

Optimal input states for quantifying the performance of continuous-variable unidirectional and bidirectional teleportation

Hemant K. Mishra ,^{1,2} Samad Khabbazi Oskouei ,³ and Mark M. Wilde ,^{1,2}

¹Hearne Institute for Theoretical Physics, Department of Physics and Astronomy, and Center for Computation and Technology, Louisiana State University, Baton Rouge, Louisiana 70803, USA

²School of Electrical and Computer Engineering, Cornell University, Ithaca, New York 14850, USA

³Department of Mathematics, Varamin-Pishva Branch, Islamic Azad University, Varamin 33817-7489, Iran



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Continuous-variable (CV) teleportation is a fundamental protocol in quantum information science. A number of experiments have been designed to simulate ideal teleportation under realistic conditions. In this paper, we detail an analytical approach for determining optimal input states for quantifying the performance of CV unidirectional and bidirectional teleportation. The metric that we consider for quantifying performance is the energy-constrained channel fidelity between ideal teleportation and its experimental implementation, and along with this, our focus is on determining optimal input states for distinguishing the ideal process from the experimental one. We prove that, under certain energy constraints, the optimal input state in unidirectional as well as bidirectional teleportation is a finite entangled superposition of twin-Fock states saturating the energy constraint. Moreover, we also prove that, under the same constraints, the optimal states are unique; that is, there is no other optimal finite entangled superposition of twin-Fock states.

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I. INTRODUCTION

Quantum teleportation is a foundational protocol in quantum information science that has no classical analog [1] (see also Ref. [2]). It consists of transmitting an unknown quantum state from one place to another by using shared entanglement and local operations and classical communication (LOCC). Quantum teleportation plays an important role in quantum technologies such as quantum information processing protocols [3], quantum computing [4,5], and quantum networks [6]. Since the invention of this protocol, various modifications have been proposed, such as probabilistic teleportation [7–9], controlled teleportation [10–13], and bidirectional teleportation [14–22]. There has also been significant progress in implementing quantum teleportation in laboratories around the world in the last three decades [23]. Several experiments have implemented the teleportation protocol for simple quantum systems [24–29], and attempts are being made to extend them to more complex quantum systems [30–33].

The first theoretical proposal for quantum teleportation was for two-level quantum systems, also commonly called qubits [1]. Later, continuous-variable (CV) teleportation was devised as an extension of the original protocol to quantum systems described by infinite-dimensional Hilbert spaces [14,34]. This was followed by many experimental implementations of CV teleportation, which include teleportation of collective spins of atomic ensembles [35,36], polarization states of photon beams [33], coherent states [37], etc. In standard CV teleportation, the entangled resource state shared between the sender and receiver—Alice and Bob, respectively—is a two-mode squeezed vacuum (TMSV) state. The protocol begins with Alice mixing an unknown input state with her share of the entanglement on a balanced beam splitter and then performing

homodyne detection of complementary quadratures. Based on the classical measurement outcomes received by Alice and subsequently transmitted to Bob, he then performs displacement operations on his share of the TMSV state and recovers an approximation of the original state [34].

An ideal implementation of CV teleportation in principle allows for perfect transmission of quantum states and hence simulates an ideal quantum channel. However, an ideal implementation also demands the unphysical conditions of noiseless homodyne detection and infinite squeezing in the TMSV state, which is not possible in practice because both noiseless homodyne detection and infinite squeezing require infinite energy. Any experimental implementation of CV teleportation accounts for an unideal detection and finite squeezing, which results in an imperfect transmission of quantum states, and hence simulates a noisy quantum channel [34]. It is therefore important for experimentalists to employ performance metrics, as well as quantify the performance, for any experimental simulation of ideal teleportation.

Several works on characterizing the performance of experimental implementations of the teleportation protocol have been conducted for finite-dimensional quantum systems in the past few years [38–44], including a more recent work on bidirectional teleportation, which benchmarks the performance in terms of normalized diamond distance and channel infidelity for transmission of arbitrary quantum states [45]. There have also been many theoretical and experimental works on quantifying the performance of experimental implementations of the CV teleportation protocol. However, most of them study the performance by evaluating specific classes of quantum states, such as coherent states [46–49], pure single-mode Gaussian states [50,51], squeezed states [52], cat states [53], etc. All such evaluations are incomplete, in the sense that they test the

performance by transmitting specific states rather than arbitrary unknown states. A true quantifier for CV unidirectional teleportation was given in Ref. [54], which benchmarks the performance of an experimental implementation in terms of the energy-constrained channel fidelity between ideal teleportation and its experimental implementation. We also note here that Ref. [54] is foundational for the present paper.

In this paper, we quantify the performance of any experimental implementation of CV unidirectional, as well as bidirectional, teleportation, under certain energy constraints. The performance metric that we consider is the energy-constrained channel fidelity between an ideal teleportation and its experimental implementation. We explicitly find *optimal* input states, i.e., quantum states whose output fidelity corresponding to the ideal channel and its experimental approximation is the same as the energy-constrained channel fidelity between the two channels. Our method is purely analytical, employing optimization techniques from multivariable calculus. The optimal states for unidirectional, as well as bidirectional, teleportation are finite entangled superpositions of twin-Fock states saturating the energy constraint. Furthermore, we prove that the optimal input states are unique; i.e., there is no other optimal finite entangled superposition of twin-Fock states.

Our results on bidirectional teleportation are also related to one of the most interesting mathematical problems in quantum information theory: the study of additive and multiplicative properties of measures associated with quantum channels [55–58]. Much progress has been made in addressing these additivity issues [59–63], settling some of the questions posed in Refs. [64,65]. The fidelity of quantum states is well known to be multiplicative for tensor-product quantum states [63]. This induces an inequality for energy-constrained channel fidelity between two tensor-product channels. As a consequence of our work, we give examples where the induced inequality is strict; that is, our results also imply that the energy-constrained fidelity between the identity channel and an additive-noise channel is strictly submultiplicative.

The rest of our paper is organized as follows. In Sec. II, we review some definitions. We present a derivation of the optimal input state for CV unidirectional teleportation in Sec. III and for CV bidirectional teleportation in Sec. IV. We show in Sec. V that the energy-constrained fidelity between the ideal swap channel and the tensor product of two additive-noise channels is strictly submultiplicative. We then discuss possible extensions and generalizations of the present work in Sec. VI. Finally, in Sec. VII we summarize our results and outline questions for future work.

The appendices contain necessary calculations for deriving the results. In Appendix A, we provide proofs of some preliminary results required to derive the optimal input state for CV unidirectional teleportation. Similarly, we prove some preliminary results in Appendix B that are used to derive the optimal input state for CV bidirectional teleportation.

II. PRELIMINARIES

Let \mathcal{H} be a separable Hilbert space, and let T be an operator acting on \mathcal{H} . The adjoint of T is the unique operator T^\dagger acting on \mathcal{H} defined by $\langle \phi | T | \psi \rangle = \overline{\langle \psi | T^\dagger | \phi \rangle}$ for all $|\phi\rangle, |\psi\rangle \in \mathcal{H}; T$

is said to be self-adjoint if $T = T^\dagger$. If $\text{Tr}(\sqrt{T^\dagger T}) < \infty$ then T is said to be a trace-class operator, and its trace norm is defined as $\|T\|_1 := \text{Tr}(\sqrt{T^\dagger T})$. A quantum state is a positive semidefinite, trace-class operator with trace norm equal to one. We denote by $\mathcal{D}(\mathcal{H})$ the set of all quantum states or density operators acting on \mathcal{H} . Let $\rho, \sigma \in \mathcal{D}(\mathcal{H})$. The fidelity between ρ and σ is defined by [66]

$$F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1^2. \quad (1)$$

If one of the quantum states is pure, i.e., say, $\rho = |\psi\rangle\langle\psi|$, then $F(\rho, \sigma) = \text{Tr}(\rho\sigma)$. The sine distance between ρ and σ is given by [67–70]

$$C(\rho, \sigma) := \sqrt{1 - F(\rho, \sigma)}. \quad (2)$$

The following inequalities relate the fidelity, sine distance, and trace distance [71, Theorem 1]:

$$1 - \sqrt{F(\rho, \sigma)} \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq C(\rho, \sigma). \quad (3)$$

The set of bounded operators on \mathcal{H} forms a C^* -algebra under the operator norm, and we denote it by $\mathcal{L}(\mathcal{H})$. Let \mathcal{H}_A denote the Hilbert space corresponding to a quantum system A . A quantum channel from a quantum system A to a quantum system B is a completely positive, trace-preserving linear map from $\mathcal{L}(\mathcal{H}_A)$ to $\mathcal{L}(\mathcal{H}_B)$. Let $\mathcal{M}_{A \rightarrow B}$ and $\mathcal{N}_{A \rightarrow B}$ be quantum channels. Let H_A be a Hamiltonian corresponding to the quantum system A , and let R denote a reference system. The energy-constrained channel fidelity between $\mathcal{M}_{A \rightarrow B}$ and $\mathcal{N}_{A \rightarrow B}$ for $E \in [0, \infty)$ is defined by [72,73]

$$F_E(\mathcal{M}_{A \rightarrow B}, \mathcal{N}_{A \rightarrow B}) := \inf_{\rho_{RA}: \text{Tr}(H_A \rho_{RA}) \leq E} F(\mathcal{M}_{A \rightarrow B}(\rho_{RA}), \mathcal{N}_{A \rightarrow B}(\rho_{RA})), \quad (4)$$

where $\rho_{RA} \in \mathcal{D}(\mathcal{H}_R \otimes \mathcal{H}_A)$, $\rho_A = \text{Tr}_R(\rho_{RA})$, and it is implicit that the identity channel \mathcal{I}_R acts on the reference system R . Furthermore, the optimization in (4) is taken over every possible reference system R . Similarly, the energy-constrained sine distance between $\mathcal{M}_{A \rightarrow B}$ and $\mathcal{N}_{A \rightarrow B}$ for $E \in [0, \infty)$ is defined by [72,73]

$$C_E(\mathcal{M}_{A \rightarrow B}, \mathcal{N}_{A \rightarrow B}) := \sup_{\rho_{RA}: \text{Tr}(H_A \rho_{RA}) \leq E} C(\mathcal{M}_{A \rightarrow B}(\rho_{RA}), \mathcal{N}_{A \rightarrow B}(\rho_{RA})). \quad (5)$$

Although the optimizations in (4) and (5) are over arbitrary mixed states and arbitrary reference systems, it suffices to restrict the optimization over pure states such that the reference system R is isomorphic to the channel input system A . This is a consequence of purification, the Schmidt decomposition, and data processing [74, Sec. 3.5.4]. We thus have

$$F_E(\mathcal{M}_{A \rightarrow B}, \mathcal{N}_{A \rightarrow B}) = \inf_{\phi_{RA}: \text{Tr}(H_A \phi_{RA}) \leq E} F(\mathcal{M}_{A \rightarrow B}(\phi_{RA}), \mathcal{N}_{A \rightarrow B}(\phi_{RA})), \quad (6)$$

$$C_E(\mathcal{M}_{A \rightarrow B}, \mathcal{N}_{A \rightarrow B}) = \sup_{\phi_{RA}: \text{Tr}(H_A \phi_{RA}) \leq E} C(\mathcal{M}_{A \rightarrow B}(\phi_{RA}), \mathcal{N}_{A \rightarrow B}(\phi_{RA})), \quad (7)$$

where the optimizations (6) and (7) are taken over pure states ϕ_{RA} with reference system R isomorphic to system A .

III. OPTIMAL INPUT STATE FOR CV UNIDIRECTIONAL TELEPORTATION

The CV quantum teleportation protocol describes how to transmit an unknown quantum state from Alice to Bob when their systems are in CV modes and they share a prior entangled state known as a *resource state* [34]. In this protocol, Alice mixes the unknown quantum state with her share of the resource state (TMSV state) and performs homodyne detection. The homodyne detection destroys the input state on Alice's end. Alice then communicates the classical outcomes of the detection to Bob, based on which he performs unitary operations on his share of the resource state to generate an approximation of the input state. Let A denote the input mode, and let B denote the output mode. An ideal teleportation protocol requires noiseless homodyne detection and infinite squeezing in the TMSV state, and it induces the identity channel $\mathcal{I}_{A \rightarrow B}$ on the input states [1,34] (see Ref. [75] for further clarification of the convergence of the protocol to the identity channel). However, an experimental implementation of CV teleportation has a noisy detector and finite squeezing in the resource state which makes the experimental implementations of teleportation perform less than ideal. It realizes an additive-noise channel $\mathcal{T}_{A \rightarrow B}^{\xi}$, where the noise parameter $\xi > 0$ encodes unideal detection and finite squeezing [34,76]. The additive-noise channel $\mathcal{T}_{A \rightarrow B}^{\xi}$ is a composition of the quantum-limited amplifier $\mathcal{A}^{1/\eta}$ with gain parameter $1/\eta$ and the pure-loss channel \mathcal{L}^{η} with transmissivity η , where $\eta = 1/(1 + \xi)$ [77,78]. See Ref. [79, Sec. II.B] for more details.

By taking the performance metric to be the energy-constrained channel fidelity between ideal teleportation and the additive-noise channel, the performance of experimental implementations has been studied in Ref. [54]. By choosing the Hamiltonian H_A to be the photon number operator $\hat{n}_A = \sum_{n=0}^{\infty} n|n\rangle\langle n|_A$, the energy-constrained channel fidelity in (6) for the identity channel $\mathcal{I}_{A \rightarrow B}$ and the additive-noise channel $\mathcal{T}_{A \rightarrow B}^{\xi}$ can be further simplified, as a consequence of phase averaging and joint phase covariance of these channels [54,73], as

$$F_E(\mathcal{I}_{A \rightarrow B}, \mathcal{T}_{A \rightarrow B}^{\xi}) = \inf_{\psi_{RA}} F(\mathcal{I}_{A \rightarrow B}(\psi_{RA}), \mathcal{T}_{A \rightarrow B}^{\xi}(\psi_{RA})), \quad (8)$$

where the infimum is taken over pure and entangled superpositions of twin-Fock states $\psi_{RA} = |\psi\rangle\langle\psi|_{RA}$ such that

$$|\psi\rangle_{RA} = \sum_{n=0}^{\infty} \lambda_n |n\rangle_R |n\rangle_A, \quad (9)$$

$\lambda_n \in \mathbb{R}^+$ for all n , $\sum_{n=0}^{\infty} \lambda_n^2 = 1$, and $\sum_{n=0}^{\infty} n\lambda_n^2 \leq E$. An analytical solution to the energy-constrained channel fidelity in (8), using Karush–Kuhn–Tucker conditions, was given in Ref. [54] for small values of ξ and arbitrary values of E . The optimal input state so obtained was

$$|\psi\rangle_{RA} = \sqrt{1 - \{E\}} |E\rangle_R |E\rangle_A + \sqrt{\{E\}} |E\rangle_R |E\rangle_A, \quad (10)$$

where $\{E\} := E - \lfloor E \rfloor$. Another contribution of Ref. [54] was to provide a method, using a combination of numerical and analytical techniques, for finding optimal input states to test

the performance of unidirectional CV teleportation under the energy-constrained channel fidelity measure.

In this section, we show that an optimal input state for the energy-constrained channel fidelity (8) is a finite entangled superposition of twin-Fock states saturating the energy constraint for arbitrary values of ξ and E satisfying $E \leq (1 + \xi)/(1 + 3\xi)$, and it is given by

$$|\psi\rangle_{RA} = \sqrt{1 - E} |0\rangle_R |0\rangle_A + \sqrt{E} |1\rangle_R |1\rangle_A. \quad (11)$$

Observe that the optimal state in (11) is the same as that in (10) under the common conditions of $E \leq (1 + \xi)/(1 + 3\xi)$ and small ξ . Our method also shows that the optimal state in (11) is unique; i.e., there is no other optimal finite entangled superposition of twin-Fock states for (8). We emphasize that our method is purely analytical. We use optimization techniques from multivariable calculus, and the constraint $E \leq (1 + \xi)/(1 + 3\xi)$ is needed in our analysis in the proof of Proposition 2 that plays a major role in establishing the result. We also note that it is still an open problem to find the optimal state for larger values of E analytically.

In order to compute the energy-constrained channel fidelity between the ideal channel $\mathcal{I}_{A \rightarrow B}$ and its experimental implementation $\mathcal{T}_{A \rightarrow B}^{\xi}$, we define the *M-truncated* energy-constrained channel fidelity between $\mathcal{I}_{A \rightarrow B}$ and $\mathcal{T}_{A \rightarrow B}^{\xi}$ as

$$F_{E,M}(\mathcal{I}_{A \rightarrow B}, \mathcal{T}_{A \rightarrow B}^{\xi}) := \inf_{\psi_{RA}} F(\mathcal{I}_{A \rightarrow B}(\psi_{RA}), \mathcal{T}_{A \rightarrow B}^{\xi}(\psi_{RA})), \quad (12)$$

where the infimum is taken over pure states $\psi_{RA} = |\psi\rangle\langle\psi|_{RA}$ of the form

$$|\psi\rangle_{RA} = \sum_{n=0}^M \sqrt{p_n} |n\rangle_R |n\rangle_A, \quad (13)$$

such that $p_n \geq 0$ for all n , $\sum_{n=0}^M p_n = 1$, and $\sum_{n=0}^M n p_n \leq E$. In Proposition 1 in Appendix A, we show that

$$\begin{aligned} & F(\mathcal{I}_{A \rightarrow B}(\psi_{RA}), \mathcal{T}_{A \rightarrow B}^{\xi}(\psi_{RA})) \\ & \geq \frac{1}{(1 + \xi)} \left[\left(\sum_{n=0}^M \frac{p_n}{(1 + \xi)^n} \right)^2 + \left(\sum_{n=1}^M \frac{p_n \xi}{(1 + \xi)^n} \right)^2 \right] \end{aligned} \quad (14)$$

for every state of the form in (13). Define the real-valued function

$$f_{M,\xi}(p) := \frac{1}{(1 + \xi)} \left[\left(\sum_{n=0}^M \frac{p_n}{(1 + \xi)^n} \right)^2 + \left(\sum_{n=1}^M \frac{p_n \xi}{(1 + \xi)^n} \right)^2 \right] \quad (15)$$

for all $p \in \mathbb{R}^{M+1}$. By (14) and (15), we thus have

$$F(\psi_{RA}, \mathcal{T}_{A \rightarrow B}^{\xi}(\psi_{RA})) \geq f_{M,\xi}(p). \quad (16)$$

The minimizer of the function $f_{M,\xi}$ subject to $p_n \geq 0$ for all $n \in \{0, \dots, M\}$,

$$\sum_{n=0}^M p_n = 1, \quad \sum_{n=0}^M n p_n \leq E, \quad (17)$$

is the unique point given by $p_0 = 1 - E$, $p_1 = E$, and $p_n = 0$ for all $n \geq 2$, whenever $E \leq (1 + \xi)/(1 + 3\xi)$. See Proposition 2 in Appendix A. It thus follows from (14) that the optimal input state to the M -truncated energy-constrained channel fidelity is unique, and it is given by (11), whenever $E \leq (1 + \xi)/(1 + 3\xi)$. See Lemma 3 in Appendix A. From the solution to the M -truncated energy-constrained channel fidelity, and the inequality (A35) in Appendix A, it follows that the optimal input state in (8) is given by (11), whenever $E \leq (1 + \xi)/(1 + 3\xi)$. The uniqueness follows from the uniqueness of the optimal state for the M -truncated energy-constrained channel fidelity. See Theorem 4 in Appendix A.

We compare two classes of experimentally relevant quantum states, namely, coherent states and TMSV states, with the optimal state under the given energy constraint. See Ref. [80] for further background on CV quantum information. Let $|\alpha\rangle$ denote a coherent state, which is given by

$$|\alpha\rangle := e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (18)$$

The energy of the coherent state $|\alpha\rangle$ is $E = |\alpha|^2$, and its covariance matrix is I_2 , the 2×2 identity matrix. The covariance matrix of $\mathcal{T}^\xi(|\alpha\rangle\langle\alpha|)$ is $(1 + 2\xi)I_2$. Let $\psi(\bar{n})_{RA} = |\psi(\bar{n})\rangle\langle\psi(\bar{n})|_{RA}$ be the TMSV state given by

$$|\psi(\bar{n})\rangle_{RA} := \frac{1}{\sqrt{\bar{n}+1}} \sum_{n=0}^{\infty} \sqrt{\left(\frac{\bar{n}}{\bar{n}+1}\right)^n} |n\rangle_R |n\rangle_A. \quad (19)$$

The energy of its reduced state is $E = \bar{n}$, and its covariance matrix is

$$V_{\psi(\bar{n})_{RA}} = \begin{bmatrix} (2\bar{n}+1)I_2 & 2\sqrt{\bar{n}(\bar{n}+1)}\sigma_z \\ 2\sqrt{\bar{n}(\bar{n}+1)}\sigma_z & (2\bar{n}+1)I_2 \end{bmatrix}, \quad (20)$$

where σ_z is the Pauli- z matrix. The covariance matrix of $\mathcal{T}^\xi[\psi(\bar{n})_{RA}]$ is given by

$$V_{\mathcal{T}^\xi(\psi(\bar{n})_{RA})} = \begin{bmatrix} (2\bar{n}+1)I_2 & 2\sqrt{\bar{n}(\bar{n}+1)}\sigma_z \\ 2\sqrt{\bar{n}(\bar{n}+1)}\sigma_z & (2\bar{n}+1+2\xi)I_2 \end{bmatrix}. \quad (21)$$

We then have

$$F(|\alpha\rangle\langle\alpha|, \mathcal{T}^\xi(|\alpha\rangle\langle\alpha|)) = \frac{2}{\sqrt{\text{Det}(2(1 + \xi)I_2)}} \quad (22)$$

$$= \frac{1}{1 + \xi}, \quad (23)$$

and also

$$F(\psi(\bar{n})_{RA}, \mathcal{T}^\xi(\psi(\bar{n})_{RA})) = \frac{2^2}{\sqrt{\text{Det}(V_{\psi(\bar{n})_{RA}} + V_{\mathcal{T}^\xi(\psi(\bar{n})_{RA})})}} \quad (24)$$

$$= \frac{1}{1 + (2\bar{n} + 1)\xi} \quad (25)$$

$$= \frac{1}{1 + (2E + 1)\xi}. \quad (26)$$

These fidelity expressions are evaluated using Eq. (4.51) of Ref. [80].

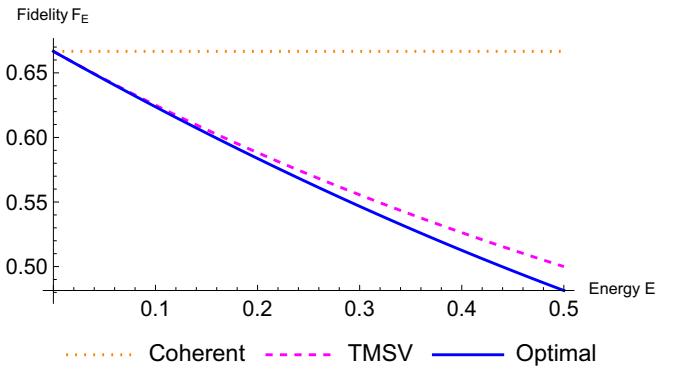


FIG. 1. The graph plots the output fidelity F_E between the ideal channel and an additive-noise channel versus the input energy E , corresponding to a coherent state, a TMSV state, and the optimal state (each having input energy E). The noise parameter for the additive-noise channel is taken as $\xi = 0.5$ and the states have energy $E \in [0, 0.5]$. The dotted (orange), dashed (magenta), and solid (blue) lines represent the output fidelity for the coherent state, the TMSV state, and the optimal state, respectively.

In Fig. 1, we plot the output fidelity F_E between the ideal channel and an additive-noise channel versus the input energy E , corresponding to a coherent state, a TMSV state, and the optimal state, with input energy $E \in [0, 0.5]$. The noise parameter is taken as $\xi = 0.5$. In order, the dotted (orange), dashed (magenta), and solid (blue) lines indicate the output fidelity for the coherent state (23), the TMSV state (26), and the optimal state (A33). The graph indicates that coherent and TMSV states are not optimal states in general. Interestingly, however, the TMSV state is very close to being an optimal input state for CV unidirectional teleportation. This was observed in a different regime for the energy constraint, in Fig. 2 of Ref. [54].

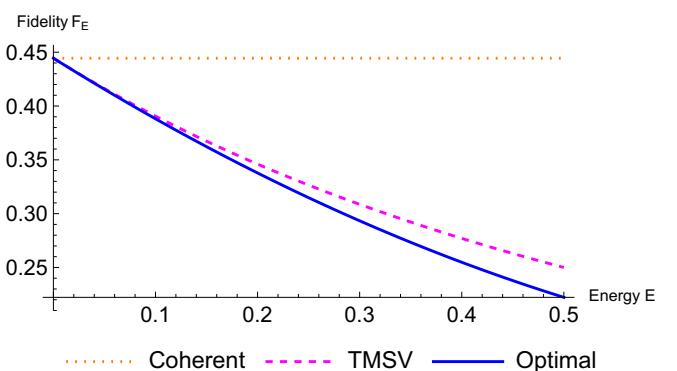


FIG. 2. The graph plots the output fidelity F_E between the ideal swap channel and the tensor product of two identical additive-noise channels versus input energy E , corresponding to a tensor product of coherent states, a tensor product of TMSV states, and the optimal state (each having input energy E). The noise parameter for the additive-noise channels is taken as $\xi = 0.5$ and the states have energy $E \in [0, 0.5]$. The dotted (orange), dashed (magenta), and solid (blue) lines represent the output fidelity for the coherent state, the TMSV state, and the optimal state, respectively.

IV. OPTIMAL INPUT STATE FOR CV BIDIRECTIONAL TELEPORTATION

The CV bidirectional teleportation protocol consists of a two-way transmission of unknown quantum states between Alice and Bob. One implementation of the protocol, which we consider here, allows for a simulation of two ideal CV unidirectional quantum channels with the help of shared entanglement and LOCC. See Ref. [45] for a discussion of more general implementations. The protocol that we consider here can be thought of as a combination of two CV unidirectional teleportations, one from Alice to Bob and the other from Bob to Alice. Let A and B denote the input modes for Alice and Bob, and let A' and B' denote the output modes for Alice and Bob, respectively. An ideal CV bidirectional teleportation between Alice and Bob is represented by the following unitary swap channel

$$\mathcal{S}_{AB \rightarrow A'B'}(\cdot) := \text{SWAP}(\cdot) \text{SWAP}^\dagger, \quad (27)$$

where the unitary swap operator SWAP is defined as

$$\text{SWAP} := \sum_{m,n=0}^{\infty} |m\rangle_{A'}\langle n|_A \otimes |n\rangle_{B'}\langle m|_B. \quad (28)$$

Here $\{|m\rangle_A\}_{m=0}^{\infty}$ is the photonic number basis corresponding to the system A , and so on. The swap channel acts on product states by swapping them, i.e.,

$$\mathcal{S}_{AB \rightarrow A'B'}(\phi_A \otimes \psi_B) = \psi_{A'} \otimes \phi_{B'}. \quad (29)$$

Thus, the swap channel can be thought of as the tensor product of the ideal channels $\mathcal{I}_{A \rightarrow B'}$ and $\mathcal{I}_{B \rightarrow A'}$. An experimental implementation of CV bidirectional teleportation realizes an approximate swap channel given by the tensor product of two additive-noise channels $\mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}$. The Hamiltonian for the composite system AB is the total photon number operator:

$$\hat{n}_{AB} := \hat{n}_A \otimes I_B + I_A \otimes \hat{n}_B. \quad (30)$$

Given any state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, the inequality $\text{Tr}(\hat{n}_{AB}\rho_{AB}) \leq 2E$ implies that the average photon number in ρ_{AB} over each of the modes A and B is at most E . So, the energy-constrained channel fidelity (6) between the ideal bidirectional teleportation $\mathcal{S}_{AB \rightarrow A'B'}$ and its experimental

implementation $\mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}$ is given by

$$\begin{aligned} & F_E(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}) \\ &= \inf_{\phi_{RAB}: \text{Tr}(\hat{n}_{AB}\phi_{RAB}) \leq 2E} F(\mathcal{S}_{AB \rightarrow A'B'}(\phi_{RAB}), \\ & \quad \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\phi_{RAB})), \end{aligned} \quad (31)$$

where ϕ_{RAB} is a pure state and $\phi_{AB} = \text{Tr}_R(\phi_{RAB})$. As a consequence of the joint phase covariance of $\mathcal{S}_{AB \rightarrow A'B'}$ and $\mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}$, and the arguments given for (8), the infimum in (31) can be recast as

$$\begin{aligned} & F_E(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}) \\ &= \inf_{\psi_{RAB}} F(\mathcal{S}_{AB \rightarrow A'B'}(\psi_{RAB}), \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\psi_{RAB})), \end{aligned} \quad (32)$$

where $\psi_{RAB} = |\psi\rangle\langle\psi|_{RAB}$ is a pure, entangled superposition of twin-Fock states given by

$$|\psi\rangle_{RAB} = \sum_{m,n=0}^{\infty} \lambda_{m,n} |m, n\rangle_R |m, n\rangle_{AB}, \quad (33)$$

such that $\lambda_{m,n} \geq 0$ for all m and n , $\sum_{m,n=0}^{\infty} \lambda_{m,n}^2 = 1$, and $\sum_{m,n=0}^{\infty} (m+n)\lambda_{m,n}^2 \leq 2E$. We further simplify the fidelity expression in (32) as follows. Since we are working with CV modes, the systems A, B, A', B' are all isomorphic to each other. So, the energy constrained channel fidelity between $\mathcal{S}_{AB \rightarrow A'B'}$ and $\mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}$ must be the same as that of $\mathcal{I}_{A \rightarrow A'} \otimes \mathcal{I}_{B \rightarrow B'}$ and $\mathcal{T}_{A \rightarrow A'}^\xi \otimes \mathcal{T}_{B \rightarrow B'}^{\xi'}$. For simplicity of notations, we shall denote any channel $\mathcal{M}_{C \rightarrow C'}$ by \mathcal{M}_C , where C and C' are isomorphic systems. Recall that an additive-noise channel \mathcal{T}^ξ can be written as a composed channel $\mathcal{A}^{1/\eta} \circ \mathcal{L}^\eta$ for $\eta = 1/(1+\xi)$. Also, the adjoint of the quantum-limited amplifier is related to the pure-loss channel by $(\mathcal{A}^{1/\eta})^\dagger = \eta \mathcal{L}^\eta$ [81]. For the given pure state in (33), we have

$$\begin{aligned} & F(\mathcal{S}_{AB \rightarrow A'B'}(\psi_{RAB}), \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\psi_{RAB})) \\ &= F((\mathcal{I}_A \otimes \mathcal{I}_B)(\psi_{RAB}), (\mathcal{T}_A^\xi \otimes \mathcal{T}_B^{\xi'})(\psi_{RAB})) \end{aligned} \quad (34)$$

$$= \text{Tr}(\psi_{RAB} (\mathcal{T}_A^\xi \otimes \mathcal{T}_B^{\xi'})(\psi_{RAB})) \quad (35)$$

$$= \text{Tr}(\psi_{RAB} ((\mathcal{A}_A^{1/\eta} \otimes \mathcal{L}_A^\eta) \otimes (\mathcal{A}_B^{1/\eta'} \otimes \mathcal{L}_B^{\eta'}))(\psi_{RAB})) \quad (36)$$

$$= \text{Tr}((\mathcal{A}_A^{1/\eta} \otimes \mathcal{A}_B^{1/\eta'})^\dagger(\psi_{RAB}) (\mathcal{L}_A^\eta \otimes \mathcal{L}_B^{\eta'})(\psi_{RAB})) \quad (37)$$

$$= \eta\eta' \text{Tr}((\mathcal{L}_A^\eta \otimes \mathcal{L}_B^{\eta'})(\psi_{RAB}) (\mathcal{L}_A^\eta \otimes \mathcal{L}_B^{\eta'})(\psi_{RAB})) \quad (38)$$

$$= \eta\eta' \text{Tr}(((\mathcal{L}_A^\eta \otimes \mathcal{L}_B^{\eta'})(\psi_{RAB}))^2) \quad (39)$$

$$= \eta\eta' \text{Tr} \left\{ \left[(\mathcal{L}_A^\eta \otimes \mathcal{L}_B^{\eta'}) \left(\sum_{m,n,m',n'=0}^{\infty} \lambda_{m,n} \lambda_{m',n'} |m, n\rangle \langle m', n'|_R \otimes |m, n\rangle \langle m', n'|_{AB} \right) \right]^2 \right\} \quad (40)$$

$$= \eta\eta' \text{Tr} \left\{ \left[\sum_{m,n,m',n'=0}^{\infty} \lambda_{m,n} \lambda_{m',n'} |m, n\rangle \langle m', n'|_R \otimes \mathcal{L}_A^\eta(|m\rangle \langle m'|_A) \otimes \mathcal{L}_B^{\eta'}(|n\rangle \langle n'|_B) \right]^2 \right\} \quad (41)$$

$$= \eta\eta' \text{Tr} \left\{ \left[\sum_{m,n,m',n'=0}^{\infty} \lambda_{m,n} \lambda_{m',n'} |m, n\rangle \langle m', n'|_R \otimes \mathcal{L}_A^\eta(|m'\rangle \langle m'|_A) \otimes \mathcal{L}_B^{\eta'}(|n'\rangle \langle n'|_B) \right] \times \left[\sum_{m'',n'',m''',n'''=0}^{\infty} \lambda_{m'',n''} \lambda_{m''',n'''} |m'', n''\rangle \langle m''', n'''|_R \otimes \mathcal{L}_A^\eta(|m'''\rangle \langle m'''|_A) \otimes \mathcal{L}_B^{\eta'}(|n'''\rangle \langle n'''|_B) \right] \right\} \quad (42)$$

$$= \eta n' \sum_{m,n,m',n'=0}^{\infty} \lambda_{m,n}^2 \lambda_{m',n'}^2 \text{Tr} [\mathcal{L}_A^\eta (|m\rangle\langle m'|_A) \mathcal{L}_A^\eta (|m'\rangle\langle m'|_A)] \text{Tr} [\mathcal{L}_B^{\eta'} (|n\rangle\langle n'|_B) \mathcal{L}_B^{\eta'} (|n'\rangle\langle n'|_B)] \quad (43)$$

$$= \frac{1}{(1+\xi)(1+\xi')} \sum_{m,n,m',n'=0}^{\infty} \lambda_{m,n}^2 \lambda_{m',n'}^2 \text{Tr} [\mathcal{L}_A^{\frac{1}{1+\xi}} (|m\rangle\langle m'|_A) \mathcal{L}_A^{\frac{1}{1+\xi}} (|m'\rangle\langle m'|_A)] \text{Tr} [\mathcal{L}_B^{\frac{1}{1+\xi'}} (|n\rangle\langle n'|_B) \mathcal{L}_B^{\frac{1}{1+\xi'}} (|n'\rangle\langle n'|_B)]. \quad (44)$$

Let $p_{m,n} := \lambda_{m,n}^2$ for all $m, n \geq 0$, and

$$T_{\xi}^{mm'} := \text{Tr} [\mathcal{L}_A^{\frac{1}{1+\xi}} (|m\rangle\langle m'|_A) \mathcal{L}_A^{\frac{1}{1+\xi}} (|m'\rangle\langle m'|_A)], \quad (45)$$

$$T_{\xi'}^{nn'} := \text{Tr} [\mathcal{L}_B^{\frac{1}{1+\xi'}} (|n\rangle\langle n'|_B) \mathcal{L}_B^{\frac{1}{1+\xi'}} (|n'\rangle\langle n'|_B)]. \quad (46)$$

This gives

$$\begin{aligned} F(\mathcal{S}_{AB \rightarrow A'B'}(\psi_{RAB}), \mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\psi_{RAB})) \\ = \frac{1}{(1+\xi)(1+\xi')} \sum_{m,n,m',n'=0}^{\infty} p_{m,n} p_{m',n'} T_{\xi}^{mm'} T_{\xi'}^{nn'}. \end{aligned} \quad (47)$$

Let p denote the infinite vector $p = (p_{m,n})_{m,n=0}^{\infty}$. Define

$$f_{\xi,\xi'}(p) := \frac{1}{(1+\xi)(1+\xi')} \sum_{m,n,m',n'=0}^{\infty} p_{m,n} p_{m',n'} T_{\xi}^{mm'} T_{\xi'}^{nn'}. \quad (48)$$

Therefore, (32) reduces to the following quadratic optimization problem in terms of an infinite number of variables:

$$\begin{aligned} F_E(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}) \\ = \begin{cases} \inf_p f_{\xi,\xi'}(p) \\ \text{subject to } p_{m,n} \geq 0 \ \forall m, n \geq 0, \\ \sum_{m,n=0}^{\infty} (m+n)p_{m,n} \leq 2E, \\ \sum_{m,n=0}^{\infty} p_{m,n} = 1. \end{cases} \end{aligned} \quad (49)$$

In this section, we find an optimal input state for the energy-constrained channel fidelity (49). We consider two cases. The first case is when $\xi = \xi'$ and $2E \leq (1+\xi)/(2+3\xi)$, which corresponds to having identical additive-noise channels in both directions and states with low energy. The second case is when $\xi' \geq 1$ and $2E \leq \min\{(\xi'^2 - 1)/[\xi'(3\xi' - 1)], (1+\xi)/(2\xi)\}$, which corresponds to experimental implementations with minimum excess noise in one of the quantum channels and low-energy states. In both cases, we show that the optimal input state is a finite entangled superposition of twin-Fock states saturating the energy constraint. Furthermore, our method shows that such optimal states are unique; i.e., there is no other optimal finite entangled superposition of twin-Fock states that achieves this performance. Our method is again purely analytical, similar to the unidirectional case, which uses optimization techniques from multivariable calculus. Also, the given constraints on E, ξ , and ξ' are consequences of our analysis in the proofs of Proposition 6 and Proposition 9 used to establish the main results. It still remains to solve (49) for larger values of E ; we note here that numerical solutions can be obtained using truncation.

In order to find the solution to the energy-constrained channel fidelity (49), we define the M -truncated energy-constrained channel fidelity between $\mathcal{S}_{AB \rightarrow B'A'}$ and $\mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}$ as

$$\begin{aligned} F_{E,M}(\mathcal{S}_{AB \rightarrow B'A'}, \mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}) \\ := \inf_{\psi_{RAB}} F(\mathcal{S}_{AB \rightarrow B'A'}(\psi_{RAB}), \mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\psi_{RAB})), \end{aligned} \quad (50)$$

where the infimum is taken over pure bipartite states $|\psi\rangle_{RAB} = \sum_{m,n=0}^M \sqrt{p_{m,n}} |m, n\rangle_R |m, n\rangle_{AB}$, where $p_{m,n} \geq 0$ for all m, n ,

$$\sum_{m,n=0}^M p_{m,n} = 1, \quad \sum_{m,n=0}^M (m+n)p_{m,n} \leq 2E. \quad (51)$$

It turns out that the energy-constrained channel fidelity (49) can be bounded from above and from below by the M -truncated energy-constrained channel fidelity (50), which gives a way to approximate the former in terms of the latter. See Proposition 5 of Appendix B.

We show in Lemma 7 of Appendix B 1 that the unique optimal input state for the M -truncated energy-constrained channel fidelity is given by

$$\begin{aligned} |\psi\rangle_{RAB} = \sqrt{1-2E} |0, 0\rangle_R |0, 0\rangle_{AB} \\ + \sqrt{E} |0, 1\rangle_R |0, 1\rangle_{AB} + \sqrt{E} |1, 0\rangle_R |1, 0\rangle_{AB}, \end{aligned} \quad (52)$$

whenever $\xi = \xi'$ and $2E \leq (1+\xi)/(2+3\xi)$. We then use the bounds in (B1) to show that (52) is also an optimal input state for the energy-constrained channel fidelity (49) under the same conditions $\xi = \xi'$ and $2E \leq (1+\xi)/(2+3\xi)$, and it is unique, as described earlier. See Theorem 8 in Appendix B 1 for a detailed proof. Using similar methods, we also find an optimal input state for (49) in the case $\xi' \geq 1$ and $2E \leq \min\{(\xi'^2 - 1)/[\xi'(3\xi' - 1)], (1+\xi)/(2\xi)\}$. The optimal input state in this case is

$$\begin{aligned} |\psi\rangle_{RAB} = \sqrt{1-2E} |0, 0\rangle_R |0, 0\rangle_{AB} \\ + \sqrt{2E - p_E} |0, 1\rangle_R |0, 1\rangle_{AB} + \sqrt{p_E} |1, 0\rangle_R |1, 0\rangle_{AB}, \end{aligned} \quad (53)$$

where p_E is given by (B73) in Appendix B 2. Again, the state in (53) is unique. See Theorem 11 in Appendix B 2.

We compare the optimal state with a tensor product of two coherent states $|\alpha\rangle \otimes |\alpha\rangle$ given by (18) and a tensor product of two TMSV states $\psi(\bar{n})_{RA} \otimes \psi(\bar{n})_{RA}$ given by (19) under

the given conditions of Theorem 8. We find that

$$F(\mathcal{S}(|\alpha\rangle\langle\alpha| \otimes |\alpha\rangle\langle\alpha|), \mathcal{T}^\xi \otimes \mathcal{T}^\xi (|\alpha\rangle\langle\alpha| \otimes |\alpha\rangle\langle\alpha|)) = (F(|\alpha\rangle\langle\alpha|, \mathcal{T}^\xi (|\alpha\rangle\langle\alpha|)))^2 \quad (54)$$

$$= \frac{1}{(1 + \xi)^2}. \quad (55)$$

Also,

$$F(\mathcal{S}(\psi(\bar{n})_{RA} \otimes \psi(\bar{n})_{RA}), \mathcal{T}^\xi \otimes \mathcal{T}^\xi (\psi(\bar{n})_{RA} \otimes \psi(\bar{n})_{RA})) = F(\psi(\bar{n})_{RA}, \mathcal{T}^\xi (\psi(\bar{n})_{RA}))^2 \quad (56)$$

$$= \frac{1}{(1 + (2E + 1)\xi)^2}. \quad (57)$$

Figure 2 plots the output fidelity F_E between the ideal swap channel and the tensor product of two identical additive-noise channels versus the input energy E , corresponding to a tensor product of coherent states, a tensor product of TMSV states, and the optimal state, with input energy $E \in [0, 0.5]$. The noise parameter is taken as $\xi = 0.5$. In order, the dotted (orange), dashed (magenta), and solid (blue) lines indicate the output fidelity corresponding to the coherent state (55), the TMSV state (57), and the optimal state (B70). Similar to unidirectional teleportation, neither coherent nor TMSV states are optimal. However, we have proven here that entanglement between the channel uses and a reference system has the optimal performance in distinguishing the two identity channels from the two additive-noise channels [recall (52)].

V. MULTIPLICATIVITY OF ENERGY-CONSTRAINED CHANNEL FIDELITY

Let \mathcal{M}_i and \mathcal{N}_i be quantum channels from quantum subsystem A_i to subsystem B_i , and let ρ_i and σ_i be quantum states in A_i for $i \in \{1, \dots, n\}$. The multiplicative property of the fidelity of tensor-product states follows from the definition in (1), i.e.,

$$F\left(\left(\bigotimes_{i=1}^n \rho_i\right), \left(\bigotimes_{i=1}^n \sigma_i\right)\right) = \prod_{i=1}^n F(\rho_i, \sigma_i). \quad (58)$$

This implies that

$$F\left(\left(\bigotimes_{i=1}^n \mathcal{M}_i\right)\left(\bigotimes_{i=1}^n \rho_i\right), \left(\bigotimes_{i=1}^n \mathcal{N}_i\right)\left(\bigotimes_{i=1}^n \sigma_i\right)\right) = \prod_{i=1}^n F(\mathcal{M}_i(\rho_i), \mathcal{N}_i(\sigma_i)). \quad (59)$$

The energy-constrained channel fidelity between the tensor-product channels $\bigotimes_{i=1}^n \mathcal{M}_i$ and $\bigotimes_{i=1}^n \mathcal{N}_i$ is

$$\begin{aligned} & F_E\left(\left(\bigotimes_{i=1}^n \mathcal{M}_i\right), \left(\bigotimes_{i=1}^n \mathcal{N}_i\right)\right) \\ &= \inf_{\psi_{RA_1 \dots A_n} : \text{Tr}(\hat{H}_n \psi_{RA_1 \dots A_n}) \leq nE} F\left(\left(\bigotimes_{i=1}^n \mathcal{M}_i\right)(\psi_{RA_1 \dots A_n}), \left(\bigotimes_{i=1}^n \mathcal{N}_i\right)(\psi_{RA_1 \dots A_n})\right). \end{aligned} \quad (60)$$

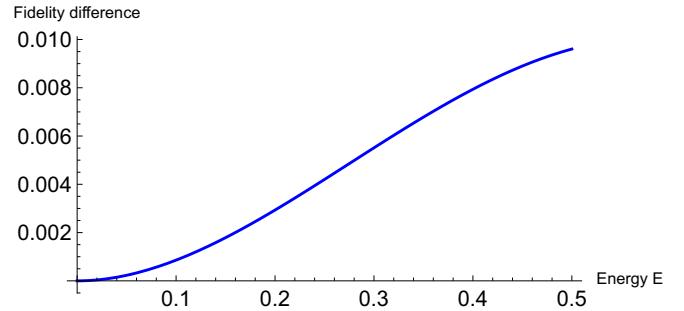


FIG. 3. The graph plots the difference between the right-hand and the left-hand side of (66) versus the energy $E \in [0, 0.5]$ for $\xi = 0.5$.

In (60), \hat{H}_n is the Hamiltonian operator acting on the composite system $A_1 \dots A_n$ given by

$$\hat{H}_n := H_1 \otimes \dots \otimes I_n + \dots + I_1 \otimes \dots \otimes H_n, \quad (61)$$

where H_1, \dots, H_n are Hamiltonian operators, and I_1, \dots, I_n are the identity operators for the subsystems $A_1 \dots A_n$, respectively. The reference system R can be taken to be $R = R_1 \dots R_n$, where each reference subsystem R_i is isomorphic to the input subsystem A_i . By taking $\psi_{RA_1 \dots A_n}$ as the n -tensor-product state

$$\psi_{RA_1 \dots A_n} = \psi_{R_1 A_1} \otimes \dots \otimes \psi_{R_n A_n}, \quad (62)$$

we know from (59) that

$$\begin{aligned} & F\left(\left(\bigotimes_{i=1}^n \mathcal{M}_i\right)(\psi_{RA_1 \dots A_n}), \left(\bigotimes_{i=1}^n \mathcal{N}_i\right)(\psi_{RA_1 \dots A_n})\right) \\ &= \prod_{i=1}^n F(\mathcal{M}_i(\psi_{R_i A_i}), \mathcal{N}_i(\psi_{R_i A_i})). \end{aligned} \quad (63)$$

If each state $\psi_{R_i A_i}$ satisfies the energy constraint $\text{Tr}(H_i \psi_{A_i}) \leq E$, then $\psi_{RA_1 \dots A_n}$ satisfies the energy constraint in (60). Thus, by taking the appropriate infimum on each side of (63), we get

$$F_E\left(\left(\bigotimes_{i=1}^n \mathcal{M}_i\right), \left(\bigotimes_{i=1}^n \mathcal{N}_i\right)\right) \leq \prod_{i=1}^n F_E(\mathcal{M}_i, \mathcal{N}_i). \quad (64)$$

By using the fact that $\mathcal{S}_{AB \rightarrow A'B'} \equiv \mathcal{I}_{A \rightarrow B'} \otimes \mathcal{I}_{B \rightarrow A'}$, from (64) we get

$$\begin{aligned} & F_E(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^\xi) \\ & \leq F_E(\mathcal{I}_{A \rightarrow B'}, \mathcal{T}_{A \rightarrow B'}^\xi) F_E(\mathcal{I}_{B \rightarrow A'}, \mathcal{T}_{B \rightarrow A'}^\xi). \end{aligned} \quad (65)$$

The *submultiplicative* property (65) is well known. An interesting consequence of Theorems 4 and 8 is that the inequality in (65) can be strict. In particular, we have

$$\begin{aligned} & F_E(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^\xi) \\ & < F_E(\mathcal{I}_{A \rightarrow B'}, \mathcal{T}_{A \rightarrow B'}^\xi) F_E(\mathcal{I}_{B \rightarrow A'}, \mathcal{T}_{B \rightarrow A'}^\xi), \end{aligned} \quad (66)$$

whenever $2E \leq (1 + \xi)/(2 + 3\xi)$. This is illustrated in Fig. 3, which plots the difference between the right-hand and the left-hand side of (66) for $\xi = 0.5$. Thus, a consequence of our result on bidirectional teleportation is that the energy-constrained channel fidelity is not multiplicative, i.e.,

entanglement between a reference and the channel inputs provides a benefit in distinguishing an identity channel from an additive-noise channel, when there is an energy constraint.

VI. DISCUSSION

A characterization of the performance of any experimental implementation of CV unidirectional teleportation in terms of the energy-constrained channel fidelity is known for arbitrary values of the noise parameter ξ and the energy constraint E [54]. In our work, we characterize the performance of an experimental implementation of CV bidirectional teleportation under certain conditions on the noise parameters ξ and ξ' , as well as the energy constraint E . It still remains to solve the problem in the most general case, i.e., for arbitrary values of the noise parameters and every energy constraint.

It is also an interesting question for future work to consider the following quantity, and which states achieve the optimal value:

$$F_E(\mathcal{I}^{\otimes n}, (\mathcal{T}^\xi)^{\otimes n}). \quad (67)$$

This quantity applies to a generalization of CV bidirectional teleportation in which there are n parties involved, and the i th party is trying to communicate quantum information to the $i+1$ st party for $i \in \{1, \dots, n-1\}$ and the n th party is trying to do so to the 1st party. Based on the findings of our paper, in particular, the structures of the optimal states (11) and (53) for unidirectional and bidirectional teleportation, we suspect that for a sufficiently low-energy constraint E , the following state optimizes the value in (67):

$$\begin{aligned} & \sqrt{1-nE}|0\rangle_R|00\dots00\rangle_{A_1A_2\dots A_{n-1}A_n} \\ & + \sqrt{E}|1\rangle_R|10\dots00\rangle_{A_1A_2\dots A_{n-1}A_n} \\ & + \sqrt{E}|2\rangle_R|01\dots00\rangle_{A_1A_2\dots A_{n-1}A_n} \\ & \quad \vdots \\ & + \sqrt{E}|n-1\rangle_R|00\dots10\rangle_{A_1A_2\dots A_{n-1}A_n} \\ & + \sqrt{E}|n\rangle_R|00\dots01\rangle_{A_1A_2\dots A_{n-1}A_n}. \end{aligned} \quad (68)$$

Note that the reduced state on systems $A_1 \dots A_n$ has total energy nE . To support the conjecture for $n = 3$, we compare the output fidelity corresponding to a tensor product of coherent states, a tensor product of TMSV states, and the conjectured optimal state, and show that the conjectured optimal state behaves the same as the optimal states for unidirectional and bidirectional teleportation. For the coherent state $\alpha = |\alpha\rangle\langle\alpha|$ given by (18), we have

$$F(\mathcal{I}^{\otimes 3}(\alpha^{\otimes 3}), (\mathcal{T}^\xi)^{\otimes 3}(\alpha^{\otimes 3})) = (F(\alpha, \mathcal{T}^\xi(\alpha)))^3 \quad (69)$$

$$= \frac{1}{(1+\xi)^3}. \quad (70)$$

For the TMSV state $\psi(\bar{n})$ given by (19), we have

$$\begin{aligned} & F(\mathcal{I}^{\otimes 3}(\psi(\bar{n})^{\otimes 3}), (\mathcal{T}^\xi)^{\otimes 3}(\psi(\bar{n})^{\otimes 3})) \\ & = (F(\psi(\bar{n}), \mathcal{T}^\xi(\psi(\bar{n}))))^3 \end{aligned} \quad (71)$$

$$= \frac{1}{(1+(1+2E)\xi)^3}. \quad (72)$$

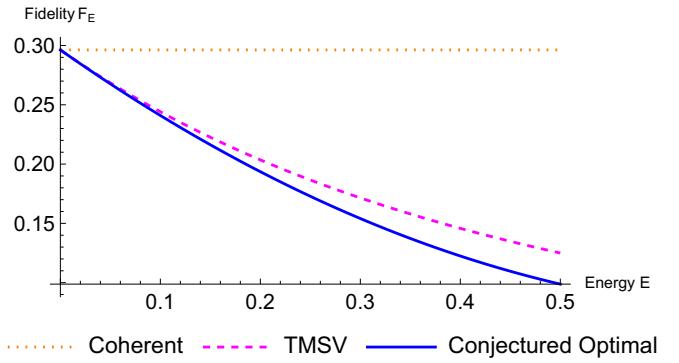


FIG. 4. The graph plots the output fidelity F_E between the tensor product of three identity channels and the tensor product of three identical additive-noise channels versus the input energy E , corresponding to a tensor product of coherent states, a tensor product of TMSV states, and the conjectured optimal state (each having the input energy E). The noise parameter for the additive-noise channels is taken as $\xi = 0.5$ and the states have energy $E \in [0, 0.5]$. The dotted (orange), dashed (magenta), and solid (blue) lines represent the output fidelity for the coherent state, the TMSV state, and the optimal state, respectively.

Let ϕ be the conjectured optimal state given by (68) for $n = 3$. By following the same arguments given in (34)–(44), we get

$$\begin{aligned} & F(\mathcal{I}^{\otimes 3}(\phi^{\otimes 3}), (\mathcal{T}^\xi)^{\otimes 3}(\phi^{\otimes 3})) \\ & = \frac{1}{(1+\xi)^3} \left(1 - 6 \left(\frac{\xi}{1+\xi} \right) E + 12 \left(\frac{\xi}{1+\xi} \right)^2 E^2 \right). \end{aligned} \quad (73)$$

In Fig. 4, we plot the output fidelity F_E corresponding to (70), (72), and (73) versus the input energy $E \in [0, 0.5]$ for $\xi = 0.5$. The plot shows similar behavior of the conjectured optimal state as the plots for the optimal states for unidirectional and bidirectional teleportation, which is expected.

VII. CONCLUSION

We have characterized the performance of CV unidirectional and bidirectional teleportation in terms of the energy-constrained channel fidelity between ideal CV teleportation and its experimental approximation. Through a purely analytical method, using optimization techniques from multi-variable calculus, we explicitly determined the optimal input state for CV unidirectional as well as bidirectional teleportation. We showed that, in both the protocols, the optimal input state is a finite entangled superposition of twin-Fock states. Furthermore, the optimal states are unique; i.e., there is no other optimal finite entangled superposition of twin-Fock states. As an application of our results, we have shown that the energy-constrained channel fidelity of two tensor-product channels is strictly submultiplicative.

Another metric to quantify performance of any experimental implementation of CV unidirectional and bidirectional protocol is the Shirokov–Winter energy-constrained diamond distance [82,83] between ideal CV teleportation and its experimental approximation. We leave the study of this quantity for future work. Additionally, it remains an intriguing open

question to determine optimal input states for multidirectional CV teleportation, which we also leave for future work.

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APPENDIX A: OPTIMAL STATE FOR CV UNIDIRECTIONAL TELEPORTATION

We shall use the following Kraus representation for the pure-loss channel [81]:

$$\mathcal{L}_{A \rightarrow B}^{\eta}(|m\rangle\langle m'|_A) = \sum_{k=0}^{\min\{m, m'\}} \sqrt{\binom{m}{k} \binom{m'}{k}} \eta^{\frac{1}{2}(m+m'-2k)} \times (1-\eta)^k |m-k\rangle\langle m'-k|_B. \quad (\text{A1})$$

For $\eta = 1/(1+\xi)$, we have

$$F(\mathcal{I}_{A \rightarrow B}(\psi_{RA}), \mathcal{T}_{A \rightarrow B}^{\xi}(\psi_{RA})) = \eta \text{Tr}((\mathcal{L}_{A \rightarrow B}^{1-\eta}(\psi_A))^2) \quad (\text{A4})$$

$$\geq \eta \text{Tr} \left[\left(\sum_{n=0}^M p_n \sum_{k=0}^n \binom{n}{k} (1-\eta)^k \eta^{n-k} |k\rangle\langle k|_B \right)^2 \right] \quad (\text{A5})$$

$$= \eta \text{Tr} \left[\left(\sum_{k=0}^M \left(\sum_{n=k}^M p_n \binom{n}{k} (1-\eta)^k \eta^{n-k} \right) |k\rangle\langle k|_B \right)^2 \right] \quad (\text{A6})$$

$$= \eta \sum_{k=0}^M \left\{ \sum_{n=k}^M p_n \binom{n}{k} (1-\eta)^k \eta^{n-k} \right\}^2 \quad (\text{A7})$$

$$\geq \eta \sum_{k=0}^1 \left\{ \sum_{n=k}^M p_n \binom{n}{k} (1-\eta)^k \eta^{n-k} \right\}^2 \quad (\text{A8})$$

$$= \eta \left(\sum_{n=0}^M p_n \eta^n \right)^2 + \eta \left(\sum_{n=1}^M p_n n (1-\eta) \eta^{n-1} \right)^2 \quad (\text{A9})$$

$$\geq \eta \left(\sum_{n=0}^M p_n \eta^n \right)^2 + \eta \left(\sum_{n=1}^M p_n (1-\eta) \eta^{n-1} \right)^2 \quad (\text{A10})$$

$$= \frac{1}{(1+\xi)} \left[\left(\sum_{n=0}^M \frac{p_n}{(1+\xi)^n} \right)^2 + \left(\sum_{n=1}^M \frac{p_n \xi}{(1+\xi)^n} \right)^2 \right]. \quad (\text{A11})$$

The equality (A4) is given in Ref. [54], the inequality (A5) follows by truncating the infinite sum in (A3) to M , the equality (A6) is obtained by interchanging the indices of the sum in (A5), and (A7) follows by definition of trace. We get (A8) by truncating the outer sum in (A7) to 1, and the inequality (A10) is obtained by replacing n with 1 in the multiples of p_n in the second term of (A9). We use the relation $\eta = 1/(1+\xi)$ in the last equality. Furthermore, each inequality is saturated if $p_n = 0$ for all $n \geq 2$. \blacksquare

Proposition 1. Let $p = (p_0, p_1, p_2, \dots)$ be an arbitrary infinite-dimensional probability vector, and let ψ_{RA} be a pure state given by $|\psi\rangle_{RA} = \sum_{n=0}^{\infty} \sqrt{p_n} |n\rangle_R |n\rangle_A$. Then

$$\begin{aligned} & F(\mathcal{I}_{A \rightarrow B}(\psi_{RA}), \mathcal{T}_{A \rightarrow B}^{\xi}(\psi_{RA})) \\ & \geq \frac{1}{(1+\xi)} \left[\left(\sum_{n=0}^M \frac{p_n}{(1+\xi)^n} \right)^2 + \left(\sum_{n=1}^M \frac{p_n \xi}{(1+\xi)^n} \right)^2 \right]. \end{aligned} \quad (\text{A2})$$

The inequality in (A2) is saturated if $p_n = 0$ for all $n \geq 2$.

Proof. Let $\psi_A = \text{Tr}_R(\psi_{RA}) = \sum_{n=0}^{\infty} p_n |n\rangle\langle n|_A$. From (A1) we get

$$\mathcal{L}_{A \rightarrow B}^{1-\eta}(\psi_A) = \sum_{n=0}^{\infty} p_n \sum_{k=0}^n \binom{n}{k} (1-\eta)^k \eta^{n-k} |k\rangle\langle k|_B. \quad (\text{A3})$$

For $\eta = 1/(1+\xi)$, we have

$$F(\mathcal{I}_{A \rightarrow B}(\psi_{RA}), \mathcal{T}_{A \rightarrow B}^{\xi}(\psi_{RA})) = \eta \text{Tr}((\mathcal{L}_{A \rightarrow B}^{1-\eta}(\psi_A))^2) \quad (\text{A4})$$

$$\geq \eta \text{Tr} \left[\left(\sum_{n=0}^M p_n \sum_{k=0}^n \binom{n}{k} (1-\eta)^k \eta^{n-k} |k\rangle\langle k|_B \right)^2 \right] \quad (\text{A5})$$

$$= \eta \text{Tr} \left[\left(\sum_{k=0}^M \left(\sum_{n=k}^M p_n \binom{n}{k} (1-\eta)^k \eta^{n-k} \right) |k\rangle\langle k|_B \right)^2 \right] \quad (\text{A6})$$

$$= \eta \sum_{k=0}^M \left\{ \sum_{n=k}^M p_n \binom{n}{k} (1-\eta)^k \eta^{n-k} \right\}^2 \quad (\text{A7})$$

$$\geq \eta \sum_{k=0}^1 \left\{ \sum_{n=k}^M p_n \binom{n}{k} (1-\eta)^k \eta^{n-k} \right\}^2 \quad (\text{A8})$$

$$= \eta \left(\sum_{n=0}^M p_n \eta^n \right)^2 + \eta \left(\sum_{n=1}^M p_n n (1-\eta) \eta^{n-1} \right)^2 \quad (\text{A9})$$

$$\geq \eta \left(\sum_{n=0}^M p_n \eta^n \right)^2 + \eta \left(\sum_{n=1}^M p_n (1-\eta) \eta^{n-1} \right)^2 \quad (\text{A10})$$

$$= \frac{1}{(1+\xi)} \left[\left(\sum_{n=0}^M \frac{p_n}{(1+\xi)^n} \right)^2 + \left(\sum_{n=1}^M \frac{p_n \xi}{(1+\xi)^n} \right)^2 \right]. \quad (\text{A11})$$

Proposition 2. The minimum value of $f_{M,\xi}$, as defined in (15), subject to $p_n \geq 0$ for all $n \in \{0, \dots, M\}$,

$$\sum_{n=0}^M p_n = 1, \quad \sum_{n=0}^M n p_n \leq E, \quad (\text{A12})$$

is attained at the unique point $p = (p_n)_{n=0}^M$ given by $p_0 = 1 - E$, $p_1 = E$, and $p_n = 0$ for all $n \in \{2, \dots, M\}$, whenever $E \leq (1+\xi)/(1+3\xi)$.

Proof. We divide the proof into two parts. Let $E_0 \in [0, E]$. In the first part, we show that the minimum of $f_{M,\xi}$, subject to $p_n \geq 0$ for all $n \in \{0, \dots, M\}$ and the equality constraints

$$\sum_{n=0}^M p_n = 1, \quad \sum_{n=0}^M np_n = E_0, \quad (\text{A13})$$

is uniquely attained at $p_0 = 1 - E_0$, $p_1 = E_0$, and $p_n = 0$ for all $n \in \{2, \dots, M\}$. In the second part, we show that the minimum value obtained in the first part is a strictly decreasing function of E_0 .

Part 1. Let $p \in \mathbb{R}^{M+1}$ be any vector satisfying the equality constraints (A13), which gives

$$p_0 = 1 - E_0 + \sum_{n=2}^M (n-1)p_n, \quad p_1 = E_0 - \sum_{n=2}^M np_n. \quad (\text{A14})$$

Substitute the values of p_0 and p_1 in (15), and use the relation $\eta = 1/(1 + \xi) \in [0, 1]$ to get

$$f_{M,\xi}(p) = \eta \left(1 - (1 - \eta)E_0 + \sum_{n=2}^M [n(1 - \eta) - 1 + \eta^n]p_n \right)^2 + \eta(1 - \eta)^2 \left(E_0 - \sum_{n=2}^M (n - \eta^{n-1})p_n \right)^2. \quad (\text{A15})$$

Let $g: \mathbb{R}^{M-1} \rightarrow \mathbb{R}$ be the function defined by

$$g(x_2, \dots, x_M) := \eta \left(1 - (1 - \eta)E_0 + \sum_{n=2}^M [n(1 - \eta) - 1 + \eta^n]x_n \right)^2 + \eta(1 - \eta)^2 \left(E_0 - \sum_{n=2}^M (n - \eta^{n-1})x_n \right)^2. \quad (\text{A16})$$

From (A15), we thus have $f_{M,\xi}(p) = g(p_2, \dots, p_M)$. We show that g is a strictly increasing function in each variable over

$$\mathbb{R}_+^{M-1} = \{(x_2, \dots, x_M) : \forall k \in \{2, \dots, M\}, x_k \geq 0\}. \quad (\text{A17})$$

This implies that the unique global minimizer of g over \mathbb{R}_+^{M-1} is at the origin. For all $k \in \{2, \dots, M\}$, differentiate g partially with respect to x_k . We get

$$\begin{aligned} \frac{\partial g(x_2, \dots, x_M)}{\partial x_k} &= 2\eta(\eta^k - k\eta + k - 1) \left[1 - E_0 + \eta E_0 + \sum_{n=2}^M [n(1 - \eta) - 1 + \eta^n]x_n \right] \\ &\quad + 2\eta(1 - \eta)^2(\eta^{k-1} - k) \left[E_0 + \sum_{n=2}^M (\eta^{n-1} - n)x_n \right]. \end{aligned} \quad (\text{A18})$$

Simplifying (A18) gives

$$\begin{aligned} \frac{\partial g(x_2, \dots, x_M)}{\partial x_k} &= 2\eta [[k(1 - \eta) - (1 - \eta^k)][1 - (1 - \eta)E_0] - (k - \eta^{k-1})(1 - \eta)^2 E_0] \\ &\quad + 2\eta \sum_{n=2}^M [[k(1 - \eta) - (1 - \eta^k)][n(1 - \eta) - (1 - \eta^n)] + (1 - \eta)^2(k - \eta^{k-1})(n - \eta^{n-1})]x_n. \end{aligned} \quad (\text{A19})$$

The coefficients of x_n are positive because for all $k \geq 2$,

$$k(1 - \eta) - (1 - \eta^k) = (1 - \eta) \sum_{i=0}^{k-1} (1 - \eta^i) > 0. \quad (\text{A20})$$

Also, the remaining term in (A19) is positive:

$$\begin{aligned} 2\eta [[k(1 - \eta) - (1 - \eta^k)][1 - (1 - \eta)E_0] - (k - \eta^{k-1})(1 - \eta)^2 E_0] \\ \geq 2\eta \left[[k(1 - \eta) - (1 - \eta^k)] \left(1 - \frac{1 - \eta}{3 - 2\eta} \right) - \frac{(k - \eta^{k-1})(1 - \eta)^2}{3 - 2\eta} \right] \end{aligned} \quad (\text{A21})$$

$$= \frac{2\eta}{3 - 2\eta} [[k(1 - \eta) - 1 + \eta^k](2 - \eta) - (k - \eta^{k-1})(1 - \eta)^2] \quad (\text{A22})$$

$$= \frac{2\eta}{3 - 2\eta} [k(1 - \eta) - 1 + \eta^k + [k(1 - \eta) - 1 + \eta^k](1 - \eta) - k(1 - \eta)^2 + \eta^{k-1}(1 - \eta)^2] \quad (\text{A23})$$

$$= \frac{2\eta}{3 - 2\eta} [k(1 - \eta) - 1 + \eta^k + k(1 - \eta)^2 - (1 - \eta) + \eta^k(1 - \eta) - k(1 - \eta)^2 + \eta^{k-1}(1 - \eta)^2] \quad (\text{A24})$$

$$= \frac{2\eta}{3 - 2\eta} [k(1 - \eta) + \eta^k + \eta + \eta^{k-1}(1 - \eta)] \quad (\text{A25})$$

$$= \frac{2\eta}{3-2\eta} [k(1-\eta) + \eta + \eta^{k-1}] \quad (A26)$$

$$> 0. \quad (A27)$$

The inequality (A21) follows from the fact that $E_0 \leq E \leq (1+\xi)/(1+3\xi) = 1/(3-2\eta)$, and (A27) follows from (A20). We thus have $\partial g(x_2, \dots, x_M)/\partial x_k > 0$, and hence g is a strictly increasing function of each of its variables in \mathbb{R}_+^{M-1} .

Part 2. We know from the first part of the proof that the minimum value of $f_{M,\xi}$ subject to the constraints (A14) is

$$f_{M,\xi}(1-E_0, E_0, 0, \dots, 0) = \frac{1}{(1+\xi)} \left[1 - 2 \left(\frac{\xi}{1+\xi} \right) E_0 + 2 \left(\frac{\xi}{1+\xi} \right)^2 E_0^2 \right]. \quad (A28)$$

The quadratic polynomial $1 - 2[\xi/(1+\xi)]E_0 + 2[\xi/(1+\xi)]^2E_0^2$ is a strictly decreasing function in the interval $[0, (1+\xi)/(2\xi)]$, and from the hypothesis we have $E_0 \leq E \in [0, (1+\xi)/(2\xi)]$. Therefore, the minimum value of $f_{M,\xi}$ subject to the constraints (A12) is obtained at $p_0 = 1-E$, $p_1 = E$, and $p_k = 0$ for all $k \geq 2$. \blacksquare

Lemma 3. The M -truncated energy-constrained channel fidelity (12) has the unique optimal state given by (11), whenever $E \leq (1+\xi)/(1+3\xi)$. Furthermore,

$$F_{E,M}(\mathcal{I}_{A \rightarrow B}, \mathcal{T}_{A \rightarrow B}^\xi) = \frac{1}{(1+\xi)} \left[1 - 2 \left(\frac{\xi}{1+\xi} \right) E + 2 \left(\frac{\xi}{1+\xi} \right)^2 E^2 \right]. \quad (A29)$$

In particular, $F_{E,M}(\mathcal{I}_{A \rightarrow B}, \mathcal{T}_{A \rightarrow B}^\xi)$ is independent of M .

Proof. Let us consider any pure bipartite state $|\phi\rangle_{RA} = \sum_{n=0}^M \sqrt{p_n} |n\rangle_R |n\rangle_A$ with $p_n \geq 0$ for all $n \in \{0, \dots, M\}$, such that

$$\sum_{n=0}^M p_n = 1, \quad \sum_{n=0}^M n p_n \leq E. \quad (A30)$$

From Propositions 1 and 2, we have

$$F(\phi_{RA}, \mathcal{T}_{A \rightarrow B}^\xi(\phi_{RA})) \geq f_{M,\xi}(1-E, E, 0, \dots, 0) = \frac{1}{(1+\xi)} \left[1 - 2 \left(\frac{\xi}{1+\xi} \right) E + 2 \left(\frac{\xi}{1+\xi} \right)^2 E^2 \right]. \quad (A31)$$

By taking the infimum in (A31) over ϕ_{RA} , and by definition (12), we get

$$F_{E,M}(\mathcal{I}_{A \rightarrow B}, \mathcal{T}_{A \rightarrow B}^\xi) \geq \frac{1}{(1+\xi)} \left[1 - 2 \left(\frac{\xi}{1+\xi} \right) E + 2 \left(\frac{\xi}{1+\xi} \right)^2 E^2 \right]. \quad (A32)$$

Also, the inequality in (A31) is saturated for the state in (11). This means that (A32) is actually an equality. The uniqueness of the optimal state follows from Proposition 2. \blacksquare

Theorem 4. The energy-constrained channel fidelity (8) has an optimal input state given by (11), whenever $E \leq (1+\xi)/(1+3\xi)$. Moreover, the optimal state is unique in the sense that there is no other optimal finite entangled superposition of twin-Fock states. The value of the energy-constrained channel fidelity is

$$F_E(\mathcal{I}_{A \rightarrow B}, \mathcal{T}_{A \rightarrow B}^\xi) = \frac{1}{(1+\xi)} \left[1 - 2 \left(\frac{\xi}{1+\xi} \right) E + 2 \left(\frac{\xi}{1+\xi} \right)^2 E^2 \right]. \quad (A33)$$

Proof. From the inequalities (B31) of Ref. [54], we have

$$1 - \left[2\sqrt{\frac{E}{M+1}} + \sqrt{1 - F_{E,M}(\mathcal{I}_{A \rightarrow B}, \mathcal{T}_{A \rightarrow B}^\xi)} \right]^2 \leq F_E(\mathcal{I}_{A \rightarrow B}, \mathcal{T}_{A \rightarrow B}^\xi) \leq F_{E,M}(\mathcal{I}_{A \rightarrow B}, \mathcal{T}_{A \rightarrow B}^\xi). \quad (A34)$$

By Lemma 3, we thus get

$$\begin{aligned} 1 - \left[2\sqrt{\frac{E}{M+1}} + \sqrt{1 - \frac{1}{(1+\xi)} \left(1 - 2 \left(\frac{\xi}{1+\xi} \right) E + 2 \left(\frac{\xi}{1+\xi} \right)^2 E^2 \right)} \right]^2 &\leq F_E(\mathcal{I}_{A \rightarrow B}, \mathcal{T}_{A \rightarrow B}^\xi) \\ &\leq \frac{1}{(1+\xi)} \left[1 - 2 \left(\frac{\xi}{1+\xi} \right) E + 2 \left(\frac{\xi}{1+\xi} \right)^2 E^2 \right]. \end{aligned} \quad (A35)$$

(A33) is obtained by taking the limit $M \rightarrow \infty$, and the optimal state is given by (11), which follows from the proof of Lemma 3.

Any finite entangled superposition of twin-Fock states that is optimal for $F_E(\mathcal{I}_{A \rightarrow B}, \mathcal{T}_{A \rightarrow B}^\xi)$ is also optimal for $F_{E,M}(\mathcal{I}_{A \rightarrow B}, \mathcal{T}_{A \rightarrow B}^\xi)$ for large M . We know by Lemma 3 that $F_{E,M}(\mathcal{I}_{A \rightarrow B}, \mathcal{T}_{A \rightarrow B}^\xi)$ has the same unique optimal state for large M . This implies the uniqueness of the optimal state (11) in the given sense. \blacksquare

APPENDIX B: OPTIMAL STATE FOR THE BIDIRECTIONAL TELEPORTATION PROTOCOL

The proof of the following result is based on the ideas of Refs. [54, Appendix B] and [79, Proposition 2].

Proposition 5. The energy-constrained channel fidelity (49) and its truncated version (50) satisfy the inequalities

$$\begin{aligned} 1 - \left[2\sqrt{1 - \left(1 - \frac{2E}{M+1}\right)^2} + \sqrt{1 - F_{E,M}(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'})} \right]^2 \\ \leq F_E(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}) \leq F_{E,M}(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}). \end{aligned} \quad (\text{B1})$$

Proof. By definition, we have

$$F_E(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}) \leq F_{E,M}(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}). \quad (\text{B2})$$

We now establish the inequality in the other direction. Let Π_{AB}^M be the $(M+1)^2$ -dimensional projection operator defined as

$$\Pi_{AB}^M := \sum_{m,n=0}^M |m, n\rangle\langle m, n|_{AB}. \quad (\text{B3})$$

Let ψ_{RAB} be an arbitrary pure state in (33). We then have

$$\text{Tr}(\Pi_{AB}^M \psi_{RAB}) = 1 - \sum_{\substack{m,n=0 \\ \max\{m,n\} \geq M+1}}^{\infty} \langle m, n | \psi_{AB} | m, n \rangle \quad (\text{B4})$$

$$= 1 - \sum_{\substack{m,n=0 \\ \max\{m,n\} \geq M+1}}^{\infty} \lambda_{m,n}^2 \quad (\text{B5})$$

$$\geq 1 - \sum_{\substack{m,n=0 \\ \max\{m,n\} \geq M+1}}^{\infty} \left(\frac{m+n}{M+1} \right) \lambda_{m,n}^2 \quad (\text{B6})$$

$$\geq 1 - \frac{2E}{M+1}. \quad (\text{B7})$$

(B6) follows from the fact $(m+n)/(M+1) \geq 1$, and (B7) is a consequence of the constraint $\sum_{m,n=0}^{\infty} (m+n) \lambda_{m,n}^2 \leq 2E$. Let ψ_{RAB}^M be the truncated state given by

$$\psi_{RAB}^M := \frac{\Pi_{AB}^M \psi_{RAB} \Pi_{AB}^M}{\text{Tr}(\Pi_{AB}^M \psi_{RAB})}. \quad (\text{B8})$$

We have

$$F(\psi_{RAB}, \psi_{RAB}^M) \geq \left(1 - \frac{1}{2} \|\psi_{RAB} - \psi_{RAB}^M\|_1 \right)^2 \quad (\text{B9})$$

$$\geq \left(1 - \sqrt{\frac{2E}{M+1}} \right)^2. \quad (\text{B10})$$

The inequality (B9) follows from (3), and (B10) follows from the gentle measurement lemma (see, e.g., Ref. [63, Lemma 9.4.1]). We have

$$\begin{aligned} & C(\mathcal{S}_{AB \rightarrow A'B'}(\psi_{RAB}), \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\psi_{RAB})) \\ & \leq C(\mathcal{S}_{AB \rightarrow A'B'}(\psi_{RAB}), \mathcal{S}_{AB \rightarrow A'B'}(\psi_{RAB}^M)) + C(\mathcal{S}_{AB \rightarrow A'B'}(\psi_{RAB}^M), \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\psi_{RAB}^M)) \\ & \quad + C(\mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\psi_{RAB}^M), \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\psi_{RAB})) \end{aligned} \quad (\text{B11})$$

$$\leq 2C(\psi_{RAB}, \psi_{RAB}^M) + C(\mathcal{S}_{AB \rightarrow A'B'}(\psi_{RAB}^M), \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\psi_{RAB}^M)) \quad (\text{B12})$$

$$= 2\sqrt{1 - F(\psi_{RAB}, \psi_{RAB}^M)} + \sqrt{1 - F(\mathcal{S}_{AB \rightarrow A'B'}(\psi_{RAB}^M), \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'})} \quad (B13)$$

$$\leq 2\sqrt{1 - \left(1 - \frac{2E}{M+1}\right)^2} + \sqrt{1 - F_{E,M}(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'})}. \quad (B14)$$

The inequalities (B11) and (B12) are consequences of the triangle inequality and monotonicity of the sine distance, respectively; (B14) follows from (B10) and the fact that ψ_{RAB}^M is a legitimate finite dimensional state that satisfies the energy constraint $\text{Tr}(\hat{n}_{AB}\psi_{RAB}^M) \leq 2E$. The inequality (B14) is true for arbitrary ψ_{RAB} in (33). So, we get

$$C_E(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}) \leq 2\sqrt{1 - \left(1 - \frac{2E}{M+1}\right)^2} + \sqrt{1 - F_{E,M}(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'})}. \quad (B15)$$

Squaring and then rearranging (B15) gives the desired inequality. \blacksquare

Define $f_{M,\xi,\xi'} : \mathbb{R}^{(M+1)^2} \rightarrow \mathbb{R}$ as

$$f_{M,\xi,\xi'}(p) := \frac{1}{(1+\xi)(1+\xi')} \sum_{m,n,m',n'=0}^M p_{m,n} p_{m',n'} T_\xi^{mm'} T_{\xi'}^{nn'}. \quad (B16)$$

We recall from (47) that for any *probability vector* p in $\mathbb{R}^{(M+1)^2}$, a pure state of the form $|\phi\rangle_{RAB} = \sum_{m,n=0}^M \sqrt{p_{m,n}} |m, n\rangle_R |m, n\rangle_{AB}$ satisfies

$$F(\mathcal{S}_{AB \rightarrow A'B'}(\phi_{RAB}), \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}) = f_{M,\xi,\xi'}(p). \quad (B17)$$

By (47) and (50) we thus have

$$F_{E,M}(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}) = \inf_p f_{M,\xi,\xi'}(p), \quad (B18)$$

where the minimum is taken over probability vectors p in $\mathbb{R}^{(M+1)^2}$ satisfying $\sum_{m,n=0}^M (m+n)p_{m,n} \leq 2E$. Let

$$D_M := \{(m, n, m', n') \in \mathbb{Z}_+^4 : m, n, m', n' \leq M, m' + n' \geq 2, m + n \geq 2\}, \quad (B19)$$

where $\mathbb{Z}_+^4 \subset \mathbb{Z}^4$ is the set of 4-tuples of nonnegative integers. We can rewrite $f_{M,\xi,\xi'}$ as

$$\begin{aligned} (1+\xi)(1+\xi')f_{M,\xi,\xi'}(p) &= \sum_{\substack{m,n,m',n'=0 \\ (m,n,m',n') \notin D_M}}^M p_{m,n} p_{m',n'} T_\xi^{mm'} T_{\xi'}^{nn'} + \sum_{\substack{m,n,m',n'=0 \\ (m,n,m',n') \in D_M}}^M p_{m,n} p_{m',n'} T_\xi^{mm'} T_{\xi'}^{nn'} \\ &= \sum_{\substack{m,n,m',n'=0 \\ m+n \leq 1 \\ m'+n' \leq 1}}^M p_{m,n} p_{m',n'} T_\xi^{mm'} T_{\xi'}^{nn'} + \sum_{\substack{m,n,m',n'=0 \\ m+n \leq 1 \\ m'+n' \geq 2}}^M p_{m,n} p_{m',n'} T_\xi^{mm'} T_{\xi'}^{nn'} \\ &\quad + \sum_{\substack{m,n,m',n'=0 \\ m+n \geq 2 \\ m'+n' \leq 1}}^M p_{m,n} p_{m',n'} T_\xi^{mm'} T_{\xi'}^{nn'} + \sum_{\substack{m,n,m',n'=0 \\ (m,n,m',n') \in D_M}}^M p_{m,n} p_{m',n'} T_\xi^{mm'} T_{\xi'}^{nn'}. \end{aligned} \quad (B20)$$

From (A1), using $\eta = 1/(1+\xi)$ we get

$$\mathcal{L}^{\frac{1}{1+\xi}}(|m\rangle\langle m'|) \mathcal{L}^{\frac{1}{1+\xi}}(|m'\rangle\langle m|) = \sum_{k=0}^{\min\{m,m'\}} \binom{m}{k} \binom{m'}{k} \frac{\xi^{2k}}{(1+\xi)^{m+m'}} |m-k\rangle\langle m-k|. \quad (B21)$$

By taking the trace on both sides of (B21), and from (45) we get

$$T_\xi^{mm'} = \sum_{k=0}^{\min\{m,m'\}} \binom{m}{k} \binom{m'}{k} \frac{\xi^{2k}}{(1+\xi)^{m+m'}}. \quad (B22)$$

In particular, $T_\xi^{0k} = 1/(1+\xi)^k$ and $T_\xi^{1k} = 1/(1+\xi)^{k+1} + k\xi^2/(1+\xi)^{k+1}$ for all $k \geq 0$. Similarly,

$$T_{\xi'}^{nn'} = \sum_{k=0}^{\min\{n,n'\}} \binom{n}{k} \binom{n'}{k} \frac{\xi'^{2k}}{(1+\xi')^{n+n'}}. \quad (B23)$$

We also note that $T_{\xi}^{mm'} = T_{\xi}^{m'm}$ and $T_{\xi'}^{nn'} = T_{\xi'}^{n'n}$. From (B20) we thus get

$$f_{M,\xi,\xi'}(p) = \frac{1}{(1+\xi)(1+\xi')} \left[\sum_{\substack{m,n,m',n'=0 \\ m+n \leq 1 \\ m'+n' \leq 1}}^M p_{m,n} p_{m',n'} T_{\xi}^{mm'} T_{\xi'}^{nn'} + 2 \sum_{\substack{m,n,m',n'=0 \\ m+n \leq 1 \\ m'+n' \geq 2}}^M p_{m,n} p_{m',n'} T_{\xi}^{mm'} T_{\xi'}^{nn'} + \sum_{\substack{m,n,m',n'=0 \\ (m,n,m',n') \in D_M}}^M p_{m,n} p_{m',n'} T_{\xi}^{mm'} T_{\xi'}^{nn'} \right]. \quad (\text{B24})$$

1. Optimal input state for the energy-constrained channel fidelity for $\xi = \xi'$

Proposition 6. The minimum of $f_{M,\xi,\xi}$, subject to $p_{m,n} \geq 0$ for all $m, n \in \{0, \dots, M\}$,

$$\sum_{m,n=0}^M p_{m,n} = 1, \quad \sum_{m,n=0}^M (m+n)p_{m,n} \leq 2E, \quad (\text{B25})$$

is attained at the unique point $p = (p_{m,n})_{m,n=0}^M$ in $\mathbb{R}^{(M+1)^2}$, given by

$$p_{m,n} = \begin{cases} 1-2E & \text{if } m=n=0, \\ E & \text{if } m+n=1, \\ 0 & \text{if } m+n \geq 2, \end{cases} \quad (\text{B26})$$

whenever $2E \leq (1+\xi)/(2+3\xi)$.

Proof. We divide the proof into two parts, similar to the proof of Proposition 2. Let $E_0 \in [0, E]$. In the first part, we show that the minimum of $f_{M,\xi,\xi}$, subject to $p_{m,n} \geq 0$ for all $m, n \in \{0, \dots, M\}$, and the equality constraints

$$\sum_{m,n=0}^M p_{m,n} = 1, \quad \sum_{m,n=0}^M (m+n)p_{m,n} = 2E_0 \quad (\text{B27})$$

is uniquely attained at p as given in (B26). In the second part, we show that the minimum value obtained in the first part is a strictly decreasing function of E_0 . It then follows that the minimizer of $f_{M,\xi,\xi}$ subject to (B25) is given by (B26).

Part 1. In (B24), substitute the values of $T_{\xi}^{mm'}$ and $T_{\xi'}^{nn'}$ from (B22) and (B23) to get

$$\begin{aligned} f_{M,\xi,\xi}(p) &= \eta^2 \left[p_{0,0}^2 + 2\eta p_{0,0}p_{0,1} + 2\eta p_{0,0}p_{1,0} + 2\eta^2 p_{0,1}p_{1,0} + (\eta^2 + (1-\eta)^2)(p_{0,1}^2 + p_{1,0}^2) \right. \\ &\quad + 2 \sum_{\substack{m,n=0 \\ m+n \geq 2}}^M \{ \eta^{m+n} p_{0,0} + \eta^m (\eta^{n+1} + n\eta^{n-1}(1-\eta)^2) p_{0,1} + \eta^n (\eta^{m+1} + m\eta^{m-1}(1-\eta)^2) p_{1,0} \} p_{m,n} \\ &\quad \left. + \sum_{\substack{m,n,m',n'=0 \\ (m,n,m',n') \in D_M}}^M p_{m,n} p_{m',n'} T_{\xi}^{mm'} T_{\xi'}^{nn'} \right]. \end{aligned} \quad (\text{B28})$$

Here we used the relation $\eta = 1/(1+\xi)$ for making the following calculations convenient. From (B27), we can write $p_{0,0}$ and $p_{0,1}$ as

$$p_{0,1} = 2E_0 - p_{1,0} - \sum_{\substack{m,n=0 \\ m+n \geq 2}}^M (m+n)p_{m,n}, \quad (\text{B29})$$

$$p_{0,0} = 1 - 2E_0 + \sum_{\substack{m,n=0 \\ m+n \geq 2}}^M (m+n-1)p_{m,n}. \quad (\text{B30})$$

Using the relations (B29) and (B30), we can treat $p_{0,0}, p_{0,1}$ as dependent variables so that $f_{M,\xi,\xi}(p)$ is a function of $(M+1)^2 - 2$ independent variables $\{p_{r,s} : r, s \in \{0, \dots, M\} \setminus \{(0,0), (0,1)\}\}$. We have

$$\frac{\partial p_{0,0}}{\partial p_{r,s}} = r+s-1, \quad \frac{\partial p_{0,1}}{\partial p_{r,s}} = -(r+s). \quad (\text{B31})$$

We now argue that a necessary condition for a minimizer is $p_{r,s} = 0$ whenever $r + s \geq 2$. Differentiate $f_{M,\xi,\xi}$ partially with respect to $p_{r,s}$ for $r + s \geq 2$. We get

$$\begin{aligned} \frac{1}{2\eta^2} \frac{\partial f_{M,\xi,\xi}(p)}{\partial p_{r,s}} &= (r+s-1)p_{0,0} + \eta[-(r+s)p_{0,0} + (r+s-1)p_{0,1}] + \eta[(r+s-1) - \eta(r+s)]p_{1,0} \\ &\quad - [\eta^2 + (1-\eta)^2](r+s)p_{0,1} + \eta^{r+s}p_{0,0} + \eta^r[\eta^{s+1} + s\eta^{s-1}(1-\eta)^2]p_{0,1} + \eta^s[\eta^{r+1} + r\eta^{r-1}(1-\eta)^2]p_{1,0} \\ &\quad + \sum_{\substack{m,n=0 \\ m+n \geq 2}}^M \{(r+s-1)\eta^{m+n} - (r+s)\eta^m[\eta^{n+1} + n\eta^{n-1}(1-\eta)^2] + T_{\xi}^{mr}T_{\xi}^{ns}\}p_{m,n}. \end{aligned} \quad (\text{B32})$$

Further simplification gives

$$\begin{aligned} \frac{1}{2\eta^2} \frac{\partial f_{M,\xi,\xi}(p)}{\partial p_{r,s}} &= [(r+s-1) - \eta(r+s) + \eta^{r+s}]p_{0,0} \\ &\quad + [\eta(r+s-1) - (\eta^2 + (1-\eta)^2)(r+s) + \eta^{r+s+1} + s\eta^{r+s-1}(1-\eta)^2]p_{0,1} \\ &\quad + [\eta((r+s-1) - \eta(r+s)) + \eta^{r+s+1} + r\eta^{r+s-1}(1-\eta)^2]p_{1,0} \\ &\quad + \sum_{\substack{m,n=0 \\ m+n \geq 2}}^M [(r+s-1)\eta^{m+n} - (r+s)(\eta^{m+n+1} + n\eta^{m+n-1}(1-\eta)^2) + T_{\xi}^{mr}T_{\xi}^{ns}]p_{m,n}. \end{aligned} \quad (\text{B33})$$

Substitute the values of $p_{0,0}$ and $p_{0,1}$ in (B33) from (B29) and (B30), and simplify to get

$$\begin{aligned} \frac{1}{2\eta^2} \frac{\partial f_{M,\xi,\xi}(p)}{\partial p_{r,s}} &= (r+s)(1-\eta) - (1-\eta^{r+s}) - 2E_0(1-\eta)[(2(r+s) - s\eta^{r+s-1})(1-\eta) - (1-\eta^{r+s})] \\ &\quad + (1-\eta)^2[(r+s) + (r-s)\eta^{r+s-1}]p_{1,0} + \sum_{\substack{m,n=0 \\ m+n \geq 2}}^M \Gamma_{m,n,r,s,\eta}p_{m,n}, \end{aligned} \quad (\text{B34})$$

where

$$\begin{aligned} \Gamma_{m,n,r,s,\eta} &= (1-\eta)^2 \left[2(m+n)(r+s) - \frac{1-\eta^{r+s}}{1-\eta}(m+n) - \frac{1-\eta^{m+n}}{1-\eta}(r+s) \right] + 1 - \eta^{m+n} - \eta^{r+s} + T_{\eta}^{mr}T_{\eta}^{ns} \\ &\quad - (1-\eta)^2[s(m+n)\eta^{r+s-1} + n(r+s)\eta^{m+n-1}]. \end{aligned} \quad (\text{B35})$$

The following arguments show that $\Gamma_{m,n,r,s,\eta} > 0$. We have

$$T_{\eta}^{mr} \geq \eta^{m+r}, \quad (\text{B36})$$

$$T_{\eta}^{ns} \geq \eta^{n+s} \left[1 + ns \left(\frac{1-\eta}{\eta} \right)^2 \right]. \quad (\text{B37})$$

This implies

$$T_{\eta}^{mr}T_{\eta}^{ns} \geq \eta^{m+n}\eta^{r+s} \left[1 + [(r+s)-r][(m+n)-m] \left(\frac{1-\eta}{\eta} \right)^2 \right]. \quad (\text{B38})$$

Let $\alpha = r+s$ and $\beta = m+n$. Using the inequality (B38), from (B35) we get

$$\begin{aligned} \Gamma_{m,n,r,s,\eta} &\geq (1-\eta)^2 \left[2\alpha\beta - \frac{1-\eta^{\alpha}}{1-\eta}\beta - \frac{1-\eta^{\beta}}{1-\eta}\alpha \right] + 1 - \eta^{\beta} - \eta^{\alpha} + \eta^{\alpha+\beta} \left[1 + (\alpha-r)(\beta-m) \left(\frac{1-\eta}{\eta} \right)^2 \right] \\ &\quad - (1-\eta)^2[(\alpha-r)\beta\eta^{\alpha-1} + (\beta-m)\alpha\eta^{\beta-1}] \end{aligned} \quad (\text{B39})$$

$$\begin{aligned} &= (1-\eta)^2 \left[2\alpha\beta - \frac{1-\eta^{\alpha}}{1-\eta}\beta - \frac{1-\eta^{\beta}}{1-\eta}\alpha \right] + 1 - \eta^{\beta} - \eta^{\alpha} + \eta^{\alpha+\beta} \\ &\quad + (1-\eta)^2[(\alpha-r)(\beta-m)\eta^{\alpha+\beta-2} - (\alpha-r)\beta\eta^{\alpha-1} - (\beta-m)\alpha\eta^{\beta-1}] \end{aligned} \quad (\text{B40})$$

$$\begin{aligned} &= (1-\eta)^2 \left[2\alpha\beta - \frac{1-\eta^{\alpha}}{1-\eta}\beta - \frac{1-\eta^{\beta}}{1-\eta}\alpha \right] + (1-\eta^{\alpha})(1-\eta^{\beta}) \\ &\quad + (1-\eta)^2[(\alpha\beta - \beta r - \alpha m + \beta m)\eta^{\alpha-1}\eta^{\beta-1} - (\alpha\beta - \beta r)\eta^{\alpha-1} - (\alpha\beta - \alpha m)\eta^{\beta-1}] \end{aligned} \quad (\text{B41})$$

$$\begin{aligned}
&= (1-\eta)^2 \left[2\alpha\beta - \frac{1-\eta^\alpha}{1-\eta}\beta - \frac{1-\eta^\beta}{1-\eta}\alpha \right] + (1-\eta^\alpha)(1-\eta^\beta) + (1-\eta)^2\alpha\beta[\eta^{\alpha-1}\eta^{\beta-1} - \eta^{\alpha-1} - \eta^{\beta-1}] \\
&\quad + (1-\eta)^2[\beta r\eta^{\alpha-1}(1-\eta^{\beta-1}) + \alpha m\eta^{\beta-1}(1-\eta^{\alpha-1}) + rm\eta^{\alpha-1}\eta^{\beta-1}] \tag{B42}
\end{aligned}$$

$$\geq (1-\eta)^2 \left[2\alpha\beta - \frac{1-\eta^\alpha}{1-\eta}\beta - \frac{1-\eta^\beta}{1-\eta}\alpha \right] + (1-\eta^\alpha)(1-\eta^\beta) + (1-\eta)^2\alpha\beta[\eta^{\alpha-1}\eta^{\beta-1} - \eta^{\alpha-1} - \eta^{\beta-1}] \tag{B43}$$

$$= (1-\eta)^2 \left[\alpha\beta(1 + \eta^{\alpha-1}\eta^{\beta-1} - \eta^{\alpha-1} - \eta^{\beta-1}) + \alpha\beta - \frac{1-\eta^\alpha}{1-\eta}\beta - \frac{1-\eta^\beta}{1-\eta}\alpha + \left(\frac{1-\eta^\alpha}{1-\eta}\right)\left(\frac{1-\eta^\beta}{1-\eta}\right) \right] \tag{B44}$$

$$= (1-\eta)^2 \left[\alpha\beta(1 - \eta^{\alpha-1})(1 - \eta^{\beta-1}) + \left(\alpha - \sum_{i=0}^{\alpha-1} \eta^i\right)\left(\beta - \sum_{i=0}^{\beta-1} \eta^i\right) \right] > 0. \tag{B45}$$

Now, the constant term is nonnegative if

$$2E_0(1-\eta)[(2\alpha - s\eta^{\alpha-1})(1-\eta) - (1-\eta^\alpha)] \leq \alpha(1-\eta) - (1-\eta^\alpha). \tag{B46}$$

Using the fact that $s \geq 0$, the inequality (B46) holds if

$$2E_0(1-\eta)[2\alpha(1-\eta) - (1-\eta^\alpha)] \leq \alpha(1-\eta) - (1-\eta^{r+s}). \tag{B47}$$

By basic real analysis it is easy to verify that the coefficient of E_0 in (B47) is positive. So, the inequality (B47) is equivalent to

$$2E_0 \leq \frac{\alpha(1-\eta) - (1-\eta^\alpha)}{(1-\eta)[2\alpha(1-\eta) - (1-\eta^\alpha)]}. \tag{B48}$$

The right-hand side expression in (B48) is an increasing function of $\alpha \geq 2$. To verify this, it suffices to show that

$$\frac{\alpha(1-\eta) - (1-\eta^\alpha)}{(1-\eta)[2\alpha(1-\eta) - (1-\eta^\alpha)]} - \frac{2(1-\eta) - (1-\eta^2)}{(1-\eta)[4(1-\eta) - (1-\eta^2)]} \geq 0 \iff \tag{B49}$$

$$\frac{\alpha(1-\eta) - (1-\eta^\alpha)}{2\alpha(1-\eta) - (1-\eta^\alpha)} - \frac{1-\eta}{3-\eta} \geq 0 \iff \tag{B50}$$

$$1 - \frac{\alpha(1-\eta)}{2\alpha(1-\eta) - (1-\eta^\alpha)} - \frac{1-\eta}{3-\eta} \geq 0 \iff \tag{B51}$$

$$\frac{2}{3-\eta} - \frac{\alpha(1-\eta)}{2\alpha(1-\eta) - (1-\eta^\alpha)} \geq 0 \iff \tag{B52}$$

$$4\alpha(1-\eta) - 2(1-\eta^\alpha) - \alpha(1-\eta)(3-\eta) \geq 0 \iff \tag{B53}$$

$$4\alpha - 4\alpha\eta - 2 + 2\eta^\alpha - [3\alpha - 4\alpha\eta + \alpha\eta^2] \geq 0 \iff \tag{B54}$$

$$\alpha - 2 + 2\eta^\alpha - \alpha\eta^2 \geq 0 \iff \tag{B55}$$

$$\alpha(1-\eta^2) - 2(1-\eta^\alpha) \geq 0 \iff \tag{B56}$$

$$\alpha(1+\eta) - 2\left(\frac{1-\eta^\alpha}{1-\eta}\right) \geq 0 \iff \tag{B57}$$

$$\alpha + \alpha\eta - 2 - 2\sum_{i=1}^{\alpha-1} \eta^i \geq 0 \iff \tag{B58}$$

$$\alpha + \alpha\eta - 2 - 2(\alpha-1)\eta + 2\eta \sum_{i=1}^{\alpha-1} (1-\eta^{i-1}) \geq 0 \iff \tag{B59}$$

$$\alpha - \alpha\eta - 2 + 2\eta + 2\eta \sum_{i=1}^{\alpha-1} (1-\eta^{i-1}) \geq 0 \iff \tag{B60}$$

$$(\alpha - 2)(1-\eta) + 2 \sum_{i=1}^{\alpha-1} \eta(1-\eta^{i-1}) \geq 0. \tag{B61}$$

The inequality (B61) holds because $\alpha = r + s \geq 2$. A sufficient condition on E for the inequality (B47) is thus obtained by keeping $2E$ not more than the minimum value of the right hand side of (B48) which is attained for $\alpha = 2$. This is given by

$$2E \leq \frac{1}{3-\eta} = \frac{1+\xi}{2+3\xi}. \quad (\text{B62})$$

We have thus shown that $\partial f_{M,\xi,\xi}(p)/\partial p_{r,s} > 0$, whenever $p_{r,s} > 0$ and $r + s \geq 2$. So, $f_{M,\xi,\xi}$ is a strictly increasing function of the variables $p_{r,s}$ such that $r + s \geq 2$. Let q be a minimizer of $f_{M,\xi,\xi}$, which exists because $f_{M,\xi,\xi}$ is a continuous function over the compact set (B27). From the necessary condition derived earlier, we must have $q_{r,s} = 0$ for all $r + s \geq 2$. Thus, we get

$$q_{0,0} = 1 - 2E_0, \quad q_{0,1} = 2E_0 - q_{1,0} \quad (\text{B63})$$

from (B29) and (B30). This gives

$$f_{M,\xi,\xi}(q) = \eta^2 [1 - 4(1-\eta)E_0 + 4(1-4\eta+2\eta^2)E_0^2 - 4(1-\eta)^2E_0q_{1,0} + 2(1-\eta)^2q_{1,0}^2]. \quad (\text{B64})$$

This is a convex polynomial in $q_{1,0}$, which has the unique minimizer $q_{1,0} = E_0$. In other words, the minimizer of $f_{M,\xi,\xi}$, subject to the constraints (B27), is given by $q_{0,0} = 1 - 2E_0$, $q_{0,1} = q_{1,0} = E_0$, and $q_{m,n} = 0$ for all m, n with $m + n \geq 2$.

Part 2. By evaluating (B64) at the minimizer q obtained in the first part, and resubstituting $\eta = 1/(1+\xi)$, we get

$$f_{M,\xi,\xi}(q) = \frac{1}{(1+\xi)^2} \left[1 - 4 \left(\frac{\xi}{1+\xi} \right) E_0 + 6 \left(\frac{\xi}{1+\xi} \right)^2 E_0^2 \right], \quad (\text{B65})$$

which is a strictly decreasing function of E_0 in the interval $[0, (1+\xi)/(3\xi)]$. Also, we have $E_0 \leq E \in [0, (1+\xi)/(3\xi)]$, which follows from the hypothesis $E \leq (1+\xi)/(2+3\xi) < (1+\xi)/(3\xi)$. This completes the proof. \blacksquare

Lemma 7. The M -truncated energy-constrained channel fidelity (50) has the unique optimal state given by (52), whenever $\xi = \xi'$ and $2E \leq (1+\xi)/(2+3\xi)$. Furthermore, we have

$$F_{E,M}(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^\xi) = \frac{1}{(1+\xi)^2} \left[1 - 4 \left(\frac{\xi}{1+\xi} \right) E + 6 \left(\frac{\xi}{1+\xi} \right)^2 E^2 \right]. \quad (\text{B66})$$

In particular, the right-hand side of (B66) is independent of the truncation parameter M .

Proof. Let ψ_{RAB} be a pure state given by $|\psi\rangle_{RAB} = \sum_{m,n=0}^M \sqrt{p_{m,n}} |m, n\rangle_R |m, n\rangle_{AB}$, where $p = (p_{m,n})_{m,n=0}^M$ is a probability vector such that $\sum_{m,n=0}^M (m+n)p_{m,n} \leq 2E$. From (B17), we have

$$F(\mathcal{S}_{AB \rightarrow A'B'}(\phi_{RAB}), \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^\xi(\phi_{RAB})) = f_{M,\eta,\eta}(p). \quad (\text{B67})$$

By Proposition 6, for p in (B26), we have

$$F(\mathcal{S}_{AB \rightarrow A'B'}(\psi_{RAB}), \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^\xi(\psi_{RAB})) \geq f_{M,\xi,\xi}(p) = \frac{1}{(1+\xi)^2} \left[1 - 4 \left(\frac{\xi}{1+\xi} \right) E + 6 \left(\frac{\xi}{1+\xi} \right)^2 E^2 \right]. \quad (\text{B68})$$

By taking infimum in (B68) over ψ_{RAB} , we get

$$F_{E,M}(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^\xi) \geq \frac{1}{(1+\xi)^2} \left[1 - 4 \left(\frac{\xi}{1+\xi} \right) E + 6 \left(\frac{\xi}{1+\xi} \right)^2 E^2 \right]. \quad (\text{B69})$$

The inequality in (B68) is saturated for the state (52), which corresponds to the minimizer of $f_{M,\xi,\xi}$ by Proposition 6. Therefore, the inequality (B69) is actually an equality. The uniqueness of the optimal state follows from Proposition 6. \blacksquare

Theorem 8. The energy-constrained channel fidelity (49) has an optimal input state given by (52), whenever $\xi = \xi'$ and $2E \leq (1+\xi)/(2+3\xi)$. Moreover, the optimal state is unique in the sense that there is no other optimal finite entangled superposition of twin-Fock states. The energy-constrained channel fidelity is given by

$$F_E(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^\xi) = \frac{1}{(1+\xi)^2} \left[1 - 4 \left(\frac{\xi}{1+\xi} \right) E + 6 \left(\frac{\xi}{1+\xi} \right)^2 E^2 \right]. \quad (\text{B70})$$

Proof. The equality (B70) follows directly by substituting the value of $F_{E,M}(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^\xi)$ in (B1), and then by taking limit the limit $M \rightarrow \infty$. It thus follows by Lemma 7 that the state in (52) is an optimal state.

Any optimal finite entangled superposition of twin-Fock states for $F_E(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^\xi)$ is also optimal for $F_{E,M}(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^\xi)$ for large M . Moreover, for all M , $F_{E,M}(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^\xi)$ has the same unique optimal state given by (52), which follows from Lemma 7. This implies the uniqueness of the optimal state for $F_E(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^\xi \otimes \mathcal{T}_{B \rightarrow A'}^\xi)$, in the given sense. \blacksquare

2. Optimal input state for energy-constrained channel fidelity for $\xi' \geq 1$ and arbitrary ξ

Proposition 9. The minimum of $f_{M,\eta,\eta'}$, subject to $p_{m,n} \geq 0$ for all $m, n \in \{0, \dots, M\}$, and the equality constraints

$$\sum_{m,n=0}^M p_{m,n} = 1, \quad \sum_{m,n=0}^M (m+n)p_{m,n} = 2E, \quad (\text{B71})$$

is attained at the unique $p = (p_{m,n})_{m,n=0}^M$ in $\mathbb{R}^{(M+1)^2}$, given by

$$p_{m,n} = \begin{cases} 1 - 2E & \text{if } m = n = 0, \\ 2E - p_E & \text{if } m = 0, n = 1, \\ 0 & \text{if } m + n \geq 2, \end{cases} \quad (\text{B72})$$

whenever $\xi' \geq 1$ and $2E \leq (\xi'^2 - 1)/[\xi'(3\xi' - 1)]$. In (B72), we have

$$p_E = \begin{cases} 0 & \text{if } 2E < \frac{(\xi - \xi')(1 + \xi')}{\xi'^2(1 + \xi)}, \\ 2E & \text{if } 2E < \frac{(\xi' - \xi)(1 + \xi)(1 + \xi')}{\xi^2(1 + \xi')^2 + (\xi - \xi')^2}, \\ \frac{(\xi - \xi')(1 + \xi)(1 + \xi') + 2E\xi'^2(1 + \xi)^2}{2((\xi - \xi')^2 + \xi\xi'(1 + \xi)(1 + \xi'))} & \text{otherwise.} \end{cases} \quad (\text{B73})$$

Proof. From (B24), and using the relations $\eta = 1/(1 + \xi)$ and $\eta' = 1/(1 + \xi')$, we get

$$f_{M,\xi,\xi'}(p) \geq \frac{1}{(1 + \xi)(1 + \xi')} \sum_{\substack{m,n,m',n'=0 \\ m+n \leq 1, \\ m'+n' \leq 1}}^M p_{m,n} p_{m',n'} T_{\xi}^{mm'} T_{\xi'}^{nn'}. \quad (\text{B74})$$

Simplify the right-hand side of (B74) using $\eta = 1/(1 + \xi)$ and $\eta' = 1/(1 + \xi')$ to get

$$f_{M,\xi,\xi'}(p) \geq \eta\eta' [p_{0,0}^2 + 2p_{0,0}(\eta'p_{0,1} + \eta p_{1,0}) + 2\eta\eta' p_{0,1}p_{1,0} + (\eta^2 + (1 - \eta')^2)p_{0,1}^2 + (\eta^2 + (1 - \eta)^2)p_{1,0}^2]. \quad (\text{B75})$$

Let us denote the right-hand side of (B75) by $g_{M,\eta,\eta'} : \mathbb{R}^{(M+1)^2} \rightarrow \mathbb{R}$,

$$g_{M,\eta,\eta'}(p) := \eta\eta' [p_{0,0}^2 + 2p_{0,0}(\eta'p_{0,1} + \eta p_{1,0}) + 2\eta\eta' p_{0,1}p_{1,0} + (\eta^2 + (1 - \eta')^2)p_{0,1}^2 + (\eta^2 + (1 - \eta)^2)p_{1,0}^2]. \quad (\text{B76})$$

We thus have

$$f_{M,\eta,\eta'}(p) \geq g_{M,\eta,\eta'}(p). \quad (\text{B77})$$

We will show that $g_{M,\eta,\eta'}$ has a unique minimizer, and then show that it is also the minimizer of $f_{M,\xi,\xi'}$. Differentiate $g_{M,\eta,\eta'}$ with respect to $p_{r,s}$ for $r + s \geq 2$. We get

$$\frac{1}{2\eta\eta'} \frac{\partial g_{M,\eta,\eta'}(p)}{\partial p_{r,s}} = ((r+s)(1 - \eta') - 1)p_{0,0} - ((r+s)(1 - 2\eta')(1 - \eta') + \eta')p_{0,1} + \eta((r+s)(1 - \eta') - 1)p_{1,0}. \quad (\text{B78})$$

Substitute the values of $p_{0,0}$ and $p_{0,1}$ from (B29) and (B30) in (B78), and simplify to get

$$\begin{aligned} \frac{1}{2\eta\eta'} \frac{\partial g_{M,\eta,\eta'}(p)}{\partial p_{r,s}} &= (r+s)(1 - \eta') - 1 - 2E((1 - \eta')(2(r+s)(1 - \eta') - 1)) \\ &\quad + [\eta((r+s)(1 - \eta') - 1) + (r+s)(1 - 2\eta')(1 - \eta')]p_{1,0} \\ &\quad + \sum_{\substack{m,n=0 \\ m+n \geq 2}}^M [(1 - \eta')(2(r+s)(1 - \eta') - 1)(m+n) + (r+s)(1 - \eta')(2(1 - \eta') - 1) + \eta']p_{m,n}. \end{aligned} \quad (\text{B79})$$

The coefficients of $p_{m,n}$ are positive which follows from the assumption $\eta' = 1/(1 + \xi') \leq 1/2$. The coefficient of $p_{1,0}$ is nonnegative. Also, the remaining term of (B79) is nonnegative if

$$2E \leq \frac{(r+s)(1 - \eta') - 1}{(1 - \eta')(2(r+s)(1 - \eta') - 1)}. \quad (\text{B80})$$

The right-hand side of Eq. (B80) is an increasing function of $r + s \geq 2$, which follows because its derivative with respect to $r + s$ is given by $[1 - 2(r+s)(1 - \eta')]^{-2}$ and thus is nonnegative for all $r + s$. Therefore, a sufficient condition on E for

$\partial g_{M,\eta,\eta'}(p)/\partial p_{r,s} > 0$ is obtained from (B80) by substituting $r+s=2$ in the right-hand side. A sufficient condition is

$$2E \leq \frac{2(1-\eta')-1}{(1-\eta')(4(1-\eta')-1)} \quad (B81)$$

$$= \frac{1-2\eta'}{(1-\eta')(3-4\eta')} \quad (B82)$$

$$= \frac{\xi'^2-1}{\xi'(3\xi'-1)}, \quad (B83)$$

which holds by hypothesis. So, $g_{M,\eta,\eta'}$ is a strictly increasing function of $p_{r,s}$ for all r,s with $r+s \geq 2$. Therefore, a necessary condition for any of its minimizer $p \in \mathbb{R}^{(M+1)^2}$ is $p_{r,s} = 0$ for all r,s with $r+s \geq 2$. Also, we know by (B24) that $f_{M,\xi,\xi'} = g_{M,\eta,\eta'}$ at such points. Therefore, it follows by (B77) that the minimum value of $f_{M,\eta,\eta'}$ is obtained at a point p for which $p_{r,s} = 0$ for all r,s with $r+s \geq 2$. Let q be such a point in the feasible region. From (B25) we have $q_{0,0} = 1-2E$, $q_{0,1} = 2E-q_{1,0}$. Note that $q_{1,0} \in [0, 2E]$. This gives

$$f_{M,\xi,\xi'}(q) = g_{M,\eta,\eta'}(q) \quad (B84)$$

$$= (1-2E)^2 + 2(1-2E)(\eta'(2E-q_{1,0}) + \eta q_{1,0}) + 2\eta\eta'(2E-q_{1,0})q_{1,0} + (\eta^2 + (1-\eta)^2)(2E-q_{1,0})^2 + (\eta^2 + (1-\eta)^2)q_{1,0}^2 \quad (B85)$$

$$= (1-2E)(1-2E+4\eta'E) + 4E^2(\eta'^2 + (1-\eta')^2) + [2(1-2E)(\eta-\eta') + 4\eta\eta'E - 4E(\eta'^2 + (1-\eta')^2)]q_{1,0} + 2[(\eta-\eta')^2 + (1-\eta)(1-\eta')]q_{1,0}^2. \quad (B86)$$

It is a quadratic polynomial in $q_{1,0}$, and the coefficient of $q_{1,0}^2$ is positive. The global minimum of a quadratic polynomial $c+bx+ax^2$ with $a > 0$ is attained by $x = -b/(2a)$. Also, the polynomial is a decreasing function of $x < -b/(2a)$ and an increasing function of $x > -b/(2a)$. So, the minimum of $c+bx+ax^2$ over $[0, 2E]$ occurs at

$$x = \begin{cases} 0 & \text{if } -b/2a < 0, \\ 2E & \text{if } -b/2a > 2E, \\ -\frac{b}{2a} & \text{otherwise.} \end{cases} \quad (B87)$$

By comparing the quadratic polynomial (B86) with $c+bx+ax^2$, and from (B87), the minimizer of $f_{M,\xi,\xi'}$ is given by (B72). \blacksquare

Lemma 10. The M -truncated energy-constrained channel fidelity (50) has the unique solution given by (53), whenever $\xi' \geq 1$ and $2E \leq \min\{(\xi'^2-1)/[\xi'(3\xi'-1)], (1+\xi)/(2\xi)\}$. Also, the energy-constrained channel fidelity is

$$F_{E,M}(\mathcal{S}_{AB \rightarrow B'A'}, \mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}) = \begin{cases} 1 - 4\left(\frac{\xi'}{1+\xi'}\right)E + 8\left(\frac{\xi'}{1+\xi'}\right)^2 E^2 & \text{if } 2E < \frac{(\xi-\xi')(1+\xi')}{\xi'^2(1+\xi)}, \\ 1 - 4\left(\frac{\xi}{1+\xi}\right)E + 8\left(\frac{\xi}{1+\xi}\right)^2 E^2 & \text{if } 2E < \frac{(\xi'-\xi)(1+\xi)(1+\xi')}{\xi^2(1+\xi')^2 + (\xi-\xi')^2}, \\ \frac{4ac-b^2}{4a} & \text{otherwise,} \end{cases} \quad (B88)$$

where

$$a = \frac{2(\xi'-\xi)^2}{(1+\xi)^2(1+\xi')^2} + \frac{2\xi\xi'}{(1+\xi)(1+\xi')}, \quad (B89)$$

$$b = \frac{2(\xi'-\xi)}{(1+\xi')(1+\xi)} + \frac{4\xi'(\xi(1-\xi')-2\xi')E}{(1+\xi)(1+\xi')^2}, \quad (B90)$$

$$c = 1 - \frac{4\xi'E}{(1+\xi')} + \frac{8(\xi'E)^2}{(1+\xi')^2}. \quad (B91)$$

In particular, the the right-hand side of (B88) is independent of the truncation parameter M .

Proof. Let ϕ_{RAB} be any pure state given by $|\phi\rangle_{RAB} = \sum_{m,n=0}^M \sqrt{q_{m,n}} |m,n\rangle_R |m,n\rangle_{AB}$, where $q = (q_{m,n})_{m,n=0}^M$ is a probability vector such that $\sum_{m,n=0}^M (m+n)q_{m,n} = 2E_0 \leq 2E$, and let ψ_{RAB} be the pure state given in (53). We know from (B17) that

$$F(\mathcal{S}_{AB \rightarrow A'B'}(\phi_{RAB}), \mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\phi_{RAB})) = f_{M,\xi,\xi'}(q). \quad (B92)$$

By Proposition 9, we thus get

$$F(\mathcal{S}_{AB \rightarrow A'B'}(\phi_{RAB}), \mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\phi_{RAB})) \geq \eta\eta' (h_2(E_0) + h_1(E_0)p_{E_0} + h_0(E_0)p_{E_0}^2), \quad (B93)$$

where $\eta = 1/(1 + \xi)$, $\eta' = 1/(1 + \xi')$, p_{E_0} is given by (B73), and h_0, h_1, h_2 are polynomial functions defined by

$$h_0(x) = 2((\eta - \eta')^2 + (1 - \eta)(1 - \eta')), \quad (\text{B94})$$

$$h_1(x) = 2(\eta - \eta') + 4(1 - \eta')(2\eta' - 1 - \eta)x, \quad (\text{B95})$$

$$h_2(x) = 1 - 4(1 - \eta')x + 8(1 - \eta')^2x^2. \quad (\text{B96})$$

The polynomials $h_0(x), h_1(x), h_2(x)$ are decreasing functions of x for $x < 1/[4(1 - \eta')]$. From the given hypothesis, and the relation $\xi' = (1 - \eta')/\eta'$, we have

$$E \leq \frac{\xi'^2 - 1}{2\xi'(3\xi' - 1)} \quad (\text{B97})$$

$$= \frac{1 - 2\eta'}{2(1 - \eta')(3 - 4\eta')} \quad (\text{B98})$$

$$= \frac{1}{2(1 - \eta')}\left(1 - \frac{2(1 - \eta')}{3 - 4\eta'}\right) \quad (\text{B99})$$

$$< \frac{1}{2(1 - \eta')}\left(1 - \frac{2(1 - \eta')}{4 - 4\eta'}\right) \quad (\text{B100})$$

$$= \frac{1}{4(1 - \eta')}. \quad (\text{B101})$$

This implies $h_i(E_0) \geq h_i(E)$ for $i \in \{0, 1, 2\}$.

Our proof is divided into three cases, based on the conditions in (B88). We use Proposition 9 in each case.

Case 1. Suppose $2E < (\xi - \xi')(1 + \xi')/[\xi'^2(1 + \xi)]$.

The minimum value of the right-hand side term of (B93) occurs at $p_{1,0} = 0$. We thus get

$$F(\mathcal{S}_{AB \rightarrow A'B'}(\phi_{RAB}), \mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\phi_{RAB})) \geq \eta\eta' h_2(E_0) \quad (\text{B102})$$

$$\geq \eta\eta' h_2(E) \quad (\text{B103})$$

$$= F(\mathcal{S}_{AB \rightarrow A'B'}(\psi_{RAB}), \mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\psi_{RAB})). \quad (\text{B104})$$

Case 2. Suppose $2E < (\xi' - \xi)(1 + \xi)(1 + \xi')/[\xi^2(1 + \xi')^2 + (\xi - \xi')^2]$.

The minimum value of the right-hand side term of (B93) occurs at $p_{1,0} = 2E$. By substituting $p_{1,0} = 2E_0$ in (B93) and then by simplifying, we get

$$F(\mathcal{S}_{AB \rightarrow A'B'}(\phi_{RAB}), \mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\phi_{RAB})) \geq \eta\eta'(1 - 4E_0(1 - \eta) + 8(1 - \eta)^2E_0). \quad (\text{B105})$$

The polynomial $1 - 4(1 - \eta)x + 8(1 - \eta)^2x^2$ is decreasing for $x \leq 1/[4(1 - \eta)]$. Since we have $E \leq (1 + \xi)/(4\xi) = 1/[4(1 - \eta)]$, by (B105) we get

$$F(\mathcal{S}_{AB \rightarrow A'B'}(\phi_{RAB}), \mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\phi_{RAB})) \geq \eta\eta'(1 - 4(1 - \eta)E + 8(1 - \eta)^2E^2) \quad (\text{B106})$$

$$= F(\mathcal{S}_{AB \rightarrow A'B'}(\psi_{RAB}), \mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\psi_{RAB})). \quad (\text{B107})$$

Case 3. In this case we have

$$2E \geq \max \left\{ \frac{(\xi - \xi')(1 + \xi')}{\xi'^2(1 + \xi)}, \frac{(\xi' - \xi)(1 + \xi)(1 + \xi')}{\xi^2(1 + \xi')^2 + (\xi - \xi')^2} \right\}. \quad (\text{B108})$$

Now, for all $x \geq 0$, we have

$$h_2(E_0) + h_1(E_0)x + h_0(E_0)x^2 \geq h_2(E) + h_1(E)x + h_0(E)x^2 \quad (\text{B109})$$

$$\geq h_2(E) + h_1(E)p_E + h_0(E)p_E^2. \quad (\text{B110})$$

In (B109), we used the fact that $h_i(E_0) \geq h_i(E)$ for all $i \in \{0, 1, 2\}$; (B110) follows because the global minimum of the polynomial $h_2(E) + h_1(E)x + h_0(E)x^2$ is attained at $x = p_E$. We thus have

$$h_2(E_0) + h_1(E_0)p_E + h_0(E_0)p_E^2 \geq h_2(E) + h_1(E)p_E + h_0(E)p_E^2. \quad (\text{B111})$$

From (B93) and (B111), we thus get

$$F(\mathcal{S}_{AB \rightarrow A'B'}(\phi_{RAB}), \mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\phi_{RAB})) \geq F(\mathcal{S}_{AB \rightarrow A'B'}(\psi_{RAB}), \mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\psi_{RAB})). \quad (\text{B112})$$

In all the cases, we proved that

$$F(\mathcal{S}_{AB \rightarrow A'B'}(\phi_{RAB}), \mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\phi_{RAB})) \geq F(\mathcal{S}_{AB \rightarrow A'B'}(\psi_{RAB}), \mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'}(\psi_{RAB})). \quad (\text{B113})$$

This means that the state in (53) is optimal. Also, the value of the energy-constrained channel fidelity (B88) can be obtained by direct substitution of the minimizer p_E from (B73). \blacksquare

Theorem 11. An optimal state for the energy-constrained channel fidelity (49) is given by (53), whenever $\xi' \geq 1$ and $2E \leq \min\{(\xi'^2 - 1)/[\xi'(3\xi' - 1)], (1 + \xi)/(2\xi)\}$. Moreover, the optimal state is unique in the sense that there is no other optimal finite entangled superposition of twin-Fock states. The value of the energy-constrained channel fidelity $F_E(\mathcal{S}_{AB \rightarrow A'B'}, \mathcal{T}_{A \rightarrow B'}^{\xi} \otimes \mathcal{T}_{B \rightarrow A'}^{\xi'})$ is the same as the value of any of its M -truncated counterpart given in (B88). \blacksquare

Proof. The proof follows similar arguments as given in the proof of Theorem 8, and using Lemma 10. \blacksquare

[1] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels, *Phys. Rev. Lett.* **70**, 1895 (1993).

[2] A. Furusawa and N. Takei, Quantum teleportation for continuous variables and related quantum information processing, *Phys. Rep.* **443**, 97 (2007).

[3] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[4] D. Gottesman and I. L. Chuang, Demonstrating the viability of universal quantum computation using teleportation and single-qubit operations, *Nature (London)* **402**, 390 (1999).

[5] E. Knill, R. Laflamme, and G. J. Milburn, A scheme for efficient quantum computation with linear optics, *Nature (London)* **409**, 46 (2001).

[6] S. L. N. Hermans, M. Pompili, H. K. C. Beukers, S. Baier, J. Borregaard, and R. Hanson, Qubit teleportation between non-neighbouring nodes in a quantum network, *Nature (London)* **605**, 663 (2022).

[7] Y. Feng-Li and D. He-Wei, Probabilistic teleportation of an unknown two-particle state with a four-particle pure entangled state and positive operator valued measure, *Chin. Phys. Lett.* **23**, 17 (2006).

[8] A. K. Pati and P. Agrawal, Probabilistic teleportation of a qudit, *Phys. Lett. A* **371**, 185 (2007).

[9] F. Yan and T. Yan, Probabilistic teleportation via a non-maximally entangled GHZ state, *Chin. Sci. Bull.* **55**, 902 (2010).

[10] F.-G. Deng, C.-Y. Li, Y.-S. Li, H.-Y. Zhou, and Y. Wang, Symmetric multiparty-controlled teleportation of an arbitrary two-particle entanglement, *Phys. Rev. A* **72**, 022338 (2005).

[11] L. Xi-Han, D. Fu-Guo, and Z. Hong-Yu, Controlled teleportation of an arbitrary multi-qudit state in a general form with d -dimensional Greenberger–Horne–Zeilinger states, *Chin. Phys. Lett.* **24**, 1151 (2007).

[12] P. Zhou, X.-H. Li, F.-G. Deng, and H.-Y. Zhou, Multiparty-controlled teleportation of an arbitrary m -qudit state with a pure entangled quantum channel, *J. Phys. A: Math. Theor.* **40**, 13121 (2007).

[13] Z.-X. Man, Y.-J. Xia, and N. B. An, Genuine multiqubit entanglement and controlled teleportation, *Phys. Rev. A* **75**, 052306 (2007).

[14] L. Vaidman, Teleportation of quantum states, *Phys. Rev. A* **49**, 1473 (1994).

[15] S. F. Huelga, J. A. Vaccaro, A. Chefles, and M. B. Plenio, Quantum remote control: Teleportation of unitary operations, *Phys. Rev. A* **63**, 042303 (2001).

[16] X.-W. Zha, Z.-C. Zou, J.-X. Qi, and H.-Y. Song, Bidirectional quantum controlled teleportation via five-qubit cluster state, *Int. J. Theor. Phys.* **52**, 1740 (2013).

[17] Y.-H. Li and L.-P. Nie, Bidirectional controlled teleportation by using a five-qubit composite GHZ-Bell state, *Int. J. Theor. Phys.* **52**, 1630 (2013).

[18] A. Yan, Bidirectional controlled teleportation via six-qubit cluster state, *Int. J. Theor. Phys.* **52**, 3870 (2013).

[19] H.-Z. Fu, X.-L. Tian, and Y. Hu, A general method of selecting quantum channel for bidirectional quantum teleportation, *Int. J. Theor. Phys.* **53**, 1840 (2014).

[20] S. Hassanpour and M. Houshmand, Bidirectional teleportation of a pure EPR state by using GHZ states, *Quant. Info. Proc.* **15**, 905 (2016).

[21] G. Yang, B.-W. Lian, M. Nie, and J. Jin, Bidirectional multi-qubit quantum teleportation in noisy channel aided with weak measurement, *Chin. Phys. B* **26**, 040305 (2017).

[22] J. Chen, D. Li, M. Liu, and Y. Yang, Bidirectional quantum teleportation by using a four-qubit GHZ state and two Bell states, *IEEE Access* **8**, 28925 (2020).

[23] T. Liu, The applications and challenges of quantum teleportation, *J. Phys.: Conf. Ser.* **1634**, 012089 (2020).

[24] D. Bouwmeester, J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Experimental quantum teleportation, *Nature (London)* **390**, 575 (1997).

[25] M. Riebe, H. Häffner, C. F. Roos, W. Hänsel, J. Benhelm, G. P. T. Lancaster, T. W. Körber, C. Becher, F. Schmidt-Kaler, D. F. V. James *et al.*, Deterministic quantum teleportation with atoms, *Nature (London)* **429**, 734 (2004).

[26] R. Ursin, T. Jennewein, M. Aspelmeyer, R. Kaltenbaek, M. Lindenthal, P. Walther, and A. Zeilinger, Quantum teleportation across the Danube, *Nature (London)* **430**, 849 (2004).

[27] X.-L. Wang, X.-D. Cai, Z.-E. Su, M.-C. Chen, D. Wu, L. Li, N.-L. Liu, C.-Y. Lu, and J.-W. Pan, Quantum teleportation of multiple degrees of freedom of a single photon, *Nature (London)* **518**, 516 (2015).

[28] J.-G. Ren, P. Xu, H.-L. Yong, L. Zhang, S.-K. Liao, J. Yin, W.-Y. Liu, W.-Q. Cai, M. Yang, L. Li *et al.*, Ground-to-satellite quantum teleportation, *Nature (London)* **549**, 70 (2017).

[29] T. Feng, Q. Xu, L. Zhou, M. Luo, W. Zhang, and X. Zhou, Quantum information transfer between a two-level and a four-level quantum system, *Photon. Res.* **10**, 2854 (2022).

[30] X.-M. Hu, C. Zhang, B.-H. Liu, Y. Cai, X.-J. Ye, Y. Guo, W.-B. Xing, C.-X. Huang, Y.-F. Huang, C.-F. Li, and G. C. Guo, Experimental High-Dimensional Quantum Teleportation, *Phys. Rev. Lett.* **125**, 230501 (2020).

[31] Y.-H. Luo, H.-S. Zhong, M. Erhard, X.-L. Wang, L.-C. Peng, M. Krenn, X. Jiang, L. Li, N.-L. Liu, C.-Y. Lu, A. Zeilinger, and J. W. Pan, Quantum Teleportation in High Dimensions, *Phys. Rev. Lett.* **123**, 070505 (2019).

[32] X.-S. Ma, S. Kropatschek, W. Naylor, T. Scheidl, J. Kofler, T. Herbst, A. Zeilinger, and R. Ursin, Experimental quantum teleportation over a high-loss free-space channel, *Opt. Express* **20**, 23126 (2012).

[33] X.-M. Jin, J.-G. Ren, B. Yang, Z.-H. Yi, F. Zhou, X.-F. Xu, S.-K. Wang, D. Yang, Y.-F. Hu, S. Jiang *et al.*, Experimental free-space quantum teleportation, *Nat. Photon.* **4**, 376 (2010).

[34] S. L. Braunstein and H. J. Kimble, Teleportation of Continuous Quantum Variables, *Phys. Rev. Lett.* **80**, 869 (1998).

[35] H. Krauter, D. Salart, C. A. Muschik, J. M. Petersen, H. Shen, T. Fernholz, and E. S. Polzik, Deterministic quantum teleportation between distant atomic objects, *Nat. Phys.* **9**, 400 (2013).

[36] J. F. Sherson, H. Krauter, R. K. Olsson, B. Julsgaard, K. Hammerer, I. Cirac, and E. S. Polzik, Quantum teleportation between light and matter, *Nature (London)* **443**, 557 (2006).

[37] M. Yukawa, H. Benichi, and A. Furusawa, High-fidelity continuous-variable quantum teleportation toward multistep quantum operations, *Phys. Rev. A* **77**, 022314 (2008).

[38] M. Christandl, F. Leditzky, C. Majenz, G. Smith, F. Speelman, and M. Walter, Asymptotic performance of port-based teleportation, *Commun. Math. Phys.* **381**, 379 (2021).

[39] J. Bang, J. Ryu, and D. Kaszlikowski, Fidelity deviation in quantum teleportation, *J. Phys. A: Math. Theor.* **51**, 135302 (2018).

[40] S. Roy and A. Ghosal, Rating the performance of noisy teleportation using fluctuations in fidelity, *Phys. Rev. A* **102**, 012428 (2020).

[41] M. Studziński, M. Mozrzymas, P. Kopszak, and M. Horodecki, Efficient multi port-based teleportation schemes, *IEEE Trans. Inf. Theory* **68**, 7892 (2022).

[42] R. Wagner, Jr. and J. P. Clemens, Performance of a quantum teleportation protocol based on collective spontaneous emission, *J. Opt. Soc. Am. B* **26**, 541 (2009).

[43] R. Wagner, Jr. and J. P. Clemens, Performance of a quantum teleportation protocol based on temporally resolved photodetection of collective spontaneous emission, *Phys. Rev. A* **79**, 042322 (2009).

[44] T. Holdsworth, V. Singh, and M. M. Wilde, Quantifying the performance of approximate teleportation and quantum error correction via symmetric two-PPT-extendibility, *Phys. Rev. A* **107**, 012428 (2023).

[45] A. U. Siddiqui and M. M. Wilde, Quantifying the performance of bidirectional quantum teleportation, [arXiv:2010.07905](https://arxiv.org/abs/2010.07905).

[46] K. Hammerer, M. M. Wolf, E. S. Polzik, and J. I. Cirac, Quantum Benchmark for Storage and Transmission of Coherent States, *Phys. Rev. Lett.* **94**, 150503 (2005).

[47] T. J. Johnson, S. D. Bartlett, and B. C. Sanders, Continuous-variable quantum teleportation of entanglement, *Phys. Rev. A* **66**, 042326 (2002).

[48] F. Grosshans and P. Grangier, Quantum cloning and teleportation criteria for continuous quantum variables, *Phys. Rev. A* **64**, 010301(R) (2001).

[49] S. L. Braunstein, C. A. Fuchs, and H. J. Kimble, Criteria for continuous-variable quantum teleportation, *J. Mod. Opt.* **47**, 267 (2000).

[50] G. Chiribella and G. Adesso, Quantum Benchmarks for Pure Single-Mode Gaussian States, *Phys. Rev. Lett.* **112**, 010501 (2014).

[51] I. Kogias, S. Ragy, and G. Adesso, Continuous-variable versus hybrid schemes for quantum teleportation of Gaussian states, *Phys. Rev. A* **89**, 052324 (2014).

[52] G. Adesso and G. Chiribella, Quantum Benchmark for Teleportation and Storage of Squeezed States, *Phys. Rev. Lett.* **100**, 170503 (2008).

[53] K. P. Seshadreesan, J. P. Dowling, and G. S. Agarwal, Non-Gaussian entangled states and quantum teleportation of Schrödinger-cat states, *Phys. Scr.* **90**, 074029 (2015).

[54] K. Sharma, B. C. Sanders, and M. M. Wilde, Optimal tests for continuous-variable quantum teleportation and photodetectors, *Phys. Rev. Res.* **4**, 023066 (2022).

[55] G. G. Amosov, A. S. Holevo, and R. F. Werner, On the additivity conjecture in quantum information theory, *Prob. Peredachi Inf.* **36**, 25 (2000) [Probl. Inf. Transm. 36, 305 (2000)].

[56] A. S. Holevo, The additivity problem in quantum information theory, in *Proceedings of the International Congress of Mathematicians* (European Mathematical Society, 2006), pp. 999–1018.

[57] A. S. Holevo, Multiplicativity of p-norms of completely positive maps and the additivity problem in quantum information theory, *Russ. Math. Surv.* **61**, 301 (2006).

[58] A. S. Holevo, Gaussian optimizers and the additivity problem in quantum information theory, *Russ. Math. Surv.* **70**, 331 (2015).

[59] T. Cubitt, A. W. Harrow, D. Leung, A. Montanaro, and A. Winter, Counterexamples to additivity of minimum output p-Rényi entropy for p close to 0, *Commun. Math. Phys.* **284**, 281 (2008).

[60] M. B. Hastings, Superadditivity of communication capacity using entangled inputs, *Nat. Phys.* **5**, 255 (2009).

[61] G. Aubrun, S. Szarek, and E. Werner, Hastings’s additivity counterexample via Dvoretzky’s theorem, *Commun. Math. Phys.* **305**, 85 (2011).

[62] M. Fukuda, Revisiting additivity violation of quantum channels, *Commun. Math. Phys.* **332**, 713 (2014).

[63] M. M. Wilde, *Quantum Information Theory* (Cambridge University Press, Cambridge, 2019).

[64] O. Krueger and R. F. Werner, Some open problems in quantum information theory, [arXiv:quant-ph/0504166](https://arxiv.org/abs/quant-ph/0504166).

[65] M. B. Ruskai, Open problems in quantum information theory, [arXiv:0708.1902](https://arxiv.org/abs/0708.1902).

[66] A. Uhlmann, The “transition probability” in the state space of a^* -algebra, *Rep. Math. Phys.* **9**, 273 (1976).

[67] A. E. Rastegin, Relative error of state-dependent cloning, *Phys. Rev. A* **66**, 042304 (2002).

[68] A. E. Rastegin, A lower bound on the relative error of mixed-state cloning and related operations, *J. Opt. B: Quantum Semiclass. Opt.* **5**, S647 (2003).

[69] A. Gilchrist, N. K. Langford, and M. A. Nielsen, Distance measures to compare real and ideal quantum processes, *Phys. Rev. A* **71**, 062310 (2005).

[70] A. E. Rastegin, Sine distance for quantum states, [arXiv:quant-ph/0602112](https://arxiv.org/abs/quant-ph/0602112).

[71] C. A. Fuchs and J. Van De Graaf, Cryptographic distinguishability measures for quantum-mechanical states, *IEEE Trans. Inf. Theory* **45**, 1216 (1999).

[72] M. E. Shirokov, Uniform continuity bounds for information characteristics of quantum channels depending on input dimension and on input energy, *J. Phys. A: Math. Theor.* **52**, 014001 (2019).

[73] K. Sharma, M. M. Wilde, S. Adhikari, and M. Takeoka, Bounding the energy-constrained quantum and private capacities of phase-insensitive bosonic Gaussian channels, *New J. Phys.* **20**, 063025 (2018).

[74] S. Khatri and M. M. Wilde, *Principles of Quantum Communication Theory: A Modern Approach*, 2020.

[75] M. M. Wilde, Strong and uniform convergence in the teleportation simulation of bosonic Gaussian channels, *Phys. Rev. A* **97**, 062305 (2018).

[76] S. L. Braunstein, Error Correction for Continuous Quantum Variables, *Phys. Rev. Lett.* **80**, 4084 (1998).

[77] R. García-Patrón, C. Navarrete-Benlloch, S. Lloyd, J. H. Shapiro, and N. J. Cerf, Majorization Theory Approach to the Gaussian Channel Minimum Entropy Conjecture, *Phys. Rev. Lett.* **108**, 110505 (2012).

[78] F. Caruso, V. Giovannetti, and A. S. Holevo, One-mode bosonic Gaussian channels: A full weak-degradability classification, *New J. Phys.* **8**, 310 (2006).

[79] K. Sharma and M. M. Wilde, Characterizing the performance of continuous-variable Gaussian quantum gates, *Phys. Rev. Res.* **2**, 013126 (2020).

[80] A. Serafini, *Quantum Continuous Variables: A Primer of Theoretical Methods* (CRC Press, Boca Raton, 2017).

[81] J. S. Ivan, K. K. Sabapathy, and R. Simon, Operator-sum representation for bosonic Gaussian channels, *Phys. Rev. A* **84**, 042311 (2011).

[82] M. E. Shirokov, Energy-constrained diamond norms and their use in quantum information theory, *Probl. Inf. Transm.* **54**, 20 (2018).

[83] A. Winter, Energy-constrained diamond norm with applications to the uniform continuity of continuous variable channel capacities, December 2017.