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Quantum injectivity of multi-window Gabor frames in finite dimensions

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Abstract

Finite Gabor frames for $\mathbb{C}^N$ have been extensively studied in the context of signal processing, in particular, phase-retrieval in recent years. While the phase-retrieval problem asks to distinguish the pure states from their quantum measurements with a positive operator valued measure (POVM), the quantum detection problem asks to distinguish all the states from their measurements. Inspired by some recent work on the quantum detection problem by (discrete) frames and continuous frames, in this paper we examine the quantum detection problem with multi-window Gabor frames. We firstly obtain a necessary and sufficient condition in terms of the window vectors for a multi-window Gabor frame to be quantum injective. This generalizes the known result for the single-window case. As a consequence of this characterization, the set of all the multi-window generators $(\varphi_1, \dots, \varphi_s)$ for injective Gabor frames is Zariski dense in $\mathbb{C}^N \oplus \cdots \oplus \mathbb{C}^N$, and consequently every generic multi-window Gabor frame is injective and the set of all the injective s-window Gabor frames is stable under perturbation. In particular, we present a quantitative stability result for one of the metrics. Some examples are also provided to demonstrate the necessity of such a characterization for multi-window Gabor frames.

Keywords Quantum detection · Quantum injectivity · Multi-window Gabor frames · Frame POVM

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1 Introduction

The problem of quantum detection is to uniquely determine a state ρ (a density operator, i.e., a positive trace-one operator) from quantum measurements described as positive operator-valued measures (POVMs) acting on a state via $\{\mathrm{tr}(\nu(E)\rho)\}_{E\in\Sigma}$, where ν is a POVM (from σ-algebra Σ and taking value in the linear space of bounded linear operators on some Hilbert space $\mathcal{H}$). This problem was recently settled in Botelho-Andrade et al. [2, 3] mainly for finite or infinite but discrete frames and Han et al. [17] for continuous frames by constructing some kinds of frame POVMs. The purpose of this paper is to investigate the quantum detection in finite dimensional spaces by multi-window Gabor frames.

We firstly recall some background about the quantum detection problem and its mathematical formulations. The quantum detection problem was originally from quantum state tomography [20] which asks to recover a state from the probability of observing outcomes from quantum measurements on this state. The central issue is to find a specific POVM ν that distinguishes states from their measurements, i.e., for two given states ρ_1, ρ_2, $\mathrm{tr}(\nu(E)\rho_1) = \mathrm{tr}(\nu(E)\rho_2)$ for every measurable set E implies $\rho_1 = \rho_2$. Such POVMs are often referred as informationally complete quantum measurements [21]. In this paper, we focus on the POVMs that are induced by Hilbert space frame theory [8, 10], and we directly call them frame POVMs [18]. The problem of characterizing the frames for quantum detections will be referred to as the frame quantum detection problem.

Recall that the notation of (discrete) frames was first introduced by Duffin and Shadffer [11], viewed as some kind of "overcomplete basis" as each element can be represented via a frame but the representation might not be unique. The concept was later generalized to the continuous frame, called frame associated with measure spaces, or generalized frame [1, 14]. The class of frames in finite dimensional spaces, i.e., finite frames [10], is very important due to its significant relevance to applications and suitability for computation. Moreover, frames with good structures such as group representation frames [22] are of particular interests for both theoretical development and applications. Gabor frames, collections of modulations and translations of a single vector or multiple vectors, is such a typical example that has been extensively studied in Gabor analysis [16] and time-frequency analysis [13]. The focus of this paper is to examine the frame quantum detection problem using multi-window generated Gabor frames.

As discussed in [2], the frame quantum detection problem actually asks whether the phaselift operator associated with a frame POVM is injective on the quantum state space. For a multi-window Gabor frame, by characterizing the kernel of the phaselift operator we derive the necessary and sufficient condition for a multi-windowed Gabor frame to be injective on the quantum state space. Due to the irreducibility of the Gabor representation, this happens to be equivalent to the injectivity on the entire space $M_n(\mathbb{C})$ (Proposition 2.4). After defining a metric between frames, we get that the injective full Gabor frames is stable under perturbation and for each

s, the set of all the multi-window generators $(\varphi_1, \ldots, \varphi_s)$ for injective Gabor frames is Zariski dense in $\mathbb{C}^N \oplus \cdots \oplus \mathbb{C}^N$ (s-copies), and so every generic multi-window Gabor frame is injective.

The remainder of this paper is organized as follows: In Sect. 2, we introduce some necessary notations and the frame quantum detection problem, and point out that state-injectivity and $M_n(\mathbb{C})$-injectivity are the same if a frame contains a tight subframe. This clearly applies to finite Gabor frames since every single (nonzero) window generated sequence is a tight frame. We characterize all the injective full multi-widow Gabor frames, and show that the set of all such injective frames is open and dense in Sect. 3. At the end of this paper, we provide a few examples to show the existence of injective Gabor frames with window vectors $\varphi_1, \ldots, \varphi_s$ such that every $s-1$ of these window vectors do not generate an injective Gabor frame, and to demonstrate that, even for the single-window case, it is much more effective to our characterization than using some other known injectivity criterion or the definition of injectivity.

2 Preliminary

2.1 Notations

We will be working in $\mathbb{C}^N$. Here is a list of notations that will be used in this paper.

- Vectors $x \in \mathbb{C}^n$ will be treated as column vectors and written as $x = (x(i))_{i=0}^{N-1}$. The inner product (for convenience, we always assume that the inner produce is linear in the first place and conjugate linear in the second) between vectors x and y is defined as

$$\langle x, y \rangle = \sum_{i=0}^{N-1} x(i)\overline{y}(i).$$

- $M_{n,m}(\mathbb{C})$—the space of $n \times m$ complex matrices, abbreviated as $M_{n,m}$ ($M_N = M_{N,N}(\mathbb{C})$). M_N is a Hilbert space equipped with the Hilbert–Schmidt inner product

$$\langle A, B \rangle_{HS} = \mathrm{tr}(AB^*) = \sum_{i=0}^{N-1} \langle Ae_i, Be_i \rangle,$$

where $\left\{e_i\right\}_{i=0}^{N-1}$ is an orthonormal basis of $\mathbb{C}^N$.
- $\mathcal{S}(\mathcal{H})$—the set of states or density operators on some Hilbert space $\mathcal{H}$, i.e., $\{\rho : \rho \geq 0, \mathrm{tr}(\rho) = 1\}$. A rank-one state is called pure state.
- $\omega = e^{\frac{2\pi i}{N}}$—the Nth root of unity.
- $\mathbb{Z}_N = \{0, \ldots, N-1\}$, the ring of integers mod N. $\mathbb{Z}_N^2 = \mathbb{Z}_N \times \mathbb{Z}_N = \{(k, l) : k, l \in \mathbb{Z}_N\}$.
- For any set Λ, denote $\sharp\Lambda$ as the order of the set.

Following [18], we give the definition of operator-valued measures.

Definition 2.1 Let (Ω, Σ) be a measurable space and $\mathcal{H}$ be a separable Hilbert space. A map $\nu : \Sigma \to B(\mathcal{H})$ is an operator-valued measure (OVM) if for all collections $\{E_k\}_{k\in\mathbb{N}} \subseteq \Sigma$ with $E_i \cap E_j = \emptyset$ for $i \neq j$ we have

$$\nu\Big(\bigcup_{k\in\mathbb{N}} E_k\Big) = \sum_{k\in\mathbb{N}} \nu(E_k).$$

where the convergence on the right side of the equation above is with respect to the weak operator topology of $B(\mathcal{H})$. We say ν is

1. bounded if $\sup\{\|\nu(E)\| : E \in \Sigma\} < \infty$.
2. self-adjoint if $\nu(E)^* = \nu(E)$, for all $E \in \Sigma$.
3. positive if $\nu(E) \in B(\mathcal{H})_+$, for all $E \in \Sigma$.

In mathematical physics, ν is call a positive operator-valued measure (POVM) if it is positive and additionally $\nu(\Omega) = \mathrm{id}_{\mathcal{H}}$. We focus on the frame POVMs that are derived from the Hilbert space frame theory [8, 10].

Recall that a sequence $\{f_i\}_{i\in\mathcal{J}}$ in a finite or infinite dimensional Hilbert space $\mathcal{H}$ is called a frame if there exists $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i\in\mathcal{J}} \left|\langle x, f_i\rangle\right|^2 \leq B\|x\|^2, \quad \forall x \in \mathcal{H}.$$

The constants A, B above are the lower and upper bounds of the frame, respectively. The frame is called tight if $A = B$, and Parseval if $A = B = 1$. In finite-dimensional space case, for example, complex N-dimensional Hilbert space $\mathbb{C}^N$, it is conventional to deal with finite frames $\{f_i\}_{i=1}^M$ which are exactly the span sets, i.e., $\mathrm{span}\{f_i\}_{i=1}^M = \mathbb{C}^N$.

If $\{f_i\}_{i\in\mathcal{J}}$ is a Parseval frame, we define the operator $f_i \otimes f_i$ on $\mathcal{H}$ by $(f_i \otimes f_i)(x) = \langle x, f_i\rangle f_i$, actually $f_i \otimes f_i$ is $f_i f_i^*$. Let Σ be the σ-algebra of all subsets of $\mathcal{J}$, then define $\nu : \Sigma \to B(\mathcal{H})$ by

$$\nu(E) = \sum_{i\in E} f_i \otimes f_i = \sum_{i\in E} f_i f_i^*, \quad \forall E \in \Sigma.$$

It is easy to check that ν is a POVM. In particularly, in finite-dimensional cases, typical as $\mathbb{C}^N$, consider the POVM induced by finite frames $\{f_i\}_{i=1}^M$. Since the corresponding σ-algebra is just the set of all subsets of $\{1, 2, \dots, M\}$, it is enough to handle each singleton $1 \leq i \leq M$, that is, $f_i \otimes f_i$, rather than all the elements in the σ -algebra when dealing with the POVM as defined above.

Let $k \in \mathbb{Z}_N, l \in \mathbb{Z}_N$. The translation operator T_k and the modulation operator M_l : $\mathbb{C}^N \to \mathbb{C}^N$ are defined by

$$(T_k x)(n) = x(n-k), \quad (M_l x)(n) = \omega^{ln} x(n).$$

To avoid confusion, indexes always have to be assumed modulo N. We will use $\pi(k,l)$ to denote $M_l T_k$. Some useful facts are worth pointing out, such as the well-known commutation relations between translations and modulations [12]

$$M_l T_k = \omega^{kl} T_k M_l, \quad \pi(m,n)\pi(k,l) = \omega^{kn-ml}\pi(k,l)\pi(m,n). \tag{1}$$

Then it follows that

$$\pi(m,n)\pi(k,l) = \omega^{-nk}\pi(m+k,n+l), \quad \forall\,(m,n),(k,l) \in \mathbb{Z}_N^2,$$

which implies that π is a projective unitary representation of $\mathbb{Z}_N^2$ with the multiplier $m((m,n),(k,l)) = \omega^{-nk}$ [22]. Moreover, $\{\frac{1}{\sqrt{N}}\pi(m,n)\}_{(m,n)\in\mathbb{Z}_N^2}$ forms an orthonormal basis of M_N as

$$\langle \pi(m,n), \pi(k,l)\rangle_{HS} = N\delta_{m,k}\delta_{n,l}. \tag{2}$$

Furthermore, for every nonzero vector $\varphi \in \mathbb{C}^N$,

$$\begin{aligned}\sum_{(m,n)\in\mathbb{Z}_N^2} \left|\langle \pi(m,n)\varphi, x\rangle\right|^2 &= \sum_{(m,n)\in\mathbb{Z}_N^2} \left|\langle \pi(m,n), x\varphi^*\rangle_{HS}\right|^2 \\ &= N\|x\varphi^*\|_{HS}^2 = \left(N\|\varphi\|^2\right)\|x\|^2,\end{aligned}$$

that is, the collection $\{\pi(m,n)\varphi\}_{(m,n)\in\mathbb{Z}_N^2}$ is a tight frame with frame bound $N\|\varphi\|^2$ hence spans $\mathbb{C}^N$, which implies the irreducibility of π. Such a frame is called a (single-window) Gabor frame.

2.2 The frame quantum detection problem

As mentioned before, the quantum detection problem is to explore the existence of the quantum measurement performed by a POVM ν, which can distinguish states on some Hilbert space $\mathcal{H}$, that is, for $\rho_1, \rho_2 \in \mathcal{S}$, if

$$\mathrm{tr}(\rho_1 \nu(E)) = \mathrm{tr}(\rho_2 \nu(E)), \quad \forall E \in \Sigma,$$

it follows that $\rho_1 = \rho_2$. Let $B(\Sigma, \mathbb{R})$ denote the set of bounded functions on Σ. Then the quantum detection problem asks if there exists a POVM ν ensures the following map

$$P : \mathcal{S}(\mathcal{H}) \to B(\Sigma,\mathbb{R}) \quad P(\rho)(E) = \mathrm{tr}(\rho\nu(E)), \quad \forall E \in \Sigma.$$

to be injective.

The frame method to quantum detection problem is to find a Parseval frame which induces a frame POVM such that the corresponding map P is injective. We consider the generalization of quantum detection problem, that is, we directly work with general operators and frames may not be Parseval.

In the finite dimensional cases, we will directly work on $\mathbb{C}^N$, and assume that $\{f_i\}_{i=1}^M$ is a finite frame in $\mathbb{C}^N$. Since $\langle T, f_i f_i^* \rangle_{HS} = \langle Tf_i, f_i \rangle$, the frame quantum detection problem is formulated as: under what properties of a frame $\{f_i\}_{i=1}^M$ is the following map

$$\mathcal{P} : M_N \to \mathbb{C}^M, \quad T \mapsto (\langle Tf_i, f_i \rangle)_{1=1}^M \tag{3}$$

injective? Clearly, such a property for a frame is preserved under a linear isomorphism [2], and it is a reasonable relaxation to work with a general frame as every frame is similar to a Parseval frame [8, 10].

Definition 2.2 We say that a frame $\{f_i\}_{i\in\mathcal{J}}$ gives quantum injectivity (or is quantum injective, simply as, injective, abbreviated by QI) if the map $\mathcal{P}$ associated with $\{f_i\}_{i\in\mathcal{J}}$ is injective. Or, if

$$\langle Tf_i, f_i \rangle = 0, \quad \forall i \in \mathcal{J}$$

implies $T = 0$. Moreover, the $\mathcal{P}$ is called the phaselift operator.

Remark 2.3 (i) In the finite-dimensional complex Hilbert space case, a finite frame $\{f_i\}_{i=1}^M$ gives quantum injectivity if and only if the frame has the maximal span property [6] in the sense that $\operatorname{span}\{f_i f_i^*\} = M_N(\mathbb{C})$. Such a frame clearly distinguishes pure states, which is often referred to as a phase-retrievable frame [5]. However, a phase-retrievable frame does not necessarily give quantum injectivity.

(ii) Similarly, while a finite frame $\{f_i\}_{i=1}^M$ that gives quantum injectivity also gives state injectivity, i.e., distinguishes all the states in $\mathcal{S}(H)$, and the converse is not true. However, the following proposition points out that if $\{f_i\}_{i=1}^M$ contains a tight subframe, then these two injectivities are the same. This also explains why we can directly work with general operators but not necessarily positive trace-one operators as mentioned before since single-window Gabor frames are all tight.

Proposition 2.4 *Let $\{f_i\}_{i=1}^M$ be a frame for $\mathbb{C}^N$. If there exists a subset Λ of $\{1, \ldots, M\}$ such that $\{f_i\}_{i\in\Lambda}$ is a tight frame for $\mathbb{C}^N$ and $\{f_i\}_{i=1}^M$ is state-injective, then $\{f_i\}_{i=1}^M$ has the maximal span property, i.e., quantum-injective.*

Proof Let $T \in M_N(\mathbb{C})$ be such that $\langle T, f_i f_i^* \rangle_{HS} = 0$ for $j = 1, \ldots, N$. Write

$$T = (T_1^+ - T_1^-) + i(T_2^+ - T_2^-),$$

where T_1^+, T_1^-, T_2^+ and T_2^- are positive operators. Then $\langle T, f_i f_i^* \rangle_{HS} = 0$ implies that $\langle T_1^+, f_i f_i^* \rangle_{HS} = \langle T_1^-, f_i f_i^* \rangle_{HS}$ and $\langle T_2^+, f_i f_i^* \rangle_{HS} = \langle T_2^-, f_i f_i^* \rangle_{HS}$ for all $i = 1, \ldots, m$. Since $\{f_i\}_{i\in\Lambda}$ is a tight frame, $\sum_{i\in\Lambda} f_i f_i^* = a \cdot \mathrm{id}$ for some $a > 0$. Therefore

$$a\mathrm{Tr}(T_1^+) = \sum_{i\in\Lambda} \langle T_1^+, f_i f_i^* \rangle_{HS} = \sum_{i\in\Lambda} \langle T_1^-, f_i f_i^* \rangle_{HS} = a\mathrm{Tr}(T_1^-)$$

which implies that $\mathrm{Tr}(T_1^+) = \mathrm{Tr}(T_1^{-1})$. Similarly, $\mathrm{Tr}(T_2^+) = \mathrm{Tr}(T_2^{-1})$. Thus $T_1^+ = T_1^-$ and $T_2^+ = T_2^-$ since $\{f_i\}_{i=1}^M$ is state-injective. This implies that $T = 0$, and so $\{f_i\}_{i=1}^M$ is quantum injective. □

3 Injectivity of full Gabor frames

Recall that a *multi-window Gabor frame* is a Gabor frame $\left\{\pi(m,n)\varphi_r\right\}_{(m,n)\in\Lambda}^{1\le r\le s}$ generated by s window vectors $\varphi_1, \dots, \varphi_s$ with $\Lambda \subseteq \mathbb{Z}_N^2$. We always assume that all of the window vectors are nonzero. Moreover, if $\Lambda = \mathbb{Z}_N^2$, it is called a *full Gabor frame*. We have pointed out that $\{\pi(m,n)\varphi : (m,n) \in \mathbb{Z}_N^2\}$ is a tight frame for every nonzero window function φ. Thus, by Proposition 2.4, the state-injectivity of a full multi-window Gabor frame is the same as the quantum injectivity of the frame.

Lemma 3.1 *Let*

$$B = (\omega^{(ml-kn)})_{N^2_{(m,n)} \times N^2_{(k,l)}}.$$

Then, for each submatrix $\tilde{B}$ consisting of v columns of B, $\mathrm{rank}(\tilde{B}) = v$. *Particularly, B is invertible.*

Proof Denote that $\tilde{B} \cdot \tilde{B}^* = (a_{(p,q)(r,s)})_{v\times v}$. Then by a little arithmetic, we have

$$\begin{aligned}
a_{(p,q)(r,s)} &= \sum_{0\le k,l\le N-1} \omega^{(qk-pl)} \cdot \omega^{(rl-sk)} \\
&= \sum_{0\le k,l\le N-1} \omega^{((q-s)k+(p-r)l)} \\
&= \sum_{0\le k\le N-1} \omega^{(q-s)k} \cdot \sum_{0\le l\le N-1} \omega^{(p-r)l} \\
&= N^2 \cdot \delta(q-s) \cdot \delta(p-r).
\end{aligned}$$

Thus $\tilde{B} \cdot \tilde{B}^* = N^2 \cdot \mathrm{id}_{v\times v}$, which implies that $\mathrm{rank}(\tilde{B}) = v$. □

Remark 3.2 As is emerged in the proof above, B is unitary. Moreover, $B^2 = N^2 \mathrm{id}_{N^2\times N^2}$ provided by the observation that $B = B^*$.

The following characterizes all injective full multi-window Gabor frames.

Theorem 3.3 *Let $\left\{\pi(m,n)\varphi_r\right\}_{(m,n)\in\mathbb{Z}_N^2,\ 1\le r\le s}$ be a full multi-window Gabor frame generated by window vectors $\varphi_1, \dots, \varphi_s$ with $\mathbb{Z}_N^2$. Then the following are equivalent.*

1. $\left\{\pi(m,n)\varphi_r\right\}_{(m,n)\in\mathbb{Z}_N^2,\ 1\le r\le s}$ *is quantum injective*;
2. $\forall(k,l)\in\mathbb{Z}_N^2,\ \langle\pi(k,l)\varphi_r,\varphi_r\rangle\neq 0$ *for some* $1\le r\le s$.

Proof (1) $\Rightarrow$ (2): Suppose that $\exists(p,q)\in\mathbb{Z}_N^2$ such that $\langle\pi(p,q)\varphi_r,\varphi_r\rangle=0$ for all $1\le r\le s$.

Then, Eq. (1) implies that

$$\begin{aligned}&\langle\pi(p,q)\pi(m,n)\varphi_r,\pi(m,n)\varphi_r\rangle\\&\quad=\omega^{(pn-qm)}\langle\pi(m,n)\pi(p,q)\varphi_r,\pi(m,n)\varphi_r\rangle\\&\quad=\omega^{(pn-qm)}\langle\pi(p,q)\varphi_r,\varphi_r\rangle\\&\quad=0\end{aligned}$$

for all $(m,n)\in\mathbb{Z}_N^2$ and $1\le r\le s$.

Taking conjugation of both of the two sides of the equation, we have

$$\langle\pi(p,q)^*\pi(m,n)\varphi_r,\pi(m,n)\varphi_r\rangle=0.$$

Since $2\pi(p,q)=\pi(p,q)+\pi(p,q)^*+\pi(p,q)-\pi(p,q)^*\neq 0$, assume that $\pi(p,q)+\pi(p,q)^*\neq 0$, otherwise we can choose $i(\pi(p,q)-\pi(p,q)^*)$ for further discussion. Set $T=\pi(p,q)+\pi(p,q)^*$. Then $T\neq 0$ is self-adjoint. But $\langle T\cdot\pi(m,n)\varphi_r,\pi(m,n)\varphi_r\rangle=0$ for all $(m,n)\in\mathbb{Z}_N^2$ and $1\le r\le s$, which means that $\left\{\pi(m,n)\varphi_r\right\}_{(m,n)\in\mathbb{Z}_N^2,\ 1\le r\le s}$ is not injective, which leads to the contradiction.

(2) $\Rightarrow$ (1): Recall that $\left\{\frac{1}{\sqrt{N}}\pi(k,l)\right\}_{0\le k,l\le N-1}$ forms an orthonormal basis of M_N. Thus, for all T, we have

$$T=\frac{1}{N}\cdot\sum_{0\le k,l\le N-1}\langle T,\pi(k,l)\rangle_{HS}\cdot\pi(k,l).$$

Thus, by Eqs. (1) and (2) and a little arithmetic, we have

$$\begin{aligned}&(\langle T\pi(m,n)\varphi_r,\pi(m,n)\varphi_r\rangle)_{N^2\times 1}\\&\quad=\left(\left\langle\frac{1}{N}\cdot\sum_{0\le k,l\le N-1}\langle T,\pi(k,l)\rangle_{HS}\cdot\pi(k,l)\cdot\pi(m,n)\varphi_r,\pi(m,n)\varphi_r\right\rangle\right)_{N^2\times 1}\\&\quad=\frac{1}{N}(\langle\pi(k,l)\cdot\pi(m,n)\varphi_r,\pi(m,n)\varphi_r\rangle)_{N^2_{(m,n)}\times N^2_{(k,l)}}\cdot(\langle T,\pi(k,l)\rangle_{HS})_{N^2\times 1}\\&\quad=\frac{1}{N}\cdot B\cdot(\mathrm{diag}\,\langle\pi(k,l)\varphi_r,\varphi_r\rangle)_{N^2\times N^2}\cdot(\langle T,\pi(k,l)\rangle_{HS})_{N^2\times 1}\\&\quad=\frac{1}{N}\cdot B\cdot(\langle\pi(k,l)\varphi_r,\varphi_r\rangle\cdot\langle T,\pi(k,l)\rangle_{HS})_{N^2\times 1},\end{aligned}$$

where $B=(\omega^{(ml-kn)})_{N^2_{(m,n)}\times N^2_{(k,l)}}$.

Then, for all $1\le r\le s$, if we let $(\langle T\pi(m,n)\varphi_r,\pi(m,n)\varphi_r\rangle)_{N^2\times 1}=0$, it means that

$$\begin{pmatrix} \langle T\pi(0,0)\varphi_r, \pi(0,0)\varphi_r\rangle \\ \vdots \\ \langle T\pi(m,n)\varphi_r, \pi(m,n)\varphi_r\rangle \\ \vdots \\ \langle T\pi(N-1,N-1)\varphi_r, \pi(N-1,N-1)\varphi_r\rangle \end{pmatrix}_{N^2\times 1} = \frac{1}{N}\cdot B\cdot \begin{pmatrix} \varphi^r_{0,0}\cdot \mathcal{T}_{0,0} \\ \vdots \\ \varphi^r_{k,l}\cdot \mathcal{T}_{k,l} \\ \vdots \\ \varphi^r_{N-1,N-1}\cdot \mathcal{T}_{N-1,N-1} \end{pmatrix}_{N^2\times 1} = 0, \tag{4}$$

in which $B = (\omega^{(ml-kn)})_{N^2_{(m,n)}\times N^2_{(k,l)}}$, $\varphi^r_{m,n} = \langle \pi(m,n)\varphi_r, \varphi_r\rangle$ and $\mathcal{T}_{k,l} = \langle T, \pi(k,l)\rangle_{HS}$.

Furthermore, Eq. (4) can be rewritten as follows.

$$\begin{cases} \sum_{0\le k,l\le N-1} \varphi^r_{k,l}\mathcal{T}_{k,l} = 0 \\ \vdots \\ \sum_{0\le k,l\le N-1} \omega^{(ml-kn)}\varphi^r_{k,l}\mathcal{T}_{k,l} = 0 \\ \vdots \\ \sum_{0\le k,l\le N-1} \omega^{((N-1)l-k(N-1))}\varphi^r_{k,l}\mathcal{T}_{k,l} = 0 \end{cases}$$

Since the system of equations consists of N^2 equations with no more than N^2 variables, by Lemma 3.1, the system of equations is solvable, which means that $\mathcal{T}_{k,l} = \langle T, \pi(k,l)\rangle_{HS} = 0$ for those (k,l) such that $\langle \pi(k,l)\varphi_r, \varphi_r\rangle \ne 0$.

Since $\forall (k,l) \in \mathbb{Z}_N^2$, $\langle \pi(k,l)\varphi_r, \varphi_r\rangle \ne 0$ for some $1 \le r \le s$, we have

$$\langle T, \pi(k,l)\rangle_{HS} = 0$$

for all $0 \le k, l \le N-1$, which implies that $T = 0$.

Therefore, $\left\{\pi(m,n)\varphi_r\right\}_{(m,n)\in\mathbb{Z}_N^2,\ 1\le r\le s}$ is injective. □

Remark 3.4 (i) Recall from [21] that a POVM generated by a finite frame in $\mathbb{C}^N$ is called informationally complete if the corresponding rank-one matrices $\{\pi(m,n)\varphi\cdot(\pi(m,n)\varphi)^*\}_{(m,n)\in\mathbb{Z}_N^2}$ form a basis of $M_N(\mathbb{C})$, which coincides with the definition of quantum injectivity. Different from the linear algebraic method, we can deliver a more conceptual proof for Theorem 3.3 inspired by [15] based on the characterization of the spectrum of Gramian matrices of the rank-one projectors $\{\pi(m,n)\varphi\cdot(\pi(m,n)\varphi)^*\}_{(m,n)\in\mathbb{Z}_N^2}$. Firstly, for the single window case, denote a full Gabor frame as $\{\pi(m,n)\varphi\}_{(m,n)\in\mathbb{Z}_N^2}$. Then, the rank of the Gramian matrix

$$G_\varphi = \Big(\langle \pi(m,n)\varphi\cdot(\pi(m,n)\varphi)^*, \pi(k,l)\varphi\cdot(\pi(k,l)\varphi)^*\rangle_{HS}\Big)_{N^2\times N^2}$$

is the same with the dimension of $\mathrm{span}_{(m,n)\in\mathbb{Z}_N^2}\{(\pi(m,n)\varphi\cdot(\pi(m,n)\varphi)^*\}$. Thus it is equivalent to prove that $\mathrm{rank}(G_\varphi) = N^2$. Goldberger et al. showed in [15] that the Gramian matrix G_φ is diagonalizable with eigenvalues $\{N|\langle \pi(m,n)\varphi, \varphi\rangle|^2\}_{(m,n)\in\mathbb{Z}^2}$, then

$$\mathrm{rank}(G_\varphi) = \sharp\big\{(m,n) \in \mathbb{Z}_N^2 \,:\, \langle \pi(m,n)\varphi, \varphi\rangle \neq 0\big\},$$

which also leads to the conclusion. While for a multi-window Gabor frame $\{\pi(m,n)\varphi_r\}_{(m,n)\in\mathbb{Z}^2}^{1\le r\le s}$. Let V_r be the $N^2 \times N^2$ matrix whose columns are $\pi(m,n)\varphi_r \cdot (\pi(m,n)\varphi_r)^*$ written as vectors. Define

$$V = [V_1 \cdots V_r \cdots V_s]_{N^2\times sN^2}.$$

Then $\{\pi(m,n)\varphi_r\}_{(m,n)\in\mathbb{Z}^2}^{1\le r\le s}$ is quantum injective if and only if $\mathrm{rank}(V) = N^2$. To see that, we have

$$\mathrm{rank}(V) = \mathrm{rank}(VV^*) = \mathrm{rank}\Big(\sum_{1\le r\le s} V_r V_r^*\Big).$$

Note that $V_r^* V_r = G_{\varphi_r}$, then the matrix $V_r V_r^*$ has the same eigenvalues $\{N|\langle \pi(m,n)\varphi_r, \varphi_r\rangle|^2\}\}_{(m,n)\in\mathbb{Z}^2}$ as G_{φ_r}. The matrices $\{V_r V_r^*\}_{1\le r\le s}$ have common diagonalizing basis $\{u(m,n)_{i,j} = \delta(i-j-n)\omega^{im}\}_{(m,n)\in\mathbb{Z}^2}$. Thus VV^* is diagonalizable with eigenvalues $\{N\sum_{r=1}^{r=s}|\langle \pi(m,n)\varphi_r, \varphi_r\rangle|^2\}_{(m,n)\in\mathbb{Z}^2}$and

$$\mathrm{rank}(V) = \mathrm{rank}(VV^*) = \sharp\Big\{(m,n) \in \mathbb{Z}^2 \,:\, \sum_{1\le r\le s} |\langle \pi(m,n)\varphi_r, \varphi_r\rangle|^2 \neq 0\Big\},$$

which leads to the conclusion.

(ii) For the single window vector case, the above characterization has been known and it has been generalized to any irreducible projective representation for finite abelian groups [7, 9, 19]. It would be interesting to know if such a generalization also applies to the multi-window case.

A conclusion given by [15] says that for arbitrary $S \subseteq Z_N$ with $\sharp S > \frac{N}{2}$, there exists a unit vector g with $\mathrm{supp}(g) = S$ that gives quantum injectivity, which immediately implies the existence of injective finite Gabor frames. Since the window vector $\varphi = (\varphi_1, \dots, \varphi_s)$ that generates an injective Gabor frame is characterized by the zero sets of a finite collections of nonzero polynomials, we immediately get

Corollary 3.5 *The Gabor frame $\{\pi(k,l)\varphi_r\}_{(k,l)\in\mathbb{Z}_N^2,\ 1\le r\le s}$ is injective for every generic window vector $\varphi = (\varphi_1, \dots, \varphi_s)$ in $\mathbb{C}^N \oplus \cdots \oplus \mathbb{C}^N$, and hence the set of all such window vectors is dense in $\mathbb{C}^N \oplus \cdots \oplus \mathbb{C}^N$.*

Next we examine the stability of the injective property of Gabor frames under small perturbations. For this purpose, a metric between frames is needed. There are various metrics defined in [4]. For the same reason as in Corollary 3.5, the injective Gabor frames are preserved under small perturbation with respect to the norm $\sum_{i=1}^{s}\|\varphi_i-\psi_i\|^2$ in $\mathbb{C}^N\oplus\cdots\oplus\mathbb{C}^N$. Here we will use the following metric and present a quantitative version of the stability result with respect to this metric.

Definition 3.6 Let $\mathcal{F}_1=\left\{\pi(m,n)\varphi_r\right\}_{(m,n)\in\mathbb{Z}_N^2}^{1\le r\le s}$, $\mathcal{F}_2=\left\{\pi(m,n)\psi_r\right\}_{(m,n)\in\mathbb{Z}_N^2}^{1\le r\le s}$ be two full Gabor frames generated by s window vectors for each with $\mathbb{Z}_N^2$. Let P be the set of permutations of $\{1,\dots,s\}$. The distance between $\mathcal{F}_1$, $\mathcal{F}_2$ is defined by

$$d_{\mathcal{G}}^2(\mathcal{F}_1,\ \mathcal{F}_2)=\min_{\sigma\in P}\left(\sum_{i=1}^{s}\|\varphi_{\sigma(i)}-\psi_i\|^2\right).$$

Let $\mathcal{F}_1=\{\pi(m,n)\varphi_r\}_{(m,n)\in\mathbb{Z}_N^2,\ 1\le r\le s}$ be an injective full Gabor frame for $\mathbb{C}^N$. Set $\delta'=\min_{(k,l)\in\mathbb{Z}_N^2}\left\{\sum_{r=1}^{s}|\langle\pi(k,l)\varphi_r,\varphi_r\rangle|\right\}$ and

$$\delta=\sqrt{\sum_{r=1}^{s}\|\varphi_r\|^2+\delta'}-\sqrt{\sum_{r=1}^{s}\|\varphi_r\|^2}.$$

Proposition 3.7 *Let $\mathcal{F}_1=\{\pi(m,n)\varphi_r\}_{(m,n)\in\mathbb{Z}_N^2,\ 1\le r\le s}$ be an injective full Gabor frame for $\mathbb{C}^N$. Then every full multi-window Gabor frame $\mathcal{F}_2=\{\pi(m,n)\psi_r\}_{(m,n)\in\mathbb{Z}_N^2,\ 1\le r\le s}$ satisfying the condition $d_{\mathcal{G}}(\mathcal{F}_1,\mathcal{F}_2)<\delta$ is also injective.*

Proof A slight variation of Theorem 3.3 shows that $\mathcal{F}_1=\{\pi(m,n)\varphi_r\}_{(m,n)\in\mathbb{Z}_N^2}^{1\le r\le s}$ is injective if and only if for all $(k,l)\in\mathbb{Z}_N^2$,

$$\sum_{r=1}^{s}|\langle\pi(k,l)\varphi_r,\varphi_r\rangle|\ne 0.$$

Let σ be the permutation on the index set of $\mathcal{F}_1$ that attains the distance of $\mathcal{F}_1$ and $\mathcal{F}_2$. Then for each $(k,l)\in\mathbb{Z}_N^2$, by Cauchy–Schwarz inequality and a little arithmetic, we get that

$$
\begin{aligned}
&\sum_{r=1}^{s} |\langle \pi(k,l)\psi_r, \psi_r\rangle| \\
&\quad = \sum_{r=1}^{s} |\langle \pi(k,l)((\psi_r - \varphi_{\sigma(r)}) + \varphi_{\sigma(r)}), (\psi_r - \varphi_{\sigma(r)}) + \varphi_{\sigma(r)}\rangle| \\
&\quad = \sum_{r=1}^{s} \Big(|\langle \pi(k,l)\varphi_{\sigma(r)}, \varphi_{\sigma(r)}\rangle + \langle \pi(k,l)(\psi_r - \varphi_{\sigma(r)}), \psi_r - \varphi_{\sigma(r)}\rangle \\
&\qquad + \langle \pi(k,l)(\psi_r - \varphi_{\sigma(r)}), \varphi_{\sigma(r)}\rangle + \langle \pi(k,l)\varphi_{\sigma(r)}, \psi_r - \varphi_{\sigma(r)}\rangle| \Big) \\
&\quad \geq \sum_{r=1}^{s} \Big(|\langle \pi(k,l)\varphi_{\sigma(r)}, \varphi_{\sigma(r)}\rangle| - |\langle \pi(k,l)(\psi_r - \varphi_{\sigma(r)}), \psi_r - \varphi_{\sigma(r)}\rangle| \\
&\qquad - |\langle \pi(k,l)(\psi_r - \varphi_{\sigma(r)}), \varphi_{\sigma(r)}\rangle| - |\langle \pi(k,l)\varphi_{\sigma(r)}, \psi_r - \varphi_{\sigma(r)}\rangle| \Big) \\
&\quad \geq \sum_{r=1}^{s} |\langle \pi(k,l)\varphi_r, \varphi_r\rangle| - \Big(\sum_{r=1}^{s} \|\psi_r - \varphi_{\sigma(r)}\|^2 \Big) \\
&\qquad - 2\Big(\sum_{r=1}^{s} \|\varphi_{\sigma(r)}\| \cdot \|\psi_r - \varphi_{\sigma(r)}\| \Big) \\
&\quad \geq \sum_{r=1}^{s} |\langle \pi(k,l)\varphi_r, \varphi_r\rangle| - \Big(\sum_{r=1}^{s} \|\psi_r - \varphi_{\sigma(r)}\|^2 \Big) \\
&\qquad - 2\Big(\Big(\sum_{r=1}^{s} \|\varphi_r\|^2 \Big)^{\frac{1}{2}} \cdot \Big(\sum_{r=1}^{s} \|\psi_r - \varphi_{\sigma(r)}\|^2 \Big)^{\frac{1}{2}} \Big) \\
&\quad \geq \delta' - d_{\mathcal{G}}^2(\mathcal{F}_1, \mathcal{F}_2) - 2\Big(\sum_{r=1}^{s} \|\varphi_r\|^2 \Big)^{\frac{1}{2}} d_{\mathcal{G}}(\mathcal{F}_1, \mathcal{F}_2) \\
&\quad > 0,
\end{aligned}
$$

which implies that

$$\sum_{r=1}^{s} |\langle \pi(k,l)\psi_r, \psi_r\rangle| \neq 0, \ \forall\ (k,l) \in \mathbb{Z}_N^2.$$

Hence $\mathcal{F}_2$ is injective. □

Remark 3.8 Since we have

$$
\begin{aligned}
&\langle \pi(0,0)\varphi, \varphi\rangle = \|\varphi\|^2, \\
&\langle \pi(k,l)\varphi, \varphi\rangle = \omega^{kl} \cdot \overline{\langle \pi(N-k, N-l)\varphi, \varphi\rangle}
\end{aligned}
$$

for all $\varphi \in \mathbb{C}^N$ and $(k,l) \in \mathbb{Z}_N^2$ and the assumption that all of the window vectors are nonzero, it is enough for us to only check for $k = 0,\ 1 \leq l \leq \frac{N}{2}$; $1 \leq k \leq \frac{N-1}{2},\ 0 \leq l \leq N-1$ and $\frac{N-1}{2} < k \leq \frac{N}{2},\ 0 \leq l \leq \frac{N}{2}$ when checking the needed conditions of injective Gabor frames. Therefore, the criterion given in this paper is much more efficient than it appears.

Since $\{\pi(m,n)\varphi : (m,n) \in \mathbb{Z}_N^2\}$ is injective for every generic vector φ, it raises a natural question about the existence of injective multi-window Gabor frames such that none of the full Gabor frames generated by proper subsets of the window vectors gives the injectivity, or, specifically, how many window vectors $(\varphi_r)_{r=1}^s$ are allowed so that the injective multi-window Gabor frames $\{\pi(k,l)\varphi_r\}_{(k,l)\in\mathbb{Z}_N^2,\ 1\le r\le s}$ for $\mathbb{C}^N$ is exact, where *exact* means that it fails to be injective when removing any single window Gabor frame $\{\pi(k,l)\varphi_r\}_{(k,l)\in\mathbb{Z}_N^2}$. If we have an exact s-window Gabor frame $\{\varphi_r\}_{r=1}^s$ as desired, then each window vectors φ_r must posses a pair $(m,\ n)$ with $\varphi(m,n) := \langle \pi(m,n)\varphi, \varphi\rangle \neq 0$, which does not appear in any other windows. To estimate s, we need some observations. Firstly, due to the fact given in [15], for an arbitrary vector $\varphi \in \mathbb{C}^N \setminus \{0\}$, we have

$$\sharp\left\{(m,n) \in \mathbb{Z}_N^2 : \varphi(m,n) \neq 0\right\} \geq N.$$

Meanwhile, by Remark 3.8, we have $\varphi(m,n) \neq 0$ if and only if $\varphi(N-m, N-n) \neq 0$. If N is odd, then we have one window φ_1 with N pairs $\varphi(m,n) \neq 0$ and each $\varphi_i(2 \le i \le s)$ must add at least 2 new zero pairs, thus $N + 2(s-1) \le N^2$. While when N is even, this is same except the pairs $(0, N/2)$, $(N/2, 0)$, $(N/2, N/2)$ that may appear alone as 3 new pairs for three windows, thus $N + 2(s-1-3) + 3 \le N^2$. To conclude, if $\{\pi(k,l)\varphi_r\}_{(k,l)\in\mathbb{Z}_N^2}^{1\le r\le s}$ is an exact injective s–window Gabor frame, as s is an integer, then

$$s \le \begin{cases} \frac{N^2-N}{2} + 1, & \text{if } N \text{ is odd} \\ \frac{N^2-N}{2} + 2, & \text{otherwise}. \end{cases}$$

is necessary. However, we can still have several examples in dimension 2 and 3 as follows.

Example Consider two window vectors with their frame matrices of single-window Gabor frames with $\Lambda = \mathbb{Z}_2^2$ and

$$\varphi_1 = \begin{bmatrix} e^{\frac{\pi}{3}i} \\ e^{\frac{\pi}{2}i} \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then we have

$$\begin{aligned} \langle \pi(0,1)\varphi_1, \varphi_1\rangle &= 0, \\ \langle \pi(1,0)\varphi_1, \varphi_1\rangle &= e^{\frac{\pi}{6}i} + e^{-\frac{\pi}{6}i} \neq 0, \\ \langle \pi(1,1)\varphi_1, \varphi_1\rangle &= e^{\frac{\pi}{6}i} - e^{-\frac{\pi}{6}i} \neq 0 \end{aligned}$$

and

$$\langle \pi(0,1)\varphi_2, \varphi_2 \rangle = 1,$$
$$\langle \pi(1,0)\varphi_2, \varphi_2 \rangle = 0,$$
$$\langle \pi(1,1)\varphi_2, \varphi_2 \rangle = 0.$$

Thus, by Theorem 3.3, neither the Gabor frame generated by φ_1 nor φ_2 is injective, and clearly the bi-window Gabor frame generated by both of them is injective.

Next is an example with 3-window vectors.

Example Consider window vectors

$$\varphi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \varphi_3 = \begin{bmatrix} e^{\frac{\pi}{4}i} \\ e^{-\frac{\pi}{4}i} \end{bmatrix}.$$

If we compute $\langle \pi(k,l)\varphi_r, \varphi_r \rangle$ for $(k,l) \in \mathbb{Z}_2^2$, then a bit arithmetic shows that:

- $\langle \pi(k,l)\varphi_1, \varphi_1 \rangle \neq 0$ only for $(k,l) = (0,1)$;
- $\langle \pi(k,l)\varphi_2, \varphi_2 \rangle \neq 0$ only for $(k,l) = (1,0)$;
- $\langle \pi(k,l)\varphi_3, \varphi_3 \rangle \neq 0$ only for $(k,l) = (1,1)$;

which means that the full Gabor frame generated by any two of these window vectors is not injective while the full triple-window Gabor frame is injective.

The following is an example for $\mathbb{C}^3$ with $s = 4$

Example Consider window vectors

$$\varphi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}, \quad \varphi_3 = \begin{bmatrix} \omega \\ \omega \\ 1 \end{bmatrix}, \quad \varphi_4 = \begin{bmatrix} \omega \\ 1 \\ 1 \end{bmatrix}.$$

As an analogue of Example 3, we can compute $\langle \pi(k,l)\varphi_r, \varphi_r \rangle$ for $k = 0,\ l = 1$ and $k = 1,\ 0 \leq l \leq 2$ provided by Remark 3.8. Then we can get the following results by a little arithmetic.

- $\langle \pi(k,l)\varphi_1, \varphi_1 \rangle \neq 0$ only for $(k,l) = (0,1)$;
- $\langle \pi(k,l)\varphi_2, \varphi_2 \rangle \neq 0$ only for $(k,l) = (1,0)$;
- $\langle \pi(k,l)\varphi_3, \varphi_3 \rangle \neq 0$ only for $(k,l) = (1,1)$;
- $\langle \pi(k,l)\varphi_4, \varphi_4 \rangle \neq 0$ only for $(k,l) = (1,2)$.

Therefore, it can be concluded by Theorem 3.3 that only all of the four vectors $\{\varphi_r\}_{1 \leq r \leq 4}$ can generate an injective full Gabor frame.

The final two examples demonstrate that, even for the single-window case, it is much more effective to the characterization in Theorem 3.3 than using some other known injectivity criterion or the definition of injectivity.

Example Let $\varphi = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{C}^2$. Then the full Gabor frame generated by the window vector φ is given by

$$\begin{pmatrix} x & x & y & y \\ y & -y & x & -x \end{pmatrix}.$$

If we set $\langle \pi(k,l)\varphi, \varphi \rangle \neq 0$ for all $0 \leq k, l \leq 1$, then we have

$$\begin{aligned} &|x|^2 + |y|^2 \neq 0 \quad |x|^2 - |y|^2 \neq 0 \\ &y\bar{x} + x\bar{y} \neq 0 \quad y\bar{x} - x\bar{y} \neq 0, \end{aligned}$$

i.e.,

$$\begin{aligned} &|x|^2 + |y|^2 \neq 0 \quad |x|^2 - |y|^2 \neq 0 \\ &\mathrm{Re}(\bar{x}y) \neq 0 \quad \mathrm{Im}(\bar{x}y) \neq 0. \end{aligned}$$

Recall that a characterization of injectivity obtained in [3] is as follows: Given $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and let

$$\begin{aligned} &\tilde{x} \\ &= (|x_1|^2_, \ldots, \mathrm{Re}(\bar{x}_1 x_n), \mathrm{Im}(\bar{x}_1 x_n), |x_2|^2_, \ldots, \mathrm{Re}(\bar{x}_2 x_n), \mathrm{Im}(\bar{x}_2 x_n); \ldots; |x_n|^2)^{\mathrm{T}}. \end{aligned}$$

Then $\{y_k\}_{k=1}^m$ is a injective frame for $\mathbb{C}^n$ if and only if $\{\tilde{y}_k\}$ spans $\mathbb{R}^{n^2}$. Applying this method to our frame, we can construct 4 vectors $\{v_i\}_{i=1}^4$, if the frame is injective hence $\{v_i\}_{i=1}^4$ spans $\mathbb{R}^4$, that is, $\{v_i\}_{i=1}^4$ is linearly independent and the determinant of the matrix $(v_1, \ldots, v_4)$ is not 0. A direct computation show that

$$\left|(v_1, \ldots, v_4)\right| = -8\left(|x|^2 + |y|^2\right) \cdot \left(|x|^2 - |y|^2\right) \cdot \mathrm{Re}(\bar{x}y) \cdot \mathrm{Im}(\bar{x}y).$$

While both ways can give the same condition for a frame in $\mathbb{C}^2$ to be injective, the Gabor frames method provides a much more efficient way to check its injectivity.

Example Let $\varphi = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{C}^3$. Then the full Gabor frame generated by the window vector φ is given by

$$\begin{pmatrix} x & x & x & y & y & y & z & z & z \\ y & \omega y & \omega^2 y & z & \omega z & \omega^2 z & x & \omega x & \omega^2 x \\ z & \omega^2 z & \omega z & x & \omega^2 x & \omega x & y & \omega^2 y & \omega y \end{pmatrix}.$$

If we set $\langle \pi(k,l)\varphi, \varphi \rangle \neq 0$ for $k = 0,\ 0 \leq l \leq 1$ and $k = 1,\ 0 \leq l \leq 2$, then we get

$$\begin{aligned} |x|^2+|y|^2+|z|^2 \neq 0 \quad & x\bar{y}+y\bar{z}+z\bar{x} \neq 0 \\ |x|^2+\omega|y|^2+\omega^2|z|^2 \neq 0 \quad & \omega^2 x\bar{y}+\omega y\bar{z}+z\bar{x} \neq 0 \\ & \omega x\bar{y}+\omega^2 y\bar{z}+z\bar{x} \neq 0. \end{aligned}$$

Let $T=\begin{bmatrix} a & d & e \\ \bar{d} & b & f \\ \bar{e} & \bar{f} & c \end{bmatrix} \in M_3(\mathbb{C})$. Then the condition $\langle T\pi(k,l)\varphi, \pi(k,l)\varphi\rangle = 0$ for all $0 \le k, l \le 2$ are given by the following messy equations:

$$\begin{aligned} a|x|^2+b|y|^2+c|z|^2+2\mathrm{Re}(d\omega^i\bar{x}y)+2\mathrm{Re}(f\omega^i\bar{y}z)+2\mathrm{Re}(\bar{e}\omega^i\bar{z}x)=0 \\ b|x|^2+c|y|^2+a|z|^2+2\mathrm{Re}(f\omega^i\bar{x}y)+2\mathrm{Re}(\bar{e}\omega^i\bar{y}z)+2\mathrm{Re}(d\omega^i\bar{z}x)=0 \\ c|x|^2+a|y|^2+b|z|^2+2\mathrm{Re}(\bar{e}\omega^i\bar{x}y)+2\mathrm{Re}(d\omega^i\bar{y}z)+2\mathrm{Re}(f\omega^i\bar{z}x)=0, \end{aligned}$$

where $i = 0, 1, 2$. Therefore, again it is truly complicated to check the injectivity of a Gabor frame using the definition.

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