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Orbit-injective covariant quantum channels

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ABSTRACT

The purpose of this paper is to investigate the quantum channels that preserve and also separate the orbits of pure states under the action of a group unitary representation π. Such a quantum channel will be called π-orbit injective. We prove that for finite group and complex Hilbert space cases, such a channel necessarily separates all the pure states. However, this is no longer true for quantum channels acting on real Hilbert spaces, or quantum channels acting on complex Hilbert spaces with (infinite) compact group representations. In both cases, we obtain necessary and/or sufficient conditions under which the quantum channel is orbit injective. These conditions are given in terms of the so called property (H) of characters (more generally, irreducible representations) of the group, and characterizations of property (H) are presented for real and complex valued multiplicative characters.

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1. Preliminaries

Quantum channels are essential components in quantum communications and quantum computings. They also have a wide range of applications such as quantum cryptography etc. Given two quantum systems H and K which are usually considered as finite dimensional Hilbert spaces. A quantum channel Φ is a completely positive trace-preserving (CPTP for short) linear map from $B(H)$ and $B(K)$, and (π, σ)-covariant quantum channel with respect to two unitary representations π and σ of a group G is a quantum channel that satisfies the condition $\Phi(\pi(g)T\pi(g^{-1})) = \sigma(g)\Phi(T)\sigma(g^{-1})$ for every $T \in B(H)$ and every $g \in G$. Covariant quantum channels form important class of channels since many challenging problems in quantum information theory are usually more tractable when certain symmetries are imposed on the channel. We refer to for example [2,6,8–11,19,22,24,25,28] for some recent progresses on theoretical studies of covariant quantum channels. In particular, in their recent work [22], M. Mozrzymas, M. Studzi'nski and N. Datta investigated the structure of covariant quantum channels with respect to an irreducible representation π for a finite group G, and obtained spectral decomposition of such a covariant quantum channels in terms of representation characteristics of the group G.

If $\sigma(g) = I_K$ is the trivial representation, where I_K is the identity map on K, a (π, I_K)-covariant quantum channel preserves the π-*orbit invariant* of any pure state $\rho_x := x \otimes x \in B(H)$, i.e.,

$$\Phi(\rho_x) = \Phi(\pi(g)\rho_x\pi(g^{-1}))$$

for all $g \in G$ and $x \in H$. If we view each orbit $[\rho_x]_\pi := \{\pi(g)\rho_x\pi(g^{-1}) : g \in G\}$ as a class of pure states of interests, then naturally we would like to know under what condition does a π-orbit invariant quantum channel separate these orbits, i.e., the condition $\Phi(\rho_x) = \Phi(\rho_y)$ implies that $[\rho_y]_\pi = [\rho_x]_\pi$. Such a quantum channel will be called *orbit injective* for pure states. This type of channels allow a quantum channel to map different symmetric related pure states to different output states that can be used to distinguish input pure states by their symmetries. The concept of group-invariant and orbit injective maps also recently finds important applications on max filtering in machine learning cf. [1,20,21].

The purpose of this paper is to obtain necessary and/or sufficient conditions for orbit injective quantum channels. The theory of quantum information could have some conceptual differences based on real and complex Hilbert spaces cf. [3,26]. In our case, we do have some subtle differences between finite and infinite groups, and between complex and real Hilbert space representations. Therefore for the purpose of clarity we will first treat the finite group case in section 2 and then briefly discuss the case for complex Hilbert space representations of (infinite) compact groups in section 3. Almost all of our characterizations involve a special family of irreducible representations whose characterizations will be discussed in the last section of the paper.

Here is a list of standard notations we will use in this paper:

- H, K – finite dimensional Hilbert spaces over $\mathbb{C}$ or $\mathbb{R}$, $B(H, K)$ – the space of all the linear operators from H to K, write $B(H) = B(H, K)$ if $H = K$. In the case that $H = \mathbb{C}^n$ and $K = \mathbb{C}^m$, $B(H, K) = M_{m\times n}(\mathbb{C})$ and we use $M_n(\mathbb{C})$ for the case when $m = n$. We use I_H (or I if no confusion from the context) to denote the identity operator on H.
- $U(n)$ or $U(H)$ – the group of unitary operators on an n-dimensional complex Hilbert space.
- For a subset $\mathcal{A}$ of $B(H)$, the commutant $\mathcal{A}' = \{T \in B(H) : TA = AT, \forall A \in \mathcal{A}\}$.
- Let $x \in H, y \in K$. We will use $\rho_{x,y}$ (write ρ_x if $x = y$) or $x\otimes y$ to denote the rank-one operator defined by $z \mapsto \langle z, y\rangle x$ for $z \in K$. Occasionally, $x \otimes y$ is also used to denote the tensor product in $H \otimes K$ and the readers should be able to tell from the context.
- Let π be a unitary representation of a group G, we use π^m to denote the representation $\pi \oplus ... \oplus \pi$ (m-copies). For one-dimensional unitary representation π of a group G, we also write $\pi = \chi$ and $\pi^m = \chi I_m$, where $\chi = tr(\pi(g))$ is the corresponding multiplicative character.
- Let π be a unitary representation of a group G acting on a Hilbert space H and $x \in H$. We use $[\rho_x]_\pi$ to denote the orbit $\{\pi(g)\rho_x\pi(g^{-1}) : g \in G\}$ of ρ_x.

1.1. *Unitary representations [23,27,29]*

For a compact group G, a continuous function $\pi : G \to U(H)$ is called a (finite dimensional) unitary representation if $\pi(gh) = \pi(g)\pi(h)$. A subspace V of H is called invariant if $\pi(g)x \in V$ for all $g \in G$ and $x \in V$. A representation π is called irreducible if $\{0\}$ and H are the only invariant subspaces. It is well-known that any unitary representation π on a finite-dimensional Hilbert space H is the direct sum of irreducible representations. Given a pair of unitary representation (π, H_π) and (σ, H_σ). Intertwining space is the space

$$Hom_G(\pi, \sigma) = \{A \in B(H_\sigma, H_\pi) : A\pi(g) = \sigma(g)A,\ \forall g \in G\},$$

where each element in this space is an intertwining operator. It is well-known that π and σ are unitarily equivalent if and only if $Hom_G(\pi, \sigma)$ contains a unitary matrix A (hence $\dim H_\pi = \dim H_\sigma$).

We remark that while all the results in this paper are presented for the case that π is a unitary representation, it is straightforward to check that all the results remain valid for projective unitary representations of G: A projective unitary representation π for a group G is a mapping $g \mapsto \pi(g)$ from G into the group $U(H)$ such that the map $g \to \alpha_g$ is a group homomorphism of G into the automorphism group of the operator algebra $B(H)$, where $\alpha_g(T) = \pi(g)T\pi(g)^*$. Or equivalently, there is a 2-cocycle μ of G such that $\pi(g)\pi(h) = \mu(g, h)\pi(gh)$. Recall that $\mu : G \times G\to\mathbb{T}$ is a 2-cocycle if it satisfies

(i) $\mu(g_1, g_2g_3)\mu(g_2, g_3) = \mu(g_1g_2, g_3)\mu(g_1, g_2)$ for all $g_1, g_2, g_3 \in G$, and (ii) $\mu(g, e) = \mu(e, g) = 1$ for all $g \in G$, where e denotes the group unit of G and $\mathbb{T}$ denotes the unit circle.

1.2. Covariant quantum channels

A quantum channel is a model for a particular snapshot of the time evolution of a density matrix, and especially for the evolution of pure into mixed states. Let H, K be finite-dimensional Hilbert spaces. A quantum channel is a completely positive trace-preserving (CPTP for short) linear map $\Phi : B(H) \to B(K)$, and a quantum channel in the Heisenberg picture is described by its adjoint map. By the famous theorem of Kraus (1971), a CP map Φ from $B(H)$ to $B(K)$ has a simple Kraus representation:

$$\Phi(T) = \sum_{i=1}^{r} A_i T A_i^*, \ \forall T \in B(H)$$

for some operators $A_1, ..., A_r \in B(H, K)$. In this representation, $A_1, ..., A_r$ are also referred to as the Kraus representation operators of the CP map Φ. Clearly, such a map Φ is trace-preserving if and only if $\sum_{i=1}^{r} A_i^* A_i = I_H$, and it is unital (i.e., $\Phi(I_H) = I_K$) if and only if $\sum_{i=1}^{r} A_i A_i^* = I_K$.

The *Choi rank* of a CP map Φ (denoted by $Cr(\Phi)$) is defined to be the smallest r such that $\Phi(T) = \sum_{j=1}^{r} A_j T A_j^*$ for some $A_j \in B(H, K)$. Let $\{e_i\}_{i=1}^{n}$ be a basis of H and E_{ij} be the rank-one operator $e_i \otimes e_j$. Then the Choi-Jamiolkowski matrix C_Φ is the linear operator on $H \otimes K$ defined by

$$C_\Phi = \sum_{i,j=1}^{n} E_{ij} \otimes \Phi(E_{ij})$$

In the case that $H = \mathbb{C}^n$, $K = \mathbb{C}^m$ and $\{e_i\}_{i=1}^{n}$ is the standard orthonormal basis, $C_\Phi = [\Phi(e_i \otimes e_j)]_{n \times n}$. The following are well known (cf. [4,5,7,18]).

Theorem 1.1. *(Choi's first theorem) Let $\Phi : B(H) \to B(K)$ be a linear map. Then Φ is CP if and only if C_Φ is positive. Moreover, $Cr(\Phi) = \mathrm{rank}(C_\Phi)$ if Φ is a CP map.*

Theorem 1.2. *(Choi's second theorem) Suppose that $\Phi : M_d(\mathbb{C}) \to M_k(\mathbb{C})$ is completely positive and $Cr(\Phi) = r$. Write $\Phi(T) = \sum_{j=1}^{r} A_j T A_j^*$ with $A_j \in M_{k\times d}(\mathbb{C})$. Let $B_j \in M_{k\times d}(\mathbb{C})$. Then $\Phi(T) = \sum_{j=1}^{m} B_j T B_j^*$ if and only if there exists matrix $U = (u_{ij}) \in M_{m\times r}(\mathbb{C})$ such that $U^*U = I_r$ and $A_i = \sum_{j=1}^{r} u_{ij} B_j$ for $i = 1, ..., m$ and $span\{B_1, ..., B_m\} = span\{A_1, ..., A_r\}$.*

The following group representation induced quantum channel plays an essential role in our investigation: Let G be a compact group and let μ be the unique Haar measure such that $\mu(G) = 1$. In the case that G is finite, then μ is the normalized counting measure. Given a pair of continuous unitary representations π, σ on Hilbert spaces H and K, respectively. We define

$$\Phi_{\pi,\sigma}(T) = \int_G \pi(g)T\sigma(g^{-1})d\mu(g), \quad T \in B(H),$$

and write $\Phi_\pi = \Phi_{\pi,\pi}$. Clearly, if G is a finite group then

$$\Phi_\pi(T) = \frac{1}{|G|}\sum_{g\in G} \pi(g)T\pi(g^{-1}).$$

We have the following properties:
(i) Φ_π is a quantum channel, and $range(\Phi_\pi) = \pi(G)'$.
(ii) $\Phi_\pi(T) = \frac{1}{\dim H} tr(T)I_H$ if and only if π is irreducible.
(iii) $\Phi_{\pi,\sigma} = 0$ if π and σ are inequivalent irreducible unitary representations.
(iv) $\Phi_\pi(T) = T$ if $\pi = \chi I_H$ for some multiplicative character χ.

Definition 1.1. Let π and σ be unitary representations of a group G on $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. We say that Φ is (π, σ)-covariant if

$$\Phi(\pi(g)T\pi(g^{-1})) = \sigma(g)\Phi(T)\sigma(g^{-1})$$

holds for every $g \in G$.

There is a natural way, called channel twirling, to produce a (π, σ)-covariant quantum channel from any given quantum channel. Let $\Phi : B(H) \to B(K)$ be a quantum channel, and π, σ be two continuous unitary representations of a group G on H and K, respectively. Then

$$\Psi(T) = \int_G \sigma(g^{-1})\Phi(\pi(g)T\pi(g^{-1}))\sigma(g)d\mu(g)$$

is a (π, σ)-covariant quantum channel. Again

$$\Psi(T) = \frac{1}{|G|}\sum_{g\in G} \sigma(g^{-1})\Phi(\pi(g)T\pi(g^{-1}))\sigma(g)$$

if G is finite.

Remark 1.1. (i) Since $\Phi(T) = \Psi(T)$ if Φ is (π, σ)-covariant, we immediately get that a quantum channel Φ is (π, σ)-covariant if and only if it is obtained by channel twirling.
(ii) If $\sigma(g) = I$ for every $g \in G$, then $\Psi(T) = \Phi(\int_G \pi(g)T\pi(g^{-1})d\mu(g)) = \Phi(\Phi_\pi(T))$.

(iii) Suppose that $\Phi(T) = \sum_{i=1}^{r} A_i T A_i^*$ and $Cr(\Phi) = r$. If Φ is (π, I)-covariant, then $span\{A_1^*, ..., A_r^*\}$ is π-invariant, i.e., $\pi(g)A_i^* \in span\{A_1^*, ..., A_r^*\}$ for every $g \in G$ and $i = 1, .., r$. Indeed, if Φ is (π, I)-covariant, then by Theorem 1.2 there is a unitary matrix $U = [u_{ij}]$ such $A_i\pi(g) = \sum_{j=1} u_{ij} A_j$ which is in turn equivalent to $\pi(g^{-1})A_i^* = \sum_{j=1}^{r} \bar{u}_{ij} A_j^*$ and hence $span\{A_1^*, ..., A_r^*\}$ is π-invariant. The converse is also true if we assume that $\{A_j A_k^*\}$ is linear independent: Assume that $A_i\pi(g) = \sum_{j=1} u_{ij} A_j$. Then

$$I = \sum_{i=1}^{r} (A_i\pi(g))(A_i\pi(g))^* = \sum_{i=1}^{r} (\sum_{j=1}^{r} u_{ij} A_j)(\sum_{j=1}^{r} \bar{u}_{ij} A_j^*) = \sum_{j,k=1}^{r} (\sum_{i=1}^{r} u_{ij}\bar{u}_{ik}) A_j A_k^*$$

Since $\{A_j A_k^*\}$ is linear independent, we have that $\sum_{i=1}^{r} u_{ij}\bar{u}_{ik} = \delta_{jk}$. Thus, by Theorem 1.2 again,

$$\Phi(\pi(g)T\pi(g^{-1})) = \sum_{i=1}^{r} A_i\pi(g)T(A_i\pi(g))^* = \sum_{i=1}^{r} A_i T A_i^* = \Phi(T)$$

for every $T \in B(H)$.

Definition 1.2. Let π be a unitary representation of G on a Hilbert space H, and $\Phi : B(H) \to B(K)$ be a quantum channel. We call that Φ is

(i) *pure-state injective* if $\Phi(\rho_x) = \Phi(\rho_y)$ implies that $\rho_x = \rho_y$,

(ii) *π-orbit invariant* if

$$\Phi(\pi(g)(\rho_x)\pi(g^{-1})) = \Phi(\rho_x)$$

for any $x \in H$ and any $g \in G$.

(iii) *π-orbit-injective* if it is π-orbit invariant and that $\Phi(\rho_x) = \Phi(\rho_y)$ implies $[\rho_x]_\pi = [\rho_y]_\pi$.

Remark 1.2. (i) We point out that π-orbit injectivity of Φ does not necessarily imply the orbit injectivity of Φ over $B(H)$ (i.e., $\Phi(S) = \Phi(T) \Rightarrow S \in \{\pi(g)T\pi(g^{-1}) : g \in G\}$). For example, let $\pi = I$ be the trivial representation, it is easy to construct examples that are pure-state injective (and hence π-orbit injective) but $\ker(\Phi)$ is nontrivial.

(ii) For quantum channels over complex Hilbert space case, π-orbit invariant is the same as (π, I)-covariant since every operator T is a linear combination of pure states ρ_x. This is no longer true for real Hilbert space cases.

Proposition 1.3. *Let H, K be finite dimensional complex Hilbert spaces and π be a unitary representation of a group G on H. Then we have*

(i) $\Phi : B(H) \to B(K)$ is π-orbit invariant if and only if $range(\Phi^*) \subseteq \pi(G)'$

(ii) If $Cr(\Phi) = 1$, then $\Psi = \Phi \circ \Phi_\pi$ is orbit injective if and only if Φ_π is orbit injective.

Proof. (i) Suppose that Φ is π-orbit invariant. Then for any $T \in B(K)$ and $g \in G$ we have

$$\begin{aligned}\langle \pi(g)\Phi^*(T)\pi(g^{-1}), \rho_x \rangle &= \langle \Phi^*(T), \pi(g^{-1})\rho_x\pi(g)\rangle \\ &= \langle T, \Phi(\pi(g^{-1})\rho_x\pi(g))\rangle = \langle T, \Phi(\rho_x)\rangle \\ &= \langle \Phi^*(T), \rho_x\rangle\end{aligned}$$

for every $x \in H$. Thus $\pi(g)\Phi^*(T)\pi(g^{-1}) = \Phi^*(T)$ since $span\{\rho_x : x \in H\} = B(H)$ when H is a complex Hilbert space. Conversely, if $range(\Phi^*) \subseteq \pi(G)'$, then $\langle \pi(g)\Phi^*(T)\pi(g^{-1}), \rho_x\rangle = \langle \Phi^*(T), \rho_x\rangle$ for any $g \in G, x \in H$ and $T \in B(H)$ which is equivalent to $\langle T, \Phi(\pi(g^{-1})\rho_x\pi(g))\rangle = \langle T, \Phi(\rho_x)\rangle$ and hence $\Phi(\pi(g^{-1})\rho_x\pi(g)) = \Phi(\rho_x)$ for all $g \in G$ and $x \in H$. Therefore Φ is π-orbit invariant.

(ii) Let $\Phi(T) = VTV^*$ where $V : H \to K$ is an isometry. Then the statement follows from the fact that $\Psi(\rho_x) = \Psi(\rho_y)$ if and only if $\Phi_\pi(\rho_x) = \Phi_\pi(\rho_y)$ □

We point out that the condition in Proposition 1.3 (ii) is still necessary for any Φ. However, it is not sufficient in general. Nevertheless, it is clear that Φ_π will play an essential role in our characterization of orbit injective quantum channels.

2. Orbit-injective quantum channel for finite groups

An essential tool in our characterizations of orbit injective quantum channels involves some special properties of multiplicative characters. For our convenience, we introduce the following concept:

Definition 2.1. A family of multiplicative characters $\chi_1, .., \chi_k$ of a compact group G is called to have the *Hadamard property* (*property (H)*, for short) if for any k-tuple $(a_1, ...a_k)$ of modulus one entries, there exists $g \in G$ such that $a_i\bar{a}_j = \chi_i(g)\bar{\chi_j}(g)$ for all $i, j = 1, ..., k$.

The characterizations of orbit injective quantum channels for finite groups are different for real and complex Hilbert space representations and we treat them separately in two subsections. In this section, we always assume that $|G| < \infty$.

2.1. Complex space case

For finite groups, we will show that there is no nontrivial orbit injective quantum channels acting on complex Hilbert spaces other than the pure-state injective ones.

Lemma 2.1. *Let $\pi = \pi_1 \oplus ... \oplus \pi_k$ be a unitary representation of G on a (real or complex) Hilbert space H, where $\pi_1, ..., \pi_k$ are irreducible representations. If $\Phi_\pi(T) = \frac{1}{|G|}\sum_{g\in G} \pi(g)T\pi(g^{-1})$ is orbit-invariant, then each π_i is one-dimensional.*

Proof. Assume, for example, that $d_1 = \dim H_{\pi_1} \geq 2$. Let u be a fixed unit vector in H_{π_1} and $v \in H_{\pi_1}$ be an arbitrary unit vector. Set $x = u \oplus 0 \oplus ... \oplus 0$ and $y = v \oplus 0 \oplus ... \oplus 0$. Then

$$\Phi_{\pi_1}(\rho_u) = \frac{1}{|G|} tr(\rho_u) I = \frac{1}{|G|} tr(\rho_v) I = \Phi_{\pi_1}(\rho_v),$$

which implies that $\Phi_\pi(\rho_x) = \Phi_\pi(\rho_y)$. Thus $[\rho_x]_\pi = [\rho_y]_\pi$, which in turn implies that $\rho_v \in \{\pi(g)\rho_u\pi(g^{-1}) : g \in G\}$ for any unit vector $v \in H_{\pi_1}$. This is impossible since $d_1 \geq 2$ and $\{\pi(g)\rho_u\pi(g^{-1}) : g \in G\}$ is a finite set. Therefore each π_i must be a one-dimensional representation. □

Proposition 2.2. *Let π be a unitary representation of G on H. Then the following are equivalent:*

(i) Φ_π is pure-state injective;

(ii) Φ_π is orbit-injective;

(iii) $\pi = \sigma^n$ for some one-dimensional representation σ.

Proof. Clearly $(i) \Rightarrow (ii)$, and $(iii) \Rightarrow (i)$ follows from fact that if $\pi(g) = \sigma(g) I_n$, then

$$\Phi_\pi(x \otimes x) = \frac{1}{|G|} \sum_{g \in G} \pi(g)x \otimes \pi(g)x = x \otimes x.$$

$(ii) \Rightarrow (iii)$: Assume that Φ_π is obit-injective. Then, by Lemma 2.1, we get that $\pi = \pi_1 \oplus ... \oplus \pi_n$ such that each π_i is one-dimensional and so we can assume that each π_i is a multiplicative character acting on the one-dimensional space $\mathbb{C}$. Suppose, for example, that π_1 and π_2 are inequivalent. Without losing the generality we can assume that $\pi = \pi_1 \oplus \pi_2$. Note that $S := \{\pi_1(g)\pi_2(g^{-1}) : g \in G\}$ is a finite set. Let $a, b \in \mathbb{C}$ be such that $|a| = |b| = 1$ and $a\bar{b} \notin S$. Let $x = x_1 \oplus x_2$ and $y = ax_1 \oplus bx_2$ be two unit vectors such that both x_1 and x_2 are nonzero. Note that $\sum_{g \in G} \pi_1(g)\pi_2(g^{-1}) = 0$. So we get

$$\Phi_\pi(x \otimes x) = \Big[\frac{1}{|G|} \sum_{g \in G} \pi_i(g)(x_i \otimes x_j)\pi_j(g^{-1})\Big]_{2\times 2} = \begin{bmatrix} |x_1|^2 & 0 \\ 0 & |x_2|^2 \end{bmatrix}$$

Thus $\Phi_\pi(x \otimes x) = \Phi_\pi(y \otimes y)$. However, from

$$\pi(g)(x \otimes x)\pi(g^{-1}) = \begin{bmatrix} |x_1|^2 & \pi_1(g)\pi_2(g^{-1})x_1\bar{x}_2 \\ \pi_2(g)\pi_1(g^{-1})x_2\bar{x}_1 & |x_2|^2 \end{bmatrix}$$

and

$$y \otimes y = \begin{bmatrix} |x_1|^2 & a\bar{b}x_1\bar{x}_2 \\ \bar{b}ax_2\bar{x}_1 & |x_2|^2 \end{bmatrix}$$

we obtain that $y \otimes y \neq \pi(g)(x \otimes x)\pi(g^{-1})$ for any $g \in G$, which contradicts with the assumption that Φ_π is orbit-injective. Therefor all the π_i's are unitarily equivalent which implies (iii). □

With the help of the above result we get:

Theorem 2.3. *Let Φ be a π-invariant quantum channel over a complex Hilbert space H. Then Φ is π-orbit injective if and only if Φ is pure-state injective.*

Proof. Since Φ is π-invariant, we get

$$\Phi(\rho_x) = \frac{1}{|G|}\sum_{g\in G}\Phi(\pi(g)\rho_x\pi(g^{-1})) = \Phi(\Phi_\pi(\rho_x)), \quad \forall x \in H.$$

We only need to prove the necessary part. Assume that Φ is π-orbit injective. First we claim that $\pi = \chi I$ for some multiplicative character χ. Indeed, if this is not the case, then by Proposition 2.2, Φ_π is not orbit-injective. Thus there exists $x, y \in H$ such that $\rho_y \notin [\rho_x]_\pi$ and $\Phi_\pi(x \otimes x) = \Phi_\pi(y \otimes y)$. This implies that

$$\Phi(x \otimes x) = \Phi(\Phi_\pi(x \otimes x)) = \Phi(\Phi_\pi(y \otimes y)) = \Phi(y \otimes y),$$

which leads to a contradiction since Φ is π-orbit injective. Thus $\pi = \chi I$ for some multiplicative character χ. Now suppose that $\Phi(x \otimes x) = \Phi(y \otimes y)$. Then $\rho_y \in [\rho_x]_\pi$ and hence $x \otimes x = y \otimes y$ since $\pi = \chi I$. Therefore Φ is pure-state injective □

2.2. Real space case

Unlike the complex space case, the next example shows that orbit-injective quantum channels that are not pure-state injective do exist in real Hilbert space setting.

Example 2.1. Let $G = \{I, U\}$ where $U = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ Then $U^2 = I$. Let π be the identity map. Then $\Phi_\pi(T) = \frac{1}{2}(T + UTU^*)$. By Proposition 2.2, the complex quantum channel $\Phi_\pi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is not orbit-injective. However, the real quantum channel $\Phi_\pi : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ is orbit-injective.

Proof. Assume that $\Phi_\pi(x \otimes x) = \Phi_\pi(y \otimes y)$. If x and Ux are linearly dependent, then $\Phi_\pi(x \otimes x)$ is a rank-one operator and hence $y = cx$ for some $c \in \mathbb{R}$. This implies that $y \otimes y = x \otimes x$ since $||x|| = ||y||$. Now assume that x and Ux are linearly independent. We write

$$y = ax + bUx$$

for some $a, b \in \mathbb{R}$. This implies that $Uy = bx + aUx$ since $U^2 = I$. From $\Phi_\pi(x \otimes x) = \Phi_\pi(y \otimes y)$ we get that

$$x \otimes x + Ux \otimes Ux = (|a|^2 + |b|^2)(x \otimes x + Ux \otimes Ux) + 2ab(x \otimes Ux + Ux \otimes x),$$

which implies that

$$x \otimes [(1 - (a^2 + b^2))x - 2abUx] + Ux \otimes [(1 - b^2 + b^2))Ux - 2abx] = 0.$$

Since x, Ux are linearly independent, we get that $1 = a^2 + b^2$ and $2ab = 0$. So $a = 0$ or $b = 0$ which implies that either $y \otimes y = x \otimes x$ or $y \otimes y = Ux \otimes Ux$. Thus $[y]_\pi = [x]_\pi$. $\square$

The reason for the above example to work is simply because that we have a decomposition $\pi = \chi_1 \oplus \chi_2$ such that $\{\chi_1, \chi_2\}$ has property (H).

Theorem 2.4. *Let π be a unitary representation of G on a real Hilbert space H. Then Φ_π is orbit-injective if and only if $\pi = \chi_1 I_{m_1} \oplus ... \oplus \chi_k I_{m_k}$ such that $\chi_1, ..., \chi_k$ are multiplicative characters with property (H).*

Proof. ($\Rightarrow$). Assume that Φ_π is orbit-injective. By Lemma 2.1, we know that $\pi = \chi_1 I_{m_1} \oplus ... \oplus \chi_k I_{m_k}$ such $\chi_1, ..., \chi_k$ are distinct multiplicative characters. If $\chi_1, ..., \chi_k$ fail to satisfy property (H), then there exists a vector $(a_1, ..., a_k)$ such that $a_i = \pm 1$ and $[a_i a_j]_{k\times k} \neq [\chi_i(g)\chi_j(g)]_{k\times k}$ for any $g \in G$. Now let $x = x_1 \oplus ... \oplus x_k$ and $y = a_1 x_1 \oplus ... \oplus a_k x_k$ such that $x_i \neq 0$ for each $1 \leq i \leq k$. Then we get $\Phi_\pi(x \otimes x) = \Phi_\pi(y \otimes y)$ and $y \otimes y \notin \{\pi(g)x \otimes \pi(g)x : g \in G\}$, which contradicts with the orbit-injectivity assumption of Φ_π.

($\Leftarrow$) Assume that $\pi = \chi_1 I_{m_1} \oplus ... \oplus \chi_k I_{m_k}$ such that $\chi_1, ..., \chi_k$ satisfy property (H). Suppose that $\Phi_\pi(x \otimes x) = \Phi_\pi(y \otimes y)$. Write $x = x_1 \oplus ... \oplus x_k$ and $y = y_1 \oplus ... \oplus y_k$. Then we get $\Phi_{\pi_i}(x_i \otimes x_i) = \Phi_{\pi_i}(y_i \otimes y_i)$ for $i = 1, ..., k$, where $\pi_i = \chi_i I_{m_i}$. This implies that $x_i \otimes x_i = y_i \otimes y_i$ since $\Phi_{\pi_i}(T) = T$ for any $T \in B(H_{\pi_i})$. Thus $x_i = a_i y_i$ for some $a_i = \pm 1$. Since $\chi_1, ..., \chi_k$ satisfy property (H), there exists $g \in G$ such that $\alpha_i a_j = \chi_i(g)\chi_j(g)$ for all $i, j = 1, ..., k$. This implies that

$$x \otimes x = [x_i \otimes x_j]_{k\times k} = [a_i a_j y_i \otimes y_j]_{k\times k} = [\chi_i(g) y_i \otimes \chi_j(g) y_j]_{2\times 2} = \pi(g)(y \otimes y)\pi(g^{-1}).$$

Thus Φ_π is orbit-injective. $\square$

Remark 2.1. From the definition, it is easy to see that a family of real valued multiplicative characters $\{\chi_1, ..., \chi_k\}$ has property (H) if and only if there exists a subset Λ of G such that $|\Lambda| = 2^{k-1}$ and $(\chi_1(g), ..., \chi_k(g) \neq \pm(\chi_1(h), ..., \chi_k(h))$ for any distinct $g, h \in \Lambda$. In particular, if χ_1 and χ_2 are two distinct multiplicative characters, then they are orthogonal to each other, which implies that the matrix $[\chi_1(g), \chi_2(g)]_{|G|\times 2}$ has rank-2 and hence has two linearly independent rows. Thus any two distinct real valued multiplicative characters always satisfy property (H). So we get

Corollary 2.5. *(i) If* $\pi = \chi_1 I_{m_1} \oplus \chi_2 I_{m_2}$ *is a unitary representation over a real Hilbert space* H*, then* Φ_π *is orbit-injective.*

(ii) Suppose that G *has at most two real valued multiplicative characters and* π *be a unitary representation of* G *over a real Hilbert space* H*. Then* Φ_π *is orbit injective over* H *if and only if* π *is a direct sum of some one-dimensional representations.*

Remark 2.2. Many groups (e.g. Frobenius group F_5, symmetric groups S_4, S_5, S_{13}, Dihedral groups D_9, D_{11} etc.) have at most two real-valued multiplicative characters.

Example 2.2. (i) Let $G = (\mathbb{Z}/2\mathbb{Z})^2$ (Klein four-group). The four characters $\chi_1, \chi_1, \chi_3, \chi_4$ are listed as column vectors in the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

(ii) Let $G = D_4$ (Dihedral group). The four multiplicative characters $\chi_1, \chi_1, \chi_3, \chi_4$ are given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Clearly, in both cases, any three of the four characters satisfy property (H). Thus we get that Φ_π is orbit-injective on a real Hilbert space if and only if $\pi = \chi_1 I_{m_1} \oplus ... \oplus \chi_k I_{m_k}$ for some $k \leq 3$.

Now we move to the general π-orbit invariant quantum channel case. Let $\pi = \chi_1 I_{m_1} \oplus ... \oplus \chi_k I_{m_k}$ be a unitary representation acting on $H = H_1 \oplus ... \oplus H_k$, where I_{m_i} is the identity operator on H_i and $m_i = \dim H_i$. For a quantum channel Φ acting on H, let Φ_i be the restricting of Φ on $B(H_i)$.

Theorem 2.6. *Let* π *be a unitary representation of* G *on a real Hilbert space* H*. Then the following claims hold.*

(i) If Φ *is* π*-orbit-injective, then* $\pi = \chi_1 I_{m_1} \oplus ... \oplus \chi_k I_{m_k}$ *such that* $\chi_1, ..., \chi_k$ *are multiplicative characters with property (H) and each* Φ_i *is pure-state injective.*

(ii) Suppose that $\pi = \chi_1 I_{m_1} \oplus ... \oplus \chi_k I_{m_k}$ *such that* $\chi_1, ..., \chi_k$ *are multiplicative characters with property (H), and each* $A_i = diag(A_{i1}, ..., A_{ik})$ *is block-diagonal with* $A_{ij} \in B(H_j)$*, then* Φ *is* π*-orbit injective if and only if each* Φ_i *is pure-state injective*

Proof. (i) Assume that Φ is π-orbit-injective. We first show that $\pi = \chi_1 I_{m_1} \oplus ... \oplus \chi_k I_{m_k}$ such that $\chi_1, ..., \chi_k$ are multiplicative characters with property (H). If this is false, then, by Theorem 2.4, Φ_π is not orbit injective. Thus there exists $x, y \in H$ such that $\rho_y \notin [\rho_x]_\pi$ and $\Phi_\pi(x \otimes x) = \Phi_\pi(y \otimes y)$. This implies that

$$\Phi(x \otimes x) = \Phi(\Phi_\pi(x \otimes x)) = \Phi(\Phi_\pi(y \otimes y)) = \Phi(y \otimes y),$$

which leads to the contradiction to the orbit-injectivity of Φ. Now we show that each Φ_i is pure-state injective. Let $x_i, y_i \in H_i$ be such that $\Phi_i(x_i \otimes x_i) = \Phi_i(y_i \otimes y_i)$. Set

$$x = 0 \oplus .. \oplus 0 \oplus x_i \oplus 0 \oplus ... \oplus 0$$

and

$$y = 0 \oplus .. \oplus 0 \oplus y_i \oplus 0 \oplus ... \oplus 0.$$

Then $\Phi(x \otimes x) = \Phi(y \otimes y)$. Thus $y \otimes y = \pi(g)x \otimes \pi(g)x$ for some $g \in G$, which implies that $y_i \otimes y_i = \chi_i(g)x_i \otimes \chi_i(g)x_i = x_i \otimes x_i$. Therefore Φ_i is pure-state injective.

(ii) By (i), we only need to prove the condition is sufficient. Since $\pi = \chi_1 I_{m_1} \oplus ... \oplus \chi_k I_{m_k}$ such that $\chi_1, ..., \chi_k$ are multiplicative characters with property (H), again by Theorem 2.4, Φ_π is orbit-injective. Now suppose that $\Phi(x \otimes x) = \Phi(y \otimes y)$. Write $x = (x_1, ..., x_k)$, $y = (y_1, ..., y_k)$. Note that

$$\Phi_\pi(x \otimes x) = \begin{bmatrix} x_1 \otimes x_1 & 0 & 0 & \cdots & 0 \\ 0 & x_2 \otimes x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & x_k \otimes x_k \end{bmatrix}$$

Since each A_i is block-diagonal,

$$\Phi(x \otimes x) = \Phi(\Phi_\pi(x \otimes x)) = \begin{bmatrix} \Phi_1(x_1 \otimes x_1) & 0 & 0 & \cdots & 0 \\ 0 & \Phi_2(x_2 \otimes x_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \Phi_k(x_k \otimes x_k) \end{bmatrix}.$$

Thus $\Phi(x \otimes x) = \Phi(y \otimes y)$ implies that $\Phi_i(x_i \otimes x_i) = \Phi_i(y_i \otimes y_i)$ for $i = 1, ..., k$. Since each Φ_i is pure-state injective, we get that $x_i \otimes x_i = y_i \otimes y_i$ for each i, and hence $\Phi_\pi(x \otimes x) = \Phi_\pi(y \otimes y)$. Since Φ_π is orbit injective, we get that $\rho_y \in [\rho_x]_\pi$. Therefore Φ is π-orbit injective. □

Example 2.3. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for $\mathbb{R}^n$ and $\pi(g) = diag(\chi_1(g), ..., \chi_n(g))$ such that $\chi_1, ..., \chi_k$ are multiplicative characters satisfying property (H). Let

$A_i = e_i \otimes e_i$. Then, by the above theorem, the quantum channel defined by $\Phi(T) = \sum_{i=1}^n A_i T A_i^*$ is π-orbit injective.

The above example can be slightly generalized. Recall that a sequence $\{x_1, ..., x_d\}$ in a Hilbert space H is called a *phase-retrievable frame* for H if $|\langle x, x_i\rangle| = |\langle y, x_i\rangle|$ for all i implies that $x \otimes x = y \otimes y$. A Parseval frame is a frame $\{x_i\}$ such that $\sum_{i=1}^d x_i \otimes x_i = I$. We refer to [12–15] for some background materials related to the connections among phase retrievable frames, group representations and quantum channels.

Example 2.4. Suppose that $\pi = \chi_1 I_{H_1} \oplus ... \oplus \chi_k I_{H_k}$ such that $\chi_1, ..., \chi_k$ are different multiplicative characters with property (H), where $\mathbb{R}^n = H_1 \oplus ... \oplus H_k$. For each i, let $\{x_{i1}..., x_{id_i}\}$ be a phase-retrievable frame for H_i such that $\{||x_{ij}||x_{ij}\}_{j=1}^{d_i}$ is a Parseval frame for H_i and $\{A_{ij}\}_{j=1}^{d_i}$ is linearly independent, where $A_{ij} = x_{ij} \otimes x_{ij}$ for $1 \leq j \leq d_i$ and $1 \leq i \leq k$. Define a quantum channel

$$\Phi(T) = \sum_{i=1}^{k} \sum_{j=1}^{d_i} A_{ij} T A_{ij}^*.$$

Then Φ is π-orbit injective.

Proof. Suppose that $\Phi(u \otimes u) = \Phi(v \otimes v)$ for some $u = (u_1, ..., u_k)$ and $v = (v_1, ..., v_k)$ in $\mathbb{R}^n = H_1 \oplus ... \oplus H_k$ with $v_i, u_i \in H_i$. Then we get for each i, $\sum_{j=1}^{d_i} A_{ij}(u \otimes u)A_{ij}^* = \sum_{j=1}^{d_i} A_{ij}(v \otimes v)A_{ij}^*$ which implies that $\sum_{j=1}^{d_i} A_{ij}(u_i \otimes u_i)A_{ij}^* = \sum_{j=1}^{d_i} A_{ij}(v_i \otimes v_i)A_{ij}^*$, or equivalently,

$$\sum_{j=1}^{d_i} |\langle u_i, x_{ij}\rangle|^2 A_{ij} = \sum_{j=1}^{d_i} |\langle v_i, x_{ij}\rangle|^2 A_{ij}.$$

Thus $|\langle u_i, x_{ij}\rangle|^2 = |\langle u_i, x_{ij}\rangle|^2$ for $1 \leq j \leq d_i$ due to the linear independence of $\{A_{ij}\}_{j=1}^{d_i}$. Since $\{x_{ij}\}_{j=1}^{d_i}$ is phase-retrievable for H_i, we get that $u_i \otimes u_i = v_i \otimes v_i$ for each i. Finally, using the property (H) for $\chi_1, ..., \chi_k$, we get that $[\rho_u]_\pi = [\rho_v]_\pi$ and hence Φ is π-orbit injective. $\square$

Remark 2.3. If $2m_i - 1 \leq d_i \leq \frac{m_i(m_i+1)}{2}$, then it is known that every generic Parseval frame $\{x_{ij}\}_{j=1}^{d_i}$ satisfies the requirements in the above example. Similar examples can also be easily constructed without requiring that each A_{ij} is a rank-one operator.

3. Orbit injective quantum channels for compact groups

3.1. Characterizations

In this section we assume that G is a compact group and π is a continuous unitary representation on a finite dimensional complex Hilbert space case. Unlike the finite group case, n-tuples ($n \geq 2$) of complex characters with property (H) does exist for some infinite compact groups (see section 4). By using the fact that $\Phi(x \otimes x) = \Phi(\Phi_\pi(x \otimes x))$ for any π-orbit invariant quantum channel Φ, we also get the following two results similar to the real-valued multiplicative character case. The proofs are almost indentical to that of Theorem 2.4 and Theorem 2.6, and we leave the proofs to the interested readers.

Theorem 3.1. *Suppose that $\pi = \chi_1 I_{m_1} \oplus ... \oplus \chi_k I_{m_k}$ such that $\chi_1, ..., \chi_k$ are distinct multiplicative characters of G. Then*

(i) Φ_π is orbit injective if and only if $\chi_1, ..., \chi_k$ satisfy property (H).

(ii) If

$$\Phi(T) = \sum_{i=1}^{r} A_i T A_i^*$$

is a π-orbit invariant quantum channel such that $A_i = diag(A_{i1}, ..., A_{ik})$ is block-diagonal with $A_{ij} \in B(H_j)$, then Φ is π-orbit injective if and only if $\chi_1, ..., \chi_k$ satisfy property (H) and each Φ_i is pure-state injective.

The above theorem allows us to construct a rich class of orbit injective quantum channels. In particular we point out the following:

Corollary 3.2. *Let $\pi = \chi_1 I_{m_1} \oplus ... \oplus \chi_k I_{m_k}$ be a unitary representation of a group G on H such that $\chi_1, ..., \chi_k$ satisfy property (H). Suppose that $\Psi : B(H) \to B(K)$ is an injective quantum channel (i.e., one to one, but not necessarily π-orbit invariant), and let*

$$\Phi(T) = \Psi(\Phi_\pi(T)) \quad T \in B(H).$$

Then Φ is π-orbit injective.

Proof. By construction we know that Φ is π-orbit invariant. Suppose that $\Phi(\rho_x) = \Phi(\rho_y)$ for some $x, y \in H$. Since Ψ is injective, we get that $\Phi_\pi(\rho_x) = \Phi_\pi(\rho_y)$. By Theorem 3.1 (i), we get that $[\rho_x]_\pi = [\rho_y]_\pi$ and hence Φ is π-orbit injective. □

While the above results provide a large class of orbit injective quantum channels for unitary representations that are direct sum of one-dimensional representations, unlike the finite group case this one-dimensional decomposition is no longer necessary for infinite compact groups. For example, let $G = U(n)$ and $\pi(g) = g$. Then π is an irreducible representation on $\mathbb{C}^n$. Since $[\rho_x]_\pi = \{g(x \otimes x)g^{-1} : g \in G\} = \{y \otimes y : y \in H, ||y|| = \|x\|\}$, the quantum channel $\Phi(T) = \frac{1}{n} tr(T)I$ is π-orbit invariant and orbit injective.

Now we examine the cases when the irreducible subrepresentations of π is not necessarily one dimensional. We have the following simple observation for the case that $\Phi = \Phi_\pi$ and π is irreducible.

Lemma 3.3. *Let π be an irreducible unitary representation of G and let x be a fixed unit vector in H. Then Φ_π is orbit injective if and only if $y \in \mathbb{T}\pi(G)x$ for any unit vector y.*

Proof. Since π is irreducible we get that $\Phi_\pi(T) = \frac{1}{\dim H} tr(T)I$. So $\Phi_\pi(x \otimes x) = \Phi_\pi(y \otimes y)$ if and only if $||x|| = ||y||$. Thus Φ_π is orbit injective if and only if $[x]_\pi = \{y \otimes y : \|y\| = 1\}$, which is equivalent to the condition that $y \in \mathbb{T}\pi(G)x$ for any unit vector y. □

Remark 3.1. Clearly, the equality $\{y \otimes y : \|y\| = 1\} = \mathbb{T}\pi(G)x$ holds if $\mathbb{T}\pi(G) = U(n)$. So the inclusion representation $\pi : SU(n) \hookrightarrow U(n)$ has the property in the above lemma.

Naturally we need to generalize the property (H) for multiplicative characters to more general irreducible representations.

Definition 3.1. Let G be a compact group. We say that a family of irreducible representations $\pi_1, ..., \pi_k$ satisfy property (H) if the following is true: For any k-tuples $(g_1, ..., g_k)$ and $(x_1, ..., x_k)$, there exist $g \in G$ and $t \in \mathbb{T}$ such that $\pi_i(g_i)x_i = t\pi_i(g)x_i$ for every i.

Remark 3.2. Let $G = U(d_1) \oplus ... \oplus U(d_k)$ and $\pi_i(G) = g_i$ for every $g = g_1 \oplus ... \oplus g_k \in G$. Then it is easy to verify that $\pi_1, ..., \pi_k$ satisfy property (H). More generally, we say that a family of irreducible representations $\pi_1, ..., \pi_k$ satisfy property (H_+) if for any k-tuples $(g_1, ..., g_k)$, there exist $g \in G$ such that $\pi_i(g_i) = \pi_i(g)$ for every i. Clearly property (H_+) implies property (H).

(i) Let $\pi = \pi_1 \oplus ... \oplus \pi_k$. Then property ($H_+$) holds if and only if π induces a surjection of G onto $\pi_1(G) \oplus ... \oplus \pi_k(G)$. That is, the natural map $G \to G/\ker(\pi_1) \oplus \cdots \oplus G/\ker(\pi_k)$ is surjective.

(ii) We construct an example such that property (H) holds but property (H_+) does not hold. Let $G = \mathbb{T} \times G_1 \times G_2$ with $G_i = SU(n_i)$. Let $\rho_i : SU(n_i) \to U(n_i)$ be the inclusion map. Let π_i be the representation of G such that $\pi_i(t, g_1, g_2) = t^i \rho_i(g_i)$ for $t \in \mathbb{T}$ and $g_i \in G_i$. It is easy to verify that for any k-tuples $(g_1, ..., g_k)$, there exist $g \in G$ and $t \in \mathbb{T}$ such that $\pi_i(g_i) = t\pi_i(g)$ for $i = 1, 2$. Hence $\pi_1,\ \pi_2$ satisfy property (H). They do not satisfy property (H_+) by (i).

Theorem 3.4. *Suppose that $\pi = \pi_1^{m_1} \oplus ... \oplus \pi_k^{m_k}$ such that $\pi_1, ..., \pi_k$ are inequivalent irreducible representations of G, and let $\sigma_i = \pi_i^{m_i}$.*

(i) If Φ_π is π-orbit injective, then $\pi_1, ..., \pi_k$ satisfy property (H) and each Φ_{σ_i} is σ_i-orbit injective.

(ii) If $\pi_1, ..., \pi_k$ satisfy property (H_+) and each Φ_{σ_i} is σ_i-orbit injective, then Φ_π is π-orbit injective.

(iii) If $m_1 = ... = m_k = 1$, then Φ_π is π-orbit injective if and only if $\pi_1, ..., \pi_k$ satisfy property (H) and each Φ_{π_i} is π_i-orbit injective.

Proof. (i) Assume that Φ_π is π-orbit injective. Clearly Φ_{π_i} is π_i-orbit injective by restricting Φ_π to the vectors $x = (0, .., 0, x_i, 0, ...0) \in H$. If $\pi_1, ..., \pi_k$ do not satisfy property (H), then there exist two k-tuple $(g_1, ..., g_k)$ and $(x_1, ..., x_k)$ such that $(\pi_1(g_1)x_1, ..., \pi_k(g_k)x_k) \notin \mathbb{T}\pi(G)x$, where $x = (\tilde{x}_1, ..., \tilde{x}_k) \in H$, $x_i \in H_{\pi_i}$ and $\tilde{x}_i = (x_i, 0, ..., 0) \in H_{\sigma_i}$. Let $y_i = \pi_i(g_i)x_i$ and $y = (\tilde{y}_1, ..., \tilde{y}_k)$. Then $y \notin [x]_\pi$. Note that $\Phi_{\pi_i}(x_i \otimes x_i) = \Phi_{\pi_i}(y_i \otimes y_i)$ and hence $\Phi_{\sigma_i}(\tilde{x}_i \otimes \tilde{x}_i) = \Phi_{\sigma_i}(\tilde{y}_i \otimes \tilde{y}_i)$. Since $\pi_1, ..., \pi_k$ are inequivalent irreducible representations, we get that

$$\int_G \sigma_i(g)u \otimes \sigma_j(g)v = 0$$

for $i \neq j$ and $u \in H_{\sigma_i}, v \in H_{\sigma_j}$. Thus we get that $\Phi(y \otimes y) = \Phi(x \otimes x)$, which contradicts to the π-orbit injectivity of Φ.

(ii) Assume that $\pi_1, ..., \pi_k$ satisfy property (H_+) and each Φ_{σ_i} is σ_i-orbit injective. If $\Phi(y \otimes y) = \Phi(x \otimes x)$, then

$$\Phi_{\sigma_i}(x_i \otimes x_i) = \Phi_{\sigma_i}(y_i \otimes y_i),$$

where $x_i, y_i \in H_{\sigma_i}$ and $x = (x_1, ..., x_k), y = (y_1, ..., y_k) \in H$. Since Φ_{σ_i} is σ_i-orbit injective, there is $g_i \in G$ such that $y_i \otimes y_i = \sigma_i(g_i)x_i \otimes \sigma_i(g_i)x_i$. Since $\pi_1, ..., \pi_k$ satisfy property (H_+), we get that there exist $g \in G$ and $t \in \mathbb{T}$ such that $\pi_i(g_i) = t\pi_i(g)$ for all i. This implies that

$$y \otimes y = [\sigma_i(g_i)x_i \otimes \sigma_j(g_j)x_j]_{k\times k} = [\sigma_i(g)x_i \otimes \sigma_j(g)x_j]_{k\times k} = \sigma(g)x \otimes \sigma(g)x.$$

Thus Φ_π is orbit injective.

(iii) By (i) we only need to prove the sufficient part which is similar to the proof of (ii): If $\Phi(y \otimes y) = \Phi(x \otimes x)$, then

$$\Phi_{\pi_i}(x_i \otimes x_i) = \Phi_{\pi_i}(y_j \otimes y_i)$$

for every i. Since π_i-orbit injective, there is $g_i \in G$ such that $y_i \otimes y_i = \pi_i(g_i)x_i \otimes \pi_i(g_i)x_i$, which implies by property (H) that there exist $g \in G$ and $t \in \mathbb{T}$ such that $\pi_i(g_i)x_i = t\pi_i(g)x_i$ for all i. Thus

$$y \otimes y = [\pi_i(g_i)x_i \otimes \pi_j(g_j)x_j]_{k\times k} = [\pi_i(g)x_i \otimes \pi_j(g)x_j]_{k\times k} = \pi(g)x \otimes \pi(g)x,$$

and so Φ_π is orbit injective. □

Corollary 3.5. *Suppose that $\pi = \pi_1^{m_1} \oplus ... \oplus \pi_k^{m_k}$ such that $\pi_1, ..., \pi_k$ are inequivalent irreducible representations of G. If there exists a π-orbit injective quantum channel Φ, then $\pi_1, ..., \pi_k$ satisfy property (H).*

Proof. Since Φ is π-orbit invariant, we have that $\Phi(x \otimes x) = \Phi(\Phi_\pi(x \otimes x))$ for every $x \in H$. So if Φ is orbit injective, then so is Φ_π. Thus, by Theorem 3.1, $\pi_1, ..., \pi_k$ satisfy property (H). □

3.2. Orbit injectivity for compatible quantum channels

Quantum channel compatibility is one of the fundamental issues in quantum information which asks if two quantum channels are the marginal channels of a joint channel. In the Heisenberg picture, two quantum channels $\Phi_1 : B(H_1) \to B(K)$ and $\Phi_2 : B(H_2) \to B(K)$ are *compatible* if there is a quantum channel $\Phi : B(H_1 \otimes H_2) \to B(K)$ such that $\Phi_1(A) = \Phi(A \otimes I_{H_2})$ and $\Phi_2(B) = \Phi(I_{H_1} \otimes B)$, where $A \in B(H_1)$ and $B \in B(H_2)$. It is natural to impose covariant structure on Φ if both Φ_i are covariant channels. Let σ_i be unitary representations on H_i and π be a unitary representation on K of a group G. Let $\sigma(g) = \sigma_1(g) \otimes \sigma_2(g)$ be the unitary representation of G acting on $H_1 \otimes H_2$. It is clear that if Φ is (σ, π)-covariant, then Φ_i is (σ_i, π)-covariant for $i = 1, 2$. But the converse is not necessarily true. However, there exists always a quantum (σ, π)-covariant quantum channel Ψ such that Φ_1 and Φ_2 are the two marginal channels of Ψ. This can be done by channel twirling or covariantizing [10]:

$$\tilde{\Phi}(C) = \int_G \pi(g)^* \Phi(\sigma(g) C \sigma(g)^*) \pi(g) d\mu(g), \quad C \in B(H_1 \otimes H_2).$$

Clearly $\tilde{\Phi}$ is (σ, π)-covariant and for any $A \in B(H_1)$ we have

$$\begin{aligned}
\tilde{\Phi}(A \otimes I_{H_2}) &= \int_G \pi(g)^* \Phi(\sigma(g)(A \otimes I_{H_2})\sigma(g)^*)\pi(g) d\mu(g) \\
&= \int_G \pi(g)^* \Phi(\sigma_1(g) A \sigma_1(g)^* \otimes I_{H_2})\pi(g) d\mu(g) \\
&= \int_G \pi(g)^* \Phi_1(\sigma_1(g) A \sigma_1(g)^*)\pi(g) d\mu(g) \\
&= \int_G \Phi_1(A) d\mu(g) = \Phi_1(A)
\end{aligned}$$

and similarly $\tilde{\Phi}(I_{H_1} \otimes B) = \Phi_2(B)$ for $B \in B(H_2)$. So we can always assume that the joint channel Φ is already (σ, π)-covariant.

If we particularly let σ_i be the identity map, then we get $range(\Phi) \subset \pi(G)'$ whenever $range(\Phi_i) \subset \pi(G)'$ for $i = 1, 2$. This implies, by Proposition 1.3, that Φ_i^* and Φ^* all are π-orbit invariant. Moreover,

$$\Phi_1^*(T) = Tr_{H_2} \Phi^*(T)$$

and

$$\Phi_2^*(T) = Tr_{H_1} \Phi^*(T)$$

for $T \in B(K)$ (cf., [17]). Thus we immediately have the following

Proposition 3.6. *Let Φ, Φ_i be as above. If one of Φ_i^* is π-orbit injective, then so is Φ^*.*

We point out that it could happen only one of the Φ_i^* is π-orbit injective: Consider the case $H = K_1 = K_2$ and $\dim H = n$. Let $\Psi : B(H) \to B(H)$ be a π-orbit injective quantum channel, where π is a unitary representation of a finite group G on H. Let $\Phi^*(A) = \frac{1}{n} I_H \otimes \Psi(A)$. Then $\Phi_1^*(A) = \frac{1}{n} Tr(\Psi(A)) I_H$ and $\Phi_2^*(A) = \Psi(A)$. While clearly Φ_2^* is π-orbit injective, Φ_1^* is not π-orbit injective. Indeed, for any unit vectors $x \in H$, we have

$$\Phi_1^*(\rho_x) = \frac{1}{n} Tr(\Psi(\rho_x)) I_H = \frac{1}{n} Tr(\rho_x) I_H = \frac{1}{n} I_H$$

which implies that $\Phi_1^*(\rho_x) = \Phi_1^*(\rho_y)$ for all unit vectors $x, y \in H$. But $[\rho_x]_\pi = [\rho_y]_\pi$ is not necessarily true since G is finite.

The next example shows that the converse of Proposition 3.6 is false.

Example 3.1. Let $n = \dim H$ and let $\Psi_i : B(H) \to B(H)$ $(i = 1, 2)$ be quantum channels such that each of them is not π-orbit injective but they jointly orbit injective, i.e., $\Psi_i(\rho_x) = \Psi_i(\rho_y)$ for both $i = 1, 2$ imply that $[\rho_x]_\pi = [\rho_y]_\pi$ (such a pair can be easily constructed). Define

$$\Phi^*(A) = \frac{1}{n} \Psi_1(A) \otimes I_H + \frac{1}{n} I_H \otimes \Psi_2(A), \quad A \in B(H).$$

Then

$$\Phi_1^*(A) = \Psi_1(A) + \frac{1}{n} Tr(\Psi_2(A)) I_H = \Psi_1(A) + \frac{1}{n} Tr(A) I_H$$

and

$$\Phi_2^*(A) = \frac{1}{n} Tr(\Psi_1(A)) I_H + \Psi_2(A) = \frac{1}{n} Tr(A) I_H + \Psi_2(A).$$

If $\Phi^*(\rho_x) = \Phi^*(\rho_y)$ for some $x, y \in H$, then we get

$$\frac{1}{n} \Psi_1(\rho_x) \otimes I_H + \frac{1}{n} I_H \otimes \Psi_2(\rho_x) = \frac{1}{n} \Psi_1(\rho_y) \otimes I_H + \frac{1}{n} I_H \otimes \Psi_2(\rho_y)$$

which implies by taking partial traces that $\Psi_1(\rho_x) = \Psi_1(\rho_y)$ and $\Psi_2(\rho_x) = \Psi_2(\rho_y)$. Thus $[\rho_x]_\pi = [\rho_y]_\pi$ and therefore Φ^* is π-orbit injective. However, neither of its marginal channels is π-orbit injective since neither Ψ_1 nor Ψ_2 is π-orbit injective.

3.3. The dual picture of orbit injective channels

Let $\Phi : B(H) \to B(K)$ be a quantum channel. Then $\Phi^* : B(K) \to B(H)$ is unital and completely positive. In fact, $\Phi(S) = \sum_{i=1}^{r} A_i^* S A_i$ if $\Phi(T) = \sum_{i=1}^{r} A_i T A_i^*$. We say that Φ does *phase retrieval* if there exists a collection of orthogonal projections (observables) $\{P_\alpha\}$ in $B(K)$ such that the measurements $\{tr(\rho_x \Phi^*(P_\alpha))\}_\alpha$ uniquely determine the pure state ρ_x in $B(H)$ in the sense that $tr(\rho_x \Phi^*(P_\alpha)) = tr(\rho_y \Phi^*(P_\alpha))$ for each α implies that $\rho_x = \rho_y$. Some characterizations for such channels have been examined in [16]. The following tells us that a π-orbit injective quantum channel is not phase retrievable unless π is trivial (i.e., $\pi = \chi I_H$)

Proposition 3.7. *Let $\Phi : B(H) \to B(K)$ be a π-orbit invariant quantum channel from $B(H)$ to $B(K)$. Then the following are equivalent:*

(i) Φ is orbit injective.

(ii) there exists a collection of observables $\{P_\alpha\}$ in $B(K)$ such that the measurements $\{tr(\rho_x \Phi^*(P_\alpha))\}_\alpha$ uniquely determine the pure state $[\rho_x]_\pi$.

In particular, if $\Phi = \Phi_\pi$, then Φ is orbit injective if and only if the orbit $[\rho_x]_\pi$ is uniquely determined by $\{tr(\rho_x \Phi(P_\alpha))\}_\alpha$.

Proof. $(i) \Rightarrow (ii)$: Suppose that Φ is orbit injective. Let $\{P_\alpha\}$ be a family of orthogonal projections such that $span\{P_\alpha\} = B(K)$. If $tr(\rho_x \Phi^*(P_\alpha)) = tr(\rho_y \Phi^*(P_\alpha))$ for every α, then we get $\langle \Phi(\rho_x), P_\alpha \rangle = \langle \Phi(\rho_y), P_\alpha \rangle$ for each α and hence $\Phi(\rho_x) = \Phi(\rho_y)$ since $span\{P_\alpha\} = B(K)$. Therefore, by the orbit injectivity of Φ, $[\rho_x]_\pi = [\rho_y]_\pi$.

$(ii) \Rightarrow (i)$: Suppose that $\Phi(\rho_x) = \Phi(\rho_y)$. Then the same argument implies that

$$tr(\rho_x \Phi^*(P_\alpha)) = tr(\rho_y \Phi^*(P_\alpha))$$

for every α and hence $[\rho_x]_\pi = [\rho_y]_\pi$. Therefore Φ is orbit injective. □

4. Characters with property (H)

In this section we characterize the family of characters that have property (H).

4.1. The real case

We study real valued multiplicative characters which are used in Theorem 2.4 and Theorem 2.6 to characterize orbit injective quantum channels over real Hilbert spaces.

Let G be a finite group. Let $\chi : G \to \{\pm 1\}$ be a character. Then χ factors through G^{ab} (the abelianization of G) and then factors through $G^{ab}/(G^{ab})^2$ as $\{\pm 1\}$ has order 2. Without loss of generality, we may assume that G is an elementary finite abelian 2-group, i.e. $G \cong (\mathbb{Z}/2\mathbb{Z})^k$. Then $\widehat{G} \cong (\mathbb{Z}/2\mathbb{Z})^k$ and we view it as an $\mathbb{F}_2$-vector space of dimension k. Fix an isomorphism $\iota : \widehat{G} \to G$. This is equivalent to fixing a pairing on $G \times G$.

Let χ_1, χ_2,..., χ_k be an $\mathbb{F}_2$-basis of $\widehat{G}$. Then $\widehat{G} = \langle\chi_1\rangle \oplus \cdots \oplus \langle\chi_k\rangle$. Let G_i be the orthogonal complement of the kernel of χ_i with respect to the above paring. Then $G = G_1 \oplus \cdots \oplus G_k$. We claim that for any tuple $(a_i)_{1\leq i\leq k}$ with $a_i \in \{\pm 1\}$, there exists an element $g \in G$ such that $a_i a_j = \chi_i(g)\chi_j(g)$. Indeed, $\langle\chi_i\rangle$ is the dual group of G_i and one may simply take $g_i \in G_i$ with $\chi_i(g_i) = a_i$. Then $\chi_j(g_i) = 1$ for all $j \neq i$. Let $g = (g_i)_{1\leq i\leq k} \in G$, then this g satisfies $a_i = \chi_i(g)$ hence satisfies the required property. The element $1 - g = (1 - g_i)_{1\leq i\leq k} \in G$ satisfies $a_i = -\chi_i(1 - g)$. They are the only two elements with $a_i a_j = \chi_i(g)\chi_j(g)$.

Now let $\chi_1, \cdots, \chi_l$ be l characters with $l > k$. Then there are 2^l possibilities for the tuple $(a_j)_{1\leq j\leq l}$, but only 2^k possibilities for the tuple $(\chi_j(g))_{1\leq j\leq l}$. In general the expected element g does not exist. We have the following result.

Proposition 4.1. *Let G be a finite group. Let χ_1,..., χ_l be real valued characters of G. Let $k = \log_2(|G^{ab}/(G^{ab})^2|)$ and $s = \dim_{\mathbb{F}_2} span\{\chi_1, \ldots, \chi_l\}$. Note that $s \leq k$. Then the characters χ_1,..., χ_l have property (H) if and only if one of the following two holds:*

(1) *$s = l$;*
(2) *$s + 1 = l$ and the only relation involves odd number of characters.*

Proof. Let $H := G^{ab}/(G^{ab})^2$. Then $H \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus k} \cong \widehat{H}$. View $\widehat{H}$ as an $\mathbb{F}_2$-vector space of dimension k and fix an isomorphism $H \to \widehat{H}$.

Let χ_1,..., χ_s be a maximal linearly independent subset of χ_1,..., χ_l. Let H_i be the orthogonal complement of the kernel of χ_i as before. Then

$$\sum_{i=1}^{s}(\ker(\chi_i)) = \sum_{i=1}^{l}\ker(\chi_i).$$

Note that $H_i \cong \mathbb{Z}/2\mathbb{Z}$ and $\widehat{H}_i = \langle\chi_i\rangle$. Write $H = H_1 \oplus \cdots \oplus H_s \oplus H^c$. Then $\chi_i(g) = \chi_i(g + x)$ for all $x \in H^c$ and $1 \leq i \leq l$. Therefore, changing g by an element $x \in H^c$ does not affect our discussion. We may assume that H^c is trivial and $s = k$.

If $s = l$, then χ_1,..., χ_l satisfy property (H).

If $s + 2 \leq l$, then χ_1,..., χ_l do not satisfy property (H) by counting the possibilities.

If $s + 1 = l$, write $\chi_{s+1} = \prod_{i=1}^{s} \chi_i^{e_i}$ ($e_i = 0$ or 1). Let $(a_i)_{i=1}^{s+1} \in \{\pm 1\}^{s+1}$. Without χ_{s+1}, there exists a unique $g \in H = H_1 \oplus \cdots \oplus H_s$ with $\chi_i(g) = a_i$ $(1 \leq i \leq s)$. Then $\chi_i(1 - g) = -a_i$ $(1 \leq i \leq s)$. These are the only two elements with $a_i a_j = \chi_i(x)\chi_j(x)$ $(1 \leq i \leq s)$. (Recall that we assume that H^c is trivial as its elements do not affect the values of $\chi_i(g)$.) Note that

$$\chi_{s+1}(g) = \prod_{i=1}^{s} a_i^{e_i}, \quad \chi_{s+1}(1 - g) = (-1)^{\sum_{i=1}^{s} e_i} \prod_{i=1}^{s} a_i^{e_i}.$$

- Case that $\sum_{i=1}^{s} e_i$ is even. In this case, if $a_{s+1} = \prod_{i=1}^{s} a_i^{e_i}$, then g satisfies $\chi_i(g) = a_i$ for $1 \le i \le l$; if $a_{s+1} = -\prod_{i=1}^{s} a_i^{e_i}$, then $1-g$ satisfies $\chi_i(1-g) = -a_i$ for $1 \le i \le l$. Here we have an element $x \in H$ with $a_i a_j = \chi_i(x)\chi_j(x)$ $(1 \le i \le l)$.
- Case that $\sum_{i=1}^{s} e_i$ is odd. In this case, if $a_{s+1} = \prod_{i=1}^{s} a_i^{e_i}$, then g satisfies $\chi_i(g) = a_i$ for $1 \le i \le l$; if $a_{s+1} = -\prod_{i=1}^{s} a_i^{e_i}$, then we have no solution for $a_i a_j = \chi_i(x)\chi_j(x)$ $(1 \le i \le l)$.

The proposition follows. □

Example 4.1. Let $G = (\mathbb{Z}/2\mathbb{Z})^k$ and let $\chi_1, \chi_2, \ldots, \chi_k$ be an $\mathbb{F}_2$-basis of $\widehat{G}$. The characters $\chi_1, \ldots, \chi_k$ satisfy property (H). Therefore if $\pi = \chi_{i_1}^{m_1} \oplus \ldots \oplus \chi_{i_\ell}^{m_\ell}$ for some $i_1, \ldots, i_\ell \in \{1, 2, \ldots, k\}$, then Φ_π is orbit-injective by Theorem 2.4. See also Example 2.2.

4.2. The complex case

The second characterization is about complex valued multiplicative characters for (infinite) compact groups. In this case, the unit circle $\mathbb{T}$ plays the role of $\mathbb{Z}/2\mathbb{Z}$ and the essential case is $\mathbb{T}^k = \mathbb{T} \times \cdots \times \mathbb{T}$. For example, for a Lie group G, a character $\chi : G \to \mathbb{T}$ factors through G^{ab} and its connected component would be a product of copies of $\mathbb{R}$ and copies of $\mathbb{T}$.

We could see the strategy in the following explicit case. Let $G = \mathbb{T} \times \cdots \times \mathbb{T} = \mathbb{T}^k$. Let ψ_i be the projection of G to the i-th component. Then $\widehat{G}$ is a free $\mathbb{Z}$-module with basis $\psi_1, \ldots, \psi_k$. Certainly, $\psi_1, \ldots, \psi_k$ satisfy property (H) in the sense that for any $(a_i) \in \mathbb{T}^k$ there exist $g \in G$ with $a_i \bar{a}_j = \psi_i(g)\bar{\psi}_j(g)$. This g is unique if we require $a_i = \psi_i(g)$ as $\psi_1 \oplus \cdots \oplus \psi_k$ is an isomorphism. The other elements with $a_i \bar{a}_j = \chi_i(g)\bar{\chi}_j(g)$ are those of the form tg where $t \in \mathbb{T}$ and we view it as an element of G by diagonal embedding. (Indeed, $\psi_i(tg) = t\psi_i(g) = ta_i$.) It is possible to add one more character to the family of characters. Let $\psi_{k+1} = \prod_{i=1}^{k} \psi_i^{d_i}$. Let $(a_1, \ldots, a_k, a_{k+1}) \in \mathbb{T}^{k+1}$. Let $g \in G$ be the unique element such that $\psi_i(g) = a_i$ $(1 \le i \le k)$. Then

$$\psi_{k+1}(g) = \prod_{i=1}^{k} a_i^{d_i} = c a_{k+1}, \quad c \in \mathbb{T}.$$

If $c = 1$ (i.e. $\prod_{i=1}^{k} a_i^{d_i} = a_{k+1}$), then g satisfies $a_i = \psi_i(g)$ for all i, hence satisfies the equations $a_i \bar{a}_j = \psi_i(g)\bar{\psi}_j(g)$ $(1 \le i, j \le k+1)$.

If $c \ne 1$, we look for $d \in \mathbb{T}$ with $\psi_i(dg) = da_i$ $(d \in \mathbb{T},\ 1 \le i \le k+1)$. The only problem is for $i = k+1$, which gives us the equation

$$d^{\sum_{i=1}^{k} d_i} c = d.$$

Therefore, characters $\psi_1, \ldots, \psi_k$, $\psi_{k+1} = \prod_{i=1}^{k} \psi_i^{d_i}$ satisfy property (H) if and only if $\sum_{i=1}^{k} d_i \ne 1$.

For the general case, let $\chi_1,\ldots,\chi_l$ be a family of characters of G. Note that if $l \geq k+2$, then these characters do not satisfy property (H) by comparing the dimensions. Let s be the rank of the subgroup of $\widehat{G}$ generated by these χ. This subgroup corresponds to a subtorus of G with dimension s. By replacing G with this subtorus, we may assume that $s = k$. As for finite group case, we have two cases $l = k$ and $l = k+1$. Similar argument gives us the following result.

Proposition 4.2. *Let $G = \mathbb{T}^k$. Let $\chi_1,\ldots,\chi_l$ be characters of G such that the rank of the group generated by them is s. The characters $\chi_1,\ldots,\chi_l$ satisfy property (H) if and only if one of the following two holds.*

(1) *$s = l$.*
(2) *$s+1 = l$, and suppose that $\chi_1,\ldots,\chi_s$ form a set of generators, in the relation $\chi_l = \prod_{i=1}^{s} \chi_i^{d_i}$, $\sum_{i=1}^{s} d_i \neq 1$.*

Proof. As explained above, we may assume that $s = k$. Any nontrivial character of G is surjective. If $\chi_1,\ldots,\chi_s$ are $\mathbb{Q}$-linearly independent in $\widehat{G} \otimes \mathbb{Q} \cong \mathbb{Q}^k$, then $\chi_1 \oplus \cdots \oplus \chi_s : G \to \mathbb{T}^s$ is surjective. If $l = s$, then $\chi_1,\ldots,\chi_l$ satisfy property (H).

If $l = s+1 = k+1$ and $\chi_1,\ldots,\chi_{l-1}$ form a set of generators, for a given $(a_i) \in \mathbb{T}^l$, there exist $g \in G$ such that $\chi_i(g) = a_i$ $(1 \leq i \leq l-1 = k)$. From the assumption on $\chi_1,\ldots,\chi_{l-1}$, the natural map $\chi_1 \oplus \cdots \oplus \chi_{l-1} : G \to \mathbb{T}^{l-1}$ is surjective. Therefore for any $t \in \mathbb{T}$, there exists at least one $h_t \in G$ such that $\chi_i(h_t) = t$ for all $1 \leq i \leq l-1$. Write $\chi_l = \prod_{i=1}^{s} \chi_i^{d_i}$, similar argument as before shows that $\chi_1,\ldots,\chi_l$ satisfy property (H) if and only if $\sum_{i=1}^{s} d_i \neq 1$. This completes the proof of the proposition. □

Example 4.2. Let $G = U(n_1) \oplus \cdots \oplus U(n_k)$. Then each character of G factors through $G^{ab} \cong \mathbb{T} \oplus \cdots \oplus \mathbb{T} = \mathbb{T}^k$ via the determinant map. One could then easily construct orbit injective quantum channels by combining Theorem 3.1 and Proposition 4.2.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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