Editors' Suggestion

Partition Function for a Volume of Space

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We consider the quantum gravity partition function that counts the dimension of the Hilbert space of a spatial region with topology of a ball and fixed proper volume, and evaluate it in the leading order saddle point approximation. The result is the exponential of the Bekenstein-Hawking entropy associated with the area of the saddle ball boundary, and is reliable within effective field theory provided the mild curvature singularity at the ball boundary is regulated by higher curvature terms. This generalizes the classic Gibbons-Hawking computation of the de Sitter entropy for the case of positive cosmological constant and unconstrained volume, and hence exhibits the holographic nature of nonperturbative quantum gravity in generic finite volumes of space.

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Introduction.—Gibbons and Hawking (GH) [1] proposed in 1977 that the thermal partition function Z in quantum gravity can be approximated by a Euclidean saddle point of a path integral over spacetime geometries. When applied to spacetimes without a boundary, using the Einstein-Hilbert action I with a positive cosmological constant Λ, they found that there is a saddle corresponding to a round Euclidean four-sphere, which can be obtained by analytic continuation of the time coordinate in a static patch of Lorentzian de Sitter (dS) spacetime. This method yields

$$Z \approx \exp(-I_{\text{saddle}}/\hbar) = \exp(A/4\hbar G), \tag{1}$$

where A is the area of the dS horizon for the saddle and G is the gravitational constant (in units with the speed of light equal to 1). Each "time" slice of the Euclidean saddle is a spatial geometry that is identical to the spatial geometry of a static slice bounded by the event horizon of the corresponding Lorentzian geometry. In the Euclidean solution these time slices all coincide at this event horizon surface, which we refer to as the "Euclidean horizon," or just "horizon."

This and other results strongly suggested that the concept of Bekenstein-Hawking entropy $A/4\hbar G$ for black hole horizons [2,3] applies also to de Sitter horizons [4]. Two decades later Fischler [5] and Banks [6] argued that, since the sphere partition function is an integral unconstrained by any boundary conditions, this Z must represent the dimension of the Hilbert space of *all* states in this theory describing a volume of space, whose size in the saddle point approximation is determined by the value of the cosmological constant. That is, $\log Z$ is the entropy of the maximally mixed state in that Hilbert space. This interpretation of the "entropy of de Sitter space" has since received support from several directions [7–12].

Numerous lines of evidence indicate that gravitational entropy is associated not only to the area of a de Sitter or black hole horizon, but also to the area of any boundary separating a region of space (see, for example, [13,14]), in particular to the bounding area of a topological ball of space. We present a derivation of the "entropy of a volume of space" from a quantum gravity partition function, without or with a cosmological constant, in the vein of the original GH calculation. In our case, the size of the spatial region is determined not by a cosmological constant, but by a volume constraint imposed on the states. The constraint modifies the saddle point condition, and we find that the Einstein-Hilbert action of the saddle geometry is $-A/4G$, where A is the boundary area. Our derivation involves no classical background spacetime as input; rather, it is in principle a fully nonperturbative quantum gravity calculation from which a classical saddle arises as output. On the other hand, like the GH calculation, it is formulated within general relativity, which is the low-energy effective theory of some ultraviolet-complete theory of quantum gravity.

The microstates counted by the Bekenstein-Hawking entropy are arguably related to vacuum fluctuations [15–19], but their precise nature is not resolvable within the low-energy effective theory. Moreover, the notion of a Hilbert space of a subregion in quantum gravity requires explanation. Because of the diffeomorphism constraints, the Hilbert space of quantum gravity is not decomposable as a tensor product of subregions of Hilbert spaces. This is similar to the case of Yang-Mills theory, for which the entire Hilbert space can nevertheless be realized as a gauge-invariant *subspace* of a tensor product of subregion Hilbert spaces that contain "edge state" degrees of freedom [20,21]. We presume that the Hilbert space of a ball of space in quantum gravity admits (at least in some approximate description) a similar subregion interpretation, and that horizon entropy can be thought of as the logarithm of the dimension of the space of edge states. The remarkable thing is that, even without having the microscopic theory of the edge states, their contribution to the entropy per unit horizon area may be encoded in the value of the low-energy effective gravitational constant [22,23].

Sphere partition function.—The GH partition function for the case of dS—which has no boundary—can be interpreted as the trace of the identity operator on the Hilbert space of *all* states of a topological ball. This interpretation has recently been justified [10] by first introducing an artificial internal boundary sphere of radius R_B, and inverse temperature β at that boundary, and considering the canonical partition function $Z = \mathrm{Tr}\exp(-\beta H_{\mathrm{BY}})$, where H_{BY} is the Brown-York Hamiltonian for that system. In the limit $R_B \to 0$ the boundary disappears and $H_{\mathrm{BY}} \to 0$, assuming the geometry is regular inside the shrinking boundary and assuming $D \geq 3$ spacetime dimensions. Therefore, in this limit, $Z \to \mathrm{Tr}\,\exp(0) = \mathrm{Tr} I_{\mathcal{H}}$, where $\mathcal{H}$ is the Hilbert space of the ball of space. That is, the no-boundary canonical ensemble is maximally mixed, and Z counts the dimension of the entire Hilbert space.

The paths in a path integral representation of Z are periodic in time because the path integral is computing the trace. At each time the configuration is a Riemannian metric on a topological $(D-1)$ ball of space, whose surface is a $(D-2)$ sphere. In the saddle D geometry the time translation becomes a Euclidean signature rotation encircling the ball surface, which is a fixed point set of the rotation. The manifold generated by rotating the ball through a complete time circle in the about the ball surface is topologically a D sphere, S^D. This is easy to visualize in the $D = 2$ case (see Fig. 1), where a complete rotation of a line segment about its endpoints sweeps out a topological 2-sphere. The way it works for $D = 3$ is explained in detail in Sec. 5 of [10], which includes a possibly useful figure. To see it in a general dimension D one can invoke the topological fact that the $(D-1)$ ball is the one point compactification of the half-space $\frac{1}{2}\mathbb{R}^{(D-1)}$, and rotating this through an extra dimension around its boundary yields the one point compactification of $\mathbb{R}^D$, which is S^D. The boundaryless D-sphere topology for the paths in effect removes the need for a boundary condition at the surface of the spatial ball, and presumably corresponds to the condition that the spatial ball is smoothly embedded into the ambient space, like the dS static patch.

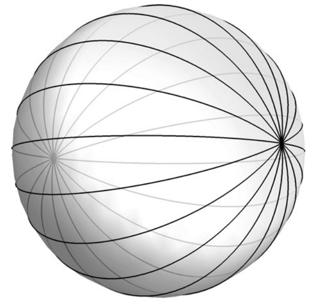

FIG. 1. Illustration of the topology of the *round* Euclidean sphere "path" S^D for the case $D = 2$. The meridian arcs (half circles) each correspond to a patch of space at one time. They are topologically 1-balls, which all share the same 0-sphere boundary consisting of the two poles. Together they comprise a foliation of the 2-sphere that is degenerate at the poles which constitute the Euclidean horizon.

The partition function is dominated and thus well-approximated by the saddle—i.e., the solution to the vacuum Einstein equation—with minimal action. In the presence of a positive cosmological constant Λ, the saddle with minimal action is a round D sphere of radius $L = \sqrt{(D-1)(D-2)/(2\Lambda)}$, a.k.a., Euclidean dS spacetime [10] (for a proof see Appendix A in Supplemental Material [24]). As mentioned above, the action of this saddle is $-A/4G$, which (since there is no temperature dependence) implies that the entropy is given by $A/4\hbar G$. The cosmological constant (together with $\hbar G$) thus determines the dimension of the Hilbert space of all states of a topological ball of space in pure gravity.

In the limit $\Lambda \to 0$, the de Sitter sphere saddle becomes a sphere of infinite radius and the entropy is therefore infinite. We believe this result is correct, but it does not tell us what is the entropy of a ball of a *fixed size*. Unlike the case in which a particular size is selected by the cosmological constant, any other size must be specified as an external constraint that restricts the ensemble of states.

Since the entropy of a ball is expected to be proportional to the boundary area one might think that the natural way to restrict the size would be to restrict the boundary area. A fixed area constraint can be imposed using a Lagrange multiplier term in the action, which contributes to the field equations an effective energy-momentum tensor of a cosmic membrane at the ball boundary. The solutions to the resulting field equation have a conical defect at the membrane. It seems unlikely, however, that the presence of such a conical defect allows for a Euclidean solution to the $\Lambda = 0$ vacuum Einstein equation on S^D. For instance, in $D = 3$ dimensions such a metric would be locally flat everywhere except at the S^1, where there would be a conical defect, but it can be shown that this is not possible [29]. The (presumed) absence of a saddle for the fixed area ensemble suggests that the rigidity of this constraint entails quantum fluctuations that are too large to be compatible with a semiclassical description of the ensemble. According to the uncertainty relation, at fixed area one expects that the conjugate variable, which is the angle at which the maximal surface meets the edge of the causal diamond [30], is totally uncertain.

One could instead fix the Euclidean spacetime volume, and this constrained partition function admits a saddle; however, this is not a physically valid way to implement the constraint that the ball has a particular "size." Since we are trying to count the dimension of the Hilbert space of the system, the size constraint should be imposed on the system itself, whose states are enumerated *at one time*. Moreover, the Euclidean spacetime geometry is relevant only in the saddle point approximation to the partition function being computed; the domain of integration in the path integral representation of the partition function is not Euclidean geometries [10,31–34]. That said, the fixed spacetime volume partition function can perhaps be understood as a weighted sum dominated by spatial balls of a given size. We are currently exploring this interpretation [35].

Partition function at fixed spatial volume.—An apparently sensible way to fix the size of the system is to constrain its *spatial* volume. The dimension of the Hilbert space of a quantum gravitational ball at fixed volume V can be approximated as a path integral over metrics on the topological D sphere that admit a (degenerate) foliation by $(D-1)$ balls with volume V whose $(D-2)$-sphere boundaries all coincide. The paths are weighted by the exponential of minus the Einstein-Hilbert action with cosmological constant Λ together with a Lagrange multiplier term implementing the volume constraint

$$Z[V,\Lambda] = \int \mathcal{D}\lambda\mathcal{D}g\exp\left[\frac{1}{16\pi\hbar G}\int d^D x\sqrt{g}(R-2\Lambda) + \frac{1}{\hbar}\int d\phi\lambda(\phi)\left(\int d^{D-1}x\sqrt{\gamma}-V\right)\right], \tag{2}$$

where the contour of integration for the metric is assumed to pass through a Euclidean saddle. (See Ref. [10] for a discussion of the nature of the required contour deformation.) Here ϕ is a periodic Euclidean coordinate, and $\gamma_{ab} = g_{ab} - N^2\phi_{,a}\phi_{,b}$ is the induced metric on a constant ϕ slice, where $N \coloneqq (g^{ab}\phi_{,a}\phi_{,b})^{-1/2}$, so that $Nd\phi$ is a unit 1-form. The integral of the Lagrange multiplier $\lambda(\phi)$ is for each ϕ over a contour parallel to the imaginary axis, and thus introduces a Dirac delta function that imposes the constraint that the spatial volume of each ϕ slice is equal to V. Following Gibbons and Hawking, we shall estimate Z as in Eq. (1), where I_{saddle} is the Einstein-Hilbert action evaluated at the stationary point with the lowest action. The constraint term vanishes when the constraint is satisfied, so does not contribute to the saddle action.

The saddle point equations are given by the volume constraint, together with the Euclidean Einstein equation sourced by a perfect fluid stress-energy tensor T_{ab} with vanishing "energy density," arising from the variation of the volume element in the volume constraint term

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \quad \text{with} \quad T_{ab} = \frac{\lambda}{N}\gamma_{ab} \eqqcolon P\gamma_{ab}, \tag{3}$$

where P is the effective fluid pressure, and the λ contour has been deformed so as to pass through the required real value at the saddle. For each value of V, there is now a saddle, even if $\Lambda = 0$. This is referred to as the method of constrained instantons [36,37]. By analogy with the unconstrained $\Lambda > 0$ case, where the saddle with the lowest action (i.e., the greatest volume) is the sphere, we presume that the dominating saddle is the most symmetrical one. We therefore look for a static, spherically symmetric solution with ϕ as a Killing coordinate, such that $\lambda(\phi) = \lambda =$ constant. Our metric ansatz for the Euclidean saddle thus takes the form

$$ds^2 = N^2(r)d\phi^2 + h(r)dr^2 + r^2 d\Omega^2_{D-2}, \tag{4}$$

where $d\Omega^2_{D-2}$ is the line element on a unit $(D-2)$ sphere, and we choose the period of the ϕ coordinate to be $\Delta\phi = 2\pi$. With this symmetry ansatz, the equations to be solved are the Euclidean version of those of a static, spherically symmetric fluid star in D spacetime dimensions, but with different boundary conditions.

The spatial metric function $h(r)$ is determined by the $\phi\phi$ component of the Einstein equation, which receives no contribution from the fluid stress tensor since that has vanishing "energy density" $T_{\phi\phi}$. The solution for $h(r)$ with $\Lambda > 0$ is thus the same as for D-dimensional dS space,

$$h(r) = (1 - r^2/L^2)^{-1}, \tag{5}$$

while that with $\Lambda = 0$ corresponds to $L \to \infty$ and that with $\Lambda < 0$ to $L \to iL$. The equation for the metric function $N(r)$ is the (Euclidean, D-dimensional) Tolman-Oppenheimer-Volkoff equation, which for vanishing energy density is identical to the corresponding Lorentzian equation. The boundary conditions for $N(r)$ arise from the requirement that the ball boundary be a regular Euclidean horizon, i.e., a fixed point set of the periodic ϕ translation symmetry where the metric is locally flat. This implies that the lapse $N(r)$ must vanish at some value $r = R_V$ (determined by the volume constraint) that defines the location of the horizon. According to Eq. (3) the pressure $P(r)$ and curvature must therefore diverge at the horizon, unless $\lambda = 0$. Furthermore, absence of a conical singularity at R_V requires that the line element in the $r-\phi$ subspace there takes the form of the Euclidean plane in polar coordinates, $l^2 d\phi^2 + dl^2$, where l is the proper radial distance from the Euclidean horizon. This fixes the coefficient of the linear term in a Taylor expansion of the lapse N about R_V,

$$\left.\frac{dN}{dl}\right|_{l=0} = 1. \tag{6}$$

We first solve the saddle point equations for $\Lambda = 0$, and then generalize the solution to nonzero Λ.

$\Lambda = 0$ *case*.—For $\Lambda = 0$, Eq. (5) reduces to $h(r) = 1$, so the spatial metric for the saddle is flat, and the Tolman-Oppenheimer-Volkoff equation with zero energy density is [38]

$$\frac{dP}{dr} = -\frac{8\pi G}{D-2} P^2 r. \tag{7}$$

The general solution to this equation is

$$P(r) = -\frac{D-2}{4\pi G} \frac{1}{R_V^2 - r^2}, \tag{8}$$

where R_V^2 is an integration constant. The condition that P diverges somewhere requires that $R_V^2 > 0$, and the horizon is located at $r = R_V$. The value of R_V is set by the volume constraint $[\Omega_{D-2}/(D-1)]R_V^{D-1} = V$, where Ω_{D-2} is the volume of a unit $(D-2)$ sphere. The lapse is given by $N(r) = \lambda/P(r)$, so the condition, Eq. (6), implies that $\lambda = -1/(dP^{-1}/dl)_{l=0}$, i.e.,

$$\lambda = -\frac{1}{8\pi G} \frac{D-2}{R_V}. \tag{9}$$

The Euclidean saddle metric is given by

$$ds^2 = \frac{1}{4R_V^2} (R_V^2 - r^2)^2 d\phi^2 + dr^2 + r^2 d\Omega_{D-2}^2. \tag{10}$$

To estimate $Z[V]$, Eq. (2), we evaluate the on-shell Einstein-Hilbert action with $\Lambda = 0$

$$I_{\text{saddle}} = -\frac{1}{16\pi G} \int d^D x \sqrt{g} R = \frac{D-1}{D-2} 2\pi\lambda V = -\frac{A_V}{4G}, \tag{11}$$

where A_V is the area of the $(D-2)$ sphere that forms the boundary of the spatial volume, i.e., the horizon of the Euclidean "diamond" with spatial volume V. This action yields an approximation to the partition function in the zero-loop saddle point approximation

$$Z[V] \approx \exp(A_V/4\hbar G). \tag{12}$$

The logarithm of the dimension of the Hilbert space is thus given by the Bekenstein-Hawking entropy formula, with the horizon area equal to the boundary area of the saddle point ball metric. This result for the action can also be seen from the fact that the Euclidean Einstein-Hilbert action with Gibbons-Hawking-York boundary term—evaluated on a

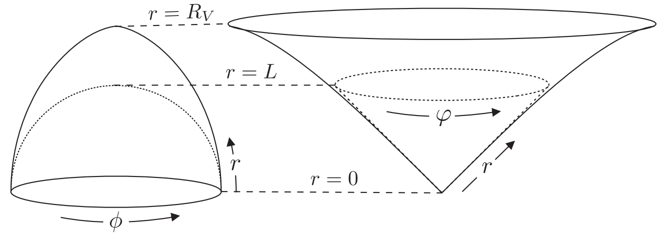

FIG. 2. The Euclidean saddles for de Sitter with $\Lambda > 0$ (dotted) and for fixed volume with $\Lambda = 0$ (solid), with $R_V = 2L$ so that the Euclidean time periods match at the center of the saddle (the cap perimeter on the left). The hemispherical caps are flat space embeddings of the intrinsic geometries of the time-radius $(\phi - r)$ discs, with the horizons at the tips of the caps. At each point of the cap there is a round $(D-2)$ sphere, shown here on the right as a circular section of a cone with azimuthal angle φ, whose radius in the de Sitter case is equal to the vertical embedding height on the hemispherical cap and in the $\Lambda = 0$ case is equal to the radial distance along the cap. The fixed volume saddle has a mild curvature singularity at the horizon, whereas the de Sitter saddle is a round D sphere that is everywhere smooth.

configuration that is independent of a periodic Euclidean "time" coordinate, has a Euclidean horizon of area A, and satisfies the vacuum Hamiltonian constraint—generally takes the form $I_{\text{saddle}} = \beta E_{\text{BY}} - A/4G$, with β the Euclidean time period, and E_{BY} the Brown-York energy [39]. This applies in the present case, and E_{BY} vanishes as the boundary size goes to zero, so the action, Eq. (11), is reproduced in that limit.

Let us discuss some properties of the saddle. The saddle has topology S^D, is conformally flat [40], spatially flat, and spherically symmetric and has a rotational Killing symmetry along the Euclidean time direction (see Fig. 2). Further, the effective fluid pressure, Eq. (8), is negative, and diverges as the inverse proper distance to the edge of the ball (horizon), $P \sim -1/(R_V - r)$. Hence, the energy-momentum tensor, and therefore the Ricci tensor, is singular on the horizon in the sense that it has eigenvalues that diverge as $1/(R_V - r)$. However, this curvature divergence [which can be traced to a nonzero quadratic term in the Taylor expansion of the lapse function $N(r)$ about the horizon] is sufficiently mild that the on-shell Einstein-Hilbert action is finite and given by Eq. (11).

Nevertheless, since the curvature diverges one should take into account higher derivative terms in the effective Lagrangian $L \sim R + \ell^2 R^2 + \cdots$, with relative coefficients determined by some UV length scale ℓ. It is possible that such higher derivative terms allow for a regular saddle with everywhere finite curvature, but with divergent derivatives of curvature that match the divergence of the effective energy-momentum tensor. If they do not, then our use of effective field theory appears inadequate to treat the problem. However, if they do, the entropy will be given by the Bekenstein-Hawking term with the area of the horizon of the regular saddle, plus the higher curvature corrections to the horizon entropy functional [42–44]. This computation would be under control in the effective field theory provided that the higher curvature terms are systematically suppressed. To estimate their contribution, note that the contribution of the R^2 term to the field equation behaves as $\sim\ell^2\partial_r^2 R$, and the curvature of the uncorrected saddle diverges as $(R_V\rho)^{-1}$, where $\rho := R_V - r$, so the contribution from the R^2 terms scales as $\sim(\ell/\rho)^2 R$, which becomes comparable to the Einstein term when $\rho \lesssim \ell$. At that point the curvature is $\sim(R_V\ell)^{-1}$. If the curvature saturates at this value, the contribution of the R^2 term to the entropy functional, which is the integral of $\sim\ell^2 R$ over the horizon, would be of order ℓ/R_V relative to that of the Bekenstein-Hawking area term. Moreover, the contributions of higher curvature terms would be suppressed by an additional factor of ℓ/R_V for each additional power of curvature in the Lagrangian, so that it would be consistent to truncate the effective field theory.

$\Lambda \neq 0$ *case.*—For nonzero Λ a similar saddle exists, with action again given by $I_{\text{saddle}} = -A_V/4G$. The most significant difference occurs for $\Lambda > 0$, in which case the spatial geometry is a ball of volume V embedded in a round $(D-1)$ sphere of radius L. A_V grows with V until V reaches half the volume of that $(D-1)$ sphere, and then decreases to zero as V reaches the full $(D-1)$-sphere volume. (For details, see the appendix.)

Discussion.—To sum up, the quantum gravity partition function that counts the dimension of the Hilbert space of a spatial region with topology of a ball and fixed proper volume is approximated, in the leading order saddle point approximation, by $\exp(A_V/4\hbar G)$, where A_V is the area of the saddle ball boundary, up to higher curvature corrections suppressed by a UV length scale divided by the ball radius. For positive cosmological constant and for volume $V > V_{\text{dS}}$ (where V_{dS} is the volume of the de Sitter static patch), A_V and hence the dimension of the Hilbert space *decreases* as V increases. The Hilbert space becomes one dimensional for $V = 2V_{\text{dS}}$, and no saddle exists for $V > 2V_{\text{dS}}$. Hence, more space does not always imply more states. The integral over all volumes, $\int dV \exp(A_V/4\hbar G)$, agrees, in the leading order saddle point approximation, with the Gibbons-Hawking result, Eq. (1), for a ball of unconstrained volume because the ensemble is dominated by ball geometries with surface area equal to that of the de Sitter horizon.

That the logarithm of the Hilbert space dimension in our nonperturbative framework matches the horizon entropy attributed to a semiclassical static patch in de Sitter space and other causal diamonds supports the notion that the total dimension of the Hilbert space is captured already at leading order by the exponential of the semiclassical entropy. From this follows the surprising conclusion that the semiclassical, gravitationally dressed vacuum state of such a causal diamond is close to a maximally mixed state, a notion that has already been advanced for the case of a de Sitter static patch [5–12].

We have explicitly considered only the states of the gravitational field in a ball, but matter fields could be included without any modification of the saddle calculation, provided the matter fields vanish in the saddle configuration. The existence of extra states due to the matter fields would be accounted for at leading order by the value of the low energy effective gravitational constant G, which figures in the denominator of the Bekenstein-Hawking entropy [22,23]. The entropy match discussed above thus lends nonperturbative quantum gravitational support to the "maximal vacuum entanglement hypothesis" [45]—that entanglement entropy of matter and gravity in small balls at fixed volume is maximized in the semiclassical, gravitationally dressed vacuum state. To elevate that support to full justification would require mastery of the corrections to the saddle point approximation.

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Appendix: Derivation of the $\Lambda \neq 0$ *saddle.*—In the presence of a positive cosmological constant, $h(r)$ is given by Eq. (5), so the spatial metric is that of a round $(D-1)$ sphere of radius L and is half covered by the range $r \in [0, L]$. The Tolman-Oppenheimer-Volkoff equation for Einstein gravity with $\Lambda > 0$ and zero energy density is [38,46]

$$\frac{dP}{dr} = -P\,\frac{\left(\frac{8\pi G L^2}{D-2}P - 1\right)r}{L^2 - r^2}. \tag{A1}$$

To uniquely label points on the $(D-1)$ sphere, we switch from r to the polar angle coordinate $\chi \in [0, \pi]$ (with $r = L\sin\chi$), in terms of which the proper radial distance is $L d\chi$. The case of $\Lambda < 0$ is obtained by the replacement $(L, \chi) \to (iL, -i\chi)$ in the following formulas. For notational brevity we momentarily adopt units with $8\pi G/(D-2) = 1$. Then Eq. (A1) becomes

$$\frac{dP}{d\chi} = -P(L^2P - 1)\tan\chi, \tag{A2}$$

and the solution is

$$P(\chi) = \frac{L^{-2}}{1 - \cos\chi/\cos\chi_V}, \tag{A3}$$

where $\cos\chi_V$ is an integration constant whose value is determined by the volume constraint,

$$V = V(\chi_V) \coloneqq L^{D-1}\Omega_{D-2}\int_0^{\chi_V} d\chi(\sin\chi)^{D-2}. \tag{A4}$$

The pressure is negative for $\chi_V < \pi/2$, and positive for $\chi_V > \pi/2$, and blows up as the reciprocal of the proper distance to the horizon at $\chi = \chi_V$. The Lagrange multiplier λ is fixed by the condition, Eq. (6), at the horizon,

$$\lambda = -L^{-1}\cot\chi_V. \tag{A5}$$

The Euclidean saddle metric is thus

$$ds^2 = L^2\left[\left(\frac{\cos\chi - \cos\chi_V}{\sin\chi_V}\right)^2 d\phi^2 + d\chi^2 + \sin^2\chi d\Omega_{D-2}^2\right], \tag{A6}$$

with $\chi \in [0, \chi_V]$. This metric is also conformally flat, and it has a Killing horizon at $\chi = \chi_V$. The saddle again has topology S^D, and a constant ϕ slice is a ball of volume V embedded in a round $(D-1)$ sphere of radius L. In the limit $L \to \infty$ we have $\chi_V \to 0$, and holding $L\chi$ fixed, Eq. (A6), becomes the spatially flat metric, Eq. (10). For $\chi_V = \pi/2$ the metric becomes that of Euclidean de Sitter space (the round D sphere of radius L) and $\lambda = 0$, so the volume constraint plays no role in determining the saddle geometry. In the zero-loop saddle point approximation, the partition function is thus unaffected by the constraint if the volume is set equal to that of (a maximal slice of) the dS static patch.

The Euclidean action of the saddle can be computed directly by using the saddle point equations, Eq. (3), the pressure, Eq. (A3), and the metric, Eq. (A6). In Appendix C of the Supplemental Material [24], we present an explicit computation of the on-shell action using the above coordinate system, and in Appendix D we derive the on-shell action as well as a Smarr formula and first law using the Noether charge formalism [43,47] (see also [48–50]). Alternatively, as explained below, Eq. (11), we know from the general properties of the saddle that the action is given by $I_{\text{saddle}} = -A_V/4$. The horizon area of the saddle geometry is determined by the parameters of the ensemble V and Λ. At fixed V the area decreases as the cosmological constant increases. At fixed Λ, the area increases with volume until it reaches a maximum when $V = V(\chi_V = \pi/2)$, such that the $(D-1)$ ball covers half of a $(D-1)$ sphere of radius L, after which it decreases, reaching zero when $V = V(\chi_V = \pi)$.

In the case of $\Lambda < 0$ the trig functions become hyperbolic trig functions, the saddle ball is embedded in hyperbolic space, and there is no upper limit to χ or to the size of the saddle ball as the volume grows.

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