Hamiltonian paths, unit-interval complexes, and determinantal facet ideals

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Abstract

We study d-dimensional generalizations of three mutually related topics in graph theory: Hamiltonian paths, (unit) interval graphs, and binomial edge ideals. We provide partial high-dimensional generalizations of Ore and Pósa's sucient conditions for a graph to be Hamiltonian. We introduce a hierarchy of combinatorial properties for simplicial complexes that generalize unit-interval, interval, and co-comparability graphs. We connect these properties to the already existing notions of determinantal facet ideals and (tight and weak) Hamiltonian paths in simplicial complexes. Some important consequences of our work are:

(1) Every unit-interval strongly-connected d-dimensional simplicial complex is traceable. (This extends the well-known result \unit-interval connected graphs are traceable".)

(2) Every unit-interval d-complex that remains strongly connected after the deletion of d or less vertices, is Hamiltonian. (This extends the fact that \unit-interval 2-connected graphs are Hamiltonian".)

(3) Unit-interval complexes are characterized, among traceable complexes, by the property that the minors dening their determinantal facet ideal form a Gröbner basis for a diagonal term order which is compatible with the traceability of the complex. (This corrects a recent theorem by Ene et al., extends a result by Herzog and others, and partially answers a question by Almousa{Vandebogert.)

(4) Only the d-skeleton of the simplex has a determinantal facet ideal with linear resolution. (This extends the result by Kiani and Saeedi-Madani that \only the complete graph has a binomial edge ideal with linear resolution".)

(5) The determinantal facet ideals of all under-closed and semi-closed complexes have a square-free initial ideal with respect to lex. In characteristic p, they are even F-pure.

Introduction

The rst Combinatorics paper in History is apparently Leonhard Euler's 1735 solution of the Königsberg bridge problem. In that article, Euler introduced the notion of graph, and studied cycles (now called 'Eulerian') that touch all edges exactly once. Euler proved that the graphs admitting them, are exactly those graphs with all vertices of even degree. Hamiltonian cycles

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are instead cycles that touch all vertices exactly once; they are named after sir William Rowan Hamilton, who in 1857 invented a puzzle game which asked to nd one such cycle in the icosahedron. Unlike for the Eulerian case, guring out if a graph admits a Hamiltonian cycle or not is a hard problem, now known to be NP-complete [Kar72].

Even if simple characterizations are o the table, in the 1950s and 1960s Dirac, Ore, Pósa and others were able to obtain simple conditions on the vertex degrees (in the spirit of Euler's work) that are sucient for a graph to admit Hamiltonian cycles [Dir52, Ore60, Pós62]. Ore's theorem, for example, says, \Any graph with n vertices such that $\deg u + \deg v$ n for all non-adjacent vertices $u; v$, admits a Hamiltonian cycle". Ore's condition is far from being necessary: In any cycle, no matter how large, one has $\deg u + \deg v = 4$ for all $u; v$.

In the same years, the two papers [LB62] and [GH64] initiated the study of unit-interval graphs. This very famous class consists, as the name suggests, of all intersection graphs of a bunch of length-one open intervals on the real line. (That is, we place a node in the middle of each interval, and we connect two nodes with an arc if and only if the corresponding intervals overlap). Bertossi's theorem says that if they are connected, such graphs always admit Hamiltonian paths, i.e. paths that touch all vertices once [Ber83]. Chen{Chang{Chang's theorem states that 2-connected unit-interval graphs admit Hamiltonian cycles [CCC97]. For these results, the length-one request can be weakened to \pairwise not-nested", but it cannot be dismissed: Within the larger world of interval graphs, one encounters connected graphs such as $K_{1;3}$ that do not admit Hamiltonian paths, and also 2-connected graphs like the G_5 of Remark 45 that do not admit Hamiltonian cycles.

In the 1970s, the work of Stanley and Reisner established a fundamental bridge between Combinatorics and Commutative Algebra, namely, a natural bijection between labeled simplicial complexes on n vertices and radical monomial ideals in a polynomial ring with n variables. This correspondence lead Stanley to prove the famous Upper Bound Theorem for triangulated spheres [Sta14]. After this success, many authors have investigated ways to encode graphs into monomial ideals. In 2010, Herzog et al. [H&10] rst considered a natural way to encode graphs into binomial ideals, the so-called binomial edge ideals. The catch is that all such binomial edge ideals are radical [H&10]. In the process, Herzog et al. re-discovered unit-interval graphs, characterizing them as the graphs whose binomial edge ideals have quadratic Gröbner bases with respect to a diagonal term order [H&10, Theorem 1.1].

So far, we sketched three graph-theoretic topics from three dierent centuries: Hamiltonian paths, (unit) interval graphs, binomial edge ideals. In the last years, there has been an increasing interest in expanding these three notions to higher dimensions. Specically:

- Katona{Kierstead [KK99] and many others [HS10, K&10, RSR08] have studied \tight Hamiltonian paths" and \loose Hamiltonian paths" in d-dimensional simplicial complexes; both notions for $d = 1$ boil down to ordinary Hamiltonian paths. The good news is that extremal combinatorics provides a non-trivial way to extend Dirac's theorem for d-complexes with a very large number of vertices that satisfy certain ridge-degree conditions. The bad news is that already Ore and Pósa's theorems seem very hard to extend.
- Ene et al. [E&13] introduced \determinantal facet ideals", which directly generalize binomial edge ideals, and \closed d-complexes", which generalize 'unit-interval graphs'. The good news is that the denitions are rather natural. The bad news is that determinantal facet ideals are not radical in general (see Example 73), and they are hard to manipulate; alas, the two main results of the paper [E&13] are incorrect, cf. Remark 85.

In the present paper we take a new, unied look at these approaches. In Chapter 1, we introduce a notion of 'weakly-Hamiltonian paths' for d-dimensional simplicial complexes that for $d = 1$ also boils down to ordinary Hamiltonian paths. This weaker notion enables us to obtain a rst, partial extension of Dirac, Ore and Posa's theorem to higher dimensions:

Main Theorem I (Higher-dimensional Ore and Dirac, cf. Proposition 18 and Corollary 20). Let be any traceable d-complex on $n > 2d$ vertices. If in some labeling that makes traceable the two (d 1)-faces and formed by the rst d and the last d vertices, respectively, have facet degrees summing up to at least n, then admits a weakly-Hamiltonian cycle.
In particular, if in a traceable pure d-complex with n vertices, every (d 1)-face belongs to at least $\frac{n}{2}$ facets, then the complex admits a weakly-Hamiltonian cycle.

Main Theorem II (Higher-dimensional Posa, cf. Proposition 23). Let be any traceable pure d-complex on n vertices, $n > 2d$. Suppose that with any labeling in which has a weakly-Hamiltonian path, is traceable. Let $_1; _2; : : : ; _s$ be the (d 1)-faces of , ordered so that d_1 d_2 $: : :$ d_s, where $d_i = d_{_i}$ is the number of d-faces containing $_i$. If for every d k < $\frac{n}{2}$ one has $d_{k \; d+1} > k$, then admits a weakly-Hamiltonian cycle.

As you can see these results are conditional: 'Traceability', i.e. the existence of a tight Hamiltonian path, must be known a priori, in order to infer the existence of a weakly-Hamiltonian cycle. This sounds like a bad deal, but in the one-dimensional case our results above still immediately imply the original theorems by Ore and Posa for graphs. Moreover, since no extremal combinatorics is used in the proof, there is an advantage: Main Theorems I and II do not require the number of vertices to be extremely large. On the contrary: In the two-dimensional case, they already apply to complexes with ve vertices.

In Chapter 2, we introduce a hierarchy of four natural properties that progressively weaken (for strongly-connected complexes) the notion of \closed d-complexes", as originally proposed in [E&13]. We introduce \unit-interval", \under-closed", and \weakly-closed" complexes, as natural combinatorial higher-dimensional generalizations of unit-interval graphs, of interval graphs, and of co-comparability graphs, respectively. The forth property, called \semi-closed", is intermediate between \under-closed" and \weakly-closed"; it is also dened very naturally, but it seems to be new already for graphs. We will see its algebraic consequence in Main Theorem VI below. The main goal of Chapter 2 is to connect this hierarchy to the notions of Chapter 1:

Main Theorem III (Higher-dimensional Bertossi, Theorem 56). Every unit-interval strongly-connected d-dimensional simplicial complex is traceable.

Main Theorem IV (Higher-dimensional Chen{Chan{Chang, Theorem 60). Every unit-interval d-dimensional simplicial complex that remains strongly connected after the deletion of d or less vertices, however chosen, is Hamiltonian.

Finally, Chapter 3 is dedicated to the connection with commutative algebra. For a homogeneous ideal of polynomials, having a square-free Gröbner degeneration is a strong and desirable property. In 2020, Conca and the third author proved Herzog's conjecture that if a homogeneous ideal I has a square-free initial ideal in(I), then the extremal Betti numbers of I and in(I) are the same [CV20]. This allows us to infer the depth, the Castelnuovo{Mumford regularity, and many other invariants of the ideals I with squarefree initial-ideal, simply by computing these invariants on the initial ideal | which is a much simpler task, because the aforementioned Stanley{Reisner correspondence activates techniques from combinatorial topology. Building on the very recent work of the second author [Sec21], we are able to revise one of the results claimed in Ene et al [E&13] as follows:

Main Theorem V (Theorem 82 and 87). A strongly-connected d-dimensional simplicial complex is unit-interval if and only if the complex is traceable and with respect to the same labeling, the minors dening the determinantal facet ideal of form a Gröbner basis with respect to any diagonal term order.

We conclude our work with a result that provides a broad class of determinantal facet ideals that are radical, and even F-pure (if the characteristic is positive):

Main Theorem VI (Theorem 77). The determinantal facet ideals of all semi-closed complexes are radical. Indeed, they have a square-free initial ideal with respect to any diagonal term order. Moreover, in characteristic $p > 0$, the quotients by these ideals are all F-pure.

The proof relies once again on the recent work by the second author [Sec21]. Since all shifted complexes are under-closed, and in particular semi-closed, Theorem 77 immediately implies that the determinantal facet ideals of shifted complexes admit a square-free Grӧbner degeneration and, in positive characteristic, dene F-pure rings. As a consequence of Main Theorem VI, we can extend to all dimensions the result by Kiani and Saeedi-Madani that \among all graphs, only complete graphs have a binomial edge ideal with a linear resolution" [SK12]. Namely, we prove that among all d-dimensional simplicial complexes with n vertices, only the d-skeleta of simplices have a determinantal facet ideal with a linear resolution (Corollary 81).

Notation

Throughout $d; n$ are positive integers, with $d < n$. We denote by ^{d} the d-simplex, and by ^d_n the d-skeleton of $^{n-1}$. We write each face of ^d_n by listing its vertices in increasing order. We describe simplicial complexes by listing their facets in any order, e.g. $= 123; 124; 235$. For any d-face $F = a_0 a_1 \, a_d$ of ^d_n, we call gap of F the integer $\text{gap}(F) \overset{\text{def}}{=} a_d \;\; a_0 \;\; d$, which counts the integers i strictly between a_0 and a_d that are not present in F. For each i in $f1; 2; :::; n \;\; dg$, we call H_i the d-face of ^d_n with vertices $i; i+1; :::; i+d$. Clearly, $H_1; H_2; :::; H_{n \; d}$ are exactly those faces of ^d_n that have gap zero. With abuse of notation, we extend the denition of H_i also to $i \; 2 \; fn \;\; d+1; :::; ng$ using \congruence modulo n". Namely, by \n + 1" we mean vertex 1, by \n + 2" we mean vertex 2, and so on. So H_n will be the d-face adjacent to H_1 and of vertices $fn; 1; 2; 3; \; ; dg$, which we write down in increasing order, so $H_n = 123 \, d \, n$. Note that $\text{gap}(H_i) > 0$ when $i > n \;\; d$.

Denition 1 (traceable, Hamiltonian). A complex is (tight-) traceable if it has a labeling such that $H_1; :::; H_{n \; d}$ are in . It is (tight-) Hamiltonian if it has a labeling such that all of $H_1; :::; H_n$ are in .

Clearly, Hamiltonian implies traceable. For $d = 1$, Denition 1 boils down to the classical notions of traceable and Hamiltonian graphs, that is, graphs that admits a Hamiltonian path and a Hamiltonian cycle, respectively. In fact, nobody prevents us from relabeling the vertices in the order in which we encounter them along such path (or cycle).

Recall that two facets of a pure simplicial d-complex are adjacent if their intersection has cardinality d, or equivalently, dimension $d \;\; 1$. For example, each H_i is adjacent to H_{i+1}. The dual graph of a pure simplicial d-complex has nodes corresponding to the facets of ; two nodes are connected by an arc if and only if the corresponding facets of are adjacent. A pure simplicial d-complex is strongly-connected if its dual graph is connected. For $d \;\; 1$, every strongly-connected d-complex is connected, and when $d = 1$ the two notions coincide. According to our convention, all strongly-connected simplicial complexes are pure.

Remark 2. The statement \the dual graph of any Hamiltonian d-complex is Hamiltonian" holds true only for $d = 1$: For example, the Hamiltonian simplicial complex

$$_1 = 123; 234; 345; 456; 567; 678; 789; 189; 129; 147$$

is not even strongly connected, because the facet 147 is isolated in the dual graph. The deletion of vertex 1 from $_1$ yields a simplicial complex that is not even pure.

1 Weakly-traceable/Hamiltonian complexes and ridge degrees

In this section, we introduce two weaker notions of traceability and Hamiltonicity that first appeared in [K&10], and we study their nontrivial relationship with the "ridge degree", i.e. how many d-faces contain any given $(d-1)$-face. This relationship has a long history, beginning in 1952 with one of the most classical results in graph theory, due to Gabriel Dirac [Dir52], the son of Nobel Prize physicist Paul Dirac:

Theorem 3 (Dirac [Dir52]). Let G be a graph with n vertices. If $\deg v \geq \frac{n}{2}$ for every vertex v, then G is Hamiltonian.

Later Øystein Ore [Ore60] improved Dirac's result and extended it to traceable graphs:

Theorem 4 (Ore [Ore60]). Let G be a graph with n vertices.

(A) If $\deg u + \deg v \geq n$ for all non-adjacent vertices u, v, the graph G is Hamiltonian.

(B) If $\deg u + \deg v \geq n - 1$ for all non-adjacent vertices u, v, the graph G is traceable.

Two years later Pósa extended Ore's condition (A) much further:

Theorem 5 (Pósa [Pos62]). Let G be a graph with n vertices. Order the vertices $v_1, \ldots, v_n$ so that the respective degrees are weakly increasing, $d_1 \leq d_2 \leq \ldots \leq d_n$.

(C) If for every $k < \frac{n}{2}$ one has $d_k > k$, the graph G is Hamiltonian.

These theorems have been generalized in five main directions, over the course of more than a hundred papers (see also Li [Li13] for a survey with a different perspective than ours):

1. Bondy and Chvátal [Bon69, Bo71a, Chv84, BC71] weakened the antecedent in the implication (C) of Pósa's theorem (see [Far99] for an application to self-complementary graphs);
2. Bondy [Bo71b] strengthened the conclusion of Ore's theorem, from Hamiltonian to pancyclic (=containing cycles of length ℓ for any $3 \leq \ell \leq n$); later Schmeichel–Hakimi [SH74] showed that Pósa and Chvátal's theorems can be strengthened in the same direction;
3. Fan [Fan84] showed that for 2-connected graphs, it suffices to check Ore's condition for vertices u and v at distance 2; and even more generally, it suffices to check that for any two vertices at distance two, at least one of them has degree $\geq \frac{n}{2}$. With these weaker assumptions he was still able to achieve a pancyclicity conclusion. See [BCS93], [LLF07], [CSZ14] for recent extensions of Fan's work.
4. A forth line of generalizations of Ore's theorem involved requiring certain vertex sets to have large neighborhood unions, rather than large degrees: Compare Broersma–van den Heuvel–Veldman [BHV93] and Chen–Schelp [CS92].

Here we are interested in the fifth main direction, namely, the generalization to higher dimensions. This is historically a rather difficult task: As of today, no straightforward extension of Ore's theorem or of Posa's theorem is known. However, some elegant positive results were obtained in 1999 by Katona and Kierstead [KK99], who applied extremal graph theory to generalize Dirac's theorem to simplicial complexes with a huge number of vertices. Building on the work by Katona and Kierstead [KK99], Rödl, Szemeredi, and Ruciński [RSR08] were able in 2008 to prove the following 'extremal' version of Dirac's theorem:

Theorem 6 (Rödl–Szemeredi–Ruciński [RSR08]). For all integers $d \geq 2$ and for every $\varepsilon > 0$ there exists a (very large) integer n_0 such that every d-dimensional simplicial complex Δ with more than n_0 vertices, and such that every $(d-1)$-face of Δ is in at least $n(\frac{1}{2} + \varepsilon)$ facets, is Hamiltonian.

Now we are ready to introduce the main definition of the present section. Recall that two facets of a pure simplicial d-complex are incident if their intersection is nonempty.

Definition 7 (weakly-traceable, weakly-Hamiltonian). A d-dimensional simplicial complex Δ is weakly-traceable if it has a labeling such that Δ contains faces $H_{i_1}, \ldots, H_{i_k}$ from $\{H_1, \ldots, H_{n-d}\}$ that altogether cover all vertices, and such that H_{i_j} is incident to $H_{i_{j+1}}$ for each $j \in \{1, \ldots, k-1\}$. In this case, we call the list $H_{i_1}, \ldots, H_{i_k}$ a weakly-Hamiltonian path.

A d-dimensional simplicial complex Δ is weakly-Hamiltonian if it has a labeling such that Δ contains faces $H_{i_1}, \ldots, H_{i_k}$ from $H_1, \ldots, H_n$ that altogether cover all vertices, such that H_{i_j} is incident to $H_{i_{j+1}}$ for each $j \in \{1, \ldots, k-1\}$, and in addition H_{i_k} is incident to H_{i_1}. In this case, we call the list $H_{i_1}, \ldots, H_{i_k}$ a weakly-Hamiltonian cycle.

Remark 8. These notions are not new. For what we called "weakly-Hamiltonian", Keevash et al. [K&10] use the term "generic Hamiltonian". Their paper [K&10] focuses however on the stronger notion of "loose-Hamiltonian" complexes, which are weakly-Hamiltonian complexes where all of the intersections $H_{i_j} \cap H_{i_{j+1}}$ consist of a single point (with possibly one exception). By definition, all Hamiltonian complexes are loose-Hamiltonian, and all loose-Hamiltonian complexes are weakly-Hamiltonian. For $d = 1$ all these different notions converge: "Weakly-Hamiltonian 1-complexes" are simply "graphs with a Hamiltonian cycle", and "weakly-traceable 1-complexes" are "graphs with a Hamiltonian path". In 2010 Han–Schacht [HS10] and independently Keevash et al. [K&10] proved the following extension of Theorem 6 above:

Theorem 9 (Han-Schacht [HS10], Keevash et al. [K&10]). For all integers $d \geq 2$ and for every $\varepsilon > 0$ there exists a (very large) integer n_0 such that every d-dimensional simplicial complex Δ with more than n_0 vertices, and such that every $(d-1)$-face of Δ is in at least $n(\frac{1}{2d} + \varepsilon)$ facets, is loose-Hamiltonian, and in particular weakly-Hamiltonian.

Remark 10. In Definition 7, note that if Δ is weakly-traceable, necessarily $i_1 = 1$ and $i_k = n-d$, because otherwise 1 and n would not be covered. So equivalently, in Def. 7 we could demand

$$\{i_2, \ldots, i_{k-1}\} \subseteq \{2, \ldots, n-d-1\}.$$

Note also that if a labeling $v_1, \ldots, v_n$ makes Δ (weakly-) traceable, so does the "reverse labeling" $v_n, \ldots, v_1$. As for Hamiltonian complexes: If a labeling $v_1, \ldots, v_n$ makes Δ weakly-Hamiltonian, so does its reverse, and also $v_{i_1}, \ldots, v_{i_n}$, where $(i_1, \ldots, i_n)$ is any cyclic permutation of $(1, \ldots, n)$. So we may assume that $i_1 = 1$. Or we may assume that $i_k = n-d$. But as the next remark shows, we cannot assume both.

Remark 11. When $d > 1$, not all weakly-Hamiltonian d-complexes are weakly-traceable. For $d = 2$, a simple counterexample is given by

$$\Delta_0 = 123, 156, 345.$$

The weakly-Hamiltonian cycle is of course H_1, H_3, H_5. Any labeling that makes Δ_0 weakly-Hamiltonian is either the reverse or a cyclic shift (or both) of the labeling above. For parity reasons, in any labeling that makes Δ_0 weakly-Hamiltonian, only one of H_1 and H_4 is in Δ_0.

Remark 12. Weakly-traceable complexes are obviously connected. Weakly-Hamiltonian complexes are even 2-connected, in the sense that the deletion of any vertex leaves them connected. The converses are well-known to be false already for $d = 1$. In fact, let $n \geq 4$. Let A_{n-2} be the edgeless graph on $n-2$ vertices. Let x, y be two new vertices. The "suspension"

$$\mathrm{susp}(A_{n-2}) \stackrel{\text{def}}{=} A_{n-2} \cup \{x \ast v : v \in A_{n-2}\} \cup \{y \ast v : v \in A_{n-2}\}$$

is a 2-connected graph on n vertices that is not Hamiltonian for $n \geq 5$, and not even traceable for $n \geq 6$. In higher dimensions, the Δ_3 of Lemma 44 is d-connected, but neither weakly-traceable nor weakly-Hamiltonian.

We start with a few Lemmas that are easy, and possibly already known; we include nonetheless a proof for the sake of completeness. For the following lemma, a subword of a word is a subsequence formed by consecutive letters of a word: So for us "word" is a subword of "subword", whereas "sword" is not.

Lemma 13. Let $d \geq 2$. If a d-complex Δ is weakly-Hamiltonian (resp. weakly traceable), then for any $k \in \{1, \ldots, d\}$ the k-skeleton of Δ is weakly-Hamiltonian (resp. weakly-traceable).

Proof. Given a weakly-Hamiltonian path/cycle, replace any d-face H_1 with its $(k+1)$-letter subwords, ordered lexicographically. The result, up to canceling possible redundancies, will be a weakly-Hamiltonian path/cycle for the k-skeleton. □

For example: if $d = 3$ and $k = 1$, suppose that a 3-complex on 8 vertices admits the Hamiltonian path

$$1234, \quad 2345, \quad 5678.$$

Then the 1-skeleton admits the Hamiltonian path

$$12, 23, 34, \quad 23, 34, 45, \quad 56, 67, 78.$$

The next Lemma is an analog to the fact that Hamiltonian complexes are traceable.

Lemma 14. Let Δ be a d-dimensional complex that has a weakly-Hamiltonian cycle $H_{i_1}, \ldots, H_{i_k}$, with $k \geq 3$. For any j in $\{1, \ldots, k\}$, let m_j be the number of vertices of H_{i_j} that are neither contained in $H_{i_{j-1}}$ nor in $H_{i_{j+1}}$ (where by convention $i_{k+1} \stackrel{\text{def}}{=} i_1$).

- If $m_j > 0$, the deletion of those m_j vertices from Δ yields a weakly-traceable complex.
- If $m_j = 0$, and in addition $H_{i_{j-1}}$ and $H_{i_{j+1}}$ are disjoint, then Δ itself is weakly-traceable.

Proof. Fix j in $\{1, \ldots, k\}$. If $m_j > 0$, the m_j vertices that belong to H_{i_j} and to no other facet of the cycle are labeled consecutively. So up to relabeling the vertices cyclically, we can assume that they are the vertices $n - m_j + 1, n - m_j + 2, \ldots, n - 1, n$. Thus the facet in the cycle they all belong to is the last one, H_{i_k}. Now let D be the complex obtained from Δ by deleting these m_j vertices. It is easy to see that

$$H_1 = H_{i_1}, H_{i_2}, \ldots, H_{i_{k-1}}$$

is a weakly-Hamiltonian path for D.
The case $m_j = 0$ is similar: Up to relabeling the vertices cyclically, $i_{j+1} = 1$ and thus $j = k$. By assumption $H_{i_{k-1}}$ and H_1 are disjoint. But since $m_k = 0$, and vertex n does not belong to H_1, it must belong to $H_{i_{k-1}}$. Therefore $H_{i_{k-1}} = H_{n-d}$. So

$$H_1 = H_{i_1}, H_{i_2}, \ldots, H_{i_{k-1}}$$

is a weakly-Hamiltonian path for Δ itself. □

The next Lemma can be viewed as a d-dimensional extension of the fact that the cone over the vertex set of a graph G is a Hamiltonian graph if and only if the starting graph G is traceable.

Lemma 15. Let Δ be any d-complex on n vertices. Let σ^{d-1} be the $(d-1)$-simplex. Let Δ' be the d-complex obtained by adding to Δ a d-face $v * \sigma^{d-1}$ for every vertex v in Δ. Then

$$\Delta \text{ is weakly-traceable} \iff \Delta' \text{ is weakly-Hamiltonian.}$$

Proof. \) " : If $H_{i_1}; : : : ; H_{i_k}$ is a list of facets proving that is weakly-traceable, then the list $H_{i_1}; : : : ; H_{i_k}; H_n; H_{n+1}$ shows that is weakly-Hamiltonian.

\(" : Pick a labeling that makes weakly-Hamiltonian. By how the complex is constructed, the vertices of ^{d} ^{1} must be labeled consecutively; so without loss, we may assume that they are n + 1; : : : ; n + d. Take a weakly-Hamiltonian cycle for and delete from the list all the d-faces containing any vertex whose label exceeds n. □

Remark 16. The following statements are valid only for d = 1.

(i) \ is weakly-traceable ( ) [w (d-1)-skel() is weakly-Hamiltonian."

(ii) \Deleting a single vertex from a weakly-Hamiltonian d-complex yields a weakly-traceable complex."

(iii) \Deleting (the interior of) any of the H_i's from a weakly-Hamiltonian d-complex yields a weakly-traceable complex."

Simple counterexamples in higher dimensions are:

(i) $_1$ = 126; 234; 456; 489; 678 is not weakly-traceable, yet $_2$ = $_1$ $[^{def}$(10 2-skel($_1$)) admits the weakly-Hamiltonian cycle 234; 456; 678; 89 10; 12 10. This is a counterexample to \(" . In contrast, the direction \ ) " holds in all dimensions.

(ii) If from the $_2$ above we delete vertex 10, we get back to $_1$, not weakly-traceable.

(iii) $_3$ = 1234; 2345; 5678; 167 10; 189 10 is weakly-Hamiltonian, but the deletion of (the interior) of 5678 yields a complex that is not weakly-traceable.

Our rst non-trivial result is an \Ore-type result": We shall see later that in some sense it extends 'most' of the proof of Ore's theorem 4, part (A), to all dimensions.

Denition 17. Let be a pure d-dimensional simplicial complex, and let be any (d 1)-face of . The degree d of is the number of d-faces of containing .

Proposition 18. Let be a traceable d-dimensional simplicial complex on n vertices, n > 2d. If in some labeling that makes traceable the two (d 1)-faces and formed by the rst d and the last d vertices, respectively, satisfy d + d n, then is weakly-Hamiltonian.

Proof. Since n > 2d, the two faces and are disjoint. Let J $\stackrel{def}{=}$ fd + 2; d + 3; : : : ; n dg. For every i in J, which has cardinality n 2d 1, consider the two d-faces of $_n$ ^{d}

$$S_i \stackrel{def}{=} i \quad \text{and} \quad T_i \stackrel{def}{=} (i \quad 1) :$$

Now there are two cases, both of which will result in a weakly-Hamiltonian cycle:

Case 1: For some i, both S_i, T_i are in . We are going to introduce a new vertex labeling '$_1$; : : : ; '$_n$. The \consecutive facets of the new labeling" will be called L_1 = '$_1$ '$_d$ '$_{d+1}$, L_2 = '$_2$ '$_{d+2}$, and so on. The following describes a weakly-Hamiltonian cycle:

- Start with the rst i 1 vertices in the same order: That is, set '$_1$ $\stackrel{def}{=}$ 1, : : :, '$_i$ $_1$ $\stackrel{def}{=}$ i 1. Hence L_1 = H_1; L_2 = H_2; : : : ;, up until $L_{i \ d \ 1}$ = $H_{i \ d \ 1}$, which (since is traceable) is the rst of the H_i's that contains the vertex i 1.
- Then set $L_i \stackrel{def}{=} T_i$. The vertices of are to be relabeled by '$_i$; '$_{i+1}$; : : : ; '$_{i+d}$: Specically, label by '$_i$ the vertex that is in $H_{n \ d}$ but not in $H_{n \ d \ 1}$, by '$_{i+2}$ the vertex in in $H_{n \ d \ 1}$ but not in in $H_{n \ d \ 2}$, and so on. Facet-wise, we are traveling in reverse order across the last facets of the original labeling. Stop until you get to relabel vertex i by '$_n$. (Or equivalently, if you prefer to think about facets, stop once you reach facet H_i.)
- The weakly-Hamiltonian cycle gets then concluded with S_i, which is adjacent to L_1 = H_1 via . The facets previously called $H_{i \ d}$, $H_{i \ d+1}$, : : :, $H_{i \ 1}$ are not part of the new weakly-Hamiltonian cycle.

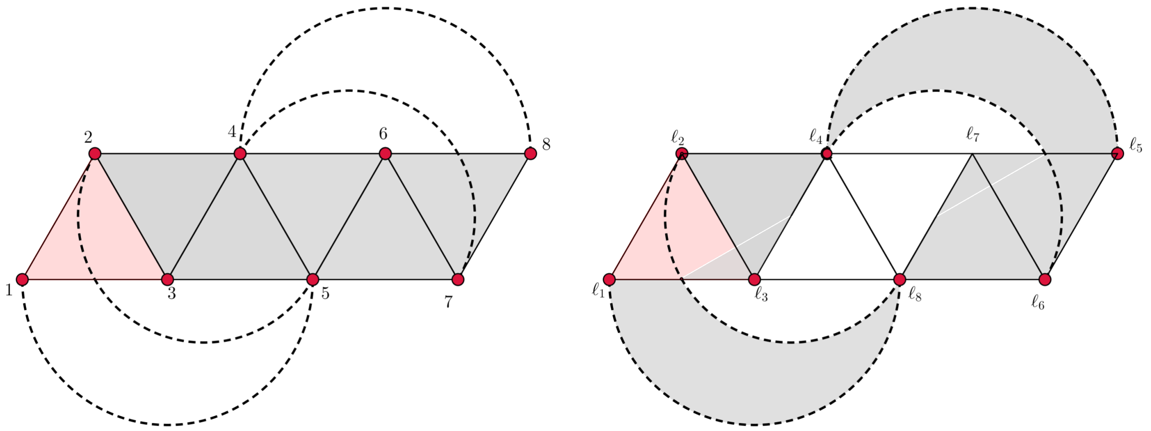


Figure 1: left: The dashed triangles are S_5 and T_5. Were they both in , then one could relabel the vertices and create a weakly-Hamiltonian cycle (right).

Case 2: For all i, at most one of S_i, T_i is in . Since the two sets $\{i \in J : i \in \;\}$ and $\{i \in J : (i-1) \in \;\}$ are disjoint, the sum of their cardinalities is the cardinality of their union, which is contained in J. So

$$|\{i \in J : i \in S\}| + |\{i \in J : (i - 1) \in \;\}| \quad |J| = n - 2d + 1: \tag{1}$$

Now, we claim that either $\;n$ or $1\;$ is a face of . From the claim the conclusion follows immediately, as such face creates a weakly-Hamiltonian cycle. We prove the claim by contradiction. Suppose contains neither $\;n$ nor $1\;$. Every d-face containing is of the form $\;v$, where v is either in J or in the set $\{d; n - d + 1; n - d + 2; \dots; n - 1\}$ (which has size d). So

$$d \quad |\{i \in J : i \in \;\}| + d: \tag{2}$$

Symmetrically, the d-faces containing are of the form $w\;$, with w either in J or in the size-d set $\{2; 3; \dots; d; n - d + 1\}$. So

$$d \quad |\{i \in J : (i - 1) \in \;\}| + d: \tag{3}$$

Putting together inequalities 1, 2 and 3, we reach a contradiction:

$$d + d \quad (n - 2d - 1) + d + d = n - 1: \qquad \square$$

Corollary 19. Let be a traceable d-dimensional simplicial complex on n vertices, $n > 2d$. If for any two disjoint $(d - 1)$-faces and one has $d + d \quad n$, then is weakly-Hamiltonian.

Corollary 20. Let be a traceable d-dimensional simplicial complex on n vertices, $n > 2d$. If every $(d - 1)$-face of belongs to at least $\frac{n}{2}$ facets of , then is weakly-Hamiltonian.

Example 21. Let $n > 2d$. Let $_4$ be the simplicial complex on n vertices obtained from ^d_n by removing the interior of the d-faces H_{n-d+1}, H_{n-d+2}, $\dots$, H_n. By construction $_4$ is traceable, but the given labeling (as well as any labeling obtained from it by reversing or cyclic shifting) fails to prove that $_4$ is weakly-Hamiltonian. Now, in $_4$, the $(d - 1)$-faces $_i \overset{def}{=} H_i \setminus H_{i+1}$, with $i \in \{n - d + 1; n - d + 2; \dots; n - 1\}$, have degree $n - d - 2$. All other $(d - 1)$-faces $_j$ contained in one of $H_{n-d+1}; H_{n-d+2}; \dots; H_n$ have degree $n - d - 1$. Finally, all $(d - 1)$-faces not contained in any of $H_{n-d+1}; H_{n-d+2}; \dots; H_n$ have degree $n - d$. Therefore:

- If n 2d + 4, Corollary 20 tells us that $_4$ is weakly-Hamiltonian, because n d 2 $\frac{n}{2}$. • If n = 2d+3 or n = 2d+2, any two of the $_i$'s are incident, and any $_j$ is incident to all of the $_i$'s. Hence, for any two disjoint (d 1)-faces and , we do have d+d 2n 2d 2 n. So we can still conclude that $_4$ is weakly-Hamiltonian via Corollary 19.
- If n = 2d+1, then the assumptions of Corollaries 20 and 19 are not met, but Proposition 18 is still applicable. In fact, for the facets resp. formed by the rst resp. the last vertices of the given labeling, one has d + d = (n d) + (n d 1) = 2n (2d + 1) = n.

So in all cases, $_4$ is weakly-Hamiltonian. The proof of Proposition 18 also suggests a relabeling that works: $\ell_1 \stackrel{def}{=} 1;\ \ell_2 \stackrel{def}{=} 2; \ldots ;\ \ell_{d+1} \stackrel{def}{=} d+1;\ \ell_{d+2} = n;\ \ell_{d+3} \stackrel{def}{=} n \quad 1; \ldots ;\ \ell_n \stackrel{def}{=} d+2$.

To see in what sense Proposition 18 is a higher-dimensional version of Ore's theorem 4, part (A), the best is to give a proof of the latter using the former:

Proof of Ore's theorem 4, part (A). By contradiction, let G be a non-Hamiltonian graph satisfying deg u + deg v n for all non-adjacent vertices u; v. Add edges to it until you reach a maximal non-Hamiltonian graph G. Since any further edge between the existing vertices would create a Hamiltonian cycle, G is traceable, and obviously it still satises deg u + deg v n. By Proposition 18 G is (weakly-)Hamiltonian, a contradiction. □

It is possible that the bound of Proposition 18 can be improved. But in any case, the possible improvement could only be small, as the following construction shows.

Non-Example 22. Let d < m be positive integers. Take the disjoint union of two copies $A^0; A^{00}$ of ^d_m. Let be any facet of ^d_m and let $^0;^{00}$ be its copies in A^0 and A^{00}, respectively. Glue to A^0 [A^{00} a triangulation without interior vertices of the prism [0; 1], so that the lower face f0g is identied with ^{0}, and the upper face f1g is identied with ^{00}. Let $_5$ be the resulting d-complex on n = 2m vertices. This $_5$ is traceable: the added prism, triangulated as a path of d-faces, serves as \bridge" to move between the two copies of ^d_m. However, this bridge can only be traveled once, so $_5$ is not weakly-Hamiltonian. For the labeling that makes it traceable, d + d = (m d) + (m d) = n 2d.

Our next result is a \Posa{type" result, in the sense that it extends most of Nash{Williams' proof [Nas66] of Posa's theorem [Pos62] to all dimensions. We focus on complexes with the property that any labeling that makes them weakly-traceable, makes them also traceable. Such class is nonempty: for example, it contains all 1-dimensional complexes and all trees of d-simplices (i.e. all triangulations of the d-ball whose dual graph is a tree).

Proposition 23. Let be any traceable pure d-complex on n vertices, n > 2d. Suppose that any labeling that makes weakly-traceable makes it also traceable.

Let $_1;_2; \ldots ;_s$ be an ordering of the (d 1)-faces of , such that the respective degrees $d_i \stackrel{def}{=} d_i$ are weakly-increasing, d_1 d_2 ::: d_s. If for every d k < $\frac{n}{2}$ one has d_k $_{d+1}$ > k, then is weakly-Hamiltonian.

Proof. Among all possible labelings that make weakly-traceable (and thus traceable, by assumption), choose one that maximizes d + d, where is the (d 1)-face of H_1 spanned by the rst d vertices (that is, 1; 2; d) and is the (d 1)-face of $H_{n \quad d}$ spanned by the last d vertices (that is, n d + 1; : : : ; n). Since n > 2d, the faces and are disjoint. If d + d n, using the proof of Proposition 18 we get that is weakly-Hamiltonian, and we are done. If not, then one of , has degree < $\frac{n}{2}$. Up to reversing the labeling, which would swap and , we can assume that d < $\frac{n}{2}$. Now let J $\stackrel{def}{=}$ fd + 2; d + 3; : : : ; n dg. For every i in J, which has cardinality n 2d 1, consider the two d-faces of ^d_n

$$S_i \stackrel{def}{=} i \quad \text{and} \quad T_i \stackrel{def}{=} (i \quad 1)\ :$$

We may assume that at most one of these two faces is in , otherwise a weakly-Hamiltonian cycle arises, exactly as in the proof of Proposition 18. Now for each i in $J_1 \overset{\text{def}}{=}$ fi 2 J : i 2 g, consider the (d 1)-face $_i$ with vertices fi d; i d + 1; : : : ; i 1g.

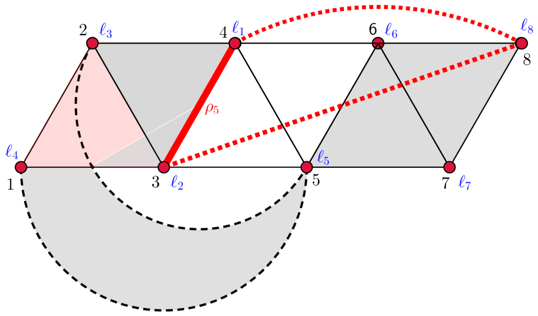


Figure 2: A higher-dimensional \Posa ip": Since 125 is in , the vertex 5 is in J_1. Now the red triangle $_5$ w = 348 cannot be in , or else we would have a weakly-Hamiltonian cycle with the blue labeling. The blue labeling makes weakly-traceable, with $_5$ playing the role of the \rst" (d 1)-face; the \last" (d 1)-face is the same as before. By how our original labeling was chosen, d_5 d < $\frac{n}{2}$.

If for some i in J_1 the d-face $_i$ w is in , then there is a new relabeling ‘$_1$; : : : ; ‘$_n$ of the vertices for which we have a weakly-Hamiltonian cycle: see Figure 2 above. (The proof is essentially identical to that of Proposition 18, up to replacing T_i with $T_i^0 \overset{\text{def}}{=}$ $_i$ w, reversing the order, and permuting it cyclically, so that $_i$ is the rst face.) So also in this case, we are done. It remains to discuss the case in which for all i 2 $J_1 \overset{\text{def}}{=}$ fi 2 J : i 2 g, the d-face $_i$ w is not in . In this case the relabeling ‘$_1$; : : : ; ‘$_n$ introduced above makes weakly-traceable, and thus traceable by assumption. For such relabeling, the (d 1)-faces spanned by the rst and the last d vertices are $_i$ and , respectively. So by the way our original labeling was chosen, d_i + d d + d, and in particular

$$d_i \quad d < \frac{n}{2}:$$

Now, any d-face containing is of the form v, where v is either in the set J_1 or in the set $Z \overset{\text{def}}{=}$ fd + 1; n d + 1; : : : ; n 1g, which has cardinality d. So d jJ_1 [Zj. Since J and Z are disjoint, and J_1 J, the sets J_1 and Z are also disjoint and we have

$$d \quad d = d \quad jZj \; jJ_1 \; [\; Zj \quad jZj = jJ_1j + jZj \quad jZj = jJ_1j:$$

So the set f$_i$: i 2 J_1g contains at least d d faces of dimension d 1 and degree d. If we count also , we have in at least d d + 1 faces of dimension d 1 and degree d. But then, setting $k \overset{\text{def}}{=}$ d, we obtain

$$d_k \quad {}_{d+1} \quad k < \frac{n}{2};$$

which contradicts the assumption. □

Again, to see in what sense Proposition 23 is a higher-dimensional version of Posa's Theorem 5, perhaps the best is to see how easily the latter follows from the former:

Proof of Posa's theorem 5. By contradiction, if G is not Hamiltonian, we can add edges to it until we reach a maximal non-Hamiltonian graph G, which still satises the degree conditions and is traceable. By Proposition 23, G is (weakly-)Hamiltonian, a contradiction. □

A natural question is whether one can generalize to higher dimensions also part (B) of Ore's theorem 4. The answer is positive, although some extra work is required. In fact, for graphs part (B) of Ore's theorem can be quickly derived from part (A) by means of a coning trick. This trick however does not extend to higher dimensions, as we explained in Remark 16, so we'll have to take a long detour, which makes the proof three times as long. The bored reader may skip directly to the next section.

Denition 24. A d-dimensional complex Δ is quasi-traceable if there exists a vertex labeling for which $\Delta \cup H_j$ is weakly-traceable, and moreover, with respect to the same labeling,

- (a) if $j = 1$, then Δ contains all of $H_2, \ldots, H_{n-d}$ (i.e., $\Delta \cup H_1$ is traceable);
- (b) if $j \in \{2, \ldots, n-2d\}$, then Δ already contains all of $H_1, \ldots, H_{j-1}$ and $H_{j+d}, \ldots, H_{n-d}$ (i.e., $\Delta \cup H_j \cup \ldots \cup H_{j+d-1}$ is traceable);
- (c) if $j \in \{n-2d+1, \ldots, n-d-1\}$, then Δ contains all of $H_1, \ldots, H_{j-1}$ and also H_{n-d} (i.e., $\Delta \cup H_j \cup \ldots \cup H_{n-d-1}$ is traceable);
- (d) if $j = n-d$, then Δ already contains all of $H_1, \ldots, H_{n-d-1}$ (i.e., $\Delta \cup H_{n-d}$ is traceable).

Example 25. The complex $\Delta_6 = 123, 234, 567, 678, 789$ is quasi-traceable, although not weakly-traceable. In fact, Δ_6 becomes weakly-traceable if we add one of the facets 345 and 456, and it becomes even traceable if we add both.

Denition 24 allows the "added faces" to be already present in Δ. In particular, all traceable complexes are quasi-traceable. Here comes our high-dimensional version of Theorem 4, part (B):

Proposition 26. Let Δ be a quasi-traceable d-dimensional simplicial complex on n vertices, $n > 2d$. If in some labeling that makes Δ quasi-traceable the two $(d-1)$-faces α and ω formed by the rst d and the last d vertices satisfy $d_\alpha + d_\omega \geq n - 1$, then Δ is weakly-traceable.

Proof. By contradiction, suppose Δ is not weakly-traceable; we treat the four cases of Denition 24 separately.

Case (a) is symmetric to Case (d), so we will leave it to the reader.

Case (b) is the main case. Since $j \in \{2, \ldots, n-2d\}$, by denition Δ contains all of $H_1, \ldots, H_{j-1}$ and also $H_{j+d}, \ldots, H_{n-d}$. Since Δ is not weakly-traceable, it does not contain H_j. Moreover, $\omega * (d+j)$ cannot be a facet of Δ, otherwise the two "halfpaths" above would be connected into a weakly-Hamiltonian path. For the same reason, since $(d+j-1) \in H_{j-1}$, the d-face $\alpha * (d+j-1)$ cannot be in Δ. So let $J' \overset{\text{def}}{=} \{d+2, d+3, \ldots, n-d\} \setminus \{d+j\}$. For every i in J', which has cardinality $n-2d-2$, consider the two d-faces of Δ^d_n

$$S_i \overset{\text{def}}{=} \alpha * i \quad \text{and} \quad T_i \overset{\text{def}}{=} \omega * (i-1) .$$

Now there are two subcases: Either there exists an i such that S_i, T_i are both in Δ, or not.

Case (b.1): For some i, both S_i and T_i are in Δ. There are two subsubcases, according to whether i is "before the gap" or "after the gap".

- Case (b.1.1): $i < d+j$. A weakly-Hamiltonian path arises from a relabeling as follows: We start at the beginning of the second halfpath, with the facets previously called H_{j+d}, H_{j+d+1}, etc., until we reach H_{n-d}. Then we use T_i to get back to the vertex previously labeled by $i-1$. Next, we use in reverse order the facets previously called $H_{i-d-1}, H_{i-d-2}, \ldots, H_2, H_1$. Finally use S_i to jump forward to the vertex previously called i, and conclude the path with the facets previously called $H_i, H_{i+1}, \ldots, H_{j-1}$.

– Case (b.1.2): i > d + j. A weakly-Hamiltonian path arises from a relabeling as fol-lows: We start at the beginning of the second halfpath, with the facets previously called $H_{j+d}; H_{j+d+1}$; etc., until $H_{i \ d \ 1}$. Then we use T_i to jump forward. As next faces, we use in reverse order the facets previously called $H_{n \ d}; H_{n \ d \ 1}; \ldots; H_2; H_i$. Finally, we use S_i to jump back to H_1, and conclude the path with the facets previously called $H_1; H_2; \ldots; H_{j \ 1}$. So also in this case is weakly-traceable, a contradiction.

Case (b.2): For all i, at most one of S_i and T_i is in . Since the two sets fi 2 J^0 : i 2 g and fi 2 J^0 : (i 1) 2 g are disjoint, we obtain a numerical contradiction:

$$d + d \quad d + \ jfi\ 2\ J^0 : \ i\ 2\ gj + d + \ jfi\ 2\ J^0 : (i \quad 1) \ 2\ gj = = 2d + \ jfi\ 2\ J^0 : \ i\ 2\ g[\ fi\ 2\ J^0 : (i \quad 1) \ 2\ gj \quad 2d + jJ^0j = 2d + n \quad 2d \quad 2 = n \quad 2:$$

Case (c) is the easiest. If j 2 fn 2d + 1; : : : ; n d 1g, then $H_{j \ 1}$ intersects $H_{n \ d}$. Since contains $H_1; \ldots; H_{j \ 1}$ and also $H_{n \ d}$, it is weakly-traceable, a contradiction.

Case (d) is the last one. So, assume j = 1 and set $J^{00} \stackrel{\text{def}}{=}$ fd + 2; d + 3; : : : ; n dg. For every i in J^{00}, which has cardinality n 2d 1, consider the two d-faces of ^d_n

$$S_i \stackrel{\text{def}}{=} \quad i \qquad \text{and} \qquad T_i \stackrel{\text{def}}{=} (i \quad 1) \ :$$

Now there are two subcases: Either there exists an i such that $S_i; T_i$ are both in , or not.

Case (d.1): For some i, both S_i and T_i are in . Then we obtain a weakly-Hamiltonian path as follows: Starting with $‘_1$ = 1, rst we use the face i, then $H_2; \ldots; H_{i \ d \ 1}$ in their order, then we use (i 1) n to jump forward, and then we come back with $H_{n \ d}; \ldots; H_i$.

Case (d.2): For all i, at most one of S_i and T_i is in . We know by that d = H_1 is not in because we are treating the case j = 1, and we know that n is not in otherwise we would have a weakly-Hamiltonian path. Thus any d-face containing is of the form v, where v is either in J^{00} or in the disjoint set fn d + 1; : : : ; n 1g, which has cardinality d 1. In contrast, any d-face containing is of the form (i 1) , where i is either in J^{00} or in the set f2; : : : ; d + 1g, which has cardinality d. Since the two sets fi 2 J^{00} : i 2 g and fi 2 J^{00} : (i 1) 2 g are disjoint, the sum of their cardinality is equal to the cardinality of their union, which is a subset of J^{00}. So also in this case we obtain a contradiction

$$d + d \quad d \quad 1 + \ jfi\ 2\ J^{00} : \ i\ 2\ gj + d + \ jfi\ 2\ J^{00} : (i \quad 1) \ 2\ gj = = 2d \quad 1 + \ jfi\ 2\ J^{00} : \ i\ 2\ g[\ fi\ 2\ J^{00} : (i \quad 1) \ 2\ gj \quad 2d \quad 1 + jJ^{00}j = 2d \quad 1 + n \quad 2d \quad 1 = n \quad 2:$$

□

Example 27. Let $_7$ be the simplicial complex on 5 vertices obtained from $_5$ by removing the interior of the two triangles 123 and 124. Clearly $_7$ is quasi-traceable with j = 1, because $_7$ [H_1 is traceable. Since $d_{12} + d_{45}$ = 4 = n 1, by Proposition 26 $_7$ is weakly-traceable. In fact, the reader may verify that $_7$ is even Hamiltonian with the relabeling $‘_1$ = 1, $‘_2$ = 2, $‘_3$ = 5, $‘_4$ = 3, $‘_5$ = 4.

For completeness, we conclude this section by showing how Proposition 26 implies part (B) of Ore’s theorem 4:

Proof of Ore’s theorem 4, part (B). By contradiction, let G be a non-traceable graph satis-fying deg u + deg v n 1 for all non-adjacent vertices u; v. Add edges to it until we reach a max-imal non-traceable graph G. This G is quasi-traceable and still satises deg u + deg v n 1. By Proposition 26 G is (weakly-)traceable, a contradiction. □

2 Interval graphs and semiclosed complexes

In the present section,

(1) we introduce \weakly-closed d-complexes", generalizing co-comparability graphs;

(2) we create a hierarchy of properties between closed and weakly-closed complexes, among which a d-dimensional generalization of interval graphs; and

(3) we connect such hierarchy to traceability and chordality.

2.1 A foreword on interval graphs and related graph classes

Interval graphs are the intersection graphs of intervals of $\mathbb{R}$. They have long been studied in combinatorics, since the pioneering papers by Lekkerkerker{Boland [LB62] and Gilmore{Homan [GH64], and have a tremendous amount of applications; see e.g. [Gol80, Ch. 8, Sec. 4] for a survey. Unit-interval graphs, also known as \indierence graphs" [Rob69] or \proper interval graphs", are the intersection graphs of unit intervals, or equivalently, the intersection graphs of sets of intervals no two of which are nested. The claw $K_{1,3}$ is the classical example of a graph that can be realized as intersection of four intervals, three of which contained in the forth; but it cannot be realized as intersection of unit intervals.

Bertossi noticed in 1983 that connected unit-interval graphs are traceable [Ber83], whereas connected interval graphs in general are not: The claw strikes. All 2-connected unit-interval graphs are Hamiltonian [CCC97][PD03]; again, this does not extend to 2-connected interval graphs. That said, for interval graphs (and even co-comparability graphs, see below for the denition) the Hamiltonian Path Problem and the Longest Path Problem can be solved in polynomial time [DS52] [MC12], whereas for arbitrary graphs both problems are well known to be NP-complete, cf. [Kar72].

Given a nite set of intervals in the horizontal real line, we can swipe them \left-to-right", and thus order them by increasing left endpoint. This so-called \canonical labeling" of the vertices of an interval graph obviously satises the following property: for all $a < b < c$,

$$ac \in G \Longrightarrow ab \in G. \tag{4}$$

This \under-closure" is a characterization: It is easy to prove by induction that any graph with n vertices labeled so that (4) holds can be realized as the intersection graph of n intervals. This result was rst discovered by Olario, cf. [LO93, Proposition 4].

There is a \geometrically dual argument" to the one above: Given a nite set of intervals in $\mathbb{R}$, we could also swipe them right-to-left, thereby ordering the intervals by decreasing right endpoint. This yields a vertex labeling that again satises (4), for the same geometric reasons. In general, since some of the intervals may be nested, this \dual labeling" bears no relation with the canonical one. But if we start with a nite set of unit intervals, then the dual labeling is simply the reverse of the canonical labeling. Thus in unit-interval graphs, not only the canonical labeling is under-closed, but also its reverse is. Or equivalently, in unit-interval graphs, the canonical labeling is closed 'both below and above': in mathematical terms, for all $a < b < c$,

$$ac \in G \Longrightarrow ab, bc \in G. \tag{5}$$

Again, it is not dicult to prove by induction that any graph with n vertices, labeled so that (5) holds, can be realized as the intersection graph of n unit intervals [LO93, Theorem 1]; see Gardi [Gar07] for a computationally-ecient construction.

Recently Herzog et al. [H&10, E&13] rediscovered unit-interval graphs from an algebraic perspective, which will be discussed in the next chapter. They called them closed graphs and expanded the notion to higher dimensions as well (\closed d-complexes"). Later Matsuda [Mat18] extended this algebraic approach to the broader class of “co-comparability graphs” (or “weakly-closed graphs”), that we shall now describe in terms of their complement.

Any graph can be given an acyclic orientation by choosing a vertex labeling and then by directing all edges from the smaller to the larger endpoint. Every acyclic orientation can be induced this way. (This is not a bijection: different labelings may induce the same orientation). The *drawings of posets*, also called comparability graphs, admit also transitive orientations, namely, orientations such that if $\vec{ab}$ and $\vec{bc}$ are present, so is $\vec{ac}$. Let us rephrase this in terms of a vertex labeling, which happens to be the same as a choice of a linear extension of the poset: Comparability graphs are those graphs $\overline{G}$ that admit a labeling such that, for all $a < b < c$,

$$ab \in \overline{G} \text{ and } bc \in \overline{G} \Longrightarrow ac \in \overline{G}.$$

Not all graphs admit transitive orientations: The pentagon, for example, does not.

Co-comparability graphs, also called *weakly-closed graphs* in [Mat18], are by definition the complements of comparability graphs. So they have a labeling that satisfies the contrapositive of the property above: Namely, for all $a < b < c$,

$$ac \in G \Longrightarrow ab \in G \text{ or } bc \in G. \tag{6}$$

By comparing (4) and (6), it is clear that all interval graphs are co-comparability.

We should mention other two famous properties that all interval graphs enjoy. A graph is *perfect* if its chromatic number equals the size of the maximum clique. For example, even cycles are perfect, but odd cycles are not, because they have chromatic number 3 and maximal cliques of size 2. Note that in poset drawings, a clique (resp. an independent set) is just a chain (resp. an antichain) in the poset, whereas a coloring represents a partition of the poset into antichains. Thus Dilworth's theorem (“for every partially ordered set, the maximum size of an antichain equals the minimum number of chains into which the poset can be partitioned” [Dil50] – see Fulkerson [Ful56] for an easy proof) can be equivalently stated as “every co-comparability graph is perfect”. Not all perfect graphs are co-comparability, as shown by large even cycles.

Last property: A graph is *chordal* if it has no induced subcycles of length ≥ 4. One can characterize chordality in the same spirit of (4), (5) and (6): Namely, a graph is chordal if and only if it admits a labeling such that, for all $a < b < c$,

$$ac, bc \in G \Longrightarrow ab \in G. \tag{7}$$

In fact, if a graph G has a labeling that satisfies (7), then G is obviously chordal, because if c is the highest-labeled vertex in any induced cycle, then its neighbors a and b in the cycle must be connected by a chord by (7). The converse, first noticed by Fulkerson–Gross [FG65], follows recursively from Dirac's Lemma that every chordal graph has a “simplicial vertex”, i.e. a vertex whose neighbors form a clique (cf. [Gol80, p. 83] for a proof). In fact, let us pick any simplicial vertex and label it by n. Then, in the (chordal!) subgraph induced on the unlabeled vertices, let us pick another simplicial vertex and label it by $n-1$; and so on. The result is a labeling that satisfies (7). See [Gol80, pp. 84–87] for two algorithmic implementations.

Now, if the same labeling satisfies (6) & (7), then it trivially satisfies (4); and conversely, if (4) holds, then also (6) & (7) trivially hold. Thus it is natural to conjecture that interval graphs are the same as the co-comparability chordal graphs. The conjecture is true, although the ‘obvious’ proof does not work: Some labelings on chordal graphs satisfy (6) but not (4), like $13, 23, 24$ on the three-edge path. However, Gilmore–Hoffman proved that any labeling that satisfies (6) on a chordal graph (or more generally, on a graph that lacks induced 4-cycles) can be modified in a way that ‘linearly orders’ all maximal cliques [Gol80, Theorem 8.1] and thus satisfies (4). For more characterizations, and a proof that all chordal graphs are perfect, see Golumbic [Gol80, Chapter 4].

2.2 Higher-dimensional analogs and a hierarchy

A d-dimensional extension of Characterization (7) of chordality was provided in 2010 by Emtander [Emt10], and is equivalent to the following:

Denition 28 (chordal). Let be a pure d-dimensional simplicial complex with n vertices. is called chordal if there exists a labeling $1; : : : ; n$ of its vertices (called a \PEO" or \Perfect Elimination Ordering") such that for any two facets $F = a_0a_1 \; a_d$ and $G = b_0 \; b_d$ of with $a_d = b_d$, the complex contains the full d-skeleton of the simplex on the vertex set F [G.

In 2013, Characterization (5) of unit-interval graphs was generalized as well:

Denition 29 (closed [E&13]). Let be a pure d-dimensional simplicial complex with n vertices. is called closed if there exists a labeling $1; : : : ; n$ of its vertices such that for any two facets $F = a_0a_1 \; a_d$ and $G = b_0 \; b_d$ of with $a_i = b_i$ for some i, the complex contains the full d-skeleton of the simplex on the vertex set F [G.

Obviously, closed implies chordal. We now present four notions that in the strongly connected case are progressive weakenings of the closed property (see Theorem 50 and Proposition 54 for the proofs); the rst property still implies chordality, whereas the last three do not. In Section 2.3, we connect all these notions to traceability (Theorem 63). One of these properties is \new" even for $d = 1$: We will see its importance in Chapter 3.

Denition 30 (unit-interval). Let be a pure d-dimensional simplicial complex with n vertices. The complex is called unit-interval if there exists a labeling $1; : : : ; n$ of its vertices such that for any d-face $F = a_0a_1 \; a_d$ of , the complex contains the whole d-skeleton of the simplex with vertex set $fa_0; a_0 + 1; a_0 + 2; : : : ; a_dg$.

Denition 31 (under-closed). Let be a pure d-dimensional simplicial complex with n vertices. The complex is called under-closed if there exists a labeling $1; : : : ; n$ of its vertices such that for any d-face $F = a_0a_1 \; a_d$ of the following condition holds:

- all faces $a_0i_1i_2 : : : i_d$ of $_n^d$ with $i_1 \; a_1; i_2 \; a_2; : : : ; i_d \; a_d$, are in .

Denition 32 (semi-closed). Let be a pure d-dimensional simplicial complex with n vertices. The complex is called semi-closed if there exists a labeling of its vertices such that for any d-face $F = a_0a_1 : : : a_d$ of , at least one of the two following conditions holds:

(i) either all faces $a_0i_1i_2 : : : i_d$ of $_n^d$ with $i_1 \; a_1; i_2 \; a_2; : : : ; i_d \; a_d$, are in , (ii) or all faces $i_0i_1 : : : i_{d \; 1}a_d$ of $_n^d$ with $i_0 \; a_0; i_1 \; a_1; : : : ; i_{d \; 1} \; a_{d \; 1}$ are in .

Denition 33 (weakly-closed). Let be a pure d-dimensional simplicial complex with n ver-tices. is called weakly-closed if there exists a labeling $1; : : : ; n$ of its vertices such that for each d-face $F = a_0a_1 \; a_d$ 2 , for every integer $g \not 2 F$ with $a_0 < g < a_d$, there exists a d-face $G = b_0b_1 \; b_d$ in such that G contains g, G is adjacent to F, and at least one of the following two conditions hold:

(i) either $b_d = a_d$,

(ii) or $b_0 = a_0$.

Remark 34. For $d = 1$, and assuming connectedness:

- \closed 1-complexes" and \unit-interval 1-complexes" are the same as the unit interval graphs; compare Looges{Olario [LO93, Theorem 1] and Matsuda [Mat18, Prop. 1.3].

Several dierent d-dimensional generalizations of chordality exist in the literature, e.g. toric chordality [ANS16] or ridge-chordality, cf. e. g. [BB21]. Emtander chose the name \d-chordal" for what here we call \chordal".

- "under-closed 1-complexes" are the same as the interval graphs, cf. [LO93, Proposition 4].
- "weakly-closed 1-complexes" are the same as the co-comparability graphs; this is clear from the definition we gave, but a proof is also in Matsuda [Mat18, Theorem 1.9].

We will see that "semi-closed 1-complexes" are an intermediate class between the previous two. For example, such class contains the 4-cycle but not the complement of long even cycles, as we will prove in Theorem 50.

Remark 35 ("unit-interval" vs. "chordal"). Suppose F and G are two faces of a complex with $\min F = \min G$. Then any of the two conditions " is closed", " is unit-interval" forces to contain the full d-skeleton of the simplex on the vertex set $F \cup G$. (Instead, the condition " is under-closed" does not suffice: See Remark 36 below). Symmetrically, if F and G are d-faces of with $\max F = \max G$, and is either closed or unit-interval, then must contains the full d-skeleton of the simplex on the vertex set $F \cup G$. For this reason, all unit-interval d-dimensional complexes are chordal.

Remark 36 ("Under-closed" vs. "chordal"). Not all chordal complexes are under-closed: Already for $d = 1$, the chordal graph $G = 12, 13, 14, 23, 25, 36$, known as "3-sun" or "net graph", is neither interval nor co-comparability. However, while all interval graphs are chordal (and co-comparability), the statement "all under-closed d-complexes are chordal" is false for $d > 1$. In fact, we leave it to the reader to verify that the smallest counterexample is the 2-complex

$$\stackrel{\text{def}}{=} 123, 124, 234, 235.$$

The other direction in Gillmore–Hoffman's theorem (namely, "all chordal co-comparability graphs are interval graphs") does not extend to $d > 1$ either, as the next Proposition shows.

Proposition 37. (i) Some chordal simplicial complexes are semi-closed, but not under-closed.

(ii) If a simplicial complex is chordal and semi-closed with respect to the same labeling, then with respect to that labeling the complex is also under-closed.

Proof. (i) The example we found is the complex

$$= 123, 124, 134, 135, 167, 234, 246.$$

The labeling above is a PEO, so is chordal. A convenient relabeling (we leave it to the reader to figure out the bijection from the vertex degrees) allows us to rewrite it as

$$= 123, 256, 345, 346, 347, 356, 456.$$

With this new labeling we see that is weakly- and semi-closed. However, with the help of a software designed by Pavelka [Pav21], we verified that is not under-closed.

(ii) Let be a simplicial complex with a labeling that is a PEO and makes semi-closed. Let $F = a_0 \cdots a_d$ be a face of with gap $F > 0$. Let $G = a_0 b_1 \cdots b_d$ be a different d-face of $_n$ such that $G \leq F$ (componentwise) and $\min G = \min F$. We claim that for any b_i not in F, there exists a d-face A_i of $_n$ that contains b_i, such that $A_i \leq F$ (componentwise) and $\max A_i = \max F$. In fact, by construction $a_0 < b_i \leq b_d \leq a_d$. Since b_i is not in F, there exists a unique $j \in \{0, \ldots, d-1\}$ such that $a_j < b_i < a_{j+1}$. Thus if we set

$$A_i \stackrel{\text{def}}{=} a_0 \cdots a_{j-1} b_i a_{j+1} \cdots a_d$$

the claim is proven. Now, either F satisfies condition (i) of the semi-closed definition, and then $G \in$; or F satisfies condition (ii), in which case all A_i's are in . But by construction, the maximum of all these A_i's is a_d, the same maximum of F. So by chordality, must contain all the d-faces of $_n$ with vertex set contained in

$$F \cup \bigcup_{i \text{ s.t. } b_i \notin F} A_i = \{a_0, a_1, \ldots, a_d\} \cup \{b_1, \ldots, b_d\} = F \cup G.$$

So also in this case $G \in$. □

Remark 38. Part (ii) of Proposition 37 is false if one replaces the assumption "semi-closed" with "weakly-closed": The subcomplex ^{0} = 123; 124; 134; 135; 234 of is weakly-closed and chordal with respect to this labeling, but to prove it under-closed, we need to change labeling.

Remark 39 ("Under-closed" vs. "Shifted"). Recall that a simplicial complex on n vertices is called shifted if for every face F of , and for every face G of the simplex on n vertices, if dim F = dim G and F G componentwise, then also $G \in$. Shifted complexes are obviously under-closed. The converse is false, as shown by the graph 12; 23; 34.

Remark 40. Being shifted is maintained under taking cones, by assigning label 1 to the new vertex. In contrast, G = 12; 13; 23 is closed and chordal, but the cone over it is neither closed nor chordal. In fact, none of the five properties (closed, unit-interval, under-closed, semi-closed, weakly-closed) is maintained under taking cones. A counterexample for all is the unit-interval graph G = 12; 34; 56; 78. The cone over G is the U_4^2 of Lemma 43 below.

Let us start exploring the relations between all the new properties with some Lemmas.

Lemma 41. Let d k 1 be integers. If a pure d-dimensional simplicial complex is unit-interval (resp. under-closed, resp. semi-closed, resp. weakly-closed), then its k-skeleton is also unit-interval (resp. under-closed, resp. semi-closed, resp. weakly-closed).

Proof. It suffices to prove the claim for k = d 1; the general claim follows then by iterating. We prove only the weakly-closed case; the others are easier. Let be a pure weakly-closed d-complex. Let = $a_0 \ldots a_{d-1}$ be a (d 1)-face of . Let $g \notin$ be an integer such that $a_0 < g < a_{d-1}$. Since is pure, there exists a d-face F of that contains . Let v be the vertex of F not in . If v = g, i.e. if F = $\{g\} \cup$, then all the d facets of different than are adjacent to and contain g; if we choose one of these d facets that has either different minimum or different maximum than , we are done. So let us assume that v = g, or equivalently, that F does not contain g. By the weakly-closed assumption, there exists a d-face G in such that G contains g, G is adjacent to F, and G and F do not have same minimum and maximum. If G contains the entire face , i.e. G = $\cup$ g, then again we could conclude as above, choosing some facet of G different than . So we can assume that G does not contain the whole of , or in other words, that the vertex v is present in G. Let be the unique face of G that does not contain v. By construction, and are adjacent, and $g \in$. If and had same minimum and maximum, then also F and G would, because F and G are obtained by adding to and , respectively, the same element v. Hence, the (d 1)-skeleton of is weakly-closed. □

Lemma 42. Let d 2. Let C^{d+1} be the (d + 1)-dimensional simplicial complex with facets H_1 and H_2. The boundary S^d of C^{d+1} is strongly-connected, semi-closed, but not under-closed. The d-skeleton B^d of C^{d+1} is traceable, strongly-connected, unit-interval, but not closed.
In particular, the k-skeleton of a closed complex need not be closed.

Proof. Note that S^d is B^d minus a d-face, so since d 2 the 1-skeleta of B^d and of S^d coincide. The vertices of B^d (respectively, of S^d) can be partitioned with respect to the number of edges containing them, as follows: exactly two vertices have degree $d + 1$, and we shall call them \apices"; the remaining $d + 1$ have degree $d + 2$, and we shall call them \basepoints". The crucial remark is that in B^d (resp. S^d) the two apices are not connected by any edge. We claim that any labeling that makes B^d or S^d closed must assign labels 1 and $d + 3$ to the two apices. In fact:

- If the label 1 is assigned to a basepoint, let $b_1; :::; b_d$ be the other d basepoints and let $v; w$ be the apices, with $v < w$. Then B^d (resp. S^d) contains a d-face F of vertices $f1; b_1; :::; b_{d\ 1}; vg$ and a d-face G of vertices $f1; b_1; :::; b_{d\ 1}; wg$. Note that 1 is in the same position in F and G, yet B^d (or S^d) does not contain the whole d-skeleton of the simplex on F [G, because vw is missing. So the closed condition is not satised.
- Symmetrically, if $d + 3$ is assigned to a basepoint, call $b_1; :::; b_d$ the other basepoints and $v; w$ the apices, with $v < w$. Then B^d (resp. S^d) contains a d-face F of vertices $fv; b_1; :::; b_{d\ 1}; d + 3g$ and a d-face G of vertices $fv; b_1; :::; b_{d\ 1}; d + 3g$. So $d + 3$ is the maximum of both faces, and again B^d (resp. S^d) does not contain the edge vw, so the closed condition is not met.

Next, we claim that any labeling that makes S^d under-closed must assign labels 1 and $d + 3$ to the two apices. (Caveat: This claim is valid only for S^d, since already B^2 is under-closed with the labeling 123; 124; 134; 234; 125; 135; 235, where the apices are 4 and 5.) In fact:

- If the label 1 is assigned to a basepoint, then any other vertex is contained in a facet that contains also 1. The same is true if $d + 3$ is assigned to a basepoint. So either way, there is a face H containing both 1 and $d + 3$. Thus gap $H = 2$. But then if the labeling is under-closed, the complex must contain all three facets $12\,d\,j$, with j 2 $fd + 1; d + 2; d + 3g$. So we found in S^d three dierent facets containing the $(d\ \ 1)$-face $\overset{\text{def}}{=} 12\,d$. This is a contradiction because S^d is topologically a sphere: Every $(d\ \ 1)$-face in it lies in exactly two d-faces.

Thus the two claims are proven. So up to a rotation that does not aect the list of facets, both for B^d and S^d we may focus on the labeling that we introduced from the start. With respect to that labeling, S^d is clearly semi-closed, but it is not under-closed, because the d-face with vertices $2; 3; :::; d + 1; d + 2$ is missing. Similarly, with respect to that labeling, B^d is traceable and unit-interval, but it is not closed for the following reason. Let F (resp. G) be the face of vertices $1; 3; 4; :::; d + 1; d + 2$ (resp. $2; 3; 4; :::; d + 1; d + 3$). Since F (resp. G) is contained in the facet H_1 (resp. H_2) of C^{d+1}, it is in B^d. Yet vertex 3 appears in second position in both F and G. However, the face H_3 of vertices $1; 3; 4; :::; d + 1; d + 3$ contains the edge connecting the two apices, so H_3 is not in B^d. □

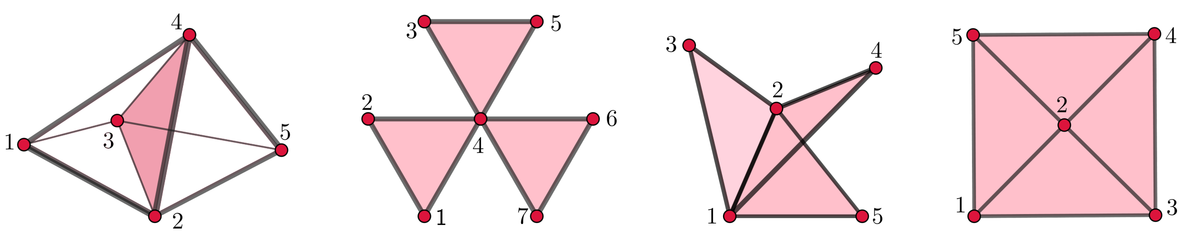


Figure 3: (i) A 2-complex B^2 = 123; 124; 134; 234; 235; 245; 345 that is unit-interval, but not closed; if we remove the triangle 234, we get a 2-complex S^2 that is semi-closed, not under-closed, cf. Lemma 42. (ii) A 2-complex U^2 = 124; 345; 467 that is closed, but not weakly-closed, cf. Lemma 43.
(iii) A 2-complex $^2_3$ = 123; 124; 125 that is under-closed, but not unit-interval, cf. Lemma 44.
(iv) A 2-complex Q^2 = 123; 125; 234; 245 that is weakly-closed, but not semi-closed, cf. Lemma 46.

Lemma 43. Let d and k be positive integers. Let U^d_k be a one-point union of k copies of ^{d}. Then U^d_k is closed if and only if k $d + 1$, and it is weakly-closed if and only if k 2. In particular, for all d 2, the d-complex U^d_{d+1} is closed, but not weakly-closed.

Proof. Let v be the vertex common to all facets. When $k > d + 1$, by the pigeonhole principle there are two facets in which v appears in the same position; were U^d_k closed, its dual graph would have to contain a clique, which is not the case. When k $d + 1$, we force the closed property by giving v a label so that v appears in a dierent position in all facets. We show an algorithm to do this in case $k = d + 1$, leaving the case $k < d + 1$ to the reader. We label v by $f_d = \frac{d+1}{2}^{def} + 1$. We label the vertices of the rst facet by $123\,d\,f_d$: so in the rst facet, v comes last. Then for all $i = 2; 3; \dots; k = d + 1$, we label the i-th facet by using the next available d i integers below f_d, then f_d, then the rst i 1 available integers after f_d. This way in the i-th facet, v comes \i-th last". For example, the labeling we construct for U^3_4, since $f_3 \overset{def}{=} \frac{4}{2} + 1 = 7$, is $U^3_4 = 1237; 4578; 67910; 7111213$.

Finally, suppose that U^d_k is weakly-closed. No face of U^d_k has an adjacent facet. Hence, the labeling satisfying the weakly-closed condition must consist only of gap-0 faces. But labeling all facets with consecutive vertices is possible if and only if k 2. □

Lemma 44. Let k 1 and d 2 be integers. Let ^d_k be the d-dimensional complex on $d + k$ vertices obtained by joining the $(d$ $1)$-simplex $^{d\ 1}$ to a 0-complex consisting of k points. Then

(a) ^d_k is under-closed for all k.

(b) ^d_k is closed, if and only if it is unit-interval, if and only if it is (weakly) traceable, if and only if k 2.

Proof. Let us label the vertices of $^{d\ 1}$ by $1; 2; \dots; d$. This labeling immediately shows that ^d_k is under-closed. Moreover, the d-complex ^d_k is strongly-connected. It has exactly $d + k$ vertices and k facets. When k 2 its dual graph is a path, so clearly the obvious, consecutive labeling makes ^d_k a closed, unit-interval, and traceable complex. But when k 3, the \path of k d-simplices" is not a subcomplex of $_d$. Hence, for k 3 the complex ^d_k is not traceable, not weakly-traceable, and not weakly-Hamiltonian. The fact that ^d_k is neither unit-interval nor closed can be veried either directly, or using Proposition 54 and Theorem 56 below. □

Remark 45. The 1-skeleton of $^2_3$ = $123; 124; 125$ (cf. Figure 3) is the graph

$$G_5 = 12; 13; 14; 15; 23; 24; 25$$

which is under-closed by Lemma 41. It is not dicult to see that G_5 is the smallest 2-connected interval graph that is not Hamiltonian.

Lemma 46. Let d 2 be an integer. Let Q^d be the d-dimensional complex on $d + 3$ vertices obtained by taking d 1 consecutive cones over the square. Then Q^d is weakly-closed, but not semi-closed.

Proof. Both $Q^2 = 123; 125; 234; 245$ and $Q^3 = 1236; 1256; 2346; 2456$ are weakly-closed. If we label further coning vertices using consecutive labels after 6, we claim that the weakly-closed property is maintained. (This is not obvious, as the weakly-closed property is not maintained under arbitrary cones, cf. Remark 40.) In fact, since every face F of Q^3 contains 6, the gap of F equals the gap of F [f7g, and the missing integers are the same, so the calculations proving weakly-closedness end up being the same for Q^3 and Q^4. For the same reasons, one can show that if some Q^d is semi-closed with a labeling that assigns consecutive labels to two apices, then $Q^{d\ 1}$ is semi-closed too. But if d 7, Q^d has 10 vertices, and only 4 of them are not apices; so necessarily two apices are assigned consecutive labels. So to complete the proof we only need to show that $Q^2; Q^3; Q^4; Q^5$ and Q^6 are not semi-closed, which can be veried with [Pav21]. □

Lemma 47. Let be a pure d-complex where every vertex is in at most k facets.

(1) In any labeling that makes weakly-closed, every facet has gap $2k$ 2.

(2) In any labeling that makes semi-closed, every facet has gap k 1.
If in addition $d = 1$ and is a k-regular graph, then in any labeling that makes semi-closed, the k edges of the type $1j$, with 2 j $k + 1$, are all in ; and so are all the k edges of the type in, with n k i n 1.

(3) In any labeling that makes unit-interval, every facet has gap g, where g is the largest integer such that $\binom{g+d}{d}$ k; in particular, every facet has gap $\sqrt[d]{kd!}$ 1.

Proof. For any vertex v of , let $\deg v$ be the number of facets of containing it. For any facet F of , let S_F be the set of integers $i \notin F$ such that $\min F < i < \max F$. By definition, S_F has cardinality equal to $\operatorname{gap} F$. For brevity, set $a \stackrel{def}{=} \min F$ and $b \stackrel{def}{=} \max F$.

(1) For every i in S_F, there is a face G_i adjacent to F that contains the vertex i and exactly d vertices of F, among which exactly one of $a; b$. Clearly as i ranges over S_F, the G_i's are all different. So $\deg a + \deg b$ $\operatorname{gap} F + 2$. (The summand 2 is due to the fact that we should count also F itself, once contributing to $\deg a$ and once to $\deg b$). Since k $\deg a$ and k $\deg b$, we conclude that $\operatorname{gap} F$ $2k$ 2.

(2) For every i in S_F, either contains the n_a $\operatorname{gap} F + 1$ facets (including F itself) with minimum a that are componentwise F, or contains the n_b $\operatorname{gap} F + 1$ facets (including F itself) with maximum b that are componentwise F. Either way, there is a vertex v (either a or b) with $\deg v$ $\operatorname{gap} F + 1$. Since $\deg v$ k by assumption, we conclude that $\operatorname{gap} F$ k 1. So the first claim is settled. From this applied to $d = 1$, it follows that

$$\{ \text{ edges of containing } 1 \, \} \quad \{ 1j \text{ such that } 2 \; j \; k + 1 \}:$$

The two sets above have size $\deg 1$ and k, respectively. If is k-regular, the two quantities are equal, hence the sets coincide. The same argument applies to the edges containing n.

(3) For every i in S_F, by definition of unit-interval, contains the $\binom{\operatorname{gap} F+d}{d}$ d-faces that contain vertex i and have vertices in $\{a; a+1; \dots; b\}$. So we must have $\binom{\operatorname{gap} F+d}{d}$ k. In particular, since $\binom{g+d}{d}$ $\frac{(g+1)^d}{d!}$ for all positive integers $g; d$, we cannot have $\frac{(\operatorname{gap} F+1)^d}{d!} > k$. □

Our next Lemma is a d-dimensional version of the well-known fact that cycles of length 5 or more are not co-comparability, cf. Matsuda [Mat18].

Lemma 48. For n $2d+3$, the d-dimensional annulus $A_n^d \stackrel{def}{=} H_1; H_2; \dots; H_n$ and any k-skeleton of it are not weakly-closed.

Proof. By Lemma 41, it suffices to prove that the 1-skeleton G of A_n^d is not weakly-closed. By contradiction, let $a_1; \dots; a_n$ be a re-labeling of the vertices $1; \dots; n$ (respectively) that proves G weakly-closed. Up to rotating the labeling cyclically, we can assume that a_1 is the smallest of the a_i's. Since n $2d + 3$, in particular n $d > d + 2$, so the labels $a_{n-d}; a_{n-d+1}; \dots; a_n; a_1; a_2; \dots; a_{d+2}$ are all distinct. Were $a_{d+2} < a_n$, we would have a contradiction with the weakly-closed assumption: $a_1 a_n$ is in G, but neither $a_1 a_{d+2}$ nor $a_{d+2} a_n$ is. So $a_n < a_{d+2}$. Symmetrically, were $a_{d+1} > a_{n-d}$, we would have a contradiction: $a_1 a_{d+1}$ is in G, but neither $a_1 a_{n-d}$ nor $a_{n-d} a_{d+1}$ is. So $a_{d+1} < a_{n-d}$. Now let us compare a_{d+1} and a_n:

– If $a_{d+1} > a_n$, then $a_n < a_{d+1} < a_{n-d}$ by what we said above; so we get a contradiction, because the edge $a_n a_{n-d}$ is in G, but neither $a_n a_{d+1}$ nor $a_{d+1} a_{n-d}$ is.

– If $a_{d+1} < a_n$, then $a_{d+1} < a_n < a_{d+2}$ by what we said above; so symmetrically we get another contradiction, because $a_{d+1} a_{d+2}$ is in G, but neither $a_{d+1} a_n$ nor $a_n a_{d+2}$ is. □

Remark 49. A_6^2 is not weakly-closed, even if its 1-skeleton is semi-closed [Pav21]. (A_5^2 instead is weakly-closed.) So the bound $n \;\; 2d + 3$ of Lemma 48 is best possible in general, but if one only cares about A_n^d and not about its skeleta, then it can be improved.

Theorem 50. For each $d \;\; 1$, for (pure) simplicial d-complexes, one has the hierarchy

$$f \text{ unit-interval } g \;(\;\; f \text{ under-closed } g \;(\;\; f \text{ semi-closed } g \;(\;\; f \text{ weakly-closed } g \;(\;\; f \text{ all } g:$$

Proof. All inclusions are obvious except perhaps the third one. Let $F = a_0 a_1 ::: a_d$ be a face of . If F satises condition (i) in the denition of semi-closed, and there is a g such that $a_i < g < a_{i+1}$, then $G^0 \overset{\text{def}}{=} a_0 a_1 \; a_i \, g \, a_{i+1} \; a_{d \;\; 1}$ is componentwise F and thus belongs to ; moreover, since $\max G^0 < \max F$, the face G^0 satises condition (i) in the denition of weakly-closed. If instead F satises condition (ii) in the denition of semi-closed, and $a_i < g < a_{i+1}$ for some g, then $G^{00} \overset{\text{def}}{=} a_1 \; a_i \, g \, a_{i+1} \; a_d$ is componentwise F, so G^{00} is in ; and since $\min G^{00} > \min F$, this G^{00} satises condition (ii) in the denition of weakly-closed.

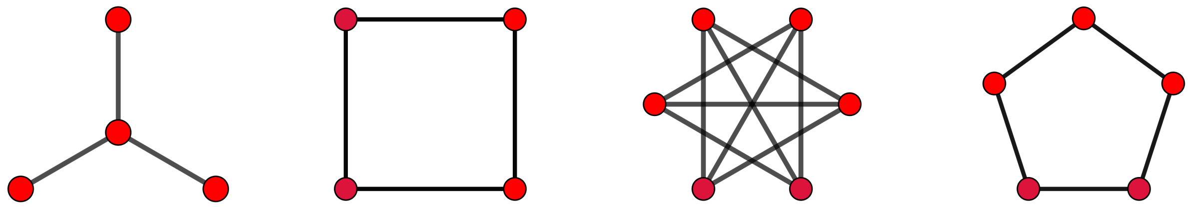


Figure 4: One-dimensional simplicial complexes that are: (i) Not unit-interval, but under-closed. (ii) Not under-closed, but semi-closed. (iii) Not semi-closed, but weakly-closed. (iv) Not even weakly-closed.

Next, we discuss the strictness of the inclusions, which is the interesting part of the theorem.

(i) For $d = 1$, the claw graph $12; 13; 14$ is under-closed only with this labeling, which is not unit-interval because for example 23 is missing.
For $d \;\; 2$, strictness follows by Lemma 44.

(ii) For $d = 1$, the 4-cycle is semi-closed with the labeling $12; 13; 24; 34$. By Lemma 47, part (2), only this labeling makes the 4-cycle semi-closed. This labeling is not under-closed, because 24 is an edge, but 23 is not. More generally, for any $n \;\; 4$, one can show that the graph $\operatorname{susp}(A_{n \;\; 2})$ of Remark 12 is semi-closed (with the suspension apices labeled by 1 and n), but not under-closed.
For $d \;\; 2$, the strictness of the inclusion follows by Lemma 42.

(iii) For $d = 1$: Since C_{2k} is a comparability graph (it is the nonempty-face poset of the k-gon), $\overline{C_{2k}}$ is co-comparability. We claim that $\overline{C_{2k}}$ is not semi-closed for any $k \;\; 3$. For notational simplicity, we give the proof for $k = 3$; the case of arbitrary k has a completely analogous proof. Suppose by contradiction that $\overline{C_6}$ has a semi-closed labeling. Since C_6 is 2-regular, its complement is $(6 \;\; 1 \;\; 2)$-regular, i.e. 3-regular. By Lemma 47, part (2), all of $12; 13; 14$ and $36; 46; 56$ are edges. In contrast, 15, 16 and 26 are not edges, again by Lemma 47. But then 25 must be an edge of $\overline{C_6}$, for otherwise 15, 16, 26 and 25 would form a 4-cycle inside the complement, which is C_6. We claim that this edge 25 cannot satisfy the semi-closed condition. In fact, if all of $23; 24; 25$ were edges, together with 12 we would have 4 edges containing vertex 2, contradicting 3-regularity; and similarly, if all of $25; 35; 45$ were edges, counting also 56 we would have 4 edges containing vertex 5. This shows strictness of the inclusion for $d = 1$; the case $d \;\; 2$ is settled by Lemma 46.

(iv) For any $d \;\; 1$, this is settled by Lemma 48. □

2.3 Shortest dual paths and relation with traceability

As we saw in Lemma 43, there exist complexes like U_3^2 = 124; 345; 467 that are closed but not weakly-closed. So at this point we owe the reader some explanation: Why did we (and before us, Matsuda [Mat18] and others, in the 1-dimensional case) choose to call \weakly-closed" a property not implied by \closed"? Here is the reason. We are going to show that all strongly-connected closed complexes are unit-interval (Proposition 54), so in particular under-closed, semi-closed, and weakly-closed. We will then prove that all such complexes are traceable (Theorem 56), which can be viewed as a higher-dimensional generalization of the graph-theoretical results by Bertossi [Ber83] and Herzog et al's [H&10, Proposition 1.4]. The key to our generalization is to focus on shortest paths in the dual graph.

Denition 51. Let F be a facet a pure d-dimensional simplicial complex . Let v be a vertex of . A shortest path between F and v is a path in the dual graph of of minimal length from F to some facet containing v. The distance between F and v is the length of a shortest path, if any exists, or + 1 , otherwise.

Denition 52. Let be a pure d-dimensional simplicial complex, with vertices labeled from 1 to n. A path $F_0; F_1; :::; F_\ell$ in the dual graph of is called ascending, if each F_i is obtained from F_{i-1} by replacing the smallest vertex of F_{i-1}, with a vertex greater than all remaining vertices of F_{i-1}. A path is called descending, if the reverse path is ascending.

For example, suppose that a 2-complex contains the facets 124; 245; 456, and 356. The dual path they form is not ascending { or better, it is ascending, except for the last step. Such dual path demonstrates that the vertex v = 3 is at distance 3 from 124. Now suppose that we know in advance that is closed: Then from 356; 456 2 , we immediately derive that must contain the whole 2-skeleton of the simplex 3456. Note that the same conclusion could be reached also if we knew in advance that is unit-interval, rather than closed. Either way: contains the facet G = 345 which contains 3 and is adjacent to 245. So 124; 245; 345 yields a \shortcut" to the original path, thereby proving that v = 3 is actually at distance 2 from 124. And it gets even better: Since 245 and 345 are in , by the closed assumption (or the unit-interval assumption) on , we may conclude that contains the whole 2-skeleton of the simplex 2345. So also 234 is in , which means that v = 3 is at distance 1 from 124.

This example generalizes as follows, in what can be viewed as a higher-dimensional version of Cox{Erskine's narrowness property [CE15]:

Lemma 53. Let be a pure d-dimensional simplicial complex, with a labeling that makes it either closed or unit-interval. Let $F = a_0 a_1 a_d$ be a facet of . Let v be a vertex. If the distance between F and v is a nite number ℓ 2, then

- either there is a shortest path from F to v that is ascending (and thus $v > a_d$),
- or there is a shortest path from F to v that is descending (and thus $v < a_0$).

If instead $a_0 < v < a_d$, and some facet containing v is in the same strongly-connected component of F , then the distance between F and v is at most one, and contains the whole d-skeleton of the simplex on the vertex set F [fvg.

Proof. Let

$$F = F_0; :::; F_{i-1}; F_i; F_{i+1}$$

be a shortest path from F to a vertex v 2 F_{i+1}. Suppose the path is ascending until F_i, but it stops being ascending when passing from F_i to F_{i+1}. This means that max F_i = max F_{i+1}. By Remark 35, contains the whole d-skeleton of the simplex with vertex set F_i [F_{i+1}. In particular, if we set $\sigma \overset{def}{=} F_{i-1} \cap F_i$, the complex Δ contains $G \overset{def}{=} \sigma \cup v$. But since G is a d-face that contains v and is already adjacent to F_{i-1},

$$F = F_0, \ldots, F_{i-1}, G$$

is a shorter path from F to v than the one we started with, a contradiction. The same argument applies to descending paths. If instead $a_0 < v < a_d$, clearly there cannot be any ascending or descending path from F to v. So either $v \in F$, in which case the distance from F to v is 0 and there is nothing to prove, or $v \notin F$, in which case the distance is 1. In the latter case, F and the adjacent face G containing v have same maximum, so again by Remark 35 the complex Δ contains the d-skeleton of the simplex on $F \cup G = F \cup \{v\}$. □

Proposition 54. All strongly-connected closed simplicial complexes are unit-interval.

Proof. Let Δ be a strongly-connected d-dimensional simplicial complex that is closed with respect to some-labeling. Let $F = a_0 a_1 \cdots a_d \in \Delta$. We claim the following:

(*) If there exist $m \in \{1, \ldots, d\}$ and $g_1, \ldots, g_m$ not in F, with $a_0 < g_1 < g_2 < \cdots < g_m < a_d$, then Δ contains the d-skeleton of the simplex with vertex set $\{a_0, \ldots, a_d, g_1, \ldots, g_m\}$.

If $\mathrm{gap}(F) = 0$, then the implication is trivially true, because the antecedent is never verified. So suppose $\mathrm{gap}(F) > 0$, and let us proceed by induction on m.

For $m = 1$: Pick a vertex g of Δ not in F, with $a_0 < g < a_d$. Since Δ is strongly connected, by the second part of Lemma 53 the complex Δ has a facet G that contains g and is adjacent to F. Had G neither same minimum nor same maximum of F, then either $G = a_1 a_2 \cdots a_d g$ or $G = g a_0 a_1 \cdots a_{d-1}$. But both cases contradict the assumption $a_0 < g < a_d$. Hence, F and G have either same minimum or same maximum (or both), so they share at least one vertex in the same position. Since Δ is closed, Δ contains the d-skeleton of the simplex on $F \cup G = F \cup \{g\}$.

For $m > 1$: let H be a subset of $\{a_0, \ldots, a_d, g_1, \ldots, g_m\}$ of cardinality $d + 1$. If H contains at most $m - 1$ elements of $\{g_1, \ldots, g_m\}$, then we know that $H \in \Delta$ by the inductive assumption. If $g_1, \ldots, g_m$ are all vertices of H, let us consider a new face H' with exactly the same vertices of H, except for one replacement, to be decided as follows:

- If $\min H = a_0$ and $\max H = a_d$, we shall replace g_1 with any vertex v of F that is not in H. This way, since $a_0 \le v \le a_d$, we have that as real intervals

$$(\min H, \max H) = (a_0, a_d) = (\min H', \max H').$$

- If $\min H = g_1$, or if $\min H = a_i$ for some $i > 0$, we shall replace g_1 with a_0. This way

$$(\min H, \max H) \subsetneq (a_0, \max H) = (\min H', \max H').$$

- If $\max H = g_m$, or $\max H = a_i$ for some $i < d$, we shall replace g_m with a_d. This way

$$(\min H, \max H) \subsetneq (\min H, a_d) = (\min H', \max H').$$

In all three cases, if w is the only element that belongs to H but not to H', then w is either g_1 or g_m, and we have

$$\min H' < w < \max H'.$$

Moreover, H' contains at most $m - 1$ elements of $\{g_1, \ldots, g_m\}$, so by the inductive assumption H' is in Δ. But since $\min H' < w < \max H'$, by the second part of Lemma 53 we conclude that also H is in Δ. By the genericity of H, this proves Claim (*). From the Claim the conclusion follows immediately, by choosing m maximal. □

Remark 55. The converse is false: The complex with k disjoint d-simplices is obviously not strongly-connected, yet it is unit-interval with the natural labeling below:

$$= H_1; H_{d+2}; H_{2d+3}; \ldots; H_{(k \quad 1)d+k}:$$

For connected graphs, it is obvious that \closed" and \unit-interval" are the same: This is noticed also in Matsuda [Mat18, Proposition 1.3] and Crupi{Rinaldo [CR14]. However, as we saw in Lemma 42, higher-dimensional complexes that are both strongly-connected and unit-interval might not be closed.

We have arrived to the main result of this section, the generalization of Bertossi's theorem:

Theorem 56 (Higher-dimensional Bertossi). Let be a pure d-dimensional simplicial complex that is either closed or unit-interval. Then

$$\text{is strongly-connected} \quad ( ) \quad \text{is traceable.}$$

Proof.
(: Let F be a d-face of . We want to nd a walk from to H_1 in the dual graph. If gap F = 0, then $F = H_j$ for some j, and $H_1; H_2; \ldots; H_j$ is the desired path. If gap $F > 0$, let $i \stackrel{\text{def}}{=} \min F$. Since F and H_i have same minimum, by Remark 35 contains the whole d-skeleton of the simplex on F [H_i. But the d-skeleton of a higher-dimensional simplex is strongly-connected, which means that in the dual graph of we can walk from F to H_i. And since H_i has gap 0, we can walk from it to H_1.

) : Fix a labeling for which is (almost-)closed. We are going to show by induction on j that with the same labeling, every H_j is in . For $j = 1$, since is pure, it contains a face $F = a_0 a_1 \, a_d$ with $a_0 = 1$, and then it is easy to derive (either directly, or using that the labeling satises the under-closed condition by Theorem 50) that H_1 is in . Now suppose that contains H_j and let us show that contains H_{j+1}. By Lemma 53, has a d-face H^0 that contains $d + j + 1$ and is adjacent to H_j. Such H^0 has the same vertices of H_j, with the exception of a single vertex i that was replaced by $d + j + 1$. Now either $i = j$, in which case $H^0 = H_{j+1}$ and we are done; or $i > j$. If $i > j$, then j was not replaced, so it is still present in H^0. Hence H^0 and H_j are adjacent faces with the same minimum, namely, j. By Remark 35, this implies that H_{j+1} is in . □

Remark 57. If the \unit-interval" assumption is weakened to \under-closed", then the direction \ ) " of Theorem 56 no longer holds, with $K_{1;3}$ playing the usual role of the counterexample. The direction \ (" instead is still valid. We claim in fact that all weakly-closed traceable complexes are strongly-connected. To see this, it suces to show that from any d-face F of positive gap we can walk in the dual graph to some gap-0 face. But the weakly-closed denition tells us how to move in the dual graph from F to a face F_0 of smaller gap than F. So if we iterate this, eventually we get from F to a gap-0 face. (The same type of argument is carried out in details in the proof of Theorem 63, item (5), below.) That said, the \weakly-closed" assumption is needed for \ (". In fact, for any d 2, if $G_d \stackrel{\text{def}}{=}$ f1; $d + 2; 2d + 3; \ldots; (k \quad 1)d + k; kd + (k+1); \ldots; d^2 + d + 1$g, then the traceable d-complex with $d^2 + d + 1$ vertices $= H_1; H_2; \ldots; H_{d^2}; H_{d^2+1}; G_d$ is not strongly-connected. Its dual graph is a path of length $d^2 + 1$ plus an isolated vertex.

Generalizing a result by Chen, Chang, and Chang [CCC97, Theorem 2], we can push Theorem 56 a bit further. If D is a simplicial complex obtained from by deleting some vertices $v_1; \ldots; v_k$, then any labeling of naturally induces a compressed labeling for D, just by ordering the vertices of D in the same way as they are ordered inside . For example, if $= 123; 134; 345$, the compressed labeling for $D = \text{del}(2;)$ is $123; 234$. A priori, this D need not be pure.

Lemma 58. Let ^{0} be a d-dimensional simplicial complex obtained by deleting some vertices from a d-dimensional simplicial complex . If is unit-interval (resp. under-closed, resp. semi-closed), then so is ^{0}.

Proof. If the original labeling satised the unit-interval (resp. under-closed, resp. semi-closed) condition, so does the compressed labeling. □

Lemma 59. Let be a d-dimensional strongly-connected simplicial complex, with a labeling that makes it unit-interval. The following are equivalent:

(a) The deletion of d or less vertices, however chosen, yields a d-complex that is strongly connected.

(b) The deletion of d or less vertices, however chosen, yields a pure d-complex that with the compressed labeling is traceable.

(c) contains all faces of gap d.

Proof. (a) , (b): By Lemma 58 the compressed labeling satises the unit-interval condition. Via Theorem 56, we conclude.

(b)) (c): By deleting zero vertices we notice that is itself traceable. Let F = $a_0 a_d$ be any d-face of ${}^d_n$ that has gap d. If gap(F) = 0, then F is one of $H_1; :::; H_{n \; d}$, so F is in by denition of traceable. Otherwise, set S_F = f j $\overset{\text{def}}{2}$ F such that $a_0 < j < a_d$g. Let ^{0} be the complex obtained from by deleting the vertices in S_F, which are at most d. By assumption, ^{0} is traceable with the \compressed labeling". So ^{0} contains a gap-0 face of minimum a_0. But by how the compressed labeling is dened, this face has exactly the vertices that in the original labeling for were called $a_0; a_1; :::; a_d$. So F is in .

(c)) (b): Let ^{0} be the d-complex resulting from the deletion. With the compressed labeling, ^{0} is traceable, because any gap-0 d-face of ^{0} with the compressed labeling, is a d-face of that had gap d in the original labeling. It remains to see that ^{0} is pure. We prove that ^{0} has no facets of dimension d 1, leaving the case of facets of even lower dimensions to the reader. We claim that every (d 1)-face of lies in at least d + 1 distinct d-faces of . From the claim the conclusion follows via the pigeonhole principle: If we delete d vertices, however chosen, then at least one of the d-faces containing will survive the deletion, which implies that is not a facet in ^{0}.
So let us prove the claim. Let = $b_0 b_{d \; 1}$. If $b_{d \; 1}$ b_0 d + 1 = $\overset{\text{def}}{\text{gap}}$() d, then $b_{d \; 1}$ + 1 b_0 + 2d. So for each i in the (d + 1)-element set

$$T \overset{\text{def}}{=} fb_0; b_0 + 1; :::; b_{d \; 1}; b_{d \; 1} + 1; :::; b_0 + 2dg \; n \; fb_0; b_1; :::; b_{d \; 1}g$$

the d-face [fig has gap d, and thus is in by assumption. If instead gap() d + 1, we use the unit-interval assumption: for every i in S $\overset{\text{def}}{=}$fi 2 such that $b_0 < i < b_{d \; 1}$g, the d-face [fig is in . So either way the claim is proven. □

Theorem 60 (Higher-dimensional Chen{Chang{Chang). Let be a pure d-dimensional simplicial complex.

- If is unit-interval and the deletion of d vertices, however chosen, yields a strongly-connected d-complex, then is Hamiltonian.
- If is weakly-closed and Hamiltonian, the deletion of 1 vertices, however chosen, yields a strongly-connected d-complex.

Proof. For the second claim: Up to a cyclic reshuing, the vertex we wish to delete is n. The argument of Remark 57 yields a dual path in from each d-face F to H_1. If F does not contain n, none of the d-faces in such dual path does, so the path belongs to the dual graph of the deletion of n from .

Now we prove the rst claim. By Lemma 59, contains all d-faces of gap d. In particular: • for any odd i such that 1 i n 2d, contains the gap-d face O_i formed by i and by the rst d consecutive odd integers after i;

- for any even j such that 2 j n 2d, contains the gap-d face E_j formed by j and by the rst d consecutive even integers after j;
- contains the gap-(d 1) face F = 1; 2; 4; : : : ; 2d formed by 1 and by the d smallest even natural numbers;
- contains the gap-(d 1) face G formed by the largest even integer n and by the d largest odd integers n.

Now consider the following sequence C of d-faces in : First all O_i's in increasing order, then G, then all E_j's in decreasing order, then F. Note that any two O_i's are adjacent, and the last of them is adjacent to G; symmetrically, any two E_j's are adjacent, and F is adjacent to E_2. We claim that this sequence would form a weakly-Hamiltonian cycle if we relabeled the vertices of rst by listing the odd ones increasingly, and then the even ones decreasingly.

Formally, if n is odd, we introduce the new labeling

$$‘_1 \overset{\text{def}}{=} 1;\ ‘_2 \overset{\text{def}}{=} 3;\ ‘_3 \overset{\text{def}}{=} 5; : : : ;\ ‘_{\frac{n+1}{2}} \overset{\text{def}}{=} n;\ ‘_{\frac{n+1}{2}+1} \overset{\text{def}}{=} n \quad 1;\ ‘_{\frac{n+1}{2}+2} \overset{\text{def}}{=} n \quad 3; : : : ;\ ‘_{n \quad 1} \overset{\text{def}}{=} 4;\ ‘_n \overset{\text{def}}{=} 2:$$

And if instead n is even, we introduce the new labeling

$$‘_1 \overset{\text{def}}{=} 1;\ ‘_2 \overset{\text{def}}{=} 3;\ ‘_3 \overset{\text{def}}{=} 5; : : : ;\ ‘_{\frac{n}{2}} \overset{\text{def}}{=} n \quad 1;\ ‘_{\frac{n}{2}+1} \overset{\text{def}}{=} n;\ ‘_{\frac{n+1}{2}+2} \overset{\text{def}}{=} n \quad 2; : : : ;\ ‘_{n \quad 1} \overset{\text{def}}{=} 4;\ ‘_n \overset{\text{def}}{=} 2:$$

Let us set $L_1 \overset{\text{def}}{=} ‘_1 ‘_2 ‘_{d+1}$, $L_2 \overset{\text{def}}{=} ‘_2 ‘_3 ‘_{d+2}$, and so on. Then the sequence C described above is equal (whether n is even or odd) to

$$L_1; L_2; : : : ; L_{b\frac{n+1}{2}c \quad (d \quad 1)}; L_{b\frac{n+1}{2}c+1}; L_{b\frac{n+1}{2}c+2}; : : : ; L_{n \quad d}; L_{n \quad (d \quad 1)}:$$

This shows that with the new labeling is weakly-Hamiltonian. It remains to show for d 2 that our weakly-Hamiltonian cycle can indeed be 'completed' to a Hamiltonian cycle, in the sense that the L_i's that were not mentioned in C are anyway contained in . First of all, note that with the original labeling contained all the d-faces of gap d, so in particular it contained all d-faces containing 1 and with vertex set contained in F [O_1. This shows that with the new labeling, $L_{n \quad (d \quad 2)}$, : : :, L_n are all in . So it remains to consider the missing L_i's from the 'center' of the sequence C. For the \n odd" case (the case for n even is analogous), we have to see whether contains also the d 1 facets

$$L_{\frac{n+1}{2} \quad d+2};\ L_{\frac{n+1}{2} \quad d+3};\ : : : ;\ L_{\frac{n+1}{2}}:$$

When we translate these d-faces back into the old labeling, it is easy to see that the face with the largest gap is the last one, which has gap d 1. So all these faces are in by assumption. □

Example 61. Let be an unit-interval 3-complex on n = 9 vertices that contains all tetrahedra with gap 3. With the notation of Theorem 60 the complex contains the sequence C below:

$$O_1 = 1357;\ O_2 = 3579;\ G = 5789;\ E_2 = 2468;\ F = 1246:$$

If we relabel the vertices as in the proof of Theorem 60, the list above becomes

$$L_1;\ L_2;\ L_3;\ L_6;\ L_7:$$

Thus is weakly-Hamiltonian. To prove that it is Hamiltonian, we need to check that L_4; L_5 and L_8; L_9 are in . Translated into the original labeling, this means checking that 6789; 4689 and 1234; 1235 are in , which is clearly the case because they all have gap 2.

Remark 62. For $d = 1$, Theorem 60 boils down to Chen{Chang{Chang's result that \unit interval graphs are Hamiltonian if and only if they are 2-connected" [CCC97, Theorem 2]. The G_5 of Remark 45 is 2-connected and not Hamiltonian; hence the \unit-interval" assumption in the rst claim of Theorem 60 is necessary. As for the second claim, the \weakly-closed" assumption is necessary for $d > 1$, because we saw in Remark 2 that some Hamiltonian d-complexes are not strongly-connected.

We may condense most of the results of this chapter in the following summary:

Theorem 63. Let be a d-dimensional simplicial complex.
(1) If is closed (or unit-interval) and strongly connected, then is traceable.
(2) If is closed (or unit-interval), and the deletion of d or less vertices, however chosen, yields a strongly connected complex, then is Hamiltonian.
(3) If is under-closed, it contains H_1. If in addition has a face of minimum i for each i 2 f2; : : : ; n dg, then is traceable.
(4) If is semi-closed, then for every face $F = a_0 a_d$ of either H_{a_0} or H_{a_d} is in . (5) If is weakly-closed, then contains at least one of the H_i's.
If in addition contains H_1, plus a face with minimum i and of gap smaller than d for each i in f2; : : : ; n dg, then is weakly-traceable.

Proof. (1) This is given by Proposition 54 and Theorem 56 above.
(2) This is given by Proposition 54 and Theorem 60 above.
(3) By denition of under-closed, if has a face of minimum i, then contains H_i. The fact that has a face of minimum 1 follows from the assumption that is pure.
(4) This is straightforward from the denition of semi-closed.
(5) Let $F = a_0 a_1 a_d$ be any facet of with $\text{gap}(F) > 0$. Let g 2 F such that $a_0 < g < a_d$. By denition of \weakly-closed", some face $G = b_0 b_1 b_d$ of contains g, is adjacent to F, and has either $b_0 = a_0$ or $b_d = a_d$. Thus $\text{gap}\, G < \text{gap}\, F$. Iterating the process, eventually we nd in a gap-0 face, which has to be one of

$$H_{a_0};\ H_{a_0+1}; : : : ; H_{a_d \quad d}:$$

As for the second claim: By assumption, contains H_1. Also, contains $H_{n \quad d}$, because no other face has minimum n d. Now let $H^0 = 2a_1 a_d$ be a face of with minimum 2 and gap d 1. By the argument above, we know that must contain at least one of

$$H_2; H_3; : : : ; H_{a_d \quad d}:$$

Let us call this face H_{i_2}. By how H^0 was chosen,

$$2 \quad i_2 \quad a_d \qquad d = \text{gap}(H^0) + 2 \quad d + 1:$$

But since H_1 contains all vertices from 1 to $d + 1$, in particular it contains i_2. So H_{i_2} is incident with H_1. Now let $H^{00} = a_0 a_1 a_d$ be a face of with gap smaller than d, and minimum $a_0 = i_2 + 1$. Repeating the argument above, contains one of

$$H_{i_2+1}; H_{i_2+2}; : : : ; H_{a_d \quad d}:$$

Call this facet H_{i_3}; as above, it must intersect H_{i_2}. And so on. Eventually, we obtain a list $H_1 = H_{i_1}; H_{i_2} : : : ; H_{i_k \ 1}; H_{i_k} = H_{n \quad d}$ of facets of that makes it weakly-traceable. □

Remark 64. In the previous theorem, a relabeling was necessary only to prove item (2). For all other items, the original labeling was already suitable for the desired conclusion. So for item (1) we proved a slightly stronger statement: \If is strongly-connected, then any labeling that makes unit-interval automatically makes traceable". Same for items (3), (4), (5).

3 Algebraic motivation

In this section, we review Ene et al's definition of determinantal facet ideals [E&13]. We find out a large class of them that are radical. In fact, we prove the following:

- If a simplicial complex is semi-closed, then its determinantal facet ideal has a square-free Gröbner degeneration (and in particular is radical), and the quotient by such ideal in positive characteristic is F-pure (Theorem 77).
- If the simplicial complex is unit-interval, then the natural generators of its determinantal facet ideal form a Gröbner basis with respect to a diagonal term order (Theorem 82). Moreover, the converse is true if with respect to the same labeling, the simplicial complex is traceable (Theorem 87).

3.1 A foreword on F-pure rings, F-split rings, and Knutson ideals

Let p be a prime number. Let R be a ring of characteristic p. Recall that the Frobenius map is the ring homomorphism from R to itself that maps an element $r \in R$ to r^p. We denote by F_*R the R-module defined as follows: $F_*R \stackrel{def}{=} R$ as additive group, and $r \cdot x \stackrel{def}{=} r^p x$ for all $r \in R$ and $x \in F_*R$. This allows us to view the Frobenius map as a map of R-modules,

$$F : R \longrightarrow F_*R$$
$$r \longmapsto r^p.$$

The ring R is reduced if and only if F is injective. So the following definitions are natural:

Definition 65. R is F-pure if $F \otimes 1_M : M \to F_*R \otimes_R M$ is injective for any R-module M.

Definition 66. R is F-split if there exists a homomorphism $\phi : F_*R \to R$ of R-modules such that $\phi \circ F = 1_R$. Such a ϕ is called an F-splitting of R.

If a ring is F-split, it is clearly F-pure. The converse does not hold in general. However, the two concepts are equivalent in a number of cases, for example:

Lemma 67. Let $R = \bigoplus_{i \in \mathbb{Z}} R_i$ be a Noetherian graded ring of characteristic p having a unique homogeneous ideal m that is maximal with respect to inclusion. Furthermore, assume that the Noetherian local ring R_0 is complete. Then the following are equivalent:

(a) R is F-split.

(b) R is F-pure.

(c) $F \otimes 1_E : E \to F_*R \otimes_R E$ is injective, where E is the injective hull of R/m.

Proof. (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) are obvious implications. To see (c) $\Longrightarrow$ (a): the map

$$F \otimes 1_E : E \longrightarrow F_*R \otimes_R E$$

is injective if and only if the corresponding map

$$\mathrm{Hom}_R(F_*R; \mathrm{Hom}_R(E; E)) = \mathrm{Hom}_R(F_*R \otimes_R E; E) \longrightarrow \mathrm{Hom}_R(E; E)$$

is surjective. Hence, by [BH93, Corollary 3.6.7, Proposition 3.6.16, Theorem 3.6.17], the corresponding map $\Phi : \mathrm{Hom}_R(F_*R; R) \to R$ is surjective. So there exists $\phi \in \mathrm{Hom}_R(F_*R; R)$ such that $\Phi(\phi) = 1$. On the other hand, by construction $\Phi(\phi) = \phi(F(1))$, so $\phi \circ F = 1_R$. □

Since we want to study homogeneous quotients of a polynomial ring over a field, by Lemma 67 we may as well regard the F-split notion and the F-pure notion as equivalent.

In the following the concept of Knutson ideal will be fundamental. The name arises from the work of Knutson [Knu09], later systematically investigated by the second author [Sec20], who extended several properties from $\mathbb{Z}/p\mathbb{Z}$ to any field. The result from [Sec20] that we shall need is the following:

Theorem 68 (Seccia [Sec20]). *Let K be a field. Let $g \in S = K[x_1, \dots, x_n]$ be a polynomial with $\mathrm{in}_<(g)$ square-free for some term order $<$ on S. Let $\mathcal{C}_g$ be the smallest set of ideals of S containing (g) and such that:*

1. $I \in \mathcal{C}_g \implies I : h \in \mathcal{C}_g$ *whenever* $h \in S$,
2. $I, J \in \mathcal{C}_g \implies I + J \in \mathcal{C}_g,\ I \cap J \in \mathcal{C}_g$.

If $I \in \mathcal{C}_g$, then $\mathrm{in}_<(I)$, and therefore I, is radical. Furthermore, if $I, J \in \mathcal{C}_g$, then $\mathrm{in}_<(I + J) = \mathrm{in}_<(I) + \mathrm{in}_<(J)$ and $\mathrm{in}_<(I \cap J) = \mathrm{in}_<(I) \cap \mathrm{in}_<(J)$. Finally, if K has positive characteristic, S/I is F-pure whenever $I \in \mathcal{C}_g$.

Example 69. It can be shown that, if $g = x_1 x_2 \cdots x_n$, then $\mathcal{C}_g$ is the set of squarefree monomial ideals.

3.2 Determinantal facet ideals: basic properties

Let d, n be positive integers with $d + 1 \leq n$. Let $S \stackrel{\text{def}}{=} K[x_{ij} : i = 1, \dots, n, j = 0, \dots, d]$ be a polynomial ring in $(d + 1)n$ variables over some field K. Set

$$X = \begin{bmatrix} x_{01} & x_{02} & \cdots & x_{0n} \\ x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{d1} & x_{d2} & \cdots & x_{dn} \end{bmatrix}.$$

Given $1 \leq r \leq d$, and integers $0 \leq a_0 < a_1 < \dots < a_r \leq d$ and $1 \leq b_0 < \dots < b_r \leq n$, an $(r + 1)$-minor of X is any element of the form

$$[a_0 a_1 \dots a_r | b_0 b_1 \dots b_r] \stackrel{\text{def}}{=} \det \begin{bmatrix} x_{a_0 b_0} & x_{a_0 b_1} & \cdots & x_{a_0 b_r} \\ x_{a_1 b_0} & x_{a_1 b_1} & \cdots & x_{a_1 b_r} \\ \vdots & \vdots & \cdots & \vdots \\ x_{a_r b_0} & x_{a_r b_1} & \cdots & x_{a_r b_r} \end{bmatrix}.$$

If $r = d$, the row indices are forced to be $a_0 = 0, a_1 = 1, \dots, a_d = d$. For this reason we denote $[01 \dots d | b_0 b_1 \dots b_d]$ simply by $[b_0 b_1 \dots b_d]$. The ideal of S generated by the $r + 1$-minors of X is denoted by $I_{r+1}(X)$. This ideal defines the variety of $(d + 1)n$ matrices with entries in K and with rank at most r. The set Π of all the minors of X can be partially ordered by the relation

$$[a_0 a_1 \dots a_r | b_0 b_1 \dots b_r] \leq [c_0 c_1 \dots c_s | d_0 d_1 \dots d_s] \stackrel{\text{def}}{\iff} r \geq s,\ a_i \leq c_i \text{ and } b_i \leq d_i\ \forall\, i = 0, \dots, s.$$

In particular, for maximal minors the previous definition restricts to

$$[a_0 a_1 \dots a_d] \leq [b_0 b_1 \dots b_d] \iff a_0 \leq b_0, a_1 \leq b_1, \dots, a_d \leq b_d.$$

It is not our intent to review the theory of Algebras with Straightening Law here, as the interested reader can learn it directly from the standard source [BV88]. However, we wish to introduce a few concepts for the sake of clarity. The starting observation is that the polynomial ring S is generated by Π as a K-algebra. In fact, a basis of S as K-vector space is given by

$$\{\pi_1 \cdots \pi_m : m \in \mathbb{N},\ \pi_i \in \Pi,\ \pi_1 \leq \pi_2 \leq \cdots \leq \pi_m\}.$$

The elements of this K-basis are called standard monomials. It may happen that the product of two standard monomials is not a standard monomial. However, such product will be uniquely writable as K-linear combination of standard monomials, which is in some sense compatible with the poset structure on . This is what is known as 'Straightening Law'; compare [BV88, Theorem 4.11]. What we wish to outline is that the ideals of S generated by poset ideals of (i.e. subsets such that for all ! 2 , 2 , ! =) 2) are particularly nice.

Example 70. For any r d, the ideal $I_{r+1}(X)$ is generated by the poset ideal $_{r+1}$ of all t-minors of X with t r + 1. This $_{r+1}$ has a unique maximal element, [d r ::: djn r ::: n].

Some new notation: if 1 i < j n, by $X_{[i;j]}$ we mean the matrix

$$X_{[i;j]} = \begin{bmatrix} x_{0i} & x_{0;i+1} & \cdots & x_{0j} \\ x_{1i} & x_{1;i+1} & \cdots & x_{1j} \\ \vdots & \vdots & \cdots & \vdots \\ x_{di} & x_{d;i+1} & \cdots & x_{dj} \end{bmatrix};$$

so $I_{r+1}(X_{[i;j]})$ is the ideal of S generated by the r+1-minors of $X_{[i;j]}$, whenever r minfd; j ig.

Eventually, we say that a term order < on S is a diagonal term order if, for all 1 r d and integers 0 $a_0 < a_1 < \cdots < a_r$ d and 1 $b_0 < \cdots < b_r$ n, $in_<([a_0 a_1 \dots a_r | b_0 b_1 \dots b_r]) = x_{a_0 b_0} x_{a_1 b_1} x_{a_r b_r}$. For example, the lexicographic term order on S extending the linear order of the variables given by $x_{ij} > x_{hk}$ if and only if i < h or i = h and j < k is a diagonal term order. We will use the following result from [Stu90]:

Theorem 71 (Sturmfels [Stu90]). If < is a diagonal term order, 1 i < j n and r minfd; j ig, then $f[a_0 a_1 \dots a_r | b_0 b_1 \dots b_r] : 0$ $a_0 < a_1 < \cdots < a_r$ d and i $b_0 < \cdots < b_r$ jg is a Grœbner basis of the $I_{r+1}(X_{[i;j]})$.

So far, by a \simplicial complex on n vertices" we have always implicitly assumed that each vertex i = 1; : : : ; n appears in the complex. From now on, we will drop this convention, i.e. henceforth a simplicial complex on a set A is also a simplicial complex on any nite set B A.

Denition 72. Let be a d-dimensional simplicial complex on n vertices. Let K be any eld. Let $S = K[x_{ij} : i = 1; \dots; n; j = 0; \dots; d]$. The determinantal facet ideal of is the ideal

$$J := ([a_0 a_1 \dots a_d] : a_0 a_1 \dots a_d \ 2 \ ) \ S:$$

When d = 1, then is a graph, and J is the binomial edge ideal of . Binomial edge ideals have been intensively studied in the recent literature: Among the many papers on this topic, see for example [H&10], [Oht11], [MM13], [Mat18]. Unlike binomial edge ideals, determinantal facet ideals are not always radical { not even if the complex is weakly-closed:

Example 73. Consider the weakly-closed 2-dimensional simplicial complex on ve vertices

$$= 124; 145; 234; 345:$$

Thus in the polynomial ring with 15 variables $x_{i;j}$, for i 2 f0; 1; 2g and j 2 f1; : : : ; 5g, the ideal J is generated by the four degree-3 polynomials

$$x_{0;4}x_{1;2}x_{2;1} + x_{0;2}x_{1;4}x_{2;1} + x_{0;4}x_{1;1}x_{2;2} \quad x_{0;1}x_{1;4}x_{2;2} \quad x_{0;2}x_{1;1}x_{2;4} + x_{0;1}x_{1;2}x_{2;4};$$

$$x_{0;5}x_{1;4}x_{2;1} + x_{0;4}x_{1;5}x_{2;1} + x_{0;5}x_{1;1}x_{2;4} \quad x_{0;1}x_{1;5}x_{2;4} \quad x_{0;4}x_{1;1}x_{2;5} + x_{0;1}x_{1;4}x_{2;5};$$

$$x_{0;4}x_{1;3}x_{2;2} + x_{0;3}x_{1;4}x_{2;2} + x_{0;4}x_{1;2}x_{2;3} \quad x_{0;2}x_{1;4}x_{2;3} \quad x_{0;3}x_{1;2}x_{2;4} + x_{0;2}x_{1;3}x_{2;4};$$

$$x_{0;5}x_{1;4}x_{2;3} + x_{0;4}x_{1;5}x_{2;3} + x_{0;5}x_{1;3}x_{2;4} \quad x_{0;3}x_{1;5}x_{2;4} \quad x_{0;4}x_{1;3}x_{2;5} + x_{0;3}x_{1;4}x_{2;5}:$$

It can be checked using the software Macaulay 2 [GSm2] that J is not radical.

Determinantal facet ideals are multi-graded. To see this, we endow S with the multi-grading defined by $\deg(x_{ij}) = e_j \in \mathbb{N}^n$ for all $i = 0,\dots,d$; $j = 1,\dots,n$. Here e_j is the vector with a one in position j, and zeroes everywhere else. With such grading J_Δ is homogeneous, and S/J_Δ admits a multi-graded minimal free resolution

$$0 \to \bigoplus_{v\in\mathbb{N}^n} S(-v)^{\beta_{p,v}} \to \cdots \to \bigoplus_{v\in\mathbb{N}^n} S(-v)^{\beta_{1,v}} \to S \to S/J_\Delta \to 0,$$

where p is the projective dimension of S/J_Δ. We set $|v| = v_1+\dots+v_n$ for each $v = (v_1,\dots,v_n) \in \mathbb{N}^n$; this way the graded Betti numbers with respect to the standard grading are

$$\beta_{i,j} = \sum_{\substack{v\in\mathbb{N}^n\\ |v|=j}} \beta_{i,v}.$$

In particular, $\operatorname{reg}(S/J_\Delta) = \max\{|v| - i : \beta_{i,v} \neq 0\}$. In the next result, inspired by [MM13, Lemma 2.1] $\operatorname{supp}(v) = \{i : v_i \neq 0\} \subseteq [n]$ for each $v = (v_1,\dots,v_n) \in \mathbb{N}^n$. For each subset $W \subseteq \{1,\dots,n\}$, by Δ_W we denote the subcomplex of Δ induced on W.

Proposition 74. Let Δ be a d-dimensional simplicial complex on n vertices and $W \subseteq [n]$. Whenever $v \in \mathbb{N}^n$ is such that $\operatorname{supp}(v) \subseteq W$,

$$\beta_{i,v}(S/J_\Delta) = \beta_{i,v}(S/J_{\Delta_W}) \qquad \forall\, i \in \mathbb{N}.$$

In particular, $\operatorname{reg}(S/J_\Delta) \geq \operatorname{reg}(S/J_{\Delta_W})$.

Proof. Let F be the multi-graded minimal free resolution of S/J_Δ:

$$F: 0 \to \bigoplus_{v\in\mathbb{N}^n} S(-v)^{\beta_{p,v}} \to \cdots \to \bigoplus_{v\in\mathbb{N}^n} S(-v)^{\beta_{1,v}} \to S \to 0.$$

Consider the complex of multi-graded S-modules

$$F^0: 0 \to \bigoplus_{\substack{v\in\mathbb{N}^n\\ \operatorname{supp}(v)\subseteq W}} S(-v)^{\beta_{p,v}} \to \cdots \to \bigoplus_{\substack{v\in\mathbb{N}^n\\ \operatorname{supp}(v)\subseteq W}} S(-v)^{\beta_{1,v}} \xrightarrow{\varphi} S \to 0.$$

The cokernel of φ is S/J_{Δ_W}, hence all we need to show is that F^0 is acyclic. But since the minimal generators of the free S-modules in F^0 involve only the variables x_{ij} with $j \in W$, to show that F^0 is acyclic is enough to show that F^0_u is acyclic for any $u \in \mathbb{N}^n$ with $\operatorname{supp}(u) \subseteq W$. On the other hand, for any $v \in \mathbb{N}^n$, $S(-v)_u$ is nonzero if and only if $u - v \in \mathbb{N}^n$: in particular $S(-v)_u \neq 0$ implies $\operatorname{supp}(v) \subseteq \operatorname{supp}(u) \subseteq W$, hence $F^0_u = F_u$ whenever $\operatorname{supp}(u) \subseteq W$. We conclude since F_u is acyclic for any $u \in \mathbb{N}^n$. □

3.3 Many radical and many F-pure determinantal facet ideals

Let us warm up by studying the algebraic counterpart of the traceability of Δ:

Proposition 75. Let Δ be a traceable d-dimensional simplicial complex on n vertices. Then $\operatorname{height}(J_\Delta) = n - d$. Furthermore, if J_Δ is radical and unmixed, then it admits a square-free initial ideal. If in addition K has positive characteristic, then S/J_Δ is even F-pure.

Proof. Let us fix a labeling for which Δ is traceable. Set

$$C \overset{\text{def}}{=} ([1 \ldots d+1], [2 \ldots d+2], \ldots, [n-d \ldots n]) \subseteq J_\Delta.$$

Let us fix a diagonal term order $<$ on S. Note that

$$\mathrm{in}_<([i \ldots i+d]) = x_{0i}x_{1(1+i)}\cdots x_{d(d+i)} \quad \text{and} \quad \mathrm{in}_<([j \ldots j+d]) = x_{0j}x_{1(1+j)}\cdots x_{d(d+j)}$$

are coprime if $i \neq j$. So $\{[1 \ldots d+1], [2 \ldots d+2], \ldots, [n-d \ldots n]\}$ is a Gröbner basis of C and

$$\mathrm{in}_<(C) = (x_{01}x_{12}\cdots x_{d(d+1)}, x_{02}x_{13}\cdots x_{d(d+2)}, \ldots, x_{0(n-d)}x_{1(1+n-d)}\cdots x_{dn})$$

is a complete intersection of height $n-d$. Hence C is a complete intersection of height $n-d$ inside J_Δ, which implies $\mathrm{height}(J_\Delta) \geq n-d$. On the other hand $\mathrm{height}(J_\Delta) \leq n-d$ because J_Δ is contained in $I_{d+1}(X)$, which has height equal to $n-d$. As for the final claim, set $g = [1 \ldots d+1]\cdots[n-d \ldots n]$. Notice that $\mathrm{in}_<(g)$ is square-free. Obviously, we also have $C \in \mathcal{C}_g$. But if J_Δ is radical and unmixed, since $\mathrm{height}(J_\Delta) = \mathrm{height}(C)$ by the previous part, then J_Δ must be of the form $C : h$ for some $h \in S$. Thus $J_\Delta \in \mathcal{C}_g$ and we conclude via Theorem 68. □

The next lemma will help us identify a large class of complexes whose determinantal facet ideal is indeed radical.

Lemma 76. Let $1 \leq a_0 < a_1 < \cdots < a_d \leq n$, and Δ_a the simplicial complex generated by the facets $a_0 i_1 \ldots i_d$ with $i_j \leq a_j$ for all $j = 1, \ldots, d$. Then

$$J_{\Delta_a} = I_{d+1}(X_{[a_0,a_d]}) \cap I_d(X_{[a_0,a_{d-1}]}) \cap I_{d-1}(X_{[a_0,a_{d-2}]}) \cap \cdots \cap I_1(X_{[a_0,a_0]}).$$

Analogously, if Δ^a is the simplicial complex generated by the facets $i_0 i_1 \ldots a_d$ with $i_j \geq a_j$ for all $j = 0, \ldots, d-1$, then

$$J_{\Delta^a} = I_{d+1}(X_{[a_0,a_d]}) \cap I_d(X_{[a_1,a_d]}) \cap I_{d-1}(X_{[a_2,a_d]}) \cap \cdots \cap I_1(X_{[a_{d-1},a_d]}).$$

Proof. Since the two identities are symmetric, we will only prove the first one. The containment "$\subseteq$" is obvious; so let us show "$\supseteq$". To make the notation lighter, we make the harmless assumption that $a_0 = 1$. Note that J_{Δ_a} is generated by a poset ideal, namely by

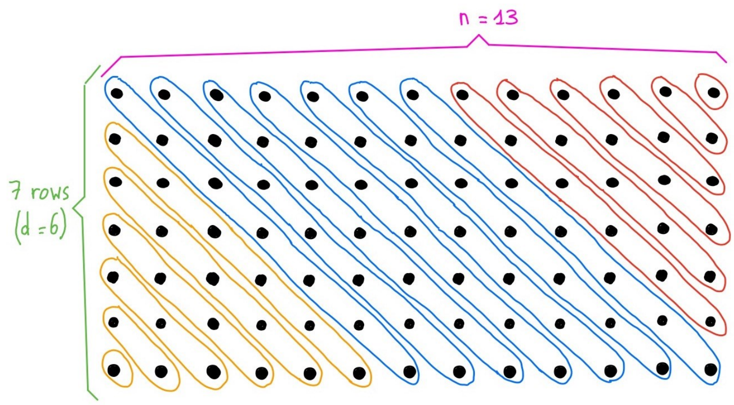
$$\Omega = \{\pi \in \Pi : \pi \leq [a_0 \ldots a_d]\}.$$

Similarly, for all $j = 0, \ldots, d$, the ideal $I_{j+1}(X_{[1,a_j]})$ is generated by the poset ideal

$$\Omega_j = \{\pi \in \Pi : \pi \leq [d-j \ldots d | a_j - j \ldots a_j]\}.$$

Since it is easy to check that $\Omega = \bigcap_{j=0}^{d} \Omega_j$, via [BV88, Proposition (5.2)] we obtain $J_{\Delta_a} = I_{d+1}(X_{[1,a_d]}) \cap I_d(X_{[1,a_{d-1}]}) \cap I_{d-1}(X_{[1,a_{d-2}]}) \cap \cdots \cap I_1(X_{[1,1]})$. □

Now, let $\gamma \in S$ be the product of the minors whose main diagonals are illustrated in the 7×13 matrix below.



More precisely,

$$\overset{\text{def}}{=} [dj1][d \quad 1; dj1; 2] [1; 2; :::; d \quad 1; dj1; 2; :::; d \quad 1; d]$$
$$[1; 2; :::; d; d + 1] [n \quad d; n \quad d + 1; :::; n \quad 1; n]$$
$$[n \quad d + 1; n \quad d + 2; :::; n \quad 1; nj0; 1; :::; d \quad 2; d \quad 1] [n \quad 1; nj0; 1][nj0]:$$

The reason we dened this way is that if $<$ is a diagonal term order, we have

$$\mathrm{in}_<() = \prod_{i=0}^{d} \prod_{j=1}^{n} x_{ij}:$$

Using this , we are now ready to prove the rst main result of this Chapter.

Theorem 77. Let be a d-dimensional semi-closed simplicial complex on n vertices. Then J is a radical ideal. Moreover:

(1) For any diagonal term order (compatible with the labeling which makes semi-closed), in(J) is a squarefree term ideal.
(2) If the eld K has positive characteristic, S=J is F-pure.

Proof. We will prove that if is semi-closed with respect to the given labeling then J () 2 C, whence both claims follow by Theorem 68. Let $1 \; a_0 < a_1 < ::: < a_d \; n$. Using the notation of Lemma 76, since is semi-closed, either $_a$ or ^{a} is contained in whenever $a_0a_1 \, a_d$ 2 . For any $a_0a_1 \, a_d$ 2 , set $_a = \; _a$ if $_a$, and $_a = \; ^a$ otherwise. Then

$$= \bigcup_{a_0a_1a_d 2} {}_a:$$

In particular,

$$J() = \sum_{a_0a_1a_d 2} J(_a):$$

Since C is closed under sums, in order to show that J () 2 C we only need to check that each $J(_a)$ 2 C. To verify this, we use a result in [Sec21]: The ideal $I_{r+1}(X_{[ij]})$ 2 C whenever 1 i < j n and 0 r minfd; j ig. Since C is closed under intersections, Lemma 76 guarantees that $J(_a)$ 2 C, as desired. □

Remark 78. The assumption "semi-closed" is best possible: if we replace it with "weakly-closed", the theorem no longer holds, cf. Example 73. That said, the converse of Theorem 77 is false. To see this, consider the non-weakly-closed complex $U_3 = 124; 345; 467$ of Figure 3. If g = [124][345][467] then for a diagonal term order in(g) = $x_{01}x_{12}x_{24}x_{03}x_{14}x_{25}x_{04}x_{16}x_{27}$, which is squarefree. Obviously [124]; [345]; [467] 2 C_g, hence J 2 C_g. So in(J) is squarefree, and, in the positive characteristic case, S=J is F-pure by Theorem 68. On the other hand, when d = 1 Theorem 77 is true for all weakly closed graphs, via the main result of Matsuda [Mat18]. This shows that the techniques used in [Mat18] do not generalize to higher dimensions.

Remark 79. Suppose that K has positive characteristic. Theorem 77 implies that, whenever is a poset ideal of consisting only of maximal minors, then the corresponding ASL is F-pure. On the other hand, some ASLs are not F-pure, as explained in [KV21, Remark 5.2]. We do not know whether all the ASLs on a poset ideal of are F-pure.

Remark 80. In positive characteristic, having a square-free initial ideal or an F-pure quotient are unrelated properties. Many ideals, like $I = (x^2 + xy + y^2)$ $S \stackrel{def}{=} Z{=}pZ[x; y]$, for p prime, have the property that $S{=}I$ is F-pure even if in(I) is not square-free for any term order. On the other hand, the binomial edge ideal of a 5-cycle is not F-pure in characteristic 2 [Mat18, Example 2.7], even if it admits a squarefree initial ideal. See [KV21] for a discussion on the relationship between the two properties of being F-pure and having a squarefree initial ideal.

Theorem 77 allows us to characterize the determinantal facet ideals having a linear resolution: It turns out that there is only one. This extends to all dimensions the result for graphs by Saeedi-Madani and Kiani [SK12].

Corollary 81. Let be a pure d-dimensional simplicial complex on n vertices.

$$J \text{ has a linear resolution} \quad (\ ) \quad = {}_n: \ {}^d$$

Proof. \(": If is the d-skeleton of the (n 1)-simplex, J is the ideal of maximal minors of the matrix X. This ideal is resolved by the Eagon-Northcott complex [EN67], which is linear.
\)": By contradiction, suppose there is a subset W [n] of cardinality d + 2 such that $_W$ is not the d-skeleton of the (d + 1)-simplex on W. We can re-label the vertices so that W = f1; 2; : : : ; d + 2g and

$$_W = 12 : : : (d + 1);\ 12 : : : d(d + 2);\quad : : : \ ;\ 1 : : : i(i + 2)(i + 3) : : : (d + 2)$$

where 2 i d. With respect to such a labeling $_W$ is semi-closed. So by Theorem 77, in(J_W) is a squarefree monomial ideal for any diagonal term order. Hence, by the work of Conca{Varbaro [CV20], reg($S{=}J_W$) = reg($S{=}$in(J_W)). But by Lemma 76

$$J_W = I_i(X_{[1;i]}) \setminus I_{d+1}(X_{[1;d+2]});$$

so by Theorem 68 in(J_W) = in($I_i(X_{[1;i]})$) \ in($I_{d+1}(X_{[1;d+2]})$). Via Theorem 71, it is easy to check that the monomial $(x_{d\ \ i+1;1}x_{d\ \ i+2;2}\ x_{d;i})(x_{0;2}x_{1;3}\ x_{d;d+2})$ is a minimal generator of in($I_i(X_{[1;i]})$) \ in($I_{d+1}(X)$). Hence in(J_W) has a minimal generator of degree i + d + 1. In particular,

$$\text{reg}(S{=}J)\ \ \text{reg}(S{=}J_W) = \text{reg}(\text{in}(S{=}J_W))\ \ i + d > d:$$

So by Proposition 74, reg($S{=}J$) reg($S{=}J_W$) > d. So J cannot have a linear resolution. □

3.4 Determinantal facet ideals dened by a Grobner basis

If is a closed simplicial complex, it is easy to see that the minors generating J form a Grobner basis with respect to a diagonal monomial order, corresponding to the labeling that makes closed: See [E&13]. In [E&13] it has been incorrectly claimed that the converse of the above statement holds true. The following result, which is a consequence of [Sec21, Corollary 2.4], shows that there are many other complexes for which the minors generating J form a Grobner basis:

Theorem 82. Let be a d-dimensional simplicial complex, with a labeling that makes it unit-interval. The set f[$a_0 : : : a_d$] : $a_0 : : : a_d$]g is a Grobner basis of J with respect to any diagonal term order. If in addition the eld K has positive characteristic, then $S{=}J$ is F-pure.

Proof. By denition, is the union of d-skeleta of simplices on consecutive vertices. We can choose these d-skeleta to be maximal with respect to inclusion. This yields a decomposition

$$= {}^{d}_{[i_1;j_1]} \cup {}^{d}_{[i_2;j_2]} \cup : : : \cup {}_{[i_l;j_l]}; \ {}^d$$

where $\Delta^d_{[i_k;j_k]}$ denotes the d-skeleton of the simplex on vertices $i_k, i_k+1, i_k+2, \ldots, j_k$. Therefore

$$J_\Delta = I_{d+1}(X_{[i_1;j_1]}) + I_{d+1}(X_{[i_2;j_2]}) + \cdots + I_{d+1}(X_{[i_l;j_l]}).$$

So by [Sec21, Corollary 2.4]

$$\mathrm{in}_<(J_\Delta) = \mathrm{in}_<(I_{d+1}(X_{[i_1;j_1]})) + \mathrm{in}_<(I_{d+1}(X_{[i_2;j_2]})) + \cdots + \mathrm{in}_<(I_{d+1}(X_{[i_l;j_l]})).$$

By Theorem 71, $\{[a_0, \ldots, a_d] \mid a_0 \cdots a_d \in \Delta\}$ is a Gröbner basis for J_Δ. Finally, the F-purity claim in the case of positive characteristic follows again from [Sec21, Corollary 2.4]. □

Remark 83. That the set $\{[a_0 \cdots a_d] : a_0 \cdots a_d \in \Delta\}$ is a Gröbner basis when Δ is unit-interval has been independently proved, using a completely different method, in Almousa–Vandebogert [AV21, Theorem 2.16]. They also obtained the analogous result for r-determinantal facet ideals (a more general concept than determinantal facet ideals) of unit-interval simplicial complexes. We were not aware of the paper [AV21] of Almousa and Vandebogert before posting the first version of the present work on the arXiv. (We coordinated efforts to adopt the same name \unit-interval complexes" in the two papers.) For the sake of completeness, we point out that [Sec21, Corollary 2.4] implies that also r-determinantal facet ideals of unit-interval simplicial complexes define F-pure quotients in positive characteristic. We do not know, however, whether the (r-)determinantal facet ideals of \lcm-closed" complexes, as defined in [AV21], or whether those of \closed complexes", as defined here, are all F-pure.

Remark 84. The converse of Theorem 82 is false: as explained above, any closed but not unit-interval complex is a counterexample. For a more interesting example, consider

$$W = \langle 123, 124, 134, 234, 235, 245, 345, 568, 789, 8\,10\,11 \rangle$$

corresponding to a one-point union of the B^2 and the U^2_3 of Figure 3. This complex W is not unit-interval, not closed, and not even weakly-closed [Pav21]. However, one can verify with Macaulay2 [GSm2] or via [AV21, Theorem 2.15] that $\{[a_0, a_1, a_2] : a_0 \cdots a_d \in \Delta\}$ form a Gröbner basis of J_W for any diagonal term order.

Remark 85. Two of the results of [E&13] are incorrect because of the following counterexamples. As we already mentioned, the complex B_d of Lemma 42 (cf. Figure 3) is not closed, but the set of all the minors $[abc]$, where abc ranges over all facets of B^d, is a Gröbner basis of J_{B^d} for any diagonal term order by Theorem 82. Thus one direction of [E&13, Theorem 1.1] is incorrect for all $d > 1$. Moreover, the graph $G_0 = \langle 12, 13, 23, 24, 34 \rangle$ is closed, but one can verify that S/J_{G_0} is not Cohen-Macaulay. Thus [E&13, Corollary 1.3] is incorrect already for $d = 1$.

The final part of our work is dedicated to the delicate quest for some partial converse for Theorem 82. To increase the chances of success, we restrict ourselves to traceable complexes. The traceable assumption is rather natural in this case, as we have anyway seen in Theorem 56 that all strongly-connected unit-interval complexes are traceable. We start off with a Lemma:

Lemma 86. Let Δ be a simplicial complex such that $GB \overset{\text{def}}{=} \{[a_0, \ldots, a_d] \mid a_0 \cdots a_d \in \Delta\}$ is a Gröbner basis of J_Δ for some diagonal term order. Let $F = a_0 \cdots a_d$ and $G = b_0 \cdots b_d$ be two facets of Δ. If for some integer $l \in \{0, \ldots, d-1\}$

(i) $a_i = b_i$ for all $i \in \{0, \ldots, l\}$,

(ii) $a_{l+1} > a_l + 1$,

(iii) $b_{l+k} = b_l + k$ for all $k \geq 1$,

then the facet $a_0 \cdots a_{l-1}(a_l+1)\, a_{l+1} \cdots a_d$ is also in Δ. Symmetrically, if for some $l \in \{1, \ldots, d\}$

(iv) $a_i = b_i$ for all $i \in \{l, \dots, d\}$,

(v) $a_{l-1} < a_l - 1$,

(vi) $b_{l-k} = b_l - k$ for all $k \in \{1, \dots, l\}$,

then the facet $a_0 \dots a_{l-1}(a_l - 1)\, a_{l+1} \dots a_d$ is also in Δ.

Proof. It is harmless to assume that the term order $<$ is the lexicographic term order defined before, cf. Theorem 71. Let F and G be two facets of Δ satisfying (i), (ii) and (iii). Let us compute the initial term of the polynomial

$$f \stackrel{\text{def}}{=} [l+1 \dots d \mid a_{l+1} \dots a_d][b_0 \dots b_d] - [l+1 \dots d \mid b_{l+1} \dots b_d][a_0 \dots a_d].$$

If we set

$$p \stackrel{\text{def}}{=} [a_0 \dots a_d]; \quad p' \stackrel{\text{def}}{=} [l+1 \dots d \mid a_{l+1} \dots a_d]$$
$$q \stackrel{\text{def}}{=} [b_0 \dots b_d]; \quad q' \stackrel{\text{def}}{=} [l+1 \dots d \mid b_{l+1} \dots b_d]$$

then $f = p'q - pq'$, and by Laplace expansion we have

$$p'q = \underbrace{(x_{0b_0} \cdots x_{l-1\, b_{l-1}} x_{l b_l} p' q')}_{h} + g_1; \qquad \gamma_1 < \gamma \quad \forall \gamma_1 \in \operatorname{supp}(g_1);\ \forall \gamma \in \operatorname{supp}(h);$$
$$pq' = \underbrace{(x_{0a_0} \cdots x_{l-1\, a_{l-1}} x_{l a_l} p' q')}_{h} + g_2; \qquad \gamma_2 < \gamma \quad \forall \gamma_2 \in \operatorname{supp}(g_2);\ \forall \gamma \in \operatorname{supp}(h).$$

Furthermore

$$\mathrm{in}_<(g_1) = (x_{l+1\, a_{l+1}} \cdots x_{d a_d})(x_{0b_0} \cdots x_{l-1\, b_{l-1}} x_{l\, b_{l+1}} x_{l+1\, b_l} x_{l+2\, b_{l+2}} \cdots x_{d b_d});$$
$$\mathrm{in}_<(g_2) = (x_{l+1\, b_{l+1}} \cdots x_{d b_d})(x_{0a_0} \cdots x_{l-1\, a_{l-1}} x_{l\, a_{l+1}} x_{l+1\, a_l} x_{l+2\, a_{l+2}} \cdots x_{d a_d}).$$

Since $\mathrm{in}_<(g_2)$ is smaller than $\mathrm{in}_<(g_1)$, we conclude that

$$\mathrm{in}_<(f) = \mathrm{in}_<(g_1 - g_2) = (x_{l+1\, a_{l+1}} \cdots x_{d a_d})(x_{0b_0} \cdots x_{l-1\, b_{l-1}} x_{l\, b_{l+1}} x_{l+1\, b_l} x_{l+2\, b_{l+2}} \cdots x_{d b_d}).$$

In addition $f \in J$ because $F, G \in \Delta$. Thus, there must be a minor $g = [c_0 \dots c_d]$ in GB such that $\mathrm{in}_<(g)$ divides $\mathrm{in}_<(f)$. Note that for $c_0, \dots, c_l$ we only have one option, namely,

$$\begin{cases} c_0 & = b_0 = a_0 \\ & \vdots \\ c_{l-1} & = b_{l-1} = a_{l-1} \\ c_l & = b_{l+1} = b_l + 1 = a_l + 1. \end{cases}$$

For c_{l+1} we have a priori two possibilities: either $c_{l+1} = b_l$ or $c_{l+1} = a_{l+1}$. But $b_l < b_{l+1} = c_l$, so it must be $c_{l+1} = a_{l+1}$. Similarly, for c_{l+2} we have a priori two options: Either $c_{l+2} = b_{l+2}$, or $c_{l+2} = a_{l+2}$. But by the assumptions, we have that $b_{l+2} \geq a_{l+1} = c_{l+1}$, so since $c_{l+2} > c_{l+1}$ it must be $c_{l+2} = a_{l+2}$. In general, for any $k \geq 2$ we have $b_{l+k} \geq a_{l+k-1} = c_{l+k-1}$. Since $c_i > c_{i-1}$, arguing recursively we obtain that the only possible option is $c_{l+k} = a_{l+k}$ for all $k \geq 2$. Hence we have proved that

$$g = [c_0, \dots, c_d] = [a_0 \dots a_{l-1}(a_l + 1) a_{l+1} \dots a_d].$$

Since g is an element of GB, we conclude that $a_0 \dots a_{l-1}(a_l + 1) a_{l+1} \dots a_d \in \Delta$.

The proof of the second part of the lemma is symmetric; namely, one considers the polynomial

$$f' \stackrel{\text{def}}{=} [0 \dots l-1 \mid a_0 \dots a_{l-1}][b_0 \dots b_d] - [0 \dots l-1 \mid b_0 \dots b_{l-1}][a_0 \dots a_d] \in J$$

whose leading term is

$$\mathrm{in}_<(f') = (x_{0a_0} x_{l-1 a_{l-1}})(x_{0b_0} x_{l-2 b_{l-2}} x_{l-1 b_l} x_{l b_{l-1}} x_{l+1 b_{l+1}} \cdots x_{d b_d});$$

and one proceeds analogously to the argument above. □

Theorem 87. Let Δ be a d-dimensional simplicial complex. If with respect to the same labeling Δ is traceable and the set $\{[a_0 \ldots a_d] : a_0 \ldots a_d\}$ is a Grœbner basis of J_Δ with respect to some diagonal term order, then such labeling makes Δ unit-interval.

Proof. Let $F = a_0 \ldots a_d$ be a facet of Δ with $\mathrm{gap}(F) = k$. We proceed by induction on k. For $k = 0$ there is nothing to prove, so we assume $k > 0$. Let $g_1, \ldots, g_k$ be the vertices not in F, and such that $a_0 < g_1 < \ldots < g_k < a_d$. We want to show that Δ contains the d-skeleton of $\{a_0, \ldots, a_d, g_1, \ldots, g_k\}$. The strategy is to first show that $\Delta^d_{[a_0+1, a_d]}, \Delta^d_{[a_0, a_d-1]}$ by inductive assumption, and then to prove that Δ contains also the facets of the form $a_0 c_1 \ldots c_{d-1} a_d$. So let us proceed. Let l be the greatest integer such that $a_l < g_1$, so that $g_1 = a_l + 1$. Consider the two facets F and H_{a_0} of Δ. They satisfy the assumptions of Lemma 86, so

$$F' = a_0 \ldots a_{l-1}\, g_1\, a_{l+1} \ldots a_d \in \Delta.$$

If $l = 0$, then $\mathrm{gap}(F') = k - 1$, so by the inductive assumption $\Delta^d_{[g_1, a_d]} = \Delta^d_{[a_0+1, a_d]}$. Otherwise, since $\mathrm{gap}(F') = k$, we cannot apply the inductive assumption yet. However, we have "shifted" the first gap to the left and now the first missing vertex is $a_l = a_{l-1} + 1$. We can apply again Lemma 86 to the facets F_0 and H_{a_0} and we get

$$F'' = a_0 \ldots a_{l-2}\, a_l\, g_1\, a_{l+1} \ldots a_d \in \Delta.$$

If $l = 1$, then $\mathrm{gap}(F'') = k - 1$, so by the inductive assumption $\Delta^d_{[a_1, a_d]} = \Delta^d_{[a_0+1, a_d]}$. Otherwise, once again $\mathrm{gap}(F'') = k$ and the first missing vertex $a_{l-1} = a_{l-2} + 1$ has been shifted by one to the left. Iterating this procedure, we eventually get that

$$(a_0 + 1) \ldots a_l\, g_1\, a_{l+1} \ldots a_d \in \Delta.$$

This face has gap equal to $k - 1$ and we can finally apply induction: We get $\Delta^d_{[a_0+1, a_d]}$.

To prove that $\Delta^d_{[a_0, a_d-1]}$ we use a similar argument. Let l be the smaller integer such that $g_k < a_l$, so that $g_k = a_l - 1$, and consider the two facets of Δ

$$F = a_0 \ldots a_{l-1}\, a_l \ldots a_d$$
$$H_{a_d - d} \overset{\mathrm{def}}{=} H_{a_d - d} = (a_d - d)(a_d - d + 1) \ldots g_k\, a_l \ldots a_d.$$

Iteratively applying the second part of Lemma 86, we can shift the last missing vertex to the right until we reach the facet

$$a_0 \ldots a_{l-1}\, g_k\, a_l \ldots a_{d-1} \in \Delta;$$

which has gap $k - 1$. So by induction $\Delta^d_{[a_0, a_d-1]}$.

It remains to prove that all the facets of the form $G = a_0 c_1 \ldots c_{d-1} a_d$ are in Δ. To do so, we start from $F = a_0 a_1 \ldots a_d$ and we replace one by one each a_i with the corresponding c_i. In detail: For $i = 1$, we have three possibilities:

- $c_1 = a_1$, or
- $a_0 < c_1 < a_1$, or
- $c_1 > a_1$.

If $c_1 = a_1$ there is nothing to do. If $a_0 < c_1 < a_1$, consider the two facets

$$F = a_0 \; a_1 \cdots a_d$$
$$\tilde{F} = (a_1 - 1) \; a_1 \cdots a_d.$$

Since $a_0 < c_1 < a_1$, we have that $a_1 - 1 > a_0$. Hence $\tilde{F} \in \Delta^d_{[a_0+1; a_d]}$. So by Lemma 86 $a_0(a_1 - 1)a_2 \cdots a_d \in \Delta$. If $c_1 = a_1 - 1$ we stop, otherwise we repeat the same argument. At each iteration of Lemma 86, the second vertex in the facet decreases by one unit; eventually, we obtain that $a_0 c_1 a_2 \cdots a_d \in \Delta$.

As for the third possibility ($c_1 > a_1$), we claim that we can simply dismiss it without loss of generality. In fact, for every $i \in \{1, \ldots, d\}$, we can always "flatten all the vertices after a_{i-1} to the right": that is, we can always replace F with another face in Δ of the form

$$F_i = a_0 a_1 \cdots a_{i-1}(a_d - d - i) \cdots (a_d - 1) a_d.$$

To see it, let $0 \le l \le d-1$ be the largest index for which $a_l + 1 < a_{l+1}$ (such an l must exist because $\mathrm{gap}(F) > 0$). Applying Lemma 86 to the facets F and $F' = a_0 \cdots a_l (a_l+1)(a_l+2) \cdots a_l + (d - l)$ in $\Delta^d_{[a_0; a_d+1]}$, we get that the facet $a_0 \cdots a_{l-1}(a_l+1)a_{l+1} \cdots a_d$ is in Δ. Proceeding this way we end up with the face

$$F_l = a_0 \cdots a_{l-1}(a_{l+1} - 1)a_{l+1} \cdots a_d \in \Delta.$$

Replacing F with F_l, and arguing the same way, we infer that $F_i \in \Delta$ for all $i = 0, \ldots, d-1$. In particular, for $i = 1$, we could replace F with a face with same minimum and maximum

$$F_1 = a_0 \, (a_d - d + 1) \, (a_d - d + 2) \cdots a_d \in \Delta.$$

Note that $c_1 \le a_d - d + 1$. So our claim is proven: Up to replacing F with F_1, we can assume that $c_1 \le a_1$.

So the case $i = 1$ is settled. Consider now $i = 2$. If $c_2 = a_2$, there is nothing to do. Otherwise, flattening the vertices after c_1 of $a_0 c_1 a_2 \cdots a_d$ to the right, we may assume that $c_2 < a_2$. Consider the two facets

$$F = a_0 \; c_1 \; a_2 \cdots a_d \in \Delta$$
$$\tilde{F} = (a_2 - 2) \; (a_2 - 1) \; a_2 \cdots a_d \in \Delta.$$

Since $c_2 < a_2$, we have that $c_1 < a_2 - 1$, so applying Lemma 86 we obtain that

$$a_0 \, c_1 \, (a_2 - 1) \, a_3 \cdots a_d \in \Delta.$$

If $c_2 = a_2 - 1$ we stop, otherwise we repeat the same argument. At every iteration of Lemma 86, the third vertex in the facet decreases by one unit; eventually, we obtain that $a_0 c_1 c_2 a_3 \cdots a_d \in \Delta$. Iterating this procedure for all i's, we conclude that

$$G = a_0 c_1 c_2 \cdots c_{d-1} a_d \in \Delta. \qquad \square$$

Remark 88. Very recently Almousa and Vandebogert [AV21] introduced a technical property of simplicial complexes, called "lcm-closed", that simultaneously generalizes the two properties of being "closed" and being "unit-interval". They asked [AV21, Question 2.19] whether such property for simplicial complexes would characterize the fact that the minors of the determinantal facet ideal form a Gröbner basis with respect to any diagonal term order. With a little ingenuity, one can see that for traceable complexes, "lcm-closed" is simply equivalent to "unit-interval". Thus Theorem 87 answers Almousa–Vandebogert's question positively, for complexes that with respect to the same labeling are traceable.

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