SPATIAL ASYMPTOTICS OF GREEN'S FUNCTION AND APPLICATIONS

By

SERGEY A. DENISOV*

Abstract. We study the spatial asymptotics of Green's function for the 1d Schrödinger operator with operator-valued decaying potential. The bounds on the entropy of the spectral measures are obtained. They are used to establish the presence of the a.c. spectrum.

1 Introduction and the main result

In this note, we revisit the spectral theory of Schrödinger operators with long-range potentials. In dimension one, the quest for the minimal assumptions on the decay of potential that guarantee the preservation of absolutely continuous spectrum resulted in the theorem (Deift–Killip [1], see also [13]), which says:

If $V \in L^2(\mathbb{R}^+)$*, then* $\sigma_{\rm ac}(-\partial^2_{rr} + V) = [0, \infty)$ *where* $\sigma_{\rm ac}$ *denotes the a.c. spectrum of the operator with Dirichlet boundary condition at zero.*

In the case of the Dirac equation, an analogous result was obtained by M. Krein already in 1955 (see [15] and [2]). The L^2-condition is sharp: it is known [14] that $V \in L^p(\mathbb{R}^+), p > 2$ can lead to an empty a.c. spectrum. In higher dimension, one again is interested in finding the minimal assumptions on the decay of V in $-\Delta + V$, $x \in \mathbb{R}^d$, $d \geq 2$ that guarantee "scattering" which can be understood either in the sense of preservation of the a.c. spectrum or as the existence of wave operators in Schrödinger dynamics. Some results were obtained for decaying potentials that oscillate (see [4, 17] for their surveys). However, if the oscillation condition is dropped and no additional smoothness (see, e.g., [16, 19] for various classes of potentials) is assumed, then the identity $\sigma_{\rm ac}(-\Delta + V) = [0, \infty)$ is not known even for V obeying fairly strong constraints, such as $|V(x)| \leq C(1+|x|)^{-1+\epsilon}$,

*The work was supported by NSF DMS-1764245, NSF DMS-2054465, and Van Vleck Professorship Research Award.

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 148 (2022)
DOI 10.1007/s11854-022-0236-1

$0 < \epsilon \ll 1$. Notice that the last assumption is only slightly weaker than the short-range condition of the classical scattering theory [11]. In this paper, we make progress on a related problem.

Consider the Hilbert space $\mathcal{H} := \bigoplus_{n=1}^{\infty} L^2(\mathbb{R}^+)$ with the inner product defined by

$$\langle F, G\rangle_{\mathcal{H}} = \int_0^\infty \langle F, G\rangle dr = \sum_{n=1}^{\infty} \int_0^\infty f_n \overline{g_n} dr,$$

where $F = (f_1, f_2, \ldots)$, $G = (g_1, g_2, \ldots)$. We define the 1-d Schrödinger operators

$$H = -\partial^2_{rr} + V, \quad H^{(0)} = -\partial^2_{rr}, \quad x \geq 0 \tag{1.1}$$

with Dirichlet boundary condition at the origin and operator-valued potential V. It satisfies $V(r) = V^*(r)$ for a.e. $r > 0$ and $\|V\| \in L^\infty[0, \infty)$. By the general theory of symmetric operators, H defines the self-adjoint operator with the domain

$$\mathcal{D}(H) = \mathcal{D}(H^{(0)}) = \bigoplus_{n=1}^{\infty} \mathcal{H}^2_0(\mathbb{R}^+),$$

where $\mathcal{H}^2_0(\mathbb{R}^+) := \{f : f, f'' \in L^2(\mathbb{R}^+), f(0) = 0\}$ is the standard $\mathcal{H}^2(\mathbb{R}^+)$ Sobolev space of functions vanishing at the origin. Denote the Green's function of H by $G(r, \rho, z)$, i.e.,

$$R_z F = (H - z)^{-1} F = \int_{\mathbb{R}^+} G(r, \rho, z) F(\rho) d\rho, \quad F \in \mathcal{H}.$$

We let $z \in \mathbb{C}^+$ and $k = \sqrt{z} \in \{k \in \mathbb{C}^+, \operatorname{Im} k > 0, \operatorname{Re} k > 0\}$. The Green's function of the unperturbed operator will be called $G^{(0)}$. Notice that

$$G^{(0)}(r, \rho, k^2) = \frac{i}{2k}\left(e^{ik|r-\rho|} - e^{ik(r+\rho)}\right). \tag{1.2}$$

Let $u := R_{k^2} F$, $\psi := e^{-ikr} u$. We have $-u'' + Vu = k^2 u + F$, $u(0, k) = 0$ and

$$-\psi'' - 2ik\psi' + V\psi = Fe^{-ikr}. \tag{1.3}$$

In this note, we develop the perturbative theory which partially controls the spatial asymptotics of u when F has compact support, $r \to +\infty$ and $z \in \mathbb{C}^+$ is taken close to $\mathbb{R}^+$. Our analysis allows the direct study of $G(r, \rho, z)$ when ρ is fixed and $r \to \infty$ but u has a better regularity and we will work with it instead. The following theorem showcases the typical application of our analysis to the study of spectral type.

Theorem 1.1. *Suppose $\gamma > \frac{2}{3}$, $\lambda > 0$, and $\|V\| \leq \lambda(1 + r)^{-\gamma}$. Then,*

$$\mathbb{R}^+ \subseteq \sigma_{\rm ac}(H).$$

Later in the text, we can assume that γ is fixed in the range $\gamma \in (\frac{2}{3}, 1)$. Many constants the reader encounters in this text depend on γ and λ but we might not explicitly mention that.

Remark. The proof of the theorem employs elementary properties of subharmonic functions and a few apriori integral estimates obtained directly from the equation itself. We avoid ODE asymptotical methods so this technique can potentially be applied to study elliptic partial differential equations and difference operators on graphs.

The connection between σ_F, the spectral measure of $F \in \mathcal{H}$, and the asymptotics of u at infinity is revealed in the following lemma.

Lemma 1.1. *Suppose* $T > 1$, $\operatorname{supp} V \subset [0, T]$, $F \in \mathcal{H}$, *and* $\operatorname{supp} F \subset [0, 1]$. *Then,* σ_F *is absolutely continuous on* $\mathbb{R}^+$ *and*

$$\sigma_F'(k^2) = k\pi^{-1} \| \psi(\infty, k)\|^2 \tag{1.4}$$

for $k \in \mathbb{R}^+$.

Proof. Under the assumption of the lemma, the so-called absorption principle holds (see, e.g., [8, 9, 10] for the Weyl–Titchmarsh theory of the operator-valued Schrödinger operator). In particular, for every interval $I \subset (0, \infty)$ and every positive r, the function $u(r, k) = (R_{k^2} F)(r)$ has continuous extension in k from $R_{I,1} := I \times (0, 1)$ to the interval I and this u satisfies $-u'' + Vu = k^2 u + F$, $u(0, k) = 0$ for $k \in \overline{R_{I,1}}$. Thus, $\psi(r, k) = e^{-ikr} u(r, k)$ is defined as well for $k \in I$ and $\psi(r, k) = \psi(\infty, k)$ if $r > T$. That explains why the right-hand side in (1.4) is well-defined. The absorption principle also implies that σ_F is purely a.c. on $\mathbb{R}^+$. Next, we take $k \in R_{I,1}$ and write $-u'' + Vu = k^2 u + F$. Take an inner product with u and integrate over $[0, T]$. Subtracting the resulting identity from its conjugate gives us

$$\langle u'(T, k), u(T, k)\rangle - \langle u(T, k), u'(T, k)\rangle = (\overline{k}^2 - k^2) \int_0^T \|u\|^2 d\rho + \langle R_{k^2} F, F\rangle - \langle F, R_{k^2} F\rangle.$$

Due to the absorption principle, we can take $\operatorname{Im} k \to 0$ in the last formula. This gives (1.4) after we take into account that $u(r, k) = e^{ikr} \psi(\infty, k)$ for $r > T$. □

Remark. One of the key ideas in the proof of Theorem 1.1 is based on the following observation. Taking the logarithm of both sides in (1.4) gives

$$\log \sigma_F'(k^2) = \log(k\pi^{-1}) + 2 \log \| \psi(\infty, k)\|.$$

The function $\log \| \psi(\infty, k)\|$ is subharmonic in $R_{I,1} = I \times (0, 1)$ for every closed interval $I \subset (0, \infty)$. Thus, rough bounds for $\log \| \psi(\infty, k)\|$ in $R_{I,1}$ can provide the lower bounds for the entropy $\int_{I'} \log \sigma_F'(k^2)dk$, $I' \subset I$ by application of the mean-value inequality for subharmonic functions. The uniform control over the logarithmic integral implies the a.c. spectral type by the standard argument. A serious obstacle we will face is that the good control of $\|\psi(\infty, k)\|$ is only possible when $\operatorname{Im} k$ is very small. The development of strategy that overcomes this difficulty was the main motivation to write this note.

Some previous results. In [20], the reader can find an overview of one-dimensional results related to the topic. The survey papers [4, 17] discuss the higher-dimensional case. See also [5, 19, 18] for more recent advances. The one-dimensional Schrödinger with operator-valued potential was extensively studied in the past and a thorough account of the literature can be found in [8, 9, 10]. The a.c. spectrum of the operator-valued Schrödinger with decaying potential was studied in the context of hyperbolic pencils in [3]. In particular, it was established that $\mathbb{R}^+ \subseteq \sigma_{\rm ac}(-\partial_{rr}^2 + tV)$ for a.e. $t \in \mathbb{R}$, provided that the operator-valued potential V satisfies $\|V\| \in L^2(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$.

Motivation. To relate (1.1) to multidimensional problems, consider the three-dimensional Schrödinger operator $-\Delta + V$, $x \in \mathbb{R}^3$ which allows the representation

$$-\partial_{rr}^2 - \frac{B}{r^2} + V(r,\theta) \tag{1.5}$$

in the spherical coordinates $(r,\theta) \in \mathbb{R}^+ \times \mathbb{S}^2$. Here, B stands for the Laplace–Beltrami operator on $\mathbb{S}^2$ and the Dirichlet boundary condition is assumed at the origin. If the higher spherical modes can be neglected, one considers

$$H = -\partial_{rr}^2 - \frac{P_{\le r^\kappa} B}{r^2} + V(r,\theta) \tag{1.6}$$

instead of (1.5), where $P_{\le r^\kappa}$ is an orthogonal projection to the first $[r^\kappa]$ spherical harmonics. Assuming $|V(x)| \le C(1+|x|)^{-\gamma}$ with $\gamma > \frac{2}{3}$ and choosing κ in a suitable way, we reduce (1.6) to the form (1.1).

Structure of the paper. The second section contains some a priori estimates for the solutions to equation (1.3). In the third section, we give the proof of Theorem 1.1. Some useful estimates on subharmonic functions are collected in Appendix 1. The second appendix contains general bounds on Green's function.

Notation.

- If I is a closed interval on $\mathbb{R}$, c_I denotes its center and $|I|$ denotes its length; I_r stands for the interval centered at zero with radius r; $\mathbb{R}^+ = (0,\infty)$.
- If ψ is a vector in Hilbert space $\ell^2(\mathbb{N})$, then $\|\psi\|$ denotes its norm. If V is a bounded linear operator acting in $\ell^2(\mathbb{N})$, then $\|V\|$ denotes its operator norm.
- If I is a closed interval in $\mathbb{R}^+$ and $\delta > 0$, then $R_{I,\delta} := I \times (0,\delta)$.

- If $\phi, \psi \in \ell^2(\mathbb{N})$, then $\langle \phi, \psi \rangle$ refers to the inner product in $\ell^2(\mathbb{N})$.
- For $a > 0$, we define $\log_+ a = \max\{0, \log a\}$, $\log_- a = \min\{0, \log a\}$.
- The symbol C_α will indicate a positive constant whose dependence on a parameter α we want to emphasize. The actual value of this constant can change from one formula to another.
- For two non-negative functions $f_{1(2)}$, we write $f_1 \lesssim f_2$ if there is an absolute constant C such that $f_1 \leq Cf_2$ for all values of the arguments of $f_{1(2)}$. We define $\gtrsim$ similarly and say that $f_1 \sim f_2$ if $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$ simultaneously. If the constant C depends on parameter α, we might write $f_1 \lesssim_\alpha f_2$.
- For the set $\Delta \subset \mathbb{R}$, we denote $\Delta^2 = \{E^2 : E \in \Delta\}$.

2 Two simple estimates obtained from the equation

In this section, we consider the case when $\operatorname{supp} V \subset [0, T]$ and $\|V(r)\| < \lambda(r+1)^{-\gamma}$, $\gamma \in (\frac{2}{3}, 1)$. In later discussion, we will be taking $T = 2^n$, $n \geq n_0 \gg 1$. Let, e.g., F be such that

$$F = (f, 0, \ldots), \quad \|f\|_{L^2(\mathbb{R}^+)} = 1, \quad \operatorname{supp} f \subset [0, 1], \quad f \not\equiv 0. \tag{2.1}$$

Let σ_F be the spectral measure of F, i.e.,

$$\langle R_z F, F\rangle_{\mathcal{H}} = \int \frac{d\sigma_F(E)}{E - z}, \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

Recall that σ_F is a probability measure and that $u = R_z F$. Rewrite equation (1.3) for ψ as

$$\psi' = i\frac{\psi''}{2k} - i\frac{V\psi}{2k}, \quad r > 1. \tag{2.2}$$

Lemma 2.1. *If I is any closed interval in $\mathbb{R}^+$, $\alpha \in (0, 1)$ and $k \in R_{I,T^{-\alpha}}$, then*

$$\sup_{r>0} \|\psi(r, k)\| \leq C_{I,\alpha} \exp(2(\operatorname{Im} k)^{-(1-\alpha)/\alpha}).$$

Proof. Since $V(r) = 0$ for $r > T$, $\psi(r, k) = \psi(T, k)$ if $r > T$ and we can assume that $r \leq T$. Because $\|u\|_{L^2[0,\infty)} \leq C_I(\operatorname{Im} k)^{-1}$ we have $\|u''\|_{L^2[0,\infty)} \leq C_I(\operatorname{Im} k)^{-1}$ from the equation $-u'' + Vu = k^2 u + F$. Then, $\|u\|_{L^\infty[0,\infty)} \leq C_I(\operatorname{Im} k)^{-1}$ as follows from the standard Sobolev's embedding. Since $\|\psi(r, k)\| = e^{(\operatorname{Im} k)r}\|u(r, k)\|$, this gives us the statement of the lemma because

$$(\operatorname{Im} k)r \leq (\operatorname{Im} k)T \leq (\operatorname{Im} k)^{-(1-\alpha)/\alpha}$$

and

$$(\operatorname{Im} k)^{-1} \exp((\operatorname{Im} k)^{-(1-\alpha)/\alpha}) \leq C_{\alpha,I} \exp(2(\operatorname{Im} k)^{-(1-\alpha)/\alpha}). \qquad \square$$

Remark. Notice that this lemma only requires that $\|V\| \in L^\infty(\mathbb{R}^+)$ and $\operatorname{supp} V \subset [0, T]$.

Next, we will study $\psi(r, k)$ when $r \in [T/2, T]$. In particular, we will be interested in how $\|\psi(r, k)\|$ deviates from $\|\psi(T/2, k)\|$ when $r > T/2$, $k \in R_{I,1}$, and $\operatorname{Im} k$ is small. Our basic tool is the following integral identity.

Lemma 2.2. *Let* $1 < a < b$ *and* $\operatorname{Re} k > 0$, $\operatorname{Im} k > 0$. *Then*

$$\|\psi(b,k)\|^2+\frac{\operatorname{Im} k}{|k|^2}\int_a^b \|\psi'\|^2 d\rho = \|\psi(a,k)\|^2+Q_1-Q_2-\frac{\operatorname{Im} k}{|k|^2}\int_a^b \langle V\psi, \psi\rangle d\rho \tag{2.3}$$

where

$$Q_1 := \frac{i}{2k}\langle \psi'(b,k), \psi(b,k)\rangle - \frac{i}{2k}\langle \psi(b,k), \psi'(b,k)\rangle$$

and

$$Q_2 := \frac{i}{2k}\langle \psi'(a,k), \psi(a,k)\rangle - \frac{i}{2k}\langle \psi(a,k), \psi'(a,k)\rangle .$$

Proof. Take the inner product of both sides in (2.2) with ψ and integrate from a to b. Then, take the real part of the resulting identity. We get

$$\begin{aligned} &\|\psi(b,k)\|^2 \\ &\quad = \|\psi(a,k)\|^2 + \frac{i}{2k}\int_a^b \langle \psi'', \psi\rangle d\rho - \frac{i}{2k}\int_a^b \langle \psi, \psi''\rangle d\rho - \frac{\operatorname{Im} k}{|k|^2}\int_a^b \langle V\psi, \psi\rangle d\rho. \end{aligned}$$

Integration by parts gives the statement of the lemma. □

Remark. In (2.3), an additional condition $V \geq 0$ immediately provides and a priori estimate on $\int_1^\infty \|\psi'\|^2 d\rho$ with essentially no assumptions on the decay of V.

The following lemma is straightforward.

Lemma 2.3. *Let* Y *and* A *be two* $\ell^2(\mathbb{N})$*-valued functions defined on* $[a, \infty)$ *that satisfy* $\|Y\|, \|Y'\|, \|A\| \in L^2[a, \infty)$ *and*

$$Y = \frac{i}{2k}Y' + A, \qquad \operatorname{Im} k > 0.$$

Then,

$$\|Y\|_{L^\infty[a,\infty)} \lesssim \frac{|k|\|A\|_{L^2[a,\infty)}}{\sqrt{\operatorname{Im} k}}, \qquad \|Y\|_{L^2[a,\infty)} \lesssim \frac{|k|\|A\|_{L^2[a,\infty)}}{\operatorname{Im} k}. \tag{2.4}$$

Proof. We have $Y' = -2ikY + 2ikA$. If Ψ is defined by $\Psi := e^{2ikr}Y$, then $\Psi = -2ik\int_r^\infty A(s)e^{2iks}ds$. In the end, one has

$$Y = -2ike^{-2ikr}\int_r^\infty A(s)e^{2kis}ds.$$

Applying the convolution bounds, we get our lemma. □

If $T > 1$, we arrange for two positive numbers $\mathcal{L}_T$ and ℓ_T such that $\ell_T < \mathcal{L}_T$, $\ell_T := T^{1-2\gamma+2\delta_1}$ and $\mathcal{L}_T := T^{\gamma-1-\delta_1}$, where δ_1 is a positive parameter (e.g., take $\delta_1 = \frac{\gamma}{2} - \frac{1}{3}$). Its choice is possible since $\gamma \in (\frac{2}{3}, 1)$. Given any closed interval $I \subset \mathbb{R}^+$, define the set

$$PC_{I,T} := R_{I,1} \cap \{k : \ell_T \le \operatorname{Im} k \le \mathcal{L}_T\}.$$

We will refer to $PC_{I,T}$ as the **zone of perfect control**. The reader will see that this name is justified from the next two results.

Lemma 2.4. *For $k \in PC_{I,T/2}$, we have*

$$\|\psi(T,k)\|^2 = \|\psi(T/2,k)\|^2(1+\epsilon_T), \quad \epsilon_T \le C_I T^{-\delta_1} \tag{2.5}$$

where $\delta_1 > 0$.

Proof. We introduce $M := \sup_{r>T/2} \|\psi(r,k)\|$. Let $k \in R_{I,1}$. Applying Lemma 2.3 to (2.2) on the interval $[T/2, \infty)$, one has

$$\|\psi'\|_{L^\infty[T/2,\infty)} \le C_I M \frac{T^{0.5-\gamma}}{\sqrt{\operatorname{Im} k}}.$$

Hence,

$$\sup_{a,b>T/2} \|Q_{1(2)}\| \le C_I M^2 \frac{T^{0.5-\gamma}}{\sqrt{\operatorname{Im} k}}.$$

By the same Lemma 2.3,

$$\|\psi'\|_{L^2[T/2,\infty)} \le C_I M \frac{T^{0.5-\gamma}}{\operatorname{Im} k}.$$

Taking the supremum in $b \ge T/2$ in (2.3) and letting $a = T/2$, we get

$$|M^2 - \|\psi(T/2,k)\|^2| \le C_I \Big((\operatorname{Im} k) T^{1-\gamma} + \frac{T^{0.5-\gamma}}{\sqrt{\operatorname{Im} k}} + \frac{T^{1-2\gamma}}{\operatorname{Im} k}\Big) M^2.$$

Thus, one has

$$\begin{aligned} &M^2 = \|\psi(T/2,k)\|^2(1+\epsilon_T), \\ &\epsilon_T \le C_I\Big((\operatorname{Im} k)T^{1-\gamma} + \frac{T^{0.5-\gamma}}{\sqrt{\operatorname{Im} k}} + \frac{T^{1-2\gamma}}{\operatorname{Im} k}\Big) \le C_I T^{-\delta_1} \end{aligned} \tag{2.6}$$

for given k. Now, we can take $b = T$, $a = T/2$ in (2.3) and use the bound on M to get the desired statement. □

We just saw that the $\| \psi(r, k)\|$ does not change much in r when $r \in [T/2, T]$ and k is fixed in the zone of perfect control. Next, we set up the iteration scheme which will play the key role in the proof of the main result. Suppose $T_n = 2^n$, $n \geq n_0$ where n_0 is a large parameter which will be fixed later. Given $V : \|V\| \leq \lambda(1+r)^{-\gamma}$, $\gamma > \frac{2}{3}$, we let

$$V_{(n)} := V \cdot \chi_{[0,T_n]}, \quad H_{(n)} := H^{(0)} + V_{(n)}, \quad \psi_n := e^{-ikr} R_{(n),k^2} F\,, \tag{2.7}$$

where function F has been chosen in the beginning of this section and $R_{(n),z} := (H_{(n)} - z)^{-1}$. The next lemma estimates $\psi_n(\infty, k)$ in the $(n-1)$-th zone of perfect control.

Lemma 2.5. *Let I be a closed interval in $\mathbb{R}^+$. If $k \in PC_{I,T_{n-1}}$, then*

$$\| \psi_n(\infty, k)\| = \| \psi_{n-1}(\infty, k)\|(1 + \epsilon_n'), \quad |\epsilon_n'| \leq C_I T_n^{-\delta_2}$$

where δ_2 is a positive parameter.

Proof. Recall that $\psi_j(T_j, k) = \psi_j(\infty, k)$ for every j. By the previous lemma, it is enough to show that

$$\| \psi_n(T_n/2, k)\| = \| \psi_{n-1}(\infty, k)\|(1 + O(T_n^{-\delta_3})) \tag{2.8}$$

where $k \in PC_{I,T_{n-1}}$ and δ_3 is a positive fixed number independent of n. To do that, we will use Lemma 5.2. Recall that $H_{(n)} = H_{(n-1)} + V \cdot \chi_{[T_{n-1},T_n]}$ and

$$R_{(n),k^2} F = R_{(n-1),k^2} F - R_{(n),k^2}(V \cdot \chi_{[T_{n-1},T_n]}) R_{(n-1),k^2} F.$$

Multiply both sides by e^{-ikr} and recall the definition of ψ_n in (2.7). Since $\psi_{n-1}(r, k) = \psi_{n-1}(\infty, k)$ for $r \in [T_{n-1}, \infty)$ and k is in the zone of perfect control, we can apply Lemma 5.2 to $R_{(n),k^2}$. This yields

$$\begin{aligned} &\| \psi_n(T_n/2, k) - \psi_{n-1}(\infty, k)\| \\ &\quad\leq C_I \| \psi_{n-1}(\infty, k)\| \int_{T_{n-1}}^{T_n} e^{(\operatorname{Im} k)(T_{n-1} - c(\rho - T_{n-1}) - \rho)} T_n^{-\gamma} d\rho \\ &\quad\leq C_I T_n^{-\gamma} (\operatorname{Im} k)^{-1} \| \psi_{n-1}(\infty, k)\| \\ &\quad\leq C_I T_n^{\gamma - 1 - 2\delta_1} \| \psi_{n-1}(\infty, k)\|, \end{aligned} \tag{2.9}$$

because $k \in PC_{I,T_{n-1}}$. Putting together (2.8) and (2.9) gives the desired result. $\square$

3 Iteration and the proof of the main theorem

Recall that F is chosen to satisfy (2.1). First, we need an auxiliary lemma.

Lemma 3.1. *Suppose* $\|V\| \in L^\infty(\mathbb{R}^+)$ *and* ψ_n *is defined as in* (2.7)*. Then,*

$$\sup_{0\le y\le 1}\int_I \|\psi_n(\infty, x+iy)\|^2 dx < \infty, \qquad \inf_{0\le y\le 1}\int_I \log\|\psi_n(\infty, x+iy)\|^2 dx > -\infty$$

for every closed interval $I \subset \mathbb{R}^+$.

Proof. Since $V_{(n)}$ is compactly supported, $\psi_n(\infty, k)$ has continuous extension to any closed interval on the real line, and $\psi_n \not\equiv 0$. It is also analytic in k in every rectangle $R_{I,1}$ so the lemma follows from, e.g., the mean-value estimate for subharmonic function $\log\|\psi_n(\infty, k)\|$. □

To begin the iterative process which will be the key to the proof of our main result, we start with taking I, any closed interval in $\mathbb{R}^+$. Then, for this I, we choose $n_0 \in \mathbb{N}$, a fixed large parameter whose dependence on I will be specified later, and define two numbers A_{n_0} and B_{n_0} as follows:

$$(3.1)\qquad \begin{aligned} A_{n_0} &:= \sup_{0<y<\mathcal{L}_{T_{n_0}}}\int_I \|\psi_{n_0}(\infty, x+iy)\|^2 dx,\\ B_{n_0} &:= \sup_{0<y<\mathcal{L}_{T_{n_0}}}\int_I \log\|\psi_{n_0}(\infty, x+iy)\| dx. \end{aligned}$$

From the last lemma, one knows that $A_{n_0} < \infty$ and $B_{n_0} > -\infty$ for every n_0. Next, we define the sequence of intervals $\{I_{(n)}\}$, $n \ge n_0$ by conditions

$$(3.2)\qquad I_{(n_0)} := I, \quad c_{I_{(n)}} = c_I, \quad |I_{(n)}| = |I_{(n-1)}| - 2\tau_n$$

and $\tau_n = T_n^{-\upsilon}$, where $0 < \upsilon < 0.01(-\gamma + 1 + \delta_1)$ so $\mathcal{L}_n = T_n^{\gamma-1-\delta_1} \ll \tau_n = T_n^{-\upsilon}$, see Figure 1. Notice that

$$\sum_{n\ge n_0}\tau_n = \sum_{n\ge n_0} 2^{-\upsilon n} \sim C_\upsilon 2^{-\upsilon n_0}$$

and $\lim_{n_0\to\infty} 2^{-\upsilon n_0} = 0$. Therefore, if I is given, we can always arrange for n_0 large enough that $\mathcal{L}_{T_{n_0}} < 1$ and that there is $\widetilde{I}_{(n_0)}$:

$$(3.3)\qquad c_{\widetilde{I}_{(n_0)}} = c_I, \quad \widetilde{I}_{(n_0)} \subset \bigcap_{n\ge n_0} I_{(n)}, \quad \lim_{n_0\to\infty} |I\backslash \widetilde{I}_{(n_0)}| \to 0.$$

Let us collect what we already know about the sequence $\{\psi_n\}$ below:

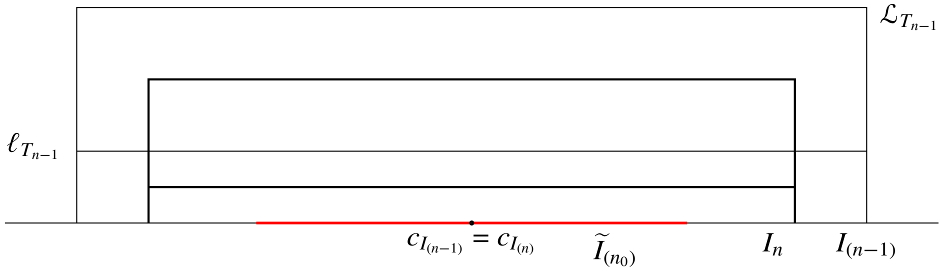

Figure 1. $R_{I_{(n-1)},T_{n-1}}$ and $R_{I_{(n)},T_n}$.

- **Rough upper bound**, Lemma 2.1:

$$\|\psi_n(\infty,k)\| \le C(I',\alpha)\exp(2(\operatorname{Im}k)^{-(1-\alpha)/\alpha}), \quad k \in R_{I',\mathcal{L}_{T_n}}, \tag{3.4}$$

where I' can be chosen as any open interval in $\mathbb{R}^+$ that contains $I_{(n_0)} = I$. The parameter α is related to γ by $\alpha = 1 + \delta_1 - \gamma$.
- **The first step**: by construction, A_{n_0} and B_{n_0} are defined for every n_0.
- **Estimate in the zone of perfect control**, Lemma 2.5: if $k \in PC(I, T_{n-1})$, then

$$\|\psi_n(\infty,k)\| = \|\psi_{n-1}(\infty,k)\|(1+\epsilon_n'), \quad |\epsilon_n'| \le C_I T_n^{-\delta_2}. \tag{3.5}$$

- **Uniform bounds on the real line**, formula (1.4): for every $I' \subset \mathbb{R}^+$, we get

$$\sup_{n\ge n_0} \int_{I'} \|\psi_n(\infty,k)\|^2 dk < C_{I'}. \tag{3.6}$$

To control $\psi_n(\infty,k)$ in $R_{I_{(n)},\mathcal{L}_{T_n}}$, one can use apriori estimates (3.4), (3.6) along with (3.5). To interpolate the bounds on $\psi_n(\infty,k)$ from the zone of perfect control all the way to $R_{I_{(n)},\mathcal{L}_{T_n}}$, we will use a few estimates on the subharmonic functions that are collected and proved in the Appendix for the reader's convenience. Our immediate goal is to prove the following lemma.

Lemma 3.2. *For every closed interval $J \subset \mathbb{R}^+$, we have the estimates*

$$\limsup_{n\to\infty} \sup_{0<y<\mathcal{L}_{T_n}} \int_J \|\psi_n(\infty,x+iy)\|^2 dx < \infty \tag{3.7}$$

and

$$\|\psi_n(\infty,x+iy)\|^2 \le C_J(1+y^{-1}+(\mathcal{L}_{T_n}-y)^{-1}), \quad x \in J,\ 0<y<\mathcal{L}_{T_n}. \tag{3.8}$$

Proof. We start with any interval I and define the sequence $\{I_{(n)}\}$ as before in (3.2). For each $n \geq n_0$, one lets

$$A_n := \sup_{0<y<\mathcal{L}_{T_n}} \int_{I_{(n)}} \|\psi_n(\infty, x+iy)\|^2 dx.$$

We will control how A_n changes when n is increased by one. Given $n-1$ and A_{n-1}, the goal is to estimate A_n. To do that, we apply (3.5) and write

$$\begin{aligned}\sup_{\ell_{T_{n-1}}<y<\mathcal{L}_{T_{n-1}}} &\int_{I_{(n-1)}} \|\psi_n(\infty, x+iy)\|^2 dx \\ &\leq (1+\epsilon_n')^2 \sup_{\ell_{T_{n-1}}<y<\mathcal{L}_{T_{n-1}}} \int_{I_{(n-1)}} \|\psi_{n-1}(\infty, x+iy)\|^2 dx \\ &\leq A_{n-1}(1+\epsilon_n')^2.\end{aligned}$$

Next, we apply (3.4), (3.6), and Lemma 4.3 with $\kappa = (1-\alpha)/\alpha$, $\delta \sim \tau_n$, $\epsilon_1 \sim \mathcal{L}_{T_{n-1}}$, $\epsilon_2 = \ell_{T_{n-1}}$ to get

$$\sup_{0<y<\ell_{T_{n-1}}} \int_{I_{(n)}} \|\psi_n(\infty, x+iy)\|^2 dx \leq C_{I'} + O(T_n^{-\delta_4}(1+C_{I'}+A_{n-1})), \quad \delta_4 > 0.$$

In the end, we have

$$A_n \leq \max\{C_{I'} + O(T_n^{-\delta_4}(1+C_{I'}+A_{n-1})), A_{n-1}(1+O(T_n^{\delta_5}))\}$$

with positive δ_4 and δ_5. That is supplemented by fixing A_{n_0}. The previous bound yields

$$A_n \leq A_{n-1}(1+O(T_n^{\delta_5})) + O(T_n^{-\delta_4})$$

and $A_n \leq C_I A_{n_0}$. Consequently,

$$\limsup_{n\to\infty} \sup_{0<y<\mathcal{L}_{T_n}} \int_{\widetilde{I}_{(n_0)}} \|\psi_n(\infty, x+iy)\|^2 dx < \infty. \tag{3.9}$$

Due to (3.3), we can start with any J, choose I that contains it and then n_0 so large that $\widetilde{I}_{(n_0)}$ contains J too. That will give us the first statement of the lemma. Now, the bound (3.8) follows from (4.8). □

Lemma 3.3. *For every closed interval $J \subset \mathbb{R}^+$, we have an estimate*

$$\liminf_{n\to\infty} \int_J \log \|\psi_n(\infty, k)\| dk > -\infty. \tag{3.10}$$

Proof. As in the previous proof, we define

$$B_n := \inf_{0<y<\mathcal{L}_{T_n}} \int_{I_{(n)}} \log \|\psi_n(\infty, x+iy)\| dx.$$

We will control how B_n changes when n is increased by one. Given B_{n-1} and the previous lemma, we want to estimate B_n. To control $\log\|\psi_n(\infty, x+iy)\|$ in the upper part of $R_{I_{(n)},T_n}$, we use estimates in $PC(I_{n-1}, T_{n-1})$. Applying (3.5), one has

$$\begin{aligned} &\inf_{\ell_{T_{n-1}}\le y\le \mathcal{L}_{T_n}} \int_{I_{(n-1)}} \log\|\psi_n(\infty, x+iy)\|dx \\ &\qquad = O(\epsilon_n') + \inf_{\ell_{T_{n-1}}\le y\le \mathcal{L}_{T_n}} \int_{I_{(n-1)}} \log\|\psi_{n-1}(\infty, x+iy)\|dx \\ &\qquad \ge O(\epsilon_n') + B_{n-1}. \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\inf_{\ell_{T_{n-1}}\le y\le \mathcal{L}_{T_n}} \int_{I_{(n)}} \log\|\psi_n(\infty, x+iy)\|dx \\ &\qquad = O(\epsilon_n') + \inf_{\ell_{T_{n-1}}\le y\le \mathcal{L}_{T_n}} \int_{I_{(n)}} \log\|\psi_{n-1}(\infty, x+iy)\|dx. \end{aligned}$$

Notice that for the chosen range of y we have

$$\begin{aligned} &\int_{I_{(n)}} \log\|\psi_{n-1}(\infty, x+iy)\|dx \\ &\qquad = \int_{I_{(n-1)}} \log\|\psi_{n-1}(\infty, x+iy)\|dx - \int_{I_{(n-1)}\backslash I_{(n)}} \log\|\psi_{n-1}(\infty, x+iy)\|dx \end{aligned}$$

and

$$-\int_{I_{(n-1)}\backslash I_{(n)}} \log\|\psi_{n-1}(\infty, x+iy)\|dx \ge -\int_{I_{(n-1)}\backslash I_{(n)}} \log_+\|\psi_{n-1}(\infty, x+iy)\|dx.$$

Then,

$$\int_{I_{(n-1)}\backslash I_{(n)}} \log_+\|\psi_{n-1}(\infty, x+iy)\|dx \lesssim_I \tau_n^{\frac{1}{2}}$$

as follows from the estimate $\log_+ t \le |t|$, Cauchy–Schwarz inequality, (3.7), and the bound $|I_{(n)}\backslash I_{(n-1)}| \lesssim \tau_n$. In the end, we get

$$\inf_{\ell_{T_{n-1}}\le y\le \mathcal{L}_{T_n}} \int_{I_{(n)}} \log\|\psi_n(\infty, x+iy)\|dx \ge B_{n-1} + O(\tau_n^{\frac{1}{2}}) + O(\epsilon_n').$$

To control the integral for the smaller values of y, i.e., when $y < \ell_{T_{n-1}}$, we apply Lemma 4.4 with $\epsilon_1 = \mathcal{L}_{T_n}$, $\epsilon_2 = 2\ell_{T_{n-1}}$ and $\delta \sim \tau_n$. The base of the smaller rectangle is $I_{(n)}$ and the base of the larger one is $I_{(n-1)}$. Given Lemma 3.2, we can write

$$\begin{aligned} &\inf_{0<y<\ell_{T_{n-1}}} \int_{I_{(n)}} \log\|\psi_n(\infty, x+iy)\|dx \\ &\qquad \ge (1+O(T_n^{-\delta_6})) \int_{I_{(n-1)}} \log\|\psi_n(\infty, x+2i\ell_{T_{n-1}})\|dx - O(T_n^{-\delta_7}). \end{aligned}$$

with positive δ_6 and δ_7. For the integral on the right-hand side, apply (3.11). In the end, one has

$$B_n \geq (1 + O(T_n^{-\delta_8}))B_{n-1} + O(T_n^{-\delta_9}), \quad \delta_8 > 0, \delta_9 > 0.$$

Consequently, $\liminf_{n\to\infty} B_n > -\infty$ and thus

$$\liminf_{n\to\infty} \int_{I_{(n)}} \log \|\psi_n(\infty, x)\| dx > -\infty.$$

Since

$$\int_{I_{(n)}} \log \|\psi_n(\infty, x)\| dx = \int_{I_{(n)}} \log_- \|\psi_n(\infty, x)\| dx + \int_{I_{(n)}} \log_+ \|\psi_n(\infty, x)\| dx$$

and (3.6) guarantees that $\limsup_{n\to\infty} \int_{I_{(n)}} \log_+ \|\psi_n(\infty, x)\| dx < \infty$, we have

$$\liminf_{n\to\infty} \int_{\widetilde{I}_{(n_0)}} \log_- \|\psi_n(\infty, x)\| dx \geq \liminf_{n\to\infty} \int_{I_{(n)}} \log_- \|\psi_n(\infty, k)\| dk > -\infty.$$

The reasoning given at the end of the proof of the previous lemma can be used again to deduce (3.10). □

The last two results provide the crucial estimates for $\|\psi_n(\infty, k)\|$ when $\operatorname{Im} k \in (0, \mathcal{L}_{T_n})$. They control the behavior of $\|(R_{(n),k^2}F)(r)\|$ for large r without giving precise asymptotics for $(R_{(n),k^2}F)(r)$. That, however, is enough to prove Theorem 1.1.

Proof of Theorem 1.1. Take any closed interval $J \subset \mathbb{R}^+$ and recall that $V_{(n)} = V \cdot \chi_{r<T_n}$. Define $\sigma_{(n),F}$, the spectral measure of F relative to $H_{(n)} = H^{(0)} + V_{(n)}$. The spectral measure of F relative to H is σ_F. Then, the previous lemma yields

$$\liminf_{n\to\infty} \int_{\Delta^2} \log \sigma'_{(n),F}(E) dE > -\infty\,.$$

Since $\lim_{n\to\infty} \|R_{(n),z}F - R_z F\|_{\mathcal{H}} = 0$, $z \in \mathbb{C}^+$, we get $\sigma_{(n),F} \to \sigma_F$ in the weak-(∗) sense. Hence (see [12], Section 5),

$$\int_{\Delta^2} \log \sigma'_F dE > -\infty$$

which implies that Δ^2 supports the a.c. spectrum of the original H. Since Δ was arbitrary, we get the statement of the theorem. □

4 Appendix 1: some estimates on subharmonic functions

For the reader's convenience, we collect some elementary estimates on subharmonic functions in this appendix. Start with the estimates for the subharmonic function of a thin isosceles trapezoid. We denote this trapezoid by $\mathcal{T}_{I,\epsilon,\beta}$ where the height is ϵ, the side angles at the lower base are both equal to π/β, and the projection of the upper base to the real line is a given interval $I \subset \mathbb{R}$. First, we will need some estimates on the harmonic measure of that trapezoid. It is instructive to start with giving the exact formula for the harmonic measure of the infinite tube which is an "infinitely long" rectangle. If $Cyl_\epsilon := \{k : 0 < \operatorname{Im} k < \epsilon\}$, then the density of the harmonic measure on its lower side is

$$\omega_k'(t) = \frac{1}{2\epsilon}\frac{\sin(\pi\epsilon^{-1}y)}{\cosh(\pi\epsilon^{-1}(x-t)) - \cos(\pi\epsilon^{-1}y)}, \quad t \in \mathbb{R},\ k = x + iy \in Cyl_\epsilon. \tag{4.1}$$

That formula can be verified directly. Let $\Gamma := \partial\mathcal{T}_{J,\epsilon,\beta} = \Gamma_1 \cup \cdots \cup \Gamma_4$, where Γ_1 is an upper base, Γ_2 the lower base, Γ_3 the left leg, and Γ_4 the right leg of the trapezoid. Denote the harmonic measure at point k by ω_k.

Lemma 4.1. *Suppose that $\Gamma_2 = [0, 2]$ and the positive parameters β, ϵ, δ are chosen such that $\beta > 2$, $\beta \sim 1$, $\epsilon < \delta^2 \ll 1$, $k = x + iy \in R_{(\delta,2-\delta),0.5\epsilon}$, and $\xi \in \Gamma$. Then, the derivative of the harmonic measure in the corresponding trapezoid with respect to its arclength satisfies*

$$\xi = s + \epsilon i \in \Gamma_1, \qquad \omega_k'(\xi) \lesssim \frac{\epsilon^{-2}y}{\cosh(\pi\epsilon^{-1}(x-s))}, \tag{4.2}$$

$$\xi = s \in \Gamma_2, \qquad \omega_k'(\xi) \le \frac{y}{\pi((s-x)^2 + y^2)}, \tag{4.3}$$

$$\xi = te^{i\pi/\beta} \in \Gamma_3, \qquad \omega_k'(\xi) \le C_\beta \frac{(xt)^{\beta-1}y}{(t^2 + x^2)^\beta}, \tag{4.4}$$

$$\xi = 2 + te^{i(\pi-\pi/\beta)} \in \Gamma_4,\ x < 1 \qquad \omega_k'(\xi) \le C_\beta y t^{\beta-1}. \tag{4.5}$$

Proof. See Figure 2.

Recall the following monotonicity property of harmonic measure. If $\Omega_1 \subset \Omega_2$ and $E \subset \partial\Omega_1 \cap \partial\Omega_2$, then $\omega_{k,\Omega_1}(E) \le \omega_{k,\Omega_2}(E)$ for $k \in \Omega_1$ ([6], p. 36) where $\omega_{k,\Omega}$ denotes the harmonic measure at point k relative to the domain Ω. This monotonicity helps us get the required upper bounds by comparing with the harmonic measure of an angle, an infinite cylinder, or a half-plane. We obtain (4.2) by comparing with an infinite cylinder, and (4.3) by comparing with the upper half-plane. The other two formulas are deduced by making a comparison with an infinite angle. □

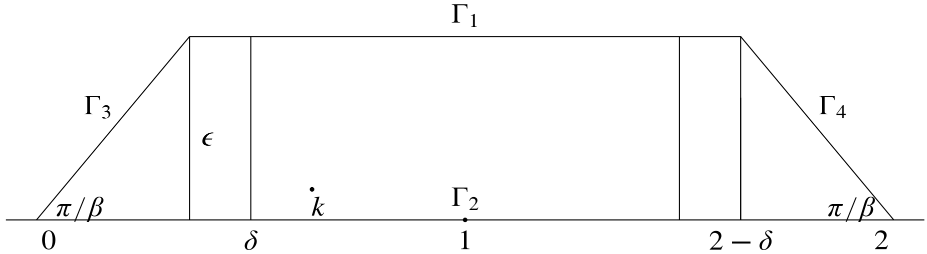

Figure 2. $\epsilon < \delta^2 \ll 1$.

Remark. The estimates in the upper part of the rectangle can be obtained in a similar way.

We will need the following result later. Recall that I_r denotes the interval on the real line with radius r centered at the origin.

Lemma 4.2. *Suppose the positive parameters $\epsilon_2, \epsilon_1, \delta$ satisfy*

$$2\epsilon_2 < \epsilon_1 < \delta^2 \ll 1$$

and let ω_k be a harmonic measure for $R_{I_{1+\delta},\epsilon_1}$. Then, for $k = x + i\epsilon_2$, we have

$$\sup_{|\xi|<1-\delta} \left| \int_{I_1} \omega'_{x+i\epsilon_2}(\xi)dx - 1 \right| \lesssim \epsilon_2\epsilon_1^{-1}. \tag{4.6}$$

Proof. The required density of harmonic measure can be written via harmonic measure of an infinite cylinder through proper extension from $I_{1+\delta}$ to $\mathbb{R}$. The resulting formula shows that the contribution from the left and right sides of the rectangle are exponentially small and the desired density can be well approximated by the density of the harmonic measure of the infinite cylinder. Then, we use formula (4.1) to obtain the required bound. □

Lemma 4.3. *Suppose the positive parameters ϵ_1, ϵ_2 and δ satisfy*

$$2\epsilon_2 < \epsilon_1 < \delta^2 \ll 1.$$

Assume that h is an $\ell^2(\mathbb{N})$-valued function holomorphic in $R_{I_2,1}$, continuous in $\overline{R_{I_2,1}}$, and

$$\|h(k)\| \le C_1 \exp(C_2(\operatorname{Im} k)^{-\kappa}), \quad k \in R_{I_2,1},\ 1 < \kappa,\ \kappa \sim 1. \tag{4.7}$$

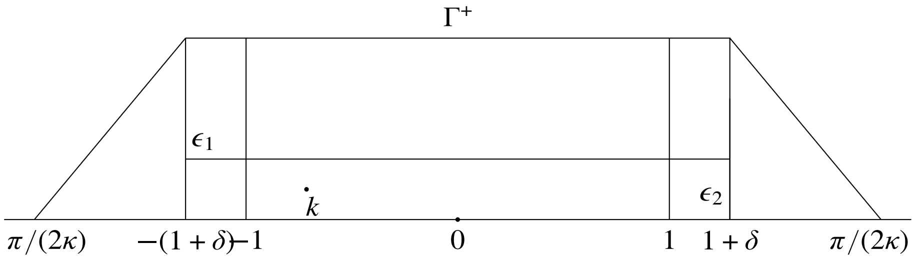

Figure 3.

Then, we have

$$\|h(x+iy)\|^2 \le C_\kappa(1+y^{-1}A+(\epsilon_1-y)^{-1}B),$$
$$A := \int_{I_2} \|h(t)\|^2dt, \quad B := \int_{-1-\delta}^{1+\delta} \|h(t+i\epsilon_1)\|^2dt, \tag{4.8}$$

provided that $k = x + iy \in R_{I_{1+\frac{\delta}{2}},\epsilon_1}$. *Moreover,*

$$\sup_{0<y<\epsilon_2} \int_{-1}^{1} \|h(x+iy)\|^2dx \le A + C_\kappa\epsilon_2\epsilon_1^{-1}(A+B+\epsilon_1). \tag{4.9}$$

Proof. See Figure 3.

We can assume $h \not\equiv 0$. Let $k = x + iy \in R_{I_{1+\frac{\delta}{2}},\epsilon_1}$. Consider the isosceles trapezoid $\mathcal{T}_{I_{1+\delta},\epsilon_1,\pi/(2\kappa)}$. Denote its upper base by Γ^+ and its lower base by Γ^-. We write the mean-value inequality for the subharmonic function $2\log_+\|h\|$ and use the estimate (4.4) on the density of harmonic measure on the legs to get

$$\begin{aligned} 2\log_+\|h(k)\| &\le 2\int_{\partial\mathcal{T}_{I_{1+\delta},\epsilon_1,\pi/(2\kappa)}} \log_+\|h\|d\omega_k \\ &\le C_\kappa y\delta^{-1-2\kappa}\epsilon_1^\kappa + 2\int_{\Gamma^+\cup\Gamma^-} \log_+\|h\|\omega_k'(\xi)d\xi \\ &\le C_\kappa + 2\int_{\Gamma^+\cup\Gamma^-} \log_+\|h\|\omega_k'(\xi)d\xi \end{aligned}$$

where we have applied the given estimates on $\|h\|$ along with $\epsilon_1 < \delta^2$. Define $Q(k) = \max\{1, \|h\|\}$ and notice that $\log Q = \log_+ Q \ge 0$ so

$$\begin{aligned} \log Q^2 &\le C_\kappa + \int_{\Gamma^+\cup\Gamma^-} (\log Q^2)\omega_k'(\xi)d\xi \le C_\kappa + \int_{\Gamma^+\cup\Gamma^-} (\log Q^2)d\mu, \\ \mu &:= \frac{\omega_k|_{\Gamma^-\cup\Gamma^+}}{\|\omega_k|_{\Gamma^-\cup\Gamma^+}\|} \ge \omega_k|_{\Gamma^-\cup\Gamma^+}. \end{aligned}$$

Taking the exponential of both sides and using Jensen's inequality

$$\exp\left(\int \log f d\mu\right) \le \int f d\mu, \quad \|\mu\| = 1$$

we get

$$Q^2 \le C_\kappa \frac{\int_{\Gamma^-\cup\Gamma^+} Q^2 \omega_k'(\xi) d\xi}{\|\omega_k|_{\Gamma^-\cup\Gamma^+}\|}.$$

For considered k, we have $\|\omega_k|_{\Gamma^-\cup\Gamma^+}\| \sim 1$. Thus,

$$\begin{aligned} Q^2 &\lesssim_\kappa 1 + \int_{-2}^{2} \frac{\pi^{-1} y}{(\xi - x)^2 + y^2} \|h(\xi)\|^2 d\xi \\ &\quad + \int_{-1-\delta}^{1+\delta} \frac{\pi^{-1}(\epsilon_1 - y)}{(\xi - x)^2 + (\epsilon_1 - y)^2} \|h(\xi + i\epsilon_1)\|^2 d\xi \\ &\lesssim_\kappa 1 + C\left(y^{-1} \int_{I_2} \|h\|^2 d\xi + (\epsilon_1 - y)^{-1} \int_{-1-\delta}^{1+\delta} \|h(\xi + i\epsilon_1)\|^2 d\xi\right). \end{aligned}$$

To obtain (4.9), we take $k \in R_{I_1,\epsilon_2}$ and apply the mean-value inequality to subharmonic function $\|h(k)\|^2$ inside the domain $R_{I_{1+\frac{\delta}{2}},\epsilon_1}$. The symbol $\Gamma^+_{I_{1+\frac{\delta}{2}},\epsilon_1}$ will stand for an upper base of this rectangle. Then,

$$\begin{aligned} \|h(k)\|^2 &\le \int_{\partial R_{I_{1+\frac{\delta}{2}},\epsilon_1}} \|h\|^2 d\omega_k \\ &\le I + \int_{I_{1+\frac{\delta}{2}}} \|h\|^2 \omega_k'(\xi) d\xi + \int_{\Gamma^+_{I_{1+\frac{\delta}{2}},\epsilon_1}} \|h\|^2 \omega_k'(\xi) d\xi. \end{aligned} \tag{4.10}$$

To estimate the first term, we use (4.8). That gives

$$\begin{aligned} I &\lesssim_\kappa \int_0^{0.5\epsilon_1} (1 + t^{-1}A + (\epsilon_1 - t)^{-1}B)\left(\frac{xyt}{(x^2 + t^2)^2}\right) dt \\ &\quad + \int_{0.5\epsilon_1}^{\epsilon_1} (1 + t^{-1}A + (\epsilon_1 - t)^{-1}B)\left(\frac{xy(\epsilon_1 - t)}{(x^2 + (\epsilon_1 - t)^2)^2}\right) dt \lesssim_\kappa (A + B + \epsilon_1) y \epsilon_1 \delta^{-3} \end{aligned}$$

as follows from (4.8) and the estimates for the harmonic measure of the rectangle. For the last term in the right hand side of (4.10), one employs the bound on the harmonic measure to write

$$\int_{\Gamma^+_{I_{1+\frac{\delta}{2}},\epsilon_1}} \|h\|^2 \omega_k'(\xi) d\xi \lesssim \int_{-1-\frac{\delta}{2}}^{1+\frac{\delta}{2}} \|h(\xi + i\epsilon_1)\|^2 \frac{\epsilon_1^{-2} y}{\cosh(\pi \epsilon_1^{-1}(x - \xi))} d\xi. \tag{4.11}$$

Next, we integrate (4.10) in $x \in I_1$. Integration of (4.11) yields

$$\int_{I_1}\left(\int_{\Gamma^+_{I_{1+\frac{\delta}{2}},\epsilon_1}} \|h\|^2\omega'_k(\xi)d\xi\right)dx$$
$$\lesssim \int_{-1-\frac{\delta}{2}}^{1+\frac{\delta}{2}} \|h(\xi+i\epsilon_1)\|^2\left(\int_{I_1} \frac{\epsilon_1^{-2}y}{\cosh(\pi\epsilon_1^{-1}(x-\xi))}dx\right)d\xi$$
$$\lesssim By\epsilon_1^{-1}.$$

The second term on the right-hand side of (4.10) contributes

$$\int_{I_1}\left(\int_{I_{1+\frac{\delta}{2}}} \|h\|^2\omega'_k(\xi)d\xi\right)dx \le \int_{I_{1+\frac{\delta}{2}}} \|h\|^2\left(\int_{I_1}\omega'_k(\xi)d\xi\right)dx \le \int_{I_2}\|h\|^2dx$$

where the estimate

$$\omega'_k(\xi) \le \frac{\pi^{-1}y}{(\xi-x)^2+y^2}$$

was used. Combining the bounds, we get (4.9) after our assumption $\epsilon_1 < \delta^2$ is taken into account. □

Lemma 4.4. *Suppose the positive parameters ϵ_1, ϵ_2 and δ are chosen such that $\epsilon_2 \le \epsilon_1|\log\epsilon_1|$, $\epsilon_1 < \delta^2$ and $\delta \ll 1$. Assume that the $\ell^2(\mathbb{N})$-valued function h is holomorphic in $R_{I_{1+\delta},\epsilon_1}$, $h \in C(\overline{R_{I_{1+\delta},\epsilon_1}})$, $h \not\equiv 0$,*

$$W := \sup_{0<y<\epsilon_1}\int_{I_{1+\delta}} \|h(x+iy)\|^2dx,$$
$$\|h(k)\|^2 \le L(y^{-1}+(\epsilon_1-y)^{-1}), \quad k = x+iy \in R_{I_{1+\delta},\epsilon_1}, \quad L > 2.$$

Then, we have

$$\inf_{0<y<\epsilon_2/2}\int_{I_{1-\delta}} \log\|h(x+iy)\|dx \ge (1+O(\epsilon_2\epsilon_1^{-1}))\left(\int_{I_1}\log\|h(x+i\epsilon_2)\|dx-\eta\right),$$
$$|\eta| < C(\epsilon_2\epsilon_1^{-1}(W^{0.5}+|\log L|+|\log\epsilon_1|)+(\delta W)^{0.5}).$$

Proof. It is enough to prove

$$\int_{I_{1-\delta}} \log\|h(x)\|dx \ge (1+O(\epsilon_2\epsilon_1^{-1}))\left(\int_{I_1}\log\|h(x+i\epsilon_2)\|dx-\eta\right). \tag{4.12}$$

Take $k = x+i\epsilon_2, x \in I_1$ and apply the mean-value inequality to the subharmonic function $\log\|h\|$ within $R_{I_{1+\delta},\epsilon_1}$. We define $\Gamma_1 = \{k : \operatorname{Re} k \in I_{1+\delta}, \operatorname{Im} k = \epsilon_1\}$, $\Gamma_2 = \{k : \operatorname{Im} k \in (0,\epsilon_1), k \in \partial R_{I_{1+\delta},\epsilon_1}\}$, $\Gamma_3 = \{k : \operatorname{Re} k \in I_{1+\delta}, \operatorname{Im} k = 0\}$. Check Figure 4.

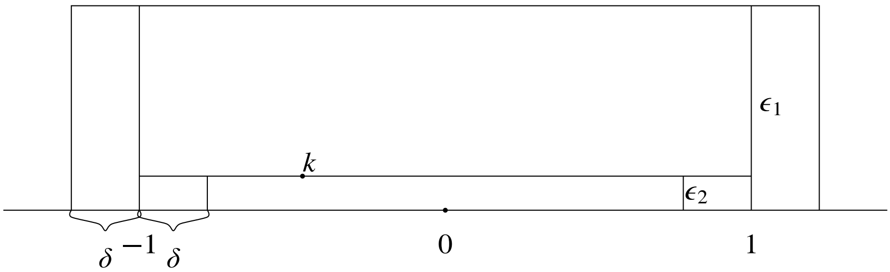

Figure 4. $\epsilon_2 \ll \epsilon_1 < \delta^2 \ll 1$.

We get

$$\int_{\Gamma_3} \log \|h\| d\omega_k \geq \log \|h(x + i\epsilon_2)\| - E_1 - E_2, \tag{4.13}$$

where

$$E_1 = \int_{\Gamma_1} \log_+ \|h\| d\omega_k, \quad E_2 = \int_{\Gamma_2} \log_+ \|h\| d\omega_k.$$

One applies the given estimates on h and the estimates on a harmonic measure to bound $E_{1(2)}$:

$$E_2 \lesssim (|\log L| + |\log \epsilon_1|)\delta^{-3}\epsilon_1^2\epsilon_2,$$
$$E_1 \lesssim \int_{I_{1+\delta}} \log_+ \|h(\xi + i\epsilon_1)\| \frac{\epsilon_1^{-2} y}{\cosh(\pi\epsilon_1^{-1}(x - \xi))} d\xi.$$

Now, we integrate (4.13) in x over I_1 and recall that $\Gamma_3 = I_{1+\delta}$. That gives

$$\int_{I_1} E_1 dx \lesssim \int_{I_{1+\delta}} \log_+ \|h(\xi + i\epsilon_1)\| \left(\int_{I_1} \frac{\epsilon_1^{-2} y}{\cosh(\pi\epsilon_1^{-1}(x - \xi))} dx \right) d\xi \leq W^{\frac{1}{2}} y \epsilon_1^{-1}.$$

Then,

$$\begin{aligned}
&\int_{I_1} \left(\int_{I_{1+\delta}} \log \|h\| d\omega_k \right) dx \\
&\qquad = \int_{I_{1+\delta}} \log \|h\| \left(\int_{I_1} \omega_k' dx \right) d\xi \\
&\qquad \leq (1 + O(\epsilon_2\epsilon_1^{-1})) \int_{I_{1-\delta}} \log \|h\| dx \\
&\qquad\quad + O(\epsilon_2\epsilon_1^{-1})) \int_{I_{1-\delta}} \log_+ \|h\| dx + \int_{I_{1+\delta}\setminus I_{1-\delta}} \log_+ \|h\| \left(\int_{I_1} \omega_k' dx \right) d\xi \\
&\qquad \leq (1 + O(\epsilon_2\epsilon_1^{-1})) \int_{I_{1-\delta}} \log \|h\| dx + C\epsilon_2\epsilon_1^{-1} W^{\frac{1}{2}} + C \int_{I_{1+\delta}\setminus I_{1-\delta}} \log_+ \|h\| d\xi
\end{aligned}$$

after we use the bound (4.6) from Lemma 4.2. Finally,

$$\int_{I_{1+\delta}\setminus I_{1-\delta}} \log_+ \|h\| d\xi \le C(\epsilon) W^{0.5}\delta^{0.5}$$

by Cauchy–Schwarz inequality. Combining the obtained estimates, we get the statement of the lemma. □

5 Appendix 2: rough bounds on Green's function

We need the following standard bounds "a la Combes-Thomas" (see, e.g., [7]) for Green's function $G(r,\rho,k^2)$ of $H = H^{(0)} + V$. In this section, we assume that I is a fixed closed interval in $\mathbb{R}^+$ and $k \in R_{I,1}$.

Lemma 5.1. *Suppose* $\|V\|_{L^\infty(\mathbb{R}^+)} < \infty$. *Then, we have*

$$\|G(r,\rho,k^2)\| \le C_I' e^{-0.5(\operatorname{Im} k)|r-\rho|} \tag{5.1}$$

for all $k \in R_{I,1}$, $\operatorname{Im} k > C_I \|V\|_{L^\infty(\mathbb{R}^+)}$ *with some* $C_I > 0$ *and* $C_I' > 0$.

Proof. This is immediate from the analysis of the perturbation identity for the Green's kernel G:

$$G(r,\rho,k^2) = G^{(0)}(r,\rho,k^2) - \int_0^\infty G^{(0)}(r,\xi,k^2)V(\xi)G(\xi,\rho,k^2)d\xi.$$

Multiply both sides by $e^{0.5(\operatorname{Im} k)|r-\rho|}$ and apply the contraction mapping principle in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$. We use (1.2) to get

$$e^{0.5(\operatorname{Im} k)|r-\rho|}\int_0^\infty e^{-(\operatorname{Im} k)|r-\xi|}\|V(\xi)\|e^{-0.5(\operatorname{Im} k)|\xi-\rho|}d\xi \le 4\|V\|_{L^\infty(\mathbb{R}^+)}(\operatorname{Im} k)^{-1}$$

and (5.1) follows provided $\operatorname{Im} k > C_I\|V\|_{L^\infty(\mathbb{R}^+)}$ with suitable C_I. □

Finally, we can focus on the lemma we need in the main text.

Lemma 5.2. *Let* $\|V\| \le \lambda(1+r)^{-\gamma}$, $H = H^{(0)} + V$, $k \in R_{I,1}$, *where* I *is a closed interval in* $\mathbb{R}^+$, $\gamma \in (0,1)$, *and* $T > 1$. *Then, there are positive* T*-independent constants* C, C_1 *and* c *such that*

$$\|G(r,\rho,k)\| < Ce^{-c(\operatorname{Im} k)|r-\rho|}$$

for $\operatorname{Im} k > C_1 T^{-\gamma}$, $0.5T < r < T$, *and* $0.5T < \rho < T$.

Proof. Define $H' = -\partial^2_{rr} + V\cdot\chi_{r>\frac{1}{4}T}$. By the previous lemma, the corresponding Green's kernel G' satisfies the bound

$$\|G'(r,\rho,k)\| \le Ce^{-0.5(\operatorname{Im} k)|r-\rho|} \tag{5.2}$$

if $\operatorname{Im} k > C_1 T^{-\gamma}$. Next, we again write the second resolvent identity

$$G(r,\rho,k^2) = G'(r,\rho,k^2) - \int_0^{\frac{1}{4}T} G(r,\xi,k^2)V(\xi)G'(\xi,\rho,k^2)d\xi.$$

For the first term, we use (5.2). To estimate the second one, we apply a general bound: for every $h \in \mathcal{H}$, one has

$$\|R_{k^2}h\|_{L^\infty(\mathbb{R}^+)} \le C(\|R_{k^2}h\|_{L^2(\mathbb{R}^+)} + \|(R_{k^2}h)''\|_{L^2(\mathbb{R}^+)}) \le C_{I,\lambda}(\operatorname{Im} k)^{-1}\|h\|_{L^2(\mathbb{R}^+)}$$

which follows from Sobolev's embedding, the equation for $R_{k^2}h$, and the Spectral Theorem. Then, since $r, \rho \in [0.5T, T]$, one deduces

$$\left\| \int_0^{\frac{1}{4}T} G(r,\xi,k^2)V(\xi)G'(\xi,\rho,k^2)d\xi \right\| \le C_{I,\lambda}(\operatorname{Im} k)^{-1}\left(\int_0^{\frac{1}{4}T} e^{-(\operatorname{Im} k)|\xi-\rho|}d\xi\right)^{\frac{1}{2}}$$
$$\le C_{I,\lambda}(\operatorname{Im} k)^{-2}e^{-0.1(\operatorname{Im} k)T}.$$

Since $\operatorname{Im} k > C_1 T^{-\gamma}$ and $\gamma \in (0,1)$, we have $(\operatorname{Im} k)^{-2}e^{-0.1(\operatorname{Im} k)T} < Ce^{-c_1(\operatorname{Im} k)T}$ with positive c_1. The result now follows because $e^{-c_1(\operatorname{Im} k)T} \le e^{-c(\operatorname{Im} k)|\rho-r|}$ with positive c provided that $0.5T < r, \rho < T$. □

REFERENCES

[1] P. Deift and R. Killip, *On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials*,Comm. Math. Phys. **203** (1999), 341–347.

[2] S. Denisov, *Continuous analogs of polynomials orthogonal on the unit circle and Kreĭn systems*, IMRS Int. Math. Res. Surv. **2006** (2006), Art. ID 54517, 148 pp.

[3] S. Denisov, *Schrödinger operators and associated hyperbolic pencils*, J. Funct. Anal. **254** (2008), 2186–2226.

[4] S. Denisov, *Multidimensional L^2 conjecture: a survey*, in *Recent Trends in Analysis*, Theta, Bucharest, 2013, pp. 101–112.

[5] S. Denisov, *Spatial asymptotics of Green's function for elliptic operators and applications: a.c. spectral type, wave operators for wave equation*, Trans. Amer. Math. Soc. **371** (2019), 8907–8970.

[6] J. Garnett and D. Marshall, *Harmonic Measure*, Cambridge University Press, Cambridge, 2008.

[7] F. Germinet and A. Klein, *Operator kernel estimates for functions of generalized Schrödinger operators*, Proc. Amer. Math. Soc. **131** (2003), 911–920.

[8] F. Gesztesy, S. Naboko, R. Weikard and M. Zinchenko, *Donoghue-type m−functions for Schrödinger operators with operator-valued potentials*. J. Analyse Math. **137** (2019), 373–427.

[9] F. Gesztesy, R. Weikard and M. Zinchenko, *Initial value problems and Weyl-Titchmarsh theory for Schrödinger operators with operator-valued potentials*. Oper. Matrices **7** (2013), 241–283.

[10] F. Gesztesy, R. Weikard and M. Zinchenko, *On spectral theory for Schrödinger operators with operator-valued potentials*. J. Differential Equations **255** (2013), 1784–1827.

[11] L. Hörmander, *The Analysis of Linear Partial Differential Operators. IV*,Springer, Berlin, 2009.

[12] R. Killip and B. Simon, *Sum rules for Jacobi matrices and their applications to spectral theory*, Ann. of Math. (2) **158** (2003), 253–321.

[13] R. Killip and B. Simon, *Sum rules and spectral measures of Schrödinger operators with L^2 potentials*, Ann. of Math. (2) **170** (2009), 739–782.

[14] A. Kiselev, Y. Last and B. Simon, *Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators*, Comm. Math. Phys. **194** (1998), 1–45.

[15] M.G. Krein, *Continuous analogues of propositions on polynomials orthogonal on the unit circle*, Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 637–640.

[16] G. Perelman, *Stability of the absolutely continuous spectrum for multidimensional Schrödinger operators*, Int. Math. Res. Not. IMRN **2005** (2005), 2289–2313.

[17] O. Safronov, *Absolutely continuous spectrum of multi-dimensional Schrödinger operators with slowly decaying potentials*, in *Spectral Theory of Differential Operators*, American Mathematical Society, Providence, RI, 2008, pp. 205–214.

[18] O. Safronov, *Absolutely continuous spectrum of the Schrödinger operator with a potential representable as a sum of three functions with special properties*. J. Math. Phys. **54** (2013), 122101, 22 pp.

[19] O. Safronov, *Absolutely continuous spectrum of a typical Schrödinger operator with a slowly decaying potential*, Proc. Amer. Math. Soc. **142** (2014), 639–649.

[20] B. Simon, *Szegő's Theorem and its Descendants. Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*, Princeton University Press, Princeton, NJ, 2011.

Sergey A. Denisov
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN-MADISON
480 NORTH LINCOLN DR.
MADISON, WI 53706, USA
email: denissov@wisc.edu

(Received March 2, 2021 and in revised form May 20, 2021)