Numerical analysis of a second order ensemble algorithm for numerical approximation of stochastic Stokes-Darcy equations[⋆]

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Abstract

Numerical approximation of stochastic Stokes-Darcy equations usually requires repeated sampling of the random hydraulic conductivity tensor and then simulating flow ensembles. In this setting, we propose an efficient, second order, ensemble algorithm for fast computation of the whole set of realizations of the stochastic Stokes-Darcy model corresponding to different random hydraulic conductivity tensor samples. The ensemble algorithm only requires the solution of two linear systems that have the same constant coefficient matrices for all realizations. We give a complete long time stability and convergence analysis for the method. Numerical experiments are presented to support theoretical results and demonstrate the application of the method.

Keywords: Stokes-Darcy equations, uncertainty quantification, ensemble algorithm, finite element method, partitioned method

1. Introduction

Effective simulations of the coupling of groundwater flows (in porous media) and surface flows are required in many engineering and geological applications. One of the major difficulties is that the hydraulic conductivity tensor $\mathcal{K}$ can not be accurately determined and uncertainties have to be taken into account using stochastic models. This leads to another challenge in numerical simulations as the numerical approximation of stochastic PDEs usually requires solution of a (usually relatively large) number of realizations corresponding to different parameter samples, which can be prohibitively expensive. To reduce the computational cost, many uncertainty quantification (UQ) methods have been developed and extensively studied, among which the ensemble-based nonintrusive methods are particularly popular because legacy codes can be directly utilized without much modifications, e.g. variants of the Monte Carlo method such as the multilevel Monte Carlo method [2] and quasi Monte Carlo method [41], alternative sampling methods such as centroidal Voronoi tessellations [55] and Latin hypercube sampling [30], non-intrusive polynomial chaos methods [29, 54], stochastic collocation methods [1, 64, 53, 6]. This direction of research has been focused on reducing the number of samples points required up to a certain accuracy. On the other hand, the problem of designing algorithms to compute ensembles more efficiently has only recently been addressed. A new direction of research is

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to speed up the ensemble computation by developing efficient ensemble algorithms. This was first studied by Jiang and Layton in [35, 36, 34] where an ensemble algorithm that results in one common coefficient matrix for all realizations was devised for computing time-dependent Navier-Stokes equations. This feature of the algorithm allows the use of iterative solvers such as Block CG [12] or Block GMRES [26], to greatly reduce the computing time and required memory. For example, in [39], the proposed stabilized SAV ensemble algorithm with a block GMRES solver for solving the corresponding linear system with multiple right hand sides was able to save 82% of the CPU time when compared with a traditional nonensemble method for computing an ensemble of 100 realizations of the Navier-Stokes flow problem. [40] provides a detailed numerical investigation of ensemble methods with block iterative solvers for evolution problems. Some recent developments on the ensemble algorithms for different flow problems can be found in [5, 13, 14, 15, 19, 20, 21, 22, 23, 27, 31, 32, 33, 37, 38, 40, 46, 47, 48, 49, 61, 62]. In [37], an efficient ensemble algorithm was proposed for fast computation of multiple realizations of the stochastic Stokes–Darcy model with a random hydraulic conductivity tensor. The algorithm results in J linear systems with the same coefficient matrix instead of J linear systems with J different coefficient matrices at each time step. Even though it is efficient, the method of [37] is only first order accurate.

In this paper, we extend the method in [37] to an efficient, second order, ensemble algorithm for fast computation of multiple realizations of the stochastic Stokes-Darcy interface model with random hydraulic conductivity, based on the second order in time Backward-differentiation (BDF2) timestepping. We give comprehensive stability analysis and error analysis of the higher order method in Sects. 3 and 4, respectively.

The Stokes-Darcy equations are well studied to model the coupling between the surface fluid flow and the groundwater flow in porous media, see for example [3, 7, 8, 9, 10, 11, 42, 44, 50, 51, 57, 45]. Let D_f denote the surface fluid flow region and D_p the porous media flow region, where $D_f, D_p \subset R^d (d = 2, 3)$ are both open, bounded domains. These two domains lie across an interface, I, from each other, and $D_f \cap D_p = \emptyset, \bar{D}_f \cap \bar{D}_p = I$, see Figure 1. The

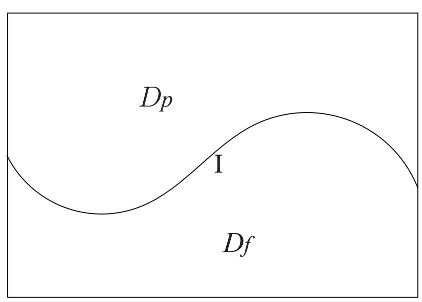


Figure 1: A sketch of the porous median domain D_p, fluid domain D_f, and the interface I.

Stokes-Darcy model is: Find fluid velocity $u(x, t)$, fluid pressure $p(x, t)$, and hydraulic head $\phi(x, t)$ that satisfy

$$\begin{aligned} u_t - \nu \Delta u + \nabla p = f_f(x, t), \nabla \cdot u = 0, \ \text{in} \ D_f, \\ S_0 \phi_t - \nabla \cdot (\mathcal{K}(x) \nabla \phi) = f_p(x, t), \ \text{in} \ D_p, \\ \phi(x, 0) = \phi_0(x), \ \text{in} \ D_p \text{ and } u(x, 0) = u_0(x), \ \text{in} \ D_f, \\ \phi(x, t) = 0, \ \text{in} \ \partial D_p \backslash I \text{ and } u(x, t) = 0, \ \text{in} \ \partial D_f \backslash I. \end{aligned} \tag{1.1}$$

Let $\widehat{n}_{f/p}$ denote the outward unit normal vector on I associated with $D_{f/p}$, where $\widehat{n}_f = -\widehat{n}_p$. The coupling conditions across I are conservation of mass, balance of forces and the Beavers-Joseph-Saffman condition on the tangential velocity:

$$u \cdot \widehat{n}_f - \mathcal{K}\nabla\phi \cdot \widehat{n}_p = 0 \text{ and } p - \nu\ \widehat{n}_f \cdot \nabla u \cdot \widehat{n}_f = g\phi \text{ on } I,$$
$$-\nu\ \nabla u \cdot \widehat{n}_f = \tfrac{\alpha_{\text{BJS}}}{\sqrt{\widehat{\tau}_i \cdot \mathcal{K}\widehat{\tau}_i}} u \cdot \widehat{\tau}_i \ \text{ on } I, \text{ for any tangential vector } \widehat{\tau}_i \text{ on } I,$$

see [4, 56]. Here, g, $\mathcal{K}$, ν and S_0 are the gravitational acceleration constant, hydraulic conductivity tensor, kinematic viscosity and specific mass storativity coefficient, respectively, which are all positive. $\mathcal{K}$ is assumed to be symmetric positive definite (SPD).

In this paper we study a second order ensemble algorithm for computing an ensemble of the Stokes-Darcy systems to account for uncertainties in initial conditions, forcing terms and the hydraulic conductivity tensor. Herein we consider computing an ensemble of J Stokes-Darcy systems corresponding to J different parameter sets $(u_j^0, \phi_j^0, f_{fj}, f_{pj}, \mathcal{K}_j)$, $j = 1, ..., J$,

$$\begin{aligned} u_{j,t} - \nu\Delta u_j + \nabla p_j = f_{f,j}(x,t),\ \nabla \cdot u_j = 0, \ \text{ in } \ D_f, \\ S_0\phi_{j,t} - \nabla \cdot (\mathcal{K}_j(x)\nabla\phi_j) = f_{p,j}(x,t), \ \text{ in } \ D_p, \\ \phi_j(x,t) = 0, \ \text{in } \partial D_p \backslash I \text{ and } u_j(x,t) = 0, \ \text{in } \partial D_f \backslash I, \end{aligned} \tag{1.2}$$

where we assume there are uncertainties in initial conditions $u^0(x)$, $\phi^0(x)$, forcing terms $f_f(x,t)$, $f_p(x,t)$ and the hydraulic conductivity tensor $\mathcal{K}(x)$, and $(u_j^0, \phi_j^0, f_{fj}, f_{pj}, \mathcal{K}_j)$ is one of the samples drawn from the respective probabilistic distributions.

The second order ensemble algorithm we propose for computing multiple realizations of the Stokes-Darcy model reads

Algorithm 1. *Find* $(u_j^{n+1}, p_j^{n+1}, \phi_j^{n+1}) \in X_f \times Q_f \times X_p$ *satisfying* $\forall\ (v, q, \psi) \in X_f \times Q_f \times X_p$,

$$\begin{cases} \left(\dfrac{3u_j^{n+1} - 4u_j^n + u_j^{n-1}}{2\Delta t}, v\right)_f + \nu(\nabla u_j^{n+1}, \nabla v)_f + \displaystyle\sum_i \int_I \bar{\eta}_i (u_j^{n+1} \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i)\ ds \\ + \displaystyle\sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)((2u_j^n - u_j^{n-1}) \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i)\ ds - \left(p_j^{n+1}, \nabla \cdot v\right)_f \\ + c_I(v, 2\phi_j^n - \phi_j^{n-1}) = (f_{f,j}^{n+1}, v)_f, \\ (q, \nabla \cdot u_j^{n+1})_f = 0, \end{cases} \tag{sub-problem 1}$$

$$\begin{cases} gS_0 \left(\dfrac{3\phi_j^{n+1} - 4\phi_j^n + \phi_j^{n-1}}{2\Delta t}, \psi\right)_p + g(\bar{\mathcal{K}}\nabla\phi_j^{n+1}, \nabla\psi)_p \\ + g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(2\phi_j^n - \phi_{j-1}^n), \nabla\psi)_p - c_I((2u_j^n - u_j^{n-1}), \psi) = g(f_{p,j}^{n+1}, \psi)_p, \end{cases} \tag{sub-problem 2}$$

where

$$\bar{\mathcal{K}} = \frac{1}{J}\sum_{j=1}^{J} \mathcal{K}_j, \quad \eta_{i,j} = \frac{\alpha_{\text{BJS}}}{\sqrt{\widehat{\tau}_i \cdot \mathcal{K}_j \widehat{\tau}_i}} \quad \textit{and} \quad \bar{\eta}_i = \frac{1}{J}\sum_{j=1}^{J} \eta_{i,j}.$$

The efficiency of the algorithm is obvious. First, this algorithm decouples the original problem into two smaller sub-physics problems, which results in two smaller linear systems to be solved at each time step reducing both storage and computational time. They can also be run in parallel to further reduce computational time. More importantly, for both subproblems, all realizations have the same coefficient matrix, which allows the use of efficient block solvers, e.g, block CG [12], block GMRES [26], or direct solvers such as LU factorization, to solve the linear systems at greatly reduced computational cost.

The layout of this paper is as follows. Section 2 gives mathematical preliminaries and defines notation. In Section 3 we prove the long time stability of the proposed method under a time step condition and two parameter conditions. In Section 4, we give a complete convergence analysis for the proposed method and prove that it is second order convergent in time. Numerical examples are given in Section 5 to illustrate our theoretical results and demonstrate the application of our ensemble algorithm incorporated with the Monte Carlo method and the sparse grid method respectively. Section 6 provides final conclusions.

2. Notation and Preliminaries

We denote the $L^2(I)$ norm by $\|\cdot\|_I$ and the $L^2(D_{f/p})$ norms by $\|\cdot\|_{f/p}$; the corresponding inner products are denoted by $(\cdot,\cdot)_{f/p}$. Further, we denote the $H^k(D_{f/p})$ norm by $\|\cdot\|_{H^k(D_{f/p})}$. The following inequalities will be used in the proofs, [44].

$$\|\phi\|_I \leq C(D_p)\sqrt{\|\phi\|_p\|\nabla\phi\|_p}, \qquad \|u\|_I \leq C(D_f)\sqrt{\|u\|_f\|\nabla u\|_f}, \tag{2.1}$$

where $C(D_{f/p}) = \mathcal{O}(\sqrt{L_{f/p}}), L_{f/p} = diameter(D_{f/p})$.

Define the function spaces:

$$\begin{aligned}
\text{Velocity} \quad &: \quad X_f := \{v \in \left(H^1(D_f)\right)^d : v = 0 \text{ on } \ \partial D_f \backslash I\},\\
\text{Pressure} \quad &: \quad Q_f \ := \left\{q \in L^2(D_f) : \int_\Omega q \ dx = 0\right\},\\
\text{Hydraulic Head} \quad &: \quad X_p := \{\psi \in H^1(D_p) : \psi = 0 \text{ on } \ \partial D_p \backslash I\}.
\end{aligned}$$

To discretize the Stokes-Darcy problem in space by the finite element method, we choose conforming velocity, pressure, hydraulic head finite element spaces based on an edge to edge triangulation ($d = 2$) or tetrahedralization ($d = 3$) of the domain $D_{f/p}$ with maximum element diameter h:

$$X_f^h \subset X_f \ , \ Q_f^h \subset Q_f, \ X_p^h \subset X_p.$$

The continuity across the interface I between the finite element meshes in the two subdomains is not assumed. The finite element spaces (X_f^h, Q_f^h) are assumed to satisfy the usual discrete inf-sup $/LBB^h$ condition for stability of the discrete pressure, see [24] for more on this condition. Taylor-Hood elements, [24], are one such choice used in the numerical tests in Section 5.

We will also consider the discretely divergence-free space:

$$V_f^h := \{v_h \in X_f^h : (q_h, \nabla\cdot v_h)_f = 0, \ \forall q_h \in Q_f^h\}.$$

Define

$$c_I(u,\phi) \ = \ g\int_I \phi u\cdot\widehat{n}_f \ ds.$$

Let $C_{P,f}$ and $C_{P,p}$ be the Poincaré constants of the indicated domains and $\bar{k}_{min}(x)$ be the minimum eigenvalue of the mean hydraulic conductivity tensor $\bar{\mathcal{K}}(x)$. Define $\bar{k}_{min} = \min_{x\in\Omega_p} \bar{k}_{min}(x)$ and two parameter-dependent constants

$$C_1 = \frac{C_{P,f}^2[gC(D_f)C(D_p)]^4}{4\nu^2}, \qquad C_2 = \frac{C_{P,p}^2 g^2[C(D_f)C(D_p)]^4}{4\bar{k}_{min}^2}.$$

We have the following estimates for the coupling term $c_I(u,\phi)$.

Lemma 1. *For any $(u,\phi) \in X_f \times X_p$ and any $\epsilon_1, \epsilon_2, \alpha_1, \beta_1 > 0$,*

$$|c_I(u,\phi)| \leq \frac{1}{4\epsilon_1}\|\phi\|_p^2 + \frac{\epsilon_1}{\alpha_1^2}C_1\|\nabla\phi\|_p^2 + \alpha_1\nu\|\nabla u\|_f^2, \tag{2.2}$$

$$|c_I(u,\phi)| \leq \frac{1}{4\epsilon_2}\|u\|_f^2 + \frac{\epsilon_2}{\beta_1^2}C_2\|\nabla u\|_f^2 + \beta_1 g\bar{k}_{min}\|\nabla\phi\|_p^2. \tag{2.3}$$

PROOF. See page 4 of [37].

The fully discrete approximation of (1.2) is:

Algorithm 2. *Find $(u_{j,h}^{n+1}, p_{j,h}^{n+1}, \phi_{j,h}^{n+1}) \in X_f^h \times Q_f^h \times X_p^h$ satisfying $\forall\ (v_h, q_h, \psi_h) \in X_f^h \times Q_f^h \times X_p^h$,*

$$\begin{cases} \left(\dfrac{3u_{j,h}^{n+1} - 4u_{j,h}^n + u_{j,h}^{n-1}}{2\Delta t}, v_h\right)_f + \nu(\nabla u_{j,h}^{n+1}, \nabla v_h)_f + \displaystyle\sum_i \int_I \bar{\eta}_i(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\ ds \\ + \displaystyle\sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)(2u_{j,h}^n - u_{j,h}^{n-1}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\ ds - \left(p_{j,h}^{n+1}, \nabla\cdot v_h\right)_f \\ + c_I(v_h, 2\phi_{j,h}^n - \phi_{j,h}^{n-1}) = (f_{f,j}^{n+1}, v_h)_f, \\ (q_h, \nabla\cdot u_{j,h}^{n+1})_f = 0, \end{cases} \tag{sub-problem 1}$$

$$\begin{cases} gS_0\left(\dfrac{3\phi_{j,h}^{n+1} - 4\phi_{j,h}^n + \phi_{j,h}^{n-1}}{2\Delta t}, \psi_h\right)_p + g(\bar{\mathcal{K}}\nabla\phi_{j,h}^{n+1}, \nabla\psi_h)_p \\ + g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(2\phi_{j,h}^n - \phi_{j,h}^{n-1}), \nabla\psi_h)_p - c_I(2u_{j,h}^n - u_{j,h}^{n-1}, \psi_h) = g(f_{p,j}^{n+1}, \psi_h)_p. \end{cases} \tag{sub-problem 2}$$

3. Stability Analysis

Let $|\cdot|_2$ denote the 2-norm of either vectors or matrices. Let $k_{j,min}(x)$, $\bar{k}_{min}(x)$ be the minimum eigenvalue of the hydraulic conductivity tensor $\mathcal{K}_j(x)$, $\bar{\mathcal{K}}(x)$ respectively, and $\rho_j'(x)$ be the spectral radius of the fluctuation of hydraulic conductivity tensor $\mathcal{K}_j(x) - \bar{\mathcal{K}}(x)$. Since both $\mathcal{K}_j(x)$ and $\bar{\mathcal{K}}(x)$ are symmetric, $|\mathcal{K}_j(x) - \bar{\mathcal{K}}(x)|_2 = \rho_j'(x)$. We then define the following quantities that will be used in our proof.

$$\eta_{i,j}'^{max} = \max_{x\in I}|\eta_{i,j}(x) - \bar{\eta}_i(x)|, \quad \eta_i'^{max} = \max_j \eta_{i,j}'^{max}, \quad \bar{\eta}_i^{min} = \min_{x\in I}\bar{\eta}_i(x), \quad k_{j,min} = \min_{x\in D_p} k_{j,min}(x),$$

$$k_{min} = \min_j k_{j,min}, \quad \bar{k}_{min} = \min_{x\in D_p}\bar{k}_{min}(x), \quad \rho_{j,max}' = \max_{x\in D_p}\rho_{j,max}'(x), \quad \rho_{max}' = \max_j \rho_{j,max}'.$$

We prove long time stability of Algorithm 2 under a time step condition and two parameter conditions

$$\Delta t \leq \min\left\{\frac{(1-\alpha_1-\alpha_2)\beta_1^2\bar{k}_{min}}{C_{P,p}^2}, \frac{(1-\beta_1-\beta_2-\frac{3\rho'_{max}}{\bar{k}_{min}})\alpha_1^2 S_0\nu}{C_{P,f}^2}\right\}\frac{\nu\bar{k}_{min}}{4g^2[C(D_f)C(D_p)]^4}, \tag{3.1}$$

$$\eta_i'^{max} \leq \frac{\bar{\eta}_i^{min}}{3}, \qquad \rho'_{max} < \frac{\bar{k}_{min}}{3}. \tag{3.2}$$

Theorem 1 (Long time stability of Algorithm 2). *If the two parameter conditions in* (3.2) *hold, and there exist* $\alpha_1, \alpha_2, \beta_1, \beta_2$ *in* $(0,1)$ *such that the time step condition* (3.1) *also holds, then Algorithm 2 is long time stable: for any* $N > 1$*,*

$$\begin{aligned}
&\frac{1}{2}\big(\|u_{j,h}^N\|_f^2 + \|2u_{j,h}^N - u_{j,h}^{N-1}\|_f^2\big) + \frac{gS_0}{2}\big(\|\phi_{j,h}^N\|_p^2 + \|2\phi_{j,h}^N - \phi_{j,h}^{N-1}\|_p^2\big) + \frac{24\Delta t^2 C_2}{\beta_1^2}\|\nabla u_{j,h}^N\|_f^2 \\
&+ \frac{8\Delta t^2 C_2}{\beta_1^2}\|\nabla u_{j,h}^{N-1}\|_f^2 + \Delta t\sum_i \bar{\eta}_i^{min}\int_I (u_{j,h}^N\cdot\widehat{\tau}_i)^2\, ds + \Delta t\sum_i \frac{\bar{\eta}_i^{min}}{3}\int_I (u_{j,h}^{N-1}\cdot\widehat{\tau}_i)^2\, ds \\
&+ \Delta t\left(\frac{24\Delta t C_1}{gS_0\alpha_1^2} + 3g\rho'_{max}\right)\|\nabla\phi_{j,h}^N\|_p^2 + \Delta t\left(\frac{8\Delta t C_1}{gS_0\alpha_1^2} + g\rho'_{max}\right)\|\nabla\phi_{j,h}^{N-1}\|_p^2 \\
&\leq \Delta t\sum_{n=1}^{N-1}\frac{C_{P,f}^2}{2\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2 + \Delta t\sum_{n=1}^{N-1}\frac{gC_{P,p}^2}{2\beta_2\bar{k}_{min}}\|f_{p,j}^{n+1}\|_p^2 + \frac{1}{2}\big(\|u_{j,h}^1\|_f^2 + \|2u_{j,h}^1 - u_{j,h}^0\|_f^2\big) \\
&+ \frac{gS_0}{2}\big(\|\phi_{j,h}^1\|_p^2 + \|2\phi_{j,h}^1 - \phi_{j,h}^0\|_p^2\big) + \frac{24\Delta t^2 C_2}{\beta_1^2}\|\nabla u_{j,h}^1\|_f^2 + \frac{8\Delta t^2 C_2}{\beta_1^2}\|\nabla u_{j,h}^0\|_f^2 \\
&+ \Delta t\sum_i \bar{\eta}_i^{min}\int_I (u_{j,h}^1\cdot\widehat{\tau}_i)^2\, ds + \Delta t\sum_i \frac{\bar{\eta}_i^{min}}{3}\int_I (u_{j,h}^0\cdot\widehat{\tau}_i)^2\, ds \\
&+ \Delta t\left(\frac{24\Delta t C_1}{gS_0\alpha_1^2} + 3g\rho'_{max}\right)\|\nabla\phi_{j,h}^1\|_p^2 + \Delta t\left(\frac{8\Delta t C_1}{gS_0\alpha_1^2} + g\rho'_{max}\right)\|\nabla\phi_{j,h}^0\|_p^2.
\end{aligned} \tag{3.3}$$

Proof. Setting $v_h = u_{j,h}^{n+1}$, $\psi_h = \phi_{j,h}^{n+1}$ in Algorithm 2 and adding all three equations yields

$$\begin{aligned}
&\frac{1}{4\Delta t}\left(\|u_{j,h}^{n+1}\|_f^2 + \|2u_{j,h}^{n+1} - u_{j,h}^n\|_f^2\right) - \frac{1}{4\Delta t}\left(\|u_{j,h}^n\|_f^2 + \|2u_{j,h}^n - u_{j,h}^{n-1}\|_f^2\right) + \frac{1}{4\Delta t}\|u_{j,h}^{n+1} - 2u_{j,h}^n \\
&+ u_{j,h}^{n-1}\|_f^2 + \nu\|\nabla u_{j,h}^{n+1}\|_f^2 + \sum_i\int_I \bar{\eta}_i(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\, ds + \frac{gS_0}{4\Delta t}\left(\|\phi_{j,h}^{n+1}\|_p^2 + \|2\phi_{j,h}^{n+1} - \phi_{j,h}^n\|_p^2\right) \\
&- \frac{gS_0}{4\Delta t}\left(\|\phi_{j,h}^n\|_p^2 + \|2\phi_{j,h}^n - \phi_{j,h}^{n-1}\|_p^2\right) + \frac{gS_0}{4\Delta t}\|\phi_{j,h}^{n+1} - 2\phi_{j,h}^n + \phi_{j,h}^{n-1}\|_p^2 + g(\bar{\mathcal{K}}\nabla\phi_{j,h}^{n+1}, \nabla\phi_{j,h}^{n+1})_p \\
&+ c_I(u_{j,h}^{n+1}, 2\phi_{j,h}^n - \phi_{j,h}^{n-1}) - c_I(2u_{j,h}^n - u_{j,h}^{n-1}, \phi_{j,h}^{n+1}) \\
&= (f_{f,j}^{n+1}, u_{j,h}^{n+1})_f + g(f_{p,j}^{n+1}, \phi_{j,h}^{n+1})_p - \sum_i\int_I (\eta_{i,j} - \bar{\eta}_i)((2u_{j,h}^n - u_{j,h}^{n-1})\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\, ds \\
&\quad - g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(2\phi_{j,h}^n - \phi_{j,h}^{n-1}), \nabla\phi_{j,h}^{n+1})_p.
\end{aligned} \tag{3.4}$$

Note that

$$
\begin{aligned}
&c_I(u_{j,h}^{n+1}, 2\phi_{j,h}^n - \phi_{j,h}^{n-1}) - c_I(2u_{j,h}^n - u_{j,h}^{n-1}, \phi_{j,h}^{n+1}) \\
&= \left[c_I(u_{j,h}^{n+1}, 2\phi_{j,h}^n - \phi_{j,h}^{n-1}) - c_I(u_{j,h}^{n+1}, \phi_{j,h}^{n+1})\right] + \left[c_I(u_{j,h}^{n+1}, \phi_{j,h}^{n+1}) - c_I(2u_{j,h}^n - u_{j,h}^{n-1}, \phi_{j,h}^{n+1})\right] \\
&= -c_I(u_{j,h}^{n+1}, \phi_{j,h}^{n+1} - 2\phi_{j,h}^n + \phi_{j,h}^{n-1}) + c_I(u_{j,h}^{n+1} - 2u_{j,h}^n + u_{j,h}^{n-1}, \phi_{j,h}^{n+1}). \qquad (3.5)
\end{aligned}
$$

Applying estimates (2.2) and (2.3) with $\epsilon_1 = \frac{\Delta t}{gS_0}$, $\epsilon_2 = \Delta t$, and using the inequality $(a + 2b + c)^2 \leq 4a^2 + 8b^2 + 4c^2$ we have

$$
\begin{aligned}
&c_I(u_{j,h}^{n+1} - 2u_{j,h}^n + u_{j,h}^{n-1}, \phi_{j,h}^{n+1}) - c_I(u_{j,h}^{n+1}, \phi_{j,h}^{n+1} - 2\phi_{j,h}^n + \phi_{j,h}^{n-1}) \\
&\geq -\frac{1}{4\Delta t}\|u_{j,h}^{n+1} - 2u_{j,h}^n + u_{j,h}^{n-1}\|_f^2 - \frac{\Delta t C_2}{\beta_1^2}\|\nabla(u_{j,h}^{n+1} - 2u_{j,h}^n + u_{j,h}^{n-1})\|_f^2 - \beta_1 g\bar{k}_{min}\|\nabla\phi_{j,h}^{n+1}\|_p^2 \\
&\quad - \frac{gS_0}{4\Delta t}\|\phi_{j,h}^{n+1} - 2\phi_{j,h}^n + \phi_{j,h}^{n-1}\|_p^2 - \frac{\Delta t}{gS_0}\frac{C_1}{\alpha_1^2}\|\nabla(\phi_{j,h}^{n+1} - 2\phi_{j,h}^n + \phi_{j,h}^{n-1})\|_p^2 - \alpha_1\nu\|\nabla u_{j,h}^{n+1}\|_f^2 \\
&\geq -\frac{1}{4\Delta t}\|u_{j,h}^{n+1} - 2u_{j,h}^n + u_{j,h}^{n-1}\|_f^2 - \frac{\Delta t C_2}{\beta_1^2}\left(4\|\nabla u_{j,h}^{n+1}\|_f^2 + 8\|\nabla u_{j,h}^n\|_f^2 + 4\|\nabla u_{j,h}^{n-1}\|_f^2\right) \\
&\quad - \beta_1 g\bar{k}_{min}\|\nabla\phi_{j,h}^{n+1}\|_p^2 - \frac{gS_0}{4\Delta t}\|\phi_{j,h}^{n+1} - 2\phi_{j,h}^n + \phi_{j,h}^{n-1}\|_p^2 - \frac{\Delta t}{gS_0}\frac{C_1}{\alpha_1^2}\big(4\|\nabla\phi_{j,h}^{n+1}\|_p^2 + 8\|\nabla\phi_{j,h}^n\|_p^2 \\
&+ 4\|\nabla\phi_{j,h}^{n-1}\|_p^2\big) - \alpha_1\nu\|\nabla u_{j,h}^{n+1}\|_f^2. \qquad (3.6)
\end{aligned}
$$

Applying Cauchy-Schwarz and Young's inequalities to the source terms, for any $\alpha_2 > 0, \beta_2 > 0$ we have

$$
\begin{aligned}
&(f_{f,j}^{n+1}, u_{j,h}^{n+1})_f + g(f_{p,j}^{n+1}, \phi_{j,h}^{n+1})_p \\
&\qquad \leq \|f_{f,j}^{n+1}\|_f\|u_{j,h}^{n+1}\|_f + g\|f_{p,j}^{n+1}\|_p\|\phi_{j,h}^{n+1}\|_p \\
&\qquad \leq C_{P,f}\|f_{f,j}^{n+1}\|_f\|\nabla u_{j,h}^{n+1}\|_f + gC_{P,p}\|f_{p,j}^{n+1}\|_p\|\nabla\phi_{j,h}^{n+1}\|_p \\
&\qquad \leq \frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2 + \alpha_2\nu\|\nabla u_{j,h}^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{4\beta_2\bar{k}_{min}}\|f_{p,j}^{n+1}\|_p^2 + \beta_2 g\bar{k}_{min}\|\nabla\phi_{j,h}^{n+1}\|_p^2. \qquad (3.7)
\end{aligned}
$$

The other two terms on the right hand side of (3.4) can be bounded as follows. Using the inequality $(2a-b)^2 \leq 6a^2 + 3b^2$, for any $\epsilon > 0$,

$$
\begin{aligned}
&-\sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)((2u_{j,h}^n - u_{j,h}^{n-1}) \cdot \widehat{\tau}_i)(u_{j,h}^{n+1} \cdot \widehat{\tau}_i)\ ds \\
&\leq \sum_i \int_I |\eta_{i,j} - \bar{\eta}_i|\left|((2u_{j,h}^n - u_{j,h}^{n-1}) \cdot \widehat{\tau}_i)(u_{j,h}^{n+1} \cdot \widehat{\tau}_i)\right| ds \\
&\leq \sum_i \eta_{i,j}'^{max} \int_I \left|((2u_{j,h}^n - u_{j,h}^{n-1}) \cdot \widehat{\tau}_i)(u_{j,h}^{n+1} \cdot \widehat{\tau}_i)\right|\ ds \\
&\leq \sum_i \left[\frac{\eta_i'^{max}}{2\epsilon}\int_I ((2u_{j,h}^n - u_{j,h}^{n-1}) \cdot \widehat{\tau}_i)^2\ ds + \frac{\epsilon\eta_i'^{max}}{2}\int_I (u_{j,h}^{n+1} \cdot \widehat{\tau}_i)^2\ ds\right] \\
&\leq \sum_i \left[\frac{3}{\epsilon}\eta_i'^{max}\int_I (u_{j,h}^n \cdot \widehat{\tau}_i)^2\ ds + \frac{3\eta_i'^{max}}{2\epsilon}\int_I (u_{j,h}^{n-1} \cdot \widehat{\tau}_i)^2\ ds + \frac{\epsilon\eta_i'^{max}}{2}\int_I (u_{j,h}^{n+1} \cdot \widehat{\tau}_i)^2\ ds\right], \qquad (3.8)
\end{aligned}
$$

and

$$
\begin{aligned}
&- g\left((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(2\phi_{j,h}^n - \phi_{j,h}^{n-1}), \nabla\phi_{j,h}^{n+1}\right)_p \\
&\leq g\int_{D_p} |\nabla\phi_{j,h}^{n+1}|_2 |\mathcal{K}_j - \bar{\mathcal{K}}|_2 |\nabla(2\phi_{j,h}^n - \phi_{j,h}^{n-1})|_2 \, dx \\
&\leq g\int_{D_p} \rho_j'(x) |\nabla\phi_{j,h}^{n+1}|_2 |\nabla(2\phi_{j,h}^n - \phi_{j,h}^{n-1})|_2 \, dx \\
&\leq g\rho_{j,max}' \int_{D_p} |\nabla\phi_{j,h}^{n+1}|_2 |\nabla(2\phi_{j,h}^n - \phi_{j,h}^{n-1})|_2 \, dx \\
&\leq g\rho_{j,max}' \|\nabla(2\phi_{j,h}^n - \phi_{j,h}^{n-1})\|_p \|\nabla\phi_{j,h}^{n+1}\|_p \\
&\leq \frac{3}{\epsilon} g\rho_{max}' \|\nabla\phi_{j,h}^n\|_p^2 + \frac{3g\rho_{max}'}{2\epsilon}\|\nabla\phi_{j,h}^{n-1}\|_p^2 + \frac{\epsilon g\rho_{max}'}{2}\|\nabla\phi_{j,h}^{n+1}\|_p^2. \qquad (3.9)
\end{aligned}
$$

Since all terms in (3.8) need to be bounded by $\sum_i \bar{\eta}_i^{min} \int_I (u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2 \, ds$, we need to minimize $(\frac{3}{\epsilon} + \frac{3}{2\epsilon} + \frac{\epsilon}{2})$ to make the time step condition sharp. This term achieves its minimum 3 when $\epsilon = 3$. Similarly, we need to take $\epsilon = 3$ in (3.9). Then (3.8) and (3.9) become

$$
\begin{aligned}
&-\sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)((2u_{j,h}^n - u_{j,h}^{n-1})\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i) \, ds \qquad (3.10) \\
&\qquad \leq \sum_i \left[\eta_i'^{max} \int_I (u_{j,h}^n\cdot\widehat{\tau}_i)^2 \, ds + \frac{\eta_i'^{max}}{2}\int_I (u_{j,h}^{n-1}\cdot\widehat{\tau}_i)^2 \, ds + \frac{3\eta_i'^{max}}{2}\int_I (u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2 \, ds\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&- g\left((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(2\phi_{j,h}^n - \phi_{j,h}^{n-1}), \nabla\phi_{j,h}^{n+1}\right)_p \qquad (3.11) \\
&\qquad \leq g\rho_{max}'\|\nabla\phi_{j,h}^n\|_p^2 + \frac{g\rho_{max}'}{2}\|\nabla\phi_{j,h}^{n-1}\|_p^2 + \frac{3g\rho_{max}'}{2}\|\nabla\phi_{j,h}^{n+1}\|_p^2.
\end{aligned}
$$

Using above estimates, equation (3.4) becomes

$$
\begin{aligned}
&\frac{1}{4\Delta t}\left(\|u_{j,h}^{n+1}\|_f^2 + \|2u_{j,h}^{n+1} - u_{j,h}^n\|_f^2\right) - \frac{1}{4\Delta t}\left(\|u_{j,h}^n\|_f^2 + \|2u_{j,h}^n - u_{j,h}^{n-1}\|_f^2\right) \qquad (3.12) \\
&+ (1 - \alpha_1 - \alpha_2 - \frac{16\Delta t C_2}{\beta_1^2\nu})\nu\|\nabla u_{j,h}^{n+1}\|_f^2 + \frac{12\Delta t C_2}{\beta_1^2}\left(\|\nabla u_{j,h}^{n+1}\|_f^2 - \|\nabla u_{j,h}^n\|_f^2\right) \\
&+ \frac{4\Delta t C_2}{\beta_1^2}\left(\|\nabla u_{j,h}^n\|_f^2 - \|\nabla u_{j,h}^{n-1}\|_f^2\right) + \sum_i \left[\frac{\bar{\eta}_i^{min}}{2} - \frac{3\eta_i'^{max}}{2}\right]\int_I (u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2 \, ds \\
&+ \sum_i \frac{\bar{\eta}_i^{min}}{2}\left[\int_I (u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2 \, ds - \int_I (u_{j,h}^n\cdot\widehat{\tau}_i)^2 \, ds\right] + \sum_i \left[\frac{\bar{\eta}_i^{min}}{3} - \eta_i'^{max}\right]\int_I (u_{j,h}^n\cdot\widehat{\tau}_i)^2 \, ds \\
&+ \sum_i \frac{\bar{\eta}_i^{min}}{6}\left[\int_I (u_{j,h}^n\cdot\widehat{\tau}_i)^2 \, ds - \int_I (u_{j,h}^{n-1}\cdot\widehat{\tau}_i)^2 \, ds\right] + \sum_i \left[\frac{\bar{\eta}_i^{min}}{6} - \frac{\eta_i'^{max}}{2}\right]\int_I (u_{j,h}^{n-1}\cdot\widehat{\tau}_i)^2 \, ds \\
&+ \frac{gS_0}{4\Delta t}\left(\|\phi_{j,h}^{n+1}\|_p^2 + \|2\phi_{j,h}^{n+1} - \phi_{j,h}^n\|_p^2\right) - \frac{gS_0}{4\Delta t}\left(\|\phi_{j,h}^n\|_p^2 + \|2\phi_{j,h}^n - \phi_{j,h}^{n-1}\|_p^2\right) \\
&+ \left(1 - \beta_1 - \beta_2 - \frac{16\Delta t C_1}{g^2 S_0 \alpha_1^2 \bar{k}_{min}} - \frac{3\rho_{max}'}{\bar{k}_{min}}\right) g\bar{k}_{min}\|\nabla\phi_{j,h}^{n+1}\|_p^2
\end{aligned}
$$

$$+\Big(\frac{12\Delta t C_1}{gS_0\alpha_1^2}+\frac{3g\rho'_{max}}{2}\Big)\big(\|\nabla\phi_{j,h}^{n+1}\|_p^2-\|\nabla\phi_{j,h}^{n}\|_p^2\big)+\Big(\frac{4\Delta t C_1}{gS_0\alpha_1^2}+\frac{g\rho'_{max}}{2}\Big)\big(\|\nabla\phi_{j,h}^{n}\|_p^2-\|\nabla\phi_{j,h}^{n-1}\|_p^2\big)$$
$$\le \frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2+\frac{gC_{P,p}^2}{4\beta_2\bar{k}_{min}}\|f_{p,j}^{n+1}\|_p^2.$$

To obtain stability, we need

$$1-\alpha_1-\alpha_2-\frac{16\Delta t C_2}{\beta_1^2\nu}\ge 0,\ \ \frac{\bar{\eta}_i^{min}}{3}-\eta_i'^{max}\ge 0,\ \ \text{and}\ \ 1-\beta_1-\beta_2-\frac{16\Delta t C_1}{g^2S_0\alpha_1^2\bar{k}_{min}}-\frac{3\rho'_{max}}{\bar{k}_{min}}\ge 0. \tag{3.13}$$

Recall that $\alpha_1,\alpha_2,\beta_1,\beta_2,\Delta t,\eta_i'^{max},\rho'_{max}$ are all positive, we then have the following constraints on these parameters.

$$0<\alpha_1<1,\quad 0<\alpha_2<1,\qquad 0<\beta_1<1,\quad 0<\beta_2<1, \tag{3.14}$$
$$\frac{\rho'_{max}}{\bar{k}_{min}}<\frac{1}{3},\qquad \frac{\eta_i'^{max}}{\bar{\eta}_i^{min}}\le\frac{1}{3}, \tag{3.15}$$
$$\Delta t\le \min\left\{\frac{(1-\alpha_1-\alpha_2)\beta_1^2\nu}{16C_2},\frac{(1-\beta_1-\beta_2-\frac{3\rho'_{max}}{\bar{k}_{min}})\alpha_1^2g^2S_0\bar{k}_{min}}{16C_1}\right\}. \tag{3.16}$$

(3.15) leads to the two parameter conditions in (3.2), and (3.16) leads to the time step condition (3.1) required for stability. Now if the time-step condition (3.1) and the two parameter conditions in (3.2) hold, (3.12) reduces to

$$\frac{1}{4\Delta t}\big(\|u_{j,h}^{n+1}\|_f^2+\|2u_{j,h}^{n+1}-u_{j,h}^{n}\|_f^2\big)-\frac{1}{4\Delta t}\big(\|u_{j,h}^{n}\|_f^2+\|2u_{j,h}^{n}-u_{j,h}^{n-1}\|_f^2\big)$$
$$+\frac{12\Delta t C_2}{\beta_1^2}\big(\|\nabla u_{j,h}^{n+1}\|_f^2-\|\nabla u_{j,h}^{n}\|_f^2\big)+\frac{4\Delta t C_2}{\beta_1^2}\big(\|\nabla u_{j,h}^{n}\|_f^2-\|\nabla u_{j,h}^{n-1}\|_f^2\big)$$
$$+\sum_i\frac{\bar{\eta}_i^{min}}{2}\left[\int_I(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2\,ds-\int_I(u_{j,h}^{n}\cdot\widehat{\tau}_i)^2\,ds\right]+\sum_i\frac{\bar{\eta}_i^{min}}{6}\left[\int_I(u_{j,h}^{n}\cdot\widehat{\tau}_i)^2\,ds-\int_I(u_{j,h}^{n-1}\cdot\widehat{\tau}_i)^2\,ds\right]$$
$$+\frac{gS_0}{4\Delta t}\big(\|\phi_{j,h}^{n+1}\|_p^2+\|2\phi_{j,h}^{n+1}-\phi_{j,h}^{n}\|_p^2\big)-\frac{gS_0}{4\Delta t}\big(\|\phi_{j,h}^{n}\|_p^2+\|2\phi_{j,h}^{n}-\phi_{j,h}^{n-1}\|_p^2\big)$$
$$+\Big(\frac{12\Delta t C_1}{gS_0\alpha_1^2}+\frac{3g\rho'_{max}}{2}\Big)\big(\|\nabla\phi_{j,h}^{n+1}\|_p^2-\|\nabla\phi_{j,h}^{n}\|_p^2\big)+\Big(\frac{4\Delta t C_1}{gS_0\alpha_1^2}+\frac{g\rho'_{max}}{2}\Big)\big(\|\nabla\phi_{j,h}^{n}\|_p^2-\|\nabla\phi_{j,h}^{n-1}\|_p^2\big)$$
$$\le \frac{C_{P,f}^2}{4\alpha_2\nu}\|f_{f,j}^{n+1}\|_f^2+\frac{gC_{P,p}^2}{4\beta_2\bar{k}_{min}}\|f_{p,j}^{n+1}\|_p^2. \tag{3.17}$$

Summing up (3.17) from $n=1$ to $N-1$ and multiply through by $2\Delta t$ to get

$$\frac{1}{2}\big(\|u_{j,h}^{N}\|_f^2+\|2u_{j,h}^{N}-u_{j,h}^{N-1}\|_f^2\big)+\frac{gS_0}{2}\big(\|\phi_{j,h}^{N}\|_p^2+\|2\phi_{j,h}^{N}-\phi_{j,h}^{N-1}\|_p^2\big)+\frac{24\Delta t^2C_2}{\beta_1^2}\|\nabla u_{j,h}^{N}\|_f^2$$
$$+\frac{8\Delta t^2C_2}{\beta_1^2}\|\nabla u_{j,h}^{N-1}\|_f^2+\Delta t\sum_i\bar{\eta}_i^{min}\int_I(u_{j,h}^{N}\cdot\widehat{\tau}_i)^2\,ds+\Delta t\sum_i\frac{\bar{\eta}_i^{min}}{3}\int_I(u_{j,h}^{N-1}\cdot\widehat{\tau}_i)^2\,ds$$
$$+\Delta t\left(\frac{24\Delta t C_1}{gS_0\alpha_1^2}+3g\rho'_{max}\right)\|\nabla\phi_{j,h}^{N}\|_p^2+\Delta t\left(\frac{8\Delta t C_1}{gS_0\alpha_1^2}+g\rho'_{max}\right)\|\nabla\phi_{j,h}^{N-1}\|_p^2$$

$$
\begin{aligned}
&\leq \Delta t \sum_{n=1}^{N-1} \frac{C_{P,f}^2}{2\alpha_2 \nu} \|f_{f,j}^{n+1}\|_f^2 + \Delta t \sum_{n=1}^{N-1} \frac{g C_{P,p}^2}{2\beta_2 \bar{k}_{min}} \|f_{p,j}^{n+1}\|_p^2 + \frac{1}{2}\big(\|u_{j,h}^1\|_f^2 + \|2u_{j,h}^1 - u_{j,h}^0\|_f^2\big) \\
&+ \frac{gS_0}{2}\big(\|\phi_{j,h}^1\|_p^2 + \|2\phi_{j,h}^1 - \phi_{j,h}^0\|_p^2\big) + \frac{24\Delta t^2 C_2}{\beta_1^2}\|\nabla u_{j,h}^1\|_f^2 + \frac{8\Delta t^2 C_2}{\beta_1^2}\|\nabla u_{j,h}^0\|_f^2 \\
&+ \Delta t \sum_i \bar{\eta}_i^{min} \int_I (u_{j,h}^1 \cdot \widehat{\tau}_i)^2 \, ds + \Delta t \sum_i \frac{\bar{\eta}_i^{min}}{3} \int_I (u_{j,h}^0 \cdot \widehat{\tau}_i)^2 \, ds \\
&+ \Delta t \left(\frac{24\Delta t C_1}{g S_0 \alpha_1^2} + 3g\rho'_{max} \right) \|\nabla \phi_{j,h}^1\|_p^2 + \Delta t \left(\frac{8\Delta t C_1}{g S_0 \alpha_1^2} + g\rho'_{max} \right) \|\nabla \phi_{j,h}^0\|_p^2. \qquad (3.18)
\end{aligned}
$$

4. Error Analysis

In this section, we analyze the error of Algorithm 2. We assume the following regularity on the true solution of the Stokes-Darcy equations.

$$
\begin{aligned}
&u_j \in L^\infty(0,T;H^{k+1}(D_f)), u_{j,t} \in L^2(0,T;H^{k+1}(D_f)), u_{j,tt} \in L^2(0,T;L^2(D_f)), \\
&\phi_j \in L^\infty(0,T;H^{m+1}(D_p)), \phi_{j,t} \in L^2(0,T;H^{m+1}(D_p)), \phi_{j,tt} \in L^2(0,T;L^2(D_p)), \\
&p_j \in L^2(0,T;H^{s+1}(D_f)).
\end{aligned}
$$

For functions $v(x,t)$ defined on $(0,T)$, we define the continuous norm

$$
\|v\|_{m,k,r} := \|v\|_{L^m(0,T;H^k(D_r))}, \; r \in \{f,p\}.
$$

Given a time step Δt, let $t_n = n\Delta t, T = N\Delta t$, $v^n = v(x,t_n)$ and define the discrete norms

$$
|||v|||_{\infty,k,r} = \max_{0\leq n\leq N} \|v^n\|_{H^k(D_r)} \quad \text{and} \quad |||v|||_{m,k,r} := \left(\sum_{n=0}^{N} \|v^n\|_{H^k(D_r)}^m \Delta t \right)^{1/m}, \; r \in \{f,p\}.
$$

We assume the finite element spaces satisfy the approximation properties of piecewise polynomials on quasiuniform meshes

$$
\begin{aligned}
\inf_{v_h \in X_f^h} \|v - v_h\|_f &\leq Ch^{k+1}\|u\|_{H^{k+1}(D_f)} && \forall v \in [H^{k+1}(D_f)]^d, && (4.1)\\
\inf_{v_h \in X_f^h} \|\nabla(v - v_h)\|_f &\leq Ch^{k}\|v\|_{H^{k+1}(D_f)} && \forall v \in [H^{k+1}(D_f)]^d, && (4.2)\\
\inf_{q_h \in Q_f^h} \|q - q_h\|_f &\leq Ch^{s+1}\|q\|_{H^{s+1}(D_f)} && \forall q \in H^{s+1}(D_f), && (4.3)\\
\inf_{\psi_h \in X_p^h} \|\psi - \psi_h\|_p &\leq Ch^{m+1}\|\psi\|_{H^{m+1}(D_p)} && \forall \psi \in H^{m+1}(D_p), && (4.4)\\
\inf_{\psi_h \in X_p^h} \|\nabla(\psi - \psi_h)\|_p &\leq Ch^{m}\|\psi\|_{H^{m+1}(D_p)} && \forall \psi \in H^{m+1}(D_p), && (4.5)
\end{aligned}
$$

where the generic constant $C > 0$ is independent of the mesh size h. An example for which both the LBB^h stability condition and the approximation properties are satisfied is the finite elements (P_{l+1}–P_l–P_{l+1}), $l \geq 1$, [24, 25, 43]. Define $e_{j,u}^n := u_j^n - u_{j,h}^n, e_{j,\phi}^n := \phi_j^n - \phi_{j,h}^n$, where $u_j^n = u_j(x,t_n), p_j^n = p_j(x,t_n), \phi_j^n = \phi_j(x,t_n)$. We prove the convergence of Algorithm 2 under a time step condition and two parameter conditions:

$$\Delta t \le \min\left\{ \frac{(1-\alpha_1-\alpha_2)\beta_1^2 \bar{k}_{min}}{C_{P,p}^2}, \frac{(1-\beta_1-\beta_2-(3+\sigma)\frac{\rho'_{max}}{\bar{k}_{min}})\alpha_1^2 S_0 \nu}{C_{P,f}^2} \right\} \frac{\nu \bar{k}_{min}}{4g^2[C(D_f)C(D_p)]^4}, \tag{4.6}$$

$$\eta_i'^{max} \le \frac{\bar{\eta}_i^{min}}{3}, \qquad \rho'_{max} < \frac{\bar{k}_{min}}{3}. \tag{4.7}$$

Theorem 2 (Error Estimate). *For any $j = 1, \ldots, J$, if the two parameter conditions in (4.7) hold, and there exist $\alpha_1, \alpha_2, \beta_1, \beta_2$ in $(0,1)$, and $\sigma > 0$ such that the time-step condition (4.6) also holds, then there is a positive constant C independent of the time step Δt and mesh size h such that*

$$\begin{aligned}
&\frac{1}{2}\|e_{j,u}^N\|_f^2 + \frac{24\Delta t^2 C_2}{\beta_1^2}\|\nabla e_{j,u}^N\|_f^2 + \frac{gS_0}{2}\|e_{j,\phi}^N\|_p^2 + \left(3\Delta t g\rho'_{max} + \frac{24\Delta t^2 C_1}{gS_0\alpha_1^2}\right)\|\nabla e_{j,\phi}^N\|_p^2 \\
&\le \frac{1}{2}(\|e_{j,u}^1\|_f^2 + \|2e_{j,u}^1 - e_{j,u}^0\|_f^2) + \frac{24\Delta t^2 C_2}{\beta_1^2}\|\nabla e_{j,u}^1\|_f^2 + \frac{8\Delta t^2 C_2}{\beta_1^2}\|\nabla e_{j,u}^0\|_f^2 \\
&+ \sum_i 3\Delta t \eta_i'^{max}\int_I (e_{j,u}^1\cdot\widehat{\tau}_i)^2\, ds + \sum_i \Delta t \eta_i'^{max}\int_I (e_{j,u}^0\cdot\widehat{\tau}_i)^2\, ds + \frac{gS_0}{2}(\|e_{j,\phi}^1\|_p^2 + \|2e_{j,\phi}^1 - e_{j,\phi}^0\|_p^2) \\
&+ \left(3\Delta t g\rho'_{max} + \frac{24\Delta t^2 C_1}{gS_0\alpha_1^2}\right)\|\nabla e_{j,\phi}^1\|_p^2 + \left(\Delta t g\rho'_{max} + \frac{8\Delta t^2 C_1}{gS_0\alpha_1^2}\right)\|\nabla e_{j,\phi}^0\|_p^2 + Ch^{2k+2}\|u_{j,t}\|_{2,k+1,f}^2 \\
&+ Ch^{2m+2}\|\phi_{j,t}\|_{2,m+1,p}^2 + Ch^{2s+2}|||p_j|||_{2,m+1,f}^2 + C\Delta t^4\|u_{j,ttt}\|_{2,0,f}^2 + C\Delta t^4\|u_{j,tt}\|_{2,1,f}^2 \\
&+ C\Delta t^4\|\phi_{j,ttt}\|_{2,0,p}^2 + C\Delta t^4\|\phi_{j,tt}\|_{2,1,p}^2 + Ch^{2k}|||u_j|||_{2,k+1,f}^2 + Ch^{2m}|||\phi_j|||_{2,m+1,p}^2.
\end{aligned} \tag{4.8}$$

Corollary 1. *Assume that $\|e_{j,u}^1\|$, $\|e_{j,u}^0\|$, $\|\nabla e_{j,u}^1\|$, $\|\nabla e_{j,u}^0\|$, $\|e_{j,\phi}^1\|$, $\|e_{j,\phi}^0\|$, $\|\nabla e_{j,\phi}^1\|$ and $\|\nabla e_{j,\phi}^0\|$ are all $O(h^2)$ accurate or better. Choosing (P_2, P_1, P_2) elements for (X_f^h, Q_f^h, X_p^h), we have*

$$\frac{1}{2}\|e_{j,u}^N\|_f^2 + \frac{24\Delta t^2 C_2}{\beta_1^2}\|\nabla e_{j,u}^N\|_f^2 + \frac{gS_0}{2}\|e_{j,\phi}^N\|_p^2 + \left(3\Delta t g\rho'_{max} + \frac{24\Delta t^2 C_1}{gS_0\alpha_1^2}\right)\|\nabla e_{j,\phi}^N\|_p^2 \le C(h^4 + \Delta t^4)\ .$$

Proof. (of Theorem 2) For $\forall v_h \in V_f^h, \forall \psi_h \in X_p^h, \forall \lambda_h^{n+1} \in Q_f^h$, the true solution (u_j, p_j, ϕ_j) satisfies

$$\begin{aligned}
&\left(\frac{3u_j^{n+1} - 4u_j^n + u_j^{n-1}}{2\Delta t}, v_h\right)_f + \nu(\nabla u_j^{n+1}, \nabla v_h)_f + \sum_i \int_I \eta_{i,j}(u_j^{n+1}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\, ds \\
&- \left(p_j^{n+1} - \lambda_h^{n+1}, \nabla\cdot v_h\right)_f + c_I(v_h, 2\phi_j^n - \phi_j^{n-1}) = (f_{f,j}^{n+1}, v_h)_f + \epsilon_{j,f}^{n+1}(v_h), \\
&gS_0\left(\frac{3\phi_j^{n+1} - 4\phi_j^n + \phi_j^{n-1}}{\Delta t}, \psi_h\right)_p + g(\mathcal{K}_j\nabla\phi_j^{n+1}, \nabla\psi_h)_p - c_I(2u_j^n - u_j^{n-1}, \psi_h) \\
&= g(f_{p,j}^{n+1}, \psi_h)_p + \epsilon_{j,p}^{n+1}(\psi_h).
\end{aligned} \tag{4.9}$$

The consistency errors $\epsilon_{j,f}^{n+1}(v_h), \epsilon_{j,p}^{n+1}(\psi_h)$ are defined by

$$\epsilon_{j,f}^{n+1}(v_h) := \left(\frac{3u_j^{n+1} - 4u_j^n + u_j^{n-1}}{2\Delta t} - u_{j,t}^{n+1}, v_h\right)_f - c_I(v_h, \phi_j^{n+1} - (2\phi_j^n - \phi_j^{n-1})),$$

$$\epsilon_{j,p}^{n+1}(\psi_h) := gS_0\left(\frac{3\phi_j^{n+1}-4\phi_j^n+\phi_j^{n-1}}{2\Delta t}-\phi_{j,t}^{n+1},\psi_h\right)_p + c_I(u_j^{n+1}-(2u_j^n-u_j^{n-1}),\psi_h).$$

Subtracting Algorithm 2.2 from (4.9) gives, for $\forall v_h \in V_f^h, \forall \psi_h \in X_p^h, \forall \lambda_h^{n+1} \in Q_f^h$,

$$\begin{aligned}
&\left(\frac{3e_{j,u}^{n+1}-4e_{j,u}^n+e_{j,u}^{n-1}}{2\Delta t},v_h\right)_f + \nu(\nabla e_{j,u}^{n+1},\nabla v_h)_f + \sum_i \int_I \bar{\eta}_i(e_{j,u}^{n+1}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\,ds\\
&+\sum_i\int_I(\eta_{i,j}-\bar{\eta}_i)\left[((2e_{j,u}^n-e_{j,u}^{n-1})\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\right]ds - \left(p_j^{n+1}-\lambda_h^{n+1},\nabla\cdot v_h\right)_f + c_I(v_h,2e_{j,\phi}^n-e_{j,\phi}^{n-1})\\
&=-\sum_i\int_I(\eta_{i,j}-\bar{\eta}_i)\left([u_j^{n+1}-(2u_j^n-u_j^{n-1})]\cdot\widehat{\tau}_i\right)(v_h\cdot\widehat{\tau}_i)\,ds + \epsilon_{j,f}^{n+1}(v_h),\\
&gS_0\left(\frac{3e_{j,\phi}^{n+1}-4e_{j,\phi}^n+e_{j,\phi}^{n-1}}{2\Delta t},\psi_h\right)_p + g(\bar{\mathcal{K}}\nabla e_{j,\phi}^{n+1},\nabla\psi_h)_p + g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla(2e_{j,\phi}^n-e_{j,\phi}^{n-1}),\nabla\psi_h)_p\\
&-c_I(2e_{j,u}^n-e_{j,u}^{n-1},\psi_h) = -g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla(\phi_j^{n+1}-(2\phi_j^n-\phi_j^{n-1})),\nabla\psi_h)_p + \epsilon_{j,p}^{n+1}(\psi_h).
\end{aligned}\tag{4.10}$$

We define

$$\begin{aligned}
e_{j,u}^{n+1} &= u_j^{n+1}-u_{j,h}^{n+1} = \left(u_j^{n+1}-U_j^{n+1}\right)+\left(U_j^{n+1}-u_{j,h}^{n+1}\right) =: \mu_{j,u}^{n+1}+\xi_{j,u}^{n+1},\\
e_{j,\phi}^{n+1} &= \phi_j^{n+1}-\phi_{j,h}^{n+1} = \left(\phi_j^{n+1}-\Phi_j^{n+1}\right)+\left(\Phi_j^{n+1}-\phi_{j,h}^{n+1}\right) =: \mu_{j,\phi}^{n+1}+\xi_{j,\phi}^{n+1},
\end{aligned}$$

where U_j^{n+1}, Φ_j^{n+1} be an interpolation of u_j^{n+1} and ϕ_j^{n+1} in V_f^h and X_p^h correspondingly.

Then (4.10) can be rewritten as

$$\begin{aligned}
&\left(\frac{3\xi_{j,u}^{n+1}-4\xi_{j,u}^n+\xi_{j,u}^{n-1}}{2\Delta t},v_h\right)_f + \nu(\nabla\xi_{j,u}^{n+1},\nabla v_h)_f + \sum_i\int_I\bar{\eta}_i(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\,ds\\
&+\sum_i\int_I(\eta_{i,j}-\bar{\eta}_i)\left[(2\xi_{j,u}^n-\xi_{j,u}^{n-1})\cdot\widehat{\tau}_i\right](v_h\cdot\widehat{\tau}_i)\,ds - \left(p_j^{n+1}-\lambda_h^{n+1},\nabla\cdot v_h\right)_f + c_I(v_h,2\xi_{j,\phi}^n-\xi_{j,\phi}^{n-1})\\
&=-\sum_i\int_I(\eta_{i,j}-\bar{\eta}_i)\left[(u_j^{n+1}-(2u_j^n-u_j^{n-1}))\cdot\widehat{\tau}_i\right](v_h\cdot\widehat{\tau}_i)\,ds + \epsilon_{j,f}^{n+1}(v_h),\\
&\quad-\left(\frac{3\mu_{j,u}^{n+1}-4\mu_{j,u}^n+\mu_{j,u}^{n-1}}{2\Delta t},v_h\right)_f - \nu(\nabla\mu_{j,u}^{n+1},\nabla v_h)_f - \sum_i\int_I\bar{\eta}_i(\mu_{j,u}^{n+1}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\,ds\\
&\quad-\sum_i\int_I(\eta_{i,j}-\bar{\eta}_i)\left[(2\mu_{j,u}^n-\mu_{j,u}^{n-1})\cdot\widehat{\tau}_i\right](v_h\cdot\widehat{\tau}_i)\,ds - c_I(v_h,2\mu_{j,\phi}^n-\mu_{j,\phi}^{n-1}),
\end{aligned}\tag{4.11}$$

$$\begin{aligned}
&gS_0\left(\frac{3\xi_{j,\phi}^{n+1}-4\xi_{j,\phi}^n+\xi_{j,\phi}^{n-1}}{2\Delta t},\psi_h\right)_p + g(\bar{\mathcal{K}}\nabla\xi_{j,\phi}^{n+1},\nabla\psi_h)_p\\
&\qquad+g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla(2\xi_{j,\phi}^n-\xi_{j,\phi}^{n-1}),\nabla\psi_h)_p - c_I(2\xi_{j,u}^n-\xi_{j,u}^{n-1},\psi_h)\\
&=-g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla(\phi_j^{n+1}-(2\phi_j^n-\phi_j^{n-1}),\nabla\psi_h)_p + \epsilon_{j,p}^{n+1}(\psi_h) - gS_0\left(\frac{3\mu_{j,\phi}^{n+1}-4\mu_{j,\phi}^n+\mu_{j,\phi}^{n-1}}{2\Delta t},\psi_h\right)_p
\end{aligned}$$

$$- g(\bar{\mathcal{K}}\nabla\mu_{j,\phi}^{n+1}, \nabla\psi_h)_p - g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(2\mu_{j,\phi}^n - \mu_{j,\phi}^{n-1}), \nabla\psi_h)_p + c_I(2\mu_{j,u}^n - \mu_{j,u}^{n-1}, \psi_h).$$

Letting $v_h = \xi_{j,u}^{n+1}, \psi_h = \xi_{j,\phi}^{n+1}$ in (4.11) and adding the two equations yields

$$\begin{aligned}
&\frac{1}{4\Delta t}(\|\xi_{j,u}^{n+1}\|_f^2 + \|2\xi_{j,u}^{n+1} - \xi_{j,u}^n\|_f^2) - \frac{1}{4\Delta t}(\|\xi_{j,u}^n\|_f^2 + \|2\xi_{j,u}^n - \xi_{j,u}^{n-1}\|_f^2) + \frac{1}{4\Delta t}\|\xi_{j,u}^{n+1} - 2\xi_{j,u}^n + \xi_{j,u}^{n-1}\|_f^2 \\
&+ \nu\|\nabla\xi_{j,u}^{n+1}\|_f^2 + \sum_i \int_I \bar{\eta}_i(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\,ds + \frac{gS_0}{4\Delta t}(\|\xi_{j,\phi}^{n+1}\|_p^2 + \|2\xi_{j,\phi}^{n+1} - \xi_{j,\phi}^n\|_p^2) \\
&- \frac{gS_0}{4\Delta t}(\|\xi_{j,\phi}^n\|_p^2 + \|2\xi_{j,\phi}^n - \xi_{j,\phi}^{n-1}\|_p^2) + \frac{gS_0}{4\Delta t}\|\xi_{j,\phi}^{n+1} - 2\xi_{j,\phi}^n + \xi_{j,\phi}^{n-1}\|_p^2 + g(\bar{\mathcal{K}}\nabla\xi_{j,\phi}^{n+1}, \nabla\xi_{j,\phi}^{n+1})_p \\
&+ c_I(\xi_{j,u}^{n+1}, 2\xi_{j,\phi}^n - \xi_{j,\phi}^{n-1}) - c_I(2\xi_{j,u}^n - \xi_{j,u}^{n-1}, \xi_{j,\phi}^{n+1}) \\
&= -\sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)\,[(2\xi_{j,u}^n - \xi_{j,u}^{n-1})\cdot\widehat{\tau}_i](\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)\,ds + \left(p_j^{n+1} - \lambda_h^{n+1}, \nabla\cdot\xi_{j,u}^{n+1}\right)_f \\
&- \sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)\,[(u_j^{n+1} - (2u_j^n - u_j^{n-1}))\cdot\widehat{\tau}_i](\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)\,ds + \epsilon_{j,f}^{n+1}(\xi_{j,u}^{n+1}) \\
&- \left(\frac{3\mu_{j,u}^{n+1} - 4\mu_{j,u}^n + \mu_{j,u}^{n-1}}{2\Delta t}, \xi_{j,u}^{n+1}\right)_f - \nu(\nabla\mu_{j,u}^{n+1}, \nabla\xi_{j,u}^{n+1})_f - \sum_i \int_I \bar{\eta}_i(\mu_{j,u}^{n+1}\cdot\widehat{\tau}_i)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)\,ds \\
&- \sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)\,[(2\mu_{j,u}^n - \mu_{j,u}^{n-1})\cdot\widehat{\tau}_i](\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)\,ds - c_I(\xi_{j,u}^{n+1}, 2\mu_{j,\phi}^n - \mu_{j,\phi}^{n-1}) \\
&- g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla[\phi_j^{n+1} - (2\phi_j^n - \phi_j^{n-1})], \nabla\xi_{j,\phi}^{n+1})_p + \epsilon_{j,p}^{n+1}(\xi_{j,\phi}^{n+1}) - g(\bar{\mathcal{K}}\nabla\mu_{j,\phi}^{n+1}, \nabla\xi_{j,\phi}^{n+1})_p \\
&- g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(2\mu_{j,\phi}^n - \mu_{j,\phi}^{n-1}), \nabla\xi_{j,\phi}^{n+1})_p + c_I(2\mu_{j,u}^n - \mu_{j,u}^{n-1}, \xi_{j,\phi}^{n+1}) \\
&- g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(2\xi_{j,\phi}^n - \xi_{j,\phi}^{n-1}), \nabla\xi_{j,\phi}^{n+1})_p - gS_0\left(\frac{3\mu_{j,\phi}^{n+1} - 4\mu_{j,\phi}^n + \mu_{j,\phi}^{n-1}}{2\Delta t}, \xi_{j,\phi}^{n+1}\right)_p. \qquad (4.12)
\end{aligned}$$

Using the same techniques in the stability proof (see (3.5) and (3.6)), we have

$$\begin{aligned}
&c_I(\xi_{j,u}^{n+1}, 2\xi_{j,\phi}^n - \xi_{j,\phi}^{n-1}) - c_I(2\xi_{j,u}^n - \xi_{j,u}^{n-1}, \xi_{j,\phi}^{n+1}) \\
=&c_I(\xi_{j,u}^{n+1} - 2\xi_{j,u}^n + \xi_{j,u}^{n-1}, \xi_{j,\phi}^{n+1}) - c_I(\xi_{j,u}^{n+1}, \xi_{j,\phi}^{n+1} - 2\xi_{j,\phi}^n + \xi_{j,\phi}^{n-1}) \\
\geq& -\frac{1}{4\Delta t}\|\xi_{j,u}^{n+1} - 2\xi_{j,u}^n + \xi_{j,u}^{n-1}\|_f^2 - \Delta t\frac{C_2}{\beta_1^2}\left(4\|\nabla\xi_{j,u}^{n+1}\|_f^2 + 8\|\nabla\xi_{j,u}^n\|_f^2 + 4\|\nabla\xi_{j,u}^{n-1}\|_f^2\right) \\
&- \beta_1 g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2 - \frac{gS_0}{4\Delta t}\|\xi_{j,\phi}^{n+1} - 2\xi_{j,\phi}^n + \xi_{j,\phi}^{n-1}\|_p^2 - \frac{\Delta t}{gS_0}\frac{C_1}{\alpha_1^2}\big(4\|\nabla\xi_{j,\phi}^{n+1}\|_p^2 + 8\|\nabla\xi_{j,\phi}^n\|_p^2 \\
&+ 4\|\nabla\xi_{j,\phi}^{n-1}\|_p^2\big) - \alpha_1\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2. \qquad (4.13)
\end{aligned}$$

Next we bound the terms on the right hand side of (4.12).

$$\begin{aligned}
&-\left(\frac{3\mu_{j,u}^{n+1} - 4\mu_{j,u}^n + \mu_{j,u}^{n-1}}{2\Delta t}, \xi_{j,u}^{n+1}\right)_f - gS_0\left(\frac{3\mu_{j,\phi}^{n+1} - 4\mu_{j,\phi}^n + \mu_{j,\phi}^{n-1}}{2\Delta t}, \xi_{j,\phi}^{n+1}\right)_p \\
&\leq \frac{CC_{P,f}^2}{\alpha_2\nu}\left\|\frac{3\mu_{j,u}^{n+1} - 4\mu_{j,u}^n + \mu_{j,u}^{n-1}}{2\Delta t}\right\|_f^2 + \frac{\alpha_2}{5}\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2 \\
&\quad + \frac{CC_{P,p}^2 gS_0^2}{\beta_2\bar{k}_{min}}\left\|\frac{3\mu_{j,\phi}^{n+1} - 4\mu_{j,\phi}^n + \mu_{j,\phi}^{n-1}}{2\Delta t}\right\|_p^2 + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2
\end{aligned}$$

$$\leq \frac{CC_{P,f}^2}{\alpha_2\nu}\frac{1}{\Delta t}\int_{t^{n-1}}^{t^{n+1}} \|\mu_{j,u,t}\|_f^2\ dt + \frac{\alpha_2}{5}\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2 + \frac{CC_{P,p}^2 gS_0^2}{\beta_2\bar{k}_{min}}\frac{1}{\Delta t}\int_{t^{n-1}}^{t^{n+1}} \|\mu_{j,\phi,t}\|_p^2\ dt$$
$$+ \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2. \tag{4.14}$$

$$-\nu(\nabla\mu_{j,u}^{n+1},\nabla\xi_{j,u}^{n+1})_f - g(\bar{\mathcal{K}}\nabla\mu_{j,\phi}^{n+1},\nabla\xi_{j,\phi}^{n+1})_p$$
$$\leq C\left(\|\nabla\mu_{j,u}^{n+1}\|_f^2 + \|\nabla\mu_{j,\phi}^{n+1}\|_p^2\right) + \frac{\alpha_2}{5}\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2 + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2. \tag{4.15}$$

By trace theorem, we obtain

$$-c_I(\xi_{j,u}^{n+1}, 2\mu_{j,\phi}^n - \mu_{j,\phi}^{n-1}) + c_I(2\mu_{j,u}^n - \mu_{j,u}^{n-1}, \xi_{j,\phi}^{n+1}) \tag{4.16}$$
$$\leq C\left(\|\nabla(2\mu_{j,u}^n - \mu_{j,u}^{n-1})\|_f^2 + \|\nabla(2\mu_{j,\phi}^n - \mu_{j,\phi}^{n-1})\|_p^2\right) + \frac{\alpha_2}{5}\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2 + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2$$
$$\leq C\left(\|\nabla\mu_{j,u}^n\|_f^2 + \|\nabla\mu_{j,u}^{n-1}\|_f^2 + \|\nabla\mu_{j,\phi}^n\|_p^2 + \|\nabla\mu_{j,\phi}^{n-1}\|_p^2\right) + \frac{\alpha_2}{5}\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2 + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2.$$

The pressure term can be bounded as follows.

$$\left(p_j^{n+1} - \lambda_h^{n+1}, \nabla\cdot\xi_{j,u}^{n+1}\right)_f \leq \frac{C}{\alpha_2\nu}\|p_j^{n+1} - \lambda_h^{n+1}\|_f^2 + \frac{\alpha_2}{5}\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2. \tag{4.17}$$

Next we bound the consistency errors.

$$\epsilon_{j,f}^{n+1}(\xi_{j,u}^{n+1}) \leq C\left\|\frac{3u_j^{n+1} - 4u_j^n + u_j^{n-1}}{2\Delta t} - u_{j,t}^{n+1}\right\|_f^2 + C\|\nabla(\phi_j^{n+1} - (2\phi_j^n - \phi_j^{n-1})\|_p^2 + \frac{\alpha_2}{5}\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2$$
$$\leq C\Delta t^3\int_{t^{n-1}}^{t^{n+1}} \|u_{j,ttt}\|_f^2\,dt + C\Delta t^3\int_{t^{n-1}}^{t^{n+1}} \|\nabla\phi_{j,tt}\|_p^2\,dt + \frac{\alpha_2}{5}\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2. \tag{4.18}$$

$$\epsilon_{j,p}^{n+1}(\xi_{j,\phi}^{n+1}) \leq C\left\|\frac{3\phi_j^{n+1} - 4\phi_j^n + \phi_j^{n-1}}{2\Delta t} - \phi_{j,t}^{n+1}\right\|_p^2 + C\|\nabla(u_j^{n+1} - (2u_j^n - u_j^{n-1}))\|_f^2 + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2$$
$$\leq C\Delta t^3\int_{t^{n-1}}^{t^{n+1}} \|\phi_{j,ttt}\|_p^2\,dt + C\Delta t^3\int_{t^{n-1}}^{t^{n+1}} \|\nabla u_{j,tt}\|_f^2\,dt + \frac{\beta_2}{4}g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2. \tag{4.19}$$

Following the discussion for (3.10) and (3.11), we have

$$-\sum_i\int_I(\eta_{i,j} - \bar{\eta}_i)\,[(2\xi_{j,u}^n - \xi_{j,u}^{n-1})\cdot\widehat{\tau}_i](\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)\ ds \tag{4.20}$$
$$\leq \sum_i\left[\eta_i'^{max}\int_I(\xi_{j,u}^n\cdot\widehat{\tau}_i)^2\ ds + \frac{\eta_i'^{max}}{2}\int_I(\xi_{j,u}^{n-1}\cdot\widehat{\tau}_i)^2\ ds + \frac{3\eta_i'^{max}}{2}\int_I(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\ ds\right],$$

and

$$-g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(2\xi_{j,\phi}^n - \xi_{j,\phi}^{n-1}), \nabla\xi_{j,\phi}^{n+1})_p \leq g\rho'_{max}\|\nabla\xi_{j,\phi}^n\|_p^2 + \frac{g\rho'_{max}}{2}\|\nabla\xi_{j,\phi}^{n-1}\|_p^2 + \frac{3g\rho'_{max}}{2}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2. \tag{4.21}$$

By (2.1) and Poincar$\acute{e}$ inequality, we have, for any $\sigma_1 > 0$

$$
\begin{aligned}
&-\sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)\left((u_j^{n+1} - (2u_j^n - u_j^{n-1}))\cdot\widehat{\tau}_i\right)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)\; ds \\
&\le \sum_i \eta_{i,j}^{\prime max} \int_I |((u_j^{n+1} - (2u_j^n - u_j^{n-1}))\cdot\widehat{\tau}_i)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)|\; ds \\
&\le \sum_i \eta_i^{\prime max}\left[\frac{1}{2\sigma_1}\int_I \left((u_j^{n+1} - (2u_j^n - u_j^{n-1}))\cdot\widehat{\tau}_i\right)^2\; ds + \frac{\sigma_1}{2}\int_I \left(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i\right)^2\; ds\right] \\
&\le \sum_i \left[\frac{\eta_i^{\prime max}}{2\sigma_1}\|u_j^{n+1} - (2u_j^n - u_j^{n-1})\|_I^2 + \frac{\sigma_1}{2}\eta_i^{\prime max}\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds\right] \\
&\le \sum_i \left[\frac{C_{P,f}C^2(D_f)}{2\sigma_1}\eta_i^{\prime max}\|\nabla(u_j^{n+1} - (2u_j^n - u_j^{n-1}))\|_f^2 + \frac{\sigma_1}{2}\eta_i^{\prime max}\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds\right] \\
&\le \sum_i \left[\frac{C_{P,f}C^2(D_f)}{2\sigma_1}\eta_i^{\prime max}\Delta t^3\int_{t^{n-1}}^{t^{n+1}}\|\nabla u_{j,tt}\|_f^2\; dt + \frac{\sigma_1}{2}\eta_i^{\prime max}\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds\right],
\end{aligned}
\tag{4.22}
$$

and

$$
\begin{aligned}
&-\sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)\left[(2\mu_{j,u}^n - \mu_{j,u}^{n-1})\cdot\widehat{\tau}_i\right](\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)\; ds \\
&\quad\le \sum_i \eta_{i,j}^{\prime max}\int_I |[(2\mu_{j,u}^n - \mu_{j,u}^{n-1})\cdot\widehat{\tau}_i](\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)|\; ds \\
&\quad\le \sum_i \eta_i^{\prime max}\left[\frac{1}{2\sigma_1}\int_I [(2\mu_{j,u}^n - \mu_{j,u}^{n-1})\cdot\widehat{\tau}_i]^2\; ds + \frac{\sigma_1}{2}\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\right]\; ds \\
&\quad\le \sum_i \left[\frac{1}{2\sigma_1}\eta_i^{\prime max}\|2\mu_{j,u}^n - \mu_{j,u}^{n-1}\|_I^2 + \frac{\sigma_1}{2}\eta_i^{\prime max}\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds\right] \\
&\quad\le \sum_i \left[\frac{C^2(D_f)C_{P,f}}{2\sigma_1}\eta_{i,j}^{\prime max}\|\nabla(2\mu_{j,u}^n - \mu_{j,u}^{n-1})\|_f^2 + \frac{\sigma_1}{2}\eta_i^{\prime max}\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds\right] \\
&\quad\le \sum_i \left[\frac{4C^2(D_f)C_{P,f}}{\sigma_1}\eta_i^{\prime max}\left(\|\nabla\mu_{j,u}^n\|_f^2 + \|\nabla\mu_{j,u}^{n-1}\|_f^2\right) + \frac{\sigma_1}{2}\eta_i^{\prime max}\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds\right].
\end{aligned}
\tag{4.23}
$$

For any $\sigma_2 > 0$

$$
\begin{aligned}
&-\sum_i \int_I \bar{\eta}_i(\mu_{j,u}^{n+1}\cdot\widehat{\tau}_i)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)\; ds \\
&\le \sum_i \left[\frac{1}{4\sigma_2}\int_I \bar{\eta}_i(\mu_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds + \sigma_2\int_I \bar{\eta}_i(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds\right] \\
&\le \sum_i \left[\frac{1}{4\sigma_2}\bar{\eta}_i^{max}\|\mu_{j,u}^{n+1}\|_I^2 + \sigma_2\int_I \bar{\eta}_i(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds\right] \\
&\le \sum_i \left[\frac{C^2(D_f)C_{P,f}}{4\sigma_2}\bar{\eta}_i^{max}\|\nabla\mu_{j,u}^{n+1}\|_f^2 + \sigma_2\int_I \bar{\eta}_i(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds\right].
\end{aligned}
\tag{4.24}
$$

For any $\sigma > 0$, we have

$$
\begin{aligned}
&-g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(\phi_j^{n+1} - (2\phi_j^n - \phi_j^{n-1})), \nabla\xi_{j,\phi}^{n+1})_p \qquad (4.25)\\
&\leq g\int_{D_p} |\nabla(\phi_j^{n+1} - (2\phi_j^n - \phi_j^{n-1}))|_2 |\mathcal{K}_j - \bar{\mathcal{K}}|_2 |\nabla\xi_{j,\phi}^{n+1}|_2 \, dx\\
&\leq g\int_{D_p} \rho_j'(x)|\nabla(\phi_j^{n+1} - \phi_j^n)|_2 |\nabla\xi_{j,\phi}^{n+1}|_2 \, dx\\
&\leq g\rho_{j,max}'\int_{D_p} |\nabla(\phi_j^{n+1} - (2\phi_j^n - \phi_j^{n-1}))|_2 |\nabla\xi_{j,\phi}^{n+1}|_2 \, dx\\
&\leq g\rho_{max}'\|\nabla(\phi_j^{n+1} - (2\phi_j^n - \phi_j^{n-1}))\|_p \|\nabla\xi_{j,\phi}^{n+1}\|_p\\
&\leq \frac{g\rho_{max}'}{2\sigma}\|\nabla(\phi_j^{n+1} - (2\phi_j^n - \phi_j^{n-1}))\|_p^2 + \frac{\sigma}{2}g\rho_{max}'\|\nabla\xi_{j,\phi}^{n+1}\|_p^2\\
&\leq \frac{Cg\rho_{max}'}{\sigma}\Delta t^3\int_{t^{n-1}}^{t^{n+1}} \|\nabla\phi_{j,tt}\|_p^2 \, dt + \frac{\sigma}{2}g\rho_{max}'\|\nabla\xi_{j,\phi}^{n+1}\|_p^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
-g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(2\mu_{j,\phi}^n - \mu_{j,\phi}^{n-1}), \nabla\xi_{j,\phi}^{n+1})_p &\leq g\int_{D_p} |\nabla(2\mu_{j,\phi}^n - \mu_{j,\phi}^{n-1})|_2 |\mathcal{K}_j - \bar{\mathcal{K}}|_2 |\nabla\xi_{j,\phi}^{n+1}|_2 \, dx \qquad (4.26)\\
&\leq g\int_{D_p} \rho_j'(x)|\nabla(2\mu_{j,\phi}^n - \mu_{j,\phi}^{n-1})|_2 |\nabla\xi_{j,\phi}^{n+1}|_2 \, dx\\
&\leq g\rho_{j,max}'\int_{D_p} |\nabla(2\mu_{j,\phi}^n - \mu_{j,\phi}^{n-1})|_2 |\nabla\xi_{j,\phi}^{n+1}|_2 \, dx\\
&\leq g\rho_{max}'\|\nabla(2\mu_{j,\phi}^n - \mu_{j,\phi}^{n-1})\|_p \|\nabla\xi_{j,\phi}^{n+1}\|_p\\
&\leq \frac{g\rho_{max}'}{2\sigma}\|\nabla(2\mu_{j,\phi}^n - \mu_{j,\phi}^{n-1})\|_p^2 + \frac{\sigma}{2}g\rho_{max}'\|\nabla\xi_{j,\phi}^{n+1}\|_p^2\\
&\leq \frac{4g\rho_{max}'}{\sigma}\left(\|\nabla\mu_{j,\phi}^n\|_p^2 + \|\nabla\mu_{j,\phi}^{n-1}\|_p^2\right) + \frac{\sigma}{2}g\rho_{max}'\|\nabla\xi_{j,\phi}^{n+1}\|_p^2.
\end{aligned}
$$

Combining all these estimates, we have the following inequality

$$
\begin{aligned}
&\frac{1}{4\Delta t}(\|\xi_{j,u}^{n+1}\|_f^2 + \|2\xi_{j,u}^{n+1} - \xi_{j,u}^n\|_f^2) - \frac{1}{4\Delta t}(\|\xi_{j,u}^n\|_f^2 + \|2\xi_{j,u}^n - \xi_{j,u}^{n-1}\|_f^2)\\
&+ \left(1 - \alpha_1 - \alpha_2 - \Delta t\frac{16C_2}{\beta_1^2\nu}\right)\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2 + \frac{12\Delta tC_2}{\beta_1^2}\left(\|\nabla\xi_{j,u}^{n+1}\|_f^2 - \|\nabla\xi_{j,u}^n\|_f^2\right)\\
&+ \frac{4\Delta tC_2}{\beta_1^2}\left(\|\nabla\xi_{j,u}^n\|_f^2 - \|\nabla\xi_{j,u}^{n-1}\|_f^2\right) + \sum_i\left((1-\sigma_2)\bar{\eta}_i^{min} - (3+\sigma_1)\eta_i'^{max}\right)\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2 \, ds\\
&+ \sum_i \frac{3}{2}\eta_i'^{max}\left(\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2 \, ds - \int_I (\xi_{j,u}^n\cdot\widehat{\tau}_i)^2 \, ds\right) + \sum_i \frac{1}{2}\eta_i'^{max}\left(\int_I (\xi_{j,u}^n\cdot\widehat{\tau}_i)^2 \, ds - \int_I (\xi_{j,u}^{n-1}\cdot\widehat{\tau}_i)^2 \, ds\right)\\
&+ \frac{gS_0}{4\Delta t}(\|\xi_{j,\phi}^{n+1}\|_p^2 + \|2\xi_{j,\phi}^{n+1} - \xi_{j,\phi}^n\|_p^2) - \frac{gS_0}{4\Delta t}(\|\xi_{j,\phi}^n\|_p^2 + \|2\xi_{j,\phi}^n - \xi_{j,\phi}^{n-1}\|_p^2)\\
&+ \left((1 - \beta_1 - \beta_2 - \Delta t\frac{16C_1}{g^2S_0\bar{k}_{min}\alpha_1^2}) - (3+\sigma)\frac{\rho_{max}'}{\bar{k}_{min}}\right)g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2
\end{aligned}
$$

$$
+\left(\frac{3}{2}g\rho'_{max}+\frac{12\Delta tC_1}{gS_0\alpha_1^2}\right)\left(\|\nabla\xi_{j,\phi}^{n+1}\|_p^2-\|\nabla\xi_{j,\phi}^{n}\|_p^2\right)+\left(\frac{1}{2}g\rho'_{max}+\frac{4\Delta tC_1}{gS_0\alpha_1^2}\right)\left(\|\nabla\xi_{j,\phi}^{n}\|_p^2-\|\nabla\xi_{j,\phi}^{n-1}\|_p^2\right)
$$
$$
\le\frac{CC_{P,f}^2}{\alpha_2\nu}\frac{1}{\Delta t}\int_{t^{n-1}}^{t^{n+1}}\|\mu_{j,u,t}\|_f^2\,dt+\frac{CC_{P,p}^2gS_0^2}{\beta_2\bar{k}_{min}}\frac{1}{\Delta t}\int_{t^{n-1}}^{t^{n+1}}\|\mu_{j,\phi,t}\|_p^2\,dt+\frac{C}{\alpha_2\nu}\|p_j^{n+1}-\lambda_h^{n+1}\|_f^2
$$
$$
+C\Delta t^3\int_{t^{n-1}}^{t^{n+1}}\|u_{j,ttt}\|_f^2\,dt+C\Delta t^3\int_{t^{n-1}}^{t^{n+1}}\|\nabla u_{j,tt}\|_f^2\,dt+C\Delta t^3\int_{t^{n-1}}^{t^{n+1}}\|\phi_{j,ttt}\|_p^2\,dt
$$
$$
+C\Delta t^3\int_{t^{n-1}}^{t^{n+1}}\|\nabla\phi_{j,tt}\|_p^2\,dt+C\left(\|\nabla\mu_{j,u}^{n+1}\|_f^2+\|\nabla\mu_{j,\phi}^{n+1}\|_p^2\right)+C\big(\|\nabla\mu_{j,u}^{n}\|_f^2+\|\nabla\mu_{j,u}^{n-1}\|_f^2+\|\nabla\mu_{j,\phi}^{n}\|_p^2
$$
$$
+\|\nabla\mu_{j,\phi}^{n-1}\|_p^2\big)+\sum_i\frac{C_{P,f}C^2(D_f)}{2\sigma_1}\eta_i^{\prime max}\Delta t^3\int_{t^{n-1}}^{t^{n+1}}\|\nabla u_{j,tt}\|_f^2\,dt+\sum_i\frac{C^2(D_f)C_{P,f}}{4\sigma_2}\bar{\eta}_i^{max}\|\nabla\mu_{j,u}^{n+1}\|_f^2
$$
$$
+\sum_i\frac{4C^2(D_f)C_{P,f}}{\sigma_1}\eta_{i,j}^{\prime max}\left(\|\nabla\mu_{j,u}^{n}\|_f^2+\|\nabla\mu_{j,u}^{n-1}\|_f^2\right)+\frac{Cg\rho'_{max}}{\sigma}\Delta t^3\int_{t^{n-1}}^{t^{n+1}}\|\nabla\phi_{j,tt}\|_p^2\,dt
$$
$$
+\frac{4g\rho'_{max}}{\sigma}\left(\|\nabla\mu_{j,\phi}^{n}\|_p^2+\|\nabla\mu_{j,\phi}^{n-1}\|_p^2\right). \tag{4.27}
$$

To obtain convergence result, we need the third, sixth and eleventh terms on the left hand side be non-negative, which implies $0<\alpha_1,\alpha_2,\sigma_2,\beta_1,\beta_2<1$, and

$$
\frac{\eta_i^{\prime max}}{\bar{\eta}_i^{min}}\le\frac{1-\sigma_2}{3+\sigma_1},\qquad\frac{\rho'_{max}}{\bar{k}_{min}}<\frac{1}{3+\sigma}. \tag{4.28}
$$

For $\forall\sigma_2\in(0,1),\forall\sigma_1>0,\forall\sigma>0$, we can derive that $\frac{1-\sigma_2}{3+\sigma_1},\frac{1}{3+\sigma}\in(0,\frac{1}{3})$. If the two parameter conditions in (4.7) are satisfied, we have $\frac{\eta_i^{\prime max}}{\bar{\eta}_i^{min}},\frac{\rho'_{max}}{\bar{k}_{min}}\in(0,\frac{1}{3})$. It is easy to check there exist $\sigma_2\in(0,1),\sigma_1>0$ such that $\frac{\eta_i^{\prime max}}{\bar{\eta}_i^{min}}=\frac{1-\sigma_2}{3+\sigma_1}$, and $\sigma>0$ such that $\frac{\rho'_{max}}{\bar{k}_{min}}<\frac{1}{3+\sigma}$. With the time-step condition (4.6) also satisfied, (4.27) reduces to

$$
\frac{1}{4\Delta t}(\|\xi_{j,u}^{n+1}\|_f^2+\|2\xi_{j,u}^{n+1}-\xi_{j,u}^{n}\|_f^2)-\frac{1}{4\Delta t}(\|\xi_{j,u}^{n}\|_f^2+\|2\xi_{j,u}^{n}-\xi_{j,u}^{n-1}\|_f^2)
$$
$$
+\frac{12\Delta tC_2}{\beta_1^2}\left(\|\nabla\xi_{j,u}^{n+1}\|_f^2-\|\nabla\xi_{j,u}^{n}\|_f^2\right)+\frac{4\Delta tC_2}{\beta_1^2}\left(\|\nabla\xi_{j,u}^{n}\|_f^2-\|\nabla\xi_{j,u}^{n-1}\|_f^2\right)
$$
$$
+\sum_i\frac{3}{2}\eta_i^{\prime max}\left(\int_I(\xi_{j,u}^{n+1}\cdot\hat{\tau}_i)^2\,ds-\int_I(\xi_{j,u}^{n}\cdot\hat{\tau}_i)^2\,ds\right)+\sum_i\frac{\eta_i^{\prime max}}{2}\left(\int_I(\xi_{j,u}^{n}\cdot\hat{\tau}_i)^2\,ds-\int_I(\xi_{j,u}^{n-1}\cdot\hat{\tau}_i)^2\,ds\right)
$$
$$
+\frac{gS_0}{4\Delta t}(\|\xi_{j,\phi}^{n+1}\|_p^2+\|2\xi_{j,\phi}^{n+1}-\xi_{j,\phi}^{n}\|_p^2)-\frac{gS_0}{4\Delta t}(\|\xi_{j,\phi}^{n}\|_p^2+\|2\xi_{j,\phi}^{n}-\xi_{j,\phi}^{n-1}\|_p^2)
$$
$$
+\left(\frac{3}{2}g\rho'_{max}+\frac{12\Delta tC_1}{gS_0\alpha_1^2}\right)\left(\|\nabla\xi_{j,\phi}^{n+1}\|_p^2-\|\nabla\xi_{j,\phi}^{n}\|_p^2\right)+\left(\frac{1}{2}g\rho'_{max}+\frac{4\Delta tC_1}{gS_0\alpha_1^2}\right)\left(\|\nabla\xi_{j,\phi}^{n}\|_p^2-\|\nabla\xi_{j,\phi}^{n-1}\|_p^2\right)
$$
$$
\le\frac{CC_{P,f}^2}{\alpha_2\nu}\frac{1}{\Delta t}\int_{t^{n-1}}^{t^{n+1}}\|\mu_{j,u,t}\|_f^2\,dt+\frac{CC_{P,p}^2gS_0^2}{\beta_2\bar{k}_{min}}\frac{1}{\Delta t}\int_{t^{n-1}}^{t^{n+1}}\|\mu_{j,\phi,t}\|_p^2\,dt+\frac{C}{\alpha_2\nu}\|p_j^{n+1}-\lambda_h^{n+1}\|_f^2
$$
$$
+C\Delta t^3\int_{t^{n-1}}^{t^{n+1}}\|u_{j,ttt}\|_f^2\,dt+C\Delta t^3\int_{t^{n-1}}^{t^{n+1}}\|\nabla u_{j,tt}\|_f^2\,dt+C\Delta t^3\int_{t^{n-1}}^{t^{n+1}}\|\phi_{j,ttt}\|_p^2\,dt
$$
$$
+C\Delta t^3\int_{t^{n-1}}^{t^{n+1}}\|\nabla\phi_{j,tt}\|_p^2\,dt+C\left(\|\nabla\mu_{j,u}^{n+1}\|_f^2+\|\nabla\mu_{j,\phi}^{n+1}\|_p^2\right)+C\big(\|\nabla\mu_{j,u}^{n}\|_f^2+\|\nabla\mu_{j,u}^{n-1}\|_f^2
$$

$$
+ \|\nabla \mu_{j,\phi}^{n}\|_p^2 + \|\nabla \mu_{j,\phi}^{n-1}\|_p^2\big) + \sum_i \frac{C_{P,f} C^2(D_f)}{2\sigma_1} \eta_i'^{max} \Delta t^3 \int_{t^{n-1}}^{t^{n+1}} \|\nabla u_{j,tt}\|_f^2 \, dt
$$

$$
+ \sum_i \frac{C^2(D_f) C_{P,f}}{4\sigma_2} \bar{\eta}_i^{max} \|\nabla \mu_{j,u}^{n+1}\|_f^2 + \sum_i \frac{4C^2(D_f) C_{P,f}}{\sigma_1} \eta_{i,j}'^{max} \left(\|\nabla \mu_{j,u}^{n}\|_f^2 + \|\nabla \mu_{j,u}^{n-1}\|_f^2\right)
$$

$$
+ \frac{Cg\rho'_{max}}{\sigma} \Delta t^3 \int_{t^{n-1}}^{t^{n+1}} \|\nabla \phi_{j,tt}\|_p^2 \, dt + \frac{4g\rho'_{max}}{\sigma} \left(\|\nabla \mu_{j,\phi}^{n}\|_p^2 + \|\nabla \mu_{j,\phi}^{n-1}\|_p^2\right). \tag{4.29}
$$

Summing up from $n = 1$ to $n = N - 1$ and multiplying through by $2\Delta t$ yields

$$
\frac{1}{2}(\|\xi_{j,u}^N\|_f^2 + \|2\xi_{j,u}^N - \xi_{j,u}^{N-1}\|_f^2) + \frac{24\Delta t^2 C_2}{\beta_1^2} \|\nabla \xi_{j,u}^N\|_f^2 + \frac{8\Delta t^2 C_2}{\beta_1^2} \|\nabla \xi_{j,u}^{N-1}\|_f^2
$$

$$
+ \sum_i 3\Delta t \eta_i'^{max} \int_I (\xi_{j,u}^N \cdot \widehat{\tau}_i)^2 \, ds + \sum_i \Delta t \eta_i'^{max} \int_I (\xi_{j,u}^{N-1} \cdot \widehat{\tau}_i)^2 \, ds + \frac{gS_0}{2} (\|\xi_{j,\phi}^N\|_p^2 + \|2\xi_{j,\phi}^N - \xi_{j,\phi}^{N-1}\|_p^2)
$$

$$
+ \left(3\Delta t g\rho'_{max} + \frac{24\Delta t^2 C_1}{gS_0\alpha_1^2}\right) \|\nabla \xi_{j,\phi}^N\|_p^2 + \left(\Delta t g\rho'_{max} + \frac{8\Delta t^2 C_1}{gS_0\alpha_1^2}\right) \|\nabla \xi_{j,\phi}^{N-1}\|_p^2
$$

$$
\le \frac{1}{2}(\|\xi_{j,u}^1\|_f^2 + \|2\xi_{j,u}^1 - \xi_{j,u}^0\|_f^2) + \frac{24\Delta t^2 C_2}{\beta_1^2} \|\nabla \xi_{j,u}^1\|_f^2 + \frac{8\Delta t^2 C_2}{\beta_1^2} \|\nabla \xi_{j,u}^0\|_f^2
$$

$$
+ \sum_i 3\Delta t \eta_i'^{max} \int_I (\xi_{j,u}^1 \cdot \widehat{\tau}_i)^2 \, ds + \sum_i \Delta t \eta_i'^{max} \int_I (\xi_{j,u}^0 \cdot \widehat{\tau}_i)^2 \, ds + \frac{gS_0}{2} (\|\xi_{j,\phi}^1\|_p^2 + \|2\xi_{j,\phi}^1 - \xi_{j,\phi}^0\|_p^2)
$$

$$
+ \left(3\Delta t g\rho'_{max} + \frac{24\Delta t^2 C_1}{gS_0\alpha_1^2}\right) \|\nabla \xi_{j,\phi}^1\|_p^2 + \left(\Delta t g\rho'_{max} + \frac{8\Delta t^2 C_1}{gS_0\alpha_1^2}\right) \|\nabla \xi_{j,\phi}^0\|_p^2
$$

$$
+ \Delta t \sum_{n=1}^{N-1} \Bigg\{ \frac{CC_{P,f}^2}{\alpha_2 \nu} \frac{1}{\Delta t} \int_{t^{n-1}}^{t^{n+1}} \|\mu_{j,u,t}\|_f^2 \, dt + \frac{CC_{P,p}^2 gS_0^2}{\beta_2 \bar{k}_{min}} \frac{1}{\Delta t} \int_{t^{n-1}}^{t^{n+1}} \|\mu_{j,\phi,t}\|_p^2 \, dt + \frac{C}{\alpha_2 \nu} \|p_j^{n+1} - \lambda_h^{n+1}\|_f^2
$$

$$
+ C\Delta t^3 \int_{t^{n-1}}^{t^{n+1}} \|u_{j,ttt}\|_f^2 \, dt + C\Delta t^3 \int_{t^{n-1}}^{t^{n+1}} \|\nabla u_{j,tt}\|_f^2 \, dt + C\Delta t^3 \int_{t^{n-1}}^{t^{n+1}} \|\phi_{j,ttt}\|_p^2 \, dt
$$

$$
+ C\Delta t^3 \int_{t^{n-1}}^{t^{n+1}} \|\nabla \phi_{j,tt}\|_p^2 \, dt + C\left(\|\nabla \mu_{j,u}^{n+1}\|_f^2 + \|\nabla \mu_{j,\phi}^{n+1}\|_p^2\right) + C\big(\|\nabla \mu_{j,u}^{n}\|_f^2 + \|\nabla \mu_{j,u}^{n-1}\|_f^2 + \|\nabla \mu_{j,\phi}^{n}\|_p^2
$$

$$
+ \|\nabla \mu_{j,\phi}^{n-1}\|_p^2\big) + \sum_i \frac{C_{P,f} C^2(D_f)}{\sigma_1} \eta_i'^{max} \Delta t^3 \int_{t^{n-1}}^{t^{n+1}} \|\nabla u_{j,tt}\|_f^2 \, dt + \sum_i \frac{C^2(D_f) C_{P,f}}{2\sigma_2} \bar{\eta}_i^{max} \|\nabla \mu_{j,u}^{n+1}\|_f^2
$$

$$
+ \sum_i \frac{8C^2(D_f) C_{P,f}}{\sigma_1} \eta_{i,j}'^{max} \left(\|\nabla \mu_{j,u}^{n}\|_f^2 + \|\nabla \mu_{j,u}^{n-1}\|_f^2\right)
$$

$$
+ \frac{Cg\rho'_{max}}{\sigma} \Delta t^3 \int_{t^{n-1}}^{t^{n+1}} \|\nabla \phi_{j,tt}\|_p^2 \, dt + \frac{8g\rho'_{max}}{\sigma} \left(\|\nabla \mu_{j,\phi}^{n}\|_p^2 + \|\nabla \mu_{j,\phi}^{n-1}\|_p^2\right) \Bigg\}. \tag{4.30}
$$

Using interpolation inequalities, we obtain

$$
\frac{1}{2}(\|\xi_{j,u}^N\|_f^2 + \|2\xi_{j,u}^N - \xi_{j,u}^{N-1}\|_f^2) + \frac{24\Delta t^2 C_2}{\beta_1^2} \|\nabla \xi_{j,u}^N\|_f^2 + \frac{8\Delta t^2 C_2}{\beta_1^2} \|\nabla \xi_{j,u}^{N-1}\|_f^2
$$

$$
+ \sum_i 3\Delta t \eta_i'^{max} \int_I (\xi_{j,u}^N \cdot \widehat{\tau}_i)^2 \, ds + \sum_i \Delta t \eta_i'^{max} \int_I (\xi_{j,u}^{N-1} \cdot \widehat{\tau}_i)^2 \, ds + \frac{gS_0}{2} (\|\xi_{j,\phi}^N\|_p^2 + \|2\xi_{j,\phi}^N - \xi_{j,\phi}^{N-1}\|_p^2)
$$

$$+\left(3\Delta t g\rho'_{max}+\frac{24\Delta t^2C_1}{gS_0\alpha_1^2}\right)\|\nabla\xi_{j,\phi}^N\|_p^2+\left(\Delta t g\rho'_{max}+\frac{8\Delta t^2C_1}{gS_0\alpha_1^2}\right)\|\nabla\xi_{j,\phi}^{N-1}\|_p^2$$
$$\leq\frac{1}{2}(\|\xi_{j,u}^1\|_f^2+\|2\xi_{j,u}^1-\xi_{j,u}^0\|_f^2)+\frac{24\Delta t^2C_2}{\beta_1^2}\|\nabla\xi_{j,u}^1\|_f^2+\frac{8\Delta t^2C_2}{\beta_1^2}\|\nabla\xi_{j,u}^0\|_f^2$$
$$+\sum_i 3\Delta t\eta_i'^{max}\int_I(\xi_{j,u}^1\cdot\widehat{\tau}_i)^2\,ds+\sum_i\Delta t\eta_i'^{max}\int_I(\xi_{j,u}^0\cdot\widehat{\tau}_i)^2\,ds+\frac{gS_0}{2}(\|\xi_{j,\phi}^1\|_p^2+\|2\xi_{j,\phi}^1-\xi_{j,\phi}^0\|_p^2)$$
$$+\left(3\Delta t g\rho'_{max}+\frac{24\Delta t^2C_1}{gS_0\alpha_1^2}\right)\|\nabla\xi_{j,\phi}^1\|_p^2+\left(\Delta t g\rho'_{max}+\frac{8\Delta t^2C_1}{gS_0\alpha_1^2}\right)\|\nabla\xi_{j,\phi}^0\|_p^2+Ch^{2k+2}\|u_{j,t}\|_{2,k+1,f}^2$$
$$+Ch^{2m+2}\|\phi_{j,t}\|_{2,m+1,p}^2+Ch^{2s+2}|||p_j|||_{2,m+1,f}^2+C\Delta t^4\|u_{j,ttt}\|_{2,0,f}^2+C\Delta t^4\|u_{j,tt}\|_{2,1,f}^2$$
$$+C\Delta t^4\|\phi_{j,ttt}\|_{2,0,p}^2+C\Delta t^4\|\phi_{j,tt}\|_{2,1,p}^2+Ch^{2k}|||u_j|||_{2,k+1,f}^2+Ch^{2m}|||\phi_j|||_{2,m+1,p}^2. \tag{4.31}$$

Applying the triangle inequality yields (4.8).

5. Numerical Illustrations

We present numerical experiments to test the proposed second order ensemble scheme herein. First, using a known exact solution we confirm the predicted convergence rates from the theory. In the second and third examples, we show how to combine our ensemble algorithm with the Monte Carlo method and the sparse grid method respectively to solve the Stokes-Darcy system with a random hydraulic conductivity tensor. The fourth example demonstrates the application of our ensemble algorithm in a realistic flow problem.

5.1. Convergence test

For the first test we consider the model problem on $D = [0,\pi]\times[-1,1]$, where $D_p = [0,\pi]\times[-1,0]$, and $D_f = [0,\pi]\times[0,1]$. We take $\alpha_{BJS}=1$, $\nu=1$, $g=1$, $S_0=1$, and

$$\mathcal{K}=\mathcal{K}_j=\begin{bmatrix}k_{11}^j & 0\\ 0 & k_{22}^j\end{bmatrix},\quad j=1,\ldots,J,$$

where $\mathcal{K}$ is the random hydraulic conductivity tensor and $\mathcal{K}_j$ is one of the samples of $\mathcal{K}$. The exact solution is given by

$$\begin{aligned}\phi_D &= (e^y-e^{-y})sin(x)e^t,\\ \overrightarrow{u}_S &= [\frac{k_{11}^j}{\pi}sin(2\pi y)cos(x),(-2k_{22}^j+\frac{k_{22}^j}{\pi^2}sin^2(\pi y))sin(x)]^Te^t,\\ p_S &= 0.\end{aligned}$$

We use Taylor-Hood elements for the approximation of the Stokes equations and the continuous piecewise quadratic finite elements for the Darcy equation. In order to check the convergence order in time, we uniformly refine the mesh size h and time step size Δt from the initial mesh size 1/4 and time step size $\Delta t = h$. In this test, we consider simulating $J = 3$ ensemble members: $k_{11}^1 = k_{22}^1 = 1.11, k_{11}^2 = k_{22}^2 = 1.21, k_{11}^3 = k_{22}^3 = 2.21$. The approximation errors of for each ensemble member at $t = T = 1$ are listed in Table 1, Table 2 and Table 3, for the velocity $\overrightarrow{u}$, the hydraulic head ϕ and the pressure p respectively, which confirm that our ensemble algorithm is second order in time convergent.

Table 1: Errors and convergence rates of the ensemble algorithm ($J = 3$) for $\triangle t = h$.

h	$\|\overrightarrow{u}_h - \overrightarrow{u}\|_0^{E,1}$	rate	$\|\overrightarrow{u}_h - \overrightarrow{u}\|_0^{E,2}$	rate	$\|\overrightarrow{u}_h - \overrightarrow{u}\|_0^{E,3}$	rate
1/4	8.3188×10^{-2}	–	8.8179×10^{-2}	–	7.9880×10^{-2}	–
1/8	1.6315×10^{-2}	2.35	2.1455×10^{-2}	2.04	1.9388×10^{-2}	2.04
1/16	4.0994×10^{-3}	2.00	5.3771×10^{-3}	2.00	4.8592×10^{-3}	2.00
1/32	1.0222×10^{-3}	2.00	1.3376×10^{-3}	2.00	1.2088×10^{-3}	2.01
h	$\|\overrightarrow{u}_h - \overrightarrow{u}\|_1^{E,1}$	rate	$\|\overrightarrow{u}_h - \overrightarrow{u}\|_1^{E,2}$	rate	$\|\overrightarrow{u}_h - \overrightarrow{u}\|_1^{E,3}$	rate
1/4	7.9014×10^{-1}	–	8.1385×10^{-1}	–	1.3989×10^{0}	–
1/8	2.0284×10^{-1}	1.96	1.9754×10^{-1}	2.04	3.4797×10^{-1}	2.01
1/16	4.9473×10^{-2}	2.04	4.9632×10^{-2}	2.00	8.7212×10^{-2}	2.00
1/32	1.2399×10^{-2}	2.00	1.2346×10^{-2}	2.01	2.1695×10^{-2}	2.01

Table 2: Errors and convergence rates of the ensemble algorithm ($J = 3$) for $\triangle t = h$.

h	$\|\phi_h - \phi\|_0^{E,1}$	rate	$\|\phi_h - \phi\|_0^{E,2}$	rate	$\|\phi_h - \phi\|_0^{E,3}$	rate
1/4	4.8649×10^{-1}	–	4.8304×10^{-1}	–	2.9751×10^{-1}	–
1/8	1.1966×10^{-1}	2.02	1.1779×10^{-1}	2.04	7.2564×10^{-2}	2.03
1/16	2.9990×10^{-2}	2.00	2.9520×10^{-2}	2.00	1.8150×10^{-2}	2.00
1/32	7.4601×10^{-3}	2.01	7.3433×10^{-3}	2.01	4.5262×10^{-3}	2.00
h	$\|\phi_h - \phi\|_1^{E,1}$	rate	$\|\phi_h - \phi\|_1^{E,2}$	rate	$\|\phi_h - \phi\|_1^{E,3}$	rate
1/4	1.1771×10^{-0}	–	9.7515×10^{-1}	–	6.2569×10^{-1}	–
1/8	2.8710×10^{-1}	2.04	2.4257×10^{-1}	2.01	1.5260×10^{-1}	2.03
1/16	7.1954×10^{-2}	2.00	6.0673×10^{-2}	2.00	3.8161×10^{-2}	2.00
1/32	1.7464×10^{-2}	2.04	1.5093×10^{-2}	2.01	9.4929×10^{-3}	2.01

5.2. Random hydraulic conductivity tensor with the Monte Carlo method

We next consider using the presented ensemble algorithm for approximating stochastic Stokes-Darcy equations with a random hydraulic conductivity tensor $\mathcal{K}(x, w)$ that depends on spatial coordinates. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space. Here Ω is the set of outcomes, $\mathcal{F} \in 2^{\Omega}$ is the $\sigma-$algebra of events, and $\mathcal{P} : \mathcal{F} \to [0, 1]$ is a probability measure. The stochastic Stokes-Darcy system considered reads: Find the functions $u : D_f \times [0, T] \times \Omega \to \mathbb{R}^d$ $(d = 2, 3)$, $p : D_f \times [0, T] \times \Omega \to \mathbb{R}$, and $\phi : D_p \times [0, T] \times \Omega \to \mathbb{R}$, such that it holds $\mathcal{P} - a.e.$ in Ω, or in other words, almost surely

$$\begin{gathered} u_t(x,t,\omega) - \nu\Delta u(x,t,\omega) + \nabla p(x,t,\omega) = f_f(x,t), \quad \nabla\cdot u(x,t,\omega) = 0, \text{ in } D_f\times\Omega \\ S_0\phi_t(x,t,\omega) - \nabla\cdot(\mathcal{K}(x,\omega)\nabla\phi(x,t,\omega)) = f_p(x,t), \text{ in } D_p\times\Omega, \\ \phi(x,0) = \phi_0(x), \text{ in } D_p, \text{ and } u(x,0) = u_0(x), \text{ in } D_f, \\ \phi(x,t,\omega) = 0, \text{ in } \partial D_p\backslash I \text{ and } u(x,t,\omega) = 0, \text{ in } \partial D_f\backslash I, \end{gathered} \tag{5.1}$$

where $f_f(x, t) \in L^2(D_f)$, $f_p(x, t) \in L^2(D_p)$. The hydraulic conductivity $\mathcal{K}(x, \omega)$ is a stochastic function, which is assumed to have continuous and bounded correlation function.

Table 3: Errors and convergence rates of the ensemble algorithm ($J = 3$) for $\triangle t = h$.

h	$\|p_h - p\|_0^{E,1}$	rate	$\|p_h - p\|_0^{E,2}$	rate	$\|p_h - p\|_0^{E,3}$	rate
1/4	1.4030×10^0	–	9.6278×10^{-1}	–	6.3049×10^{-1}	–
1/8	3.4136×10^{-1}	2.04	2.3890×10^{-1}	2.01	1.5011×10^{-1}	2.07
1/16	8.5128×10^{-2}	2.00	5.9756×10^{-2}	1.99	3.7623×10^{-2}	2.00
1/32	2.1176×10^{-2}	2.01	1.4902×10^{-2}	2.00	9.3589×10^{-3}	2.01

We construct the random hydraulic conductivity tensor that varies in the vertical direction as follows

$$\mathcal{K}(\vec{x}, \omega) = \begin{bmatrix} k_{11}(\vec{x}, \omega) & 0 \\ 0 & k_{22}(\vec{x}, \omega) \end{bmatrix},$$

and $k_{11}(\vec{x}, \omega) = k_{22}(\vec{x}, \omega) = k(\vec{x}, \omega)$ satisfy

$$log(k(\vec{x}, \omega) - 0.5) = 1 + Y_1(\omega)(\frac{\sqrt{\pi}L}{2})^{1/2} + \sum_{n=2}^{N} \zeta_n \varphi_n(\vec{x}) Y_n(\omega), \tag{5.2}$$

where

$$\zeta_n = (\sqrt{\pi}L)^{1/2} exp(\frac{-(\lfloor \frac{n}{2} \rfloor \pi L)^2}{8}), \quad \text{if} \quad n > 1,$$

$$\varphi_n(x) = \begin{cases} sin(\frac{\lfloor \frac{n}{2} \rfloor \pi \vec{x}}{L_p}) & \text{if} \quad n \text{ even}, \\ cos(\frac{\lfloor \frac{n}{2} \rfloor \pi \vec{x}}{L_p}) & \text{if} \quad n \text{ odd}. \end{cases}$$

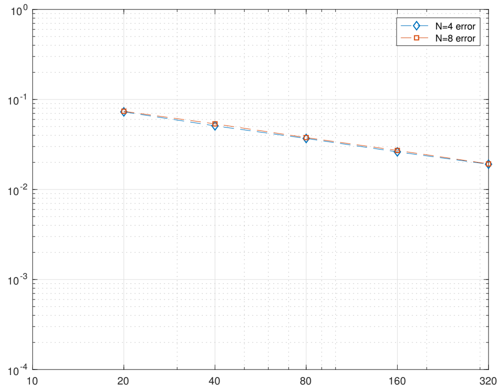
Here the random variables $\{Y_n(\omega)\}_{n=1}^{\infty}$ are independent, have zero mean and unit variance and are uniformly distributed in the interval $(-\sqrt{3}, \sqrt{3})$. In the following numerical test, for $\vec{x} \in (0, d)$, we take the desired physical correlation length $L_c = 1/64$ for the random field and the parameter $L_p = \max\{d, 2L_c\}$ and $L = L_c/L_p$.

We simulate the system over the time interval $[0, 1]$, and the uniform triangulation with mesh size $h = 1/32$ and uniform time partition with time step size $\triangle t = h$ are used. We generate a set of J random samples of $\mathcal{K}$ by the Monte Carlo sampling, and run our code for simulating the ensemble of the system associated with the J realizations. First, we need to check the rate of convergence with respect to the numbers of samples, J. As the exact solution to the stochastic Stokes-Darcy system is unknown, we take the ensemble mean of numerical solutions of $J_0 = 1000$ realizations as our exact solution (expectation), which is denoted by u_{J_0}. We also define u_h as the ensemble mean of J realizations. The numerical results with $J = 20, 40, 80, 160, 320$ realizations are listed in Table 4 and 5. Using linear regression, the errors for $d = 4$ (d is the dimension of random parameters sapce) in Table 4 satisfy

$$\|u_h - u_{J_0}\|_0 \approx 0.3051 J^{-0.4823}, \ \|\phi_h - \phi_{J_0}\|_0 \approx 1.7367 J^{-0.4845},$$

and the errors for $d = 8$ in Table 5

$$\|u_h - u_{J_0}\|_0 \approx 0.3189 J^{-0.4862}, \ \|\phi_h - \phi_{J_0}\|_0 \approx 2.3788 J^{-0.4812}.$$



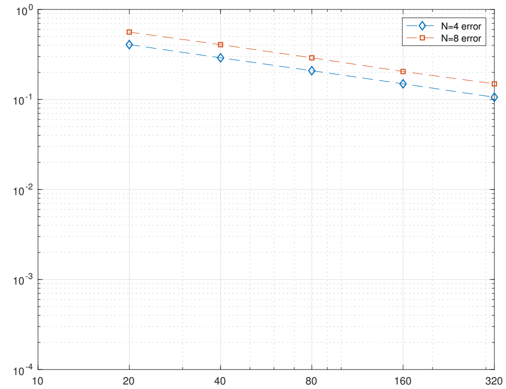


Figure 2: Ensemble simulations errors are $O(1/\sqrt{J})$ for u (left) and ϕ (right).

The values of $\|\cdot\|_0$ together with their linear regression models are plotted in Figure 2. It is seen that the rate of convergence with respect to J is close to -0.5.

Table 4: Errors of ensemble simulations for $d = 4$.

J	20	40	80	160	320
$\|u_h - u_{J_0}\|_0^E$	7.2759×10^{-2}	5.0880×10^{-2}	3.6869×10^{-2}	2.6149×10^{-2}	1.9078×10^{-2}
$\|\phi_h - \phi_{J_0}\|_0^E$	4.0653×10^{-1}	2.8996×10^{-1}	2.0861×10^{-1}	1.4911×10^{-1}	1.0575×10^{-1}

Table 5: Errors of ensemble simulations for $d = 8$.

J	20	40	80	160	320
$\|u_h - u_{J_0}\|_0^E$	7.3894×10^{-2}	5.3546×10^{-2}	3.7709×10^{-2}	2.7129×10^{-2}	1.9129×10^{-2}
$\|\phi_h - \phi_{J_0}\|_0^E$	5.6037×10^{-1}	4.0607×10^{-1}	2.9005×10^{-1}	2.0425×10^{-1}	14909×10^{-1}

5.3. Random hydraulic conductivity tensor with the sparse grid method

In this section, we present numerical results for incorporating our ensemble algorithm with the sparse grid method for approximating stochastic Stokes-Darcy equations with a random hydraulic conductivity tensor $\mathcal{K}(x, w)$. The sparse grid method was first introduced by Smolyak in 1963 [58], which constructs a multi-dimensional multilevel basis by a special truncation of the tensor product expansion of a one-dimensional multilevel basis. In this article, we follow the paper [28] by Heiss and Winschel and use Gaussian quadrature rule to construct the sparse grids. Details about the construction can be found in [28] and the corresponding open source sparse grid codes can be found in http://www.sparse-grids.de. This choice is only due to the simplicity of implementation of the available codes. There are more efficient sparse grid methods in the literature using nested quadrature rules. Interested readers are referred to [18, 17, 52, 60, 59, 63, 65] and an open source toolkit TASMANIAN (https://tasmanian.ornl.gov) for a collection of robust libraries for high dimensional integration and interpolation.

First, we take the ensemble mean of numerical solutions of $J_0 = 1000$ realizations from Monte Carlo method as our exact solution (expectation), which is denoted by u_{J_0}. By setting the finite dimensional probability space with $d = 4, 8$ and $\triangle t = h = 1/8, 1/16$, we compute the errors for SGEn method (the sparse grid method constructed by the Gaussian quadrature rule which will be exact for polynomial up to total order $2l-1$ (accuracy level is l) for d-dimensional integration+ ensemble algorithm) in Table 6-9. One can see that with only $J = 9$ and $J = 17$ for accuracy level $l = 2$ and $d = 4, 8$, we can get a good approximation of the expected value using the SGEn method. From Table 6-9, we also find that for different accuracy level l, SGEn method needs fewer nodes while it gets good approximation.

Table 6: Errors of ensemble simulations with sparse grid for different level $l = 2, 3, 4, 5$, d=4, $\triangle t = h = 1/8$

J	9	41	137	385
$\|\|u_h - u_{J_0}\|\|_0^E$	4.4871×10^{-3}	4.3589×10^{-3}	4.3612×10^{-3}	4.3611×10^{-3}
$\|\|\phi_h - \phi_{J_0}\|\|_0^E$	2.4265×10^{-2}	2.3286×10^{-2}	2.3303×10^{-2}	2.3303×10^{-2}
$\|\|p_h - p_{J_0}\|\|_0^E$	1.2783×10^{-2}	1.2281×10^{-2}	1.2289×10^{-2}	1.2289×10^{-3}

Table 7: Errors of ensemble simulations with sparse grid for different level $l = 2, 3, 4, 5$, d=8, $\triangle t = h = 1/8$

J	17	145	849	3905
$\|\|u_h - u_{J_0}\|\|_0^E$	4.8139×10^{-3}	4.6550×10^{-3}	4.6608×10^{-3}	4.6608×10^{-3}
$\|\|\phi_h - \phi_{J_0}\|\|_0^E$	4.4427×10^{-2}	3.0052×10^{-2}	3.0136×10^{-2}	3.0136×10^{-2}
$\|\|p_h - p_{J_0}\|\|_0^E$	2.9915×10^{-2}	2.5552×10^{-2}	2.5645×10^{-2}	2.5643×10^{-2}

Table 8: Errors of ensemble simulations with sparse grid for different level $l = 2, 3, 4, 5$, d=4, $\triangle t = h = 1/16$

J	9	41	137	385
$\|\|u_h - u_{J_0}\|\|_0^E$	1.0053×10^{-3}	9.9534×10^{-4}	9.9549×10^{-4}	9.9549×10^{-3}
$\|\|\phi_h - \phi_{J_0}\|\|_0^E$	5.1596×10^{-3}	5.1079×10^{-3}	5.1086×10^{-3}	5.1086×10^{-3}
$\|\|p_h - p_{J_0}\|\|_0^E$	2.7224×10^{-3}	2.7591×10^{-3}	2.7583×10^{-3}	2.7584×10^{-3}

Table 9: Errors of ensemble simulations with sparse grid for different level $l = 2, 3, 4, 5$, d=8, $\triangle t = h = 1/16$

J	17	145	849	3905
$\|\|u_h - u_{J_0}\|\|_0^E$	1.2441×10^{-3}	1.3240×10^{-3}	1.3188×10^{-3}	1.3189×10^{-3}
$\|\|\phi_h - \phi_{J_0}\|\|_0^E$	7.5489×10^{-3}	7.5844×10^{-3}	7.5623×10^{-3}	7.5631×10^{-3}
$\|\|p_h - p_{J_0}\|\|_0^E$	7.3982×10^{-3}	7.3292×10^{-3}	7.3612×10^{-3}	7.3622×10^{-3}

5.4. Applicational simulation

Next, we apply our second order ensemble algorithm to a simplified simulation of the subsurface flow in a karst aquifer. As shown in Fig. 3, the computational domain is a unit square divided into the porous media domain D_p and the free flow domain D_f. Let D_f be the polygon $\overline{ABCDEFGHIJ}$ where $A = (0,1), B = (0,3/4), C = (0,1/2), D = (1/4,1/2), E = (3/4,0), F = (1,0), G = (3/4,1/4), H = (1,1/4), I = (1,1/2)$, $J = (3/4,1/2)$ and $K = (1/2,3/4)$. Let $D_p = \Omega/D_f$, $S_0 = \overline{BC}$, $S_1 = \overline{EF}$, and $S_2 = \overline{HI}$.

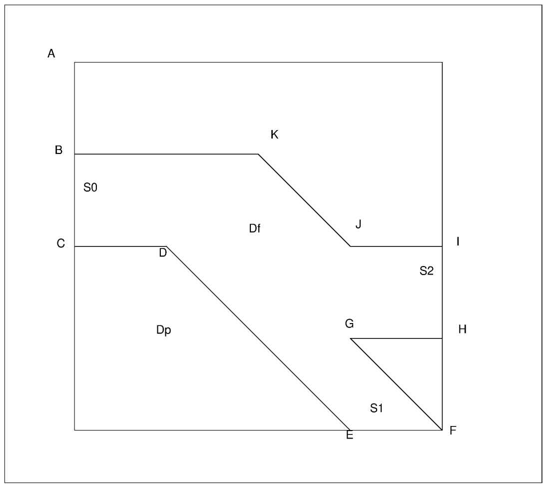


Figure 3: An illustration of the problem domain for the numerical experiment.

Set $T = 1$, $\alpha = 1$, $\nu = 1$, $g = 1$, $z = 0$. The boundary condition data and source terms are chosen to be 0 and let

$$u = \begin{cases} (U_0, 0)^T on \quad S_0 \\ (0, U_1)^T on \quad S_1 \\ (U_3, 0)^T on \quad S_2 \end{cases}$$

where U_i are constants. We subdivide Ω into rectangle of height and width $h = 1/M$, where M denotes a positive integer, and then subdivide each rectangle into two triangles by drawing a diagonal. For this numerical experiment, we choose $M = 16$ and $\triangle t = h$. In the following, we will provide the numerical results at $T = 1$ for the algorithm. We construct the random hydraulic conductivity tensor as follows

$$k(\vec{x}, \omega) = a_0 + exp\left\{ [Y_1(\omega)cos(\pi y) + Y_3(\omega)sin(\pi y)]\, e^{-\frac{1}{8}} + [Y_2(\omega)cos(\pi x) + Y_4(\omega)sin(\pi x)]\, e^{-\frac{1}{8}} \right\}.$$

where $\vec{x} = (x, y)^T, a_0 = 1/100$, and $Y_1, \ldots, Y_4$ are independent and identically distributed with zero mean and unit variance.

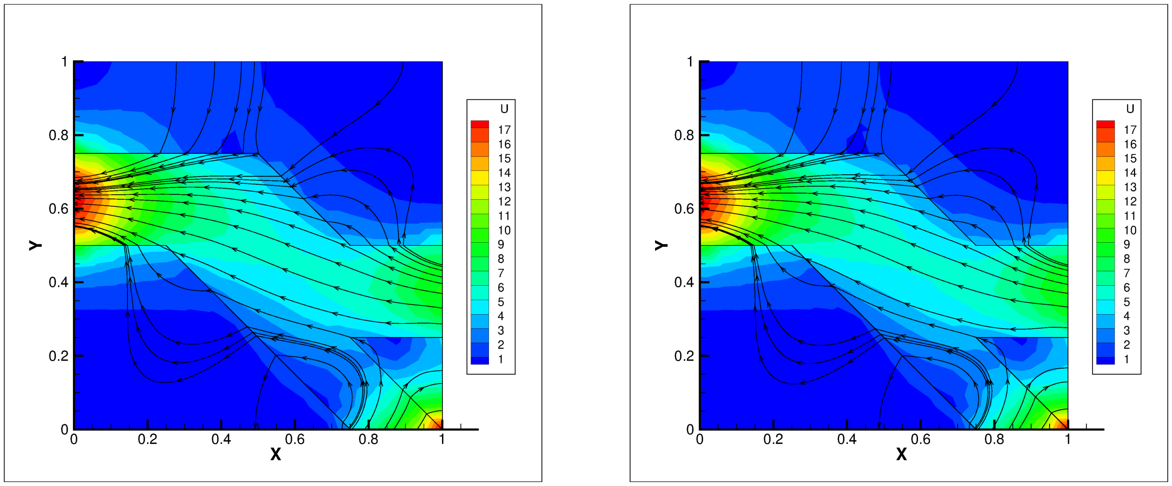


Figure 4: Plots of the ensemble mean for the higher order ensemble method (right) and traditional method (left) for $U_1 = -2$, $U_2 = -2$, and $U_0 = 2$ with $k = 10^{-2}$.

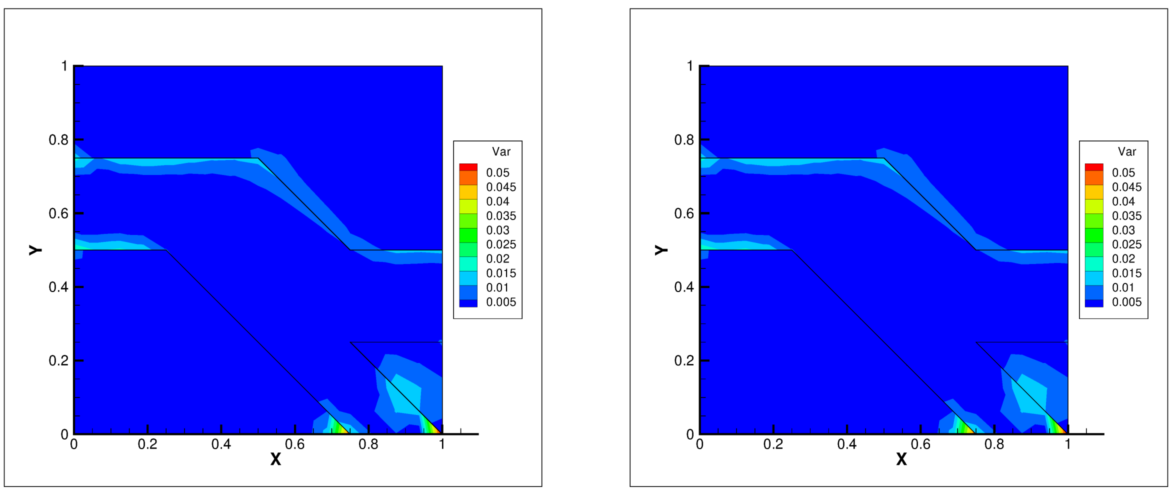


Figure 5: Plot of the variance of the higher order ensemble method (left) and traditional method (right) for $Q_1 = -2$, $Q_2 = -2$, and $Q_0 = 4$ with $k = 10^{-2}$.

In the test, we set $U_1 = U_2 = -2$ and $U_0 = 4$ so that the total inflow rate is equal to the total outflow rate. The two graphs in Figure 4 and Figure 5 illustrate the mean and variance of numerical solutions at the end time $T = 1$ for these tests. These phenomena are expected due to the chosen inflow and outflow rates for the conduit. Compared to the solutions of the traditional method, we can find they have the same general behavior of the flow while our second order ensemble algorithm is much more efficient. Furthermore, the proposed method also works well for the realistic parameter values,such as $k = 10^{-2}$ in the figures.

6. Conclusions

We developed a second order, efficient, decoupling ensemble algorithm for fast computation of multiple realizations of the stochastic Stokes-Darcy model. The ensemble algorithm decouples the original coupled system into two subproblems with two smaller linear systems to be solved at each time step. For all realizations, these two linear systems have the same coefficient matrix and one can use efficient iterative or direct solvers to greatly reduce the computational cost. We proved the algorithm is long time stable and second order in time convergent under a time-step condition and two parameter conditions. Several numerical experiments were presented to show the algorithm is second-order in time convergent and demonstrate its application in UQ applications by incorporating the ensemble algorithm with the Monte Carlo method and the sparse grid method.

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