° is a BN-curve. For this, we write f° as

$$f^\circ: (C \cup_{\Gamma'} D') \cup_{\Delta \cup \{p\}} \mathbb{P}^1 \rightarrow \mathbb{P}^r.$$

Next, note that each inequality (k) for (d, g, d', g', n) implies the same inequality (k) for $(d - r + 1, g - r - 1, d', g', n - 1)$ — except for $k = 8$ when (8) for $(d - r + 1, g - r - 1, d', g', n - 1)$ follows from (20), for $k = 10$ when (10) for $(d - r + 1, g - r - 1, d', g', n - 1)$ follows from (15), and for $k = 12$ when (12) for $(d - r + 1, g - r - 1, d', g', n - 1)$ follows from (21). Moreover, each inequality (k) for $(d - r + 1, g - r - 1, d', g', n - 2)$ implies the same inequality (k) for $(d, g, d', g', n - 1)$, except for $k \in \{5, 7, 11\}$ when (k) for $(d, g, d', g', n - 1)$ follows from (k) for (d, g, d', g', n) . Thus, $(d - r + 1, g - r - 1, d', g', n - 1)$ satisfies the inequalities of Theorem 1.4, and n is minimal with that property.

Consequently, $f^\circ|_{C \cup_{\Gamma'} D'}$ is a BN-curve by our inductive hypothesis for Theorem 1.4. Showing that f° is a BN-curve thus follows from our inductive hypothesis, since $f^\circ|_{C \cup_{\Gamma'} D'}$ admits a deformation still passing through $\Delta \cup \{p\}$ which is transverse to H along $\Gamma' \cup \{p\}$ by Lemma 2.9 together with (22).

Case 2: (13) continues to hold when n is replaced by $n - 1$, but (9) ceases to hold: Since (9) ceases to hold, we have

$$r(d - d') - (r - 1)(g - g') + (r - 1)(n - 1) - r^2 + 1 \leq -1.$$

Subtracting $(r + 1)$ times this equation from $r \cdot (5) + (8)$ and adding $r + 2 \geq 0$, we obtain upon rearrangement

$$\rho(d' - 1, g', r) = (r + 1)(d' - 1) - rg' - r(r + 1) \geq 0. \quad (23)$$

We may therefore (using Theorem 2.4) specialize $f|_C$ to a map from a reducible curve $f|_C^\circ: C' \cup_{\{p\}} \mathbb{P}^1$, with C' of genus g' and $f|_C^\circ|_{C'}$ of degree $d' - 1$; and $f|_C^\circ|_{\mathbb{P}^1}$ factoring through H of degree 1.

By Theorem 2.3 together with our assumption (7), the hyperplane section of $f|_C^\circ|_{C'}$ contains $n - 1$ general points; and by inspection, $f|_C^\circ|_{\mathbb{P}^1}$ passes through 2 general points in H . We may therefore degenerate so that $f|_C^\circ$ still passes through a set $\Gamma = \Gamma' \cup \{q_1, q_2\}$ of general points in H , with $\#\Gamma' = n - 2 > 0$ (c.f. (15)), such that $f|_C^\circ|_{C'}$ passes through Γ' and $f|_C^\circ|_{\mathbb{P}^1}$ passes through $\{q_1, q_2\}$, and such that $\Gamma' \cup \{f|_C^\circ(p)\}$ is a general set of $n - 1$ points in H .

By Lemma 2.8, it suffices to show $f^\circ: (C' \cup_{\{p\}} \mathbb{P}^1) \cup_{\Gamma' \cup \{q_1, q_2\}} D \rightarrow \mathbb{P}^r$ is a BN-curve and $d'' \geq g'' + r - 1 - (n - 2)$ (which upon rearrangement is exactly (13)).

It thus remains to show f° is a BN-curve. For this, we write f° as

$$f^\circ: C' \cup_{\Gamma' \cup \{p\}} (D \cup_{\{q_1, q_2\}} \mathbb{P}^1) \rightarrow \mathbb{P}^r.$$

Next, note that each inequality (k) for (d, g, d', g', n) implies the corresponding inequality (k') for $(d, g, d' - 1, g', n - 1)$ — except for $k = 6$ when (6') for $(d, g, d' - 1, g', n - 1)$ follows from (23), for $k = 10$ when (10') for $(d, g, d' - 1, g', n - 1)$ follows from (15), and for $k = 13$ when (13') for $(d, g, d' - 1, g', n - 1)$ follows from (13) for $(d, g, d', g', n - 1)$. Moreover, each inequality (k') for $(d, g, d' - 1, g', n - 2)$ implies the corresponding inequality (k) for $(d, g, d', g', n - 1)$, except for $k = 12$ when (12) for $(d, g, d', g', n - 1)$ follows from (12) for (d, g, d', g', n) . Thus, $(d, g, d' - 1, g', n - 1)$ satisfies the inequalities of Theorem 1.6, and n is minimal with that property.

By Theorem 2.4, $D \cup_{\{q_1, q_2\}} \mathbb{P}^1 \rightarrow H$ is a BN-curve. Our inductive hypothesis for Theorem 1.6 thus shows f° is a BN-curve as desired.

5. Proof of Theorem 1.3

To prove Theorem 1.3, we will argue by induction on n , and for fixed n by induction on $\min(\rho_1, \rho_2)$ where $\rho_i = 4d_i - 3g_i - 12$ denotes the Brill–Noether number of f_i .

Note that, since f_i passes through n general points, we have from Theorem 1.1 of [15] that $n \leq 2d_i$; by assumption one of these inequalities is strict. Note also that by assumption, $4(d_1 + d_2) - 3(g_1 + g_2 + n - 1) - 12 \geq 0$; upon rearrangement, this becomes

$$n \leq \frac{\rho_1 + \rho_2 + 15}{3}. \quad (24)$$

We will separately consider three cases:

when $\rho_1 \geq 4$ and $n \leq 2d_1 - 1$

Proof of Theorem 1.3. If $n \leq 2d_2 - 1$, then we may assume without loss of generality that $\rho_2 \leq \rho_1$. On the other hand, if $n \geq 2d_2$, then $2d_1 - 1 \geq n \geq 2d_2$, and so $d_2 \leq d_1 - 1$, which implies $\rho_2 \leq \rho(d_1 - 1, 0, 3) = 4d_1 - 16$. Either way, we have

$$\rho_2 \leq \max(\rho_1, 4d_1 - 16).$$

Combining this with (24), we obtain

$$\begin{aligned} n &\leq \frac{\rho_1 + \rho_2 + 15}{3} \leq \frac{\rho_1 + \max(\rho_1, 4d_1 - 16) + 15}{3} \\ &= \frac{\max(8d_1 - 6g_1 - 9, 8d_1 - 3g_1 - 13)}{3}. \end{aligned}$$

In particular, if $(d_1, g_1) \in \{(6, 2), (7, 4)\}$, then $n \leq 10$. Thus,

$$n - 1 \leq \begin{cases} 2(d_1 - 1) & \text{if } (d_1 - 1, g_1) \notin \{(5, 2), (6, 4)\}; \\ 9 & \text{if } (d_1 - 1, g_1) \in \{(5, 2), (6, 4)\}. \end{cases}$$

Since $\rho(d_1 - 1, g_1, 3) = \rho_1 - 4 \geq 0$, we may (using Theorem 2.4) specialize f_1 to a map from a reducible curve $f_1^\circ: C'_1 \cup_p \mathbb{P}^1 \rightarrow \mathbb{P}^3$, with C'_1 of genus g_1 , and $f_1^\circ|_{C'_1}$ of degree $d_1 - 1$, and $f_1^\circ|_{\mathbb{P}^1}$ of degree 1. By the above inequality, we may do this so f_1° still passes through a set $\Gamma = \Gamma' \cup \{x, y\}$ of n general points, such that $f_1^\circ|_{C'_1}$ passes through Γ' , and $f_1^\circ|_{\mathbb{P}^1}$ passes through $\{x, y\}$, and such that $\Gamma' \cup \{p\}$ is a general set of $n - 1$ points.

As in Lemma 2.6, it suffices to show $(C'_1 \cup_{\{p\}} \mathbb{P}^1) \cup_{\Gamma' \cup \{x, y\}} C_2 \rightarrow \mathbb{P}^3$ is a BN-curve. For this, we simply rewrite this map as $C'_1 \cup_{\Gamma' \cup \{p\}} (\mathbb{P}^1 \cup_{\{x, y\}} C_2) \rightarrow \mathbb{P}^3$, which is a BN-curve by Theorem 2.4 and our inductive hypothesis. $\square$

when $\rho_1 \geq 4$ and $n = 2d_1$

Proof of Theorem 1.3. From (24), we obtain

$$\rho_2 \geq 3n - 15 - \rho_1 = \frac{\rho_1}{2} + 3 + \frac{9}{2}g_1 \geq \frac{4}{2} + 3 = 5 \geq 4.$$

And since by assumption we do not have $n = 2d_1 = 2d_2$, we have $n \leq 2d_2 - 1$. Exchanging indices, we are thus in the previous case. $\square$

This completes the proof when $\rho_1 \geq 4$, and thus by symmetry when $\rho_2 \geq 4$. Exchanging indices if necessary, it therefore remains to consider the case $\rho_1 \leq \rho_2 \leq 3$.

when $\rho_1 \leq \rho_2 \leq 3$

Proof of Theorem 1.3. In this case, we argue by induction on ρ_1 . If $\rho_1 = 0$, then using (24), the result follows from Theorem 2.4.

For the inductive step, we therefore suppose $1 \leq \rho_1 \leq 3$ (which forces $d_1 \geq 4$ and $g_1 \geq 1$). Note that (24) gives

$$n \leq \frac{\rho_1 + \rho_2 + 15}{3} \leq \frac{3 + 3 + 15}{3} = 7,$$

with equality only if $\rho_1 = 3$ (which forces $d_1 \geq 6$). In particular,

$$n \leq \begin{cases} 2(d_1 - 1) & \text{if } (d_1 - 1, g_1 - 1) \notin \{(5, 2), (6, 4)\}; \\ 9 & \text{if } (d_1 - 1, g_1 - 1) \in \{(5, 2), (6, 4)\}. \end{cases}$$

Since $\rho(d_1 - 1, g_1 - 1, 3) = \rho_1 - 1 \geq 0$, we may (using Theorem 2.4) specialize f_1 to a map from a reducible curve $f_1^\circ: C'_1 \cup_{\{p,q\}} \mathbb{P}^1 \rightarrow \mathbb{P}^3$, with C'_1 of genus $g_1 - 1$, and $f_1^\circ|_{C'_1}$ of degree $d_1 - 1$, and $f_1^\circ|_{\mathbb{P}^1}$ of degree 1. By the above inequality, we may do this so f_1° still passes through a set $\Gamma = \Gamma' \cup \{x, y\}$ of n general points, such that $f_1^\circ|_{C'_1}$ passes through Γ' , and $f_1^\circ|_{\mathbb{P}^1}$ passes through $\{x, y\}$, and such that $\Gamma' \cup \{p, q\}$ is a general set of n points.

As in Lemma 2.6, it suffices to show $(C'_1 \cup_{\{p,q\}} \mathbb{P}^1) \cup_{\Gamma' \cup \{x,y\}} C_2 \rightarrow \mathbb{P}^3$ is a BN-curve. For this, we simply rewrite this map as $C'_1 \cup_{\Gamma' \cup \{p,q\}} (\mathbb{P}^1 \cup_{\{x,y\}} C_2) \rightarrow \mathbb{P}^3$, which is a BN-curve by Theorem 2.4, together with an application of our inductive hypothesis. $\square$

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Declarations

Conflict of Interest The author states that there is no conflict of interest.

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