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Nuclear Physics B 993 (2023) 116269
NUCLEAR PHYSICS B
www.elsevier.com/locate/nuclphysb

Quantum Field Theory and Statistical Systems

On the scaling behaviour of an integrable spin chain with $\mathcal{Z}_r$ symmetry

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Received 10 May 2023; accepted 13 June 2023
Available online 16 June 2023
Editor: Hubert Saleur

Abstract

The subject matter of this work is a 1D quantum spin - $\frac{1}{2}$ chain associated with the inhomogeneous six-vertex model possessing an additional $\mathcal{Z}_r$ symmetry. The model is studied in a certain parametric domain, where it is critical. Within the ODE/IQFT approach, a class of ordinary differential equations and a quantization condition are proposed which describe the scaling limit of the system. Some remarkable features of the CFT underlying the critical behaviour are observed. Among them is an infinite degeneracy of the conformal primary states and the presence of a continuous component in the spectrum in the case of even r.

1. Introduction

Much of the early interest in exactly solvable models in 2D classical statistical mechanics came from the study of critical phenomena. Starting from the Ising model, exact solutions were crucial for the development of key concepts such as the scaling hypothesis and universality. The latter was greatly informed by the six-vertex model [1]. Within the standard parameterization, its local Boltzmann weights depend on q, which is sometimes referred to as the anisotropy parameter. As long as q is a unimodular number, i.e., $q = \mathrm{e}^{\mathrm{i}\gamma}$, the statistical system is critical and it turns out that the corresponding critical exponents depend on $\gamma \in (0, \pi]$. This was in contradiction with the original understanding of universality. The clarification came within the framework

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https://doi.org/10.1016/j.nuclphysb.2023.116269
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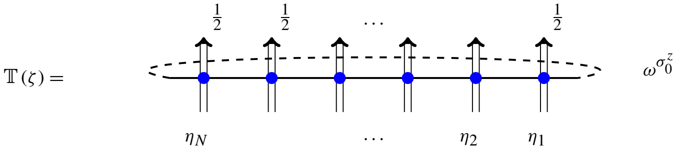

Fig. 1. A graphical representation of the transfer-matrix for the inhomogeneous six-vertex model. Twisted boundary conditions are imposed as indicated by the presence of the factor $\omega^{\sigma_0^z}$, where σ_0^z is the Pauli matrix acting in the two dimensional auxiliary space.

of QFT as an example of an exactly marginal deformation of the critical point. It was realized that the universal behaviour of the six-vertex model is governed by the massless Gaussian CFT, where the fundamental Bose field is compactified to a circle with radius $\sqrt{2\gamma/\pi}$ [2–4].

In the work [5], Baxter introduced a multiparametric statistical system, which is a generalization of the six-vertex model, and showed that it is solvable via the Bethe Ansatz technique. The corresponding set of algebraic equations takes the form

$$\prod_{J=1}^{N} \frac{\eta_J + q^{+1}\,\zeta_m}{\eta_J + q^{-1}\,\zeta_m} = -\omega^2\, q^{2S^z} \prod_{j=1}^{M} \frac{\zeta_j - q^{+2}\,\zeta_m}{\zeta_j - q^{-2}\,\zeta_m} \qquad \big(m = 1, 2, \ldots, M,\ \ S^z = \tfrac{N}{2} - M \geq 0\big)\,. \tag{1.1}$$

Having found a solution $\{\zeta_m\}_{m=1}^M$ one can compute the eigenvalues of all members of the commuting family of operators including the one-row transfer-matrix schematically depicted in Fig. 1.

Among the parameters is the anisotropy $q = \mathrm{e}^{\mathrm{i}\gamma}$ and the set of complex numbers $\{\eta_J\}_{J=1}^N$, referred to as the inhomogeneities, which are all assumed to be fixed. Together with these, the Bethe Ansatz equations also involve the parameter ω. The latter comes about as a result of imposing so-called quasi-periodic boundary conditions on the lattice and for the periodic case $\omega = 1$. With the parameters obeying certain conditions, the lattice system develops critical behaviour which can be described by a conformal field theory. We'll assume that the number of sites N is divisible by r and the inhomogeneities obey the r-site periodicity condition

$$\eta_{J+r} = \eta_J \qquad\qquad \big(J = 1, 2, \ldots, N - r\,;\ N/r \in \mathbb{Z}\big) \tag{1.2}$$

in order to ensure the presence of translational invariance for the continuous theory. In the scaling limit $N \to \infty$ while r is kept fixed.

For the investigation of the scaling limit, it is useful to switch to the Hamiltonian picture, where a central rôle belongs to the Hamiltonian $\mathbb{H}$. The latter is a member of the commuting family of operators and, furthermore, is expressible as a logarithmic derivative of the r-row transfer-matrix (see, e.g., section 6.4 of ref. [6] for details). In the case of the homogeneous model, where η_J are the same and without loss of generality may be set to one, $\mathbb{H}$ coincides with the Hamiltonian for the Heisenberg XXZ spin - $\frac{1}{2}$ chain:

$$\mathbb{H}\,|_{r=1} = -\frac{1}{2\sin(\gamma)} \sum_{m=1}^{N} \Big(\sigma_m^x\sigma_{m+1}^x + \sigma_m^y\sigma_{m+1}^y + \cos(\gamma)\,\big(\sigma_m^z\sigma_{m+1}^z - \hat{\mathbf{1}}\big)\Big)\,. \tag{1.3}$$

It should be supplemented with the quasi-periodic boundary conditions [4]

$$\sigma_{N+m}^x \pm \mathrm{i}\,\sigma_{N+m}^y = \omega^{\pm 2}\,\big(\sigma_m^x \pm \mathrm{i}\,\sigma_m^y\big)\,, \qquad\qquad \sigma_{N+m}^z = \sigma_m^z\,. \tag{1.4}$$

For $r=2$ there are two inhomogeneities η_1, η_2, but one may always restrict to the case with $\eta_2=\eta_1^{-1}$. In the parameterization $\eta_1=\mathrm{e}^{\mathrm{i}\alpha}$, the Hamiltonian reads as [8,12][1]

$$\begin{aligned}\mathbb{H}\,|_{r=2} = \frac{\cos(\gamma)}{2\sin(\gamma-\alpha)\,\sin(\gamma+\alpha)}\sum_{m=1}^{N}\bigg[\frac{\sin^2(\alpha)}{\sin(\gamma)}\,\big(\sigma_m^x\,\sigma_{m+2}^x+\sigma_m^y\,\sigma_{m+2}^y+\sigma_m^z\,\sigma_{m+2}^z-\hat{\mathbf{1}}\big)\\
-2\sin(\gamma)\left(\frac{\cos^2(\alpha)}{\cos(\gamma)}\,(\sigma_m^x\,\sigma_{m+1}^x+\sigma_m^y\,\sigma_{m+1}^y)+\sigma_m^z\,\sigma_{m+1}^z-\hat{\mathbf{1}}\right)\\
-\mathrm{i}\sin^2(\alpha)\,(\sigma_{m+2}^z-\sigma_{m-1}^z)(\sigma_m^x\sigma_{m+1}^x+\sigma_m^y\sigma_{m+1}^y)\\
+(-1)^m\sin(2\alpha)\,\Big(\tan(\gamma)\,(\sigma_m^x\sigma_{m+1}^y-\sigma_m^y\sigma_{m+1}^x)\\
+\frac{\mathrm{i}}{2\cos(\gamma)}\,(\sigma_m^x\,\sigma_{m+2}^y-\sigma_m^y\sigma_{m+2}^x)\,\sigma_{m+1}^z\\
-\frac{\mathrm{i}}{2}\,(\sigma_m^x\,\sigma_{m+1}^y-\sigma_m^y\sigma_{m+1}^x)\,(\sigma_{m+2}^z-\sigma_{m-1}^z)\Big)\bigg]\end{aligned} \tag{1.5}$$

and is subject to the quasi-periodic boundary conditions as above. When $r>2$, the explicit formula for $\mathbb{H}$ becomes cumbersome and not particularly transparent. It involves a sum over terms describing interactions of up to $r+1$ adjacent spins. All such Hamiltonians commute with the z-projection of the total spin operator

$$\mathbb{S}^z=\frac{1}{2}\sum_{m=1}^{N}\sigma_m^z\ :\qquad [\,\mathbb{S}^z,\mathbb{H}\,]=0\,. \tag{1.6}$$

The field theory provides a description of the excitations whose energy counted from that of the ground state is sufficiently low. The full phase diagram for the general lattice system, including the identification of the critical surfaces, is currently not clear. The examples of critical models that have been studied so far, since the pioneering paper of Jacobsen and Saleur [7], all correspond to the case when the anisotropy is unimodular or, in other words, γ is real and may be taken from the domain

$$q=\mathrm{e}^{\mathrm{i}\gamma}\ :\qquad \gamma\in(0,\pi\,]\,. \tag{1.7}$$

Likewise, the inhomogeneities are also unimodular, $|\eta_J|=1$, and, without loss of generality, one may always assume that

$$\prod_{J=1}^{r}\eta_J=1\,. \tag{1.8}$$

As was already mentioned, for $r=1$ the critical behaviour of the model, with quasi-periodic boundary conditions imposed, is described by a massless compact Bose field. Already for $r=2$ the model exhibits different types of critical behaviour and the phase diagram in the (γ,α) plane (the parameters of the lattice Hamiltonian (1.5)) is depicted in Fig. 2.

The different phases are already manifest in the general pattern of Bethe roots for the ground state. For the homogeneous model, all the $\{\zeta_m\}$ are real and positive. In fact, this requires the condition on the twist parameter:

[1] The Hamiltonian presented in [8] corresponds to the case $\alpha=\frac{\pi}{2}$, while the one from ref. [12] is related to (1.5) by a similarity transformation.

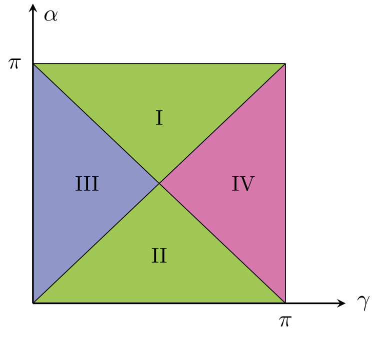

Fig. 2. The phase diagram for the model (1.5) with γ, α being real numbers lying within the segment $(0, \pi]$. Phases I and II correspond to the massless compact Bose field, while III is related to the 2D black hole sigma models [7–14]. In the parametric domain IV the critical behaviour of the statistical system is governed by a CFT consisting of a massless compact boson and two Majorana fermions [15,16].

$$\omega = \mathrm{e}^{\mathrm{i}\pi \mathtt{k}}\,, \qquad\qquad \mathtt{k} \in (-\tfrac{1}{2}, \tfrac{1}{2}] \tag{1.9}$$

with sufficiently small $|\mathtt{k}|$. In Fig. 3 depicted are the typical pattern of Bethe roots for the case $r = 2$ in phases III and IV in the complex plane of $\beta \equiv -\frac{1}{2}\log(\zeta)$. An important difference is that for phase III the roots accumulate exactly on the lines $\Im m(\beta) = 0, \frac{\pi}{2}$ independently of the value of the inhomogeneity parameter α (see, e.g., ref. [12]). For phase IV, they also accumulate along two lines. However these depend on α as $\Im m(\beta) = \pm\frac{1}{2}\alpha$ (see, e.g., ref. [16]).

The phase diagram for $r > 2$ is yet to be mapped out in general. In this work we will study the universal behaviour of the model in the domain $0 < \gamma < \frac{\pi}{r}$. It turns out to exhibit qualitatively new features compared to the cases $r = 1, 2$. In particular, the pattern of Bethe roots for the ground state do not accumulate on lines with fixed value of $\Im m(\beta)$, but on a certain locus in the complex β plane that depends on the values of the inhomogeneities (for an illustration, see Fig. 4). The exception is

$$\eta_J = (-1)^r\, \mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2J-1)}\,, \tag{1.10}$$

where the lattice model possesses an extra $\mathcal{Z}_r$ invariance. Among other things, the symmetry manifests itself in that the Bethe roots accumulate along the lines

$$\Im m(\beta_j) = 0\,,\ \frac{\pi}{r}\,,\ \frac{2\pi}{r}\,, \ldots,\ \frac{(r-1)\pi}{r} \qquad\qquad (\mathrm{mod}\ \pi)\,. \tag{1.11}$$

The most effective technique for studying the critical behaviour of an integrable lattice system is based on the ODE/IQFT correspondence [17–20]. This is demonstrated for the XXZ spin - $\frac{1}{2}$ chain in ref. [21] and for the case $r = 2$ in phase III from Fig. 2 in the work [14]. In ref. [16], the ODE/IQFT correspondence was put forward for the inhomogeneous six-vertex model with $(1 - \frac{1}{r})\pi < \gamma < \pi$.

In this paper we propose a set of differential equations that describe the scaling limit of the $\mathcal{Z}_r$ invariant spin chain subject to quasi-periodic boundary conditions (1.4), (1.9), when the inhomogeneities take the value (1.10) and

$$q = \mathrm{e}^{\mathrm{i}\gamma}\ :\quad 0 < \gamma < \frac{\pi}{r}\,. \tag{1.12}$$

The analysis turns out to be more complicated than for the $\mathcal{Z}_2$ case and a detailed study of the critical behaviour is left for future work.

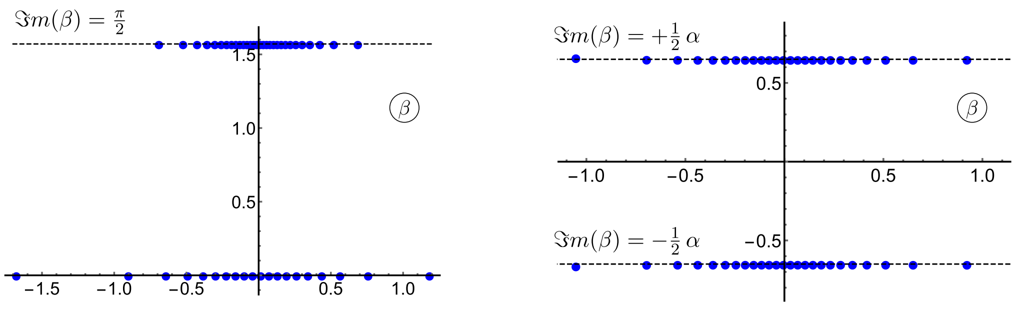

Fig. 3. The typical pattern of Bethe roots for the ground state ($S^z = 0$) of the Hamiltonian (1.5) is shown in the complex β plane with $\beta = -\frac{1}{2}\log(\zeta)$. For the left panel, the parameters were taken to be $(\gamma, \alpha) = (\frac{43\pi}{200}, \frac{13}{10})$, so that the model is in the critical phase III from Fig. 2. For the right panel $(\gamma, \alpha) = (\frac{143\pi}{200}, \frac{13}{10})$, which corresponds to phase IV. The number of sites $N = 100$ and the twist parameter $\mathtt{k}$, which enters into the boundary conditions (1.4) via $\omega = \mathrm{e}^{\mathrm{i}\pi\mathtt{k}}$ was set to $\mathtt{k} = \frac{1}{10}$.

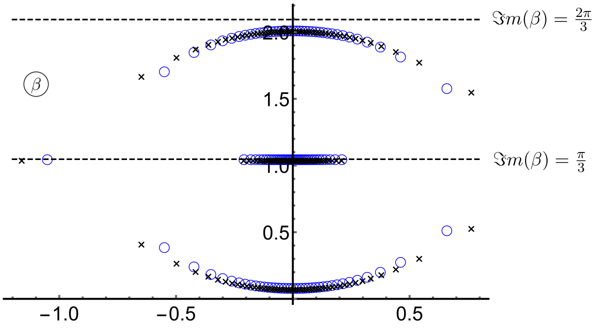

Fig. 4. The pattern of Bethe roots $\beta_j = -\frac{1}{2}\log(\zeta_j)$ for the ground state of the spin chain Hamiltonian $\mathbb{H}$ with $r = 3$ in the sector $S^z = 0$. The inhomogeneities are set to be $\eta_1 = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}}$, $\eta_2 = \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}\epsilon}$ and $\eta_3 = \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}(1-\epsilon)}$ where $\epsilon = \frac{3}{40}$. For the open blue circles, the number of lattice sites is $N = 240$, while for the black crosses $N = 324$. The remaining parameters were taken as $\gamma = \frac{\pi}{5}$, $\mathtt{k} = \frac{1}{20}$.

2. Output from numerical work

Described here are some results of our numerical analysis of the spin chain with $\mathcal{Z}_r$ symmetry in the regime (1.12).

The study of the scaling limit requires one to assign an N dependence to the low energy states, i.e., to form the RG trajectories $|\Psi_N\rangle$. It is usually clear how to do this for the ground state or, for that matter, the lowest energy states in the disjoint sectors of the Hilbert space, say, in the sector with given value of S^z (eigenvalue of $\mathbb{S}^z$ from eq. (1.6)). However forming individual RG flow trajectories for low energy stationary states is not a trivial task. In the case at hand, the procedure is facilitated by the existence of the Bethe Ansatz equations.

One starts by diagonalizing the Hamiltonian for a small number of lattice sites $N = N_{\text{in}} \lesssim 20$.[2] Each stationary state may be chosen to be a Bethe state, which is an eigenvector of the full family

[2] The explicit formula for the Hamiltonian and the other members of the commuting family may be found in sec. 6 of ref. [6].

of commuting operators. An important member of this family is the Q-operator. Its eigenvalues are polynomials in ζ, whose zeroes $\{\zeta_m\}_{m=1}^{M}$ obey the Bethe Ansatz equations (1.1). For a given low energy state $|\Psi_{N_{\rm in}}\rangle$, one can compute the eigenvalue of the Q-operator and thus determine the corresponding set of Bethe roots. With these at hand, the state $|\Psi_N\rangle$ where $N = N_{\rm in} + 2r$ may be specified without an explicit diagonalization of the Hamiltonian. One finds the solution to the Bethe Ansatz equations, whose pattern qualitatively resembles the Bethe roots for $|\Psi_{N_{\rm in}}\rangle$ (for technical details, see ref. [13]). By iterating this procedure, the RG trajectory $|\Psi_N\rangle$ for increasing N is obtained. The corresponding energy is computed from the Bethe roots labelling the Bethe state as

$$\mathcal{E} = \sum_{\ell=1}^{r} \mathcal{E}^{(\ell)}\,, \qquad\qquad \mathcal{E}^{(\ell)} = 2\mathrm{i} \sum_{m=1}^{M} \left(\frac{1}{1+\zeta_m\, q^{-1}/\eta_\ell} - \frac{1}{1+\zeta_m\, q/\eta_\ell} \right), \tag{2.1}$$

while the eigenvalue of the lattice translation operator $\mathbb{K}$ is given by[3]

$$\mathcal{K} = \prod_{\ell=1}^{r} \mathcal{K}^{(\ell)}\,, \qquad\qquad \mathcal{K}^{(\ell)} = \mathrm{e}^{\mathrm{i}\pi\mathtt{k}}\, q^{-M} \prod_{m=1}^{M} \frac{\zeta_m + \eta_\ell\, q}{\zeta_m + \eta_\ell\, q^{-1}}\,. \tag{2.2}$$

Recall that the number of Bethe roots in the sector with given value S^z coincides with $M = \frac{N}{2} - S^z$.

It should be mentioned that for $r > 1$ the lattice Hamiltonian $\mathbb{H}$ is not Hermitian with respect to the usual matrix conjugation and some of its eigenvalues turn out to be complex numbers. To order the states we use the real part of their energy. This does not interfere with the notion of a low energy state since for such states $|\Psi_N\rangle$ the imaginary part of the difference $\mathcal{E} - \mathcal{E}_{\rm vac}$ decays faster than the real part as $N \to \infty$.

2.1. Low energy spectrum for odd r

We performed a study of the large N behaviour of the low energy spectrum and found that the states organize into conformal towers as predicted from conformal field theory [22]. For the case of odd r, the low energy-momentum spectrum is described as

$$\begin{aligned} \mathcal{E} &= N\, e_\infty + \frac{2\pi r\, v_{\rm F}}{N} \left(P^2 + \bar{P}^2 - \frac{r}{12} + \mathtt{L} + \bar{\mathtt{L}} \right) + o(N^{-1}) \\ \mathcal{K} &= (-1)^{r\mathtt{w}} \exp\left(\frac{2\pi \mathrm{i} r}{N} \left(P^2 - \bar{P}^2 + \mathtt{L} - \bar{\mathtt{L}} \right) \right). \end{aligned} \tag{2.3}$$

Here the specific bulk energy and Fermi velocity are given explicitly by

$$e_\infty = -\frac{2 v_{\rm F}}{\pi} \int_0^\infty \mathrm{d}t\, \frac{\sinh\left(\frac{rt}{n}\right)}{\sinh\left(\frac{(n+r)t}{n}\right)\cosh(t)}\,, \qquad\qquad v_{\rm F} = \frac{r\,(n+r)}{n} \tag{2.4}$$

and the anisotropy parameter $q = \mathrm{e}^{\mathrm{i}\gamma}$ has been swapped for n such that

$$\gamma = \frac{\pi}{n+r} \qquad\qquad (n > 0)\,. \tag{2.5}$$

[3] The formulae for the eigenvalues of the energy and lattice shift operator are valid for any values of the inhomogeneities provided the r-site periodicity condition (1.2) is satisfied. Clearly, for the $\mathcal{Z}_r$ symmetric case (1.10) they may be simplified, but we prefer to keep them as they are.

The P and $\bar{P}$ that enter into the $1/N$ corrections take the discrete set of values,

$$P = \frac{1}{2\sqrt{n+r}}\big(S^z + (n+r)\,(\mathtt{k}+\mathtt{w})\big)\,, \qquad \bar{P} = \frac{1}{2\sqrt{n+r}}\big(S^z - (n+r)\,(\mathtt{k}+\mathtt{w})\big)\,, \tag{2.6}$$

which are characterized by the eigenvalue of $\mathbb{S}^z$ and the so-called winding number $\mathtt{w} = 0, \pm 1, \pm 2, \ldots$. Finally, the pair of non-negative integers $(\mathtt{L}, \bar{\mathtt{L}})$ give the chiral levels of the state in the conformal tower.

The Bethe states for which the asymptotic behaviour (2.3) is satisfied with $\mathtt{L} = \bar{\mathtt{L}} = 0$ will be referred to as the primary Bethe states. We observe the surprising feature that for given S^z and $\mathtt{w}$, the number of primary Bethe states grows for increasing N and in all likelihood becomes infinite in the limit $N \to \infty$. This implies the existence of an infinite number of conformal towers labelled by the same pair of conformal dimensions. To resolve such a degeneracy, one should consider the scaling limit of the spectrum of other operators from the commuting family. We used the so-called quasi-shift operators,

$$\mathbb{K}^{(\ell)} \ \ (\ell = 1, 2, \ldots, r) \ : \qquad \big[\mathbb{K}^{(\ell)}, \mathbb{H}\big] = \big[\mathbb{K}^{(\ell)}, \mathbb{K}^{(\ell')}\big] = 0\,, \tag{2.7}$$

whose definition is given in sec. 6.2 of the work [6]. Note that these operators are reshuffled under the action of the $\mathcal{Z}_r$ symmetry:

$$\hat{\mathcal{D}}^{-1}\mathbb{K}^{(\ell)}\hat{\mathcal{D}} = \mathbb{K}^{(\ell+1)} \qquad \big(\mathbb{K}^{(r+1)} \equiv \mathbb{K}^{(1)}\big)\,, \tag{2.8}$$

where the notation $\hat{\mathcal{D}}$ from ref. [6] for the generator of the symmetry is being used. Also their product coincides with the r-site translation operator:

$$\mathbb{K} = \prod_{\ell=1}^{r} \mathbb{K}^{(\ell)}\,. \tag{2.9}$$

The eigenvalues of $\mathbb{K}^{(\ell)}$ are denoted as $\mathcal{K}^{(\ell)}$ and are given in terms of the Bethe roots as in eq. (2.2).

Keeping in mind the $\mathcal{Z}_r$ symmetry, we studied the discrete Fourier transform of the logarithm of $\mathcal{K}^{(\ell)}$:[4]

$$b_a \equiv \frac{N^{1-\frac{2|a|}{r}}}{2\pi \mathrm{i} r} \sum_{\ell=1}^{r} \mathrm{e}^{\frac{\mathrm{i}\pi}{r} a(r+1-2\ell)} \log\big(\mathcal{K}^{(\ell)}\big) \qquad \big(1 \le |a| \le [\tfrac{r-1}{2}]\big)\,. \tag{2.10}$$

The factor $N^{1-\frac{2|a|}{r}}$ was introduced for the following reason; we observed that for the low energy states $b_a = b_a(N)$, defined as above, tend to finite limits as $N \to \infty$ which, in general, are non-vanishing (complex) numbers. Also, in (2.10), the symbol $[\cdots]$ stands for the integer part. It's placed here in view of the later discussion of even r, where this same definition of b_a will be utilized.

[4] For the low energy states $\mathcal{K}^{(\ell)} \to (-1)^{\mathtt{w}}$ as $N \to \infty$, where $\mathtt{w} \in \mathbb{Z}$ is the winding number. The leading term in the asymptotics $\log(\mathcal{K}^{(\ell)}) = \mathrm{i}\pi + \ldots$ for $\mathtt{w}$ odd gives no contribution to the definition of b_a since it is cancelled when the sum over ℓ in (2.10) is taken.

The limiting values of b_a (2.10) as $N \to \infty$ are important characteristics of the RG trajectories. It turns out that they, together with P and $\bar{P}$ (2.6), are sufficient to distinguish the primary Bethe states. Also, we found that b_a appear in the corrections to the scaling of the energy in eq. (2.3). In particular, for any low energy Bethe state

$$\mathcal{E} = N\, e_\infty + \frac{2\pi r\, v_{\rm F}}{N}\Big(P^2 + \bar{P}^2 - \frac{r}{12} + \mathtt{L} + \bar{\mathtt{L}}\Big) \tag{2.11}$$
$$- \frac{4\pi^2 r\, v_{\rm F}}{N^{1+\frac{2}{r}}}\cot\Big(\frac{\pi}{2n}\Big)\, b_{\frac{r-1}{2}}\, b_{\frac{1-r}{2}} + O\Big(N^{-1-\frac{6}{r}}, N^{-1-\frac{4n}{r}}\Big) \qquad (r = 3, 5, \ldots)\,.$$

2.2. *Low energy spectrum for r even*

For even r special attention needs to be paid to the operator

$$\mathbb{B} = \prod_{m=1}^{\frac{r}{2}} \mathbb{K}^{(2m-1)}\big(\mathbb{K}^{(2m)}\big)^{-1}\,. \tag{2.12}$$

A numerical study of its eigenvalues shows there are low energy states in the lattice model such that

$$b_{\frac{r}{2}} \equiv \frac{(-1)^{\frac{r}{2}+1}}{2\pi r}\,\log(\mathcal{B}) = \frac{1}{2\pi r}\sum_{\ell=1}^{r}(-1)^{\ell+\frac{r}{2}}\log\big(\mathcal{K}^{(\ell)}\big) \tag{2.13}$$

either tends to a non-zero value or decays to zero as

$$b_{\frac{r}{2}} \sim \frac{1}{\log(N)} \qquad \text{for} \qquad N \to \infty\,. \tag{2.14}$$

Such a phenomenon was already observed and extensively studied in the context of the $\mathcal{Z}_2$ invariant spin chain ($r = 2$) [8,9,11–14]. The scaling limit should be taken such that the value of $b_{\frac{r}{2}}$ is kept fixed as $N \to \infty$. This leads to the presence of a continuous component in the spectrum of the underlying conformal field theory. In the case $r = 4, 6, \ldots$ and for the low energy states, b_a defined through (2.10) remain finite and (generically) non-vanishing as $N \to \infty$. This way, for both r odd and r even, the RG trajectories are labelled by the limiting values of b_a. For even r we observed that the large N asymptotic behaviour of the energy follows a similar formula to (2.11):

$$\mathcal{E} = N\, e_\infty + \frac{2\pi r\, v_{\rm F}}{N}\Big(P^2 + \bar{P}^2 + 2n\,(b_{\frac{r}{2}})^2 - \frac{r}{12} + \mathtt{L} + \bar{\mathtt{L}}\Big) \tag{2.15}$$
$$- \frac{8\pi^2 r\, v_{\rm F}}{N^{1+\frac{4}{r}}}\cot\Big(\frac{\pi}{n}\Big)\, b_{\frac{r}{2}-1}\, b_{1-\frac{r}{2}} + O\Big(N^{-1-\frac{8}{r}}, N^{-1-\frac{4n}{r}}\Big) \qquad (r = 4, 6, \ldots)\,.$$

The following comments are in order here. In our numerical work we constructed a variety of RG trajectories for $r = 2, 3, \ldots, 7$. In all the cases our data was in full agreement with the asymptotic formula

$$\mathcal{E} = N\, e_\infty + \frac{2\pi r\, v_{\rm F}}{N}\bigg[P^2 + \bar{P}^2 + 2n\,(b_{\frac{r}{2}})^2 - \frac{r}{12} + \mathtt{L} + \bar{\mathtt{L}} \tag{2.16}$$
$$- \sum_{a=1}^{[\frac{r-1}{2}]} 2\pi\,(r-2a)\cot\big(\tfrac{\pi\,(r-2a)}{2n}\big)\,\frac{b_a\, b_{-a}}{N^{2-\frac{4a}{r}}} + O\Big(N^{-2}, N^{-\frac{4n}{r}}\Big)\bigg],$$

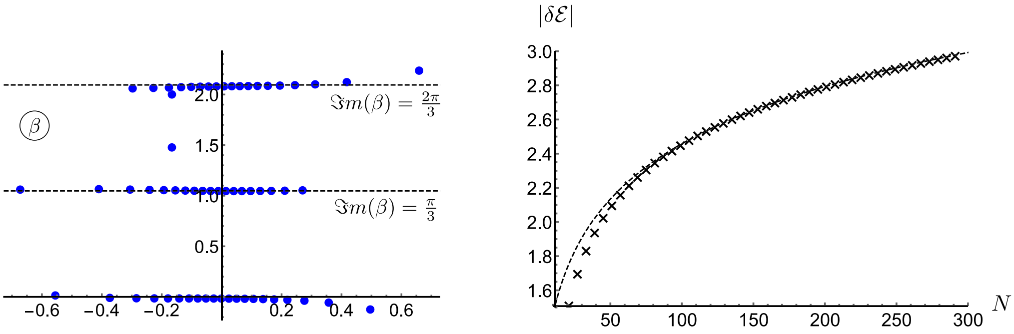

Fig. 5. Presented is numerical data for an RG trajectory of the $\mathcal{Z}_3$ invariant spin chain in the sector $S^z = \frac{1}{2}$ whose scaled energy $\delta\mathcal{E} = \frac{N}{2\pi r v_{\rm F}}\big(\mathcal{E} - Ne_\infty\big)$ grows logarithmically as $N \to \infty$. The typical pattern of Bethe roots ($N = 123$) is depicted in the complex β plane on the left panel. On the right panel, the crosses correspond to the numerical values of $|\delta\mathcal{E}|$ obtained via the solution of the Bethe Ansatz equations. The dashed line is a plot of the absolute value of the fit $\delta\mathcal{E} \approx -0.03 - 1.11\,\mathrm{i} + (0.521 + 0.099\,\mathrm{i})\log(N)$. The parameters are $n = 3$ and $\mathtt{k} = \frac{1}{20}$.

where $b_{\frac{r}{2}} \equiv 0$ for odd r and n is assumed to be generic. Also, it is worth mentioning that the summands $\mathcal{E}^{(\ell)}$ in (2.1) coincide with the eigenvalues of the mutually commuting operators

$$\mathbb{H}^{(\ell)} \ : \qquad \big[\mathbb{H}^{(\ell)}, \mathbb{H}^{(\ell')}\big] = 0\,, \qquad\qquad \mathbb{H} = \sum_{\ell=1}^{r} \mathbb{H}^{(\ell)}\,. \tag{2.17}$$

Similar to $\mathbb{H}$ these are given by a sum of terms describing interactions of up to $r + 1$ adjacent spins. However, they are not $\mathcal{Z}_r$ invariant, but rather

$$\hat{\mathcal{D}}^{-1}\mathbb{H}^{(\ell)}\hat{\mathcal{D}} = \mathbb{H}^{(\ell+1)} \qquad\qquad \big(\mathbb{H}^{(r+1)} \equiv \mathbb{H}^{(1)}\big)\,. \tag{2.18}$$

For the leading large N behaviour of their eigenvalues we found that for $r \geq 2$

$$\sum_{\ell=1}^{r} \mathrm{e}^{\frac{\mathrm{i}\pi}{r}a(r+1-2\ell)}\,\mathcal{E}^{(\ell)} = 2\pi\, v_{\rm F}\,\mathrm{sgn}(a)\,(r - 2|a|)\,N^{-1+\frac{2|a|}{r}}\,\big(b_a + o(1)\big) \qquad \big(1 \leq |a| \leq [\tfrac{r}{2}]\big)\,, \tag{2.19}$$

where $\mathrm{sgn}(a)$ denotes the sign of the integer a. Finally, Bethe states were observed for which $\delta\mathcal{E} = \frac{N}{2\pi r v_{\rm F}}\big(\mathcal{E} - Ne_\infty\big)$ grows logarithmically as $N \to \infty$, see Fig. 5. We do not count these states as being part of the low energy spectrum. A similar phenomenon was discussed in the case of the $\mathcal{Z}_2$ invariant spin chain in the original works [7,8].

3. ODEs for the scaling limit of the $\mathcal{Z}_r$ invariant spin chain

As was mentioned in the Introduction, the most effective way of studying the universal behaviour of integrable lattice systems is based on the ODE/IQFT correspondence. In this section, we propose the set of Ordinary Differential Equations which describe the scaling limit of the $\mathcal{Z}_r$ invariant spin chain where the anisotropy parameter is taken to be $q = \mathrm{e}^{\frac{\mathrm{i}\pi}{n+r}}$ with $n > 0$. Also, some numerical verifications of the proposal are discussed.

3.1. Differential equations for the primary Bethe states

The limiting values of b_a (2.10), (2.13) as $N \to \infty$ are important characteristics of an individual RG trajectory. For our purposes, it will be convenient to switch from these to

$$s_a = C_a \operatorname*{slim}_{N\to\infty} b_a\,, \qquad \bar{s}_a = -C_a \operatorname*{slim}_{N\to\infty} b_{-a}\,, \qquad 1 \le a \le [\tfrac{r}{2}] \tag{3.1}$$

with the proportionality coefficients being given by

$$C_a = (-1)^{a(r-1)}\, 2n\, \frac{\sqrt{\pi}\,\Gamma(1+\frac{r-2a}{2n})}{\Gamma(\frac{1}{2}+\frac{r-2a}{2n})} \left[\frac{\Gamma(\frac{3}{2}+\frac{r}{2n})}{\sqrt{\pi}\,\Gamma(1+\frac{r}{2n})}\right]^{\frac{r-2a}{r}}\,. \tag{3.2}$$

The symbol "slim" is used as a reminder that the limit (3.1) exists only for the class of low energy states and for even r should be taken such that $b_{\frac{r}{2}}$ is kept fixed as $N \to \infty$. Note that, by definition $b_{-\frac{r}{2}} \equiv -b_{\frac{r}{2}}$ and therefore

$$\bar{s}_{\frac{r}{2}} \equiv s_{\frac{r}{2}}\,. \tag{3.3}$$

The s_a and $\bar{s}_a$ will be combined into the two sets

$$\boldsymbol{s} = \big(s_1, \ldots, s_{[\frac{r}{2}]}\big)\,, \qquad \bar{\boldsymbol{s}} = \big(\bar{s}_1, \ldots, \bar{s}_{[\frac{r}{2}]}\big)\,. \tag{3.4}$$

Also we'll use the notations

$$p = \frac{n+r}{2}\left(\frac{S^z}{n+r} + \mathtt{k} + \mathtt{w}\right), \qquad \bar{p} = \frac{n+r}{2}\left(\frac{S^z}{n+r} - \mathtt{k} - \mathtt{w}\right). \tag{3.5}$$

Among other things, the ODE/IQFT correspondence implies a relation between the scaling limit of the eigenvalues of the Q-operator for the low energy Bethe states and the spectral determinants of a certain set of ODEs. In the case at hand, the primary Bethe states of the $\mathcal{Z}_r$ invariant spin chain are fully characterized by p, $\bar{p}$ as well as $\boldsymbol{s}$, $\bar{\boldsymbol{s}}$ and the corresponding pair of differential equations are proposed to be

$$\left[-\partial_z^2 + \frac{1}{z^2}\left(E^{-n-r}\, z^{n+r} - z^r + p^2 - \frac{1}{4} + \sum_{a=1}^{[\frac{r}{2}]} s_a\,(-z)^a\right)\right]\Psi = 0 \tag{3.6}$$

$$\left[-\partial_{\bar{z}}^2 + \frac{1}{\bar{z}^2}\left(\bar{E}^{-n-r}\, \bar{z}^{n+r} - \bar{z}^r + \bar{p}^2 - \frac{1}{4} + \sum_{a=1}^{[\frac{r}{2}]} \bar{s}_a\,(-\bar{z})^a\right)\right]\bar{\Psi} = 0\,. \tag{3.7}$$

The cases $r=1$ and $r=2$ have been extensively explored in the work [14] (see also Appendix A). In order to not overburden the text with technical details, the similar analysis for general r will not be repeated here. Instead we give a brief reminder of the construction of the spectral determinant, which is numerically efficient, focusing on eq. (3.6).

It is convenient to re-write the ODE using the variables

$$z = E\,\mathrm{e}^y\,, \qquad \Psi = \mathrm{e}^{\frac{y}{2}}\,\psi\,. \tag{3.8}$$

The differential equation (3.6) then becomes

$$\left[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} - E^r\,\mathrm{e}^{ry} + \sum_{a=1}^{[\frac{r}{2}]} s_a\,(-E\,\mathrm{e}^y)^a\right]\psi = 0\,. \tag{3.9}$$

Assuming that $\Re e(p) \geq 0$, there exists a solution which is uniquely defined by the asymptotic condition

$$\psi_p(y) \to e^{py} \qquad \text{as} \qquad y \to -\infty\,. \tag{3.10}$$

Introduce another solution, $\chi = \chi(y)$, by means of the WKB asymptotic:

$$\chi \asymp \exp\Big[-\frac{n+r}{4}\,y - \frac{2}{n+r}\,e^{(n+r)\frac{y}{2}}\,{}_2F_1\Big(-\frac{1}{2}, -\frac{1}{2}-\frac{r}{2n}, \frac{1}{2}-\frac{r}{2n}\,\Big|\,E^r e^{-ny}\Big)\Big] \tag{3.11}$$
as $y \to +\infty$.

Here ${}_2F_1$ is the conventional hypergeometric function and it is assumed that $\frac{r}{n} \neq 1, 3, \ldots$. The spectral determinant is given in terms of the Wronskian of the two solutions:

$$D_p(E) = \frac{\sqrt{\pi}}{\Gamma(1+\frac{2p}{n+r})}\,(n+r)^{-\frac{1}{2}-\frac{2p}{n+r}}\,\big(\chi\,\partial_y\psi_p - \psi_p\partial_y\chi\big) \tag{3.12}$$

with the factor out the front being chosen so that $D_p(0) = 1$ for generic values of p.

The spectral determinant is an entire function of E and hence the series

$$\log D_p(E) = -\sum_{j=1}^{\infty} J_j\,E^j \tag{3.13}$$

has a finite radius of convergence. The expansion coefficients depend on p and $\boldsymbol{s} = (s_1, \ldots, s_{[\frac{r}{2}]})$ (also, of course, on $n > 0$),

$$J_j = J_j(p, \boldsymbol{s})\,, \tag{3.14}$$

which are parameters of the differential equation. By applying perturbation theory in E to the ODE (3.9) it is possible to obtain explicit analytic formulae for the first two of them. The result for the cases $r = 1, 2$ is quoted in ref. [14]. It possesses a straightforward generalization for $r \geq 3$:

$$J_1 = s_1\;f_1\big(\tfrac{p}{n+r}, \tfrac{1}{n+r}\big)\;C \tag{3.15}$$
$$J_2 = \bigg(s_1^2\;f_2\big(\tfrac{p}{n+r}, \tfrac{1}{n+r}\big) - s_2\;2^{\frac{4}{n+r}}\;\frac{\pi\,\Gamma^2(-\frac{1}{n+r})}{\Gamma^2(\frac{1}{2}-\frac{1}{n+r})}\;f_1\big(\tfrac{p}{n+r}, \tfrac{2}{n+r}\big)\bigg)\;C^2$$

with

$$C = \frac{(n+r)^{\frac{2}{n+r}}}{\Gamma^2(-\frac{1}{n+r})} \tag{3.16}$$

(for $r = 3$, one should set s_2 identically to zero). The explicit form of the functions f_1 and f_2 are presented in Appendix B.

The eigenvalues of the lattice Q-operator are polynomials in the spectral parameter ζ, whose roots satisfy the Bethe Ansatz equations. In the notations of ref. [6],[5]

[5] As explained in that work, the Baxter TQ-relation possesses two solutions $\mathbb{Q}_\pm$. The operator $\mathbb{A}_+$ coincides with $\mathbb{Q}_+$ up to a simple factor involving fractional powers of ζ and has the advantage that its eigenvalues $A_+(\zeta)$ are polynomials in ζ of order $M \leq N/2$.

$$A_+(\zeta) = \prod_{m=1}^{M} \left(1 - \frac{\zeta}{\zeta_m}\right) \qquad\qquad (M \le N/2)\,. \tag{3.17}$$

The coefficients of the polynomial are simply expressed in terms of the sums

$$h_N^{(j)} = j^{-1} \sum_{m=1}^{M} (\zeta_m)^{-j}\,, \tag{3.18}$$

for instance,

$$\log A_+(\zeta) = -h_N^{(1)}\,\zeta - h_N^{(2)}\,\zeta^2 + \dots\,. \tag{3.19}$$

For the low energy Bethe states, the scaling behaviour of $h_N^{(j)}$ is described by J_j from the expansion (3.13) of the spectral determinant. In particular,

$$\begin{aligned} &\operatorname*{slim}_{N\to\infty} \left(\frac{N}{rN_0}\right)^{-\frac{2n}{r(n+r)}} h_N^{(1)} = J_1 \\ &\operatorname*{slim}_{N\to\infty} \left(\frac{N}{rN_0}\right)^{-\frac{4n}{r(n+r)}} h_N^{(2)} = J_2\,, \end{aligned} \tag{3.20}$$

where

$$N_0 = \frac{\sqrt{\pi}\,\Gamma(1+\frac{r}{2n})}{r\Gamma(\frac{3}{2}+\frac{r}{2n})}\,. \tag{3.21}$$

Having at hand a low energy Bethe state at fixed $N \gg 1$, the l.h.s. of eqs. (3.20) may be computed from the corresponding Bethe roots by means of (3.18). Also, from the eigenvalues of the quasi-shift operators $\mathcal{K}^{(\ell)}$ (2.2) one extracts the complex numbers b_a via the definitions (2.10) and (2.13). These determine the parameters s_a appearing on the ODE side through formula (3.1). For the primary Bethe states, the r.h.s. may be calculated using the explicit analytic expressions (3.15) for the coefficients J_1, J_2 as functions of s_a and p.

Similar relations hold true for the sums

$$\bar{h}_N^{(j)} = j^{-1} \sum_{m=1}^{M} (\zeta_m)^{j}\,. \tag{3.22}$$

They take the same form as (3.20) with $h_N^{(j)} \mapsto \bar{h}_N^{(j)}$, while $J_j(p, \boldsymbol{s})$ are replaced by $J_j(\bar{p}, \bar{\boldsymbol{s}})$. The latter occur in the Taylor series of the logarithm of the spectral determinant for the second ODE (3.7). Note that $\bar{h}_N^{(j)}$ are the expansion coefficients of $-\log\big(\prod_{m=1}^{M}(1-\zeta_m/\zeta)\big)$ in the variable ζ^{-1}:

$$\bar{A}_+(\zeta) \equiv \prod_{m=1}^{M} \left(1 - \frac{\zeta_m}{\zeta}\right) = \exp\left(-\sum_{j=1}^{\infty} \bar{h}_N^{(j)}\,\zeta^{-j}\right). \tag{3.23}$$

The polynomials $\bar{A}_+(\zeta)$ are eigenvalues of

$$\bar{\mathbb{A}}_+(\zeta) = \zeta^{\mathbb{S}^z - N/2}\,\mathbb{A}_+(\zeta)\,\big(\mathbb{A}_+^{(\infty)}\big)^{-1}\,, \tag{3.24}$$

where the operator $\mathbb{A}_+^{(\infty)}$ belongs to the commuting family and its eigenvalue for a Bethe state with corresponding Bethe roots $\{\zeta_m\}$ is given by the product:

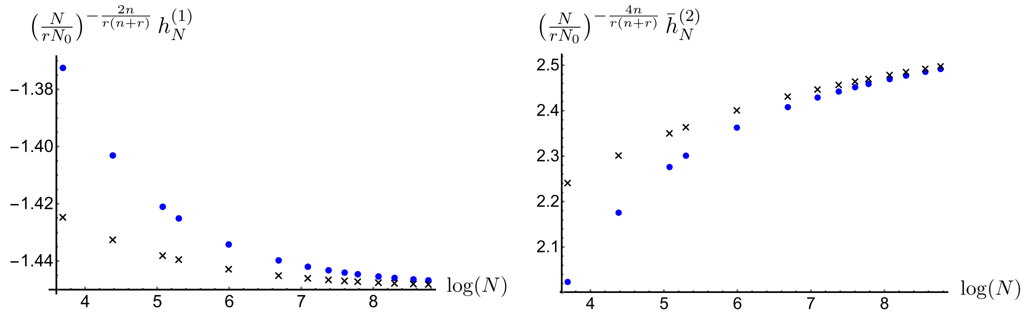

Fig. 6. Depicted is numerical data for $h_N^{(1)}$ (left panel) and $\bar{h}_N^{(2)}$ (right panel), scaled by the appropriate N dependent factor, as a function of $\log(N)$ for the $\mathcal{Z}_r$ symmetric model with $r = 4$. The Bethe state was chosen to be a primary one, and the black crosses were obtained from the corresponding Bethe roots via the definitions (3.18) and (3.22). The blue dots depict the values of $J_1(p, s_1, s_2)$ and $J_2(\bar{p}, \bar{s}_1, \bar{s}_2)$ for the left and right panels, respectively, which are given by formula (3.15). Here s_1 ($\bar{s}_1$) was substituted by $C_1\, b_1(N)$ ($-C_1\, b_{-1}(N)$), see eq. (3.1), where $b_{\pm 1}(N)$ were obtained by means of (2.10) from the eigenvalues of the quasi-shift operators. Similarly, $s_2 \equiv \bar{s}_2 \mapsto C_2\, b_2(N)$ with $b_2(N)$ being computed via (2.13). The parameters were set to be $n = \frac{1}{2}$, $\mathtt{k} = \frac{1}{10}$, while $S^z = \mathtt{w} = 0$. Note that, for computing $J_2(\bar{p}, \bar{s}_1, \bar{s}_2)$ in the case when $\bar{p} = -(n+r)\,\frac{\mathtt{k}}{2} = -\frac{9}{40} < 0$, one needs to evaluate the function f_2 entering into (3.15) when the first argument is negative. Formulae (B.2)-(B.4) remain applicable provided the integration contour over the real line is locally deformed such that it goes below the pole at $x = \mathrm{i}h$.

$$A_+^{(\infty)} = \prod_{m=1}^{M} (-\zeta_m)^{-1}\,. \tag{3.25}$$

We performed extensive numerical checks of the scaling relation (3.20) and its barred counterpart. Some of the data is presented in Fig. 6.

3.2. Eigenvalues of the Q-operator in the scaling limit

For the excited states, the ODEs are obtained from (3.6) and (3.7) following the well known procedure [20]. It was discussed in refs. [13,14] for the case of the $\mathcal{Z}_2$ invariant spin chain. With the same line of arguments, we propose that the generalization of the ODE (3.6) that describes the scaling limit of any low energy Bethe state of the $\mathcal{Z}_r$ invariant model has the form

$$\Big(-\partial_z^2 + t_0(z) + t_1(z) + E^{-n-r}\, z^{n+r-2}\Big)\,\Psi = 0\,, \tag{3.26}$$

where

$$t_0(z) = \frac{1}{z^2}\left(-z^r + p^2 - \frac{1}{4} + \sum_{a=1}^{[\frac{r}{2}]} s_a\,(-z)^a\right) \tag{3.27a}$$

$$t_1(z) = \sum_{\alpha=1}^{\mathtt{L}} \left(\frac{2}{(z-w_\alpha)^2} + \frac{n_\alpha}{z(z-w_\alpha)}\right)\,. \tag{3.27b}$$

The position of the poles w_α and the residues n_α must be taken in such a way that any solution of the ODE is single valued in the vicinity of $z = w_\alpha$ for any $\alpha = 1, \ldots, \mathtt{L}$. This condition leads to the system of algebraic equations on w_α and n_α:

$$n_\alpha\left(\tfrac{1}{4}\, n_\alpha^2 - w_\alpha^2\, t_0^{(\alpha)}\right) + w_\alpha^3\, t_1^{(\alpha)} = 0 \tag{3.28}$$
$$n_\alpha = n + r - 2 \qquad\qquad (\alpha = 1, 2, \ldots, \mathtt{L})\,.$$

Here

$$t_0^{(\alpha)} = t_0(w_\alpha) - \frac{n_\alpha}{w_\alpha^2} + \sum_{\beta\neq\alpha}\left(\frac{2}{(w_\alpha - w_\beta)^2} + \frac{n_\beta}{w_\alpha(w_\alpha - w_\beta)}\right) \tag{3.29}$$

$$t_1^{(\alpha)} = t_0'(w_\alpha) + \frac{n_\alpha}{w_\alpha^3} - \sum_{\beta\neq\alpha}\left(\frac{4}{(w_\alpha - w_\beta)^3} + \frac{n_\beta\,(2w_\alpha - w_\beta)}{w_\alpha^2\,(w_\alpha - w_\beta)^2}\right)$$

and the prime stands for the derivative w.r.t. the argument. The ODE that would generalize (3.7) takes an analogous form. It involves the additional term

$$\bar{t}_1(\bar{z}) = \sum_{\alpha=1}^{\bar{\mathtt{L}}}\left(\frac{2}{(\bar{z} - \bar{w}_\alpha)^2} + \frac{\bar{n}_\alpha}{\bar{z}(\bar{z} - \bar{w}_\alpha)}\right), \tag{3.30}$$

where the set $\bar{n}_\alpha$ and $\{\bar{w}_\alpha\}_{\alpha=1}^{\bar{\mathtt{L}}}$ satisfy the system obtained from eqs. (3.28) and (3.29) by the formal substitutions $(p, s_a, n_\alpha, w_\alpha, \mathtt{L}) \mapsto (\bar{p}, \bar{s}_a, \bar{n}_\alpha, \bar{w}_\alpha, \bar{\mathtt{L}})$.

We expect that any low energy Bethe state $|\Psi_N\rangle$ flows to the state in the conformal tower labelled by, together with the parameters p, $\bar{p}$, $\boldsymbol{s}$, $\bar{\boldsymbol{s}}$, also two sets of "apparent" singularities:

$$\boldsymbol{w} = \{w_\alpha\}_{\alpha=1}^{\mathtt{L}}\,, \qquad\qquad \bar{\boldsymbol{w}} = \{\bar{w}_\alpha\}_{\alpha=1}^{\bar{\mathtt{L}}}\,. \tag{3.31}$$

The scaling behaviour of the eigenvalue $A_+(\zeta)$ (3.17) of the Q-operator for the state $|\Psi_N\rangle$ is then described in terms of the spectral determinant $D_p(E\,|\,\boldsymbol{w})$ of the ODE (3.26) and $\bar{D}_{\bar{p}}(\bar{E}\,|\,\bar{\boldsymbol{w}})$ of the "barred" differential equation. For the excited states, formulae (3.8), (3.10)-(3.12) involved in the definition of $D_p(E\,|\,\boldsymbol{w})$ remain the same.[6] The corresponding equations for $\bar{D}_{\bar{p}}(\bar{E}\,|\,\bar{\boldsymbol{w}})$ are only notationally different. Similar to the case of the $\mathcal{Z}_2$ invariant spin chain studied in [14] we expect that

$$\operatorname*{slim}_{N\to\infty} G^{(N/r)}\big(E^r\,|\,\tfrac{r}{n+r}\big)\ A_+\Big(\big(N/(rN_0)\big)^{-\frac{2n}{r(n+r)}}\,E\Big) = D_p(E\,|\,\boldsymbol{w})\,. \tag{3.32}$$

Here the function

$$G^{(N)}(E\,|\,g) = \exp\left(\sum_{m=1}^{[\frac{1}{2(1-g)}]}\frac{(-1)^m\,N}{2m\cos(\pi m g)}\,(N/N_0)^{2m(g-1)}\,E^m\right), \qquad g\neq 1-\tfrac{1}{2k} \tag{3.33}$$

(which is the same for all the states) has been introduced to ensure the existence of the limit. Note that the result is valid for any $n>0$ provided that $\frac{r}{n}\neq 1,3,5,\ldots$. At these special values additional terms $\propto\log(N)$ need to be included in the exponent in the definition of $G^{(N)}(E\,|\,g)$ for the limits (3.32) to exist, see ref. [14] for details. Similarly, the scaling limit of the eigenvalues of the operator $\bar{\mathbb{A}}_+(\zeta)$ (3.24) is described as

$$\operatorname*{slim}_{N\to\infty} G^{(N/r)}\big(\bar{E}^r\,|\,\tfrac{r}{n+r}\big)\ \bar{A}_+\Big(\big(N/(rN_0)\big)^{\frac{2n}{r(n+r)}}\,\bar{E}^{-1}\Big) = \bar{D}_{\bar{p}}(\bar{E}\,|\,\bar{\boldsymbol{w}})\,. \tag{3.34}$$

A verification of the conjectured relations (3.32) and (3.34) may be carried out along the same lines as for the primary Bethe states. The scaling relations (3.20) for the sums $h_N^{(j)}$ (3.18) and

[6] The additional term $t_1(z)$ in the ODE does not change the leading asymptotic behaviour of the solutions ψ_p and χ in the vicinity of the singularities at $z=0$ and $z=\infty$.

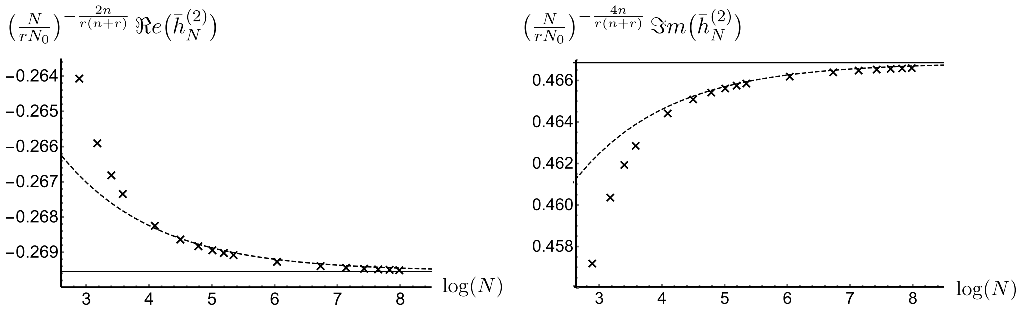

Fig. 7. Presented is numerical data for an RG trajectory of the $\mathcal{Z}_3$ invariant spin chain with $n=2$, $\mathtt{k}=\frac{1}{20}$, characterized by $S^z=1$, $\mathtt{w}=0$ and levels $(\mathtt{L},\bar{\mathtt{L}})=(0,1)$. The values of $s_1=-0.942033-1.631649\,\mathrm{i}$ and $\bar{s}_1=0.085024-0.147266\,\mathrm{i}$ were extracted by means of (3.1) using a certain interpolation procedure to take the limit $N\to\infty$. The crosses stand for the real and imaginary parts of $\big(\frac{N}{rN_0}\big)^{-\frac{2n}{r(n+r)}}\,\bar{h}_N^{(2)}$, where $\bar{h}_N^{(2)}$ denotes the sum over the Bethe roots (3.22). The solid lines represent the prediction of the barred counterpart to the scaling formula (3.20), namely, $J_2(\bar{p},\bar{\boldsymbol{s}}\,|\,\bar{\boldsymbol{w}})=-0.269543+0.466857\,\mathrm{i}$. The latter was obtained via a numerical integration of the ODE (3.26), (3.27) with the parameters replaced as $p\mapsto\bar{p}=\frac{3}{8}$, $\mathtt{L}\mapsto\bar{\mathtt{L}}=1$, $s_1\mapsto\bar{s}_1=-0.942033-1.631649\,\mathrm{i}$ and $\{w_\alpha\}\mapsto\{\bar{w}_\alpha\}$ with $\bar{w}_1=0.778812-1.348942\,\mathrm{i}$, therein. The dashed lines depict the fit $-0.269539+0.466854\,\mathrm{i}+(0.0186614-0.0323224\,\mathrm{i})\,N^{-\frac{2}{3}}$.

their barred versions still hold true provided the J_j in the r.h.s. are taken to be the expansion coefficients (3.13) of the spectral determinant for eq. (3.26). Apart from p and s, they depend now on the set $\{w_\alpha\}_{\alpha=1}^{\mathtt{L}}$, which solves the algebraic system (3.28), (3.29), i.e.,

$$J_j=J_j(p,\boldsymbol{s}\,|\,\boldsymbol{w})\,,\qquad\qquad \bar{J}_j=J_j(\bar{p},\bar{\boldsymbol{s}}\,|\,\bar{\boldsymbol{w}})\,. \tag{3.35}$$

For the case of the excited states no explicit analytical expressions are available for the expansion coefficients J_j. Nevertheless, they may be obtained via a numerical integration of the differential equations. Together with the primary states, a verification of (3.32)-(3.34) for the excited states was performed, see Fig. 7. We also analyzed the algebraic system (3.28), (3.29). It was observed that for given $\mathtt{L}$ and generic values of the parameters n, p and $\boldsymbol{s}$ the number of solutions is equal to the r coloured partitions of $\mathtt{L}$:

$$\sum_{\mathtt{L}=0}^{\infty}\mathrm{par}_r(\mathtt{L})\,\mathtt{q}^{\mathtt{L}}=\prod_{m=1}^{\infty}\frac{1}{(1-\mathtt{q}^m)^r}=1+r\mathtt{q}+\frac{1}{2}\,r\,(r+3)\,\mathtt{q}^2+\frac{1}{6}\,r\,(r+1)(r+8)\,\mathtt{q}^3+\dots\,. \tag{3.36}$$

3.3. Product rules

Being an entire function of E, the spectral determinant admits a convergent series expansion near zero. At infinity, it possesses an asymptotic expansion, whose description requires one to identify the Stokes lines, i.e., the lines which split the complex E plane into domains where the asymptotic behaviour is described differently. In the case at hand, the Stokes lines are the lines of accumulation of zeroes. Their location may be deduced from formula (3.32) for $D_p(E\,|\,\boldsymbol{w})$ and (3.34) for $\bar{D}_{\bar{p}}(\bar{E}\,|\,\bar{\boldsymbol{w}})$ from the knowledge of the pattern of Bethe roots for the low energy Bethe states of the $\mathcal{Z}_r$ invariant spin chain. According to (1.11), the Bethe roots accumulate along lines parallel to the real axis in the complex β plane with $\beta=-\frac{1}{2}\log(\zeta)$. At the edges of the distribution, the roots develop a scaling behaviour. Namely, ordering $\{\zeta_m\}_{m=1}^{M}$ w.r.t. their absolute value,

$$|\zeta_1| \le |\zeta_2| \le \ldots \le |\zeta_{M-1}| \le |\zeta_M|\,,$$

formulae (3.32) and (3.34) imply the existence of the limits

$$\operatorname*{slim}_{\substack{N\to\infty\\ j\ \text{fixed}}} \big(N/(rN_0)\big)^{\frac{2n}{r(n+r)}}\,\zeta_j = E_j\,, \qquad \operatorname*{slim}_{\substack{N\to\infty\\ M-j\ \text{fixed}}} \big(N/(rN_0)\big)^{\frac{2n}{r(n+r)}}\,\zeta_{M-j}^{-1} = \bar{E}_j\,. \tag{3.37}$$

Evidently, E_j coincide with the zeroes of the spectral determinant $D_p(E\,|\,\boldsymbol{w})$ and accumulate along the rays

$$\arg(E) = 0,\ \frac{2\pi}{r},\ \frac{4\pi}{r},\ldots,\ \frac{2\pi(r-1)}{r} \qquad (\mathrm{mod}\, 2\pi) \tag{3.38}$$

and similarly for $\bar{E}_j$.

The rays (3.38) divide the complex plane into wedges, in which the spectral determinants exhibit a different large E ($\bar{E}$) asymptotic behaviour. To write down the asymptotic formulae for the different wedges, which will be labelled by $\ell = 1, 2, \ldots, r$, we swap E and $\bar{E}$ for θ as

$$E = (-1)^{r-1}\,\mathrm{e}^{+\frac{\mathrm{i}\pi}{r}(2\ell-1)}\,\mathrm{e}^{\frac{2n\theta}{r(n+r)}}\,, \qquad \bar{E} = (-1)^{r-1}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{r}(2\ell-1)}\,\mathrm{e}^{\frac{2n\theta}{r(n+r)}}\,. \tag{3.39}$$

Then, from an analysis of the ODE (3.26) and its barred counterpart, one can show that as

$$\Re e(\theta) \to +\infty \qquad \text{and} \qquad |\Im m(\theta)| < \frac{\pi(n+r)}{2n} \tag{3.40}$$

the following asymptotics hold true

$$\begin{aligned} D_p(E\,|\,\boldsymbol{w}) &= \mathfrak{C}^{(\ell)}_{p,s}(\boldsymbol{w})\,\exp\bigg[\Big((-1)^{\ell-1}\,\mathrm{i}s - \frac{2np}{n+r}\Big)\frac{\theta}{r} + \frac{N_0}{\cos(\frac{\pi r}{2n})}\,\mathrm{e}^{\theta} + O\Big(\mathrm{e}^{-\frac{\theta}{r}}\Big)\bigg] \\ \bar{D}_{\bar{p}}(\bar{E}\,|\,\bar{\boldsymbol{w}}) &= \bar{\mathfrak{C}}^{(\ell)}_{\bar{p},\bar{s}}(\bar{\boldsymbol{w}})\,\exp\bigg[\Big((-1)^{\ell-1}\,\mathrm{i}s - \frac{2n\bar{p}}{n+r}\Big)\frac{\theta}{r} + \frac{N_0}{\cos(\frac{\pi r}{2n})}\,\mathrm{e}^{\theta} + O\Big(\mathrm{e}^{-\frac{\theta}{r}}\Big)\bigg]. \end{aligned} \tag{3.41}$$

Here, by definition,

$$s \equiv \begin{cases} 0 & \text{for odd } r \\ s_{\frac{r}{2}} & \text{for even } r \end{cases} \tag{3.42}$$

and the numerical constant N_0 is given by (3.21). For $r = 1, 2$ the coefficients $\mathfrak{C}^{(\ell)}_{p,s}(\boldsymbol{w})$ and $\bar{\mathfrak{C}}^{(\ell)}_{\bar{p},\bar{s}}(\bar{\boldsymbol{w}})$ are available in closed analytic form [25]. An efficient numerical procedure for their computation for any r is described in Appendix C.

Focusing on the cases with $r = 1$ and $r = 2$, it was observed in refs. [21] and [14], respectively, that $\mathfrak{C}^{(\ell)}_{p,s}(\boldsymbol{w})$ and $\bar{\mathfrak{C}}^{(\ell)}_{\bar{p},\bar{s}}(\bar{\boldsymbol{w}})$ appear in the scaling limit of certain products over the Bethe roots. We found that the similar relations hold true for $r \ge 3$, namely:

$$\begin{aligned} \prod_{m=1}^{M} q\,\big(\zeta_m + \eta_\ell\, q^{-1}\big)\big(\zeta_m^{-1} + \eta_{\ell'}^{-1}\, q^{-1}\big) &= \mathrm{e}^{((-1)^{\ell'}-(-1)^{\ell})\frac{\pi s}{2n}}\,\mathfrak{C}^{(\ell)}_{p,s}(\boldsymbol{w})\,\bar{\mathfrak{C}}^{(\ell')}_{\bar{p},\bar{s}}(\bar{\boldsymbol{w}}) \\ &\times\Big(\frac{N}{rN_0}\Big)^{\mathrm{i}((-1)^{\ell'}-(-1)^{\ell})\frac{s}{r}-\frac{2n(p+\bar{p})}{r(n+r)}}\,\Big(\frac{4n}{n+r}\Big)^{\frac{N}{r}}\,\Big(1+O\big(N^{-\epsilon}\big)\Big) \end{aligned} \tag{3.43}$$

Here the remainder terms fall off as a power of N with some positive exponent ϵ and η_ℓ stand for the inhomogeneities as in eq. (1.10), i.e.,

$$\eta_\ell = (-1)^r \, e^{\frac{i\pi}{r}(2\ell-1)} \qquad (\ell = 1, 2, \ldots, r)\,. \tag{3.44}$$

In addition to the relations (3.43), we also established that

$$\prod_{m=1}^{M} (\zeta_m)^r = \left(\frac{N}{rN_0}\right)^{\frac{2n(p-\bar{p})}{n+r}} \prod_{\ell=1}^{r} \frac{\bar{\mathfrak{C}}^{(\ell)}_{\bar{p},\bar{s}}(\bar{\boldsymbol{w}})}{\mathfrak{C}^{(\ell)}_{p,s}(\boldsymbol{w})} \, \Big(1 + O\big(N^{-\epsilon}\big)\Big)\,. \tag{3.45}$$

By taking the product of the l.h.s. of eq. (3.43) over $\ell = \ell' = 1, 2, \ldots, r$ and dividing/multiplying the result by $\prod_{m=1}^{M} (\zeta_m)^r$ one obtains

$$\prod_{m=1}^{M} \big(\zeta_m^{-r} + q^r\big)\big(\zeta_m^{-r} + q^{-r}\big) = \prod_{\ell=1}^{r} \Big(\mathfrak{C}^{(\ell)}_{p,s}(\boldsymbol{w})\Big)^2 \left(\frac{N}{rN_0}\right)^{-\frac{4np}{n+r}} \left(\frac{4n}{n+r}\right)^N \Big(1 + O\big(N^{-\epsilon}\big)\Big)$$

$$\prod_{m=1}^{M} \big(\zeta_m^{+r} + q^r\big)\big(\zeta_m^{+r} + q^{-r}\big) = \prod_{\ell=1}^{r} \Big(\bar{\mathfrak{C}}^{(\ell)}_{\bar{p},\bar{s}}(\bar{\boldsymbol{w}})\Big)^2 \left(\frac{N}{rN_0}\right)^{-\frac{4n\bar{p}}{n+r}} \left(\frac{4n}{n+r}\right)^N \Big(1 + O\big(N^{-\epsilon}\big)\Big)\,. \tag{3.46}$$

The advantage of the last two product rules is that they involve the asymptotic coefficients for the spectral determinant of the ODE (3.26) and its barred variant separately.

The above product rules provide additional support for the proposal for the scaling limit of the eigenvalues of the Q-operator (3.32) and (3.34). We performed numerical checks for a range of cases with some of the data being presented in Fig. 8.

4. Quantization condition

A complete description of the conformal field theory underlying the critical behaviour of the inhomogeneous six-vertex model in the regime $q = e^{\frac{i\pi}{n+r}}$ with $n > 0$ is a difficult task. The proposed ODEs provide some guidance that allows one to make progress in this problem. Our analysis suggests that under a suitable normalization, the low energy Bethe states $|\Psi_N\rangle$ for the $\mathcal{Z}_r$ invariant spin chain possess a scaling limit such that

$$\operatorname*{slim}_{N\to\infty} |\Psi_N\rangle = |\boldsymbol{\psi}_{p,s}(\boldsymbol{w})\rangle \otimes \overline{|\boldsymbol{\psi}_{\bar{p},\bar{s}}(\bar{\boldsymbol{w}})\rangle}\,. \tag{4.1}$$

We expect that the states $|\boldsymbol{\psi}_{p,s}(\boldsymbol{w})\rangle$ organize into highest weight representations of a certain chiral algebra of extended conformal symmetry and similar for $\overline{|\boldsymbol{\psi}_{\bar{p},\bar{s}}(\bar{\boldsymbol{w}})\rangle}$. In particular, the chiral components that appear in the scaling limit of the primary Bethe states would be the highest states in such representations. Immediate questions arise as to the identification of the algebra of extended conformal symmetry and under what conditions on $\boldsymbol{s}$, $\boldsymbol{w}$ and $\boldsymbol{s}'$, $\boldsymbol{w}'$ do the states $|\boldsymbol{\psi}_{p,s}(\boldsymbol{w})\rangle$ and $|\boldsymbol{\psi}_{p,s'}(\boldsymbol{w}')\rangle$ belong to the same chiral module. Moreover, what are the selection rules for the admissible values of $\boldsymbol{s}$ and $\bar{\boldsymbol{s}}$ for the states (4.1). All these points for the most part remain open to us. Here we present some analysis concerning the selection rules.

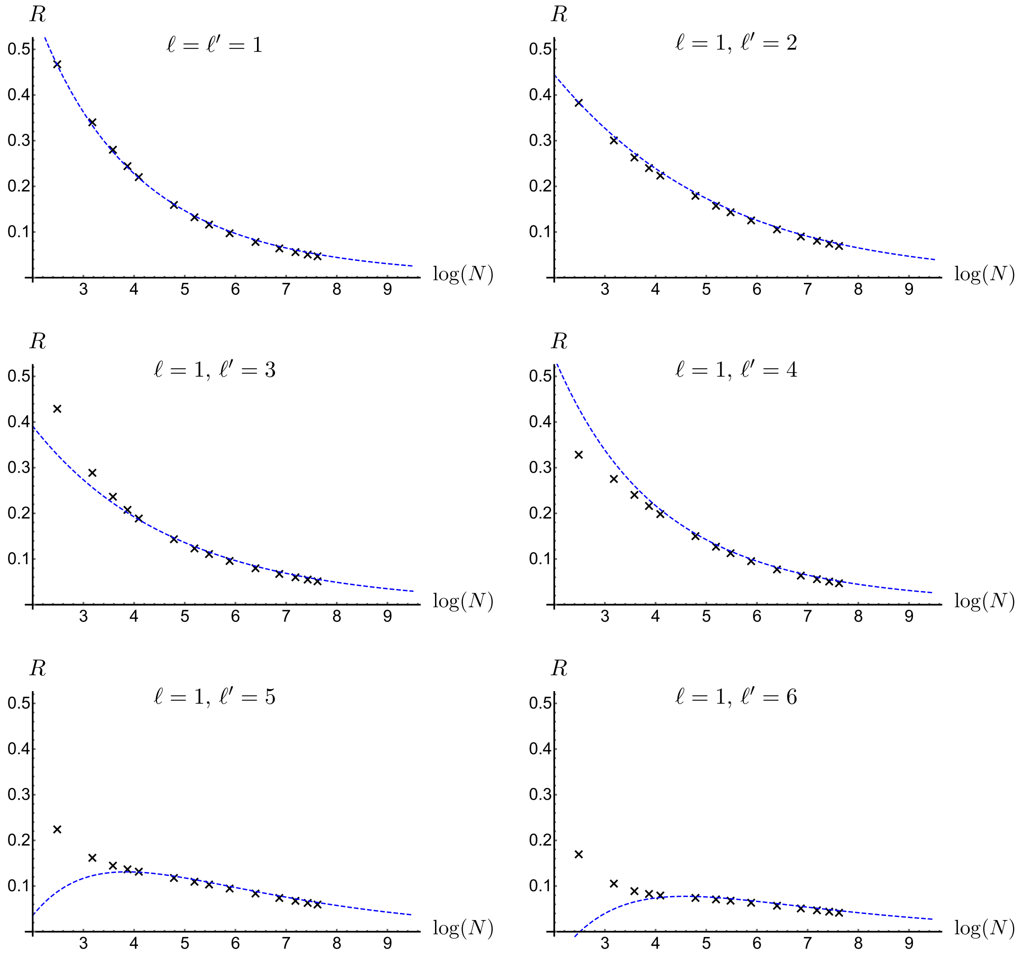

Fig. 8. Depicted is the absolute value of the logarithm of the ratio of the l.h.s. to the r.h.s. of the relation (3.43), $R = |\log(\text{l.h.s. of (3.43)}/\text{r.h.s. of (3.43)})|$, for $\ell = 1$ and $\ell' = 1, 2, \dots, 6$. This was computed for an RG trajectory with $\mathtt{w} = \mathtt{L} = \bar{\mathtt{L}} = 0$ for the $\mathcal{Z}_r$ invariant spin chain with $r = 6$, $n = 3.5$ and $\mathtt{k} = 0.1$ in the sector $S^z = 0$. The l.h.s. is obtained from the Bethe roots corresponding to the trajectory. For the r.h.s. $\boldsymbol{s} = (s_1, s_2, s_3)$ and $\bar{\boldsymbol{s}} = (\bar{s}_1, \bar{s}_2, \bar{s}_3)$ were swapped in favour of the "running couplings" $s_a \mapsto C_a\, b_a(N)$ and $\bar{s}_a \mapsto -C_a\, b_{-a}(N)$ in accordance with eq. (3.1). The $b_a(N)$ themselves are extracted from the eigenvalues of the quasi-shift operators via formulae (2.10) and (2.13). Also, in the computation of the r.h.s., the correction term $O(N^{-\epsilon})$ was ignored. The dashed lines are linear-log plots of $c_1\, N^{-\frac{1}{3}} + c_2\, N^{-\frac{2}{3}}$, where for each value of ℓ' the coefficients c_1 and c_2 were determined via a fit.

4.1. General quantization condition

The relations (3.43) yield an important consequence. It is obtained by making the specializations $(\ell, \ell') \mapsto (\ell + 1, \ell)$ and $(\ell, \ell') \mapsto (\ell, \ell + 1)$ therein, where ℓ is taken to be modulo r, and considering the ratio. The l.h.s. of the result is expressed in terms of the eigenvalues $\mathcal{K}^{(\ell)}$ and $\mathcal{K}^{(\ell+1)}$ (2.2) of the quasi-shift operators. Keeping in mind that $\log\big((-1)^{\mathtt{w}}\mathcal{K}^{(\ell)}\big) = O(N^{-\frac{1}{r}})$ for r odd and $\log\big((-1)^{\mathtt{w}}\mathcal{K}^{(\ell)}\big) = (-1)^{\ell}\,\frac{\pi}{n}\, s + O(N^{-\frac{1}{r}})$ for r even, one finds

$$\left(\frac{N}{rN_0}\right)^{\frac{4\mathrm{i}}{r}(-1)^{\ell} s}\ \frac{\mathfrak{C}^{(\ell+1)}_{p,\boldsymbol{s}}(\boldsymbol{w})}{\mathfrak{C}^{(\ell)}_{p,\boldsymbol{s}}(\boldsymbol{w})}\ \frac{\bar{\mathfrak{C}}^{(\ell)}_{\bar{p},\bar{\boldsymbol{s}}}(\bar{\boldsymbol{w}})}{\bar{\mathfrak{C}}^{(\ell+1)}_{\bar{p},\bar{\boldsymbol{s}}}(\bar{\boldsymbol{w}})} = (-1)^{\sigma}\,\mathrm{e}^{-\frac{2\pi\mathrm{i}}{r}S^z}\Big(1+O\big(N^{-\epsilon}\big)\Big) \qquad (\ell\sim\ell+r)\,, \tag{4.2}$$

where

$$\sigma=\begin{cases}0 & \text{for } \frac{N}{r} \text{ even}\\ 1 & \text{for } \frac{N}{r} \text{ odd}\end{cases}\ . \tag{4.3}$$

Recall that we always assume N is divisible by r and, in constructing an RG trajectory $|\Psi_N\rangle$, the parity of N/r must be kept fixed. Thus we conclude that the parameters $\boldsymbol{s}$, $\bar{\boldsymbol{s}}$ characterizing a low energy Bethe state are not independent, but obey $r-1$ conditions (the product of (4.2) over $\ell=1,2,\ldots,r$ is satisfied identically).

For the case $r=2$, a closed analytic formula exists for the asymptotic coefficients $\mathfrak{C}^{(\ell)}_{p,\boldsymbol{s}}(\boldsymbol{w})$ and $\bar{\mathfrak{C}}^{(\ell)}_{\bar{p},\bar{\boldsymbol{s}}}(\bar{\boldsymbol{w}})$, see sec. 3 of ref. [25]. Through the study of (4.2) it was found that the space of states of the conformal field theory underlying the $\mathcal{Z}_2$ invariant spin chain possesses both a continuous component with s taking any real values as well as a discrete one, where s belongs to a discrete set of pure imaginary numbers. When r is odd, s is by definition zero so that the N dependent factor in the l.h.s. of the above relations is absent. Then, if p and $\bar{p}$ are given, (4.2) constitute $r-1$ independent complex conditions imposed on the $r-1$ complex numbers $(s_1,\ldots,s_{\frac{r-1}{2}})$ and $(\bar{s}_1,\ldots,\bar{s}_{\frac{r-1}{2}})$. They thus play the rôle of a "quantization condition" and are expected to yield a discrete set of admissible values for $\boldsymbol{s}$ and $\bar{\boldsymbol{s}}$. For even r, the N dependent factor is present in the relations (4.2). As in the case $r=2$ we expect that together with a discrete set of values, $s=s_{\frac{r}{2}}\equiv\bar{s}_{\frac{r}{2}}$ may take any real values leading to a continuous component in the spectrum. The other $r-2$ complex numbers $(s_1,\ldots s_{\frac{r}{2}-1})$ and $(\bar{s}_1,\ldots\bar{s}_{\frac{r}{2}-1})$ belong to a certain discrete set, similar to the case of odd r. Unfortunately, a comprehensive analysis of the quantization condition (4.2) is hampered by the fact that an analytic expression for the asymptotic coefficients is absent. Nevertheless, by computing $\mathfrak{C}^{(\ell)}_{p,\boldsymbol{s}}(\boldsymbol{w})$ and $\bar{\mathfrak{C}}^{(\ell)}_{\bar{p},\bar{\boldsymbol{s}}}(\bar{\boldsymbol{w}})$ via the procedure explained in Appendix C, some numerical checks were carried out, see Table 1.

4.2. Quantization condition for the primary Bethe states

Let's first discuss some simple facts concerning the pattern of the Bethe roots for the primary Bethe states. For this purpose, it is useful to make the following observation regarding the solutions to the Bethe Ansatz equations (1.1) with the inhomogeneities η_J as in eq. (1.10). Suppose that the number of Bethe roots $M=\frac{N}{2}-S^z$ is divisible by r and assume that ζ_j have the form

$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{r}(\ell-1)}\,\mathrm{e}^{-\frac{2}{r}\alpha_c} \qquad\qquad (c=1,2,\ldots,M/r\,;\ \ell=1,2,\ldots r)\,. \tag{4.4}$$

Substituting this into (1.1), one obtains

$$\left[\frac{\cosh(\alpha_c-\frac{\mathrm{i}\tilde{\gamma}}{2})}{\cosh(\alpha_c+\frac{\mathrm{i}\tilde{\gamma}}{2})}\right]^{\frac{N}{r}} = -\mathrm{e}^{2\mathrm{i}\pi\mathtt{k}}\prod_{b=1}^{\frac{N-2S^z}{2r}}\frac{\sinh(\alpha_c-\alpha_b-\mathrm{i}\tilde{\gamma})}{\sinh(\alpha_c-\alpha_b+\mathrm{i}\tilde{\gamma})}\,. \tag{4.5}$$

These are easily recognized to be the Bethe Ansatz equations for the Heisenberg XXZ spin-$\frac{1}{2}$ chain (1.3) of length N/r and where γ is substituted for $\tilde{\gamma}$ with

Table 1
Presented is numerical data obtained from the spin chain with $r = 5$, $n = 2.5$, $\mathtt{k} = 0.1$ and in the sector $S^z = 1$ that was used for the verification of the quantization condition (4.2). The rows correspond to different RG trajectories. For all of them $\mathtt{w} = 0$, while the chiral levels $\mathtt{L}$ and $\bar{\mathtt{L}}$ are shown in the first column. The next two columns list the values of $\boldsymbol{s} = (s_1, s_2)$ and $\bar{\boldsymbol{s}} = (\bar{s}_1, \bar{s}_2)$. They were obtained from $b_a(N)$ (2.10) by interpolating data at finite N to $N = \infty$, see eq. (3.1). Also, for the states with non-zero levels, we give the sets $\boldsymbol{w}$ and $\bar{\boldsymbol{w}}$, which satisfy the system of algebraic equations (3.28) and its barred version, respectively. The absolute value of the difference between the l.h.s. and the r.h.s. of the four independent relations (4.2) with $\ell = 1, 2, 3, 4$ is displayed in the four last columns. For the r.h.s., the correction term $O(N^{-\epsilon})$ was ignored and $\sigma = 0$ since the RG trajectories were built for lattice sizes keeping $N/5$ an even integer.

			|l.h.s. − r.h.s.| of relation (4.2)			
$(\mathtt{L}, \bar{\mathtt{L}})$	$\boldsymbol{s} = (s_1, s_2),\ \boldsymbol{w} = \{w_\alpha\}_{\alpha=1}^{\mathtt{L}}$	$\bar{\boldsymbol{s}} = (\bar{s}_1, \bar{s}_2),\ \bar{\boldsymbol{w}} = \{\bar{w}_\alpha\}_{\alpha=1}^{\bar{\mathtt{L}}}$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
(0, 0)	$s_1 = -0.5482 + 1.6871\,\mathrm{i}$ $s_2 = +0.8181 + 0.5943\,\mathrm{i}$	$\bar{s}_1 = -0.4520 - 1.3912\,\mathrm{i}$ $\bar{s}_2 = +0.8017 - 0.5825\,\mathrm{i}$	1.6×10^{-4}	4.6×10^{-5}	1.8×10^{-4}	1.8×10^{-4}
(0, 1)	$s_1 = +0.0974 + 1.8226\,\mathrm{i}$ $s_2 = +0.3824 + 0.9444\,\mathrm{i}$	$\bar{s}_1 = -2.9344 - 3.7480\,\mathrm{i}$ $\bar{s}_2 = +0.7373 + 0.2855\,\mathrm{i}$ $\bar{w}_1 = +1.2595 - 0.9676\,\mathrm{i}$	3.1×10^{-4}	3.9×10^{-4}	6.9×10^{-4}	6.8×10^{-4}
(1, 0)	$s_1 = +0.2683 - 0.8257\,\mathrm{i}$ $s_2 = +0.4242 + 0.3082\,\mathrm{i}$ $w_1 = -0.6077 - 1.8703\,\mathrm{i}$	$\bar{s}_1 = -0.5075 - 1.5618\,\mathrm{i}$ $\bar{s}_2 = +0.9775 - 0.7102\,\mathrm{i}$	5.2×10^{-4}	4.4×10^{-4}	3.9×10^{-4}	3.9×10^{-4}
(2, 1)	$s_1 = -4.0464 - 1.2882\,\mathrm{i}$ $s_2 = +0.7113 + 1.7374\,\mathrm{i}$ $w_1 = +1.3211 - 1.7905\,\mathrm{i}$ $w_2 = +1.5367 - 0.9398\,\mathrm{i}$	$\bar{s}_1 = -2.6122 - 4.0725\,\mathrm{i}$ $\bar{s}_2 = +0.6267 + 0.6401\,\mathrm{i}$ $\bar{w}_1 = +1.2004 - 1.0049\,\mathrm{i}$	4.8×10^{-4}	9.6×10^{-4}	1.1×10^{-3}	1.8×10^{-3}

$$\tilde{\gamma} = \frac{\pi r}{n + r}\,. \tag{4.6}$$

The equations correspond to the sector with the z-projection of the total spin taking the value $S^z/r = 0, \frac{1}{2}, 1, \ldots$. It is well known that for the vacuum state in this sector, i.e., the state with the lowest eigenvalue of the XXZ Hamiltonian, the corresponding solution is such that all the α_c are real and distinct,

$$\alpha_1 < \alpha_2 < \ldots < \alpha_{M/r}\,, \tag{4.7}$$

provided the absolute value of the twist parameter $\mathtt{k}$ is sufficiently small. Computing the eigenvalue of the Hamiltonian $\mathbb{H}$ of the $\mathcal{Z}_r$ invariant spin chain via eq. (2.1), where the Bethe roots are taken to be as in (4.4) with this choice of α_c, turns out to yield the lowest energy in the sector with given value of S^z.

For a generic primary Bethe state of the $\mathcal{Z}_r$ invariant spin chain the pattern of the Bethe roots remains qualitatively similar. Namely, they are split into r groups such that the argument of ζ_j in each group is approximately equal to $\frac{2\pi}{r}(\ell - 1)$ with $\ell = 1, 2, \ldots, r$. However, there occurs the possibility that the number of Bethe roots with roughly equal phases is different. In other words,

$$\{\zeta_j\}_{j=1}^{M} = \{\zeta_j^{(1)}\}_{j=1}^{M_1} \cup \{\zeta_j^{(1)}\}_{j=1}^{M_2} \cup \ldots \cup \{\zeta_j^{(r)}\}_{j=1}^{M_r} \tag{4.8}$$

with

$$\arg\big(\zeta_j^{(\ell)}\big) \approx \frac{2\pi}{r}(\ell - 1) \qquad (\mathrm{mod}\ 2\pi) \tag{4.9}$$

and the integers M_ℓ are not necessarily the same. It should be mentioned that at the edges of the Bethe root distribution along each ray, the argument of some of the roots may deviate significantly from $\frac{2\pi}{r}(\ell-1)$. Then the allocation of the roots to the neighbouring groups labelled by ℓ or $\ell+1\ (\mathrm{mod}\, r)$ could become ambiguous. Ignoring this subtlety, the states may be assigned the set of numbers

$$\mathfrak{m}_1 = M_1 - \frac{N}{2r}\,, \qquad \mathfrak{m}_2 = M_2 - \frac{N}{2r}\,, \qquad \ldots\,, \qquad \mathfrak{m}_r = M_r - \frac{N}{2r}\,, \tag{4.10}$$

which are integers for N/r even and half-integers for N/r odd. They are not independent, but satisfy

$$\mathfrak{m}_1 + \mathfrak{m}_2 + \ldots + \mathfrak{m}_r = -S^z \leq 0\,, \tag{4.11}$$

since the total number of roots is equal to $\frac{N}{2} - S^z$. The states where at the edges of the root distribution the Bethe roots do not deviate much from the rays and, in addition, there are no significant gaps along the ray typically correspond to the primary Bethe states.

For the solution of the Bethe Ansatz equations of the form (4.4), corresponding to the lowest energy state ($S^z = 0$) of the $\mathcal{Z}_r$ invariant spin chain, the eigenvalues of the quasi-shift operators (2.2) are given by

$$\mathcal{K}^{(\ell)} = 1\,. \tag{4.12}$$

In turn, this implies that in the scaling limit all the parameters s_a and $\bar{s}_a$ occurring in the ODEs (3.6) and (3.7) are zero. Then, upon a change of variables, the differential equations coincide with the Schrödinger equations which describe the scaling limit of the primary states for the XXZ spin chain (see eq. (A.4)).

Consider the quantization condition (4.2) specialized to the primary Bethe states. In this case, the sets $\boldsymbol{w}$ and $\bar{\boldsymbol{w}}$ are trivial and the relation can be written as

$$\left(\frac{2^{\frac{r}{n}}\, N}{r N_0}\right)^{\frac{4\mathrm{i}}{r}(-1)^\ell s} \frac{F_p(\boldsymbol{s}^{(\ell+1)})}{F_p(\boldsymbol{s}^{(\ell)})}\, \frac{F_{\bar{p}}(\bar{\boldsymbol{s}}^{(\ell)})}{F_{\bar{p}}(\bar{\boldsymbol{s}}^{(\ell+1)})} = (-1)^\sigma\, \mathrm{e}^{-\frac{2\pi \mathrm{i}}{r} S^z} \Big(1 + O\big(N^{-\epsilon}\big)\Big)\,. \tag{4.13}$$

Here $\boldsymbol{s}^{(\ell)}$ and $\bar{\boldsymbol{s}}^{(\ell)}$ denote

$$\boldsymbol{s}^{(\ell)} = \big(s_1^{(\ell)}, \ldots, s_{[\frac{r}{2}]}^{(\ell)}\big)\,, \qquad\qquad \bar{\boldsymbol{s}}^{(\ell)} = \big(\bar{s}_1^{(\ell)}, \ldots, \bar{s}_{[\frac{r}{2}]}^{(\ell)}\big) \tag{4.14}$$

with

$$s_a^{(\ell)} = (-1)^{ar}\, \mathrm{e}^{+\frac{\mathrm{i}\pi a}{r}(2\ell-1)}\, s_a\,, \qquad\qquad \bar{s}_a^{(\ell)} = (-1)^{ar}\, \mathrm{e}^{-\frac{\mathrm{i}\pi a}{r}(2\ell-1)}\, \bar{s}_a\,. \tag{4.15}$$

The function

$$F_p(\boldsymbol{s}) \equiv F_p\big(s_1, \ldots s_{[\frac{r}{2}]}\big) \tag{4.16}$$

is a certain connection coefficient for the ODE

$$\bigg[-\partial_v^2 + \mathrm{e}^{rv} + p^2 + \sum_{a=1}^{[\frac{r}{2}]} s_a\, \mathrm{e}^{av}\bigg]\, \tilde{\psi} = 0\,. \tag{4.17}$$

To define it, one considers a Jost solution to the differential equation, which for $\Re e(p) \geq 0$ is uniquely specified by the condition:

$$\tilde{\psi}_p \to \mathrm{e}^{pv} \qquad \text{as} \qquad v \to -\infty\,. \tag{4.18}$$

Then the coefficient $F_p(\boldsymbol{s})$ occurs in the asymptotics:

$$\tilde{\psi}_p(v) \to F_p(\boldsymbol{s})\, \exp\left(-\frac{rv}{4}+\frac{sv}{2}+\frac{2}{r}\,\mathrm{e}^{\frac{rv}{2}}\right) \qquad \text{as} \qquad v\to+\infty\,. \tag{4.19}$$

Recall that s, which appears in the above equation along with (4.13), is identically zero for r odd, while $s \equiv s_{\frac{r}{2}} \equiv \bar{s}_{\frac{r}{2}}$ for even r. Clearly, $F_p(\boldsymbol{s})$ is an entire function of all the variables $(s_1,\ldots,s_{[\frac{r}{2}]})$ so that it is unambiguously defined when its arguments are any complex numbers.

Taking the logarithm of both sides of (4.13) one obtains

$$\frac{4}{r}\,(-1)^{\ell-1}\, s\log\left(\frac{2^{\frac{r}{n}}\,N}{rN_0}\right)+\mathrm{i}\log\left[\frac{F_p(\boldsymbol{s}^{(\ell+1)})}{F_p(\boldsymbol{s}^{(\ell)})}\,\frac{F_{\bar{p}}(\bar{\boldsymbol{s}}^{(\ell)})}{F_{\bar{p}}(\bar{\boldsymbol{s}}^{(\ell+1)})}\right]=2\pi\left(\mathfrak{n}_\ell+\frac{S^z}{r}\right)+O\big(N^{-\epsilon}\big)\,. \tag{4.20}$$

Here $\mathfrak{n}_\ell$ with $\ell=1,\ldots,r$ are some numbers, such that

$$\mathfrak{n}_\ell \in \begin{cases} \mathbb{Z} & \text{for } N/r \text{ even} \\ \mathbb{Z}+\frac{1}{2} & \text{for } N/r \text{ odd}\end{cases}\,. \tag{4.21}$$

In order to assign them a precise meaning, it is necessary to fix the branch of the logarithm containing the functions F_p and $F_{\bar{p}}$. An evident requirement is that the sum over $\ell=1,2,\ldots,r$ of the l.h.s. vanishes. This implies the condition

$$\mathfrak{n}_1+\ldots+\mathfrak{n}_r=-S^z\,. \tag{4.22}$$

One may expect that, with the branch of the logarithm being suitably chosen, the (half-)integers $\mathfrak{n}_\ell$ are simply related to $\mathfrak{m}_\ell$ labelling the primary Bethe states from (4.10). To explore this further, let's consider the cases $r=2,3$.

For $r=2$ the connection coefficient $F_p(\boldsymbol{s})$ admits the simple analytical expression:

$$F_p(s)=2^{\frac{s}{2}-p-\frac{1}{2}}\,\frac{\Gamma(1+2p)}{\Gamma(\frac{1}{2}+p+\frac{s}{2})}\,. \tag{4.23}$$

This is because the corresponding ODE (4.17) reduces to

$$\Big[-\partial_v^2+\mathrm{e}^{2v}+p^2+s\,\mathrm{e}^{v}\Big]\tilde{\psi}=0 \qquad\qquad (r=2)\,, \tag{4.24}$$

which may be brought to the form of the confluent hypergeometric equation. The relations (4.20) specialized to $r=2$ consist of only one independent equation. Setting $\ell=2$ and $\ell=1$ therein and taking the difference, one finds

$$-4s\log\left[\frac{N\,2^{\frac{2}{n}}\Gamma(\frac{3}{2}+\frac{1}{n})}{\sqrt{\pi}\,\Gamma(1+\frac{1}{n})}\right]-2\mathrm{i}\,\log\left[2^{-2\mathrm{i}s}\,\frac{\Gamma(\frac{1}{2}+p+\frac{\mathrm{i}s}{2})}{\Gamma(\frac{1}{2}+p-\frac{\mathrm{i}s}{2})}\,\frac{\Gamma(\frac{1}{2}+\bar{p}+\frac{\mathrm{i}s}{2})}{\Gamma(\frac{1}{2}+\bar{p}-\frac{\mathrm{i}s}{2})}\right] \\ =2\pi\,(\mathfrak{n}_2-\mathfrak{n}_1)+O\big(N^{-\epsilon}\big) \tag{4.25}$$

where N_0 was substituted for its expression (3.21) in terms of the parameter n. This is the quantization condition studied in refs. [9,13,14] in the context of the $\mathcal{Z}_2$ invariant spin chain with $\mathfrak{n}_2-\mathfrak{n}_1$ replaced by $\mathfrak{m}\equiv\mathfrak{m}_2-\mathfrak{m}_1\in\mathbb{Z}$.[7] The branch of the logarithm in this case is chosen such

[7] Our parameter $(-\frac{s}{2})$ coincides with s, which is used in those works, see also Appendix A.

that the l.h.s. is a continuous function for real s, which vanishes at $s=0$. Taking into account that $\mathfrak{n}_1+\mathfrak{n}_2=\mathfrak{m}_1+\mathfrak{m}_2=-S^z$, one concludes that

$$\mathfrak{n}_\ell=\mathfrak{m}_\ell \qquad\qquad (r=2)\,. \tag{4.26}$$

For $r\geq 3$ a simple analytic formula for $F_p(\boldsymbol{s})$, similar to (4.23), does not exist. Nevertheless this function can be analyzed within the perturbation theory and WKB approximation applied to eq. (4.17). For $r=3$ the ODE becomes

$$\Big[-\partial_v^2+\mathrm{e}^{3v}+p^2+s_1\,\mathrm{e}^{v}\Big]\,\tilde{\psi}=0 \qquad\qquad (r=3) \tag{4.27}$$

and a perturbative calculation yields the first terms of the Taylor expansion

$$\log F_p(s_1)=\log\bigg[\frac{1}{2\sqrt{\pi}}\,3^{\frac{1}{2}+\frac{2p}{3}}\,\Gamma\big(1+\tfrac{2p}{3}\big)\bigg] \\ +f_1\big(\tfrac{p}{3},\tfrac{1}{3}\big)\,C\,s_1-f_2\big(\tfrac{p}{3},\tfrac{1}{3}\big)\,C^2\,s_1^2+O(s_1^3)\,. \tag{4.28}$$

Here the functions $f_1(h,g)$ and $f_2(h,g)$ are quoted in Appendix B, while

$$C=\frac{3^{\frac{2}{3}}}{\Gamma^2(-\frac{1}{3})}\,. \tag{4.29}$$

On the other hand, the WKB approximation gives that as $|\arg(s_1)|<\pi$ and $|s_1|\to\infty$,

$$\log F_p(s_1)=\log\bigg(\frac{\Gamma(1+2p)}{2\sqrt{\pi}}\bigg)-p\,\log(s_1) \\ +\frac{\Gamma^2(\frac{1}{4})}{3\sqrt{\pi}}\,s_1^{\frac{3}{4}}+\frac{\Gamma^2(\frac{3}{4})}{8\sqrt{\pi}}\,(1-8p^2)\,s_1^{-\frac{3}{4}}+O\big(s_1^{-\frac{3}{2}}\big) \tag{4.30}$$

(in the r.h.s. of this equation and (4.28) $\log(1)=0$). The zeroes of the entire function $F_p(s_1)$ accumulate on the negative s_1 axis, which is a Stokes line. As $s_1\to-\infty$, it is possible to show that

$$F_p(s_1)=\frac{\Gamma(1+2p)}{\sqrt{\pi}}\,|s_1|^{-p} \\ \times\exp\bigg(-|s_1|^{\frac{3}{4}}\,\frac{\Gamma^2(\frac{1}{4})}{3\sqrt{2\pi}}-\frac{\Gamma^2(\frac{3}{4})}{8\sqrt{2\pi}}\,(1-8p^2)\,|s_1|^{-\frac{3}{4}}+O\big(|s_1|^{-\frac{3}{2}}\big)\bigg) \\ \times\cos\bigg(\pi p-|s_1|^{\frac{3}{4}}\,\frac{\Gamma^2(\frac{1}{4})}{3\sqrt{2\pi}}+\frac{\Gamma^2(\frac{3}{4})}{8\sqrt{2\pi}}\,(1-8p^2)\,|s_1|^{-\frac{3}{4}}+O\big(|s_1|^{-\frac{3}{2}}\big)\bigg)\,. \tag{4.31}$$

This way, the function $\log F_p(s_1)$ can be fixed at small s_1 via formula (4.28) and then extended by continuity to the wedge $|\arg(s_1)|<\pi-\delta$, where $0<\delta\ll 1$ is chosen in such a way so that the zeroes of $F_p(s_1)$ are excluded from the domain.

The quantization condition (4.20) for $r=3$ can be written as

$$\mathrm{i}\log\Big(F_p\big(\mathrm{e}^{+\frac{2\pi\mathrm{i}}{3}\ell}\,s_1\big)\Big)-\mathrm{i}\log\Big(F_p\big(\mathrm{e}^{+\frac{2\pi\mathrm{i}}{3}(\ell-1)}\,s_1\big)\Big)+\mathrm{i}\log\Big(F_{\bar{p}}\big(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}(\ell-1)}\,\bar{s}_1\big)\Big) \\ -\mathrm{i}\log\Big(F_{\bar{p}}\big(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{3}\ell}\,\bar{s}_1\big)\Big)=2\pi\,\Big(\mathfrak{n}_\ell+\frac{S^z}{r}\Big)+O\big(N^{-\epsilon}\big)\,. \tag{4.32}$$

We found that if the logarithm of $F_p(s_1)$ is fixed as above, then the (half-)integers $\mathfrak{n}_\ell$ are related to $\mathfrak{m}_\ell$ from eq. (4.10) as

$$\mathfrak{n}_\ell = \mathfrak{m}_{\ell-1} \qquad\qquad (\ell \sim \ell + r\,;\ r = 3)\,. \tag{4.33}$$

Furthermore, by means of the asymptotic formula for the logarithm of $F_p(s_1)$ (4.30), it is straightforward to show

$$\begin{aligned} |s_1| &= \pi^2 \left[\frac{3}{\Gamma^2(\frac{1}{4})}\right]^{\frac{4}{3}} \mathfrak{m}^{\frac{4}{3}} \left(1 + \frac{1-8\bar{p}^2}{9\pi\,\mathfrak{m}^2} + o\big(\mathfrak{m}^{-2}\big)\right) \\ |\bar{s}_1| &= \pi^2 \left[\frac{3}{\Gamma^2(\frac{1}{4})}\right]^{\frac{4}{3}} \mathfrak{m}^{\frac{4}{3}} \left(1 + \frac{1-8p^2}{9\pi\,\mathfrak{m}^2} + o\big(\mathfrak{m}^{-2}\big)\right). \end{aligned} \tag{4.34}$$

Here $\mathfrak{m}$ is given in terms of the triple $(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3)$ as

$$\mathfrak{m} = \sqrt{\frac{\mathfrak{m}_j^2 + \mathfrak{m}_k^2}{2}}\,, \tag{4.35}$$

where $\mathfrak{m}_j$, $\mathfrak{m}_k$ are defined via the condition

$$\mathfrak{m}_j \mathfrak{m}_k \geq 0\,. \tag{4.36}$$

Formula (4.34) is valid as $\mathfrak{m} \gg 1$ with p and $\bar{p}$ being kept fixed. Note that if all $\mathfrak{m}_1$, $\mathfrak{m}_2$ and $\mathfrak{m}_3$ have the same sign, their absolute value must be smaller than $S^z = p + \bar{p} \geq 0$. As a result, (4.34) becomes inapplicable. Also, for all the RG trajectories we were able to construct, where the (half-)integers $(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3)$ could be defined unambiguously, at least two of them were smaller or equal to zero so that $\mathfrak{m}_j, \mathfrak{m}_k \leq 0$.

The condition (4.36) does not fix $\mathfrak{m}_j, \mathfrak{m}_k$ taken from the triple $(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3)$ unambiguously. This is not important for the description of the asymptotics of $|s_1|$ and $|\bar{s}_1|$ since (4.34) involves only $\mathfrak{m}$, which is invariant under the permutation $\mathfrak{m}_j \leftrightarrow \mathfrak{m}_k$. Turning to the large $\mathfrak{m}$ asymptotic formula for the arguments of $s_1, \bar{s}_1$ we supplement (4.36) with the requirement

$$|\mathfrak{m}_j| \geq |\mathfrak{m}_k|\,. \tag{4.37}$$

Then we found that

$$\begin{aligned} &\arg(s_1) = \frac{2\pi}{3}\,\mu_\ell + (-1)^{\#(\mathcal{P})}\,\frac{4}{3}\,\arctan\left(\frac{\mathfrak{m}_j - \mathfrak{m}_k}{\mathfrak{m}_j + \mathfrak{m}_k}\right) + o\big(|\mathfrak{m}|^{-2}\big) \\ &\arg(s_1) + \arg(\bar{s}_1) = o\big(|\mathfrak{m}|^{-2}\big)\,, \end{aligned} \tag{4.38}$$

where $(-1)^{\#(\mathcal{P})}$ stands for the parity of the permutation $\mathcal{P}:\ (1,2,3) \mapsto (j,k,\ell)$, while

$$\mu_\ell = \begin{cases} 0 & \text{for} \quad \ell = 1 \\ -1 & \text{for} \quad \ell = 2 \\ +1 & \text{for} \quad \ell = 3 \end{cases}. \tag{4.39}$$

Note that for the case $\mathfrak{m}_j = \mathfrak{m}_k$, the arctangent in (4.38) vanishes and $\arg(s_1) = \frac{2\pi}{3}\,\mu_\ell + o(|\mathfrak{m}|^{-2})$ and similarly for $\arg(\bar{s}_1)$. Also, the condition (4.36) guarantees the inequality

$$-\frac{\pi}{3} < \frac{4}{3}\,\arctan\left(\frac{\mathfrak{m}_j - \mathfrak{m}_k}{\mathfrak{m}_j + \mathfrak{m}_k}\right) \leq \frac{\pi}{3}\,. \tag{4.40}$$

In order to illustrate the accuracy of (4.34) and (4.38), some numerical data for s_1 and $\bar{s}_1$ obtained from the solution of the Bethe Ansatz equations corresponding to the primary Bethe states is compared with the predictions coming from these asymptotic formulae in Table 2.

Table 2
In order to demonstrate the accuracy of the asymptotic formulae (4.34) and (4.38), primary Bethe states $|\Psi_N\rangle$ were constructed for the $\mathcal{Z}_3$ invariant spin chain that are characterized by different triples $(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3)$. The values of $s_1, \bar{s}_1$ were then obtained by computing b_1, b_{-1} (2.10) from the eigenvalues of the quasi-shift operator and then taking the scaling limits $s_1 = C_1 \operatorname{slim}_{N\to\infty} b_1$, $\bar{s}_1 = -C_1 \operatorname{slim}_{N\to\infty} b_{-1}$ with the coefficient C_1 as in eq. (3.2). The result was used to generate the numbers listed in the columns "Bethe Ansatz". Recall that $\mathfrak{m}$ is given in terms of $(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3)$ by eq. (4.35) with $\mathfrak{m}_j$, $\mathfrak{m}_k$ being defined through the conditions (4.36) and (4.37). Note that for the three primary Bethe states considered, the asymptotic formula (4.38) yields the same result for $\arg(s_1)$, which is given in the last column, and similarly for $\arg(\bar{s}_1)$. The parameters were taken to be $n = 2.5$, $\mathtt{k} = 0.05$, while for all the states $\mathtt{w} = 0$, $S^z = 2$.

	$\lvert s_1\rvert/\mathfrak{m}^{\frac{4}{3}}$		$\arg(s_1)$	
$(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3)$	Bethe Ansatz	formula (4.34)	Bethe Ansatz	formula (4.38)
$(-2, -1, 1)$	1.265280	1.280089	2.534721	
$(-4, -2, 4)$	1.351599	1.352403	2.523989	+2.523396
$(-6, -3, 7)$	1.365639	1.365794	2.523508	

	$\lvert\bar{s}_1\rvert/\mathfrak{m}^{\frac{4}{3}}$		$\arg(\bar{s}_1)$	
$(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3)$	Bethe Ansatz	formula (4.34)	Bethe Ansatz	formula (4.38)
$(-2, -1, 1)$	1.185686	1.194405	−2.522902	
$(-4, -2, 4)$	1.330539	1.330982	−2.523408	−2.523396
$(-6, -3, 7)$	1.356189	1.356273	−2.523398	

5. The spectral determinants at large E

In the context of the ODE/IQFT correspondence, an important rôle belongs to the study of the large E asymptotic expansion of the spectral determinants. The first few terms in the expansion were already presented in (3.39)-(3.41). Let's denote the r.h.s. of the last of these formulae by

$$C^{(\ell)}(\theta) \equiv \mathfrak{C}^{(\ell)}_{p,s}(\boldsymbol{w})\ \exp\left[\frac{N_0}{\cos(\frac{\pi r}{2n})}\,\mathrm{e}^{\theta} + \left((-1)^{\ell-1}\,\mathrm{i}s - \frac{2np}{n+r}\right)\frac{\theta}{r}\right]. \tag{5.1}$$

Recall that the integer $\ell = 1, 2, \ldots r$ labels the different wedges $\arg\big((-1)^{r-1}E\big) \in \big(\frac{2\pi}{r}(\ell-1), \frac{2\pi}{r}\ell\big)$ (mod 2π) in the complex E plane. An analysis of the ODE (3.26) shows that the full asymptotic expansion may be represented in the form

$$D_p(E\,|\,\boldsymbol{w}) \asymp C^{(\ell)}(\theta)\ \exp\Big(S^{(\ell)}(\theta) + I(\theta) + \tilde{H}^{(\ell)}(\theta)\Big), \tag{5.2}$$

where the formal asymptotic series $S^{(\ell)}(\theta)$, $I(\theta)$ and $\tilde{H}(\theta)$ are described as follows.

The largest corrections to (3.41) come from $S^{(\ell)}(\theta)$. In particular,

$$S^{(\ell)}(\theta) = \frac{1}{2n\sqrt{\pi}} \sum_{a=1}^{[\frac{r-1}{2}]} (\eta_\ell)^a\ \Gamma\big(\tfrac{1}{2} - \tfrac{2a-r}{2n}\big)\,\Gamma\big(\tfrac{2a-r}{2n}\big)\ s_a\ \mathrm{e}^{-(1-\frac{2a}{r})\theta} + S_1^{(\ell)}(\theta) \tag{5.3}$$

with $S_1^{(\ell)}(\theta)$ decaying faster as $\theta \to \infty$ than the explicitly displayed terms. The latter linearly depend on the parameters s_a entering into the differential equation (3.26). Within the usual interpretation s_a are the eigenvalues of certain Integrals of Motion (IM) $\mathbf{S}_a$. The subscript a coincides with the charge of these operators w.r.t. the $\mathcal{Z}_r$ symmetry transformations. Furthermore, their conformal dimensions are given by

$$\mathbf{S}_a:\qquad (\Delta, \bar{\Delta}) = \big(1 - \tfrac{2a}{r}, 0\big) \qquad \big(a = 1, \ldots, [\tfrac{r}{2}]\big). \tag{5.4}$$

Similarly the large $\bar{E}$ asymptotic of $\bar{D}_{\bar{p}}(\bar{E}\,|\,\bar{\boldsymbol{w}})$ involves the eigenvalues of the operators $\bar{\mathbf{S}}_a$, whose $\mathcal{Z}_r$ charges are $(-a)$ and conformal dimensions read as

$$\bar{\mathbf{S}}_a: \quad (\Delta, \bar{\Delta}) = \big(0, 1 - \tfrac{2a}{r}\big) \qquad \big(a = 1, \ldots, [\tfrac{r}{2}]\big)\,. \tag{5.5}$$

The formal asymptotic series $S_1^{(\ell)}(\theta)$ in (5.3) has the general structure

$$S_1^{(\ell)}(\theta) = \sum_{m=1}^{\infty} \sum_{a=1}^{[\frac{r}{2}]} (\eta_\ell)^a \; S_{m,a} \; \mathrm{e}^{-(2m+1-\frac{2a}{r})\,\theta}\,. \tag{5.6}$$

Again, the coefficients $S_{m,a}$ of the expansion are expressed in terms of the eigenvalues of certain operators whose $\mathcal{Z}_r$ charges are a and which have conformal dimensions $\Delta = 2m + 1 - \frac{2a}{r}$ and $\bar{\Delta} = 0$.

Among the coefficients $S_{m,a}$, those with $a = \frac{r}{2}$ for even r deserve special attention:

$$S_{m,\frac{r}{2}} \equiv I_{2m} \qquad (m = 1, 2, \ldots; \;\; r - \text{even})\,. \tag{5.7}$$

They are the eigenvalues of certain operators $\mathbf{I}_{2m}$ with $\mathcal{Z}_r$ charges $\frac{r}{2}\,(\mathrm{mod}\; r)$ and which have integer conformal dimensions

$$\mathbf{I}_{2m}: \quad (\Delta, \bar{\Delta}) = \big(2m, 0\big)\,. \tag{5.8}$$

The operators $\mathbf{I}_{2m}$ are expected to be local IM, i.e., they admit an expression of the form

$$\mathbf{I}_{2m} = \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi}\, T_{2m+1}(u)\,, \tag{5.9}$$

where $T_{2m+1}(u)$ are chiral local fields of Lorentz spin $2m + 1$ such that

$$\bar{\partial}\, T_{2m+1}(u) = 0\,. \tag{5.10}$$

Similarly the expansion of the spectral determinant $\bar{D}_{\bar{p}}(\bar{E}\,|\,\bar{\boldsymbol{w}})$ yields the eigenvalues of

$$\bar{\mathbf{I}}_{2m} = \int_0^{2\pi} \frac{\mathrm{d}\bar{u}}{2\pi}\, \bar{T}_{2m+1}(\bar{u})\,, \qquad \partial\, \bar{T}_{2m+1}(\bar{u}) = 0\,. \tag{5.11}$$

These have the same $\mathcal{Z}_r$ charge as $\mathbf{I}_{2m}$, i.e., $\frac{r}{2}$ (mod r).

While $\mathbf{I}_{2m}$ and $\bar{\mathbf{I}}_{2m}$ appear only when r is even, it is expected that for generic positive integer r the theory possesses the additional local IM $\mathbf{I}_{2m-1}$ and $\bar{\mathbf{I}}_{2m-1}$. For the former ones

$$\mathbf{I}_{2m-1} = \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi}\, T_{2m}(u)\,, \qquad \bar{\partial}\, T_{2m}(u) = 0\,, \tag{5.12}$$

where the chiral local density T_{2m} has Lorentz spin $2m$ and similarly for $\bar{\mathbf{I}}_{2m-1}$. These local IM are all $\mathcal{Z}_r$ invariant. The eigenvalues of $\mathbf{I}_{2m-1}$ appear in $I(\theta)$ in the formula (5.2). In particular, the leading term reads explicitly as

$$I(\theta) = -\frac{1}{\sqrt{\pi}}\, \Gamma\big(-\tfrac{r}{2n}\big)\, \Gamma\big(\tfrac{3}{2} + \tfrac{r}{2n}\big)\, I_1\, \mathrm{e}^{-\theta} + \ldots\,, \tag{5.13}$$

where

$$I_1 = \frac{p^2}{n+r} + \frac{s^2}{4n} - \frac{r}{24} + \mathtt{L} \qquad (s \equiv 0 \text{ for } r \text{ odd};\ s \equiv s_{\frac{r}{2}} \text{ for } r \text{ even}) . \tag{5.14}$$

The corresponding operator is of the form

$$\mathbf{I}_1 = \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi} \left((\partial\varphi)^2 + \ldots \right) . \tag{5.15}$$

Here $\partial\varphi$ is the chiral component of the current associated with the global U(1) symmetry of the spin chain.[8] The ellipses stand for the contribution of other degrees of freedom. Note that formula (5.15) fixes any ambiguity in the normalization of $\mathbf{I}_1$, which is chosen in such a way that its eigenvalues are given by (5.14). This follows since the eigenvalue of the zero mode $\int_0^{2\pi} \frac{\mathrm{d}u}{2\pi}\, \partial\varphi$ is $P = \frac{p}{\sqrt{n+r}}$. The CFT Hamiltonian coincides with

$$\mathbf{H}_{\mathrm{CFT}} = \mathbf{I}_1 + \bar{\mathbf{I}}_1 . \tag{5.16}$$

The full series for $I(\theta)$ takes the form

$$I(\theta) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}(n+r)^m}{2n\sqrt{\pi}\, m!}\, \Gamma\big(-(m-\tfrac{1}{2})\, \tfrac{r}{n} \big)\, \Gamma\big((m-\tfrac{1}{2})\, \tfrac{n+r}{n} \big)\, I_{2m-1}\, \mathrm{e}^{-(2m-1)\theta} . \tag{5.17}$$

Here the coefficients I_{2m-1} depend on p such that as $p \to \infty$,

$$I_{2m-1} = \left(\frac{p^2}{n+r} \right)^m + O\big(p^{2m-2} \big) . \tag{5.18}$$

These are the eigenvalues of the local IM $\mathbf{I}_{2m-1}$ from (5.12).

Finally, the term $\tilde{H}^{(\ell)}(\theta)$ from (5.2) stands for the formal asymptotic series

$$\tilde{H}^{(\ell)}(\theta) = \sum_{m=1}^{\infty} \tilde{H}_m^{(\ell)}\, \mathrm{e}^{-\frac{2nm}{r}\theta} . \tag{5.19}$$

It involves the eigenvalues of the "dual non-local" IM, whose Lorentz spin depends on n and is given by $\frac{2nm}{r}$ with $m = 1, 2, 3, \ldots$. For the case $r = 1$, such operators were first discussed in the context of the quantum KdV theory in ref. [23]. Also note that the coefficient $\mathfrak{C}_{p,s}^{(\ell)}(\boldsymbol{w})$ in (5.1) can be interpreted as the eigenvalue of the simplest non-local IM, which is related to the so-called reflection operators [24,25].

6. Conclusion

The main result of the paper is the class of second order linear differential equations and the quantization condition that describe the scaling limit of the $\mathcal{Z}_r$ invariant spin chain in the critical regime with anisotropy parameter $q = \mathrm{e}^{\frac{\mathrm{i}\pi}{n+r}}$ and $n > 0$. We can not claim to have developed an ODE/IQFT correspondence for the model as the field theory description lies beyond the scope

[8] The current $\partial\varphi$ is normalized through the operator product expansion $\partial\varphi(u_1)\, \partial\varphi(u_2) = -\frac{1}{2\,(u_1-u_2)^2} + O(1)$.

of this work. We believe that further studying the theory is worthwhile since even a superficial analysis reveals many interesting features of the CFT underlying the critical behaviour. Among them is an infinite degeneracy of the ground state (as well as all conformal primary states) and the presence of a continuous component in the spectrum which occurs in the case of even r. To conclude the paper, we would like to mention two possible directions which, in our opinion, may help to better understand the critical behaviour of the model.

The first concerns the algebra of extended conformal symmetry. A way of proceeding in its study is provided by the densities for the local IM (5.9), (5.12) $(T_j, \bar{T}_j)$ with $j = 2, 4, 6, \ldots$ for odd r and $j = 2, 3, 4, \ldots$ for even r. Since such fields are periodic, e.g., $T_j(u + 2\pi) = T_j(u)$, and occur inside an integral, they are defined up to a total derivative:

$$T_j \mapsto T_j + \partial \mathcal{O}_{j-1}\,, \qquad \bar{T}_j \mapsto \bar{T}_j + \bar{\partial} \bar{\mathcal{O}}_{j-1}\,, \tag{6.1}$$

where $\mathcal{O}_{j-1}$ $\big(\bar{\mathcal{O}}_{j-1}\big)$ are local chiral fields of Lorentz spin $j-1$ $(1-j)$. Based on the experience gained from the study of the cases $r = 1, 2$ we expect that the densities can be chosen such that they generate a closed $\mathcal{W}$-algebra. The explicit construction of the algebra of extended conformal symmetry would be an important step for describing the CFT underlying the critical behaviour of the lattice system.

The second direction is the study of the limit $n \to \infty$. As is the case for $r = 2$, one expects that it can be interpreted as a classical limit, where the CFT admits a Lagrangian description. If so, this would certainly yield valuable insights into the physical content of the theory.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

GK acknowledges discussions with Holger Frahm.

The research of GK is supported by the Deutsche Forschungsgemeinschaft under grant No. Fr 737/9-2. The final part of this work was carried out during GK's visit to the NHETC at Rutgers University. He is grateful for the support and hospitality he received during the stay.

The research of SL is partially funded by the NSF under grant number NSF-PHY-2210187.

Appendix A

The special cases $r = 1, 2$ for the proposed ODE (3.6) have already been discussed in the literature. For $r = 1$, the differential equation reads as

$$\left[-\partial_z^2 + \frac{p^2 - \frac{1}{4}}{z^2} - \frac{1}{z} + E^{-n-1}\, z^{n-1} \right] \Psi = 0 \qquad (r = 1)\,. \tag{A.1}$$

Via the change of variables:

$$z=\tfrac{1}{4}\,E_{\rm XXZ}\,x^2\,,\qquad\qquad \Psi=\sqrt{x}\,\Psi_{\rm XXZ} \tag{A.2}$$

and parameters

$$p=\sqrt{\alpha+1}\,(\ell+\tfrac{1}{2})\,,\qquad n=\alpha\,,\qquad E=2^{-\frac{2\alpha}{\alpha+1}}\,E_{\rm XXZ} \tag{A.3}$$

this ODE becomes the Schrödinger equation for the anharmonic oscillator appearing in refs. [17–19]:

$$\bigg[-\partial_x^2+\frac{\ell(\ell+1)}{x^2}+x^{2\alpha}-E_{\rm XXZ}\bigg]\Psi_{\rm XXZ}=0\,. \tag{A.4}$$

It describes the scaling behaviour of the XXZ spin chain with anisotropy parameter $q=\mathrm{e}^{\frac{\mathrm{i}\pi}{\alpha+1}}$ see, e.g., refs. [14,21].

When $r=2$, eq. (3.6) becomes

$$\bigg[-\partial_z^2+\frac{p^2-\frac{1}{4}}{z^2}-\frac{s_1}{z}-1+E^{-n-2}\,z^n\bigg]\Psi=0\qquad\qquad (r=2)\,. \tag{A.5}$$

The substitution

$$z\mapsto \mathrm{i}z\,,\qquad\qquad s_1\mapsto -2s\,,\qquad\qquad E\mapsto \mathrm{i}\lambda \tag{A.6}$$

brings it to the form of the differential equation discussed in ref. [13] in the context of the scaling limit of the $\mathcal{Z}_2$ invariant spin chain.

Appendix B

Here we present the explicit formulae for f_1, f_2, which enter into eqs. (3.15) and (4.28). We use the same notation as in ref. [14], where these functions have previously appeared.

The function f_1 is defined as

$$f_1(h,g)=\frac{\pi\,\Gamma(1-2g)}{\sin(\pi g)}\,\frac{\Gamma(g+2h)}{\Gamma(1-g+2h)}\,. \tag{B.1}$$

As for f_2, it is more complicated and is given by the integral

$$f_2(h,g)=2^{1-4g}\,\frac{\Gamma^2(1-g)}{\Gamma^2(\frac{1}{2}+g)}\,\frac{\Gamma(2g+2h)}{\Gamma(1-2g+2h)}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{2\pi}\,\frac{S_1(x)}{x+\mathrm{i}h}\qquad \big(0<g<\tfrac{1}{2},\ \Re e(h)>0\big) \tag{B.2}$$

with

$$S_1(x)=\sinh(2\pi x)\,\Gamma(1-2g+2\mathrm{i}x)\,\Gamma(1-2g-2\mathrm{i}x)\,\big(\Gamma(g+2\mathrm{i}x)\Gamma(g-2\mathrm{i}x)\big)^2\,. \tag{B.3}$$

For $\frac{1}{2}<g<1$ one has

$$f_2(h,g)=2^{1-4g}\,\frac{\Gamma^2(1-g)}{\Gamma^2(\frac{1}{2}+g)}\,\frac{\Gamma(2g+2h)}{\Gamma(1-2g+2h)}\bigg(\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{2\pi}\,\frac{S_1(x)}{x+\mathrm{i}h} \tag{B.4}$$
$$-\frac{\sin(2\pi g)\Gamma(3-4g)\Gamma^2(1-g)\Gamma^2(3g-1)}{(2h+1-2g)(2h-1+2g)}\bigg)\qquad \big(\tfrac{1}{2}<g<1,\ \Re e(h)>0\big)\,.$$

Appendix C

The coefficient $\mathfrak{C}^{(\ell)}_{p,s}(\boldsymbol{w})$, appearing in the large E asymptotic expansion of the spectral determinant (3.41), may be expressed in terms of the connection coefficients of the differential equation

$$\left[-\partial_v^2+\mathrm{e}^{rv}+p^2+\sum_{a=1}^{[\frac{r}{2}]}s_a^{(\ell)}\,\mathrm{e}^{av}+\sum_{\alpha=1}^{\mathtt{L}}\left(\frac{2}{\big(1+w_\alpha^{(\ell)}\mathrm{e}^{-v}\big)^2}+\frac{n_\alpha}{1+w_\alpha^{(\ell)}\mathrm{e}^{-v}}\right)\right]\tilde{\psi}=0\,,\quad \text{(C.1)}$$

where

$$s_a^{(\ell)}=(-1)^{ar}\,\mathrm{e}^{+\frac{\mathrm{i}\pi a}{r}(2\ell-1)}\,s_a\qquad\text{and}\qquad w_\alpha^{(\ell)}=(-1)^r\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{r}(2\ell-1)}\,w_\alpha\,. \qquad \text{(C.2)}$$

This ODE is obtained from (3.26) by changing variables,

$$z=(-1)^{r-1}\,\mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2\ell-1)}\,\mathrm{e}^{v}\,,\qquad \Psi=\mathrm{e}^{\frac{v}{2}}\,\tilde{\psi}\,, \qquad \text{(C.3)}$$

and setting E to infinity therein. Assuming that $\Re e(p)\geq 0$ we consider a solution of (C.1) such that

$$\tilde{\psi}_p\to\mathrm{e}^{pv}\qquad\text{as}\quad v\to-\infty\,. \qquad \text{(C.4)}$$

A straightforward WKB analysis yields that when $v\to+\infty$,

$$\tilde{\psi}_p(v)\to C^{(\ell)}_{p,s}(\boldsymbol{w})\,\exp\left(-\frac{rv}{4}-\mathrm{i}\,(-1)^\ell\,\frac{sv}{2}+\frac{2}{r}\,\mathrm{e}^{\frac{rv}{2}}\right) \qquad \text{(C.5)}$$

with

$$s\equiv\begin{cases}0 & \text{for odd } r\\ s_{\frac{r}{2}} & \text{for even } r\end{cases}\,. \qquad \text{(C.6)}$$

One can show that $\mathfrak{C}^{(\ell)}_{p,s}(\boldsymbol{w})$ is expressed in terms of the connection coefficient $C^{(\ell)}_{p,s}(\boldsymbol{w})$ as

$$\mathfrak{C}^{(\ell)}_{p,s}(\boldsymbol{w})=\sqrt{\frac{4\pi}{n+r}}\;(n+r)^{-\frac{2p}{n+r}}\,2^{-\mathrm{i}\,(-1)^\ell\,\frac{s}{n}}\;\frac{C^{(\ell)}_{p,s}(\boldsymbol{w})}{\Gamma\big(1+\frac{2p}{n+r}\big)}\,. \qquad \text{(C.7)}$$

The computation of $\bar{\mathfrak{C}}^{(\ell)}_{\bar{p},\bar{s}}(\bar{\boldsymbol{w}})$, which occurs in the large $\bar{E}$ asymptotic formula (3.41) of the spectral determinant $\bar{D}_{\bar{p}}(\bar{E}\,|\,\bar{\boldsymbol{w}})$ is analogous. The relevant ODE would formally coincide with (C.1) upon the substitution of the variables $(v,\tilde{\psi})\mapsto(\bar{v},\bar{\tilde{\psi}})$ and parameters $(p,s_a^{(\ell)},w_\alpha^{(\ell)},n_\alpha,\mathtt{L})\mapsto(\bar{p},\bar{s}_a^{(\ell)},\bar{w}_\alpha^{(\ell)},\bar{n}_\alpha,\bar{\mathtt{L}})$, where together with (C.2) we use the notation

$$\bar{s}_a^{(\ell)}=(-1)^{ar}\,\mathrm{e}^{-\frac{\mathrm{i}\pi a}{r}(2\ell-1)}\,\bar{s}_a\qquad\text{and}\qquad \bar{w}_\alpha^{(\ell)}=(-1)^r\,\mathrm{e}^{+\frac{\mathrm{i}\pi}{r}(2\ell-1)}\,\bar{w}_\alpha\,. \qquad \text{(C.8)}$$

The connection coefficient is extracted from the $\bar{v}\to+\infty$ asymptotic of the solution $\bar{\tilde{\psi}}_{\bar{p}}$, which is defined by the condition $\bar{\tilde{\psi}}_{\bar{p}}\to\mathrm{e}^{\bar{p}\bar{v}}$ as $\bar{v}\to-\infty$. Namely,

$$\bar{\tilde{\psi}}_{\bar{p}}(\bar{v})\to\bar{C}^{(\ell)}_{\bar{p},\bar{s}}(\bar{\boldsymbol{w}})\,\exp\left(-\frac{rv}{4}+\mathrm{i}\,(-1)^\ell\,\frac{sv}{2}+\frac{2}{r}\,\mathrm{e}^{\frac{rv}{2}}\right) \qquad \text{(C.9)}$$

(recall that $s\equiv\bar{s}_{\frac{r}{2}}$ for r even and is identically zero for r odd). Then the barred version of (C.7) reads as

$$\bar{\mathfrak{C}}^{(\ell)}_{\bar{p},\bar{s}}(\bar{\boldsymbol{w}}) = \sqrt{\frac{4\pi}{n+r}}\ (n+r)^{-\frac{2\bar{p}}{n+r}}\ 2^{+\mathrm{i}\,(-1)^{\ell}\,\frac{s}{n}}\ \frac{\bar{C}^{(\ell)}_{\bar{p},\bar{s}}(\bar{\boldsymbol{w}})}{\Gamma\big(1+\frac{2\bar{p}}{n+r}\big)}\,. \tag{C.10}$$

The following comments are in order here. For the case when there are no apparent singularities, the coefficients $C^{(\ell)}_{p,\boldsymbol{s}}$ and $\bar{C}^{(\ell)}_{\bar{p},\bar{\boldsymbol{s}}}$ are expressed in terms of the function $F_p(\boldsymbol{s}) = F_p(s_1,\ldots,s_{[\frac{r}{2}]})$ (4.16) used in sec. 4.2:

$$C^{(\ell)}_{p,\boldsymbol{s}}(\boldsymbol{\emptyset}) = F_p(\boldsymbol{s}^{(\ell)})\,, \qquad \bar{C}^{(\ell)}_{\bar{p},\bar{\boldsymbol{s}}}(\boldsymbol{\emptyset}) = F_{\bar{p}}(\bar{\boldsymbol{s}}^{(\ell)})\,, \tag{C.11}$$

where $\boldsymbol{s}^{(\ell)} = \big(s_1^{(\ell)},\ldots,s^{(\ell)}_{[\frac{r}{2}]}\big)$, $\bar{\boldsymbol{s}}^{(\ell)} = \big(\bar{s}_1^{(\ell)},\ldots,\bar{s}^{(\ell)}_{[\frac{r}{2}]}\big)$ with $s_a^{(\ell)}$ and $\bar{s}_a^{(\ell)}$ being defined in eqs. (C.2) and (C.8), respectively.

The above procedure for calculating the asymptotic coefficients, say, $\mathfrak{C}^{(\ell)}_{p,\boldsymbol{s}}(\boldsymbol{w})$ works under the assumption that $\Re e(p) \geq 0$. Nevertheless, it may be extended to any $2p \neq -1,-2,\ldots$ as follows. One notes that $\Psi_p(z) = \mathrm{e}^{\frac{v}{2}}\,\tilde{\psi}_p$ with z related to v as in (C.3) solves the differential equation

$$\Big(-\partial_z^2 + t_0(z) + t_1(z)\Big)\,\Psi_p = 0 \tag{C.12}$$

and satisfies

$$\Psi_p\big(\mathrm{e}^{2\pi\mathrm{i}}z\big) = -\mathrm{e}^{2\pi\mathrm{i}p}\,\Psi_p(z)\,. \tag{C.13}$$

Here the l.h.s. is understood as an analytical continuation along a small contour wrapping around $z=0$ in the counter-clockwise direction. Such a solution $\Psi_p(z)$ may be defined for any $2p \neq -1,-2,\ldots$. By reverting to the original variables v, $\tilde{\psi}_p$ and using (C.5), one extracts the connection coefficient $C^{(\ell)}_{p,\boldsymbol{s}}(\boldsymbol{w})$. In turn, $\mathfrak{C}^{(\ell)}_{p,\boldsymbol{s}}(\boldsymbol{w})$ is obtained via formula (C.7).

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