Applied Mathematics Letters 141 (2023) 108624

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Long-time H^1-stability of BDF2 time stepping for 2D Navier–Stokes equations

L. Rebholz [a,*], F. Tone [b]

[a] *School of Mathematical and Statistical Sciences, Clemson University, Clemson, SC 29634, USA*
[b] *Department Mathematics and Statistics, University of West Florida, Pensacola, FL 32514, USA*

ARTICLE INFO

Article history:
Received 30 November 2022
Received in revised form 13 February 2023
Accepted 14 February 2023
Available online 16 February 2023

Keywords:
Navier–Stokes equations
BDF2 scheme
Long-time stability

ABSTRACT

In this paper we study the H^1-stability for all positive time of the BDF2 scheme for the 2D Navier–Stokes equations. More precisely, we discretize in time using the backward differentiation formula (BDF2), and with the aid of the discrete Gronwall lemma and of the discrete uniform Gronwall lemma we prove that the numerical scheme admits this stability.

1. Introduction

We prove herein new analysis results for the BDF2 temporal discretization of the Navier–Stokes (NS) equations of viscous incompressible fluids. The NS equations are given on a domain $\varOmega \subset \mathbb{R}^2$ with boundary $\partial\varOmega$ of class C^2 by

$$u_t + (u \cdot \nabla)u - \nu \varDelta u + \nabla p = f, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

where u is the velocity, p is the pressure, ν is the kinematic viscosity and f represents body forces applied to the fluids. We complete these equations with the given initial condition $u(x,0) = u_0(x)$, and with the non-slip boundary condition $u|_{\partial\varOmega} = 0$ for simplicity.

The BDF2 temporal discretization is a widely used time stepping method in NS simulations due to its second order accuracy and attractive stability properties. It is the main temporal discretization method for NS and related systems in the deal.II finite element software [1], and appears in many highly cited papers and books on numerical methods for NS e.g. [2–4]. Improvements to it are still being developed including discrete

* Corresponding author.
E-mail address: rebholz@clemson.edu (L. Rebholz).

https://doi.org/10.1016/j.aml.2023.108624
0893-9659/

regularizations [5,6], variable time step sizes [7], and blending BDF2 and BDF3 [8], and its understanding through rigorous numerical analysis is also still being researched and improved [9,10]. Hence new theoretical results for BDF2 time stepping for NS are of potential wide interest in the CFD community.

In this note we prove long-time H^1 stability of BDF2 time stepping for the 2D NS equations. Long-time L^2 stability of BDF2 for NS has long been known both in 2D and 3D, see e.g. the proof in [10] which requires a very mild data-dependent restriction on the time step size, and this stability is expected since it is a discrete analogue of the (non-discretized) NS energy inequality. Long-time H^1 stability of BDF2 for NS is not expected in 3D with modern analysis tools, since such a proof is thought to also imply uniqueness of weak NS solutions and thus solve the Clay Prize problem. In 2D, long-time bounds exist on the (unique) NS solution at the PDE level, and thus one hopes that a stable and accurate time stepping scheme would also yield long-time H^1 stability. However, such results seem quite difficult to prove, and results in this direction are quite recent and include long-time H^1 stability for a BDF2-finite element scheme for the NS in velocity–vorticity form [11], for NS in usual velocity–pressure form with backward Euler time stepping [12] and for Crank–Nicolson [13] and then extended to general second order [14], and backward Euler with the NS vorticity-stream-function formulation [15].

The purpose of this note is to close the gap and prove long-time H^1 stability for BDF2 time stepping for 2D NS in usual velocity–pressure form with the restriction of only $\Delta t \leq C\nu^{-1}$. Our results show that numerical instabilities in the analogous 3D scheme must come precisely from the vortex stretching term. We point out that known higher order stability results for Crank–Nicolson or general second order methods in [13,14] require $\Delta t < O(h^2)$; while the overall proof techniques they use are similar to ours in that they both utilize Gronwall and uniform Gronwall, our analysis takes explicit advantage of the extra positivity of BDF2 to remove the CFL condition. While no time step restriction is needed for the H^1 stability in the velocity–vorticity scheme in [11], using such a scheme in practice can be more expensive and also may require access to vorticity boundary conditions. Our results are extendable to mixed Dirichlet/Neumann boundary conditions, but other types could create technical difficulties that require separate analyses.

2. Mathematical preliminaries

For the mathematical setting of the problem, we consider the following spaces:

$$V = \left\{ v \in H_0^1(\Omega)^2,\, \operatorname{div} v = 0 \right\}, \tag{2.1}$$

$$H = \left\{ v \in L^2(\Omega)^2,\, \operatorname{div} v = 0,\, v \cdot n = 0 \text{ on } \partial\Omega \right\}, \tag{2.2}$$

where n is the unit outward normal on $\partial\Omega$.

We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and inner product of L^2 and we recall the Poincaré inequality: there exists $c_P > 0$ depending only on the size of the domain and satisfying

$$\|u\| \leq c_P \|\nabla u\|, \quad \forall\, u \in V. \tag{2.3}$$

The weak formulation of the Navier–Stokes equations is obtained by multiplying (1.1) by a test function $v \in V$ and integrating by parts over Ω, using Green's formula, viz.,:

$$\frac{d}{dt}(u(t), v) + \nu(\nabla u(t), \nabla v) + b(u(t), u(t), v) = (f(t), v) \quad \forall\, v \in V, \tag{2.4}$$

where

$$b(u, v, w) := \sum_{i,j=1,2} \int_\Omega u_i(x) \frac{\partial v_j}{\partial x_i}(x) w_j(x)\, \mathrm{d}x = (u \cdot \nabla v, w). \tag{2.5}$$

The form b is trilinear continuous on $H^1(\Omega)^2$ and enjoys the following properties [16–18]:

$$|b(u,v,w)| \leq c_b\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla v\|\|w\|^{1/2}\|\nabla w\|^{1/2} \quad \forall\, u,v,w \in V, \tag{2.6}$$

$$|b(u,v,w)| \leq c_b\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla v\|^{1/2}\|\Delta v\|^{1/2}\|w\| \quad \forall\, u \in V,\ v \in H^2(\Omega)^2 \cap V,\ w \in H, \tag{2.7}$$

$$b(u,v,v) = 0, \quad \forall\, u,v \in V, \tag{2.8}$$

Our analysis will be greatly simplified by utilizing the G-stability framework, as in [19]. Hence, we define here the G-matrix

$$G = \begin{bmatrix} 1/2 & -1 \\ -1 & 5/2 \end{bmatrix},$$

and its associated G-norm

$$\|x\|_G^2 = (x, Gx) = \frac{|b|^2 + |2b-a|^2}{2}, \quad \text{for any } x = [a,b]^T \in \mathbb{R}^2.$$

It is well-known from [19] that the L^2-norm and the G-norm are equivalent in the following sense: there exist $c_l > 0$ and $c_u > 0$ such that

$$c_l\|\chi\| \leq \|\chi\|_G \leq c_u\|\chi\|. \tag{2.9}$$

We also recall from [19] that if a, b, c are in $L^2(\Omega)$, then

$$(3c - 4b + a, c) = \|[b,c]^T\|_G^2 - \|[a,b]^T\|_G^2 + \frac{\|a - 2b + c\|^2}{2}. \tag{2.10}$$

3. H^1 stability of BDF2 for Navier–Stokes

We consider now the BDF2 temporal discretization of (2.4): Given u^{-1} and u^0 in V, find $u^n \in V$ for $n = 1, 2, \ldots$ satisfying

$$\frac{1}{2\Delta t}(3u^{n+1} - 4u^n + u^{n-1}) - \nu\Delta u^{n+1} + [(2u^n - u^{n-1})\cdot\nabla]u^{n+1} = f^{n+1}. \tag{3.1}$$

We seek to obtain uniform bounds on $\|\nabla u^n\|$.

We assume that $f \in L^\infty(\mathbb{R}_+; H)$ and we set $\|f\|_\infty := \|f\|_{L^\infty(\mathbb{R}_+;H)}$. We also assume that the initial conditions are bounded as follows, where $c = O(1)$ is a constant:

$$\|u^0\| \leq \|u_0\|, \|\nabla u^0\| \leq c\|\nabla u_0\|$$
$$\|u^{-1}\| \leq \|u_0\|, \|\nabla u^{-1}\| \leq c\|\nabla u_0\|$$

We adopt the following convention: c_i denote constants that depend only on the parameters such as c_P, ν, etc; K_i depends in addition on $u(t_*)$ at some specified time t_* and on the forcing f; κ_i are bounds on the time step Δt and may depend on u_0 and f. We also set $\chi^n = [u^{n-1}, u^n]^T$, $\nabla\chi^n = [\nabla u^{n-1}, \nabla u^n]^T$, for any $n = 1, 2, \ldots$.

In proving the main result, we will need a couple of preliminary lemmas. We begin with recalling the following result from [10].

Lemma 1. *If* $0 < \Delta t \leq \frac{8c_P^2}{\nu c_l^2}$, *then* $\forall\, n \geq 0$,

$$\begin{aligned} \|\chi^n\|_G^2 + \frac{\nu}{4}\Delta t\|\nabla u^n\|^2 &\leq \left(1 + \frac{\nu c_l^2}{4c_P^2}\Delta t\right)^{-n}\left(\|\chi^0\|_G^2 + \frac{\nu}{4}\Delta t\|\nabla u^0\|^2\right) \\ &+ \frac{4c_P^4}{\nu^2 c_l^2}\|f\|_\infty^2\left[1 - \left(1 + \frac{\nu c_l^2}{4c_P^2}\Delta t\right)^{-n}\right], \end{aligned} \tag{3.2}$$

and there exists $K_1 = K_1(\|u_0\|, \|\nabla u_0\|, \|f\|_\infty) = \|\chi^0\|_G^2 + \frac{2c_P^2}{c_l^2}\|\nabla u^0\|^2 + \frac{4c_P^4}{\nu^2 c_l^2}\|f\|_\infty^2$ *such that*

$$\|\chi^n\|_G^2 \le K_1, \, \forall \, n \ge 0, \tag{3.3}$$

and

$$\nu\Delta t \sum_{j=i}^{n} \|\nabla u^j\|^2 \le K_1 + (n-i+1)\Delta t \, \frac{c_P^2}{\nu}\|f\|_\infty^2, \quad \forall \, i = 1, \dots, n. \tag{3.4}$$

Corollary 3.1. *If*

$$0 < \Delta t \le \frac{4c_P^2}{\nu c_l^2} =: \kappa_1, \tag{3.5}$$

then

$$\|\chi^n\|_G^2 \le 2\rho_0^2, \quad \forall \, n \ge N_0 := \lfloor T_0/\Delta t \rfloor, \tag{3.6}$$

where $\rho_0 := \frac{2c_P^2}{\nu c_l}\|f\|_\infty$ *and*

$$T_0 = T_0(\|\nabla u_0\|, \|f\|_\infty) := \frac{8c_P^2}{\nu c_l^2} \ln \frac{2c_u^2\|u^0\|^2 + \frac{c_P^2}{c_l^2}\|\nabla u^0\|^2}{\rho_0^2}. \tag{3.7}$$

Proof. From the bound (3.2) on $\|\chi^n\|_G^2$, we infer that

$$\|\chi^n\|_G^2 \le \left(1 + \frac{\nu c_l^2}{4c_P^2}\Delta t\right)^{-n} \left(\|\chi^0\|_G^2 + \frac{\nu}{4}\Delta t\|\nabla u^0\|^2\right) + \rho_0^2,$$

and using assumption (3.5) on Δt and the fact that $1 + x \ge \exp(x/2)$ if $x \in (0,1)$ we obtain

$$\|\chi^n\|_G^2 \le \exp\left(-n\Delta t \frac{\nu c_l^2}{8c_P^2}\right) \left(\|\chi^0\|_G^2 + \frac{\nu}{4}\Delta t\|\nabla u^0\|^2\right) + \rho_0^2.$$

For $n\Delta t \ge T_0$, the above inequality implies conclusion (3.6) of the corollary. □

We now seek to obtain uniform bounds for $\|\chi^n\|_G$ in H^1. In order to do this, we will first use the discrete Gronwall lemma to derive an upper bound on $\|\nabla\chi^n\|_G$, $n \le N$, for some $N > 0$, and then we will use the discrete uniform Gronwall lemma to obtain an upper bound on $\|\nabla\chi^n\|_G$, $n \ge N$. We begin with some preliminary inequalities.

Lemma 2. *For every* $\Delta t \le \kappa_1$ *and for every* $n \ge 1$, *we have*

$$\|\nabla\chi^{n+1}\|_G^2 - \|\nabla\chi^n\|_G^2 \le \frac{c_1}{\nu^3}K_1\Delta t\left(4\|\nabla u^n\|^2 + \|\nabla u^{n-1}\|^2\right)\|\nabla\chi^{n+1}\|_G^2 + \frac{2}{\nu}\Delta t\|f\|_\infty^2. \tag{3.8}$$

Proof. We multiply (3.1) by $-2\Delta t\Delta u^{n+1}$, integrate by parts using Green's formula, and then apply (2.10) to obtain

$$\begin{aligned} \|\nabla\chi^{n+1}\|_G^2 - \|\nabla\chi^n\|_G^2 + \frac{\|\nabla u^{n+1} - 2\nabla u^n + \nabla u^{n-1}\|^2}{2} + 2\nu\Delta t\|\Delta u^{n+1}\|^2 \\ - 2\Delta t b(2u^n - u^{n-1}, u^{n+1}, \Delta u^{n+1}) = -2\Delta t(f^{n+1}, \Delta u^{n+1}). \end{aligned} \tag{3.9}$$

Using property (2.7) of the trilinear form b and recalling (3.3), Young's inequality and utilizing the G-norm, we have the following bound of the nonlinear term,

$$\begin{aligned}
&2\Delta t b(2u^n - u^{n-1}, u^{n+1}, \Delta u^{n+1}) \\
&\quad \le 2\, c_b\, \Delta t\, \|2u^n - u^{n-1}\|^{1/2}\|2\nabla u^n - \nabla u^{n-1}\|^{1/2}\|\nabla u^{n+1}\|^{1/2}\|\Delta u^{n+1}\|^{3/2} \\
&\quad \le \frac{\nu}{2}\Delta t\|\Delta u^{n+1}\|^2 + \frac{27c_b^4}{2\nu^3}\Delta t\|2u^n - u^{n-1}\|^2\|2\nabla u^n - \nabla u^{n-1}\|^2\|\nabla u^{n+1}\|^2 \\
&\quad \le \frac{\nu}{2}\Delta t\|\Delta u^{n+1}\|^2 + \frac{c_1}{\nu^3}\Delta t\|\chi^n\|_G^2(4\|\nabla u^n\|^2 + \|\nabla u^{n-1}\|^2)\|\nabla\chi^{n+1}\|_G^2 \\
&\quad \le \frac{\nu}{2}\Delta t\|\Delta u^{n+1}\|^2 + \frac{c_1}{\nu^3}\Delta t K_1(4\|\nabla u^n\|^2 + \|\nabla u^{n-1}\|^2)\|\nabla\chi^{n+1}\|_G^2.
\end{aligned} \tag{3.10}$$

We bound the right-hand side of (3.9) by Cauchy–Schwarz,

$$2\Delta t(f^{n+1}, \Delta u^{n+1}) \le 2\Delta t\|f^{n+1}\|\|\Delta u^{n+1}\| \le \frac{\nu}{2}\Delta t\|\Delta u^{n+1}\|^2 + \frac{2}{\nu}\Delta t\|f^{n+1}\|^2. \tag{3.11}$$

Relations (3.9)–(3.11) imply

$$\begin{aligned}
&\|\nabla\chi^{n+1}\|_G^2 - \|\nabla\chi^n\|_G^2 + \frac{\|\nabla u^{n+1} - 2\nabla u^n + \nabla u^{n-1}\|^2}{2} + \nu\Delta t\|\Delta u^{n+1}\|^2 \\
&\qquad \le \frac{c_1}{\nu^3}\Delta t K_1(4\|\nabla u^n\|^2 + \|\nabla u^{n-1}\|^2)\|\nabla\chi^{n+1}\|_G^2 + \frac{2}{\nu}\Delta t\|f^{n+1}\|^2,
\end{aligned} \tag{3.12}$$

and from here (3.8) follows right away. □

Lemma 3. *For every $\Delta t \le \kappa_1$ and for every $n \ge 1$, we have*

$$\|\nabla\chi^{n+1}\|_G^2 \le K_2\|\nabla\chi^n\|_G^2 + \frac{2}{\nu}\|f\|_\infty^2, \tag{3.13}$$

where $K_2 = K_2(\|u_0\|, \|\nabla u_0\|, \|f\|_\infty) = 2(1 + \frac{c_5}{\nu^4}K_1^2)$, with K_1 being given in Lemma 1.

Proof. Multiplying (3.1) by $2\Delta t(3u^{n+1} - 4u^n + u^{n-1})$ in $L^2(\Omega)^2$, and recalling (2.10) we obtain

$$\begin{aligned}
&\|3u^{n+1} - 4u^n + u^{n-1}\|^2 + 2\nu\Delta t\left[\|\nabla\chi^{n+1}\|_G^2 - \|\nabla\chi^n\|_G^2 + \frac{\|\nabla u^{n+1} - 2\nabla u^n + \nabla u^{n-1}\|^2}{2}\right] \\
&\quad + 2\Delta t b(2u^n - u^{n-1}, u^{n+1}, 3u^{n+1} - 4u^n + u^{n-1}) = 2\Delta t(f^{n+1}, 3u^{n+1} - 4u^n + u^{n-1}).
\end{aligned} \tag{3.14}$$

Using the Cauchy–Schwarz inequality, we majorize the right-hand side of (3.14) by

$$\begin{aligned}
2\Delta t(f^{n+1}, 3u^{n+1} - 4u^n + u^{n-1}) &\le 2\Delta t\|f^{n+1}\|\|3u^{n+1} - 4u^n + u^{n-1}\| \\
&\le \frac{\|3u^{n+1} - 4u^n + u^{n-1}\|^2}{2} + 2\Delta t^2\|f^{n+1}\|^2.
\end{aligned} \tag{3.15}$$

Using (2.8) and (2.6), we bound the nonlinear term as follows:

$$\begin{aligned}
&2\Delta t b(2u^n - u^{n-1}, u^{n+1}, 3u^{n+1} - 4u^n + u^{n-1}) \\
&= 2\Delta t b(2u^n - u^{n-1}, u^{n+1}, 2(u^{n+1} - u^n) + u^{n+1} - 2u^n + u^{n-1}) \\
&= -4\Delta t b(2u^n - u^{n-1}, u^{n+1}, u^n) + 2\Delta t b(2u^n - u^{n-1}, u^{n+1}, u^{n+1} - 2u^n + u^{n-1}) \\
&\le \nu c_l^2\Delta t\|\nabla u^{n+1}\|^2 + \nu c_l^2\Delta t\|\nabla u^n\|^2 + \frac{\nu}{2}\Delta t\|\nabla u^{n+1} - 2\nabla u^n + \nabla u^{n-1}\|^2 \\
&\quad + \frac{c_2}{\nu^3}\Delta t\|2u^n - u^{n-1}\|^2\|2\nabla u^n - \nabla u^{n-1}\|^2\|u^n\|^2 \\
&\quad + \frac{c_3}{\nu^3}\Delta t\|2u^n - u^{n-1}\|^2\|2\nabla u^n - \nabla u^{n-1}\|^2\|u^{n+1} - 2u^n + u^{n-1}\|^2 \\
&\le \nu\Delta t\|\nabla\chi^{n+1}\|_G^2 + \frac{\nu}{2}\Delta t\|\nabla u^{n+1} - 2\nabla u^n + \nabla u^{n-1}\|^2 \\
&\quad + \frac{c_4}{\nu^3}\Delta t(\|u^{n+1}\|^2 + 5\|u^n\|^2 + \|u^{n-1}\|^2)\|\chi^n\|_G^2\|\nabla\chi^n\|_G^2.
\end{aligned} \tag{3.16}$$

Relations (3.14)–(3.16) yield

$$\frac{\|3u^{n+1}-4u^n+u^{n-1}\|^2}{2}+\nu\Delta t\|\nabla\chi^{n+1}\|_G^2-2\nu\Delta t\left(1+\frac{c_4}{2\nu^4}(\|u^{n+1}\|^2+5\|u^n\|^2+\|u^{n-1}\|^2)\|\chi^n\|_G^2\right)\|\nabla\chi^n\|_G^2$$
$$+\frac{\nu}{2}\Delta t\|\nabla u^{n+1}-2\nabla u^n+\nabla u^{n-1}\|^2\le 2\Delta t^2\|f^{n+1}\|^2,$$

from which (3.13) follows right away. □

In order to prove the uniform boundedness of $\|\nabla\chi^n\|_G$ we will make use of the following two lemmas, whose proofs can be found in [20] or [12]:

Lemma 4. *Given $\Delta t>0$ and positive sequences ξ_n, η_n and ζ_n such that*

$$\xi_n\le\xi_{n-1}(1+\Delta t\eta_{n-1})+\Delta t\zeta_n,\qquad for\, n\ge 1,\tag{3.17}$$

we have, for any $n\ge 2$,

$$\xi_n\le\left(\xi_0+\sum_{i=1}^{n}\Delta t\zeta_i\right)\exp\left(\sum_{i=0}^{n-1}\Delta t\eta_i\right).\tag{3.18}$$

Lemma 5. *Given $\Delta t>0$, positive integers n_0, N, positive sequences ξ_n, η_n and ζ_n such that*

$$\xi_n\le\xi_{n-1}(1+\Delta t\eta_{n-1})+\Delta t\zeta_n,\qquad for\ n\ge n_0,\tag{3.19}$$

and given the bounds

$$\sum_{n=k_0}^{N+k_0}\Delta t\eta_n\le a_1,\qquad \sum_{n=k_0}^{N+k_0}\Delta t\zeta_n\le a_2,\qquad \sum_{n=k_0}^{N+k_0}\Delta t\xi_n\le a_3,\tag{3.20}$$

for any $k_0\ge n_0$, we have $\xi_n\le\left(\frac{a_3}{N\Delta t}+a_2\right)\mathrm{e}^{a_1},\forall n\ge N+n_0$.

We are now able to prove the main result:

Theorem 1. *Let $u_0\in V$ and u^n be a solution of the numerical scheme (3.1). Also, let Δt be such that $\Delta t\le\kappa_1$. Then there exists $K_6\big(\|\nabla u_0\|,\|f\|_\infty\big)$, such that*

$$\|\nabla u^n\|\le K_6\big(\|\nabla u_0\|,\|f\|_\infty\big),\quad \forall n\ge 0.\tag{3.21}$$

Moreover, there exists $K_5=K_5\big(\|f\|_\infty\big)$, such that

$$\|\nabla u^n\|\le K_5\big(\|f\|_\infty\big),\quad \forall n\ge N+N_0+2,\tag{3.22}$$

where $N_0:=\lfloor T_0/\Delta t\rfloor$, with T_0 being given in (3.7).

Remark 3.1. The time step restriction in Theorem 1 is $\Delta t\le\kappa_1=O(\nu^{-1})$. Hence this is a data dependent time step restriction, but moreover since $\nu<1$ in most practical problems of interest, this restriction is automatically satisfied even with $\Delta t\le O(1)$. The dependence of K_5 and K_6 is exponential in the inverse of ν, i.e. exponential in the Reynolds number, which is common in higher order stability results due to the use of the uniform Gronwall inequality [12–14].

Proof. Let $T > 0$ be arbitrarily fixed and let $\Delta t \le \kappa_1$. We set $N := \lfloor T/\Delta t \rfloor$.

In order to derive a uniform bound for $\|\nabla\chi^n\|_G$ for all $n \ge 1$, we will apply (the discrete Gronwall) Lemma 4 to obtain a bound valid for $n = 1, \dots, N + N_0 + 1$, and then we will apply (the discrete uniform Gronwall) Lemma 5 to obtain a bound valid for $n \ge N + N_0 + 2$. In doing so, we first notice that using (3.13), (3.8) yields

$$\begin{aligned} \|\nabla\chi^{n+1}\|_G^2 \le \left(1 + \frac{c_1}{\nu^3} K_1 K_2 \Delta t \left(4\|\nabla u^n\|^2 + \|\nabla u^{n-1}\|^2\right)\right) \|\nabla\chi^n\|_G^2 \\ + \frac{2}{\nu}\Delta t \left[1 + \frac{c_1}{\nu^3} K_1 \left(4\|\nabla u^n\|^2 + \|\nabla u^{n-1}\|^2\right)\right] \|f\|_\infty^2, \end{aligned} \tag{3.23}$$

which we rewrite in the form

$$\xi_n \le \xi_{n-1}(1 + \Delta t \eta_{n-1}) + \Delta t \zeta_n, \tag{3.24}$$

with

$$\begin{aligned} \xi_n = \|\nabla\chi^n\|_G^2, \qquad \eta_n = \frac{c_1}{\nu^3} K_1 K_2 \left(4\|\nabla u^n\|^2 + \|\nabla u^{n-1}\|^2\right), \\ \zeta_n = \frac{2}{\nu}\left[1 + \frac{c_1}{\nu^3} K_1 \left(4\|\nabla u^n\|^2 + \|\nabla u^{n-1}\|^2\right)\right] \|f\|_\infty^2. \end{aligned} \tag{3.25}$$

Recalling (3.4), Lemma 4 gives

$$\xi_n = \|\nabla\chi^n\|_G^2 \le K_3^2\left(\|\nabla u_0\|, \|f\|_\infty, T + T_0 + \kappa_1\right), \quad \forall n = 1, \dots, N + N_0 + 1, \tag{3.26}$$

for some continuous function $K_3(\cdot,\cdot,\cdot)$, increasing in all its arguments.

We now apply Lemma 5, with $n_0 = N_0 + 2$. In computing the sums a_1, a_2 and a_3 that appear there, we note that since all those sums are taken for $n \ge N_0 + 2$ and since, by hypothesis, Δt satisfies condition (3.5) of Corollary 3.1, we can replace K_1, the bound on $\|\chi^n\|_G^2$, by $2\rho_0^2$, whenever the former appears. We thus obtain

$$\xi_n = \|\nabla\chi^n\|_G^2 \le \left(\frac{a_3}{T} + a_2\right) \mathrm{e}^{a_1} =: K_4^2(T, \|f\|_\infty), \qquad \forall n \ge N + N_0 + 2, \tag{3.27}$$

and recalling (2.9), we have (3.22). Combining (3.27) with (3.26) and (2.9), we obtain conclusion (3.21). Thus, the theorem is complete. □

4. Conclusions

Long-time H^1-stability is established for BDF2 time stepping for the 2D Navier–Stokes equations with a very mild time step restriction $\Delta t \le C\nu^{-1}$ with C independent of h. While the results are for 2D, they show that any instability in the analogous 3D scheme must come precisely from the vortex stretching term. Important future work includes extending these results to multiphysics flow problems and to other types of boundary conditions.

Data availability

No data was used for the research described in the article.

Acknowledgements

Author LR acknowledges partial support from National Science Foundation, United States grant DMS 2152623.

References

[1] B. W, D. Davydov, T. Heister, L. Heltai, G. Kanschat, M. Kronbichler, M. Maier, B. Turcksin, D. Wells, The deal.II library, version 8.4, J. Numer. Math. 24 (2016) 1–9.

[2] J. Guermond, P. Minev, J. Shen, An overview of projection methods for incompressible flows, Comput. Methods Appl. Mech. Engrg. 195 (2006) 6011–6045.

[3] V. John, Finite Element Methods for Incompressible Flow Problems, Springer, New York, 2016.

[4] M.D. Gunzburger, Finite Element Methods for Viscous Incompressible Flows: A Guide to Theory, Practice, and Algorithm, Academic Press Inc., Boston, 1989.

[5] W. Layton, N. Mays, M. Neda, C. Trenchea, Numerical analysis of modular regularization methods for the BDF2 time discretization of the Navier-Stokes equations, ESAIM Math. Model. Numer. Anal. 48 (2014) 765–793.

[6] N. Jiang, M. Mohebujjaman, L. Rebholz, C. Trenchea, An optimally accurate discrete regularization for second order timestepping methods for Navier–Stokes equations, Comput. Methods Appl. Mech. Engrg. 310 (2016) 388–405.

[7] W. Layton, W. Pei, Y. Lin, C. Trenchea, Analysis of the variable step method of Dahlquist, Liniger and Nevanlinna for fluid flow, Numer. Methods Partial Differential Equations 38 (2022) 1713–1737.

[8] N. Jiang, A second-order ensemble method based on a blended backward differentiation formula timestepping scheme for time-dependent Navier-Stokes equations, Numer. Methods Partial Differential Equations 33 (2017) 34–61.

[9] B. Garcia-Archilla, J. Novo, Robust error bounds for the Navier–Stokes equations using implicit-explicit second-order BDF method with variable steps, IMA J. Numer. Anal (2022) in Press.

[10] M. Akbas, S. Kaya, L. Rebholz, On the stability at all times of linearly extrapolated BDF2 timestepping for multiphysics incompressible flow problems, Numer. Methods Partial Differential Equations 33 (2017) 995–1017.

[11] T. Heister, M. Olshanskii, L. Rebholz, Unconditional long-time stability of a velocity–vorticity method for the 2D Navier–Stokes equations, Numer. Math. 135 (2017) 143–167.

[12] F. Tone, D. Wirosoetisno, On the long-time stability of the implicit Euler scheme for the 2D Navier–Stokes equations, SIAM J. Numer. Anal. vol. 44 (2006) 29–40.

[13] F. Tone, On the long-time stability of the Crank–Nicolson scheme for the 2D Navier-Stokes equations, Numer. Methods Partial Differential Equations vol. 23 (2007) 1235–1248.

[14] S. Breckling, S. Shields, The long time L2 and H1 stability of linearly extrapolated second order time stepping schemes for the 2D incompressible Navier-Stokes equations, Appl. Math. Comput. 342 (2019) 263–279.

[15] S. Gottlieb, F. Tone, C. Wang, X. Wang, D. Wirosoetisno, Long time stability of a classical efficient scheme for two dimensional Navier–Stokes equations, SIAM J. Numer. Anal. 50 (1) (2012) 126–150.

[16] R. Temam, Navier–Stokes Equations: Theory and Numerical Analysis, AMS Chelsea, Providence, RI, 2001, First published by North–Holland in 1977.

[17] R. Temam, Navier–Stokes Equations and Nonlinear Functional Analysis, second ed., in: vol. 66 of CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, Pennsylvania, 1995.

[18] C. Foias, O. Manley, R. Rosa, R. Temam, Navier–Stokes Equations and Turbulence, in: vol. 83 of Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Cambridge, 2001.

[19] G.W.E. Hairer, Solving Ordinary Differential II: Stiff and Differential-Algebraic Problems, second ed., Springer, 2002.

[20] J. Shen, Long time stabilities and convergences for the fully discrete nonlinear Galerkin methods, Appl. Anal. 38 (1990) 201–229.