

Direct measurement of the density matrix of a two-photon polarization qubit

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ABSTRACT

There is interest in using photon entanglement in biomedical applications. In one application, polarization-entangled photons pass through brain tissue. The effect of the brain tissue on the photon entanglement is measured via the decoherence that is imparted on the entangled state. Our current method to obtain a measure of the decoherence involves quantum state tomography, where a minimum of 16 measurements are used in conjunction with tomographic optimization to obtain the density matrix representing the state of the photons. In this work we report on a method to avoid tomographic optimization on behalf of a direct measurement of the elements of the density matrix. We make preliminary comparisons between the two methods.

Keywords: Photon entanglement, Quantum State Tomography, Polarization, Density Matrix

1. INTRODUCTION

The entanglement of two photons in the polarization degree of freedom is a popular two-qubit system that can be easily prepared in the laboratory.¹ We have used this system recently to characterize tissue sections.^{3,4} The characterization of the state of the photons is usually done by quantum state tomography, where a minimum of 16 measurements is used to determine the density matrix of the state of the photons.² The method to obtain the density matrix consists of two parts. First is a linear-algebraic operation to obtain a first version of the density matrix. The problem is that often quantum fluctuations in the various measurements can yield a density matrix that does not conform with a density matrix representing a physical system (i.e., being normalized, Hermitian and positive semidefinite). Thus, the second part involves an optimization of the matrix so that it has the properties of a physical matrix. Once the density matrix is obtained, two metrics are used to characterize the state from the density matrix: tangle and linear entropy. In our work we have found that these measures can fluctuate non-linearly with small variations in the measurements, requiring us to take a large data set to reduce signal to noise.

In this work we seek an alternative way to measure the density matrix, with the aim to reduce the fluctuations in the quantum measures. We found that the elements of the density matrix can be obtained directly from a larger set of measurements (34). In this proceeding we present the method to obtain each of the matrix elements directly from measurements, and show our preliminary results of comparisons between the direct method and state tomography.

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2. TWO-QUBIT DENSITY MATRIX

2.1 General Method

Photons pairs entangled in their polarization degree of freedom constitute a 2-qubit system. In the horizontal-vertical reference frame, the basis vectors are $|HH\rangle$, $|HV\rangle$, $|VH\rangle$ and $|VV\rangle$, where H and V represent the states of horizontal and vertical polarization, respectively, with first and second labels representing the state of the first and second photons. A general state of the two photons is given by the density matrix:

$$\hat{\rho}_\psi = |\psi\rangle\langle\psi| = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} & \rho_{1,3} & \rho_{1,4} \\ \rho_{2,1} & \rho_{2,2} & \rho_{2,3} & \rho_{2,4} \\ \rho_{3,1} & \rho_{3,2} & \rho_{3,3} & \rho_{3,4} \\ \rho_{4,1} & \rho_{4,2} & \rho_{4,3} & \rho_{4,4} \end{pmatrix}. \quad (1)$$

There are 3 types of matrix elements:

- Diagonal: $\rho_{i,i}$, where $i = 1 \dots 4$.
- Cross-diagonal: These are elements that form the other diagonal of the matrix: $\rho_{i,j}$, where $i = 1 \dots 4$ and $j = 5 - i$.
- Other off-diagonal elements $\rho_{i,j}$ with $i, j = 1 \dots 4$, which are not cross diagonal.

The general state of the two photons is measured by projections onto states $|W\rangle_1$ and $|U\rangle_2$ of the two photons (see Sec. 3). This measurement has the probability

$$P_{WU} = \text{Tr}[\hat{\rho}_\psi \hat{\rho}_{WU}], \quad (2)$$

where $\hat{\rho}_{WU} = |WU\rangle\langle WU|$.

States $|W\rangle$ and $|U\rangle$ can be any of the following projection states: horizontal (H) and vertical (V) linear polarization states:

$$|H\rangle, |V\rangle,$$

diagonal (D) and anti-diagonal (A) linear polarization states:

$$|D\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle), \quad |A\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle),$$

and right (R) and left (L) circular polarization states:

$$|L\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle), \quad |R\rangle = \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle).$$

The direct measure of a matrix element has been obtained by finding a matrix $\hat{M}$ such that

$$\rho_{i,j} = \text{Tr}[\hat{\rho}_\psi \hat{M}], \quad (3)$$

where $\hat{M}$ is a the matrix with only one non-zero element of value 1. Its elements are defined by

$$M_{u,w} = \begin{cases} 1 & \text{for } u = j, w = i \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

Using this argument we found that the matrices M can be expressed as

$$\hat{M} = \sum_k c_k |W_k U_k\rangle\langle W_k U_k|, \quad (5)$$

where c_k is a complex number of unit magnitude. This way, using Eqs. 3 and 5 we can express the matrix elements as

$$\rho_{i,j} = \sum_k c_k \text{Tr}[\hat{\rho}_\psi |W_k U_k\rangle \langle W_k U_k|]. \quad (6)$$

That is, the elements of the density matrix can be expressed as a linear combination of projective measurements.

The full density matrix can be obtained from 34 measurements. In the experiments, we measure projections to state $|WU\rangle$, obtaining N_{WU} counts for those projections. Below we will simplify the notation by calling $WU = N_{WU}$. All measurements have to be normalized. The normalization constant is obtained by

$$N = HH + HV + VH + VV \quad (7)$$

or alternatively

$$N = \frac{1}{3}(HH + HV + VH + VV + DD + DA + AD + AA + RR + RL + LR + LL), \quad (8)$$

which then requires a measurement of all 36 combinations of projections. Below we give the combination of projective measurements that give rise to the matrix elements.

2.2 Diagonal Elements

Each of the diagonal elements can be obtained from a single measurement:

$$\rho_{1,1} = \frac{1}{N} HH \quad (9)$$

$$\rho_{2,2} = \frac{1}{N} HV \quad (10)$$

$$\rho_{3,3} = \frac{1}{N} VH \quad (11)$$

$$\rho_{4,4} = \frac{1}{N} VV \quad (12)$$

2.3 Cross-Diagonal Elements

These elements are computed via the relations

$$\rho_{1,4} = -\frac{1}{2N} [(RR + LL - DD - AA) - i(RD + LA - DL - AR)] \quad (13)$$

$$\rho_{4,1} = -\frac{1}{2N} [(RR + LL - DD - AA) + i(RD + LA - DL - AR)] \quad (14)$$

$$\rho_{2,3} = \frac{1}{2N} [-(RL + LR - DD - AA) + i(DL - LD + AR - RA)] \quad (15)$$

$$\rho_{3,2} = \frac{1}{2N} [-(RL + LR - DD - AA) - i(DL - LD + AR - RA)] \quad (16)$$

2.4 Off-Diagonal Elements

The off diagonal elements are:

$$\rho_{1,2} = \frac{1}{2N} [(HD - HA) - i(HL - HR)] \quad (17)$$

$$\rho_{2,1} = \frac{1}{2N} [(HD - HA) + i(HL - HR)] \quad (18)$$

$$\rho_{1,3} = \frac{1}{2N} [(DH - AH) - i(RH - LH)] \quad (19)$$

$$\rho_{3,1} = \frac{1}{2N} [(DH - AH) + i(RH - LH)] \quad (20)$$

$$\rho_{2,4} = \frac{1}{2N} [(DV - AV) + i(RV - LV)] \quad (21)$$

$$\rho_{4,2} = \frac{1}{2N} [(DV - AV) - i(RV - LV)] \quad (22)$$

$$\rho_{3,4} = \frac{1}{2N} [(VD - VA) + i(VR - VL)] \quad (23)$$

$$\rho_{4,3} = \frac{1}{2N} [(VD - VA) - i(VR - VL)] \quad (24)$$

We note two properties of the density matrix: $\rho_{i,j} = \rho_{j,i}^*$, and $\sum_i \rho_{ii} = 1$, are incorporated automatically in this analysis.

3. COMPARISONS

The above relations have been confirmed analytically to work for pure states as well as mixed states. Let us consider two examples. First is the pure state:

$$|\psi\rangle_1 = \frac{1}{\sqrt{2}} (|HV\rangle - |VH\rangle), \quad (25)$$

with a corresponding density matrix:

$$\rho_1 = |\psi\rangle_1 \langle\psi|_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & -0.5 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (26)$$

For the other state let us consider the fully mixed state of $|RR\rangle$ and $|LL\rangle$ with probabilities 1/4 and 3/4, respectively. The density matrix is:

$$\rho_2 = \frac{1}{4} |RR\rangle \langle RR| + \frac{3}{4} |LL\rangle \langle LL| = \begin{pmatrix} 0.25 & -0.125i & -0.125i & -0.25 \\ 0.125 & 0.25 & 0.25 & -0.125i \\ 0.125 & 0.25 & 0.25 & -0.125i \\ -0.25 & 0.125 & 0.125 & 0.25 \end{pmatrix}. \quad (27)$$

Suppose that we wish to obtain elements $\rho_{2,3}$ and $\rho_{3,4}$, using the known density matrices for the entangled state of Eq. 26 and mixed state of Eq. 27. The calculated individual projections are shown in Table 1.

Thus applying Eq. 15 to states 1 and 2 gives -0.5 and 0.25 , respectively; and application of Eq. 23 to states 1 and 2 gives 0 and $-0.125i$, respectively. All are in agreement with the original matrix elements of the states. The same is true with the other matrix elements.

4. MEASUREMENTS

We have used projective measurements of two-photon states taken previously by our group to compare quantum tomography to the direct method. The data was taken with the apparatus of Fig. 1. A GaN diode laser provided pump photons with a wavelength of 405 nm. Various optical elements manipulated the polarization of the beam that was incident on a stack of beta-barium borate (BBO) crystals to prepare it in such a way to produce photon pairs in the polarization-entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|HH\rangle + |VV\rangle). \quad (28)$$

This is a Bell state that is maximally entangled. We can also prepare the other three Bell states with the suitable addition of waveplates, or any desired mixed state with the addition of a dephasing quartz plate. In previous works one of the photons would go through a tissue sample, but in this case, both photons went straight to a detection region where the state of each photon was projected by polarization filters. The photons that were

Table 1. Calculation of the probabilities involved in the projections of the states of Eqs. 26 and 27.

Projection	Value ₁	Value ₂
HH	0	0.25
HV	0.5	0.25
VH	0.5	0.25
VV	0	0.25
RL	0.5	0
LR	0.5	0
DD	0	0.25
AA	0	0.25
DL	0.25	0.375
LD	0.25	0.375
AR	0.25	0.125
RA	0.25	0.125
VD	0.25	0.25
VA	0.25	0.25
VR	0.25	0.125
VL	0.25	0.375

transmitted by the filters went through a bandpass filter that selected photons that were nearly degenerate with a wavelength of 810 nm. The photons were channeled through multimode fibers to single-photon avalanche diodes (SPAD), and the electronic signals were recorded in coincidence. Figure 2 shows an example of the plot of the real component of the density matrix obtained from measurements. The state was tuned “by hand,” and so it is not the best it can be, but that is acceptable for this test.

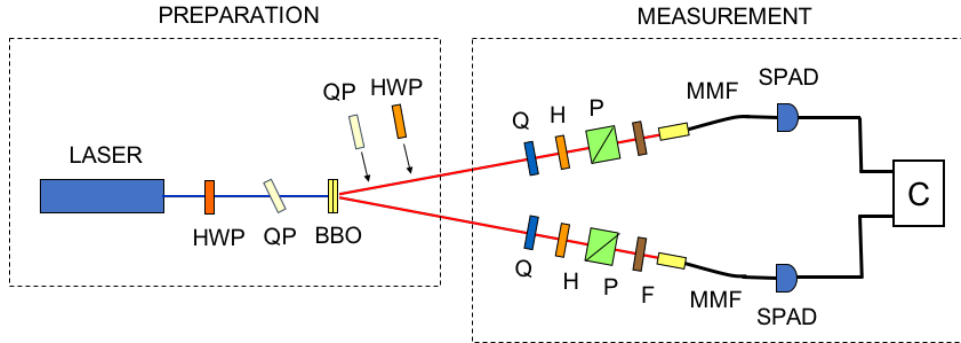


Figure 1. Apparatus to produce photon pairs in a desired state, and arrangement to measure the density matrix of the state via polarization projections. Components include a short-wavelength laser, beta-barium borate crystal (BBO), half-wave plates (HWP), quarter-wave plates (QWP), quartz plates (QP), calcite polarizers (P), bandpass filter (F), multi-mode fiber (MMF), single-photon avalanche-diode detector (SPAD), and electronic coincidence circuit (C).

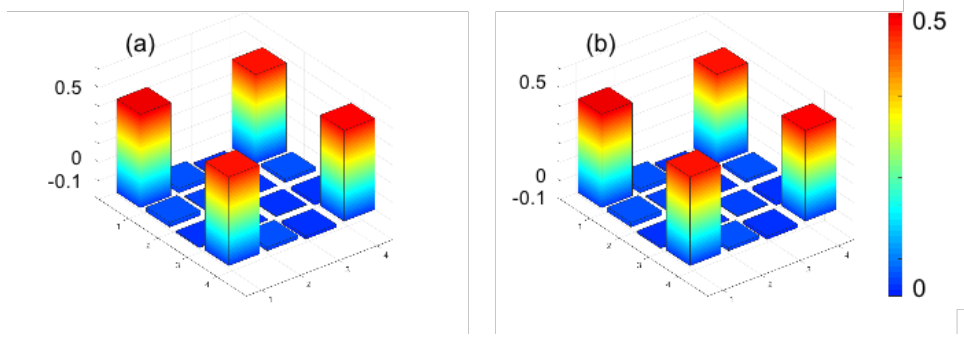


Figure 2. Pictorial representation measured data of the real component of the density matrix for state $2^{-1/2}(|HH\rangle + |VV\rangle)$ obtained via the direct method (a) and via quantum state tomography (b). The height of the columns represents the value of the matrix elements.

The results look promising. We still need to perform more tests and study the fluctuations when the data is noisy. The two density matrices are given below.

$$\hat{\rho}_{\text{DD}} = \begin{pmatrix} 0.5025 + 0.0000i & 0.0241 + 0.0173i & 0.0241 - 0.0173i & 0.0052 + 0.0000i \\ -0.0074 + 0.0165i & 0.0003 + 0.0071i & 0.4770 + 0.0461i & 0.0196 + 0.0200i \\ -0.0074 - 0.0165i & 0.4770 - 0.0461i & 0.0003 - 0.0071i & 0.0196 - 0.0200i \\ 0.0037 + 0.0000i & -0.0168 - 0.0208i & -0.0168 + 0.0208i & 0.4887 + 0.0000i \end{pmatrix} \quad (29)$$

$$\hat{\rho}_{\text{QST}} = \begin{pmatrix} 0.5031 + 0.0000i & 0.0236 + 0.0183i & 0.0236 - 0.0183i & 0.0052 + 0.0000i \\ -0.0102 - 0.0142i & 0.0005 + 0.0014i & 0.4780 + 0.0504i & 0.0178 + 0.0173i \\ -0.0102 + 0.0142i & 0.4780 - 0.0504i & 0.0005 - 0.0014i & 0.0178 - 0.0173i \\ 0.0037 + 0.0000i & -0.0181 - 0.0172i & -0.0181 + 0.0172i & 0.4879 + 0.0000i \end{pmatrix} \quad (30)$$

The tangle and linear entropy values (T, S) obtained from the above density matrices is $(0.90 \pm 0.02, 0.058 \pm 0.027)$ and $(0.92 \pm 0.02, 0.056 \pm 0.027)$ for direct method and state tomography, respectively. Both show very similar results, although the individual values are not exactly the same. The values of tangle and linear entropy differ by one standard deviation or less.

5. DISCUSSION

Obtaining the density matrix by the method provided here may give more reliable measures of the quantum state of the light and reduce the sensitivity of the quantum metrics to fluctuations in the data. We still need to do more tests to confirm our hypothesis, but the new method to calculate the density matrix stands.

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