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Free fermion six vertex model: symmetric functions and random domino tilings

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Abstract

Our work deals with symmetric rational functions and probabilistic models based on the fully inhomogeneous six vertex (ice type) model satisfying the free fermion condition. Two families of symmetric rational functions F_λ, G_λ are defined as certain partition functions of the six vertex model, with variables corresponding to row rapidities, and the labeling signatures $\lambda = (\lambda_1 \geq \ldots \geq \lambda_N) \in \mathbb{Z}^N$ encoding boundary conditions. These symmetric functions generalize Schur symmetric polynomials, as well as some of their variations, such as factorial and supersymmetric Schur polynomials. Cauchy type summation identities for F_λ, G_λ and their skew counterparts follow from the Yang–Baxter equation. Using algebraic Bethe Ansatz, we obtain a double alternant type formula for F_λ and a Sergeev–Pragacz type formula for G_λ. In the spirit of the theory of Schur processes, we define probability measures on sequences of signatures with probability weights proportional to products of our symmetric functions. We show that these measures can be viewed as determinantal point processes, and we express their correlation kernels in a double contour integral form. We present two proofs: The first is a direct computation of Eynard–Mehta type, and the second uses non-standard, inhomogeneous versions of fermionic operators in a Fock space coming from the algebraic Bethe Ansatz for the six vertex model. We also interpret our determinantal processes as random domino tilings of a half-strip with inhomogeneous domino weights. In the bulk, we show that the lattice asymptotic behavior of such domino tilings is described by a new determinantal point process on $\mathbb{Z}^2$, which can be viewed as an doubly-inhomogeneous generalization of the extended discrete sine process.

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1 Introduction

1.1 Preface

Determinantal random point processes (or *fields*) originated in random matrix theory in the 1960s and were first singled out as a class by Macchi in 1975 [71] under the name *fermion point processes*. The book of Anderson–Guionnet–Zeitouni [3, Section 4.6] provides a brief historical summary of the random matrix origins. The term *determinantal* was adopted around the year 2000, see Borodin–Olshanski [14] and Soshnikov [96]. By now there are quite a few surveys discussing various aspects of determinantal processes by Soshnikov [96], Lyons [70], Johansson [54], König [61], Hough–Krishnapur–Peres–Virag [50], Borodin [17], Kulesza–Taskar [67], and Decreusefond–Flint–Privault–Torrisi [27].

One of the most important determinantal processes is the *sine process* that goes back to Mehta–Gaudin [75] and Dyson [29]. It describes universal bulk[1] asymptotics of large determinantal systems in one space dimension, see, e.g., Yau [99] for a historical overview. For one-dimensional discrete point processes (i.e., random subsets of $\mathbb{Z}$), the corresponding universal object is the *discrete sine process* introduced by Borodin–Okounkov–Olshanski [15].

Besides being universal bulk limits in one dimension, both the continuous and the discrete sine processes admit natural extensions to two dimensions arising, respectively, from the Dyson Brownian motion, see Dyson [29], Nagao–Forrester [84], and

[1] The term “bulk” refers to the parts of the system where the space can be rescaled to form growing regions with unit particle density.

random plane partitions, see Okounkov–Reshetikhin [90], or more general dimer models, cf. Kenyon–Okounkov–Sheffield [65] and Johansson [53]. In the discrete case (which is the focus of the present paper), the *two-dimensional* (also called *extended*) *sine process* admits a natural description as a unique (cf. Sheffield [95]) translation invariant ergodic Gibbs measure on point configurations in $\mathbb{Z}^2$ of a given *slope*. The slope consists of two real or a single complex parameter that encodes the particles' densities along the two coordinate directions. In this case the Gibbs property means that the probability law of the random configuration is invariant under uniform resampling in any finite window, conditioned on the configuration on the boundary of this window. The fact that such a rich family of Gibbs measures in $\mathbb{Z}^2$ enjoys a completely explicit determinantal description of their correlations is remarkable and very rare.

The main probabilistic outcome of the present work is the introduction of a wide class of new *inhomogeneous deformations* of the extended discrete sine process. These deformations are determinantal point processes on $\mathbb{Z}^2$ with very explicit correlation kernels that depend, in addition to the complex slope parameter, on four bi-infinite sequences of real parameters associated with the horizontal and the vertical coordinate directions (two sequences per each direction). They seem to be out of reach of existing approaches to deformations of the extended sine processes such as various versions of the Schur processes, fermionic Fock space formalism with the Boson–Fermion correspondence, random matrix type ensembles, or periodic dimer models.

Free parameters varying by rows and columns is a salient feature of integrable lattice models, and those are indeed behind our construction. More precisely, we start with the *free fermion six vertex model*, show that it is described by determinantal (fermion) point processes, and in a bulk limit obtain the inhomogeneous deformations of the extended discrete sine process.

The six vertex model, first introduced as a two-dimensional model for residual entropy of water ice by Pauling in 1935 [91], is a classical model in statistical mechanics that gave birth to the domain of integrable (exactly solvable) lattice models; see the book of Baxter [5] for an introduction, and also Reshetikhin [94] for a more recent survey of the six vertex model. Integrable lattice models is a vast domain, and the present work belongs to a subdomain dealing with symmetric functions and associated stochastic systems.

The theory of symmetric functions, a classical introduction to which is Macdonald's book [73], studies remarkable families of symmetric and associated nonsymmetric polynomials with origins in diverse areas of group theory, combinatorics, representation theory, noncommutative harmonic analysis, probability, and mathematical physics. There are many works highlighting connections between symmetric functions and integrable vertex models; for some of the earlier papers see Kirillov–Reshetikhin [66], Fomin–Kirillov [36, 37], Lascoux–Leclerc–Thibon [69], Gleizer–Postnikov [44], Tsilevich [97], Lascoux [68], Zinn–Justin [102], Brubaker–Bump–Friedberg [8], Bump–McNamara–Nakasuji [13], and Korff [63].

We focus on the (asymmetric) six vertex model with vertex weights $a_1, a_2, b_1, b_2, c_1, c_2$ satisfying the *free fermion condition* $a_1a_2 + b_1b_2 = c_1c_2$. This condition corresponds to the vanishing of the quantity Δ associated to the model. See the references in Baxter [5, Ch. 8.10.III], and also Felderhof [33–35] for earlier works on the free fermion six vertex model.

We consider the six vertex model in which the free fermion condition holds at each lattice site, but otherwise the weights are fully inhomogeneous and are determined by the parameters (x_i, r_i) and (y_j, s_j) which are constant along the lattice rows and columns, respectively. The x's and y's are known as the *rapidities*, while the r's and s's are the *spin parameters*. This particular parametrization ensures that the vertex weights satisfy a version of the *Yang–Baxter equation*, which is a key algebraic property powering our results. We mainly employ the Yang–Baxter equation in the form of quadratic relations for the row operators A, B, C, D that are standard in the Algebraic Bethe Ansatz, cf. Faddeev [32], Korepin–Bogoliubov–Izergin [56, Part VII].

The structure of integrable lattice models (in particular, in our six vertex model) is very special as it is powered by connections to quantum groups. Quantum groups are deformations of universal enveloping algebras of classical Lie groups (and their generalizations), which possess certain additional structure, in particular, R-matrices satisfying the Yang–Baxter equation. In this language, the free fermion six vertex weights correspond to the R-matrix of the rank 1 quantum affine superalgebra $U_q(\widehat{\mathfrak{sl}}(1|1))$. A recent paper [1] by a subset of the authors presented a detailed study of symmetric functions related to the higher rank quantum affine superalgebras $U_q(\widehat{\mathfrak{sl}}(1|n))$ with $n > 1$. In that case fermions are no longer 'free', and most of the theory differs substantially. In particular, the results and proofs in the present work are largely independent from those of [1], although we do point to connections in a few places where those exist.

We define two families of functions F_λ, G_λ indexed by integer tuples $\lambda = (\lambda_1 \geq \ldots \geq \lambda_N \geq 0)$ as certain partition functions of the free fermion six vertex model with boundary conditions depending on λ. The functions F_λ, G_λ are rational in (a finite, λ-dependent subset of) the parameters x_i, r_i, y_j, s_j. Up to a simple product factor in F_λ, both the functions are symmetric with respect to simultaneous permutations of the row variables (x_i, r_i), which is a consequence of the Yang–Baxter equation. When the horizontal parameters y_j, s_j do not depend on j, the functions F_λ and G_λ reduce, respectively, to the ordinary symmetric Schur polynomials and the supersymmetric Schur polynomials. In another specialization of the parameters, the functions F_λ become the factorial Schur polynomials (cf. Molev [78, 101]).

We establish the following results:

- Cauchy type summation identities leading to a product form expression for $\sum_\lambda F_\lambda G_\lambda$, and their skew analogues.
- Torus biorthogonality of the functions F_λ and certain dual functions F^*_λ, with integration over the row rapidities x_j.
- A double alternant type formula for F_λ, and a Jacobi–Trudy type determinantal formula for G_λ.
- Another explicit formula for G_λ involving a summation over pairs of permutations which resembles (but does not imply) the Sergeev–Pragacz formula for the supersymmetric Schur polynomials (cf. Hamel–Goulden [49, (5)]).

By analogy with the Schur processes of [90], we define probability measures (called *FG measures*) on two-dimensional integer arrays encoded by sequences $\lambda^{(1)}, \ldots, \lambda^{(T)}$. Under an FG measure, the probability weights are expressed through the functions F_λ, G_λ and their skew analogues. Thanks to the Cauchy type summation identities,

$\lambda^{(j)}$-marginals have weights proportional to $F_{\lambda^{(j)}} G_{\lambda^{(j)}}$ for certain specializations of F and G that vary with $j = 1, \ldots, T$. We interpret the FG measures as certain ensembles of random domino tilings of a half-strip, in which the domino weights are inhomogeneous and depend on the parameters (x_i, r_i) and (y_j, s_j) varying in the two coordinate directions.

We show that the FG measures (and the corresponding random domino tilings) are *determinantal*. Namely, the random point configuration

$$\mathcal{S}^{(T)} = \{(t, \lambda_i^{(t)} + N + 1 - i) \colon 1 \le t \le T,\ 1 \le i \le N\} \subset \{1, \ldots, T\} \times \mathbb{Z}_{\ge 1}$$

has all correlation functions $\mathbb{P}[A \subseteq \mathcal{S}^{(T)}]$ (where A is finite) expressed as symmetric $|A| \times |A|$ determinants of a certain function $K(t, a; t', a')$ called the *correlation kernel*. We write K as a double contour integral which resembles (yet does not coincide with) some determinantal correlation kernels of multilevel $\beta = 2$ random matrix ensembles.

Our kernel K generalizes that of the Schur process first obtained in [90] via a vertex operator formalism in the fermionic Fock space. We obtain our double contour integral formula for K by employing an 'inhomogeneous version' of the Fock space. In particular, we establish an inhomogeneous analogue of the Boson–Fermion correspondence (cf. Kac [55, Theorem 14.10] for the homogeneous statement), which may be of independent interest. The fermionic operators in our Fock space arise as combinations of (doubly) infinite volume limits of the Algebraic Bethe Ansatz row operators A, B, C, D evaluated at certain special parameter values. We realize the commutation relations for the inhomogeneous fermionic operators, as well as the inhomogeneous Boson–Fermion correspondence, as consequences of the Yang–Baxter equation.

The double contour integral form of the correlation kernel K of the FG measures is well-suited for the asymptotic analysis in the bulk of the system by the method of steepest descent. Such analysis leads us to the generalization of the extended discrete sine kernel that was mentioned above.

Having outlined our main results, let us now proceed to describing them in greater detail.

1.2 Symmetric rational functions

Let $\lambda = (\lambda_1 \ge \ldots \ge \lambda_N \ge 0)$, $\lambda_i \in \mathbb{Z}$, be a nonincreasing integer sequence, which we call a *signature* with N parts. Central objects considered in the present work are families of rational functions $F_\lambda(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s})$ and $G_\lambda(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s})$ indexed by signatures λ. These functions depend on four sequences of (generally speaking, complex) parameters

$$\mathbf{x} = (x_1, \ldots, x_k), \qquad \mathbf{r} = (r_1, \ldots, r_k), \qquad \mathbf{y} = (y_1, y_2, \ldots), \qquad \mathbf{s} = (s_1, s_2, \ldots). \tag{1.1}$$

The functions F_λ, G_λ are defined as partition functions of the *free fermion six vertex model*. By a partition function we mean the sum of weights of all configurations of the six vertex model with given boundary conditions depending on λ, where the weight

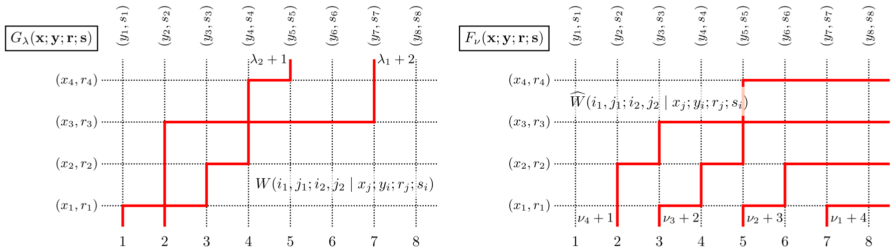
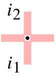
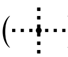
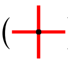
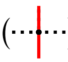
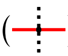
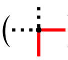


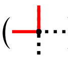
Fig. 1 Left: An example of a six vertex model configuration contributing to G_λ. The number N of parts in λ and the number k of the variables $\mathbf{x}$, $\mathbf{r}$ (1.1) may differ (here $k = 4$, $N = 2$). The boundary conditions on the left and right are empty, are $\{N, N-1, \dots, 1\}$ at the bottom, and are $\{\lambda_1+N, \lambda_2+N-1, \dots, \lambda_N+1\}$ at the top. Right: An example of a configuration contributing to F_ν. In contrast with G_λ, the number of parts in ν must be equal to k (here $k = 4$). The boundary conditions are empty on the left and at the top, fully packed on the right, and are $\{\nu_1+k, \nu_2+k-1, \dots, \nu_k+1\}$ at the bottom

of each particular configuration is equal to the product of local single-vertex weights $w_{6\text{V}}$(), where $i_1, j_1, i_2, j_2 \in \{0,1\}$, depending on the parameters x, y, r, s. The parameters x, y, r, s, in their turn, depend on the lattice coordinates of the vertex. For the definition of G_λ we take the vertex weights $w_{6\text{V}} = W$ given by

W() $= a_1 = 1$, $\quad W$() $= a_2 = \dfrac{r^{-2}x - y}{s^{-2}y - x}$,

W() $= b_1 = \dfrac{s^{-2}y - r^{-2}x}{s^{-2}y - x}$,

W() $= b_2 = \dfrac{y - x}{s^{-2}y - x}$, $\quad W$() $= c_1 = \dfrac{x(r^{-2}-1)}{s^{-2}y - x}$, (1.2)

W() $= c_2 = \dfrac{y(s^{-2}-1)}{s^{-2}y - x}$

(notation $a_1, a_2, b_1, b_2, c_1, c_2$ is the classical convention in the six vertex model weights, see, e.g., [5, 94, Ch. 8]). The functions F_λ involve the renormalized weights

$$\widehat{W}(i_1, j_1; i_2, j_2) := \frac{W(i_1, j_1; i_2, j_2)}{W(0, 1; 0, 1)}.$$

This normalization is chosen so that $\widehat{W}(0, 1; 0, 1) = 1$. One readily sees that each of the families of vertex weights W and $\widehat{W}$ satisfies the *free fermion condition* $a_1a_2 + b_1b_2 - c_1c_2 = 0$. The free fermion condition is crucial throughout our work.

Having the vertex weights, we form partition functions in the half-infinite strip $\mathbb{Z}_{\geq 1} \times \{1, \dots, k\}$ (where k is the number of variables in $\mathbf{x}$, $\mathbf{r}$ (1.1)) as in Fig. 1, and call them $G_\lambda(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s})$ and $F_\nu(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s})$.

Remark 1.1 We also define skew functions $G_{\lambda/\mu}(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s})$ and $F_{\nu/\varkappa}(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s})$ as partition functions. Namely, for $G_{\lambda/\mu}$, the signature μ encodes the bottom boundary in Fig. 1, left, so that we have $G_\lambda = G_{\lambda/(0,0,\ldots,0)}$. For $F_{\nu/\varkappa}$, the signature $\varkappa$ encodes the top boundary in Fig. 1, right, so that $F_\nu = F_{\nu/\varnothing}$. See Sect. 3.2 in the text for detailed definitions of all these functions. For brevity, in the Introduction we mostly stick to the non-skew functions.

We have normalized the weights W and $\widehat{W}$ so that the vertices occurring infinitely many times in Fig. 1 have weight 1. Therefore, the weights of individual six vertex model configurations are well-defined. Moreover, both partition functions G_λ and F_ν involve only finitely many such configurations, so there are no convergence issues. We see that $F_\nu(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s})$ and $G_\lambda(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s})$ are rational functions in (a finite subset of) the parameters (1.1).

In Sect. 4 we consider particular cases of the parameters $\mathbf{x}, \mathbf{y}, \mathbf{r}, \mathbf{s}$ under which the functions F_λ, G_λ become either the ordinary Schur symmetric polynomials [73, I.3], or their factorial or supersymmetric variations [21, 72, 78]. Here let us formulate the supersymmetric setting.

Proposition 1.2 (Proposition 4.10 in the text) *Take the horizontally homogeneous specialization $y_j = y$ and $s_j = s$ for all $j \geq 1$. Then for any signature $\lambda = (\lambda_1 \geq \ldots \geq \lambda_N \geq 0)$ we have*

$$\frac{F_\lambda(x_1,\ldots,x_N;\mathbf{y};\mathbf{r};\mathbf{s})}{F_{(0,0,\ldots,0)}(x_1,\ldots,x_N;\mathbf{y};\mathbf{r};\mathbf{s})} = s_\lambda\left(\frac{1-s^2x_1}{s^2(1-x_1)},\ldots,\frac{1-s^2x_N}{s^2(1-x_N)}\right),$$

where s_λ is the ordinary Schur symmetric polynomial [73, I.3]. *Moreover,*

$$\begin{aligned}&G_\lambda(x_1,\ldots,x_M;\mathbf{y};\mathbf{r};\mathbf{s})\\&= s_\lambda\left(\left\{\frac{s^2(1-x_j)}{1-s^2x_j}\right\}_{j=1}^{M} \Big/ \left\{\frac{s^2(x_jr_j^{-2}-1)}{1-s^2r_j^{-2}x_j}\right\}_{j=1}^{M}\right)\prod_{i=1}^{M}\left(\frac{1-s^2r_i^{-2}x_i}{1-s^2x_i}\right)^N,\end{aligned}$$

where $s_\lambda(\cdots/\cdots)$ denotes the supersymmetric Schur function [21], [72, (6.19)].

Thus, one may view our functions F_λ, G_λ as generalizations of various Schur-like symmetric functions, based on the inhomogeneous parameters y_j, s_j. In fact, many of the properties of F_λ, G_λ discussed in the rest of this subsection resemble the ones of the ordinary Schur polynomials.

The concrete parametrization of the vertex weights $W, \widehat{W}$ by x, y, r, s is chosen so that the weights satisfy the Yang–Baxter equation with the cross vertex weights independent of (y_i, s_i). These cross vertex weights are given in Fig. 5, and we refer to Sect. 2.2 in the text for a detailed formulation of the Yang–Baxter equation. In particular, the Yang–Baxter equation implies (see Proposition 3.5 in the text) that the functions

$$\begin{aligned}&G_\lambda(x_1,\ldots,x_k;\mathbf{y};r_1,\ldots,r_k;\mathbf{s}) \quad \text{and}\\&F_\nu(x_1,\ldots,x_k;\mathbf{y};r_1,\ldots,r_k;\mathbf{s})\prod_{1\leq i<j\leq k}(x_i - r_j^{-2}x_j)\end{aligned}$$

are symmetric under simultaneous permutations of the pairs of variables (x_i, r_j).

Another application of the Yang–Baxter equation (together with an explicit formula for F_λ from Theorem 1.5 below) brings the following *Cauchy type summation identity*:

Theorem 1.3 (Theorem 3.8 in the text) *Fix integers $N, k \geq 1$ and sets of complex variables $\mathbf{x} = (x_1, \dots, x_N)$, $\mathbf{r} = (r_1, \dots, r_N)$, $\mathbf{w} = (w_1, \dots, w_k)$, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$, and $\mathbf{y} = (y_1, y_2, \dots,)$, $\mathbf{s} = (s_1, s_2, \dots)$, satisfying*

$$\sup_{p\geq 1} \left| \frac{s_p^{-2} y_p - x_i}{y_p - x_i} \frac{y_p - w_j}{s_p^{-2} y_p - w_j} \right| < 1 \qquad \text{forall } 1 \leq i \leq N, 1 \leq j \leq k. \tag{1.3}$$

Then we have

$$\sum_{\lambda=(\lambda_1\geq\ldots\geq\lambda_N\geq 0)} G_\lambda(\mathbf{w}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s})\, F_\lambda(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s}) = \frac{\prod_{1\leq i\leq j\leq N}(r_i^{-2} x_i - x_j) \prod_{1\leq i<j\leq N}(s_i^{-2} y_i - y_j)}{\prod_{i,j=1}^N (y_i - x_j)} \prod_{i=1}^N \prod_{j=1}^k \frac{x_i - \theta_j^{-2} w_j}{x_i - w_j}. \tag{1.4}$$

Remark 1.4 An example of a fully inhomogeneous situation when condition (1.3) holds is $y_p = 1 - 2^{-p}$, $s_p = 1 + 2^p$, $p \geq 1$, and $\frac{1}{2} > w_j > x_i > \frac{1}{3}$ for all i, j.

Let us now discuss explicit formulas for the functions F_λ and G_λ. The first function F_λ possesses an inhomogeneous analogue of the *double alternant formula* for the Schur symmetric polynomials [73, I.(3.1)]. Define inhomogeneous analogues of the power functions by

$$\varphi_k(x \mid \mathbf{y}; \mathbf{s}) := \frac{1}{y_{k+1} - x} \prod_{j=1}^k \frac{x - s_j^{-2} y_j}{x - y_j}, \qquad k \geq 0.$$

Theorem 1.5 (Theorem 3.9 in the text) *Let λ be a signature with N parts. Then*

$$F_\lambda(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s}) = \frac{\prod_{1\leq i\leq j\leq N}(r_i^{-2} x_i - x_j)}{\prod_{1\leq i<j\leq N}(x_i - x_j)} \det \left[\varphi_{\lambda_j+N-j}(x_i \mid \mathbf{y}; \mathbf{s}) \right]_{i,j=1}^N. \tag{1.5}$$

We also obtain an explicit formula for $G_\lambda(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s})$, see Theorem 3.10 in the text. It involves summation over pairs of permutations which resembles (however, does not imply, cf. Remark 4.15) the Sergeev–Pragacz formula [49, (5)] for the supersymmetric Schur polynomials. Our formula in Theorem 3.10 is quite long so we do not reproduce it here.

We prove explicit formulas for F_λ and G_λ in Appendix A via computations with row operators (for these operators, see Sect. 1.6 below and Sect. 2.3 in the text). These computations follow [19, Section 4.5] (but are much more involved in the case of

G_λ) and are based on Algebraic Bethe Ansatz for quantum integrable systems, see, e.g., [56, Part VII]. This approach can ultimately be traced to our central tool, the Yang–Baxter equation, whose repeated application yields quadratic relations for row operators.

Remark 1.6 The inhomogeneous free fermion six vertex weights like (1.2) appeared (with a different parametrization) in [79]. Moreover, in that paper a determinantal formula like (1.5) for a partition function with F_λ-like boundary conditions was proven. This was done by an Izergin–Korepin approach, that is, by showing that both the partition function and the right-hand side of (1.5) satisfy the same list of properties which uniquely determine a function.

Along with the Sergeev–Pragacz type formula, G_λ admits another explicit expression based on the Cauchy identity and the *inhomogeneous biorthogonality* associated with the functions F_λ. Here we present a single-variable version of this biorthogonality, see Proposition 5.4 in the text for a multivariable statement involving determinants. Define

$$\psi_k(x \mid \mathbf{y}; \mathbf{s}) := \frac{y_{k+1}(s_{k+1}^{-2}-1)}{x - s_{k+1}^{-2}y_{k+1}} \prod_{j=1}^{k} \frac{x-y_j}{x-s_j^{-2}y_j}, \qquad k \geq 0.$$

Then we have for all $k, l \geq 0$ (Lemma 5.1 below):

$$\frac{1}{2\pi\mathbf{i}} \oint_\gamma \varphi_k(z \mid \mathbf{y}, \mathbf{s})\, \psi_l(z \mid \mathbf{y}, \mathbf{s})\, dz = \begin{cases} 1, & k = l; \\ 0, & k \neq l, \end{cases} \tag{1.6}$$

where the simple closed contour γ separates the sets $\{y_j\}_{j\geq 1}$ and $\{s_j^{-2}y_j\}_{j\geq 1}$ and goes around the y_j's in the positive direction.

Using (1.6), we can extract G_λ as the coefficient by F_λ from the right-hand side of the Cauchy identity (1.4). This leads to the following *Jacobi–Trudy type formula*:

Proposition 1.7 (Proposition 5.10 in the text) *Let λ be a signature with N parts. Then we have*

$$G_\lambda(\mathbf{w}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s}) = \prod_{1\leq i<j\leq N} \frac{s_i^{-2}y_i - y_j}{y_j - y_i} \det\left[h_{\lambda_i+N-i,\, j}(\mathbf{w}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s})\right]_{i,j=1}^N,$$

where

$$h_{k,m}(\mathbf{w}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s}) = \frac{1}{2\pi\mathbf{i}} \oint_{\gamma'} dz\, \frac{\psi_k(z \mid \mathbf{y}; \mathbf{s})}{y_m - z} \prod_{j=1}^{M} \frac{z - \theta_j^{-2}w_j}{z - w_j}.$$

Here the positively oriented integration contour γ' surrounds all the points y_j, w_i and leaves out all the points $s_j^{-2}y_j$.

In (1.6) and Proposition 1.7 we assume that the parameters are chosen in such a way that the integration contours γ and γ' exist.

1.3 Determinantal processes

Dividing the Cauchy identity of Theorem 1.3 by its right-hand side, we define a probability measure on the space of signatures with N parts which we call an *FG measure*:

$$\mathcal{M}(\lambda) := \frac{1}{Z} F_\lambda(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s})\, G_\lambda(\mathbf{w}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s}), \tag{1.7}$$

where Z is the normalizing constant given by the right-hand side of (1.4), and $\mathbf{x} = (x_1, \dots, x_N)$, $\mathbf{r} = (r_1, \dots, r_N)$, $\mathbf{w} = (w_1, \dots, w_k)$, and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$. This definition is analogous to that of Schur measures introduced in [86]. Further, by analogy with Schur processes [90] and Macdonald processes [9], we define (*ascending*) *FG processes* which are probability measures on sequences of signatures $\lambda^{(1)}, \dots, \lambda^{(T)}$ (each with N parts) defined as

$$\begin{aligned}\mathcal{AP}(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(T)}) = \frac{1}{Z}\, & G_{\lambda^{(1)}}(w_1; \mathbf{y}; \theta_1; \mathbf{s}) \dots G_{\lambda^{(T)}/\lambda^{(T-1)}}(w_T; \mathbf{y}; \theta_T; \mathbf{s}) \\ & F_{\lambda^{(T)}}(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s}).\end{aligned} \tag{1.8}$$

Here Z is the same normalizing constant (the right-hand side of (1.4)), and $G_{\lambda/\mu}$ are skew versions of the functions G_λ (see Remark 1.1 above, or Definition 3.2 below). For any fixed t, the marginal distribution of $\lambda^{(t)}$ under (1.8) is the FG measure (1.7) with the same $\mathbf{x}, \mathbf{r}, \mathbf{y}, \mathbf{s}$, and with $\mathbf{w} = (w_1, \dots, w_t)$, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_t)$.

Sufficient conditions under which formulas (1.7) and (1.8) define probability distributions with nonnegative probability weights are (1.3) (so that the probability weights are normalizable, i.e., the series for Z converges) and

$$x_i < y_j < r_i^{-2} x_i < s_j^{-2} y_j \quad \text{and} \quad w_i < y_j < \theta_i^{-2} w_i < s_j^{-2} y_j \quad \text{forall } i, j.$$

The latter conditions imply that all vertex weights $W(i_1, j_1; i_2, j_2)$, $\widehat{W}(i_1, j_1; i_2, j_2)$ are nonnegative, hence F_λ, G_λ and the $G_{\lambda/\mu}$'s are nonnegative, too.

We show that the probability measure $\mathcal{AP}(\lambda^{(1)}, \dots, \lambda^{(T)})$ gives rise to a *determinantal point process*, which also implies determinantal structure for the measure $\mathcal{M}$ (1.7). We refer to [17, 50, 96] for generalities on determinantal processes. Let us adapt general definitions to our setting. Let $(\lambda^{(1)}, \dots, \lambda^{(T)})$ be a sequence of random signatures with joint distribution (1.8), and define a random point configuration

$$\mathcal{S}^{(T)} := \bigcup_{t=1}^{T} \left\{(t, \lambda_1^{(t)} + N), (t, \lambda_2^{(t)} + N - 1), \dots, (t, \lambda_N^{(t)} + 1)\right\} \subset \{1, \dots, T\} \times \mathbb{Z}_{\geq 1}.$$

Let $A \subset \{1, \dots, T\} \times \mathbb{Z}_{\geq 1}$ be a fixed finite subset. A *correlation function* associated with A is, by definition, the probability $\mathbb{P}_{\mathcal{AP}}[A \subset \mathcal{S}^{(T)}]$. We show that this correlation function, for any A, is given by an $|A| \times |A|$ determinant of a fixed *correlation kernel* defined as (here $1 \le t, t' \le T$ and $a, a' \ge 1$):

$$K_{\mathcal{AP}}(t,a;t',a') = \frac{1}{(2\pi\mathbf{i})^2}\oint_{\Gamma_{y,\theta^{-2}w}} du \oint_{\Gamma_{y,w}} dv \, \frac{1}{u-v} \prod_{k=1}^{N} \frac{(u-y_k)(v-x_k)}{(u-x_k)(v-y_k)} \times \frac{y_a(1-s_a^{-2})}{v-s_a^{-2}y_a} \frac{1}{u-y_{a'}} \prod_{j=1}^{a-1} \frac{v-y_j}{v-s_j^{-2}y_j} \prod_{j=1}^{a'-1} \frac{u-s_j^{-2}y_j}{u-y_j} \prod_{d=1}^{t} \frac{v-\theta_d^{-2}w_d}{v-w_d} \prod_{c=1}^{t'} \frac{u-w_c}{u-\theta_c^{-2}w_c}, \tag{1.9}$$

where the integration contours are positively oriented circles one inside the other (the u contour is outside for $t \le t'$ while the v contour is outside for $t > t'$); the u contour encircles all the points $y_i, \theta_j^{-2}w_j$, and not x_k; and the v contour encircles all the points y_i, w_j, and not $s_k^{-2}y_k$. Here we assume that the parameters are such that the contours exist.

Theorem 1.8 (Theorem 6.7 in the text) *The ascending FG process (1.8) is determinantal with the kernel $K_{\mathcal{AP}}$. That is, for any $A = \{(t_1,a_1),\ldots,(t_m,a_m)\} \subset \{1,\ldots,T\} \times \mathbb{Z}_{\ge 1}$, we have*

$$\mathbb{P}_{\mathcal{AP}}\big[A \subset \mathcal{S}^{(T)}\big] = \det\big[K_{\mathcal{AP}}(t_i,a_i;t_j,a_j)\big]_{i,j=1}^{m}.$$

When $t = t'$, the kernel (1.8) becomes the correlation kernel $K_{\mathcal{M}}(a,a') = K_{\mathcal{AP}}(t,a;t,a')$ for the FG measure $\mathcal{M}(\lambda)$ (1.7), where $\lambda = \lambda^{(t)}$ and $\mathbf{w} = (w_1,\ldots,w_t)$, $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_t)$.

We give two proofs of Theorem 1.8. The first proof (presented in Appendix B) uses an Eynard–Mehta type approach based on [20], see also [31]. This approach is parallel to how the kernel for the Schur measures is computed in [20]. The second proof, presented in Sects. 7 and 8, is based on fermionic operators in a Fock space coming from the Algebraic Bethe Ansatz row operators. We discuss the main features of the second approach in Sect. 1.6 below.

In the horizontally homogeneous case $y_j = y, s_j = s$ for all $j \ge 1$, the correlation kernel $K_{\mathcal{M}}(a,a')$ turns into the kernel for a certain Schur measure, see Sect. 6.4 in the text. A certain inhomogeneous analogue of Schur processes (describing continuous time particle dynamics in inhomogeneous space generalizing the push-block process from [12]) was defined recently in [4]. It is likely that the probability measures of [4] could arise as degenerations of our FG processes, but we do not address this question here.

1.4 Random domino tilings

We interpret ascending FG processes (1.8) as random tilings by 1×2 dominoes of an infinite half-strip, in the spirit of the steep tiling representation of Schur processes

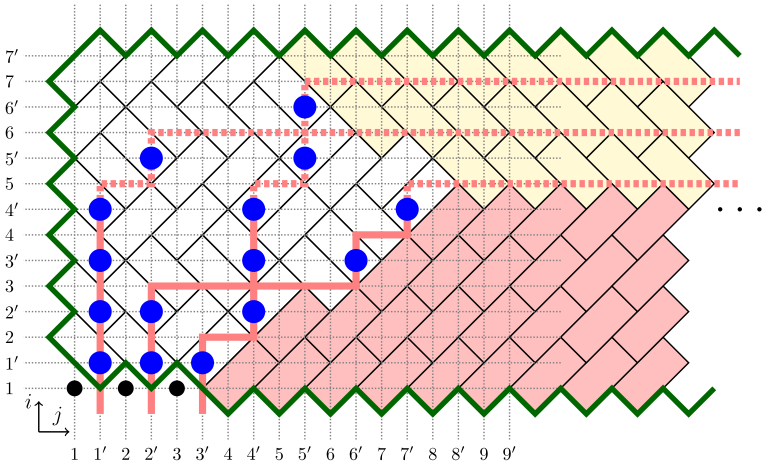


Fig. 2 An example of a domino tiling corresponding to the ascending FG process with $T = 4$, $N = 3$. The dominoes which repeat infinitely many times far to the right are shaded. The large blue dots are particles associated to the tiling, i.e., the centers of the bottom squares of dominoes which have coordinates (l', k'). They are used to identify the domino tiling with a sequence of signatures $\lambda^{(1)}, \ldots, \lambda^{(T)}$. We also display the path ensembles leading to the partition functions $G_{\lambda^{(T)}}$ (in the bottom part) and $F_{\lambda^{(T)}}$ (dashed paths in the top part)

[10]. (The connection between states of the free fermion six vertex model and random domino tilings has been long known before, cf. Elkies–Kuperberg–Larsen–Propp [30], Zinn–Justin [100], Ferrari–Spohn [40].) While our boundary conditions are not as general as those in arbitrary steep tilings in the cited work, we are able to consider dominoes with more general weights which depend on the many parameters of the FG process.

Recall that the ascending FG process is associated with two integers, N and T. Let the coordinates in the $\mathbb{Z}^2$ plane be numbered as $0' < 1 < 1' < 2 < 2' < \ldots$. Consider the infinite half-strip with vertical coordinates between $0'$ and $N + T + 1$, and with N unit squares removed from the bottom left, see Fig. 2. We consider domino tilings of this strip such that far to the right the dominoes stabilize to regular brick layers of two different directions, northeast and southeast in the regions $i \le T'$ and $i \ge T + 1$, respectively.

Let us explain how a given domino tiling corresponds to a sequence $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(T)}$ of signatures, each with N parts. Single out the dominoes for which the center of the bottom unit square has coordinates of the form (l', k'). There are only finitely many such dominoes, and in Fig. 2 we indicated the centers of the bottom squares. Let us call these points the *particles* associated with the domino tiling. More precisely, we have N dominoes containing particles in the bottom T layers, and in the top N layers there are $N - 1, N - 2, \ldots, 1, 0$ particles in each layer. For $k = 1, \ldots, T$, define the signature $\lambda^{(k)}$ so that the N particles at layer k have the horizontal coordinates

$$(\lambda_1^{(k)} + N)', (\lambda_2^{(k)} + N - 1)', \ldots, (\lambda_{N-1}^{(k)} + 2)', (\lambda_N^{(k)} + 1)'. \tag{1.10}$$

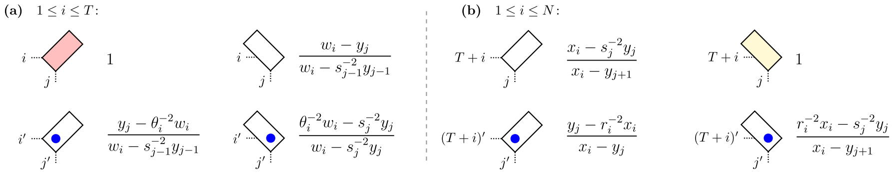


Fig. 3 Domino weights in Fig. 2 leading to the ascending FG process

For example, the sequence of signatures corresponding to the domino tiling in Fig. 2 is

$$\lambda^{(1)} = (0, 0, 0), \qquad \lambda^{(2)} = (1, 0, 0), \qquad \lambda^{(3)} = (3, 2, 0), \qquad \lambda^{(4)} = (4, 2, 0).$$

Let us now assign weights to dominoes depending on the parameters w_i, θ_i, y_j, s_j in the top part, and x_i, r_i, y_j, s_j in the bottom part, as displayed in Fig. 3. Note that the dominoes repeating infinitely often (the shaded ones in Figs. 2 and 3) have weight 1. Assuming that the weights satisfy (1.3), we see that the infinite series for the normalizing constant of this probability measure on domino tilings converges. Thus, the model of random domino tilings is well-defined.

In Sect. 9 we establish the correspondence between the random domino tiling model just described, and the ascending FG processes. This correspondence is based on the known mapping between the free fermion six vertex model and a layered free fermion five vertex model, e.g., see [98, Section 4.7].

Theorem 1.9 *The joint distribution of the signatures $\lambda^{(1)}, \dots, \lambda^{(T)}$ (each with N parts) associated via* (1.10) *to the random domino tiling as in Fig. 2 with domino weights given in Fig. 3 is described by the ascending FG process* (1.8).

Remark 1.10 One can also consider the joint distribution of all $T + N - 1$ signatures arising from Fig. 2. Namely, let $\mu^{(1)}, \dots, \mu^{(N)}$, where $\mu^{(i)}$ has i parts and $\mu^{(N)} = \lambda^{(T)}$, be constructed as in (1.10) from the coordinates (l', k') of the particles in the top N rows of the tiling. Then the sequence of signatures $(\lambda^{(1)}, \dots, \lambda^{(T)} = \mu^{(N)}, \mu^{(N-1)}, \dots, \mu^{(1)})$ has the joint distribution of a general FG process defined in Sect. 8.1 in the text. For such FG processes (and their further generalizations) we obtain the correlation kernel as a certain series coefficient using fermionic operators, see Theorem 8.9 in the text. It should be possible to rewrite the coefficient representation for the correlation kernel of the general FG processes as a contour integral. We do not pursue this here for brevity and also because the lattice (bulk) asymptotic behavior (discussed in Sect. 1.5 below) throughout the whole domino tiling in Fig. 2 is expected to be the same, up to renaming the parameters.

1.5 Bulk asymptotics and the inhomogeneous discrete sine kernel

We study bulk asymptotics of the domino tiling model described in Sect. 1.4 as $N, T \to +\infty$ and $T \gg N$. Here "bulk" means that we zoom around a global position $(\lfloor \alpha N \rfloor, \lfloor \tau N \rfloor)$, so that the lattice structure is preserved in the limit. For simplicity of the asymptotic analysis, we let the inhomogeneity parameters of the domino tiling vary only in a finite neighborhood of this global location. After taking the limit, we send the size of the finite neighborhood to infinity as well.

In this limit we observe a probability measure on the space of domino tilings of the whole plane $\mathbb{Z}^2$. This measure is determinantal. Its correlation kernel, which we call the (*two-dimensional*) *inhomogeneous discrete sine kernel* and denote by $K_{2\mathrm{d}}^{z}$ (defined below in this subsection), describes the bulk asymptotic joint distribution of the particles associated with a random domino tiling (as in Fig. 2). We take arbitrary inhomogeneity parameters around the global position $(\lfloor \alpha N \rfloor, \lfloor \tau N \rfloor)$, so that the limiting bulk kernel $K_{2\mathrm{d}}^{z}$ on $\mathbb{Z}^2$ is also inhomogeneous. Moreover, it depends on four sequences of parameters $\mathbf{w} = \{w_i\}_{i\in\mathbb{Z}}$, $\boldsymbol{\theta} = \{\theta_i\}_{i\in\mathbb{Z}}$, $\mathbf{y} = \{y_j\}_{j\in\mathbb{Z}}$, and $\mathbf{s} = \{s_j\}_{j\in\mathbb{Z}}$. Here the indexing is by $i, j \in \mathbb{Z}$ because in the bulk limit the parameters vary in all $\mathbb{Z}^2$ directions around the global scaling position which becomes the origin. Along with these four sequences, $K_{2\mathrm{d}}^{z}$ also depends on a point z in the upper half complex plane. In the homogeneous case, this point is responsible for the *slope* of the tiling, i.e., the densities of the particles in the horizontal and the vertical directions. The presence of the complex slope is typical in homogeneous two-dimensional bulk lattice asymptotics [60, 65, 90]. However, the dependence on four extra sequences of parameters is a novel feature of our kernel $K_{2\mathrm{d}}^{z}$ that is a consequence of the inhomogeneity of our model.

Theorem 1.11 (Theorem 10.9 in the text) *Fix z in the open upper half complex plane. Then there exists a choice of parameters of the ascending FG process together with a global location (α, τ) (detailed in Sect. 10.1), such that in the limit as $N, T \to +\infty$, $T \gg N$, the correlation kernel $K_{\mathcal{AP}}$ (1.9) of the ascending FG process admits the limit*

$$\lim_{N\to+\infty} K_{\mathcal{AP}}(t + \lfloor \tau N \rfloor, a + \lfloor \alpha N \rfloor; t' + \lfloor \tau N \rfloor, a' + \lfloor \alpha N \rfloor) = K_{2\mathrm{d}}^{z}(t, a; t', a'),$$

where $t, a, t', a' \in \mathbb{Z}$ are fixed.

We establish Theorem 1.11 using the steepest descent method for double contour integral correlation kernels which essentially follows [87, Sections 3.1, 3.2]. This technique is quite standard, and we refer to Sect. 10 in the text for detailed formulations.

Let us now proceed with the definition of the inhomogeneous discrete sine kernel $K_{2\mathrm{d}}^{z}$. First, we need some auxiliary notation. For any two sequences $\mathbf{b} = \{b_i\}_{i\in\mathbb{Z}}$ and $\mathbf{c} = \{c_i\}_{i\in\mathbb{Z}}$, define the following inhomogeneous analogues of the power functions $U \mapsto U^n$, $n \in \mathbb{Z}$:

$$\mathcal{P}_{n,n'}(u \mid \mathbf{b}; \mathbf{c}) := \begin{cases} \displaystyle\prod_{j=n+1}^{n'} \frac{u - b_j}{u - c_j}, & n < n'; \\ 1, & n = n'; \\ \displaystyle\prod_{j=n'+1}^{n} \frac{u - c_j}{u - b_j}, & n > n', \end{cases} \qquad n, n' \in \mathbb{Z}. \tag{1.11}$$

Assume that the sequences satisfy

$$\sup_i w_i < \inf_j y_j \le \sup_j y_j < \inf_i \theta_i^{-2} w_i \le \sup_i \theta_i^{-2} w_i < \inf_j s_j^{-2} y_j. \quad (1.12)$$

These ordering conditions are equivalent (as we show in Lemma 11.4 in the text) to the fact that all the domino weights given in Fig. 3, (a) are positive and separated from zero and infinity. We now define the two-dimensional inhomogeneous discrete sine kernel as

$$K^z_{2\mathrm{d}}(t, a; t', a') := -\frac{1}{2\pi \mathbf{i}} \int_{\bar{z}}^{z} \frac{y_a(1 - s_a^{-2})}{(u - y_a)(u - s_{a'}^{-2} y_{a'})} \mathcal{P}_{a,a'}(u \mid \mathbf{s}^{-2}\mathbf{y}; \mathbf{y})\, \mathcal{P}_{t,t'}(u \mid \mathbf{w}; \boldsymbol{\theta}^{-2}\mathbf{w})\, du, \quad (1.13)$$

where $t, a, t', a' \in \mathbb{Z}$. The integration contour is an arc from $\bar{z}$ to z which crosses the real line to the left of all w_i when $\Delta t = t' - t \ge 0$; and between $\theta_i^{-2} w_i$ and $s_j^{-2} y_j$ when $\Delta t < 0$.

From the fact that $K^z_{2\mathrm{d}}$ is a limit of $K_{\mathcal{AP}}$ (Theorem 1.11), the correlation kernel of a determinantal random point process coming from the ascending FG process, we deduce that $K^z_{2\mathrm{d}}$ with arbitrary inhomogeneity parameters $\mathbf{w}, \boldsymbol{\theta}, \mathbf{y}, \mathbf{s}$ satisfying the ordering (1.12) has the following nonnegativity property:

Theorem 1.12 (Theorem 11.3 in the text) *Under the above assumptions on the parameters and for any z in the open upper half complex plane, the kernel $K^z_{2\mathrm{d}}$ defines a determinantal random point process on $\mathbb{Z}^2$. In particular, all symmetric minors* $\det[K^z_{2\mathrm{d}}(t_i, a_i; t_j, a_j)]$ *of any order are between* 0 *and* 1.

Indeed, each symmetric minor $\det[K^z_{2\mathrm{d}}(t_i, a_i; t_j, a_j)]$ is the probability of the correlation event {therandomconfigurationcontainsallthepoints (t_i, a_i)}. This is nonnegative since the process $K^z_{2\mathrm{d}}$ is a limit of a *bona fide* determinantal random point process. We refer to [17, 50, 96] for generalities on determinantal processes.

In Sect. 11 we discuss specializations of the kernel $K^z_{2\mathrm{d}}$ leading to known determinantal correlation kernels arising in bulk lattice limits:

- the one-dimensional discrete sine kernel [15];
- one-dimensional periodic and inhomogeneous generalizations of the discrete sine kernel [16, 22, 76, 77];
- the bulk kernel arising from uniformly random domino tilings which is a bulk limit of [53, (2.21)] but also follows from the general theory of [65];
- the incomplete beta kernel [90] giving rise to the unique family of ergodic translation invariant Gibbs measures on lozenge tilings of the whole plane [65, 95] indexed by the complex slope z;
- and $k\mathbb{Z} \times \mathbb{Z}$ periodic generalizations of the incomplete beta kernel [22, 76, 77].

Our kernel $K^z_{2\mathrm{d}}$ admits a $k\mathbb{Z} \times m\mathbb{Z}$ periodic specialization for all $k, m \ge 1$ by taking the parameters (w_i, θ_i) to be m-periodic in i, and the parameters (y_j, s_j) to be k-periodic in j. For general $m \ge 2$ the arc integral representation (1.13) of such a periodic kernel is new. It would be interesting to match this arc integral to the two-dimensional torus integral representation of the doubly periodic kernels which follows from the general theory of [65], but we do not pursue this here.

We also remark that our kernel $K_{2\mathrm{d}}^{z}$ corresponds only to the so-called liquid phase of the domino tiling model. It is known [11, 25, 28, 65] that doubly periodic domino weights may lead to the appearance of gaseous phase. The gaseous phase is not present in our FG processes because our domino weights are *not* fully generic and depend on their many parameters in quite a special way. In particular, in the $2\mathbb{Z} \times 2\mathbb{Z}$ periodic case we have verified that the domino weights are gauge equivalent (in the sense of [58, Section 3.10]), in a nontrivial way, to weights periodic in only one direction.

1.6 Fermionic operators and correlation functions

In this final part of the Introduction we outline definitions and main properties of fermionic operators acting in a Fock, or "infinite wedge", space. Detailed definitions and statements are in Sects. 7 and 8 below.

Our fermionic operators are combinations of Algebraic Bethe Ansatz row operators constructed from the vertex weights W (1.2). The fermionic operators allow to compute certain generating function type series involving the correlation functions of the FG processes. The correlation functions are then extracted as series coefficients using inhomogeneous biorthogonality similar to (1.6).

Fock space and fermionic operators coming from Pieri rules for Schur functions were used in [86, 90] to compute correlation kernels of Schur measures and processes. Expressions for local operators and correlations in various quantum integrable systems through the row operators A, B, C, D also appear in, e.g., [59, 89], but our model and formulas are quite different from those. It is also worth noting that fermionic operators in the homogeneous Fock space associated to the free fermion six vertex model (and again leading to Schur functions) were considered recently in [64]. However, our inhomogeneous Fock space and the fermionic operators acting in it arising from the free fermion six vertex model seem to be new.

A subset $\mathcal{T} \subset \mathbb{Z}$ is called semi-infinite (or densely packed towards $-\infty$) if there exists $M = M(\mathcal{T}) > 0$ with $i \notin \mathcal{T}$ for all $i > M$ and $i \in \mathcal{T}$ for all $i < -M$. For a semi-infinite subset, define its charge $c(\mathcal{T}) := \#(\mathcal{T} \cap \mathbb{Z}_{>0}) - \#(\mathbb{Z}_{\le 0} \setminus \mathcal{T})$. For example, all zero-charge semi-infinite subsets are finite permutations of $\mathbb{Z}_{\le 0}$.

Let $\mathcal{F}$ be the (*fermionic*) *Fock space* spanned by $e_{\mathcal{T}}$, where $\mathcal{T}$ runs over all semi-infinite subsets of $\mathbb{Z}$. We view $\mathcal{F}$ as a subspace of the formal infinite tensor product $\bigotimes_{m=-\infty}^{+\infty} V^{(m)}$, where each $V^{(m)}$ is isomorphic to $\mathbb{C}^2$ with standard basis $e_0^{(m)}, e_1^{(m)}$. This is done by interpreting each $e_{\mathcal{T}}$ as a tensor product

$$e_{\mathcal{T}} = \bigotimes_{m=-\infty}^{+\infty} e_{k_m}^{(m)}, \qquad k_m = k_m(\mathcal{T}) = \mathbf{1}_{m \in \mathcal{T}}.$$

Here and below $\mathbf{1}_{\ldots}$ is the indicator. Sometimes in the literature the wedge product symbol $\bigwedge_{m=-\infty}^{+\infty} e_{k_m}^{(m)}$ is used instead of the tensor product, with the same meaning. In more detail, we never implicitly use the wedge commutation relation $v \wedge w = -w \wedge v$, and whenever signs are required we insert them explicitly, as in, e.g., the creation and annihilation operators ψ_j, ψ_j^* in (1.18) below.

We also need an inner product in $\mathfrak{F}$ under which the $e_{\mathcal{T}}$'s form an orthonormal basis, that is, $\langle e_{\mathcal{T}}, e_{\mathcal{R}}\rangle = \mathbf{1}_{\mathcal{T}=\mathcal{R}}$. Let us decompose $\mathfrak{F}$ into subspaces with fixed charge:

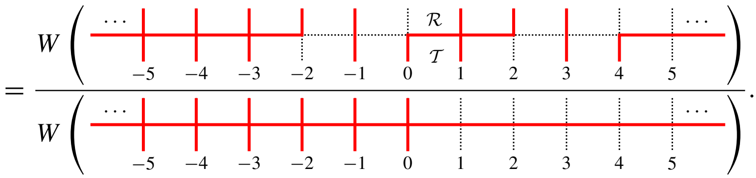
$$\mathfrak{F} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{F}_n, \qquad \mathfrak{F}_n = \operatorname{span}\{e_{\mathcal{T}} : c(\mathcal{T}) = n\}.$$

We are now in a position to define the row operators $A^{\mathbb{Z}} = A^{\mathbb{Z}}(x, r)$, $B^{\mathbb{Z}} = B^{\mathbb{Z}}(x, r)$, $C^{\mathbb{Z}} = C^{\mathbb{Z}}(x, r)$, and $D^{\mathbb{Z}} = D^{\mathbb{Z}}(x, r)$ acting in $\mathfrak{F}$. They act in the following way with respect to the charge:

$$A^{\mathbb{Z}} : \mathfrak{F}_n \to \mathfrak{F}_n, \qquad B^{\mathbb{Z}} : \mathfrak{F}_n \to \mathfrak{F}_{n-1}, \qquad C^{\mathbb{Z}} : \mathfrak{F}_n \to \mathfrak{F}_{n+1}, \qquad D^{\mathbb{Z}} : \mathfrak{F}_n \to \mathfrak{F}_n.$$

We define these operators pictorially through their matrix elements. For $A^{\mathbb{Z}}$ we have

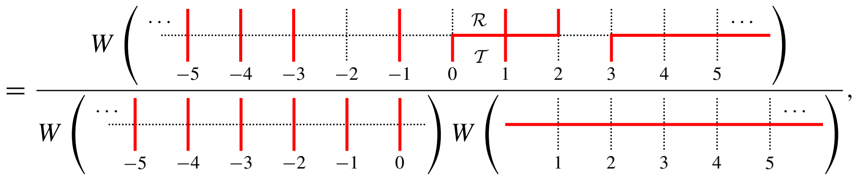
$\langle A^{\mathbb{Z}}(x, r)e_{\mathcal{T}}, e_{\mathcal{R}}\rangle$



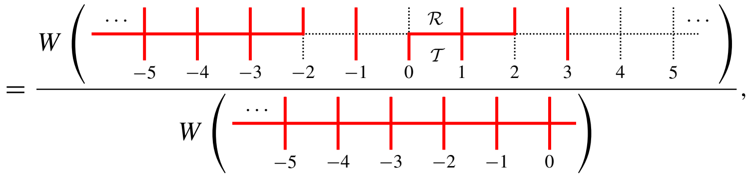
Here the numerator is a formal infinite product of vertex weights $W(\cdots \mid x; y_j; r; s_j)$ over all $j \in \mathbb{Z}$, where the bottom and top boundary conditions are $\mathcal{T}$ and $\mathcal{R}$, respectively, and the far left and far right boundary conditions are occupied. By definition, the product of the weights $W(\cdots)$ is zero if there are no six vertex model configurations with these boundary conditions. The denominator in $A^{\mathbb{Z}}$ is the normalization factor which is also a formal infinite product. The ratio is well-defined as the product of the ratios of the weights at each lattice site $j \in \mathbb{Z}$, because this product involves only finitely many factors not equal to 1.

Similarly we define the other three operators, $B^{\mathbb{Z}}$ with boundary conditions empty and full at far left and far right, $C^{\mathbb{Z}}$ with boundary conditions full and empty at far left and far right, and $D^{\mathbb{Z}}$ with empty boundary conditions on both sides:

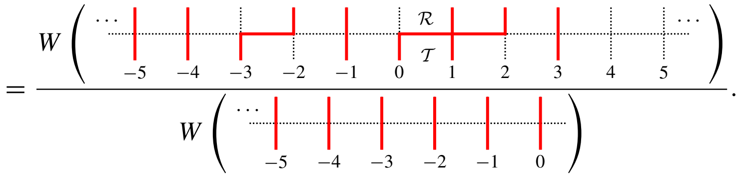
$\langle B^{\mathbb{Z}}(x, r)e_{\mathcal{T}}, e_{\mathcal{R}}\rangle$



$\langle C^{\mathbb{Z}}(x, r)e_{\mathcal{T}}, e_{\mathcal{R}}\rangle$



$\langle D^{\mathbb{Z}}(x,r)e_{\mathcal{T}}, e_{\mathcal{R}}\rangle$



We refer to Sects. 7.2 and 7.3 below for formal definitions of these operators in the Fock space via a limiting procedure $m \to -\infty$, $n \to +\infty$ starting from finite segments $\{m, m+1, \dots, n-1, n\}$.

Thanks to the Yang–Baxter equation, the row operators satisfy a number of commutation relations, for example,

$$B^{\mathbb{Z}}(x,r)D^{\mathbb{Z}}(w,\theta) = \frac{x-\theta^{-2}w}{x-w}\, D^{\mathbb{Z}}(w,\theta)B^{\mathbb{Z}}(x,r), \tag{1.14}$$

provided that x, w satisfy (1.3), which is required to avoid diverging infinite series. Other commutation relations are listed in Proposition 7.8 in the text.

In fact, relation (1.14) is closely related to the Cauchy identity (Theorem 1.3). Let λ be a signature with N parts, and set $\mathcal{T} = \{\lambda_1 + N, \lambda_2 + N - 1, \dots, \lambda_N + 1\} \cup \mathbb{Z}_{\le 0}$. Then

$$\begin{aligned} F_\lambda(x_1,\dots,x_N;\mathbf{y};r_1,\dots,r_N;\mathbf{s}) &= \langle e_{\mathcal{T}}, B^{\mathbb{Z}}(x_N,r_N)\dots B^{\mathbb{Z}}(x_1,r_1)e_{\mathbb{Z}_{\le 0}}\rangle,\\ G_\lambda(w_1,\dots,w_M;\mathbf{y};\theta_1,\dots,\theta_M;\mathbf{s}) &= \langle e_{\mathcal{T}}, D^{\mathbb{Z}}(w_M,\theta_M)\dots D^{\mathbb{Z}}(w_1,\theta_1)e_{\mathbb{Z}_{\le N}}\rangle. \end{aligned} \tag{1.15}$$

Applying (1.14) several times, we have

$$\begin{aligned} &\langle e_{\mathbb{Z}_{\le 0}}, B^{\mathbb{Z}}(x_N,r_N)\dots B^{\mathbb{Z}}(x_1,r_1)D^{\mathbb{Z}}(w_M,\theta_M)\dots D^{\mathbb{Z}}(w_1,\theta_1)e_{\mathbb{Z}_{\le N}}\rangle\\ &= \prod_{i=1}^{N}\prod_{j=1}^{M}\frac{x_i-\theta_j^{-2}w_j}{x_i-w_j}\langle e_{\mathbb{Z}_{\le 0}}, D^{\mathbb{Z}}(w_M,\theta_M)\dots D^{\mathbb{Z}}(w_1,\theta_1)\\ &\qquad B^{\mathbb{Z}}(x_N,r_N)\dots B^{\mathbb{Z}}(x_1,r_1)e_{\mathbb{Z}_{\le N}}\rangle\\ &= F_{(0,\dots,0)}(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s})\prod_{i=1}^{N}\prod_{j=1}^{M}\frac{x_i-\theta_j^{-2}w_j}{x_i-w_j}, \end{aligned} \tag{1.16}$$

where in the last step we removed the $D^{\mathbb{Z}}$ operators thanks to $e_{\mathbb{Z}_{\le 0}}$ in the inner product, and used the definition of F again. The first line of (1.16) is precisely the left-hand side of the Cauchy identity (1.4). In the third line we need an explicit product formula for $F_{(0,\dots,0)}$ which follows from Theorem 1.5 (see the proof of Theorem 3.8 for this computation), and the resulting expression becomes the right-hand side of the Cauchy identity (1.4).

Using (1.15), we may express the probability weight $\mathcal{M}(\lambda)$ (1.7) under the FG measure as the following evaluated inner product in the Fock space:

$$\mathcal{M}(\lambda) = \frac{1}{Z} \langle e_{\mathbb{Z}_{\le 0}}, B^{\mathbb{Z}}(x_N, r_N) \dots B^{\mathbb{Z}}(x_1, r_1)\, I_\lambda D^{\mathbb{Z}}(w_M, \theta_M) \dots D^{\mathbb{Z}}(w_1, \theta_1) e_{\mathbb{Z}_{\le N}} \rangle \tag{1.17}$$

Here and throughout this subsection Z denotes the right-hand side of the Cauchy identity (1.4). The operator I_λ is the rank one projection in $\mathcal{F}$ onto the semi-infinite subset corresponding to λ:

$$I_\lambda e_{\mathcal{R}} = \begin{cases} e_{\mathcal{R}}, & \text{if } \mathcal{R} = \{\lambda_1 + N, \lambda_2 + N - 1, \dots, \lambda_N + 1, 0, -1, -2, \dots\}; \\ 0, & \text{otherwise,} \end{cases}$$

for any semi-infinite subset $\mathcal{R} \subset \mathbb{Z}$. Expressions similar to (1.17) are available for FG processes as well, see Sect. 8.2 in the text. For simplicity of notation, below in this subsection we stick to the case of FG measures.

If instead of I_λ we insert into (1.17) a product of creation and annihilation operators in the Fock space $\mathcal{F}$, we would get a correlation function of the FG measure. Recall that the creation and annihilation operators are defined as

$$\begin{aligned} \psi_j e_{\mathcal{T}} &= \begin{cases} (-1)^{\#\{t \in \mathcal{T} :\, t > j\}} e_{\mathcal{T} \cup \{j\}}, & j \notin \mathcal{T}; \\ 0, & j \in \mathcal{T}, \end{cases} \\ \psi_j^* e_{\mathcal{T}} &= \begin{cases} (-1)^{\#\{t \in \mathcal{T} :\, t > j\}} e_{\mathcal{T} \setminus \{j\}}, & j \in \mathcal{T}; \\ 0, & j \notin \mathcal{T}. \end{cases} \end{aligned} \tag{1.18}$$

They satisfy the canonical anticommutation relations

$$\psi_k \psi_k^* + \psi_k^* \psi_k = 1, \qquad \psi_k \psi_\ell^* + \psi_\ell^* \psi_k = \psi_k^* \psi_\ell^* + \psi_\ell^* \psi_k^* = \psi_k \psi_\ell + \psi_\ell \psi_k = 0, \quad k \neq \ell.$$

For any finite subset $A = \{a_1, \dots, a_m\} \subset \mathbb{Z}_{\ge 1}$ we have the following expression for the correlation function:

$$\begin{aligned} &\mathbb{P}_{\mathcal{M}}\left[A \subset \{\lambda_1 + N, \lambda_2 + N - 1, \dots, \lambda_N + 1\}\right] \\ &\quad = \frac{1}{Z} \langle e_{\mathbb{Z}_{\le 0}}, B^{\mathbb{Z}}(x_N, r_N) \dots B^{\mathbb{Z}}(x_1, r_1)\, \psi_{a_m} \psi_{a_m}^* \dots \\ &\qquad \psi_{a_1} \psi_{a_1}^* D^{\mathbb{Z}}(w_M, \theta_M) \dots D^{\mathbb{Z}}(w_1, \theta_1) e_{\mathbb{Z}_{\le N}} \rangle. \end{aligned} \tag{1.19}$$

Inserting pairs of creation and annihilation operators between the D operators above produces correlation functions of ascending FG processes.

Instead of computing (1.19) directly, we replace the creation and annihilation operators with certain generating series $\Psi(u)$, $\Psi^*(v)$ containing creation and annihilation operators. The series $\Psi(u)$, $\Psi^*(v)$ themselves are operators which are expressed through the row operators with special parameters. Namely, define for any semi-infinite subset $\mathcal{T}$:

$$\begin{aligned}\Psi(u,\xi)\, e_{\mathcal{T}} &:= D^{\mathbb{Z}}(u, \sqrt{u/\xi})C^{\mathbb{Z}}(\xi, \sqrt{\xi/u})(-1)^{c(\mathcal{T})} e_{\mathcal{T}},\\ \Psi^*(\zeta, v)\, e_{\mathcal{T}} &:= D^{\mathbb{Z}}(\zeta, \sqrt{\zeta/v})B^{\mathbb{Z}}(v, \sqrt{v/\zeta})\, e_{\mathcal{T}}.\end{aligned} \tag{1.20}$$

The statement below could be viewed as an inhomogeneous analogue of the Boson–Fermion correspondence, cf. [55, Theorem 14.10] for the homogeneous version.

Theorem 1.13 (Theorem 7.11 in the text) *As operators on $\mathfrak{F}$, we have*

$$\begin{aligned}\Psi(u,\xi) &= \sum_{j\in\mathbb{Z}} \frac{y_j(1-s_j^{-2})}{u - s_j^{-2}y_j}\, \mathcal{P}_{0,j-1}(u \mid \mathbf{y}, \mathbf{s}^{-2}\mathbf{y})\, \psi_j,\\ \Psi^*(\zeta, v) &= \sum_{j\in\mathbb{Z}} \frac{v-\zeta}{v-y_j}\, \mathcal{P}_{0,j-1}(v \mid \mathbf{s}^{-2}\mathbf{y}, \mathbf{y})\, \psi_j^*,\end{aligned} \tag{1.21}$$

where we use the notation of inhomogeneous powers (1.11).

Let us set $\Psi(u) := \Psi(u, 0)$ and $\Psi^*(v) = \Psi^*(0, v)$. From (1.21) we see that $\Psi(u,\xi)$ does not depend on ξ, and $\Psi^*(v)$ is well-defined by specializing the second line of (1.21). Thanks to Theorem 1.13, operators $\Psi(u)$, $\Psi^*(v)$ satisfy the Wick determinantal formula for the "vacuum average":

$$\left\langle e_{\mathbb{Z}_{\le 0}}, \Psi(u_1)\Psi^*(v_1)\dots\Psi(u_m)\Psi^*(v_m)\, e_{\mathbb{Z}_{\le 0}}\right\rangle = \det\left[\frac{v_\alpha}{u_{\alpha'} - v_\alpha}\right]_{\alpha,\alpha'=1}^{m}. \tag{1.22}$$

See Propositions 7.12 and 7.14 in the text for details on the Wick determinantal formula.

There are two steps remaining in the computation of the correlation functions (1.19) of the FG measure $\mathcal{M}$. First, using commutation relations between the row operators and (1.22), we show the following.

Proposition 1.14 (A particular case of Proposition 8.7 in the text) *Let* $u_1, \dots, u_m, v_1, \dots, v_m$ *be independent variables satisfying certain conditions (see* (8.4) *for details). Then we have*

$$\begin{aligned}&\frac{1}{Z}\langle e_{\mathbb{Z}_{\le 0}}, B^{\mathbb{Z}}(x_N, r_N)\dots B^{\mathbb{Z}}(x_1, r_1)\Psi(u_m)\Psi^*(v_m)\dots\Psi(u_1)\Psi^*(v_1)\\ &\qquad\times D^{\mathbb{Z}}(w_M, \theta_M)\dots D^{\mathbb{Z}}(w_1,\theta_1)e_{\mathbb{Z}_{\le N}}\rangle\\ &= \prod_{i=1}^{M}\prod_{\alpha=1}^{m} \frac{(v_\alpha - \theta_i^{-2}w_i)(u_\alpha - w_i)}{(v_\alpha - w_i)(u_\alpha - \theta_i^{-2}w_i)} \prod_{\alpha=1}^{m}\prod_{j=1}^{N} \frac{u_\alpha - y_j}{v_\alpha - y_j}\frac{v_\alpha - x_j}{u_\alpha - x_j}\end{aligned}$$

$$\det \left[\frac{v_\alpha}{u_{\alpha'} - v_\alpha} \right]_{\alpha,\alpha'=1}^{m}. \tag{1.23}$$

Using (1.21), we interpret (1.23) as an inhomogeneous generating series of the correlation functions (1.19) of the FG measure $\mathcal{M}$, with the generating variables u_i, v_j. The last step is to extract the coefficients (1.19) from this generating series. The operation of a coefficient extraction is linear, and we must apply $2m$ such operations to the right-hand side of (1.23). Due to the form of this right-hand side, these operations may be put into the $m \times m$ determinant (this is essentially the Andréief identity, cf. [39]). This implies that the correlation functions have a determinantal form. In Theorem 8.9 in the text we write the resulting correlation kernel of the general FG process as such a series coefficient.

Furthermore, the coefficient extraction can be done analytically by means of contour integrals. This is an extension of the inhomogeneous biorthogonality (1.6), to inhomogeneous powers indexed by both positive and nonpositive integers, see Lemmas 8.4 and 8.5 in the text. The integration contours for u, v in the coefficient extraction must be chosen so that the commutation relations between the row operators used to obtain (1.22) and (1.23) are valid. Using this, we determine the correct integration contours for the correlation kernel $K_{\mathcal{AP}}$ (1.9) of the ascending FG process, see Theorem 8.10 in the text. As a result, we have computed the correlation kernel $K_{\mathcal{AP}}$ in two ways, via fermionic operators and via an Eynard–Mehta type approach, and both computations led to the same expression.

Part I Symmetric functions

In this part (accompanied by Appendix A) we develop symmetric rational functions based on the free fermion six vertex model.

2 Free fermion six vertex model

2.1 Vertex weights

We consider the weights $w_{6V}(i_1, j_1; i_2, j_2)$, $i_1, j_1, i_2, j_2 \in \{0, 1\}$, of the asymmetric six vertex (square ice) model:

$$\begin{aligned} &w_{6V}(0,0;0,0) = a_1, \quad &&w_{6V}(1,1;1,1) = a_2, \quad &&w_{6V}(1,0;1,0) = b_1, \\ &w_{6V}(0,1;0,1) = b_2, \quad &&w_{6V}(1,0;0,1) = c_1, \quad &&w_{6V}(0,1;1,0) = c_2, \end{aligned} \tag{2.1}$$

see Fig. 4 for the illustration. By agreement, $w_{6V}(i_1, j_1; i_2, j_2)$ is set to zero unless $i_1 + j_1 = i_2 + j_2$, which corresponds to the path preservation property: the number of paths coming into a vertex equals the number of paths coming out of it. The notation $a_1, a_2, b_1, b_2, c_1, c_2$ for the vertex weights follows the longstanding tradition, for example, see [5, 94, Ch. 8].

We further assume that our six vertex weights obey the *free fermion condition*

$$a_1 a_2 + b_1 b_2 = c_1 c_2. \tag{2.2}$$

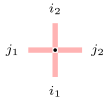
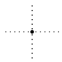
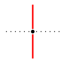
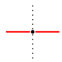
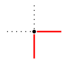
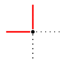
						
$w_{6V}(i_1, j_1; i_2, j_2)$	a_1	a_2	b_1	b_2	c_1	c_2
$W(i_1, j_1; i_2, j_2)$	1	$\frac{r^{-2}x - y}{s^{-2}y - x}$	$\frac{s^{-2}y - r^{-2}x}{s^{-2}y - x}$	$\frac{y - x}{s^{-2}y - x}$	$\frac{x(r^{-2} - 1)}{s^{-2}y - x}$	$\frac{y(s^{-2} - 1)}{s^{-2}y - x}$
$\widehat{W}(i_1, j_1; i_2, j_2)$	$\frac{s^{-2}y - x}{y - x}$	$\frac{r^{-2}x - y}{y - x}$	$\frac{s^{-2}y - r^{-2}x}{y - x}$	1	$\frac{x(r^{-2} - 1)}{y - x}$	$\frac{y(s^{-2} - 1)}{y - x}$

Fig. 4 Weights (2.1) of the free fermion six vertex model, and their parametrizations (2.3), (2.4) with the four parameters x, y, r, s

In other words, we impose the vanishing of the quantity $\Delta = \dfrac{a_1 a_2 + b_1 b_2 - c_1 c_2}{2\sqrt{a_1 a_2 b_1 b_2}}$ associated with the six vertex weights.

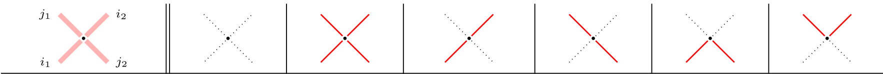
The free fermion condition (2.2) leaves five out of six independent parameters. Furthermore, in order to build symmetric functions, we normalize the weights so that one of them becomes equal to 1, and we can repeat this type of vertices infinitely many times in a configuration. The normalization leaves four independent parameters. We make two different choices which of the weights to set to 1:

- Setting the weight a_1 of the empty vertex $(0, 0; 0, 0)$ we get the weights

$$
\begin{aligned}
&W(0,0;0,0) = 1, \quad W(1,1;1,1) = \frac{r^{-2}x - y}{s^{-2}y - x}, \\
&W(1,0;1,0) = \frac{s^{-2}y - r^{-2}x}{s^{-2}y - x}, \\
&W(0,1;0,1) = \frac{y - x}{s^{-2}y - x}, \quad W(1,0;0,1) = \frac{x(r^{-2} - 1)}{s^{-2}y - x}, \\
&W(0,1;1,0) = \frac{y(s^{-2} - 1)}{s^{-2}y - x}.
\end{aligned}
\tag{2.3}
$$

- Setting the weight b_2 of the vertex $(0, 1; 0, 1)$, we get the weights

$$
\begin{aligned}
&\widehat{W}(0,0;0,0) = \frac{s^{-2}y - x}{y - x}, \quad \widehat{W}(1,1;1,1) = \frac{r^{-2}x - y}{y - x}, \\
&\widehat{W}(1,0;1,0) = \frac{s^{-2}y - r^{-2}x}{y - x}, \\
&\widehat{W}(0,1;0,1) = 1, \quad \widehat{W}(1,0;0,1) = \frac{x(r^{-2} - 1)}{y - x}, \\
&\widehat{W}(0,1;1,0) = \frac{y(s^{-2} - 1)}{y - x}.
\end{aligned}
\tag{2.4}
$$



$R(i_1, j_1; i_2, j_2)$	1	$\frac{x_2 - x_1 r_1^{-2}}{x_1 - x_2 r_2^{-2}}$	$\frac{x_1 r_1^{-2} - x_2 r_2^{-2}}{x_1 - x_2 r_2^{-2}}$	$\frac{x_1 - x_2}{x_1 - x_2 r_2^{-2}}$	$\frac{x_1(1 - r_1^{-2})}{x_1 - x_2 r_2^{-2}}$	$\frac{x_2(1 - r_2^{-2})}{x_1 - x_2 r_2^{-2}}$

Fig. 5 The cross vertex weights (2.6)

See Fig. 4 for an illustration. The four parameters which they depend on are denoted by x, y, r, s. Sometimes we will indicate this dependence explicitly as

$$W(i_1, j_1; i_2, j_2 \mid x; y; r; s), \qquad \widehat{W}(i_1, j_1; i_2, j_2 \mid x; y; r; s).$$

The concrete choice of the parametrization as in (2.3)–(2.4) is dictated by the form of the Yang–Baxter equation (see the next Sect. 2.2), and by the overall goal of constructing symmetric functions.

One readily checks that the weights W and $\widehat{W}$ satisfy the free fermion condition (2.2). We also have

$$\widehat{W}(i_1, j_1; i_2, j_2) = \frac{W(i_1, j_1; i_2, j_2)}{W(0, 1; 0, 1)} \tag{2.5}$$

for all $i_1, j_1, i_2, j_2 \in \{0, 1\}$, that is, the weights W and $\widehat{W}$ differ only by normalization.

Remark 2.1 The free fermion six vertex weights (2.3)–(2.4) possess $U_q(\widehat{\mathfrak{sl}}(1|1))$ quantum affine superalgebra symmetry, and they are a special case of the higher rank $U_q(\widehat{\mathfrak{sl}}(m|n))$ integrable weights studied in the companion work [1].

2.2 Yang–Baxter equation

Define the following cross vertex weights:

$$\begin{aligned}
&R(0, 0; 0, 0) = 1, \quad R(1, 1; 1, 1) = \frac{x_2 - x_1 r_1^{-2}}{x_1 - x_2 r_2^{-2}}, \\
&R(1, 0; 1, 0) = \frac{x_1 r_1^{-2} - x_2 r_2^{-2}}{x_1 - x_2 r_2^{-2}}, \\
&R(0, 1, 0, 1) = \frac{x_1 - x_2}{x_1 - x_2 r_2^{-2}}, \quad R(1, 0; 0, 1) = \frac{x_1(1 - r_1^{-2})}{x_1 - x_2 r_2^{-2}}, \\
&R(0, 1; 1, 0) = \frac{x_2(1 - r_2^{-2})}{x_1 - x_2 r_2^{-2}}.
\end{aligned} \tag{2.6}$$

See Fig. 5 for an illustration. These weights, together with W or $\widehat{W}$, satisfy the Yang–Baxter equation:

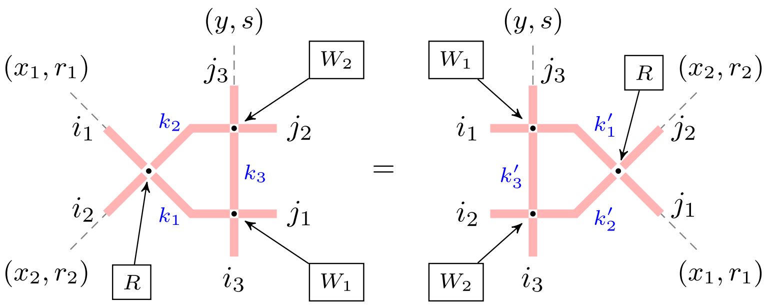


Fig. 6 Illustration of the Yang–Baxter equation. For fixed boundary values $i_1, i_2, i_3, j_1, j_2, j_3$, the partition functions in both sides are equal to each other. Here $W_j \equiv W(\cdots \mid x_j; y; r_j; s)$. The parameters (x_1, r_1), (x_2, r_2) and (y, s) correspond to lines, which is also indicated

Proposition 2.2 (Yang–Baxter equation) *For any fixed* $i_1, i_2, i_3, j_1, j_2, j_3 \in \{0, 1\}$ *we have*

$$\begin{aligned}&\sum_{k_1,k_2,k_3\in\{0,1\}} R(i_2, i_1; k_2, k_1)\, W(i_3, k_1; k_3, j_1 \mid x_1; y; r_1; s)\\ &W(k_3, k_2; j_3, j_2 \mid x_2; y; r_2; s)\\ &= \sum_{k'_1,k'_2,k'_3\in\{0,1\}} R(k'_2, k'_1; j_2, j_1)\, W(i_3, i_2; k'_3, k'_2 \mid x_2; y; r_2; s)\\ &W(k'_3, i_1; j_3, k'_1 \mid x_1; y; r_1; s),\end{aligned} \tag{2.7}$$

The same equation holds if we replace one or both of the weights

$$W(\cdots \mid x_1; y; r_1; s), \qquad W(\cdots \mid x_2; y; r_2; s)$$

everywhere by the corresponding $\widehat{W}$*, which results in three other identities. See Fig. 6 for a graphical illustration of the sums in both sides.*

Proof Identity (2.7) or, more precisely, the family of identities depending on the boundary conditions $i_1, i_2, i_3, j_1, j_2, j_3$, is checked in a straightforward way. The three other families of identities involving the weights $\widehat{W}$ are multiples of (2.7), thanks to (2.5). □

Remark 2.3 The Yang–Baxter equation of Proposition 2.2 is a consequence of the more general $U_q(\widehat{\mathfrak{sl}}(m|n))$ statement, see Proposition 5.1.4 and Example 8.1.1 in [1]. This may be considered a conceptual reason behind Proposition 2.2 since our weights are obtained from the ones in [1] by fusion.

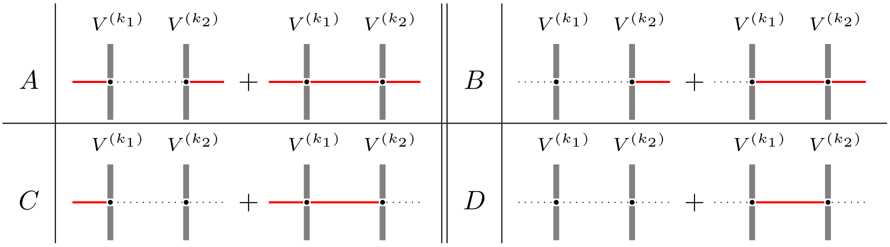


Fig. 7 Graphical illustration of the action of the operators A, B, C, D on tensor products (2.9). The sums in (2.9) correspond to various states of the inner horizontal edge. The vertical edges can have arbitrary states

2.3 Row operators

Based on the vertex weights, we define certain row operators acting in tensor powers of $\mathbb{C}^2$. Thanks to the Yang–Baxter equation, these operators satisfy certain commutation relations.

Let us fix sequences $\mathbf{s} = (s_1, s_2, \dots) \subset \mathbb{C}$ and $\mathbf{y} = (y_1, y_2, \dots) \subset \mathbb{C}$ of complex numbers. For any integer $k \geq 1$, we let $V^{(k)} \simeq \mathbb{C}^2$ denote the two-dimensional complex vector space spanned by two vectors $e_0^{(k)}$ and $e_1^{(k)}$. For notational convenience, we will also set $e_j^{(k)} = 0$ for $j \notin \{0, 1\}$.

Next, for any complex numbers $x, r \in \mathbb{C}$, we define four operators $A = A(x, r)$, $B = B(x, r)$, $C = C(x, r)$, and $D = D(x, r)$ acting from the left on any $V^{(k)}$ through the equations

$$
\begin{aligned}
Ae_i^{(k)} &= W(i, 1; i, 1 \mid x; y_k; r; s_k)\, e_i^{(k)}; \\
Be_i^{(k)} &= W(i, 0; i-1, 1 \mid x; y_k; r; s_k)\, e_{i-1}^{(k)}; \\
Ce_i^{(k)} &= W(i, 1; i+1, 0 \mid x; y_k; r; s_k)\, e_{i+1}^{(k)}; \\
De_i^{(k)} &= W(i, 0; i, 0 \mid x; y_k; r; s_k)\, e_i^{(k)},
\end{aligned}
\tag{2.8}
$$

where the weights W are given in (2.3).

We also define actions of these operators on tensor products $V^{(k_1)} \otimes V^{(k_2)} \otimes \dots \otimes V^{(k_n)}$. To do this in the case $n = 2$, set

$$
\begin{aligned}
A\big(v_1 \otimes v_2\big) &= Cv_1 \otimes Bv_2 + Av_1 \otimes Av_2; \\
B(v_1 \otimes v_2) &= Dv_1 \otimes Bv_2 + Bv_1 \otimes Av_2; \\
C(v_1 \otimes v_2) &= Cv_1 \otimes Dv_2 + Av_1 \otimes Cv_2; \\
D(v_1 \otimes v_2) &= Dv_1 \otimes Dv_2 + Bv_1 \otimes Cv_2,
\end{aligned}
\tag{2.9}
$$

for all $v_1 \in V^{(k_1)}$, $v_2 \in V^{(k_2)}$ (see Fig. 7 for an illustration). Then extend this action to $V^{(k_1)} \otimes V^{(k_2)} \otimes \dots \otimes V^{(k_n)}$ for $n > 2$ using the above relations (2.9) inductively on n. The induction step consists of taking v_1 there to be an element of $V^{(k_1)} \otimes V^{(k_2)} \otimes \dots \otimes V^{(k_{n-1})}$, and v_2 an element of $V^{(k_n)}$. This action on tensor products is associative.

The operators A, B, C, and D acting in any tensor product $V^{(k_1)} \otimes V^{(k_2)} \otimes \ldots \otimes V^{(k_n)}$ satisfy the following commutation relations:

Proposition 2.4 *For any $x_1, x_2, r_1, r_2 \in \mathbb{C}$, we have*

$$\begin{aligned}
A(x_2, r_2)A(x_1, r_1) &= A(x_1, r_1)A(x_2, r_2); && (2.10)\\
B(x_2, r_2)B(x_1, r_1) &= \frac{r_1^{-2}x_1 - x_2}{r_2^{-2}x_2 - x_1} B(x_1, r_1)B(x_2, r_2); && (2.11)\\
C(x_2, r_2)C(x_1, r_1) &= \frac{r_2^{-2}x_2 - x_1}{r_1^{-2}x_1 - x_2} C(x_1, r_1)C(x_2, r_2); && (2.12)\\
D(x_2, r_2)D(x_1, r_1) &= D(x_1, r_1)D(x_2, r_2); && (2.13)\\
B(x_2, r_2)D(x_1, r_1) &= \frac{r_1^{-2}x_1 - x_2}{x_1 - x_2} D(x_1, r_1)B(x_2, r_2) + \frac{(1 - r_2^{-2})x_2}{x_1 - x_2} D(x_2, r_2)B(x_1, r_1); && (2.14)\\
D(x_2, r_2)B(x_1, r_1) &= \frac{r_1^{-2}x_1 - x_2}{r_1^{-2}x_1 - r_2^{-2}x_2} B(x_1, r_1)D(x_2, r_2)\\
&\quad + \frac{(1 - r_1^{-2})x_1}{r_1^{-2}x_1 - r_2^{-2}x_2} B(x_2, r_2)D(x_1, r_1); && (2.15)\\
D(x_2, r_2)C(x_1, r_1) &= \frac{r_2^{-2}x_2 - x_1}{x_2 - x_1} C(x_1, r_1)D(x_2, r_2)\\
&\quad + \frac{(1 - r_2^{-2})x_2}{x_2 - x_1} C(x_2, r_2)D(x_1, r_1); && (2.16)\\
C(x_2, r_2)D(x_1, r_1) &= \frac{r_2^{-2}x_2 - x_1}{r_2^{-2}x_2 - r_1^{-2}x_1} D(x_1, r_1)C(x_2, r_2)\\
&\quad + \frac{x_1(1 - r_1^{-2})}{r_2^{-2}x_2 - r_1^{-2}x_1} D(x_2, r_2)C(x_1, r_1); && (2.17)\\
A(x_2, r_2)C(x_1, r_1) &= \frac{r_2^{-2}x_2 - x_1}{x_1 - x_2} C(x_1, r_1)A(x_2, r_2) + \frac{x_2(1 - r_2^{-2})}{x_1 - x_2} C(x_2, r_2)A(x_1, r_1), && (2.18)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{x_1(r_1^{-2} - 1)}{r_2^{-2}x_2 - x_1} D(x_2, r_2)A(x_1, r_1) + \frac{r_2^{-2}x_2 - r_1^{-2}x_1}{r_2^{-2}x_2 - x_1} C(x_2, r_2)B(x_1, r_1)\\
&\quad = \frac{x_1(r_1^{-2} - 1)}{r_2^{-2}x_2 - x_1} D(x_1, r_1)A(x_2, r_2) + \frac{x_2 - x_1}{r_2^{-2}x_2 - x_1} B(x_1, r_1)C(x_2, r_2); && (2.19)\\
&\frac{x_2(r_2^{-2} - 1)}{r_2^{-2}x_2 - x_1} A(x_2, r_2)D(x_1, r_1) + \frac{x_2 - x_1}{r_2^{-2}x_2 - x_1} B(x_2, r_2)C(x_1, r_1)\\
&\quad = \frac{x_2(r_2^{-2} - 1)}{r_2^{-2}x_2 - x_1} A(x_1, r_1)D(x_2, r_2) + \frac{r_2^{-2}x_2 - r_1^{-2}x_1}{r_2^{-2}x_2 - x_1} C(x_1, r_1)B(x_2, r_2). && (2.20)
\end{aligned}$$

Proof First, assume that the operators act in a single two-dimensional space $V^{(k)} \simeq \mathbb{C}^2$. Then all of the desired commutation relations follow from the Yang–Baxter equation

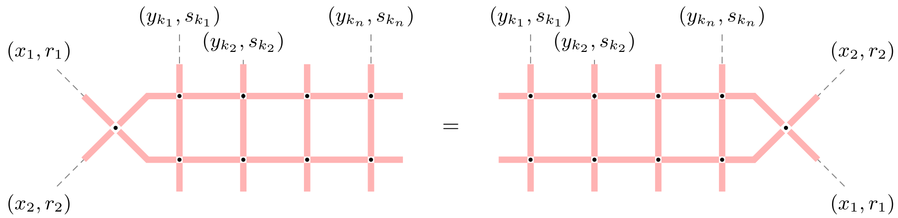


Fig. 8 The Yang–Baxter equation for a horizontal chain of two-vertex configurations

of Proposition 2.2. Let us show how to get one of these relations, say, (2.14), the others are obtained in a similar way. Write the Yang–Baxter equation (2.7) with the boundary conditions $i_1 = i_2 = 0$, $i_3 = j_1 = 1$, $j_2 = j_3 = 0$ and with the parameters (x_1, r_1) and (x_2, r_2) interchanged. In the operator form, this Yang–Baxter equation reads

$$\begin{aligned} D(x_1, r_1)B(x_2, r_2) = \frac{x_2(1 - r_2^{-2})}{x_2 - x_1 r_1^{-2}} D(x_2, r_2)B(x_1, r_1) \\ + \frac{x_2 - x_1}{x_2 - x_1 r_1^{-2}} B(x_2, r_2)D(x_1, r_1). \end{aligned} \tag{2.21}$$

Note that in the product $D(x_1, r_1)B(x_2, r_2)$, the B and D operator corresponds to the bottom and, respectively, top, vertex in the left-hand side of Fig. 6, and same for all other products in this proposition. In (2.21) we then move the term containing $D(x_2, r_2)B(x_1, r_1)$ into the left-hand side, and divide the identity by the prefactor $(x_2 - x_1)/(x_2 - x_1 r_1^{-2})$ in front of $B(x_2, r_2)D(x_1, r_1)$. This gives (2.14) for the case of the two-dimensional space $V^{(k)} \simeq \mathbb{C}^2$.

To extend the relations to arbitrary tensor products $V^{(k_1)} \otimes V^{(k_2)} \otimes \ldots \otimes V^{(k_n)}$, we apply the standard "zipper argument" to establish the Yang–Baxter equation for a horizontal chain of two-vertex configurations, see Fig. 8. This equation follows by sequentially applying the original equation (2.7) to move the cross vertex and swap the parameters (x_i, r_i). Then the Yang–Baxter equation corresponding to the space $V^{(k_1)} \otimes V^{(k_2)} \otimes \ldots \otimes V^{(k_n)}$ implies all the desired commutation relations. □

Using the weights $\widehat{W}$ (2.4), define four operators $\widehat{A} = \widehat{A}(x, r)$, $\widehat{B} = \widehat{B}(x, r)$, $\widehat{C} = \widehat{C}(x, r)$, and $\widehat{D} = \widehat{D}(x, r)$ acting from the right on each two-dimensional space $V^{(k)}$ as

$$\begin{aligned} e_i^{(k)}\, \widehat{A} &= \widehat{W}(i, 1; i, 1 \mid x; y_k; r; s_k)\, e_i^{(k)}; \\ e_i^{(k)}\, \widehat{B} &= \widehat{W}(i+1, 0; i, 1 \mid x; y_k; r; s_k)\, e_{i+1}^{(k)}; \\ e_i^{(k)}\, \widehat{C} &= \widehat{W}(i-1, 1; i, 0 \mid x; y_k; r; s_k)\, e_{i-1}^{(k)}; \\ e_i^{(k)}\, \widehat{D} &= \widehat{W}(i, 0; i, 0 \mid x; y_k; r; s_k)\, e_i^{(k)}. \end{aligned} \tag{2.22}$$

Note the difference with the operators A, B, C, D (2.8) which read vectors $e_i^{(k)}$ corresponding to a vertex $(i_1, j_1; i_2, j_2)$ "from bottom to top" (i.e., map $e_{i_1}^{(k)}$ to $e_{i_2}^{(k)}$), while

$\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}$ read vectors "from top to bottom". Note that the states of the left and right edges for the same-letter operators in (2.8) and (2.22) are same.

We extend the operators (2.22) to tensor products $V^{(k_1)} \otimes V^{(k_2)} \otimes \ldots \otimes V^{(k_n)}$ by first defining for $n = 2$,

$$\begin{aligned}(v_1 \otimes v_2)\widehat{A} &= v_1\widehat{C} \otimes v_2\widehat{B} + v_1\widehat{A} \otimes v_2\widehat{A};\\ (v_1 \otimes v_2)\widehat{B} &= v_1\widehat{D} \otimes v_2\widehat{B} + v_1\widehat{B} \otimes v_2\widehat{A};\\ (v_1 \otimes v_2)\widehat{C} &= v_1\widehat{C} \otimes v_2\widehat{D} + v_1\widehat{A} \otimes v_2\widehat{C};\\ (v_1 \otimes v_2)\widehat{D} &= v_1\widehat{D} \otimes v_2\widehat{D} + v_1\widehat{B} \otimes v_2\widehat{C},\end{aligned} \tag{2.23}$$

and then for arbitrary n by induction similarly to (2.9).

The operators $\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}$ acing in any tensor product $V^{(k_1)} \otimes V^{(k_2)} \otimes \ldots \otimes V^{(k_n)}$ satisfy commutation relations which parallel the ones in Proposition 2.4. In the next proposition, let us list a few relations which are employed in proofs later in the paper. They also follow from the Yang–Baxter equation (Proposition 2.2) and the "zipper argument", as in the proof of Proposition 2.4.

Proposition 2.5 *For any $x_1, x_2, r_1, r_2 \in \mathbb{C}$, we have*

$$\widehat{A}(x_2, r_2)\widehat{A}(x_1, r_1) = \widehat{A}(x_1, r_1)\widehat{A}(x_2, r_2); \tag{2.24}$$

$$\widehat{B}(x_2, r_2)\widehat{B}(x_1, r_1) = \frac{x_2 - r_1^{-2}x_1}{x_1 - r_2^{-2}x_2}\widehat{B}(x_1, r_1)\widehat{B}(x_2, r_2); \tag{2.25}$$

$$\begin{aligned}\widehat{B}(x_2, r_2)\widehat{D}(x_1, r_1) =& \frac{r_1^{-2}x_1 - x_2}{x_1 - x_2}\widehat{D}(x_1, r_1)\widehat{B}(x_2, r_2)\\ &+ \frac{(1 - r_2^{-2})x_2}{x_1 - x_2}\widehat{D}(x_2, r_2)\widehat{B}(x_1, r_1);\end{aligned} \tag{2.26}$$

$$\begin{aligned}\widehat{B}(x_2, r_2)\widehat{A}(x_1, r_1) =& \frac{r_1^{-2}x_1 - x_2}{x_2 - x_1}\widehat{A}(x_1, r_1)\widehat{B}(x_2, r_2)\\ &+ \frac{(1 - r_2^{-2})x_2}{x_2 - x_1}\widehat{A}(x_2, r_2)\widehat{B}(x_1, r_1);\end{aligned} \tag{2.27}$$

$$\widehat{D}(x_2, r_2)\widehat{D}(x_1, r_1) = \widehat{D}(x_1, r_1)\widehat{D}(x_2, r_2). \tag{2.28}$$

3 Symmetric functions

Here we define symmetric functions F_λ and G_λ (indexed by signatures λ) which are partition functions of certain configurations of the free fermion six vertex model.

3.1 Signature states

A (generalized) signature with N parts is a nonincreasing integer sequence

$$\lambda = (\lambda_1 \geq \ldots \geq \lambda_N), \qquad \lambda_i \in \mathbb{Z}.$$

Denote $|\lambda| := \lambda_1 + \cdots + \lambda_N$. We will mostly deal with nonnegative signatures, i.e., such that $\lambda_N \geq 0$, and will omit the word "nonnegative". To a signature λ with N parts we associate a configuration

$$\mathcal{S}(\lambda) = (\lambda_1 + N, \lambda_2 + N - 1, \ldots, \lambda_{N-1} + 2, \lambda_N + 1) \subset \mathbb{Z}_{\geq 1} \tag{3.1}$$

of distinct points in the integer half-line.

Consider the (formal) infinite tensor product $V^{(1)} \otimes V^{(2)} \otimes \ldots$, where $V^{(k)} \simeq \mathbb{C}^2$ for all k with basis $e_0^{(k)}, e_1^{(k)}$. Let $\mathcal{V}$ be the subset of the infinite tensor product spanned by the following vectors:

$$e_{\mathcal{T}} = e_{m_1}^{(1)} \otimes e_{m_2}^{(2)} \otimes e_{m_3}^{(3)} \otimes \ldots, \qquad m_i = \begin{cases} 1, & i \in \mathcal{T}; \\ 0, & \text{otherwise}, \end{cases} \tag{3.2}$$

where $\mathcal{T} \subset \mathbb{Z}_{\geq 1}$ runs over arbitrary finite sets. These basis vectors are called *finitary*. (We will sometimes use the same notation (3.2) for arbitrary subsets $\mathcal{T} \subseteq \mathbb{Z}_{\geq 1}$.) In particular, with each signature λ we associate a *signature state* $e_{\mathcal{S}(\lambda)}$. Note that all but finitely many of the tensor components of $e_{\mathcal{S}(\lambda)}$ are $e_0^{(k)}$.

By $\mathcal{V}_\ell$, $\ell = 0, 1, \ldots$, denote the subspace of $\mathcal{V}$ spanned by $e_{\mathcal{T}}$ with $\mathcal{T}$ running over all ℓ-element subsets of $\mathbb{Z}_{\geq 1}$. Let us extend some of the row operators defined in Sect. 2.3 to act in the space $\mathcal{V}$. Namely, thanks to $W(0, 0; 0, 0) = 1$, the operators $C(x, r)$ and $D(x, r)$ act in each of the $\mathcal{V}_\ell$'s, and

$$C(x, r) \colon \mathcal{V}_\ell \to \mathcal{V}_{\ell+1}, \qquad D(x, r) \colon \mathcal{V}_\ell \to \mathcal{V}_\ell. \tag{3.3}$$

Similarly, thanks to $\widehat{W}(0, 1; 0, 1) = 1$, the operators $\widehat{A}(x, r)$ and $\widehat{B}(x, r)$ act as

$$\widehat{A}(x, r) \colon \mathcal{V}_\ell \to \mathcal{V}_\ell, \qquad \widehat{B}(x, r) \colon \mathcal{V}_\ell \to \mathcal{V}_{\ell+1}. \tag{3.4}$$

These operators satisfy the commutation relations (2.24), (2.25), and (2.27). Note that while the spaces $\mathcal{V}_\ell$ involve infinite tensor products, in the action of the operators $\widehat{A}, \widehat{B}, C, D$ the boundary condition at infinity is always determined uniquely.

Remark 3.1 Later in Sect. 7 we employ a similar infinite tensor product over the whole lattice $\mathbb{Z}$, or, more precisely, a corresponding Fock space, to compute a generating function for correlations of certain probability distributions based on our symmetric functions.

3.2 Symmetric functions as partition functions

Here we define the functions $F_{\lambda/\mu}(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s})$ and $G_{\lambda/\mu}(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s})$ as certain partition functions of the free fermion six vertex model.

Definition 3.2 Fix a positive integer k, two signatures λ, μ with the same number of parts, and parameters $\mathbf{x} = (x_1, \ldots, x_k)$, $\mathbf{y} = (y_1, y_2, \ldots)$, $\mathbf{r} = (r_1, \ldots, r_k)$, $\mathbf{s} = (s_1, s_2, \ldots)$.

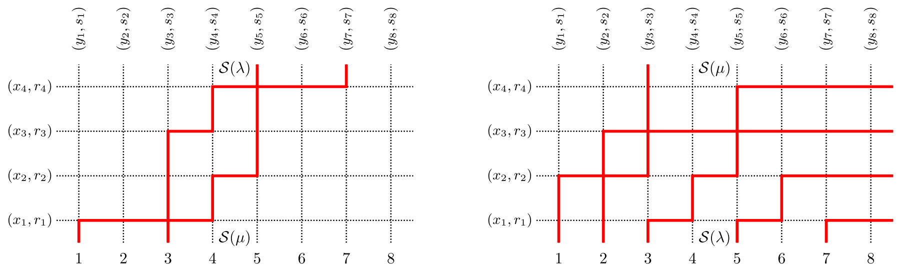


Fig. 9 Examples of path configurations contributing to the partition functions in Definitions 3.2 and 3.3. Left: $G_{\lambda/\mu}$ with $\lambda = (5, 4)$ and $\mu = (1, 0)$. Right: $F_{\lambda/\mu}$ with $\lambda = (2, 1, 0, 0, 0)$ and $\mu = (2)$

Consider the following boundary data in the half-infinite rectangle $\mathbb{Z}_{\geq 1} \times \{1, \dots, k\}$. A path vertically enters the rectangle from the bottom at each $m \in \mathcal{S}(\mu)$; a path vertically exits the rectangle at the top at each $\ell \in \mathcal{S}(\lambda)$; the left and right boundaries of the rectangle, as well as all the other boundary edges on top and bottom, are empty (see Fig. 9, left, for an illustration).

Let the vertex weight at each (i, j) in the rectangle be $W(i_1, j_1; i_2, j_2 \mid x_j; y_i; r_j; s_i)$ (2.3). That is, the parameters (x, r) are constant along the horizontal, and (y, s) are constant along the vertical direction. Denote the partition function of thus defined vertex model in the half-infinite rectangle by $G_{\lambda/\mu} = G_{\lambda/\mu}(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s})$. Even though the domain is infinite, this partition function is well-defined thanks to $W(0, 0; 0, 0) = 1$.

In the particular case $\mu = (0, \dots, 0)$ (with the same number of parts as in λ), we abbreviate $G_\lambda = G_{\lambda/\mu}$.

Definition 3.3 Within the notation of Definition 3.2, let now the number of parts in λ be $N + k$ and the number of parts in μ be N for some $N \in \mathbb{Z}_{\geq 0}$. Consider the following (different) boundary data for the half-infinite rectangle $\mathbb{Z}_{\geq 1} \times \{1, \dots, k\}$. Let a path enter vertically from the bottom at each $\ell \in \mathcal{S}(\lambda) = (\lambda_1 + N + k, \lambda_2 + N + k - 1, \dots, \lambda_{N+k} + 1)$; a path exit vertically at the top at each $m \in \mathcal{S}(\mu) = (\mu_1 + N, \mu_2 + N - 1, \dots, \mu_N + 1)$; and a path exit the rectangle far to the right at each horizontal layer. Let all the other boundary edges of the rectangle be empty (see Fig. 9, right, for an illustration).

Let the vertex weight at each (i, j) in the rectangle be $\widehat{W}(i_1, j_1; i_2, j_2 \mid x_j; y_i; r_j; s_i)$ (2.4). Denote the partition function of this vertex model by $F_{\lambda/\mu} = F_{\lambda/\mu}(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s})$. This partition function is well-defined because $\widehat{W}(0, 1; 0, 1) = 1$.

In the particular case $N = 0$ (i.e., when $\mu = \varnothing$ is the empty signature with no parts), we abbreviate $F_\lambda = F_{\lambda/\mu}$.

We extend the definitions of $G_{\lambda/\mu}$ and $F_{\lambda/\mu}$ to arbitrary pairs of signatures λ, μ (without the restrictions on the number of parts), by setting these functions to zero if there are no path configurations with the prescribed boundary data.

From the definitions it follows that the partition functions $G_{\lambda/\mu}$ and $F_{\lambda/\mu}$ are written in terms of the row operators from Sect. 2.3 acting in the subspace $\mathcal{V}$ of the infinite tensor product (as explained in Sect. 3.1):

Proposition 3.4 *The function $G_{\lambda/\mu}(x_1,\ldots,x_k;\mathbf{y};r_1,\ldots,r_k;\mathbf{s})$ is equal to the coefficient by $e_{\mathcal{S}(\lambda)}$ in $D(x_k,r_k)\ldots D(x_1,r_1)e_{\mathcal{S}(\mu)}$. Here $e_{\mathcal{S}(\lambda)}, e_{\mathcal{S}(\mu)}$ belong to the same subspace $\mathcal{V}_N$, and the product of the $D(x_i,r_i)$'s preserves this subspace by* (3.3).

Similarly, the function $F_{\lambda/\mu}(x_1,\ldots,x_k;\mathbf{y};r_1,\ldots,r_k;\mathbf{s})$ is equal to the coefficient by $e_{\mathcal{S}(\lambda)}$ in $e_{\mathcal{S}(\mu)}\widehat{B}(x_k,r_k)\ldots\widehat{B}(x_1,r_1)$. Here $e_{\mathcal{S}(\mu)}\in\mathcal{V}_N$ and $e_{\mathcal{S}(\lambda)}\in\mathcal{V}_{N+k}$, and the product of the $\widehat{B}(x_i,r_i)$'s maps $\mathcal{V}_N$ to $\mathcal{V}_{N+k}$, see (3.4).

3.3 Symmetry and branching

Let us derive a few basic properties of the functions F, G defined in the previous subsection. For each $i\geq 1$, let σ_i denote the elementary transposition of the indices $i\leftrightarrow i+1$, and define its action on (various) sets of variables as

$$\sigma_i(x_1,\ldots,x_{i-1},x_i,x_{i+1},x_{i+2}\ldots)=(x_1,\ldots,x_{i-1},x_{i+1},x_i,x_{i+2},\ldots),$$

and similarly for $\mathbf{r},\mathbf{y},\mathbf{s}$.

Proposition 3.5 (Symmetries) *For each $k\geq 1$ and $i\in\{1,\ldots,k-1\}$ we have the following symmetry properties:*

$$G_{\lambda/\mu}(\sigma_i(\mathbf{x});\mathbf{y};\sigma_i(\mathbf{r});\mathbf{s})=G_{\lambda/\mu}(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s});$$
$$F_{\lambda/\mu}(\sigma_i(\mathbf{x});\mathbf{y};\sigma_i(\mathbf{r});\mathbf{s})=\frac{x_i-x_{i+1}r_{i+1}^{-2}}{x_{i+1}-x_ir_i^{-2}}\,F_{\lambda/\mu}(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s}).$$

In other words, the functions $G_{\lambda/\mu}(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s})$ and $F_{\lambda/\mu}(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s})\prod_{1\leq i<j\leq k}(x_i-x_jr_j^{-2})$ are symmetric under simultaneous permutations of the variables (x_j,r_j).

Proof The symmetry properties of the functions G and F follow, respectively, from the commutation relations (2.13) and (2.25) for the operators acting in the subspace $\mathcal{V}$ of the infinite tensor product (see Sect. 3.1).

Proposition 3.6 (Branching) *Fix integers $M, N\geq 1$ and sets of complex variables*

$$\mathbf{x}=(x_1,\ldots,x_M),\quad \mathbf{x}'=(x_{M+1},\ldots,x_{M+N}),$$
$$\mathbf{r}=(r_1,\ldots,r_M),\quad \mathbf{r}'=(r_{M+1},\ldots,r_{M+N}),$$
$$\mathbf{y}=(y_1,y_2,y_3,\ldots,),\qquad \mathbf{s}=(s_1,s_2,s_3,\ldots).$$

Define $\mathbf{x}\cup\mathbf{x}'=(x_1,\ldots,x_{M+N})$, $\mathbf{r}\cup\mathbf{r}'=(r_1,\ldots,r_{M+N})$. Then for any signatures λ,μ we have

$$\sum_\nu G_{\lambda/\nu}(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s})\,G_{\nu/\mu}(\mathbf{x}';\mathbf{y};\mathbf{r}';\mathbf{s})=G_{\lambda/\mu}(\mathbf{x}\cup\mathbf{x}';\mathbf{y};\mathbf{r}\cup\mathbf{r}';\mathbf{s});$$
$$\sum_\nu F_{\lambda/\nu}(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s})\,F_{\nu/\mu}(\mathbf{x}';\mathbf{y};\mathbf{r}';\mathbf{s})=F_{\lambda/\mu}(\mathbf{x}\cup\mathbf{x}';\mathbf{y};\mathbf{r}\cup\mathbf{r}';\mathbf{s}).$$

Proof These identities follow from Definitions 3.2 and 3.3, respectively. For example, for the first identity consider the vertex model in $\mathbb{Z}_{\geq 1} \times \{1, \dots, M+N\}$ for the right-hand side $G_{\lambda/\mu}$. Encode the configuration of the vertical arrows between rows M and $M+1$ by a signature ν. Then the bottom and the top vertex models in $\mathbb{Z}_{\geq 1} \times \{1, \dots, M\}$ and $\mathbb{Z}_{\geq 1} \times \{M+1, \dots, N\}$ have partition functions $G_{\nu/\mu}$ and $G_{\lambda/\nu}$, respectively. Summing over ν leads to the desired identity (note that this sum is finite). The second identity is proven in the same way.

3.4 Cauchy identity

The functions F, G satisfy Cauchy-type summation identities which follow from the Yang–Baxter equation.

Proposition 3.7 (Skew Cauchy identity) *Fix two signatures λ, μ, integers $M, N \geq 1$, and sequences of complex variables*

$$\mathbf{x} = (x_1, \dots, x_M), \qquad \mathbf{r} = (r_1, \dots, r_M);$$
$$\mathbf{w} = (w_1, \dots, w_N), \qquad \boldsymbol{\theta} = (\theta_1, \dots, \theta_N);$$
$$\mathbf{y} = (y_1, y_2, y_3, \dots,), \qquad \mathbf{s} = (s_1, s_2, s_3, \dots),$$

satisfying

$$\left| \frac{y_k - s_k^2 x_i}{y_k - x_i} \frac{y_k - w_j}{y_k - s_k^2 w_j} \right| < 1 - \delta < 1 \qquad \text{forall } 1 \leq i \leq M, 1 \leq j \leq N, \text{ and } k \geq 1. \tag{3.5}$$

Then we have

$$\sum_{\nu} G_{\nu/\lambda}(\mathbf{w}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s}) F_{\nu/\mu}(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s}) = \prod_{i=1}^{M} \prod_{j=1}^{N} \frac{x_i - \theta_j^{-2} w_j}{x_i - w_j} \sum_{\varkappa} G_{\mu/\varkappa}(\mathbf{w}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s}) F_{\lambda/\varkappa}(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s}). \tag{3.6}$$

Proof With the help of of Proposition 3.6 and induction on M and N, it suffices to prove (3.6) for $M = N = 1$.

Fix λ, μ with $K+1$ and K parts, respectively, for some $K \geq 0$ (other choices for the numbers of parts of λ and μ lead to triviality of both sides). Interpret $\sum_{\varkappa} G_{\mu/\varkappa}(w; \mathbf{y}; \theta; \mathbf{s}) F_{\lambda/\varkappa}(x; \mathbf{y}; r; \mathbf{s})$ as a partition function of a vertex model in the half-infinite rectangle $\mathbb{Z}_{\geq 1} \times \{1, 2\}$, with the boundary conditions $\mathcal{S}(\lambda)$ at the bottom, $\mathcal{S}(\mu)$ at the top, empty on the left, and {full, empty} on the far right. The weights at vertices $(k, 1)$ and $(k, 2)$, $k \geq 1$, are $\widehat{W}(\cdots \mid x; y_k; r; s_k)$ and $W(\cdots \mid w; y_k; \theta; s_k)$, respectively. Due to this choice of the weights, the partition function is well-defined

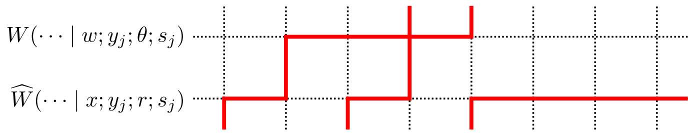


Fig. 10 The partition function corresponding to the sum over $\varkappa$ in the right-hand side of the skew Cauchy identity (3.6). Adding the empty cross vertex on the left and dragging it to the right with the help of the Yang–Baxter equation produces the sum over ν

(all vertices which are repeated infinitely often have weight 1). See Fig. 10 for an illustration.

Add the empty cross vertex with weight $R(0, 0; 0, 0) = 1$ to the left of $\mathbb{Z}_{\geq 1} \times \{1, 2\}$, and use the Yang–Baxter equation (Proposition 2.2) to move it to the right. After $L \geq \max(\lambda_1 + K + 1, \mu_1 + K)$ steps we get the identity

$$\begin{aligned}
&\sum_{\varkappa} G_{\mu/\varkappa}(w; \mathbf{y}; \theta; \mathbf{s}) F_{\lambda/\varkappa}(x; \mathbf{y}; r; \mathbf{s}) \\
&\quad = R(0, 1; 0, 1) \sum_{\nu \colon \nu_1 + K + 1 \leq L} G_{\nu/\lambda}(w; \mathbf{y}; \theta; \mathbf{s}) F_{\nu/\mu}(x; \mathbf{y}; r; \mathbf{s}) \\
&\quad\quad + \tilde{Z} \cdot R(1, 0; 0, 1) \prod_{k=\max(\lambda_1+K+1,\, \mu_1+K)+1}^{L} W(0, 1; 0, 1 \mid w; y_k; \theta; s_k) \\
&\quad\quad \widehat{W}(0, 0; 0, 0 \mid x; y_k; r; s_k).
\end{aligned}$$

Here $R(0, 1; 0, 1) = \dfrac{x - w}{x - \theta^{-2} w}$, and $\tilde{Z}$ is a quantity independent of L. Sending $L \to +\infty$, we see that the product over k in the second summand goes to zero thanks to (3.5), while in the first summand the restriction $\nu_1 + K + 1 \leq L$ disappears. Dividing both sides by $R(0, 1; 0, 1)$ produces the desired identity (3.6).

As a corollary of Proposition 3.7 and the determinantal formula for F (3.12) from the next Sect. 3.5, we also get the following identity:

Theorem 3.8 (Cauchy identity, Theorem 1.3 from Introduction) *In the setting of Proposition 3.7 we have*

$$\sum_{\nu} G_{\nu}(\mathbf{w}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s})\, F_{\nu}(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s}) = \prod_{i=1}^{M} x_i (r_i^{-2} - 1) \frac{\prod_{1 \leq i < j \leq M} (r_i^{-2} x_i - x_j)(s_i^{-2} y_i - y_j)}{\prod_{i,j=1}^{M} (y_i - x_j)} \prod_{i=1}^{M} \prod_{j=1}^{N} \frac{x_i - \theta_j^{-2} w_j}{x_i - w_j}, \tag{3.7}$$

where the summation in the left-hand side is over all signatures ν with M parts.

Proof This is a particular case of Proposition 3.7 when $\lambda = 0^M$ (this notation stands for the signature $(0, \ldots, 0)$ with 0 repeated M times) and $\mu = \varnothing$. The sum in the right-hand side of (3.6) reduces to a single term with $\varkappa = \varnothing$, and $G_{\varnothing/\varnothing} = 1$. We thus get

$$\sum_{\nu} G_{\nu}(\mathbf{w};\mathbf{y};\boldsymbol{\theta};\mathbf{s})\, F_{\nu}(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s}) = F_{0^M}(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s}) \prod_{i=1}^{M}\prod_{j=1}^{N} \frac{x_i - \theta_j^{-2} w_j}{x_i - w_j}. \tag{3.8}$$

Using Theorem 3.9 formulated below, we have

$$F_{0^M}(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s}) = \left(\prod_{i=1}^{M} x_i (r_i^{-2} - 1) \prod_{1\le i<j\le M} \frac{r_i^{-2} x_i - x_j}{x_i - x_j}\right) \det\left[\frac{1}{y_{M-j+1} - x_i} \prod_{m=1}^{M-j} \frac{y_m - s_m^2 x_i}{s_m^2 (y_m - x_i)}\right]_{i,j=1}^{M}.$$

This determinant has an explicit product form:

$$\det\left[\frac{1}{y_{M-j+1} - x_i} \prod_{m=1}^{M-j} \frac{y_m - s_m^2 x_i}{s_m^2 (y_m - x_i)}\right]_{i,j=1}^{M} = \frac{\prod_{i<j} (x_i - x_j)(s_i^{-2} y_i - y_j)}{\prod_{i,j=1}^{M} (y_i - x_j)}. \tag{3.9}$$

This can be established by induction on M using the Desnanot–Jacobi identity for determinants:

$$\det(A) = \frac{\det(A_1^1)\det(A_M^M) - \det(A_1^M)\det(A_M^1)}{\det(A_{1,M}^{1,M})}, \tag{3.10}$$

where A is the $M \times M$ matrix in the left-hand side of (3.9), and $A_{\mathcal{J}}^{\mathcal{I}}$, $\mathcal{I}, \mathcal{J} \subset \{1, \ldots, M\}$, $|\mathcal{I}| = |\mathcal{J}|$, is the matrix obtained from A by deleting rows and columns indexed by $\mathcal{I}$ and $\mathcal{J}$, respectively. Each of the matrices in the right-hand side of sizes $M-1$ and $M-2$ are essentially the ones in the left-hand side of (3.9), up to shifts in some of the parameters x_i, y_i, s_i, and diagonal factors. More precisely, denote the matrix elements by

$$a_{ij}^{(M)} = \frac{1}{y_{M-j+1} - x_i} \prod_{m=1}^{M-j} \frac{y_m - s_m^2 x_i}{s_m^2 (y_m - x_i)}.$$

One can readily check that

$$(A_M^M)_{ij} \frac{x_i - y_1}{x_i - s_1^{-2} y_1} = a_{ij}^{(M-1)}\Big|_{y_i \to y_{i+1},\ s_i \to s_{i+1}}; \qquad (A_1^1)_{ij} = a_{ij}^{(M-1)}\Big|_{x_i \to x_{i+1}};$$

$$(A_M^1)_{ij} \frac{x_{i+1} - y_1}{x_{i+1} - s_1^{-2} y_1} = a_{ij}^{(M-1)}\Big|_{x_i \to x_{i+1},\ y_i \to y_{i+1},\ s_i \to s_{i+1}}; \qquad (A_1^M)_{ij} = a_{ij}^{(M-1)};$$

$$(A_{1,M}^{1,M})_{ij}\,\frac{x_i - y_1}{x_i - s_1^{-2} y_1} = a_{ij}^{(M-2)}\Big|_{x_i \to x_{i+1},\ y_i \to y_{i+1},\ s_i \to s_{i+1}}.$$

Thus, by the induction hypothesis, all the determinants in the right-hand side of (3.10) are expressed as certain products. The induction step then follows by matching the combination (3.10) of these products with the desired right-hand side of (3.9). Putting all together produces the Cauchy identity (3.7).

3.5 Determinantal and Sergeev–Pragacz type formulas

The function F_λ admits an explicit formula involving a determinant of the single-variable functions $F_{(m)}$. The function G_λ admits a more complicated (yet still explicit) formula in terms of a summation over the product of two symmetric groups. In this section we formulate both expressions, and their proofs based on commutation relations for the row operators (Sect. 2.3) are postponed to Appendix A. See also the next Sect. 4 for simpler proofs in some particular cases.

For sequences of complex numbers $\mathbf{s} = (s_1, s_2, \ldots)$, $\mathbf{y} = (y_1, y_2, \ldots)$ and any integer $k \geq 0$ denote

$$\varphi_k(x) = \varphi_k(x \mid \mathbf{y}; \mathbf{s}) := \frac{1}{y_{k+1} - x} \prod_{j=1}^{k} \frac{x - s_j^{-2} y_j}{x - y_j}. \tag{3.11}$$

In particular, $\varphi_0(x \mid \mathbf{y}; \mathbf{s}) = 1/(y_1 - x)$. From Definition 3.3 and the formula for the vertex weights $\widehat{W}$ (2.4) we have $F_{(k)}(x; \mathbf{y}; r; \mathbf{s}) = x(r^{-2} - 1)\,\varphi_k(x \mid \mathbf{y}; \mathbf{s})$ for all $k \geq 0$.

Theorem 3.9 (Determinantal formula for F_λ, Theorem 1.5 from Introduction) *For any $N \geq 1$, complex variables*

$$\mathbf{x} = (x_1, \ldots, x_N), \qquad \mathbf{r} = (r_1, \ldots, r_N), \qquad \mathbf{y} = (y_1, y_2, \ldots), \qquad \mathbf{s} = (s_1, s_2, \ldots),$$

and any signature $\lambda = (\lambda_1 \geq \ldots \geq \lambda_N)$ we have

$$F_\lambda(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s}) = \left(\prod_{i=1}^{N} x_i (r_i^{-2} - 1) \prod_{1 \leq i < j \leq N} \frac{r_i^{-2} x_i - x_j}{x_i - x_j} \right) \det \left[\varphi_{\lambda_j + N - j}(x_i \mid \mathbf{y}; \mathbf{s}) \right]_{i,j=1}^{N}, \tag{3.12}$$

where φ_k are defined by (3.11).

This theorem is proven in Appendix A.1.

In the next theorem and throughout the text, $\mathfrak{S}_m$ denotes the group of all permutations of $\{1, \ldots, m\}$. Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_N)$ be an arbitrary signature with N parts. Let $d = d(\lambda) \geq 0$ denote the integer such that $\lambda_d \geq d$ and $\lambda_{d+1} < d + 1$. Denote by

$\ell_j = \lambda_j + N - j + 1$, $j = 1, \dots, N$, the elements of the set $\mathcal{S}(\lambda)$. Let

$$\mu = (\mu_1 < \mu_2 < \dots < \mu_d) = \{1, \dots, N\} \setminus \big(\mathcal{S}(\lambda) \cap \{1, \dots, N\}\big). \tag{3.13}$$

Theorem 3.10 (Sergeev–Pragacz type formula for G_λ) *Let $M, N > 0$ denote integers. For complex variables*

$$\mathbf{x} = (x_1, x_2, \dots, x_M), \quad \mathbf{r} = (r_1, r_2, \dots, r_M), \quad \mathbf{y} = (y_1, y_2, \dots), \quad \mathbf{s} = (s_1, s_2, \dots),$$

with the notation above, we have

$$\begin{aligned} G_\lambda(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s}) &= \prod_{j=1}^{M} \prod_{k=1}^{N} \frac{y_k - s_k^2 r_j^{-2} x_j}{y_k - s_k^2 x_j} \sum_{\substack{\mathcal{I}, \mathcal{J} \subseteq \{1, \dots, M\} \\ |\mathcal{I}| = |\mathcal{J}| = d}} \prod_{\substack{i \in \mathcal{J}^c \\ j \in \mathcal{J}}} \frac{1}{x_i - x_j} \prod_{\substack{i, j \in \mathcal{J} \\ i < j}} \frac{1}{x_j - x_i} \\ &\times \prod_{\substack{i \in \mathcal{I} \\ 1 \le j \le M}} (r_i^{-2} x_i - x_j) \prod_{\substack{i \in \mathcal{I}^c \\ j \in \mathcal{J}}} (r_i^{-2} x_i - x_j) \prod_{\substack{i \in \mathcal{I} \\ j \in \mathcal{I}^c}} \frac{1}{r_i^{-2} x_i - r_j^{-2} x_j} \prod_{\substack{i, j \in \mathcal{I} \\ i < j}} \frac{1}{r_i^{-2} x_i - r_j^{-2} x_j} \\ &\times \sum_{\sigma, \rho \in \mathfrak{S}_d} \operatorname{sgn}(\sigma\rho) \prod_{h=1}^{d} \left(\frac{y_{\ell_h} \big(1 - s_{\ell_h}^2\big)}{y_{\ell_h} - s_{\ell_h}^2 x_{j_{\rho(h)}}} \prod_{i=N+1}^{\ell_h - 1} \frac{s_i^2 \big(y_i - x_{j_{\rho(h)}}\big)}{y_i - s_i^2 x_{j_{\rho(h)}}} \right) \\ &\times \prod_{m=1}^{d} \left(\frac{s_{\mu_m}^2}{y_{\mu_m} - s_{\mu_m}^2 r_{i_{\sigma(m)}}^{-2} x_{i_{\sigma(m)}}} \prod_{k=\mu_m+1}^{N} \frac{s_k^2 \big(r_{i_{\sigma(m)}}^{-2} x_{i_{\sigma(m)}} - y_k\big)}{y_k - s_k^2 r_{i_{\sigma(m)}}^{-2} x_{i_{\sigma(m)}}} \right). \end{aligned} \tag{3.14}$$

where $\mathcal{I} = (i_1 < i_2 < \dots < i_d)$ and $\mathcal{J} = (j_1 < j_2 < \dots < j_d)$. Note that both sides of (3.14) vanish if $d(\lambda) > M$, as it should be.

This theorem is proven in Appendix A.2. The name "Sergeev–Pragacz type formula" for (3.14) is suggested by the connection between G_λ (in the horizontally homogeneous case) and supersymmetric Schur functions. Even though we could not relate (3.14) to the Sergeev–Pragacz formula (e.g., see [49, (5)]) itself, the form of our formula is sufficiently similar to justify the name. See Sect. 4.2 and in particular Remark 4.15 below for a detailed discussion.

4 Specializations to known symmetric functions

Here we discuss how our functions F and G degenerate to certain known Schur-type symmetric functions. This leads to independent proofs of Theorems 3.9 and 3.10 in some particular cases.

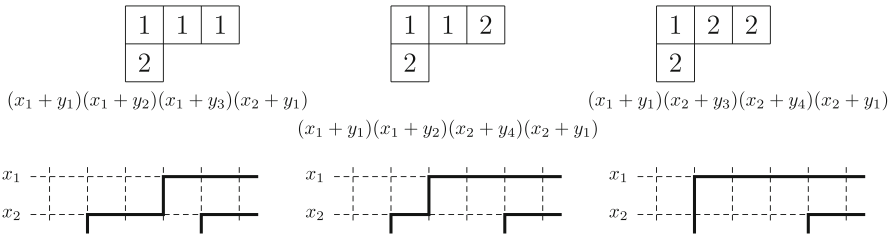


Fig. 11 All three semistandard Young tableaux for $\lambda = (3, 1)$ and $N = 2$, the corresponding summands in $s_\lambda(\mathbf{x} \mid \mathbf{y})$ given by formula (4.2), and the corresponding vertex model configurations in the proof of Lemma 4.2

4.1 Five vertex model and factorial Schur polynomials

We begin with a five vertex degeneration, when the weight of the vertex of type $(1, 1; 1, 1)$ vanishes, and connect the functions F_λ to the *factorial Schur polynomials* (also sometimes called double Schur polynomials, cf. [78]). We adopt the definition from [72, 6th variation]. The factorial Schur polynomials are indexed by signatures $\lambda = (\lambda_1 \geq \ldots \geq \lambda_N \geq 0)$, depend on the variables $\mathbf{x} = (x_1, \ldots, x_N)$ and on a sequence $\mathbf{y} = (y_1, y_2, \ldots)$ of complex parameters:

$$s_\lambda(\mathbf{x} \mid \mathbf{y}) = \frac{\det\left[(x_i \mid \mathbf{y})^{\lambda_j+N-j}\right]_{i,j=1}^{N}}{\prod_{1\leq i<j\leq N}(x_i - x_j)}, \qquad (x \mid \mathbf{y})^k := (x+y_1)\ldots(x+y_k). \tag{4.1}$$

These polynomials can also be represented as sums over semistandard Young tableaux. We use the language of Young tableaux only in this subsection, and so refer to, e.g., [73, Ch. I] for the relevant definitions. We have [72, (6.16)]

$$s_\lambda(\mathbf{x} \mid \mathbf{y}) = \sum_T (\mathbf{x} \mid \mathbf{y})^T, \qquad (\mathbf{x} \mid \mathbf{y})^T := \prod_{(i,j)\in\lambda} \left(x_{T(i,j)} + y_{T(i,j)+j-i}\right). \tag{4.2}$$

Here the sum is taken over all semistandard Young tableaux of shape λ filled with numbers from 1 to N, the product is over all boxes (i, j) in λ, where i and j are the row and column coordinates of the box, and $T(i, j)$ is the tableau entry in this box. See an example in Fig. 11.

Remark 4.1 In the particular case $\mathbf{y} = (0, 0, \ldots)$, the factorial Schur polynomials $s_\lambda(\mathbf{x} \mid \mathbf{y})$ turn into the ordinary Schur polynomials $s_\lambda(x_1, \ldots, x_N)$.

The tableaux formula (4.2) can be translated into the vertex model language. This fact is contained in either of [13, 68, 74], but for consistency we give a proof here.

Lemma 4.2 *There is a one-to-one correspondence between (see Fig. 11 for an example)*

1. *Semistandard Young tableaux of shape λ filled with numbers from* 1 *to N, and*

2. *Path configurations of the six vertex model in the half-infinite rectangle* $\mathbb{Z}_{\geq 1} \times \{1, \dots, N\}$ *with paths entering from the bottom at locations* $\mathcal{S}(\lambda) = \{\lambda_j + N - j + 1\}_{1 \leq j \leq N}$ *and exiting through the right boundary, such that the left and the top boundaries are empty. The paths must satisfy an additional* five vertex condition *that the vertex* $(1, 1; 1, 1)$ *is not present.*

Moreover, with the vertex weights at each $(i, j) \in \mathbb{Z}_{\geq 1} \times \{1, \dots, N\}$ *equal to*

$$\begin{aligned}\widehat{W}_{\mathrm{fSchur}}(0,0;0,0) &= x_{N+1-j} + y_i, \qquad \widehat{W}_{\mathrm{fSchur}}(1,1;1,1) = 0,\\ \widehat{W}_{\mathrm{fSchur}}(1,0;1,0) &= \widehat{W}_{\mathrm{fSchur}}(0,1;0,1)\\ &= \widehat{W}_{\mathrm{fSchur}}(1,0;0,1) = \widehat{W}_{\mathrm{fSchur}}(0,1;1,0) = 1,\end{aligned} \tag{4.3}$$

the partition function in the half-infinite rectangle described above is equal to the factorial Schur polynomial $s_\lambda(\mathbf{x} \mid \mathbf{y})$.

A similar bijection holds between semistandard tableaux of a skew shape λ/μ, and five vertex model configurations with top and bottom boundary conditions given by $\mathcal{S}(\mu)$, $\mathcal{S}(\lambda)$. Here the boundary conditions are the same as for the functions $F_{\lambda/\mu}$, see Definition 3.3.

Proof of Lemma 4.2 Take a semistandard tableau T of shape λ, and let ν be the shape formed by numbers from 1 to $N - 1$. These signatures interlace:

$$\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \dots \geq \lambda_{N-1} \geq \nu_{N-1} \geq \lambda_N,$$

which in the language of the configurations $\mathcal{S}(\lambda) = \left\{\lambda_j + N - j + 1\right\}$ and $\mathcal{S}(\nu) = \left\{\nu_j + N - j\right\}$ translates into

$$\begin{aligned}\lambda_1 + N > \nu_1 + N - 1 \geq \lambda_2 + N - 1 > \nu_2 + N - 2 \geq\\ \dots \geq \lambda_{N-1} + 2 > \nu_{N-1} + 1 \geq \lambda_N + 1.\end{aligned} \tag{4.4}$$

There is a unique one-layer six vertex model configuration in $\mathbb{Z}_{\geq 1}$ with boundary conditions $\mathcal{S}(\lambda)$ at the bottom and $\mathcal{S}(\nu)$ at the top, with no paths entering from the left, and a path exiting through the right boundary. The strict inequalities in (4.4) imply that this configuration satisfies the five vertex condition.

To construct the full desired bijection, look at shapes in the tableau formed by numbers from 1 to j for every $j \leq N - 1$, and argue in the same manner as above.

Finally, observe that under this bijection, for each tableau T the product $(\mathbf{x} \mid \mathbf{y})^T$ from (4.2) is the same as the product of the weights (4.4) in the corresponding five vertex model configuration. Indeed, with the notation ν as above, observe that the variable x_N enters the configuration weight only through empty vertices $(0, 0; 0, 0)$, which gives

$$\prod_{m=\nu_1+N}^{\lambda_1+N-1} (x_N + y_m) \prod_{m=\nu_2+N-1}^{\lambda_2+N-2} (x_N + y_m) \dots \prod_{m=1}^{\lambda_N} (x_N + y_m).$$

This is readily seen to be the same contribution as in (4.2). The argument similarly extends to all other variables x_i.

Remark 4.3 The weight $x_{N+1-j} + y_i$ in (4.3) may be replaced by $x_j + y_i$ because the polynomial $s_\lambda(\mathbf{x} \mid \mathbf{y})$ is symmetric in the x_j's.

We can now specialize F_λ given by Definition 3.3 into $s_\lambda(\mathbf{x} \mid \mathbf{y})$:

Proposition 4.4 *Let λ be a signature with N parts. Take complex parameters*

$$s^{-2}\mathbf{x}^{-1} = (s^{-2}x_1^{-1}, \dots, s^{-2}x_N^{-1}), \quad -\mathbf{y}^{-1} = (-y_1^{-1}, -y_2^{-1}, \dots),$$
$$\mathbf{r} = (r, \dots, r), \quad \mathbf{s} = (s, s, \dots). \tag{4.5}$$

Then we have

$$\lim_{s\to 0,\ r\to+\infty} F_\lambda(s^{-2}\mathbf{x}^{-1}; -\mathbf{y}^{-1}; \mathbf{r}; \mathbf{s}) = \frac{x_1^{N-1}x_2^{N-2}\dots x_{N-1}}{\prod_{i\ge 1} y_i^{\#\{k\in\mathcal{S}(\lambda):\, k>i\}}}\, s_\lambda(\mathbf{x} \mid \mathbf{y}). \tag{4.6}$$

The factorial Schur polynomial $s_\lambda(\mathbf{x} \mid \mathbf{y})$ is symmetric in the x_i's, and observe that the deformed symmetry in the x_i's of the right-hand side of (4.6) agrees with Proposition 3.5.

Proof of Proposition 4.4 Taking the weights $\widehat{W}$ (2.4) with the parameters (4.5) and applying the limits $s \to 0$, $r \to +\infty$ leads to the vertex weights at each $(i, j) \in \mathbb{Z}_{\ge 1} \times \{1, \dots, N\}$ (listed in the same order as in (4.3) and Fig. 4):

$$\left(1 + x_j/y_i, 0, x_j/y_i, 1, 1, x_j/y_i\right). \tag{4.7}$$

These weights are almost the same as $\widehat{W}_{\mathrm{fSchur}}$ (4.3), and after a certain renormalization we can get $s_\lambda(\mathbf{x} \mid \mathbf{y})$. Namely, denote by Z the left-hand side of (4.6). Then

$$\frac{Z}{x_1^{N-1}x_2^{N-2}\dots x_{N-1}} \prod_{i=1}^{\lambda_1+N-1} y_i^N \tag{4.8}$$

is the partition function like F_λ with the vertex weights

$$\left(x_j + y_i, 0, 1, y_i, y_i, 1\right) \tag{4.9}$$

in the part $\{1, 2, \dots, \lambda_1 + N - 1\} \times \{1, \dots, N\}$ of the half-infinite rectangle. Indeed, multiplying by the y_i's clears the all denominators, while the number of the extra x_j factors in (4.7) coming from the weights $(1, 0; 1, 0)$ and $(0, 1; 1, 0)$ is $N - j$, independently of the path configuration. Note also that to the right of $\lambda_1 + N - 1$, every path configuration contains only vertices of the type $(0, 1; 0, 1)$, and so the remaining weight is equal to 1.

Finally, one can readily check that the number of the extra y_i factors coming from the vertices of types $(0, 1; 0, 1)$ and $(1, 0; 0, 1)$ corresponding to the weights (4.9) is also independent of the path configuration, and is equal to $\prod_{i=1}^{\lambda_1+N-1} y_i^{\#\{k\in\mathcal{S}(\lambda)\setminus\{\lambda_1+N\}:\, k\le i\}}$. This allows to remove the extra y_i factors, and turn the weights (4.9) into $\widehat{W}_{\mathrm{fSchur}}$ (4.3), up to the permutation of the x_j's allowed by Remark 4.3. Combining all the extra factors leads to the result.

Let us now present an alternative derivation of the determinantal formula (3.12) for F_λ in the special factorial Schur case. This argument is simpler than in the general case discussed in Appendix A.1.

Corollary 4.5 *The determinantal formula for F_λ of Theorem 3.9 holds in the factorial Schur specialization (4.5)–(4.6).*

Alternative proof Proposition 4.4 was proven using only the partition function definition of F_λ and the tableau formula for $s_\lambda(\mathbf{x}\mid\mathbf{y})$, to compare the two functions. We can now use the determinantal formula for $s_\lambda(\mathbf{x}\mid\mathbf{y})$ (4.1), and it remains to check that the formula of Theorem 3.9 specializes to (4.1). We have

$$\lim_{s\to 0} s^{-2}\varphi_k(s^{-2}x^{-1}\mid -\mathbf{y}^{-1};\mathbf{s}) = \lim_{s\to 0}\frac{s^{-2}}{-y_{k+1}^{-1}-s^{-2}x^{-1}}\prod_{j=1}^{k}\frac{-y_j^{-1}-x^{-1}}{s^2(-y_j^{-1}-s^{-2}x^{-1})}$$
$$= -x\prod_{j=1}^{k}\frac{x+y_j}{y_j}.$$

Therefore, in the specialization of Proposition 4.4, the determinantal formula for F_λ turns into

$$\lim_{s\to 0,\, r\to+\infty}\left(\prod_{i=1}^{N} x_i^{-1}(r^{-2}-1)\prod_{1\le i<j\le N}\frac{r^{-2}x_j-x_i}{x_j-x_i}\right)$$
$$\det\left[s^{-2}\varphi_{\lambda_j+N-j}(s^{-2}x_i^{-1}\mid -\mathbf{y}^{-1};\mathbf{s})\right]_{i,j=1}^{N}$$
$$=\left(\prod_{i=1}^{N}(-x_i)^{-1}\prod_{1\le i<j\le N}\frac{x_i}{x_i-x_j}\right)\det\left[-x_i\prod_{m=1}^{\lambda_j+N-j}\frac{x_i+y_m}{y_m}\right]_{i,j=1}^{N},$$

which produces (4.1) up to the same prefactor as in (4.6).

Let us now turn to the functions G_λ, and denote

$$\check{s}_\lambda(\mathbf{x}\mid\mathbf{y}) := \frac{1}{\prod_{i\ge 1} y_i^{\#\{k\in\mathcal{S}(\lambda):\, k>i\}}}\frac{y_1^{N-1}y_2^{N-2}\cdots y_{N-1}}{(x_1\ldots x_M)^N}$$
$$\lim_{s\to 0,\,\theta\to+\infty} G_\lambda(s^{-2}\mathbf{x}^{-1};-\mathbf{y}^{-1};\boldsymbol{\theta};\mathbf{s}), \tag{4.10}$$

where λ is a signature with N parts, and the parameters are

$$s^{-2}\mathbf{x}^{-1} = (s^2x_1^{-1},\ldots,s^2x_M^{-1}),\quad -\mathbf{y}^{-1} = (y_1^{-1},y_2^{-1},\ldots),$$

$$\boldsymbol{\theta} = (\theta, \ldots, \theta), \quad \mathbf{s} = (s, s, \ldots).$$

Proposition 4.6 *The limit (4.10) exists. It is equal to the five vertex partition function in $\mathbb{Z}_{\geq 1} \times \{1, \ldots, M\}$ with boundary conditions as for G_λ (Definition 3.2), and the following vertex weight at each lattice point (i, j):*

$$\begin{aligned} &W_{\mathrm{fSchur}}(0,0;0,0) = 1, \qquad W_{\mathrm{fSchur}}(1,1;1,1) = 0, \\ &W_{\mathrm{fSchur}}(1,0;1,0) = W_{\mathrm{fSchur}}(0,1;0,1) \\ &\quad = W_{\mathrm{fSchur}}(1,0;0,1) = W_{\mathrm{fSchur}}(0,1;1,0) = \frac{1}{x_j + y_i}. \end{aligned} \tag{4.11}$$

Proof Take the weights W (2.3) used in the definition of the functions G_λ, and apply the limit transition as in (4.10). We obtain the following weights (listed in the same order as in the claim):

$$\left(1, 0, \frac{x_j}{x_j + y_i}, \frac{y_i}{x_j + y_i}, \frac{y_i}{x_j + y_i}, \frac{x_j}{x_j + y_i}\right),$$

which are almost the same as the desired W_{fSchur} (4.11). The extra factors x_j and y_i can be taken out analogously to the proof of Proposition 4.4, which results in the prefactor in (4.10).

Remark 4.7 The weights W_{fSchur} (4.11) and their partition functions (coinciding with our $\check{s}_\lambda$ for special λ) appeared in [51, 81] (see also [82, 93]) in connection with enumeration of skew standard Young tableaux.

Proposition 4.8 *The functions $s_\lambda(x_1, \ldots, x_N \mid \mathbf{y})$ and $\check{s}_\lambda(w_1, \ldots, w_M \mid \mathbf{y})$ satisfy the Cauchy summation identity*

$$\sum_{\lambda = (\lambda_1 \geq \ldots \geq \lambda_N \geq 0)} s_\lambda(x_1, \ldots, x_N \mid \mathbf{y})\, \check{s}_\lambda(w_1, \ldots, w_M \mid \mathbf{y}) = \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{1}{w_j - x_i}, \tag{4.12}$$

where $|x_i| < |w_j|$ for all i, j.

Proof This is a combination of the Cauchy identity of Theorem 3.8 and the limit transitions from Propositions 4.4 and 4.6.

When $y_j \equiv 0$, the functions $s_\lambda(\mathbf{x} \mid \mathbf{y})$ become the usual Schur polynomials $s_\lambda(\mathbf{x})$, while for $\check{s}_\lambda$ we have

$$\check{s}_\lambda(\mathbf{w} \mid 0) = (w_1 \ldots w_M)^{-N} s_\lambda(w_1^{-1} \ldots w_M^{-1}).$$

This agrees with the fact that for $y_j \equiv 0$, identity (4.12) reduces to the usual Cauchy identity for Schur functions [73, Ch. I, (4.3)].

Remark 4.9 There is another Cauchy identity involving the polynomials $s_\lambda(\mathbf{x} \mid \mathbf{y})$ together with the dual Schur functions $\widehat{s}_\lambda(\mathbf{w} \,\|\, \mathbf{y})$ [78, Theorem 3.1] (in [88] a particular case of the dual Schur functions is included into the family of dual interpolation Macdonald functions). One can check (by comparing the Cauchy identities or using the explicit determinantal formula for $\widehat{s}_\lambda$) that the $\check{s}_\lambda$'s are different from the dual Schur functions $\widehat{s}_\lambda$.

After this paper was posted, [47] studied combinatorial properties of the functions $\check{s}_\lambda$ (in their notation, these are multiples of E^λ) defined through the Cauchy identity with factorial Schur polynomials.

4.2 Horizontally homogeneous model

Throughout this subsection we set all the column parameters (in the sense of Fig. 9) to be constant:

$$s_j \equiv s, \qquad y_j \equiv 1, \qquad \text{forall } j = 1, 2, \dots. \tag{4.13}$$

Denote this specialization by $(\mathbf{y}; \mathbf{s}) = (1; s)$. In this special case we can relate the functions F_λ and G_λ to the ordinary and the supersymmetric Schur functions, respectively.

The supersymmetric Schur functions $s_\lambda(\mathbf{a}/\mathbf{b})$, where $\mathbf{a} = (a_1, \dots, a_M)$, $\mathbf{b} = (b_1, \dots, b_M)$ are two sequences of variables, may be defined as coefficients in the following Cauchy identity involving the ordinary Schur polynomials $s_\lambda(t_1, \dots, t_N)$, where N is an arbitrary large enough integer:

$$\sum_{\lambda=(\lambda_1 \geq \dots \lambda_N \geq 0)} s_\lambda(\mathbf{a}/\mathbf{b})\, s_\lambda(t_1, \dots, t_N) = \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{1 + t_i b_j}{1 - t_i a_j}. \tag{4.14}$$

The supersymmetric Schur functions are related to the factorial Schur polynomials which appeared in Sect. 4.1, but we do not need this connection here. See [21], [72, (6.19)] for details. The next statement is independent of the explicit formulas of Theorems 3.9 and 3.10 (while may also be derived as a corollary of Theorem 3.9, see Corollary 4.11 below).

Proposition 4.10 (Proposition 1.2 from Introduction) *Let λ be a signature with N parts. Under the homogeneous specialization* (4.13)*, we have*

$$F_\lambda(x_1, \dots, x_N; 1; \mathbf{r}; s) = \det\left[\left(\frac{1 - s^2 x_i}{s^2(1 - x_i)}\right)^{\lambda_j + N - j}\right]_{i,j=1}^{N}$$
$$\prod_{i=1}^{N} \frac{(r_i^{-2} - 1)x_i}{1 - x_i} \prod_{1 \leq i < j \leq N} \frac{r_i^{-2} x_i - x_j}{x_i - x_j} \tag{4.15}$$

and, in particular,

$$\frac{F_\lambda(x_1,\ldots,x_N;1;\mathbf{r};s)}{F_{0^N}(x_1,\ldots,x_N;1;\mathbf{r};s)} = s_\lambda\left(\frac{1-s^2x_1}{s^2(1-x_1)},\ldots,\frac{1-s^2x_N}{s^2(1-x_N)}\right). \tag{4.16}$$

For G_λ, we have the following expression:

$$\begin{aligned}&G_\lambda(x_1,\ldots,x_M;1;\mathbf{r};s)\\&= s_\lambda\left(\left\{\frac{s^2(1-x_j)}{1-s^2x_j}\right\}_{j=1}^{M}\Big/\left\{\frac{s^2(x_jr_j^{-2}-1)}{1-s^2r_j^{-2}x_j}\right\}_{j=1}^{M}\right)\prod_{i=1}^{M}\left(\frac{1-s^2r_i^{-2}x_i}{1-s^2x_i}\right)^N.\end{aligned} \tag{4.17}$$

Both identities (4.16) and (4.17) readily extend to skew functions thanks to the branching rules (Proposition 3.6) and the fact that Schur and supersymmetric Schur functions form bases.

Proof of Proposition 4.10 First, observe that (4.16) immediately follows from (4.15) and the formula for the Schur polynomial as a ratio of two determinants.

The claim (4.15) about F_λ follows from the results of [8]. This theorem deals with the free fermion six vertex model whose weights in the k-th row are given by

$$\begin{aligned}&w(0,0;0,0)=a_1^{(k)}; \quad w(1,0;1,0)=b_1^{(k)}; \quad w(1,0;0,1)=c_2^{(k)};\\&w(1,1;1,1)=a_2^{(k)}; \quad w(0,1;0,1)=b_2^{(k)}; \quad w(0,1;1,0)=c_1^{(k)},\end{aligned} \tag{4.18}$$

such that $a_1^{(k)}a_2^{(k)}+b_1^{(k)}b_2^{(k)}=c_1^{(k)}c_2^{(k)}$ for all k. Note the swap $c_1\leftrightarrow c_2$ in (4.18) compared to our usual conventions (2.1) which is needed to match with [8].

By [8, Theorem 9] the partition function with the same boundary conditions as for F_λ (Definition 3.3) with the weights (4.18) is equal to

$$s_\mu\left(\frac{b_2^{(1)}}{a_1^{(1)}},\ldots,\frac{b_2^{(N)}}{a_1^{(N)}}\right)\prod_{k=1}^{N}(a_1^{(k)})^{\mu_1+\lambda_N}c_2^{(k)}\prod_{1\le i<j\le N}(a_1^{(j)}a_2^{(i)}+b_1^{(i)}b_2^{(j)}), \tag{4.19}$$

where $\mu_i=\lambda_1-\lambda_{N+1-i}$. Note that here we needed to flip both the horizontal and the vertical directions for F_λ compared to the boundary conditions $\mathfrak{S}_\lambda$ in [8]. The extra factors $(a_1^{(k)})^{\lambda_N}$ come from the fact that our lattice weights for F_λ start from horizontal position 1 on the left boundary.

Let us rewrite (4.19) in a determinantal form, and specialize the weights (4.18) to our $\widehat{W}$ given by (2.4). We obtain

$$
\begin{aligned}
&F_\lambda(x_1,\ldots,x_N;1;\mathbf{r};s)\\
&\quad=\det\left[(b_2^{(i)}/a_1^{(i)})^{\lambda_1-\lambda_{N+1-j}+N-j}\right]\prod_{k=1}^{N}(a_1^{(k)})^{\lambda_1}c_2^{(k)}(a_1^{(k)})^{N-1}\\
&\quad\prod_{1\le i<j\le N}\frac{a_1^{(j)}a_2^{(i)}+b_1^{(i)}b_2^{(j)}}{a_1^{(j)}b_2^{(i)}-a_1^{(i)}b_2^{(j)}}\\
&\quad=\det\left[\left(\frac{1-s^2x_i}{s^2(1-x_i)}\right)^{\lambda_{N+1-j}+j-1}\right]\prod_{i=1}^{N}\frac{x_i(r_i^{-2}-1)}{1-x_i}\prod_{1\le i<j\le N}\frac{x_j-r_i^{-2}x_i}{x_i-x_j}.
\end{aligned}
$$

Replacing $\lambda_{N+1-j}+j-1$ by λ_j+N-j in the determinant amounts to flipping the signs in all the factors x_i-x_j in the denominator, which leads to the desired formula (4.15).

To establish the claim (4.17) about G_λ, take identity (3.8) used in the proof of Theorem 3.8:

$$
\sum_\lambda G_\lambda(w_1,\ldots,w_M;1;\boldsymbol{\theta};s)\,\frac{F_\lambda(x_1,\ldots,x_N;1;\mathbf{r};s)}{F_{0^N}(x_1,\ldots,x_N;1;\mathbf{r};s)}=\prod_{i=1}^{N}\prod_{j=1}^{M}\frac{x_i-\theta_j^{-2}w_j}{x_i-w_j}.
$$

Observe that this does not use the explicit formula for F_λ from Theorem 3.9. Employing (4.16), write this identity as

$$
\begin{aligned}
&\sum_\lambda G_\lambda(w_1,\ldots,w_M;1;\boldsymbol{\theta};s)\,s_\lambda(t_1,\ldots,t_N)\\
&\quad=\prod_{i=1}^{N}\prod_{j=1}^{M}\frac{1+s^2((t_i-1)w_j\theta_j^{-2}-t_i)}{1+s^2(t_i(w_j-1)-w_j)}\\
&\quad=\prod_{j=1}^{M}\left(\frac{1-s^2w_j\theta_j^{-2}}{1-s^2w_j}\right)^{N}\prod_{i=1}^{N}\prod_{j=1}^{M}\frac{1+t_i\frac{s^2(w_j\theta_j^{-2}-1)}{1-s^2w_j\theta_j^{-2}}}{1-t_i\frac{s^2(1-w_j)}{1-s^2w_j}},
\end{aligned}
$$

where we have denoted

$$
t_i=\frac{1-s^2x_i}{s^2(1-x_i)},\qquad\text{so that}\qquad x_i=\frac{1-s^2t_i}{s^2(1-t_i)}
$$

(the map $t_i\leftrightarrow x_i$ is an involution). Comparing the previous summation identity with (4.14) and using linear independence of the Schur polynomials (i.e., the fact that the coefficients by $s_\lambda(t_1,\ldots,t_N)$ are uniquely determined by the right-hand side), we get the claim about G_λ.

Corollary 4.11 *In the horizontally homogeneous case* (4.13) *the function* F_λ *is given by the determinantal formula* (3.12) *of Theorem* 3.9.

Alternative proof This proof does not rely on Theorem 3.9 proven in Appendix A.1. Under (4.13) we have

$$\varphi_k(x \mid \mathbf{y}; \mathbf{s}) = \frac{1}{1-x}\left(\frac{1-s^2x}{s^2(1-x)}\right)^k.$$

Thus, the claim follows from the first part of Proposition 4.10.

Remark 4.12 It should be also possible to derive the determinantal formulas of Proposition 4.10 using the Wick formula and Hamiltonian operators for the free fermion six vertex model with horizontally homogeneous weights, recently studied in [48].

Let us rewrite the explicit formula for G_λ from Theorem 3.10 in the homogeneous case. Let λ be a signature with N parts. Recall the integer $d = d(\lambda)$ for which $\lambda_d \geq d$ and $\lambda_{d+1} < d+1$. Let $\tau = (\lambda_1 - d, \ldots, \lambda_d - d)$ and $\eta = (\lambda_{d+1}, \ldots, \lambda_N)$ be two auxiliary signatures. Let η' denote the transposed signature corresponding to the reflection of the Young diagram of η with respect to the diagonal, cf. [73, I.(1.3)]. Observe that both τ and η' have d parts.

Proposition 4.13 *With the notation given before this proposition, we have*

$$\begin{aligned}
&\prod_{j=1}^{M}\left(\frac{1-s^2r_j^{-2}x_j}{1-s^2x_j}\right)^{-N} G_\lambda(\mathbf{x}; 1; \mathbf{r}; s) \\
&= \sum_{\substack{\mathcal{I},\mathcal{J}\subseteq\{1,\ldots,M\}\\ |\mathcal{I}|=|\mathcal{J}|=d}} s_\tau(\mathsf{x}_{\mathcal{J}})s_{\eta'}(\mathsf{y}_{\mathcal{I}}) \prod_{\substack{i\in\mathcal{J}^c\\ j\in\mathcal{J}}} \frac{1}{(\mathsf{x}_j-\mathsf{x}_i)} \prod_{\substack{i\in\mathcal{I}\\ j\in\mathcal{I}^c}} \frac{1}{(\mathsf{y}_i-\mathsf{y}_j)} \\
&\qquad \prod_{\substack{i\in\mathcal{I}\\ 1\leq j\leq M}} (\mathsf{x}_j+\mathsf{y}_i) \prod_{\substack{i\in\mathcal{I}^c\\ j\in\mathcal{J}}} (\mathsf{x}_j+\mathsf{y}_i),
\end{aligned} \tag{4.20}$$

where

$$\mathsf{x}_i = \frac{s^2(1-x_i)}{1-s^2x_i}, \qquad \mathsf{y}_i = \frac{s^2(x_ir_i^{-2}-1)}{1-s^2r_i^{-2}x_i}, \tag{4.21}$$

and $\mathsf{x}_{\mathcal{J}}$, $\mathsf{y}_{\mathcal{I}}$ *are subsets of the variables with indices belonging to* $\mathcal{J}$ *and* $\mathcal{I}$, *respectively.*

Proof Using notation (4.21), we have

$$\mathsf{x}_i - \mathsf{x}_j = \frac{s^2(s^2-1)(x_i-x_j)}{(1-s^2x_i)(1-s^2x_j)},$$

$$\mathsf{y}_i - \mathsf{y}_j = \frac{s^2(1-s^2)(r_i^{-2}x_i - r_j^{-2}x_j)}{(1-s^2r_i^{-2}x_i)(1-s^2r_j^{-2}x_j)},$$

$$\mathsf{x}_i + \mathsf{y}_j = \frac{s^2(s^2-1)(x_i - r_j^{-2}x_j)}{(1-s^2x_i)(1-s^2r_j^{-2}x_j)}.$$

With the help of these formulas, we can express all products in (3.14) through $\mathsf{x}_i - \mathsf{x}_\mathsf{j}$, $\mathsf{y}_\mathsf{i} - \mathsf{y}_j$, and $\mathsf{x}_j + \mathsf{y}_i$. The remaining sum over $\sigma, \rho \in \mathsf{S}_d$ turns into a product of determinants leading to Schur polynomials in d variables. This is due to the facts that in the homogeneous case (4.13) we have

$$\prod_{i=N+1}^{\ell_h-1} \frac{s_i^2\big(y_i - x_{j_{\rho(h)}}\big)}{y_i - s_i^2 x_{j_{\rho(h)}}} = \mathsf{x}_{j_{\rho(h)}}^{\lambda_h-h} = \mathsf{x}_{j_{\rho(h)}}^{\tau_h+d-h},$$

$$\prod_{k=\mu_m+1}^{N} \frac{s_k^2\big(r_{i_{\sigma(m)}}^{-2} x_{i_{\sigma(m)}} - y_k\big)}{y_k - s_k^2 r_{i_{\sigma(m)}}^{-2} x_{i_{\sigma(m)}}} = \mathsf{y}_{i_{\sigma(m)}}^{N-\mu_m} = \mathsf{y}_{i_{\sigma(m)}}^{\eta'_m+d-m}.$$

In the last equality we used the notation μ (3.13) and an observation that $N - \mu_m = \lambda'_m - m = \eta'_m + d - m$. This completes the proof.

Combining the second part of Proposition 4.10 with Proposition 4.13, we arrive at the following formula for supersymmetric Schur polynomials:

Corollary 4.14 *With the notation given before Proposition* 4.13*, we have*

$$\begin{aligned} & s_\lambda(\mathsf{x}_1, \ldots, \mathsf{x}_M / \mathsf{y}_1, \ldots, \mathsf{y}_M) \\ &= \sum_{\substack{\mathcal{I},\mathcal{J}\subseteq\{1,\ldots,M\} \\ |\mathcal{I}|=|\mathcal{J}|=d}} s_\tau(\mathsf{x}_\mathcal{J}) s_{\eta'}(\mathsf{y}_\mathcal{I}) \prod_{\substack{i\in\mathcal{J}^c \\ j\in\mathcal{J}}} \frac{1}{(\mathsf{x}_j - \mathsf{x}_i)} \\ & \prod_{\substack{i\in\mathcal{I} \\ j\in\mathcal{I}^c}} \frac{1}{(\mathsf{y}_i - \mathsf{y}_j)} \prod_{\substack{i\in\mathcal{I} \\ 1\le j\le M}} (\mathsf{x}_j + \mathsf{y}_i) \prod_{\substack{i\in\mathcal{I}^c \\ j\in\mathcal{J}}} (\mathsf{x}_j + \mathsf{y}_i). \end{aligned} \tag{4.22}$$

For $d = M$, formula (4.22) coincides with the Berele–Regev formula [21]. The latter provides an expression for supersymmetric Schur polynomials in this special case $d(\lambda) = M$:

$$s_\lambda(\mathsf{x}_1, \ldots, \mathsf{x}_M \,/\, \mathsf{y}_1, \ldots, \mathsf{y}_M) = s_\tau(\mathsf{x}_1, \ldots, \mathsf{x}_M)\, s_{\eta'}(\mathsf{y}_1, \ldots, \mathsf{y}_M) \prod_{i,j=1}^{M} (\mathsf{x}_i + \mathsf{y}_j).$$

Remark 4.15 In the general case $d(\lambda) < M$, a formula like (4.22) for factorial Grothendieck polynomials was proven using integrable lattice models in [80]. See also [38, 46] for special cases. We remark that our identity (4.22) generalizes the ones in [38, 46] in a different direction than what is shown in [80].

Moreover, there does not seem to be a direct relation between Corollary 4.14 and other known formulas for supersymmetric Schur polynomials, including the Sergeev–Pragacz formula (e.g., see [49, (5)]) and the Moens–Van der Jeugt determinantal formula [83].

5 Biorthogonality and contour integral formulas

Here we discuss torus-like biorthogonality property for the functions F_λ, and employ it to derive integral and determinantal formulas for the functions G_λ.

5.1 Biorthogonality

The functions $F_\lambda(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s})$ satisfy a certain biorthogonality property with respect to contour integration in the $\mathbf{x}$ variables. This biorthogonality extends the torus orthogonality of Schur polynomials as irreducible characters of unitary groups. To get the biorthogonality, we use the determinantal formula for F_λ of Theorem 3.9.

Fix an integer $N \geq 1$ and sequences of complex parameters $\mathbf{y} = (y_1, y_2, \ldots)$, $\mathbf{s} = (s_1, s_2, \ldots)$, $\mathbf{x} = (x_1, \ldots, x_N)$, $\mathbf{r} = (r_1, \ldots, r_N)$. Recall the functions $\varphi_k(x \mid \mathbf{y}; \mathbf{s})$ defined in (3.11). Let us also define

$$\psi_k(x) = \psi_k(x \mid \mathbf{y}; \mathbf{s}) := \frac{y_{k+1}(s_{k+1}^2 - 1)}{y_{k+1} - s_{k+1}^2 x} \prod_{j=1}^{k} \frac{s_j^2(y_j - x)}{y_j - s_j^2 x}, \qquad k \geq 0. \tag{5.1}$$

In particular, $\psi_0(x \mid \mathbf{y}, \mathbf{s}) = \frac{y_1(s_1^2-1)}{y_1 - s_1^2 x}$.

Lemma 5.1 *We have for all $k, l \geq 0$:*

$$\frac{1}{2\pi \mathbf{i}} \oint_\gamma \varphi_k(z \mid \mathbf{y}, \mathbf{s})\, \psi_l(z \mid \mathbf{y}, \mathbf{s})\, dz = \mathbf{1}_{k=l}, \tag{5.2}$$

where γ is a closed counterclockwise simple contour in the complex plane containing the points y_j for all $j \geq 1$ and not $y_j s_j^{-2}$ for all $j \geq 1$, and $\mathbf{1}_{k=l}$ is the indicator that $k = l$ (i.e., the Kronecker delta).

Remark 5.2 Here and below in this section we assume that the parameters (here, sequences $\mathbf{y}$ and $\mathbf{s}$) are such that the integration contour exists. Alternatively, one may also think of the integration *formally* as the sum of residues at all the points y_j, $j \geq 1$, which the contour must encircle.

Proof of Lemma 5.1 First, observe that at $z = \infty$ both $\varphi_k(z)$, $\psi_l(z)$ behave as const$\cdot z^{-1}$ for all $k, l \geq 0$. For $k < l$, the product $\varphi_k(z)\psi_l(z)$ has only factors of the form $y_j - s_j^2 z$ in the denominator, and thus has no poles inside the integration contour γ. Therefore, the integral (5.2) vanishes for $k < l$. For $k > l$, the product $\varphi_k(z)\psi_l(z)$ has at least two factors of the form $z - y_j$ and no factors of the form $y_j - s_j^2 z$ in the denominator. Therefore, the integrand is regular outside the contour, so the integral (5.2) vanishes for $k > l$ as well. Finally, for $k = l$ we have

$$\varphi_k(z)\psi_k(z) = \frac{1}{z - y_{k+1}} \cdot \frac{y_{k+1}(1 - s_{k+1}^2)}{y_{k+1} - s_{k+1}^2 z},$$

and the claim immediately follows.

Using Lemma 5.1, define the following counterparts of the functions F_λ:

$$F_\lambda^*(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s}) := \det\left[\psi_{\lambda_j+N-j}(x_i \mid \mathbf{y}; \mathbf{s})\right]_{i,j=1}^N \prod_{N \geq i > j \geq 1} \frac{x_j - r_i^{-2} x_i}{x_j - x_i}, \qquad (5.3)$$

where λ is a signature with N parts.

Remark 5.3 In the horizontally homogeneous case $s_j \equiv s$, $y_j \equiv 1$, the functions F_λ^* are almost the same as the F_λ's, up to a factor and a change of variables:

$$\begin{aligned} &F_\lambda^*(s^{-2}/x_1, \ldots, s^{-2}/x_N; 1; \mathbf{r}; s) \\ &= \frac{(1 - s^2)^N (s^2)^{|\lambda| + N(N-1)/2}}{\prod_{i=1}^N (r_i^{-2} - 1)} \prod_{1 \leq i < j \leq N} \frac{r_j^{-2} x_i - x_j}{r_i^{-2} x_i - x_j} F_\lambda(x_1, \ldots, x_N; \mathbf{y}; \mathbf{r}; \mathbf{s}). \end{aligned}$$

However, in general the F_λ^*'s cannot be expressed through the F_λ's.

Proposition 5.4 *For any signatures λ, μ with N parts we have*

$$\frac{1}{N!(2\pi \mathbf{i})^N} \oint_\gamma dz_1 \ldots \oint_\gamma dz_N \frac{\prod_{1 \leq i \neq j \leq N}(z_i - z_j)}{\prod_{i,j=1}^N (r_i^{-2} z_i - z_j)} F_\lambda(\mathbf{z}; \mathbf{y}; \mathbf{r}; \mathbf{s})\, F_\mu^*(\mathbf{z}; \mathbf{y}; \mathbf{r}; \mathbf{s}) = \mathbf{1}_{\lambda=\mu},$$

where the integration is over the variables $\mathbf{z} = (z_1, \ldots, z_N)$ belonging to the torus γ^N, and γ is a contour around y_j not encircling $y_j s_j^{-2}$, $j \geq 1$. Note that the integrand has no poles at $z_j = r_i^{-2} z_i$ for any i, j.

Proof This proof is similar to the well-known proof of torus orthogonality of the Schur polynomials. Cancel out the prefactors, and expand the determinants in F_λ and F_μ^* as sums over permutations:

$$\frac{1}{N!(2\pi\mathbf{i})^N}\oint_\gamma dz_1\ldots\oint_\gamma dz_N \frac{\prod_{1\le i\neq j\le N}(z_i-z_j)}{\prod_{i,j=1}^N(r_i^{-2}z_i-z_j)} F_\lambda(\mathbf{z};\mathbf{y};\mathbf{r};\mathbf{s})\, F_\mu^*(\mathbf{z};\mathbf{y};\mathbf{r};\mathbf{s})$$
$$=\frac{1}{N!(2\pi\mathbf{i})^N}\oint_\gamma dz_1\ldots\oint_\gamma dz_N$$
$$\det\left[\varphi_{\lambda_j+N-j}(z_i\mid\mathbf{y};\mathbf{s})\right]_{i,j=1}^N \det\left[\psi_{\mu_j+N-j}(z_i\mid\mathbf{y};\mathbf{s})\right]_{i,j=1}^N$$
$$=\frac{1}{N!}\sum_{\sigma,\tau\in\mathfrak{S}_N}\operatorname{sgn}(\sigma\tau)\prod_{i=1}^N\frac{1}{2\pi\mathbf{i}}\oint_\gamma \varphi_{\lambda_{\sigma(i)}+N-\sigma(i)}(z_i)\,\psi_{\mu_{\tau(i)}+N-\tau(i)}(z_i)\,dz_i.$$

Using Lemma 5.1, we see that the product of the integrals is nonzero only if $\sigma=\tau$ and $\lambda=\mu$. When the integral is nonzero, it is equal to 1. Summing $N!$ terms corresponding to each $\sigma=\tau\in\mathfrak{S}_N$, we get the result.

5.2 Contour integral formula for $F_{\lambda/\mu}$

Using the branching rule (Proposition 3.6) and the biorthogonality (Proposition 5.4), we are able to get contour integral formulas for the functions $F_{\lambda/\mu}$.

Fix $N, M\ge 1$, and signatures λ with $N+M$ parts and μ with M parts. Furthermore, fix sequences of complex numbers

$$\mathbf{x}=(x_1,\ldots,x_N),\qquad \mathbf{r}=(r_1,\ldots,r_N),\qquad \mathbf{y}=(y_1,y_2,\ldots,),\qquad \mathbf{s}=(s_1,s_2,\ldots).$$

Proposition 5.5 *With the above notation, we have*

$$F_{\lambda/\mu}(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s})=\prod_{i=1}^N x_i(r_i^{-2}-1)\prod_{1\le i<j\le N}\frac{r_i^{-2}x_i-x_j}{x_i-x_j}$$
$$\times\frac{1}{M!(2\pi\mathbf{i})^M}\oint_\gamma dz_1\ldots\oint_\gamma dz_M\prod_{i=1}^N\prod_{j=1}^M\frac{z_j-r_i^{-2}x_i}{z_j-x_i}$$
$$\times\det[\varphi_{\lambda_j+N+M-j}((\mathbf{x}\cup\mathbf{z})_i\mid\mathbf{y};\mathbf{s})]_{i,j=1}^{N+M}\det\left[\psi_{\mu_j+M-j}(z_i\mid\mathbf{y};\mathbf{s})\right]_{i,j=1}^M,\tag{5.4}$$

where

$$(\mathbf{x}\cup\mathbf{z})_i=\begin{cases}x_i, & 1\le i\le M;\\ z_{i-M}, & M+1\le i\le M+N,\end{cases}$$

the integration is over the torus γ^M, and γ is a contour around y_j not encircling $y_js_j^{-2}$, $j\ge 1$. Per Remark 5.2, we either assume that the contour γ exists, or treat the integral formally.

Proof Throughout the proof we use the notation $\mathbf{z} = (z_1, \ldots, z_M)$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_M)$. We have from the branching rule (Proposition 3.6)

$$F_\lambda(\mathbf{x} \cup \mathbf{z}; \mathbf{y}; \mathbf{r} \cup \boldsymbol{\theta}; \mathbf{s}) = \sum_\nu F_{\lambda/\nu}(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s})\, F_\nu(\mathbf{z}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s}),$$

where the sum is over all signatures with M parts. Multiply this (finite) sum by

$$\begin{aligned} &\frac{1}{M!(2\pi\mathbf{i})^M}\, F^*_\mu(\mathbf{z}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s})\, \frac{\prod_{1\le i\ne j\le M}(z_i - z_j)}{\prod_{i,j=1}^M(\theta_i^{-2} z_i - z_j)} \\ &= \frac{1}{M!(2\pi\mathbf{i})^M}\, \frac{\prod_{i<j}(z_i - z_j)}{\prod_{i\le j}(\theta_i^{-2} z_i - z_j)} \det\left[\psi_{\mu_j+M-j}(z_i \mid \mathbf{y}; \mathbf{s})\right]_{i,j=1}^M \end{aligned}$$

and integrate over $\mathbf{z} = (z_1, \ldots, z_M) \in \gamma^M$, where γ is a positively oriented contour around all y_j and not encircling $y_j s_j^{-2}$, $j \ge 1$. The integration extracts the single coefficient by F_ν with $\mu = \nu$, which together with the determinantal formulas for F_λ (3.12) and for F^*_μ (5.3) produces the desired expression.

5.3 Contour integral formula for $G_{\nu/\lambda}$

Using the skew Cauchy identity (Proposition 3.7) and the biorthogonality (Proposition 5.4), we can get contour integral formulas for the functions $G_{\nu/\lambda}$ and G_ν.

Fix integers $N, M \ge 1$ and signatures λ, ν with N parts. Also fix sequences of complex numbers

$$\mathbf{w} = (w_1, \ldots, w_M), \qquad \mathbf{y} = (y_1, y_2, \ldots), \qquad \boldsymbol{\theta} = (\theta_1, \ldots, \theta_N), \qquad \mathbf{s} = (s_1, s_2, \ldots). \tag{5.5}$$

Proposition 5.6 *With the above notation, we have the following contour integral representation for the symmetric functions $G_{\nu/\lambda}$:*

$$\begin{aligned} G_{\nu/\lambda}(\mathbf{w}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s}) = \frac{1}{N!(2\pi\mathbf{i})^N} \oint_{\gamma'} dz_1 \ldots \oint_{\gamma'} dz_N \prod_{i=1}^N \prod_{j=1}^M \frac{z_i - \theta_j^{-2} w_j}{z_i - w_j} \\ \times \det\left[\varphi_{\lambda_i+N-i}(z_j \mid \mathbf{y}; \mathbf{s})\right]_{i,j=1}^N \det\left[\psi_{\nu_i+N-i}(z_j \mid \mathbf{y}; \mathbf{s})\right]_{i,j=1}^N . \end{aligned} \tag{5.6}$$

In particular, for $\lambda = \varnothing$ *we have*

$$G_\nu(\mathbf{w}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s}) = \frac{\prod_{1\le i<j\le N}(s_i^{-2}y_i - y_j)}{N!(2\pi\mathbf{i})^N} \oint_{\gamma'} dz_1 \ldots \oint_{\gamma'} dz_N \det\left[\psi_{\nu_i+N-i}(z_j \mid \mathbf{y}; \mathbf{s})\right]_{i,j=1}^N \times \frac{\prod_{1\le i<j\le N}(z_i - z_j)}{\prod_{i,j=1}^N (y_i - z_j)} \prod_{i=1}^N \prod_{j=1}^M \frac{z_i - \theta_j^{-2} w_j}{z_i - w_j}. \tag{5.7}$$

In both formulas (5.6) *and* (5.7) *the contour* γ' *encircles all* y_j, $j \ge 1$, *and* w_i, $i = 1, \ldots, M$, *and leaves out all* $y_j s_j^{-2}$, $j \ge 1$. *Per Remark* 5.2*, we either assume the contour* γ' *exists, or treat the integrals in* (5.6)–(5.7) *formally.*

Proof Since $G_\nu = G_{\nu/0^N}$, identity (5.7) follows from (5.6) and the product formula (3.9) for F_{0^N}. Next, by Definition 3.2, for fixed λ, ν the partition function $G_{\nu/\lambda}$ is a rational function in $\mathbf{w}$, $\boldsymbol{\theta}$, as well as in a finite subfamily of the parameters $\mathbf{y}$ and $\mathbf{s}$. The integral in the right-hand side of (5.6) is also a rational function of these parameters. Therefore, in proving the proposition we are allowed to impose any open conditions on the parameters, and then identity (5.6) would hold in general thanks to analytic continuation.

Take the skew Cauchy identity (3.6) with $\mu = \varnothing$ (and hence $\varkappa = \varnothing$ in the right-hand side, which eliminates the summation):

$$\sum_\nu G_{\nu/\lambda}(\mathbf{w}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s}) F_\nu(\mathbf{z}; \mathbf{y}; \mathbf{r}; \mathbf{s}) = F_\lambda(\mathbf{z}; \mathbf{y}; \mathbf{r}; \mathbf{s}) \prod_{i=1}^M \prod_{j=1}^N \frac{z_i - \theta_j^{-2} w_j}{z_i - w_j}. \tag{5.8}$$

In fact, both sides of (5.8) contain the same factor depending on $\mathbf{r}$ which can be canceled out, see the determinantal formula for F_λ (3.12). In other words, (5.8) essentially does not depend on $\mathbf{r}$.

In (5.8), we assume that $\mathbf{w}$, $\mathbf{y}$, $\mathbf{s}$, and $\mathbf{z}$ are such that

1. All z_j belong to some contour γ' encircling all w_i and all y_i;
2. For all $z \in {}'$ and all j, k we have $\left| \dfrac{y_k - s_k^2 z}{y_k - z} \dfrac{y_k - w_j}{y_k - s_k^2 w_j} \right| < 1 - \delta < 1$. This is the condition which implies convergence in (5.8), see Proposition 3.7.

One readily sees that these restrictions on $\mathbf{w}$, $\mathbf{y}$, and $\mathbf{s}$ place them into a nonempty open set, which is sufficient for analytic continuation.

Now, multiply both sides of (5.8) by

$$\frac{1}{N!(2\pi\mathbf{i})^N} F_\nu^*(\mathbf{z}; \mathbf{y}; \mathbf{r}; \mathbf{s}) \frac{\prod_{1\le i\ne j\le N}(z_i - z_j)}{\prod_{i,j=1}^N (r_i^{-2} z_i - z_j)}$$

and integrate over $\mathbf{z} \in (\gamma')^N$. Since the sum in the right-hand side of (5.8) converges uniformly on the contours, we can interchange summation and integration and use the biorthogonality of Proposition 5.4 to extract the coefficient $G_{\nu/\lambda}(\mathbf{w}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s})$. After simplification with the help of determinantal formulas (3.12) and (5.3), the integration of the right-hand side of (5.8) yields the right-hand side of the desired identity (5.6). Observe that the dependence on the r_i's disappears, as it should be. Analytic continuation then allows to remove the restrictions stated above in the proof, and we arrive at the result.

5.4 Jacobi–Trudy type formulas for $G_{\nu/\lambda}$ and G_ν

Using the contour integral representation for $G_{\nu/\lambda}$ from Proposition 5.6, it is possible to derive a Jacobi–Trudy type determinantal formula for these symmetric functions.

For $m \geq 1$ define the shift operator sh_m acting on the sequences $\mathbf{y}, \mathbf{s}$ as $(\mathrm{sh}_m \mathbf{y})_i = y_{m+i}$, $(\mathrm{sh}_m \mathbf{s})_i = s_{m+i}$. Also define for all $l \in \mathbb{Z}$:

$$\widetilde{\mathsf{h}}_l(\mathbf{w}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s}) := \frac{\mathbf{1}_{l\geq 0}}{2\pi \mathbf{i}} \oint_{\gamma'} \frac{(s_{l+1}^2 - 1) y_{l+1}}{(y_{l+1} - s_{l+1}^2 z)(y_1 - z)} \prod_{j=1}^{l} \frac{s_j^2 (y_j - z)}{y_j - s_j^2 z} \prod_{j=1}^{M} \frac{z - \theta_j^{-2} w_j}{z - w_j} \, dz, \tag{5.9}$$

where the integration is over a contour γ' around all the points y_j and w_i, leaving outside the points $y_j s_j^{-2}$. Observe that (5.9) is symmetric under simultaneous permutations of (w_i, θ_i). Also denote

$$\mathsf{g}_{l/k}(\mathbf{w}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s}) := \widetilde{\mathsf{h}}_{l-k}(\mathbf{w}; \mathrm{sh}_k \mathbf{y}; \boldsymbol{\theta}; \mathrm{sh}_k \mathbf{s}). \tag{5.10}$$

Proposition 5.7 *Fix $N \geq 1$. For any signatures λ, ν with N parts, and sequences of complex numbers $\mathbf{w}, \mathbf{y}, \boldsymbol{\theta}, \mathbf{s}$ as in (5.5), we have*

$$G_{\nu/\lambda}(\mathbf{w}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s}) = \det\Big[\mathsf{g}_{(\nu_i+N-i)/(\lambda_j+N-j)}(\mathbf{w}; \mathbf{y}; \boldsymbol{\theta}; \mathbf{s})\Big]_{i,j=1}^{N}. \tag{5.11}$$

Remark 5.8 The shifts of the indices in $\mathbf{y}$ and $\mathbf{s}$ in (5.11) (see (5.10)) are the same as in [72, 9th variation] (see also [85]), which makes our functions $G_{\nu/\lambda}$ a particular case of the Macdonald's 9-th variation of the Schur functions.

Proof of Proposition 5.7 Observe that

$$\frac{1}{2\pi \mathbf{i}} \oint_{\gamma'} dz \, \varphi_k(z \mid \mathbf{y}; \mathbf{s}) \psi_l(z \mid \mathbf{y}; \mathbf{s}) \prod_{j=1}^{M} \frac{z - \theta_j^{-2} w_j}{z - w_j} = \widetilde{\mathsf{h}}_{l-k}(\mathbf{w}; \mathrm{sh}_k \mathbf{y}; \boldsymbol{\theta}; \mathrm{sh}_k \mathbf{s}).$$

Thus, the proposition follows by applying Andréief identity (cf. [39])

$$
\begin{aligned}
&\frac{1}{(2\pi\mathbf{i})^N}\oint_{\gamma'}\dots\oint_{\gamma'}\det[f_i(z_j)]_{i,j=1}^N\det[g_i(z_j)]_{i,j=1}^N\,dz_1\dots dz_N\\
&\quad= N!\,\det\left[\frac{1}{2\pi\mathbf{i}}\oint_{\gamma'} f_i(z)g_j(z)\,dz\right]_{i,j=1}^N
\end{aligned}
\tag{5.12}
$$

to the contour integral formula for $G_{\nu/\lambda}$ (5.6). Indeed, here we can take $f_i(z) = \varphi_{\lambda_i+N-i}(z \mid \mathbf{y};\mathbf{s})$ and $g_j(z) = \psi_{\nu_j+N-j}(z \mid \mathbf{y};\mathbf{s})\prod_{m=1}^{M}\frac{z-\theta_m^{-2}w_m}{z-w_m}$. □

Remark 5.9 Using Proposition 5.7, one can check that in the case of horizontally homogeneous parameters $y_j \equiv 1$, $s_j \equiv s$ the skew functions $G_{\lambda/\nu}(\mathbf{w};\mathbf{y};\boldsymbol{\theta};\mathbf{s})$ turn (up to a simple product factor) into the supersymmetric skew Schur functions in the variables $\frac{w_i-1}{w_i-s^{-2}}\Big/\frac{1-\theta_i^{-2}w_i}{\theta_i^{-2}w_i-s^{-2}}$. In other words, identity (4.17) extends from the non-skew case to the skew one.

For non-skew functions G_ν there is a simplification of the formula of Proposition 5.7. Define

$$
\mathsf{h}_{k,p}(\mathbf{w};\mathbf{y};\boldsymbol{\theta};\mathbf{s}) := \frac{1}{2\pi\mathbf{i}}\oint_{\gamma'} dz\,\frac{\psi_k(z \mid \mathbf{y};\mathbf{s})}{y_p - z}\prod_{j=1}^{M}\frac{z-\theta_j^{-2}w_j}{z-w_j}, \tag{5.13}
$$

where ψ_k is given by (5.1), and the integration contour γ' surrounds all the points y_j, w_i and leaves out all $y_j s_j^{-2}$. Comparing (5.9) and (5.13), we see that $\widetilde{\mathsf{h}}_l = \mathsf{h}_{l,1}$ for $l \geq 0$.

Proposition 5.10 (Proposition 1.7 from Introduction) *Fix $N \geq 1$. For any signature ν with N parts, and sequences of complex numbers* $\mathbf{w}, \mathbf{y}, \boldsymbol{\theta}, \mathbf{s}$ *as in* (5.5)*, we have*

$$
G_\nu(\mathbf{w};\mathbf{y};\boldsymbol{\theta};\mathbf{s}) = \prod_{1\leq i<j\leq N}\frac{s_i^{-2}y_i - y_j}{y_j - y_i}\,\det\left[h_{\nu_i+N-i,\,j}(\mathbf{w};\mathbf{y};\boldsymbol{\theta};\mathbf{s})\right]_{i,j=1}^N. \tag{5.14}
$$

Proof The integrand in the contour integral formula for G_ν (5.7) contains the terms which can be rewritten as a Cauchy determinant:

$$
\frac{\prod_{1\leq i<j\leq N}(z_i - z_j)}{\prod_{i,j=1}^N (y_i - z_j)} = \frac{1}{\prod_{1\leq i<j\leq N}(y_j - y_i)}\det\left[\frac{1}{y_i - z_j}\right]_{i,j=1}^N.
$$

Combining this with the other determinant $\det[\psi_{\nu_i+N-i}(z_j)]$ in (5.7) and applying Andréief identity (5.12), we arrive at the desired formula. □

Part II Determinantal processes

In this part (accompanied by Appendix B) we develop determinantal point processes based on the symmetric functions F_λ, G_λ from Part I. By analogy with Schur and Macdonald processes [9, 90], we call them the *FG processes*. We compute the correlation kernel for ascending FG processes (a particular subclass of FG processes) in a double contour integral form.

6 FG measures and processes

6.1 Specializations

Fix the parameter sequences

$$\mathbf{y} = (y_1, y_2, \ldots), \quad \mathbf{s} = (s_1, s_2, \ldots),$$

for which there exists $\varepsilon > 0$ such that

$$\varepsilon < y_j < \varepsilon^{-1}, \quad \varepsilon < s_j < 1 - \varepsilon \quad \text{forall } j. \tag{6.1}$$

Under suitable restrictions on the other parameters, the values of F_λ, G_λ become nonnegative. This leads to the following definition:

Definition 6.1 Let $N \in \mathbb{Z}_{\geq 1}$ and let $\mathbf{x} = (x_1, \ldots, x_N)$, $\mathbf{r} = (r_1, \ldots, r_N)$ be such that

$$0 < x_i < y_j < r_i^{-2} x_i < s_j^{-2} y_j, \quad \text{forall } i, j. \tag{6.2}$$

Under (6.1) and (6.2), one readily sees that all the vertex weights $W, \widehat{W}$ (2.3), (2.4) are nonnegative. This implies that the values of the symmetric functions $F_{\lambda/\mu}(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s})$ and $G_{\lambda/\mu}(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s})$ are nonnegative.

We call $(\mathbf{x}; \mathbf{r})$ a *nonnegative specialization*, and denote this by $\rho \in \mathsf{Spec}_N$ (where N indicates the number of variables). When convenient, we denote the values of our symmetric functions at ρ by $F_{\lambda/\mu}(\rho)$, $G_{\lambda/\mu}(\rho)$, and omit explicitly specifying their overall dependence on $\mathbf{y}, \mathbf{s}$.

Remark 6.2 The vertex weights $W, \widehat{W}$ (2.3), (2.4) depend only on differences between the parameters $x, y, r^{-2}x, s^{-2}y$. Therefore, conditions $x_i, y_j > 0$ in (6.1)–(6.2) may be dropped, but we keep them throughout this Part II for convenience of dealing with various inequalities on the parameters.

The empty specialization $\rho = \varnothing \in \mathsf{Spec}_0$ is nonnegative, and

$$F_{\lambda/\mu}(\varnothing) = \mathbf{1}_{\lambda=\mu}, \qquad G_{\lambda/\mu}(\varnothing) = \mathbf{1}_{\lambda=\mu}. \tag{6.3}$$

For the function $G_{\lambda/\mu}$, we also get the same delta function by substituting the zero variables, namely, $G_{\lambda/\mu}(0, \ldots, 0; \mathbf{y}; \mathbf{r}; \mathbf{s}) = \mathbf{1}_{\lambda=\mu}$. This is evident by looking at the vertex weights W (2.3), as the weight of the vertex $(1, 0; 0, 1)$ vanishes.

For two specializations $\rho = (\mathbf{x}; \mathbf{r})$ and $\rho' = (\mathbf{x}'; \mathbf{r}')$, denote by $\rho \cup \rho'$ their union (concatenation) with variables $(\mathbf{x} \cup \mathbf{x}'; \mathbf{r} \cup \mathbf{r}')$, as in Proposition 3.6.

In order to use the Cauchy identities, we need to make sure that the corresponding infinite series converge:

Definition 6.3 Two nonnegative specializations $\rho = (\mathbf{x}; \mathbf{r}) \in \mathsf{Spec}_N$ and $\rho' = (\mathbf{w}, \boldsymbol{\theta}) \in \mathsf{Spec}_M$ are called *compatible* (notation $(\rho; \rho') \in \mathsf{Comp}$) if there exists $\delta > 0$ such that

$$\left| \frac{s_k^{-2} y_k - x_i}{y_k - x_i} \frac{y_k - w_j}{s_k^{-2} y_k - w_j} \right| < 1 - \delta < 1 \qquad \text{forall}\, i, j \text{ andallsufficientlylarge}\, k > 0. \tag{6.4}$$

Compatibility depends on the parameters $(\mathbf{y}; \mathbf{s})$, which are assumed fixed. Note also that the relation $(\rho; \rho') \in \mathsf{Comp}$ is not symmetric in ρ, ρ'.

Let us denote, for $\rho = (\mathbf{x}; \mathbf{r}) \in \mathsf{Spec}_N$, $\rho' = (\mathbf{w}, \boldsymbol{\theta}) \in \mathsf{Spec}_M$,

$$\begin{aligned} \Pi(\rho; \rho') &:= \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{x_i - \theta_j^{-2} w_j}{x_i - w_j}, \\ Z(\rho) &:= \prod_{i=1}^{N} x_i (r_i^{-2} - 1) \frac{\prod_{1 \le i < j \le N} (r_i^{-2} x_i - x_j)(s_i^{-2} y_i - y_j)}{\prod_{i,j=1}^{N} (y_i - x_j)}. \end{aligned} \tag{6.5}$$

Thus, the Cauchy identities (Propositions 3.7 and 3.8) take the following form for two compatible specializations ρ, ρ':

$$\begin{aligned} \sum_{\nu} G_{\nu/\lambda}(\rho') F_{\nu/\mu}(\rho) &= \Pi(\rho; \rho') \sum_{\varkappa} G_{\mu/\varkappa}(\rho') F_{\lambda/\varkappa}(\rho), \\ \sum_{\nu} F_{\nu}(\rho) G_{\nu}(\rho') &= \Pi(\rho; \rho')\, Z(\rho). \end{aligned} \tag{6.6}$$

6.2 Probability distributions from Cauchy identities

Let $T \ge 1$, $N \ge 0$ be integers, and pick a nonnegative specialization

$$\rho = (x_1, \ldots, x_N; r_1, \ldots, r_N) \in \mathsf{Spec}_N$$

and variables $(w_1, \theta_1), \ldots, (w_T, \theta_T)$ such that each $(w_i, \theta_i) \in \mathsf{Spec}_1$ is also a nonnegative specialization in the sense of Definition 6.1. Assume that these specializations are compatible in the sense of Definition 6.3, that is,

$$\left| \frac{x_i - s_k^{-2} y_k}{x_i - y_k} \frac{w_j - y_k}{w_j - s_k^{-2} y_k} \right| < 1 - \delta < 1 \qquad \text{forall}\, i, j \text{ and sufficiently large}\, k > 0. \tag{6.7}$$

Definition 6.4 The *ascending FG process* is a probability measure on sequences of signatures $\lambda^{(1)}, \ldots, \lambda^{(T)}$ (each with N parts) defined by

$$\begin{aligned}
&\mathcal{AP}(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(T)}) \\
&\quad = \frac{1}{Z} G_{\lambda^{(1)}}(w_1; \mathbf{y}; \theta_1; \mathbf{s}) G_{\lambda^{(2)}/\lambda^{(1)}}(w_2; \mathbf{y}; \theta_2; \mathbf{s}) \ldots \\
&\quad G_{\lambda^{(T)}/\lambda^{(T-1)}}(w_T; \mathbf{y}; \theta_T; \mathbf{s}) F_{\lambda^{(T)}}(\rho),
\end{aligned} \tag{6.8}$$

where the normalizing constant is equal to

$$\begin{aligned}
Z &= Z(\rho) \prod_{j=1}^{T} \Pi(\rho; (w_j, \theta_j)) \\
&= \prod_{i=1}^{N} x_i (r_i^{-2} - 1) \frac{\prod_{1 \le i < j \le N} (r_i^{-2} x_i - x_j)(s_i^{-2} y_i - y_j)}{\prod_{i,j=1}^{N} (y_i - x_j)} \prod_{i=1}^{N} \prod_{j=1}^{T} \frac{x_i - \theta_j^{-2} w_j}{x_i - w_j}.
\end{aligned} \tag{6.9}$$

This definition is parallel to a particular case of Schur processes [90], see also [9]. Later in Sect. 9 we connect ascending FG processes to a certain dimer model.

Remark 6.5 From the explicit formula for F_λ (Theorem 3.9) it is evident that $F_{\lambda^{(T)}}$ divided by Z (6.9) does not depend on the parameters r_j, and hence the whole ascending FG process is independent of these parameters, too.

The marginal distribution of each $\lambda^{(j)}$ under the FG process has the following form:

Definition 6.6 Let the parameters $(\mathbf{y}; \mathbf{s})$ satisfying (6.1) be fixed. Let $M, N \ge 1$, and take nonnegative specializations $\rho = (\mathbf{x}; \mathbf{r}) \in \mathsf{Spec}_N$, $\rho' = (\mathbf{w}; \boldsymbol{\theta}) \in \mathsf{Spec}_M$ such that $(\rho, \rho') \in \mathsf{Comp}$. The *FG measure* is a probability distribution on signatures λ with N parts with probability weights

$$\mathcal{M}(\lambda) := \frac{F_\lambda(\rho) G_\lambda(\rho')}{Z(\rho)\Pi(\rho; \rho')}. \tag{6.10}$$

Definition 6.6 is parallel to the definition of the Schur measure [86]. Using branching (Proposition 3.6) and the first relation of (6.6), we see that for every $j = 1, \ldots, T$, the signature $\lambda^{(j)}$ (with N parts) is distributed as the FG measure with specializations $\rho = (\mathbf{x}; \mathbf{r}) \in \mathsf{Spec}_N$ and $\rho' = (w_1, \ldots, w_j; \theta_1, \ldots, \theta_j) \in \mathsf{Spec}_j$.

6.3 Determinantal correlation kernel

Take a random sequence of signatures $\lambda^{(1)}, \ldots, \lambda^{(T)}$ distributed according to the ascending FG process (6.8), and associate to it a random point configuration

$$\mathcal{S}^{(T)} := \bigcup_{t=1}^{T} \big(\{t\} \times \mathcal{S}(\lambda^{(t)})\big) \tag{6.11}$$

in $\{1,\ldots,T\}\times\mathbb{Z}_{\ge 1}$, where we use notation $\mathcal{S}(\lambda)$ from (3.1).

We are interested in *correlation functions* of the ascending FG process, which are defined for any fixed finite subset $A\subset\{1,\ldots,T\}\times\mathbb{Z}_{\ge 1}$ as the probabilities $\mathbb{P}_{\mathcal{AP}}\left[A\subset\mathcal{S}^{(T)}\right]$. We show that the ascending FG process is *determinantal*, that is, all its correlation functions are determinants of a certain *correlation kernel* $K_{\mathcal{AP}}(t,a;t',a')$, $1\le t,t'\le T$, $a,a'\ge 1$. It has the following form:

$$K_{\mathcal{AP}}(t,a;t',a')=\frac{1}{(2\pi\mathbf{i})^2}\oint_{\Gamma_{y,\theta^{-2}w}}du\oint_{\Gamma_{y,w}}dv\,\frac{1}{u-v}\prod_{k=1}^{N}\frac{(u-y_k)(v-x_k)}{(u-x_k)(v-y_k)}$$
$$\times\frac{y_a(1-s_a^{-2})}{v-s_a^{-2}y_a}\frac{1}{u-y_{a'}}\prod_{j=1}^{a-1}\frac{v-y_j}{v-s_j^{-2}y_j}$$
$$\prod_{j=1}^{a'-1}\frac{u-s_j^{-2}y_j}{u-y_j}\prod_{d=1}^{t}\frac{v-\theta_d^{-2}w_d}{v-w_d}\prod_{c=1}^{t'}\frac{u-w_c}{u-\theta_c^{-2}w_c}, \tag{6.12}$$

where the integration contours are positively oriented circles one inside the other (the u contour is outside for $t\le t'$ while the v contour is outside for $t>t'$); the u contour encircles all the points $y_i,\theta_j^{-2}w_j$ and not x_k; and the v contour encircles all the points y_i,w_j and not $s_k^{-2}y_k$. Observe that $K_{\mathcal{AP}}$ is independent of the r_j's, which agrees with Remark 6.5.

Theorem 6.7 (Theorem 1.8 from Introduction) *The random point configuration $\mathcal{S}^{(T)}$ constructed from the ascending FG process is a determinantal point process with the kernel $K_{\mathcal{AP}}$ (6.12):*

$$\mathbb{P}_{\mathcal{AP}}\left[A\subset\mathcal{S}^{(T)}\right]=\det\left[K_{\mathcal{AP}}(t_i,a_i;t_j,a_j)\right]_{i,j=1}^{m} \tag{6.13}$$

for any $A=\{(t_1,a_1),\ldots,(t_m,a_m)\}\subset\{1,\ldots,T\}\times\mathbb{Z}_{\ge 1}$.

We prove Theorem 6.7 using an Eynard–Mehta type approach (e.g., see [20]) which is possible due to determinantal formulas for our symmetric functions from Sect. 5. This approach is quite standard and is deferred to Appendix B.

Moreover, in Sect. 8 below we discuss more general FG processes (having the structure similar to the general Schur processes of [90]) and connect them to fermionic operators in the Fock space (developed in Sect. 7). We employ this connection to obtain a generating function for the correlation kernel. In Sect. 8.6 we check that the Fock space approach leads to the same correlation kernel in the ascending case.

Corollary 6.8 *The FG measure (Definition 6.6) gives rise to a determinantal point process $\mathcal{S}(\lambda)$ on $\mathbb{Z}_{\geq 1}$ with the correlation kernel*

$$K_{\mathcal{M}}(a, a') = \frac{1}{(2\pi \mathbf{i})^2} \oint_{\Gamma_{y,\theta^{-2}w}} du \oint_{\Gamma_{y,w}} dv \prod_{k=1}^{N} \frac{(u-y_k)(v-x_k)}{(u-x_k)(v-y_k)} \prod_{i=1}^{M} \frac{(u-w_i)(v-\theta_i^{-2}w_i)}{(v-w_i)(u-\theta_i^{-2}w_i)} \times \frac{1}{u-v} \frac{y_a(1-s_a^{-2})}{v-s_a^{-2}y_a} \frac{1}{u-y_{a'}} \prod_{j=1}^{a-1} \frac{v-y_j}{v-s_j^{-2}y_j} \prod_{j=1}^{a'-1} \frac{u-s_j^{-2}y_j}{u-y_j}, \tag{6.14}$$

with the integration contours are the same as in (6.12)*, and the u contour is outside the v contour.*

6.4 Horizontally homogeneous model and Schur measure

In the horizontally homogeneous case $s_i = s$, $y_i = 1$ for all $i \geq 1$, thanks to Proposition 4.10, the FG measure reduces to the Schur measure

$$\mathbb{P}_{\text{Schur}}(\lambda) = \frac{1}{Z} s_\lambda \left(\frac{1-s^2x_1}{s^2(1-x_1)}, \ldots, \frac{1-s^2x_N}{s^2(1-x_N)} \right) s_\lambda \left(\left\{ \frac{s^2(1-x_j)}{1-s^2x_j} \right\}_{j=1}^{M} \Big/ \left\{ \frac{s^2(w_j\theta_j^{-2}-1)}{1-s^2\theta_j^{-2}w_j} \right\}_{j=1}^{M} \right) \tag{6.15}$$

for a suitable normalization constant Z, and $\left| \frac{1-s^2x_i}{1-x_i} \frac{1-w_j}{1-s^2w_j} \right| < 1-\delta < 1$ for all i, j (this condition follows from Definition 6.3). Therefore, by [20, 86] its determinantal correlation kernel has a double contour integral form. Let us compare that expression with Corollary 6.8.

First, we recall the correlation kernel of the Schur measure from [20, 86]. A function that enters the kernel is

$$\Phi_{\text{Schur}}(U) = \underbrace{\prod_{i=1}^{N} \frac{1}{1-\frac{1-s^2x_i}{s^2(1-x_i)}U}}_{H_1(U)} \underbrace{\prod_{j=1}^{M} \frac{1-\frac{s^2(1-w_j)}{1-s^2w_j}U^{-1}}{1+\frac{s^2(\theta_j^{-2}w_j-1)}{1-s^2\theta_j^{-2}w_j}U^{-1}}}_{1/H_2(U^{-1})},$$

where H_1, H_2 are the generating functions associated with the two specializations of the Schur functions in (6.15). The correlation kernel is then given by

$$K_{\mathrm{Schur}}(a,a') = \frac{1}{(2\pi\mathbf{i})^2}\oint\oint \frac{dUdV}{U-V}\frac{V^{a'-N-1}}{U^{a-N}}\frac{\Phi_{\mathrm{Schur}}(U)}{\Phi_{\mathrm{Schur}}(V)}, \qquad a,a' \in \mathbb{Z}_{\geq 1}. \tag{6.16}$$

The shifts by $N+1$ in a, a' come from the fact that our encoding of the particle configurations in $\mathbb{Z}$ is different compared to the Schur measures. The integration contours are such that $|V| < |U|$ and the Taylor expansions of $H_1(U)$, $H_1(V)$, $H_2(U^{-1})$, $H_2(V^{-1})$ on the contours are into suitable generating series in U and V, respectively:

$$\left|\frac{1-s^2x_i}{s^2(1-x_i)}\,U\right| < 1, \qquad \left|\frac{s^2(\theta_j^{-2}w_j-1)}{1-s^2\theta_j^{-2}w_j}\,U^{-1}\right| < 1, \qquad \left|\frac{s^2(1-w_j)}{1-s^2w_j}\,V^{-1}\right| < 1,$$

for all i, j.

Let us change the variables in (6.16) as

$$U = \frac{u-1}{u-s^{-2}}, \qquad V = \frac{v-1}{v-s^{-2}}, \qquad \frac{dUdV}{U-V} = \frac{s^2(s^2-1)}{(1-s^2u)(1-s^2v)}\frac{dudv}{u-v},$$

which yields

$$\begin{aligned} K_{\mathrm{Schur}}(a,a') = \frac{1}{(2\pi\mathbf{i})^2}\oint\oint \frac{dudv}{u-v}\frac{(1-s^{-2})}{(u-s^{-2})(v-1)}\left(\frac{u-s^{-2}}{u-1}\right)^{a}\left(\frac{v-1}{v-s^{-2}}\right)^{a'} \\ \times\left(\frac{u-1}{v-1}\right)^{N}\prod_{i=1}^{N}\frac{v-x_i}{u-x_i}\prod_{j=1}^{M}\frac{u-w_j}{v-w_j}\frac{v-\theta_j^{-2}w_j}{u-\theta_j^{-2}w_j}, \end{aligned}$$

over the contours such that

$$\left|\frac{v-1}{v-s^{-2}}\frac{u-s^{-2}}{u-1}\right| < 1, \qquad \left|\frac{s^{-2}-x_i}{1-x_i}\frac{u-1}{u-s^{-2}}\right| < 1,$$
$$\left|\frac{\theta_j^{-2}w_j-1}{s^{-2}-\theta_j^{-2}w_j}\frac{u-s^{-2}}{u-1}\right| < 1, \qquad \left|\frac{1-w_j}{s^{-2}-w_j}\frac{v-s^{-2}}{v-1}\right| < 1.$$

One can check that these conditions hold on the contour v around 1 and w, and the contour u containing the v contour and also encircling w/θ^2. Thus, our Corollary 6.8 reduces (up to the swap $a \leftrightarrow a'$ which does not affect the determinantal point process) to the known kernel of the particular Schur measure (6.15).

7 Fermionic operators

In this section we develop fermionic operators in the Fock space which serve as inhomogeneous analogues of the operators employed in studying Schur measures and processes in [86, 90].

7.1 Simplified commutation relations

Let $V^{(k)}$, $k \in \mathbb{Z}$, be the two-dimensional complex space with basis $e_0^{(k)}, e_1^{(k)}$. We will consider tensor products of the form

$$V^{[M,N]} := V^{(M)} \otimes V^{(M+1)} \otimes \ldots \otimes V^{(N)}, \qquad M \leq N. \tag{7.1}$$

As usual, when working with row operators (see the beginning of Sect. 2.3), we think that each $V^{(k)}$ carries two parameters (y_k, s_k), $k \in \mathbb{Z}$.

Remark 7.1 This is the first time when we allow the indices of the parameters (y_k, s_k) to be nonpositive. However, when applying our computations to actual probability measures, the indices of (y_j, s_j) will always satisfy $j \in \mathbb{Z}_{\geq 1}$.

Recall the operators A, B, C, D (2.8) acting in each $V^{(k)}$. They depend on x, r, and also on the parameters (y_k, s_k) attached to $V^{(k)}$. We omit the latter in the notation, and write $A = A(x, r)$, and so on. Via (2.9), these operators also act on any tensor products of the form $V^{([M,N])}$. Our first observation is that with special values of the parameters (x, r), the operators A, B, C, D satisfy certain simplified relations:

Proposition 7.2 *For any $x, z, t \in \mathbb{C}$ we have*[2]

$$\begin{aligned} B(x, t)B(z, \sqrt{z/x}) = 0 = B(z, t)B(x, \sqrt{x/z}); \\ C(x, \sqrt{x/z})C(z, t) = 0 = C(z, \sqrt{z/x})C(x, t), \end{aligned} \tag{7.2}$$

and

$$\begin{aligned} &B(z, \sqrt{z/x})D(x, \sqrt{x/z}) + D(z, \sqrt{z/x})B(x, \sqrt{x/z}) = 0; \\ &D(z, \sqrt{z/x})C(x, \sqrt{x/z}) = C(z, \sqrt{z/x})D(x, \sqrt{x/z}); \\ &D(x, \sqrt{x/z})A(z, \sqrt{z/x}) - C(x, \sqrt{x/z})B(z, \sqrt{z/x}) \\ &\quad = D(z, \sqrt{z/x})A(x, \sqrt{x/z}) + B(z, \sqrt{z/x})C(x, \sqrt{x/z}); \\ &A(x, \sqrt{x/z})D(z, \sqrt{z/x}) - B(x, \sqrt{x/z})C(z, \sqrt{z/x}) \\ &\quad = A(z, \sqrt{z/x})D(x, \sqrt{x/z}) + C(z, \sqrt{z/x})B(x, \sqrt{x/z}). \end{aligned} \tag{7.3}$$

[2] All square roots involved in identities in this proposition and throughout the section are always squared in the action of the operators, so we do not need to specify the branches.

Proof For (7.2), we use relations (2.10), (2.11). In particular, to get the first identity in (7.2), take $(x_1, r_1) = (z, \sqrt{z/x})$ and $(x_2, r_2) = (x, t)$, which implies $(t^{-2}x - z)B(x, t)B(z, \sqrt{z/x}) = 0$. All other identities in (7.2) are established in a similar way.

Let us now turn to (7.3). For the first identity, use (2.14),

$$(x_1 - x_2)B(x_2, r_2)D(x_1, r_1) = (r_1^{-2}x_1 - x_2)D(x_1, r_1)B(x_2, r_2) \\ +x_2(1 - r_2^{-2})D(x_2, r_2)B(x_1, r_1)$$

with $(x_1, r_1) = (x, \sqrt{x/z})$ and $(x_2, r_2) = (z, \sqrt{z/x})$, which yields

$$(x - z)B(z, \sqrt{z/x})D(x, \sqrt{x/z}) = z(1 - x/z)D(z, \sqrt{z/x})B(x, \sqrt{x/z}),$$

and thus we obtain the first identity from (7.3) The second identity is analogous with the help of (2.16). The last two identities follow in a similar way from (2.19) and (2.20), respectively. □

Recall that each subset $\mathcal{T} \subseteq \{M, M + 1, \ldots, N\}$ (where $M \le N$) corresponds to a vector $e_{\mathcal{T}} \in V^{[M,N]}$ defined as

$$e_{\mathcal{T}} = e_{k_M}^{(M)} \otimes e_{k_{M+1}}^{(M+1)} \otimes \ldots \otimes e_{k_N}^{(N)}, \qquad k_i = k_i(\mathcal{T}) = \mathbf{1}_{i \in \mathcal{T}}.$$

Also recall the inner product $\langle \cdot, \cdot \rangle$ on tensor powers of $\mathbb{C}^2$ such as $V^{[M,N]}$, under which the vectors of the form $e_{\mathcal{T}}$ are orthonormal.

Propositions 7.3, 7.7 and 7.6 below show that matrix elements of $D(x, \sqrt{x/z})B(z, \sqrt{z/x})$ and $D(x, \sqrt{x/z})C(z, \sqrt{z/x})$ can be used to detect if two subsets of $\{M, \ldots, N\}$ are different by a single element. This is summarized in Theorem 7.11 below.

Proposition 7.3 *Fix nonzero $x, z \in \mathbb{C}$; integers $m \ge 0$ and $M \le N$; and two integer sets*

$$\mathcal{R} = (r_1 < r_2 < \ldots < r_m), \quad \mathcal{T} = (t_1 < t_2 < \ldots < t_m < t_{m+1}), \\ \mathcal{R}, \mathcal{T} \subset \{M, M + 1, \ldots, N\}.$$

If $\mathcal{R}$ is not a subset of $\mathcal{T}$, then

$$\left\langle e_{\mathcal{R}}, D(x, \sqrt{x/z})B(z, \sqrt{z/x})e_{\mathcal{T}} \right\rangle = 0 = \left\langle e_{\mathcal{T}}, D(x, \sqrt{x/z})C(z, \sqrt{z/x})e_{\mathcal{R}} \right\rangle. \quad (7.4)$$

Proof We only establish the first equality in (7.4), the second equality follows in a similar manner. For convenience of notation, we set $r_{m+1} = t_{m+2} = +\infty$ throughout the proof.

Since $\mathcal{R}$ is not a subset of $\mathcal{T}$, there exists an index $1 \le n \le m$ such that either $t_n < r_n < t_{n+1}$, or $t_{n+1} < r_n < t_{n+2}$. Fix such n. Define sets $\mathcal{R}' = \mathcal{R} \cap (-\infty, r_n - 1]$ and $\mathcal{R}'' = \mathcal{R} \cap [r_n, +\infty)$, and similarly $\mathcal{T}', \mathcal{T}''$.

First, assume that $t_{n+1} < r_n < t_{n+2}$. Then since $|\mathcal{R}'| = n-1 = |\mathcal{T}| - 2$, we have using $V^{[M,N]} = V^{[M,r_n-1]} \otimes V^{[r_n,N]}$ and (2.9), picking B twice for the left tensor product:

$$\begin{aligned}&\langle e_{\mathcal{R}}, D(x, \sqrt{x/z})B(z, \sqrt{z/x})e_{\mathcal{T}}\rangle\\&\quad = \langle e_{\mathcal{R}'}, B(x, \sqrt{x/z})B(z, \sqrt{z/x})e_{\mathcal{T}'}\rangle\langle e_{\mathcal{R}''}, C(x, \sqrt{x/z})A(z, \sqrt{z/x})e_{\mathcal{T}''}\rangle.\end{aligned}$$

The latter expression vanishes by identity (7.2) from Proposition 7.2.

Now let us assume that $t_n < r_n < t_{n+1}$. Define sets $\mathcal{R}_0' = \mathcal{R} \cap (-\infty, r_n]$, $\mathcal{R}_0'' = \mathcal{R} \cap [r_n+1, +\infty)$, and similarly $\mathcal{T}_0', \mathcal{T}_0''$. We have $|\mathcal{R}_0'| = |\mathcal{T}_0'| = n$, so with $V^{[M,N]} = V^{[M,r_n]} \otimes V^{[r_n+1,N]}$, in the expansion (2.9) we need to take the D operator twice. We have

$$\begin{aligned}&\langle e_{\mathcal{R}}, D(x, \sqrt{x/z})B(z, \sqrt{z/x})e_{\mathcal{T}}\rangle\\&\quad = \langle e_{\mathcal{R}_0'}, D(x, \sqrt{x/z})D(z, \sqrt{z/x})e_{\mathcal{T}_0'}\rangle\langle e_{\mathcal{R}_0''}, D(x, \sqrt{x/z})B(z, \sqrt{z/x})e_{\mathcal{T}_0''}\rangle.\end{aligned}$$

In the first factor we apply (2.9) with $V^{[M,r_n]} = V^{[M,r_n-1]} \otimes V^{(r_n)}$. Observe that $|\mathcal{R}'| = n-1 = |\mathcal{T}'| - 1$, so we obtain

$$\begin{aligned}&\langle e_{\mathcal{R}_0'}, D(x, \sqrt{x/z})D(z, \sqrt{z/x})e_{\mathcal{T}_0'}\rangle\\&\quad = \langle e_{\mathcal{R}'}, D(x, \sqrt{x/z})B(z, \sqrt{z/x})e_{\mathcal{T}'}\rangle\langle e_1^{(r_n)}, D(x, \sqrt{x/z})C(z, \sqrt{z/x})e_0^{(r_n)}\rangle\\&\qquad + \langle e_{\mathcal{R}'}, B(x, \sqrt{x/z})D(z, \sqrt{z/x})e_{\mathcal{T}'}\rangle\langle e_1^{(r_n)}, C(x, \sqrt{x/z})D(z, \sqrt{z/x})e_0^{(r_n)}\rangle.\end{aligned}$$

This expression vanishes thanks to the first two identities in (7.3). □

7.2 Normalized operators

Let us now introduce normalizations of our operators A, B, C, D, which allow to take the limit as $M \to -\infty$, $N \to +\infty$ without running into infinite products:

Definition 7.4 Fix $x, r \in \mathbb{C}$. For $M \le 0 \le N$, define the normalized operators $A^{[M,N]}(x,r)$, $B^{[M,N]}(x,r)$, $C^{[M,N]}(x,r)$, and $D^{[M,N]}(x,r)$ acting on $V^{[M,N]}$ by

$$\begin{aligned}A^{[M,N]}(x,r) &= \frac{A(x,r)}{\prod_{i=M}^{0} W_i(1,1;1,1)\prod_{j=1}^{N} W_j(0,1;0,1)}\\&= A(x,r)\prod_{i=M}^{0}\frac{r^2(y_i - s_i^2 x)}{s_i^2(x - r^2 y_i)}\prod_{j=1}^{N}\frac{y_j - s_j^2 x}{s_j^2(y_j - x)};\\B^{[M,N]}(x,r) &= \frac{B(x,r)}{\prod_{i=M}^{0} W_i(1,0;1,0)\prod_{j=1}^{N} W_j(0,1;0,1)}\\&= B(x,r)\prod_{i=M}^{0}\frac{y_i - s_i^2 x}{y_i - s_i^2 r^{-2} x}\prod_{j=1}^{N}\frac{y_j - s_j^2 x}{s_j^2(y_j - x)};\end{aligned}$$

$$C^{[M,N]}(x,r) = \frac{C(x,r)}{\prod_{i=M}^{0} W_i(1,1;1,1)} = C(x,r)\prod_{i=M}^{0}\frac{r^2(y_i - s_i^2 x)}{s_i^2(x - r^2 y_i)};$$
$$D^{[M,N]}(x,r) = \frac{D(x,r)}{\prod_{i=M}^{0} W_i(1,0;1,0)} = D(x,r)\prod_{i=M}^{0}\frac{y_i - s_i^2 x}{y_i - s_i^2 r^{-2} x}.$$

Here W_j are vertex weights (2.3) with the parameters $W_j(\cdots) = W(\cdots \mid x; y_j; r; s_j)$.

Definition 7.5 Let us define expressions Φ_j, Φ_j^* for $j \in \mathbb{Z}$ as follows:

$$\Phi_j(x,z) = \begin{cases} \dfrac{y_j(1-s_j^{-2})}{x - s_j^{-2}y_j}\displaystyle\prod_{k=1}^{j-1}\frac{x-y_k}{x-s_k^{-2}y_k}, & j>0; \\ \dfrac{y_j(1-s_j^{-2})}{x - s_j^{-2}y_j}\displaystyle\prod_{k=j}^{0}\frac{x-s_k^{-2}y_k}{x-y_k}, & j \le 0, \end{cases}$$

$$\Phi_j^*(x,z) = \begin{cases} \dfrac{z-x}{z-y_j}\displaystyle\prod_{k=1}^{j-1}\frac{z-s_k^{-2}y_k}{z-y_k}, & j>0; \\ \dfrac{z-x}{z-y_j}\displaystyle\prod_{k=j}^{0}\frac{z-y_k}{z-s_k^{-2}y_k}, & j \le 0. \end{cases}$$

Note that while $\Phi_j(x,z)$ does not depend on z, it is convenient to keep the Φ_j, Φ_j^* notation uniform.

Proposition 7.6 *Fix nonzero $x, z \in \mathbb{C}$; integers $m \ge 0$ and $M \le N$; and two integer sets*

$$\mathcal{R} = (r_1 < r_2 < \ldots < r_m), \quad \mathcal{T} = (t_1 < t_2 < \ldots < t_m < t_{m+1}),$$
$$\mathcal{R}, \mathcal{T} \subset \{M, M+1, \ldots, N\}.$$

If for some $j \in \{1,\ldots,m+1\}$ we have $\mathcal{R} = \mathcal{T} \setminus \{t_j\}$, then

$$\left\langle e_{\mathcal{R}}, D^{[M,N]}(x, \sqrt{x/z}) B^{[M,N]}(z, \sqrt{z/x}) e_{\mathcal{T}} \right\rangle = (-1)^{m-j+1}\Phi_{t_j}^*(x,z). \tag{7.5}$$

Proof Observe that by Definition 7.4,

$$D^{[M,N]}(x,\sqrt{x/z})B^{[M,N]}(z,\sqrt{z/x}) = D(x,\sqrt{x/z})B(z,\sqrt{z/x})\prod_{j=1}^{N}\frac{y_j - s_j^2 z}{s_j^2(y_j - z)}. \tag{7.6}$$

Therefore, it suffices to evaluate $D(x,\sqrt{x/z})B(z,\sqrt{z/x})$. In the action of this operator in the tensor product of the spaces $V^{(k)}$, whenever we see $D(x,\sqrt{x/z})D(z,\sqrt{z/x})$, we have

$$\left\langle e_1^{(k)}, D(x, \sqrt{x/z})D(z, \sqrt{z/x})e_1^{(k)}\right\rangle = \left\langle e_0^{(k)}, D(x, \sqrt{x/z})D(z, \sqrt{z/x})e_0^{(k)}\right\rangle = 1. \tag{7.7}$$

This means that nontrivial contributions to the left-hand side of (7.5) can only come from the configuration to the right of t_j. Without loss of the generality, we may assume that $j = 1$, and $r_i = t_{i+1}$ for $i = 1, \ldots, m$.

For any k, define $\mathcal{R}_k = \mathcal{R} \cap [M, k]$, and similarly for $\mathcal{T}_k$. If $k = t_i$ for some $i = 2, \ldots, m+1$, we have by (2.9):

$$\begin{aligned} &\left\langle e_{\mathcal{R}_k}, D(x, \sqrt{x/z})B(z, \sqrt{z/x})e_{\mathcal{T}_k}\right\rangle \\ &\quad = \left\langle e_{\mathcal{R}_{k-1}}, D(x, \sqrt{x/z})B(z, \sqrt{z/x})e_{\mathcal{T}_{k-1}}\right\rangle \left\langle e_1^{(k)}, D(x, \sqrt{x/z})A(z, \sqrt{z/x})e_1^{(k)}\right\rangle \\ &\qquad + \left\langle e_{\mathcal{R}_{k-1}}, B(x, \sqrt{x/z})D(z, \sqrt{z/x})e_{\mathcal{T}_{k-1}}\right\rangle \left\langle e_1^{(k)}, C(x, \sqrt{x/z})B(z, \sqrt{z/x})e_1^{(k)}\right\rangle. \end{aligned} \tag{7.8}$$

The first identity in (7.3) states that the D and B operators can be swapped, producing a negative sign. Applying this to the second summand in the right-hand side of (7.8), we see that

$$\begin{aligned} (7.8) &= \left\langle e_{\mathcal{R}_{k-1}}, D(x, \sqrt{x/z})B(z, \sqrt{z/x})e_{\mathcal{T}_{k-1}}\right\rangle \\ &\quad \times \left\langle e_1^{(k)}, \Big(D(x, \sqrt{x/z})A(z, \sqrt{z/x}) - C(x, \sqrt{x/z})B(z, \sqrt{z/x})\Big)\, e_1^{(k)}\right\rangle \\ &= \left\langle e_{\mathcal{R}_{k-1}}, D(x, \sqrt{x/z})B(z, \sqrt{z/x})e_{\mathcal{T}_{k-1}}\right\rangle \\ &\quad \times \left\langle e_1^{(k)}, \Big(D(z, \sqrt{z/x})A(x, \sqrt{x/z}) + B(z, \sqrt{z/x})C(x, \sqrt{x/z})\Big)\, e_1^{(k)}\right\rangle, \end{aligned}$$

where in the second equality we applied the third identity in (7.3). Now observe that the operator C maps $e_1^{(k)}$ to 0, and we can continue (evaluating the eigenaction of $D(z, \sqrt{z/x})A(x, \sqrt{x/z})$ on the vector $e_1^{(k)}$):

$$(7.8) = \frac{s_k^2(z - y_k)}{y_k - s_k^2 z} \left\langle e_{\mathcal{R}_{k-1}}, D(x, \sqrt{x/z})B(z, \sqrt{z/x})e_{\mathcal{T}_{k-1}}\right\rangle.$$

Now let us compute the same quantity if $k \notin \mathcal{T}$. Then we have by (2.9):

$$\begin{aligned} &\left\langle e_{\mathcal{R}_k}, D(x, \sqrt{x/z})B(z, \sqrt{z/x})e_{\mathcal{T}_k}\right\rangle \\ &\quad = \left\langle e_{\mathcal{R}_{k-1}}, D(x, \sqrt{x/z})B(z, \sqrt{z/x})e_{\mathcal{T}_{k-1}}\right\rangle \left\langle e_0^{(k)}, D(x, \sqrt{x/z})A(z, \sqrt{z/x})e_0^{(k)}\right\rangle \\ &\quad = \frac{s_k^2(y_k - z)}{y_k - s_k^2 z} \left\langle e_{\mathcal{R}_{k-1}}, D(x, \sqrt{x/z})B(z, \sqrt{z/x})e_{\mathcal{T}_{k-1}}\right\rangle. \end{aligned}$$

We can now compute the action of $D(x, \sqrt{x/z})B(z, \sqrt{z/x})$ by successively splitting off the tensor factors, each time we obtain the factor $\pm\dfrac{s_k^2(y_k - z)}{y_k - s_k^2 z}$. The overall number of negative signs is $(-1)^m$, which translates to $(-1)^{m-j+1}$ when dropping the assumption $j = 1$. In the last step of the splitting, at $k = t_j$, we have

$$\left\langle e_0^{(k)}, D(x, \sqrt{x/z})B(z, \sqrt{z/x})e_1^{(k)}\right\rangle = \frac{s_k^2(x - z)}{y_k - s_k^2 z}.$$

Recalling normalization (7.6), we get the desired identity. □

Proposition 7.7 *Fix nonzero $x, z \in \mathbb{C}$; integers $m \geq 0$ and $M \leq N$; and two integer sets*

$$\mathcal{R} = (r_1 < r_2 < \ldots < r_m), \quad \mathcal{T} = (t_1 < t_2 < \ldots < t_m < t_{m+1}),$$
$$\mathcal{R}, \mathcal{T} \subset \{M, M+1, \ldots, N\}.$$

If for some $j \in \{1, \ldots, m+1\}$ we have $\mathcal{R} = \mathcal{T} \setminus \{t_j\}$, then

$$\left\langle e_{\mathcal{T}}, D^{[M,N]}(x, \sqrt{x/z})C^{[M,N]}(z, \sqrt{z/x})e_{\mathcal{R}}\right\rangle = (-1)^{M-j}\Phi_{t_j}(x, z). \tag{7.9}$$

Proof The proof follows along the same lines as the proof of the previous Proposition 7.6. First we observe that (cf. Definition 7.4)

$$\begin{aligned} &D^{[M,N]}(x, \sqrt{x/z})C^{[M,N]}(z, \sqrt{z/x}) \\ &\quad = (-1)^{M+1}D(x, \sqrt{x/z})C(z, \sqrt{z/x}) \prod_{i=M}^{0} \frac{y_i - s_i^2 x}{s_i^2(y_i - x)}. \end{aligned} \tag{7.10}$$

Thus, it suffices to evaluate $D(x, \sqrt{x/z})C(z, \sqrt{z/x})$. In the action of this operator in the tensor product of the spaces $V^{(k)}$, whenever we see $D(x, \sqrt{x/z})D(z, \sqrt{z/x})$, we may use (7.7). This means that nontrivial contributions to the left-hand side of (7.9) can only come from the configuration to the left of t_j. Without loss of the generality, we may assume that $j = m + 1$, and $r_i = t_i$ for $i = 1, \ldots, m$.

For any k, define $\mathcal{R}_k = [k, N] \cap \mathcal{R}$, and similarly for $\mathcal{T}_k$. First, assuming that $k = t_i$ for some $i = 1, \ldots, m$, we have by (2.9) and the second identity in (7.3):

$$\begin{aligned} &\left\langle e_{\mathcal{T}_k}, D(x, \sqrt{x/z})C(z, \sqrt{z/x})e_{\mathcal{R}_k}\right\rangle = \left\langle e_{\mathcal{T}_k}, C(x, \sqrt{x/z})D(z, \sqrt{z/x})e_{\mathcal{R}_k}\right\rangle \\ &\quad = \left\langle e_1^{(k)}, C(x, \sqrt{x/z})B(z, \sqrt{z/x})e_1^{(k)}\right\rangle\left\langle e_{\mathcal{T}_{k+1}}, D(x, \sqrt{x/z})C(z, \sqrt{z/x})e_{\mathcal{R}_{k+1}}\right\rangle \\ &\qquad + \left\langle e_1^{(k)}, A(x, \sqrt{x/z})D(z, \sqrt{z/x})e_1^{(k)}\right\rangle\left\langle e_{\mathcal{T}_{k+1}}, C(x, \sqrt{x/z})D(z, \sqrt{z/x})e_{\mathcal{R}_{k+1}}\right\rangle. \end{aligned} \tag{7.11}$$

We next have by (7.3):

$$
\begin{aligned}
(7.11) &= \big\langle e_{\mathcal{T}_{k+1}}, C(x,\sqrt{x/z})D(z,\sqrt{z/x})e_{\mathcal{R}_{k+1}}\big\rangle \\
&\quad\times \big\langle e_1^{(k)}, \Big(A(x,\sqrt{x/z})D(z,\sqrt{z/x}) + C(x,\sqrt{x/z})B(z,\sqrt{z/x})\Big)\, e_1^{(k)}\big\rangle \\
&= \big\langle e_{\mathcal{T}_{k+1}}, C(x,\sqrt{x/z})D(z,\sqrt{z/x})e_{\mathcal{R}_{k+1}}\big\rangle \\
&\quad\times \big\langle e_1^{(k)}, \Big(A(z,\sqrt{z/x})D(x,\sqrt{x/z}) - B(z,\sqrt{z/x})C(x,\sqrt{x/z})\Big)\, e_1^{(k)}\big\rangle \\
&= \big\langle e_{\mathcal{T}_{k+1}}, C(x,\sqrt{x/z})D(z,\sqrt{z/x})e_{\mathcal{R}_{k+1}}\big\rangle \big\langle e_1^{(k)}, A(z,\sqrt{z/x})D(x,\sqrt{x/z})e_1^{(k)}\big\rangle \\
&= \frac{s_k^2(x-y_k)}{y_k - s_k^2 x}\, \big\langle e_{\mathcal{T}_{k+1}}, D(x,\sqrt{x/z})C(z,\sqrt{z/x})e_{\mathcal{R}_{k+1}}\big\rangle.
\end{aligned}
$$

Here we used the fact that the C operator maps $e_1^{(k)}$ to 0, and for the last equality we evaluated the eigenaction on $e_1^{(k)}$.

Now assume that $k \notin \mathcal{T}$. Then we have

$$
\begin{aligned}
&\big\langle e_{\mathcal{T}_k}, D(x,\sqrt{x/z})C(z,\sqrt{z/x})e_{\mathcal{R}_k}\big\rangle = \big\langle e_{\mathcal{T}_k}, C(x,\sqrt{x/z})D(z,\sqrt{z/x})e_{\mathcal{R}_k}\big\rangle \\
&\quad = \big\langle e_0^{(k)}, A(x,\sqrt{x/z})D(z,\sqrt{z/x})e_0^{(k)}\big\rangle \big\langle e_{\mathcal{T}_{k+1}}, C(x,\sqrt{x/z})D(z,\sqrt{z/x})e_{\mathcal{R}_{k+1}}\big\rangle \\
&\quad = \frac{s_k^2(y_k - x)}{y_k - s_k^2 x}\, \big\langle e_{\mathcal{T}_{k+1}}, D(x,\sqrt{x/z})C(z,\sqrt{z/x})e_{\mathcal{R}_{k+1}}\big\rangle.
\end{aligned}
$$

We can now compute the action of $D(x,\sqrt{x/z})C(z,\sqrt{z/x})$ by successively splitting off the tensor factors. Each time we obtain the factor $\pm\dfrac{s_k^2(y_k - x)}{y_k - s_k^2 x}$, and the overall number of negative signs is $(-1)^m$, which translates into $(-1)^{j-1}$ upon passing to the general case not assuming $j = m+1$. In the last splitting, at $k = t_{m+1}$, we have

$$
\big\langle e_1^{(k)}, D(x,\sqrt{x/z})C(z,\sqrt{z/x})e_0^{(k)}\big\rangle = \frac{y_k(1-s_k^2)}{y_k - s_k^2 x}.
$$

Recalling normalization (7.10), we get the desired identity. □

7.3 Fermionic operators in the Fock space

We now pass to the infinite volume limit as $M \to -\infty$ and $N \to +\infty$. As a result, from the spaces $V^{[M,N]}$ we get the *Fock space* $\mathfrak{F}$. By definition, $\mathfrak{F}$ is spanned by the vectors $e_{\mathcal{T}}$,

$$
e_{\mathcal{T}} = \bigotimes_{m=-\infty}^{+\infty} e_{k_m}^{(m)}, \qquad k_m = k_m(\mathcal{T}) = \mathbf{1}_{m\in\mathcal{T}},
$$

where $\mathcal{T}$ runs over *semi-infinite* subsets of $\mathbb{Z}$. A subset $\mathcal{T}$ is called semi-infinite (also sometimes referred to as densely packed towards $-\infty$) if there exists $C = C(\mathcal{T}) > 0$ such that $i \notin \mathcal{T}$ for all $i > C$ and $i \in \mathcal{T}$ for all $i < -C$.

In $\mathcal{F}$, we can define an inner product under which the $e_{\mathcal{T}}$'s form an orthonormal basis:

$$\langle e_{\mathcal{T}}, e_{\mathcal{R}} \rangle = \mathbf{1}_{\mathcal{T}=\mathcal{R}}.$$

We do not need to consider convergence in $\mathcal{F}$ as all computations below are done in terms of this inner product. For example, $\langle v, e_{\mathcal{T}} \rangle$ may be viewed as an operation of picking a coefficient of $e_{\mathcal{T}}$ in v, a formal infinite linear combination of the $e_{\mathcal{R}}$'s.

For a semi-infinite subset $\mathcal{T}$, define

$$c(\mathcal{T}) := \#(\mathcal{T} \cap \mathbb{Z}_{>0}) - \#(\mathbb{Z}_{\le 0} \setminus \mathcal{T}), \qquad h_j(\mathcal{T}) := \#\{t \in \mathcal{T} \colon t > j\}. \tag{7.12}$$

The quantity $c(\mathcal{T})$ is called *charge*. Define the change operator $\mathfrak{c} \colon \mathcal{F} \to \mathcal{F}$ on the basis by $\mathfrak{c}(e_{\mathcal{T}}) = c(\mathcal{T}) e_{\mathcal{T}}$, and then extend by linearity.

Clearly, $c(\mathcal{T})$ can be any integer, and we have the decomposition of $\mathcal{F}$ into subspaces with fixed charge:

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n, \qquad \mathcal{F}_n = \operatorname{span}\{e_{\mathcal{T}} \colon c(\mathcal{T}) = n\}.$$

The normalized operators $A^{[M,N]}$, $B^{[M,N]}$, $C^{[M,N]}$, and $D^{[M,N]}$ from Definition 7.4 admit matrix element-wise infinite volume limits as $M \to -\infty$, $N \to +\infty$. We denote the limiting operators by $A^{\mathbb{Z}}(x,r)$, $B^{\mathbb{Z}}(x,r)$, $C^{\mathbb{Z}}(x,r)$, $D^{\mathbb{Z}}(x,r)$. These operators act in the Fock space $\mathcal{F}$, more precisely,

$$A^{\mathbb{Z}} \colon \mathcal{F}_n \to \mathcal{F}_n, \qquad B^{\mathbb{Z}} \colon \mathcal{F}_n \to \mathcal{F}_{n-1}, \qquad C^{\mathbb{Z}} \colon \mathcal{F}_n \to \mathcal{F}_{n+1}, \qquad D^{\mathbb{Z}} \colon \mathcal{F}_n \to \mathcal{F}_n.$$

With this understanding, it is clear that the matrix elements like $\langle e_{\mathcal{T}}, A^{\mathbb{Z}} e_{\mathcal{R}} \rangle$, and so on, are well-defined for all possible values of the parameters $(x; \mathbf{y}; r; \mathbf{s})$, simply as products (= suitable partition functions) of normalized weights as in Definition 7.4, only finitely many of which differ from 1. See also Sect. 1.6 in Introduction for a pictorial definition of these operators.

The operators $A^{\mathbb{Z}}$, $B^{\mathbb{Z}}$, $C^{\mathbb{Z}}$, $D^{\mathbb{Z}}$ satisfy a number of commutation relations:

Proposition 7.8 *We have*

$$B^{\mathbb{Z}}(x_1, r_1) B^{\mathbb{Z}}(x_2, r_2) = \frac{r_2^{-2} x_2 - x_1}{r_1^{-2} x_1 - x_2} B^{\mathbb{Z}}(x_2, r_2) B^{\mathbb{Z}}(x_1, r_1); \tag{7.13}$$

$$D^{\mathbb{Z}}(x_1, r_1) D^{\mathbb{Z}}(x_2, r_2) = D^{\mathbb{Z}}(x_2, r_2) D^{\mathbb{Z}}(x_1, r_1); \tag{7.14}$$

$$B^{\mathbb{Z}}(x_1, r_1) D^{\mathbb{Z}}(x_2, r_2) = \frac{r_2^{-2} x_2 - x_1}{x_2 - x_1} D^{\mathbb{Z}}(x_2, r_2) B^{\mathbb{Z}}(x_1, r_1); \tag{7.15}$$

$$C^{\mathbb{Z}}(x_1, r_1) D^{\mathbb{Z}}(x_2, r_2) = \frac{r_1^{-2} x_1 - x_2}{r_1^{-2} x_1 - r_2^{-2} x_2} D^{\mathbb{Z}}(x_2, r_2) C^{\mathbb{Z}}(x_1, r_1); \tag{7.16}$$

$$B^{\mathbb{Z}}(x_1, r_1) C^{\mathbb{Z}}(x_2, r_2) = \frac{r_2^{-2} x_2 - r_1^{-2} x_1}{x_2 - x_1} C^{\mathbb{Z}}(x_2, r_2) B^{\mathbb{Z}}(x_1, r_1). \tag{7.17}$$

All identities are understood in the sense of matrix elements, for example, for (7.13) *we have*

$$\langle e_{\mathcal{T}}, B^{\mathbb{Z}}(x_1, r_1) B^{\mathbb{Z}}(x_2, r_2) e_{\mathcal{R}} \rangle = \frac{r_2^{-2} x_2 - x_1}{r_1^{-2} x_1 - x_2} \langle e_{\mathcal{T}}, B^{\mathbb{Z}}(x_2, r_2) B^{\mathbb{Z}}(x_1, r_1) e_{\mathcal{R}} \rangle,$$
$$e_{\mathcal{R}} \in \mathfrak{F}_n, \;\; e_{\mathcal{T}} \in \mathfrak{F}_{n-2}.$$

Identities (7.13), (7.14) *hold for arbitrary values of the parameters, but the other ones require the following restrictions. For* (7.15), *we assume*

$$\oplus_{x_1;x_2}: \left| \frac{(s_j^{-2} y_j - x_1)(y_j - x_2)}{(s_j^{-2} y_j - x_2)(y_j - x_1)} \right| < 1 - \delta < 1 \;\; \textit{for sufficiently large } j > 0. \tag{7.18}$$

For (7.16), *we assume*

$$\ominus_{r_1^{-2} x_1; r_2^{-2} x_2}: \left| \frac{(s_j^{-2} y_j - r_1^{-2} x_1)(y_j - r_2^{-2} x_2)}{(s_j^{-2} y_j - r_2^{-2} x_2)(y_j - r_1^{-2} x_1)} \right| < 1 - \delta < 1$$
$$\textit{for sufficiently small } j \le 0. \tag{7.19}$$

Finally, (7.17) *holds under both conditions* $\oplus_{x_1;x_2}$ *and* $\ominus_{r_2^{-2} x_2; r_1^{-2} x_1}$.

Proof All the desired identities follow from the Yang–Baxter equation (Proposition 2.4) applied to the operators $A^{[M,N]}$, $B^{[M,N]}$, $C^{[M,N]}$, $D^{[M,N]}$ (Definition 7.4), after taking the limit $M \to -\infty$, $N \to +\infty$. This limit is straightforward for identities (7.13) and (7.14) using (2.10) and (2.13), respectively. Let us explain how to obtain the remaining identities.

For (7.15), we use (2.14) to write

$$B^{[M,N]}(x_1, r_1) D^{[M,N]}(x_2, r_2) = \frac{r_2^{-2} x_2 - x_1}{x_2 - x_1} D^{[M,N]}(x_2, r_2) B^{[M,N]}(x_1, r_1)$$
$$+ \frac{(1 - r_1^{-2}) x_1}{x_2 - x_1} D^{[M,N]}(x_1, r_1) B^{[M,N]}(x_2, r_2) \prod_{j=1}^{N} \frac{y_j - s_j^2 x_1}{y_j - x_1} \frac{y_j - x_2}{y_j - s_j^2 x_2}.$$

Thanks to the assumptions, the second term (more precisely, its pairing with two arbitrary vectors $\langle e_{\mathcal{T}}, (\cdots) e_{\mathcal{R}} \rangle$) can be bounded in the absolute value by $C(1-\delta)^N$, and hence vanishes as $N \to +\infty$.

For (7.16), we write using (2.17):

$$C^{[M,N]}(x_1,r_1)D^{[M,N]}(x_2,r_2)=\frac{r_1^{-2}x_1-x_2}{r_1^{-2}x_1-r_2^{-2}x_2}\,D^{[M,N]}(x_2,r_2)C^{[M,N]}(x_1,r_1)$$
$$+\frac{x_2(1-r_2^{-2})}{r_1^{-2}x_1-r_2^{-2}x_2}\,D^{[M,N]}(x_1,r_1)C^{[M,N]}(x_2,r_2)\prod_{i=M}^{0}\frac{y_i-s_i^2r_1^{-2}x_1}{r_1^{-2}x_1-y_i}\frac{r_2^{-2}x_2-y_i}{y_i-s_i^2r_2^{-2}x_2}.$$

Thanks to our assumptions, the second term vanishes in the infinite volume limit. □

Finally, for (7.17), we write using (2.19):

$$B^{[M,N]}(x_1,r_1)C^{[M,N]}(x_2,r_2)=\frac{r_2^{-2}x_2-r_1^{-2}x_1}{x_2-x_1}\,C^{[M,N]}(x_2,r_2)B^{[M,N]}(x_1,r_1)$$
$$+\frac{x_1(r_1^{-2}-1)}{x_2-x_1}\Bigg(D^{[M,N]}(x_2,r_2)A^{[M,N]}(x_1,r_1)$$
$$\prod_{i=M}^{0}\frac{y_i-s_i^2r_2^{-2}x_2}{y_i-s_i^2x_2}\frac{y_i-x_1/r_1^2}{x_1-y_i/s_i^2}\prod_{j=1}^{N}\frac{s_j^2(y_j-x_1)}{y_j-s_j^2x_1}$$
$$-D^{[M,N]}(x_1,r_1)A^{[M,N]}(x_2,r_2)\prod_{i=M}^{0}\frac{y_i-s_i^2r_1^{-2}x_1}{y_i-s_i^2x_1}\frac{y_i-x_2/r_2^2}{x_2-y_i/s_i^2}\prod_{j=1}^{N}\frac{s_j^2(y_j-x_2)}{y_j-s_j^2x_2}\Bigg)$$
$$\times\prod_{i=M}^{0}\frac{y_i-s_i^2x_1}{y_i-s_i^2r_1^{-2}x_1}\frac{x_2-y_i/s_i^2}{y_i-x_2/r_2^2}\prod_{j=1}^{N}\frac{y_i-s_i^2x_1}{s_i^2(y_i-x_1)}.$$

Again, thanks to our assumptions, the terms involving the operators A, D vanish in the infinite volume limit. □

Definition 7.9 For each $j\in\mathbb{Z}$, define the fermionic *creation and annihilation* operators $\psi_j,\psi_j^*\colon\mathcal{F}\to\mathcal{F}$ on the basis by (and then extending by linearity):

$$\psi_je_{\mathcal{T}}=\begin{cases}(-1)^{h_j(\mathcal{T})}e_{\mathcal{T}\cup\{j\}}, & j\notin\mathcal{T};\\ 0, & j\in\mathcal{T},\end{cases}\qquad \psi_j^*e_{\mathcal{T}}=\begin{cases}(-1)^{h_j(\mathcal{T})}e_{\mathcal{T}\setminus\{j\}}, & j\in\mathcal{T};\\ 0, & j\notin\mathcal{T}.\end{cases}$$

Clearly, $\psi_j\colon\mathcal{F}_n\to\mathcal{F}_{n+1}$ and $\psi_j^*\colon\mathcal{F}_n\to\mathcal{F}_{n-1}$. The operators ψ_i,ψ_j^* satisfy the anticommutation relations (that are easy to check directly)

$$\psi_k\psi_k^*+\psi_k^*\psi_k=1,$$
$$\psi_k\psi_\ell^*+\psi_\ell^*\psi_k=\psi_k^*\psi_\ell^*+\psi_\ell^*\psi_k^*=\psi_k\psi_\ell+\psi_\ell\psi_k=0,\qquad k\neq\ell.$$

Moreover,

$$\psi_j\psi_j^*e_{\mathcal{T}}=\mathbf{1}_{j\in\mathcal{T}}e_{\mathcal{T}}.\tag{7.20}$$

Define the operators

$$\Psi(x,z) = D^{\mathbb{Z}}(x,\sqrt{x/z})C^{\mathbb{Z}}(z,\sqrt{z/x})(-1)^{\mathfrak{c}},$$
$$\Psi^*(x,z) = D^{\mathbb{Z}}(x,\sqrt{x/z})B^{\mathbb{Z}}(z,\sqrt{z/x}), \tag{7.21}$$

where $\mathfrak{c}$ is the charge operator. Clearly, $\Psi(x,z)\colon \mathcal{F}_n \to \mathcal{F}_{n+1}$ and $\Psi^*(x,z)\colon \mathcal{F}_n \to \mathcal{F}_{n-1}$. Let us record several relations for Ψ, Ψ^*:

Proposition 7.10 *We have*

$$B^{\mathbb{Z}}(x,r)\Psi(u,\zeta) = -\frac{u-r^{-2}x}{u-x}\,\Psi(u,\zeta)B^{\mathbb{Z}}(x,r) \quad \text{under } \oplus_{x;u}, \oplus_{x;\zeta}, \ominus_{u;r^{-2}x}; \tag{7.22}$$

$$\Psi(u,\zeta)D^{\mathbb{Z}}(w,\theta) = \frac{u-w}{u-\theta^{-2}w}\,D^{\mathbb{Z}}(w,\theta)\Psi(u,\zeta) \quad \text{under } \ominus_{u;\theta^{-2}w}; \tag{7.23}$$

$$B^{\mathbb{Z}}(x,r)\Psi^*(\kappa,v) = \frac{v-x}{r^{-2}x-v}\,\Psi^*(\kappa,v)B^{\mathbb{Z}}(x,r) \quad \text{under } \oplus_{x;\kappa}; \tag{7.24}$$

$$\Psi^*(\kappa,v)D^{\mathbb{Z}}(w,\theta) = \frac{\theta^{-2}w-v}{w-v}\,D^{\mathbb{Z}}(w,\theta)\Psi^*(\kappa,v) \quad \text{under } \oplus_{v;w}; \tag{7.25}$$

$$\Psi^*(\kappa,v)\Psi(u,\zeta) = -\Psi(u,\zeta)\Psi^*(\kappa,v) \quad \text{under } \oplus_{v;u}, \oplus_{v;\zeta}, \ominus_{u;\kappa}, \ominus_{u;v}. \tag{7.26}$$

These identities are again understood in the sense of matrix elements in the standard basis. For example, we have $\langle e_{\mathcal{T}}, \Psi^*(\kappa,v)\Psi(u,\zeta)e_{\mathcal{R}}\rangle = -\langle e_{\mathcal{T}}\Psi(u,\zeta)\Psi^*(\kappa,v)e_{\mathcal{R}}\rangle$ *for all* $e_{\mathcal{T}}, e_{\mathcal{R}} \in \mathcal{F}_n$, $n \in \mathbb{Z}$. *Conditions* $\oplus$ *and* $\ominus$ *are defined in* (7.18)–(7.19).

Proof These identities and the corresponding conditions immediately follow from Proposition 7.8. Note the extra minus signs in (7.22) and (7.26) which arise from commuting $(-1)^{\mathfrak{c}}$ with $B^{\mathbb{Z}}$. □

The next statement is one of the key results on the operators Ψ and Ψ^* in the Fock space:

Theorem 7.11 (Theorem 1.13 from Introduction) *As operators on* $\mathcal{F}$, *we have*

$$\Psi(x,z) = \sum_{j\in\mathbb{Z}} \Phi_j(x,z)\psi_j, \qquad \Psi^*(x,z) = \sum_{j\in\mathbb{Z}} \Phi_j^*(x,z)\psi_j^*,$$

where the expressions Φ_j, Φ_j^* *are given in Definition* 7.5. *In particular,* $\Psi(x,z)$ *is independent of* z, *which is evident from the formula for* $\Phi_j(x,z)$.

Proof This follows from Propositions 7.3, 7.7 and 7.6. Indeed, for $\psi^*_{t_j}$ in Proposition 7.6 we have $(-1)^{m+1-j} = (-1)^{h_{t_j}(\mathcal{T})}$, and for ψ_{t_j} in Proposition 7.7 we have $(-1)^{M-j} = (-1)^{\mathfrak{c}(\mathcal{R})+h_{t_j}(\mathcal{R})}$. Here we used the fact that the charge of any m-subset of $\{M, M+1, \dots, N\}$ for sufficiently small M and large N is equal to $m-(M+1)$. This completes the proof. □

The next statement follows from the previous Theorem 7.11 and is a key ingredient in getting determinantal correlation functions. We refer to [90, Lemma 1] or [6, Lemma B.1] for its proof which dates back to at least [42].

Proposition 7.12 (Wick's determinant) *Fix an integer $k \geq 1$ and two sequences of complex numbers $\{a_{ij}\}, \{b_{ij}\}$. Define the operators $A_i = \sum_{j\in\mathbb{Z}} a_{ij}\psi_j$ and $B_i = \sum_{j\in\mathbb{Z}} b_{ij}\psi_j^*$ (note that the operators $A_i\colon \mathfrak{F}_n \to \mathfrak{F}_{n+1}$ and $B_j\colon \mathfrak{F}_n \to \mathfrak{F}_{n-1}$ are well-defined for arbitrary coefficients a_{ij}, b_{kl}). Then*

$$\left\langle e_{\mathbb{Z}_{\leq 0}}, A_1B_1A_2B_2\dots A_kB_k e_{\mathbb{Z}_{\leq 0}}\right\rangle = \det\left[M_{ij}\right]_{i,j=1}^{k},$$
$$M_{ij} = \begin{cases} \left\langle e_{\mathbb{Z}_{\leq 0}}, A_iB_j e_{\mathbb{Z}_{\leq 0}}\right\rangle, & i \geq j; \\ -\left\langle e_{\mathbb{Z}_{\leq 0}}, B_jA_i e_{\mathbb{Z}_{\leq 0}}\right\rangle, & i < j. \end{cases}$$

7.4 Action of the Ψ operators

We now compute matrix elements of various products of the operators $\Psi(u,\zeta)$ and $\Psi^*(\kappa, v)$ on the vectors $e_{\mathbb{Z}_{\leq 0}}, e_{\mathbb{Z}_{\leq N}}$ from the Fock space $\mathfrak{F}$.

Lemma 7.13 *Under $\ominus_{u;v}$ (7.19), we have*

$$\left\langle e_{\mathbb{Z}_{\leq 0}}, \Psi(u,\zeta)\Psi^*(\kappa,v) e_{\mathbb{Z}_{\leq 0}}\right\rangle = \frac{v-\kappa}{u-v}.$$

Under $\oplus_{v;u}$, we have

$$-\left\langle e_{\mathbb{Z}_{\leq 0}}, \Psi^*(\kappa,v)\Psi(u,\zeta) e_{\mathbb{Z}_{\leq 0}}\right\rangle = \frac{v-\kappa}{u-v}.$$

Proof We only prove the first identity, the second is analogous. Observe that

$$\left\langle e_{\mathbb{Z}_{\leq 0}}, \psi_i\psi_j^* e_{\mathbb{Z}_{\leq 0}}\right\rangle = \mathbf{1}_{i=j}\mathbf{1}_{i\leq 0}.$$

Therefore, using Theorem 7.11 and Definition 7.5 we have

$$\left\langle e_{\mathbb{Z}_{\leq 0}}, \Psi(u,\zeta)\Psi^*(\kappa,v) e_{\mathbb{Z}_{\leq 0}}\right\rangle = \sum_{j=-\infty}^{0} \Phi_j(u,\zeta)\Phi_j^*(\kappa,v)$$
$$= \sum_{j=-\infty}^{0} \frac{y_j(1-s_j^2)}{y_j - s_j^2 u}\,\frac{\kappa - v}{y_j - v} \prod_{k=j}^{0} \frac{y_k - s_k^2 u}{s_k^2(y_k-u)}\,\frac{s_k^2(y_k - v)}{y_k - s_k^2 v}$$
$$= \frac{v-\kappa}{u-v}\sum_{j=-\infty}^{0}\left(1 - \frac{(y_j-v)(y_j-s_j^2u)}{(y_j-u)(y_j-s_j^2v)}\right)\prod_{k=j+1}^{0}\frac{y_k-s_k^2u}{y_k-u}\,\frac{y_k-v}{y_k-s_k^2v} = \frac{v-\kappa}{u-v}.$$

In the last equality we used the fact that the infinite sum telescopes to 1 under $\ominus_{u;v}$.

Proposition 7.14 *Fix an integer $m \geq 1$, and let u_i, v_j satisfy*

$$\begin{cases} \ominus_{u_i;v_j}, & i \geq j; \\ \oplus_{v_j;u_i}, & i < j. \end{cases}$$

Then we have

$$\begin{aligned} &\left\langle e_{\mathbb{Z}_{\leq 0}}, \Psi(u_1, \zeta_1)\Psi^*(\kappa_1, v_1)\dots\Psi(u_m, \zeta_m)\Psi^*(\kappa_m, v_m)\, e_{\mathbb{Z}_{\leq 0}} \right\rangle \\ &= \prod_{i=1}^{m}(\kappa_i - v_i) \prod_{1\leq i<j\leq m} (v_j - v_i)(u_i - u_j) \prod_{i,j=1}^{m} \frac{1}{v_j - u_i}. \end{aligned}$$

Note that $\Psi(u_i, \zeta_i)$ does not depend on ζ_i, and the ζ_i's are not present in the right-hand side, as it should be.

Proof of Proposition 7.14 Employing Proposition 7.12 and Proposition 7.13, we have

$$\left\langle e_{\mathbb{Z}_{\leq 0}}, \Psi(u_1, \zeta_1)\Psi^*(\kappa_1, v_1)\dots\Psi(u_m, \zeta_m)\Psi^*(\kappa_m, v_m)\, e_{\mathbb{Z}_{\leq 0}} \right\rangle = \det\left[\frac{v_j - \kappa_j}{u_i - v_j}\right]_{i,j=1}^{m}.$$

The determinant in the right-hand side factorizes thanks to the Cauchy determinant formula, and we arrive at the desired identity. □

8 Correlation kernel via fermionic operators

In this section we study a generalization of the ascending FG process introduced in Sect. 6.2, and compute a generating function type series for its correlation kernel $K_{\mathcal{P}}$ using fermionic operators in the Fock space developed in Sect. 7 above.

8.1 General FG processes

The following definition is parallel to the definition of the Schur process introduced in [90].

Assume that the parameters $(\mathbf{y}; \mathbf{s})$ satisfying (6.1) are fixed. Fix $T \geq 1$ and variables $(x_i; r_i)$ and $(w_i; \theta_i)$, $i = 1, \dots, T$, such that these specializations are nonnegative in the sense of Definition 6.1 and are compatible as in Definition 6.3, i.e., the variables satisfy (6.7). The *FG process* with this data is a probability measure on sequences of signatures $\lambda^{(1)}, \mu^{(1)}, \lambda^{(2)}, \dots, \mu^{(T-1)}, \lambda^{(T)}$ with probability weights

$$\begin{aligned} \mathcal{P}(\lambda^{(1)}, \mu^{(1)}, \lambda^{(2)}, \mu^{(2)}, \dots, \mu^{(T-1)}, \lambda^{(T)}) &:= \frac{1}{Z} G_{\lambda^{(1)}}(w_1; \theta_1) F_{\lambda^{(1)}/\mu^{(1)}}(x_1; r_1) \\ &\times G_{\lambda^{(2)}/\mu^{(1)}}(w_2; \theta_2) \dots F_{\lambda^{(T-1)}/\mu^{(T-1)}}(x_{T-1}; r_{T-1}) \\ &G_{\lambda^{(T)}/\mu^{(T-1)}}(w_T; \theta_T) F_{\lambda^{(T)}}(x_T; r_T), \end{aligned} \tag{8.1}$$

where the normalizing constant is computed using multiple applications of (6.6):

$$Z = \prod_{i=1}^{T} x_i (r_i^{-2} - 1) \frac{\prod_{1 \le i < j \le T} (r_i^{-2} x_i - x_j)(s_i^{-2} y_i - y_j)}{\prod_{i,j=1}^{T} (y_i - x_j)} \prod_{1 \le i \le j \le T} \frac{x_j - \theta_i^{-2} w_i}{x_j - w_i}. \tag{8.2}$$

Note that the number of parts in the signatures in (8.1) is fixed: $\lambda^{(1)}$ has T parts, and both $\mu^{(j)}$ and $\lambda^{(j+1)}$ have $T - j$ parts, $j = 1, \dots, T-1$. In (8.1) and below when convenient we omit the notation $\mathbf{y}$, $\mathbf{s}$ in the functions $F_{\lambda/\mu}(x_i; \mathbf{y}; r_i; \mathbf{s})$, $G_{\nu/\varkappa}(w_i; \mathbf{y}; \theta_i; \mathbf{s})$.

Remark 8.1 The FG process (8.1) reduces to the ascending FG process from Definition 6.4 as follows. Fix some $1 \le a \le T-1$, set $w_{a+1} = w_{a+2} = \dots = w_T = 0$, and replace each of the specializations $(x_1; r_1), \dots, (x_a; r_a)$ by $\varnothing$, the empty specialization (see (6.3)). We discuss the reduction to the ascending case in Sect. 8.6 below.

8.2 FG process via Fock space

Let us now express the probability weights under the FG process (8.1) through matrix elements of our operators acting in the Fock space $\mathcal{F}$. Fix a signature $\lambda = (\lambda_1 \ge \dots \ge \lambda_m \ge 0)$, and let I_λ be the rank one projection in $\mathcal{F}$ onto the semi-infinite subset corresponding to λ, that is, which acts as

$$I_\lambda e_{\mathcal{T}} = \begin{cases} e_{\mathcal{T}}, & \text{if } \mathcal{T} = \{\lambda_1 + m, \lambda_2 + m - 1, \dots, \lambda_m + 1, 0, -1, -2, \dots\}; \\ 0, & \text{otherwise}, \end{cases}$$

for any semi-infinite $\mathcal{T} \subset \mathbb{Z}$.

Lemma 8.2 *The probability weights of the FG process have the form*

$$\begin{aligned} &\mathcal{P}(\lambda^{(1)}, \mu^{(1)}, \lambda^{(2)}, \mu^{(2)}, \dots, \mu^{(T-1)}, \lambda^{(T)}) = \frac{1}{Z} \big\langle e_{\mathbb{Z}_{\le 0}}, B^{\mathbb{Z}}(x_T, r_T) I_{\lambda^{(T)}} \\ &\quad \times D^{\mathbb{Z}}(w_T, \theta_T) I_{\mu^{(T-1)}} \dots I_{\lambda^{(2)}} D^{\mathbb{Z}}(w_2, \theta_2) I_{\mu^{(1)}} B^{\mathbb{Z}}(x_1, r_1) I_{\lambda^{(1)}} D^{\mathbb{Z}}(w_1, \theta_1) e_{\mathbb{Z}_{\le T}} \big\rangle. \end{aligned}$$

The normalizing constant (8.2) *is*

$$\begin{aligned} Z = \big\langle e_{\mathbb{Z}_{\le 0}}, B^{\mathbb{Z}}(x_T, r_T) D^{\mathbb{Z}}(w_T, \theta_T) B^{\mathbb{Z}}(x_{T-1}, r_{T-1}) \dots \\ D^{\mathbb{Z}}(w_2, \theta_2) B^{\mathbb{Z}}(x_1, r_1) D^{\mathbb{Z}}(w_1, \theta_1) e_{\mathbb{Z}_{\le T}} \big\rangle. \end{aligned}$$

Proof The matrix element $\langle e_{\mathbb{Z}_{\le 0}}, A e_{\mathbb{Z}_{\le N}} \rangle$, where A is a product of $B^{\mathbb{Z}}$, $D^{\mathbb{Z}}$, and I_λ's, can be nonzero only if in the partition function representation of it the vertical arrows in each position $j \le 0$ move vertically straight. The normalization of the operators $B^{\mathbb{Z}}$, $D^{\mathbb{Z}}$ on $\mathbb{Z}_{\le 0}$ (Definition 7.4) ensures that this straight movement contributes the total weight 1. Next, in the positive half-line $\mathbb{Z}_{\ge 1}$, the operators $D^{\mathbb{Z}}$ are not normalized and thus yield vertex configurations for the functions $G_{\lambda/\mu}$. The normalization of the operators $B^{\mathbb{Z}}$ is equivalent to passing from the vertex weights W to the weights $\widehat{W}$, see (2.5). Therefore, the action of the operators $B^{\mathbb{Z}}$ yield vertex configurations for the functions $F_{\lambda/\mu}$. This shows the desired expressions for the probability weights of the FG process and measures.

Remark 8.3 The matrix element representation for the probability weights of the FG process in Lemma 8.2 is independent of (y_k, s_k), $k \le 0$, as it should be. □

8.3 Extracting series coefficients

In the rest of this section, we abbreviate

$$\Psi(u) := \Psi(u, 0), \qquad \Psi^*(v) := \Psi^*(0, v).$$

Note that the operator $\Psi^*(v)$ is well-defined by setting $\kappa = 0$ in the expansion of Theorem 7.11:

$$\Psi^*(v) = \sum_{j=1}^{+\infty} \left(\frac{v}{v - y_j} \prod_{k=1}^{j-1} \frac{y_k - s_k^2 v}{s_k^2(y_k - v)} \right) \psi_j^* + \sum_{j=-\infty}^{0} \left(\frac{v}{v - y_j} \prod_{k=j}^{0} \frac{s_k^2(y_k - v)}{y_k - s_k^2 v} \right) \psi_j^*.$$

Therefore, we have for any semi-infinite subset $\mathcal{T} \subset \mathbb{Z}$ and $j \ge 1$:

$$\psi_j \psi_j^* e_{\mathcal{T}} = \mathbf{1}_{j \in \mathcal{T}} e_{\mathcal{T}}. \tag{8.3}$$

The quantity $\mathbf{1}_{j \in \mathcal{T}} e_{\mathcal{T}}$ can also be written as $\left[\Phi_j(u, 0)\Phi_j^*(0, v)\right]\Psi(u)\Psi^*(v)\, e_{\mathcal{T}}$, where the notation $\left[\cdots\right]$ means extracting the coefficient in the generating series (recall Definition 7.5). The next two lemmas clarify what it means to extract such a coefficient.

Lemma 8.4 *Let $f(u) = \sum_{i \in \mathbb{Z}} c_i \Phi_i(u, 0)$, where $c_i \in \mathbb{C}$. Then for any $i \in \mathbb{Z}$ we have*

$$c_i = \frac{1}{2\pi \mathbf{i}} \oint \frac{f(u)\, du}{u - y_i} \prod_{k=1}^{i-1} \frac{u - s_k^{-2} y_k}{u - y_k}, \quad i \ge 1;$$

$$c_i = \frac{1}{2\pi \mathbf{i}} \oint \frac{f(u)\, du}{u - s_i^{-2} y_i} \prod_{k=i+1}^{0} \frac{u - y_k}{u - s_k^{-2} y_k}, \quad i \le 0.$$

In both integrals the integration contour separates the families of the points $\{y_k\}_{k \in \mathbb{Z}}$ and $\{s_k^{-2} y_k\}_{k \in \mathbb{Z}}$, and goes around the y_k's in the positive direction. Moreover, the series for $f(u)$ must converge uniformly on the contour.

Proof This is essentially the single-variable biorthogonality (Lemma 5.1). Indeed, the hypothesis of Lemma 8.4 allows to interchange summation and integration. Then one can show that for all $m \in \mathbb{Z}$ and $i \geq 1$ we have

$$\frac{1}{2\pi \mathbf{i}} \oint \frac{\Phi_m(u,0)\,du}{u - y_i} \prod_{k=1}^{i-1} \frac{u - s_k^{-2} y_k}{u - y_k} = \mathbf{1}_{m=i}.$$

Note that when m is nonpositive (the case not covered by Lemma 5.1), the u contour has no poles outside (as all the poles are of the form $u = y_k$ and are inside), so the integral vanishes. When $i \leq 0$, we similarly have for all $m \in \mathbb{Z}$:

$$\frac{1}{2\pi \mathbf{i}} \oint \frac{\Phi_m(u,0)\,du}{u - s_i^{-2} y_i} \prod_{k=i+1}^{0} \frac{u - y_k}{u - s_k^{-2} y_k} = \mathbf{1}_{m=i}.$$

This completes the proof. □

Lemma 8.5 *Let $g(v) = \sum_{j \in \mathbb{Z}} d_j \Phi_j^*(0, v)$, where $d_j \in \mathbb{C}$. Then for $j \geq 1$ and $j \leq 0$ we have, respectively,*

$$d_j = \frac{y_j(s_j^{-2} - 1)}{2\pi \mathbf{i}} \oint \frac{v^{-1} g(v)\,dv}{v - s_j^{-2} y_j} \prod_{k=1}^{j-1} \frac{v - y_k}{v - s_k^{-2} y_k};$$

$$d_j = \frac{y_j(s_j^{-2} - 1)}{2\pi \mathbf{i}} \oint \frac{v^{-1} g(v)\,dv}{v - y_j} \prod_{k=j+1}^{0} \frac{v - s_k^{-2} y_k}{v - y_k}.$$

In both integrals the integration contour separates the families of the points $\{y_k\}_{k \in \mathbb{Z}}$ and $\{s_k^{-2} y_k\}_{k \in \mathbb{Z}}$, and goes around the $s_k^{-2} y_k$'s in the positive direction. Moreover, the series for $g(v)$ must converge uniformly on the contour.

Proof This is proven in the same way as Lemma 8.4. □

Remark 8.6 Lemmas 8.4 and 8.5 imply linear independence of the products $\Phi_i(u, 0)$ $\Phi_j^*(0, v)$ for all $i, j \in \mathbb{Z}$. Therefore, the operation of extracting the coefficient

$$\left[\Phi_j(u,0)\Phi_j^*(0,v)\right] \Psi(u)\Psi^*(v)\, e_{\mathcal{T}}$$

discussed before Lemma 8.4 is indeed well-defined and can be realized by Lemmas 8.4 and 8.5.

8.4 Correlation generating function

We are now in a position to compute a generating series type expression for the correlations of the FG process. Denote

$$\boldsymbol{\Psi}(\mathbf{u}; \mathbf{v}) := \Psi(u_k)\Psi^*(v_k) \ldots \Psi(u_1)\Psi^*(v_1).$$

Applying $\boldsymbol{\Psi}(\mathbf{u};\mathbf{v})$ to a vector $e_{\mathcal{T}}$ (with semi-infinite $\mathcal{T}$) produces a linear combination of terms $\mathbf{1}_{\{j_1,\dots,j_k\}\in\mathcal{T}} e_{\mathcal{T}}$ (corresponding to the desired k-point correlations), together with some other terms. More precisely, each desired term $\mathbf{1}_{\{j_1,\dots,j_k\}\in\mathcal{T}} e_{\mathcal{T}}$, with $(j_1,\dots,j_k)\in\mathbb{Z}^k_{\geq 1}$ distinct and ordered, arises from $\psi_{j_k}\psi^*_{j_k}\dots\psi_{j_1}\psi^*_{j_1}$, where each index j_m comes from the pair of the generating functions $\Psi(u_m)\Psi^*(v_m)$. All the other terms are "parasite" and should be excluded by extracting only the appropriate coefficients as in Sect. 8.3. The generating function with all the terms put together has an explicit product form:

Proposition 8.7 *Take the following sequences of parameters for the generating series:*

$$\mathbf{u}^j=(u_1^j,\dots,u_{k_j}^j),\qquad \mathbf{v}^j=(v_1^j,\dots,v_{k_j}^j),\qquad k_j\geq 0,\qquad j=1,\dots,T.$$

Assume that for all possible indices $m=1,\dots,T$, and i,j,α,β, the parameters satisfy:

$$\begin{gathered}\oplus_{x_i;w_j},\ \oplus_{v^j_\alpha;w_i},\ \oplus_{v^i_\beta;r_j^{-2}x_j},\ \oplus_{x_j;u^i_\alpha},\ \oplus_{v^j_\beta;y_m},\ \oplus_{x_i;y_m},\ \ominus_{u^j_\alpha;\theta_i^{-2}w_i},\ \ominus_{u^i_\beta;r_j^{-2}x_j},\ \ominus_{u^i_\beta;x_j},\\ \begin{cases}\ominus_{y_j;x_i},\ \ominus_{y_j;r_i^{-2}x_i}, & 1\leq i\leq j\leq T;\\ \oplus_{x_i;r_j^{-2}x_j}, & 1\leq j<i\leq T,\end{cases}\qquad \begin{cases}\ominus_{u^i_\alpha;v^j_\beta}, & i>j \text{ or } i=j,\alpha\geq\beta;\\ \oplus_{v^j_\beta;u^i_\alpha}, & i<j \text{ or } i=j,\alpha<\beta,\end{cases}\end{gathered}\tag{8.4}$$

see (7.18), (7.19) *for the notation. Then we have*

$$\begin{aligned}&\frac{1}{Z}\big\langle e_{\mathbb{Z}_{\leq 0}},\,B^{\mathbb{Z}}(x_T,r_T)\Psi(\mathbf{u}^T;\mathbf{v}^T)D^{\mathbb{Z}}(w_T,\theta_T)\\&\qquad\times B^{\mathbb{Z}}(x_{T-1},r_{T-1})\dots\Psi(\mathbf{u}^2;\mathbf{v}^2)D^{\mathbb{Z}}(w_2,\theta_2)B^{\mathbb{Z}}(x_1,r_1)\Psi(\mathbf{u}^1;\mathbf{v}^1)D^{\mathbb{Z}}(w_1,\theta_1)e_{\mathbb{Z}_{\leq T}}\big\rangle\\&\quad=\prod_{1\leq i\leq j\leq T}\prod_{\alpha=1}^{k_j}\frac{(v^j_\alpha-\theta_i^{-2}w_i)(u^j_\alpha-w_i)}{(v^j_\alpha-w_i)(u^j_\alpha-\theta_i^{-2}w_i)}\prod_{1\leq i<j\leq T}\prod_{\alpha=1}^{k_j}\frac{(u^j_\alpha-x_i)(v^j_\alpha-r_i^{-2}x_i)}{(u^j_\alpha-r_i^{-2}x_i)(v^j_\alpha-x_i)}\\&\qquad\times\prod_{m,i=1}^{T}\prod_{\alpha=1}^{k_i}\frac{(u^i_\alpha-y_m)(v^i_\alpha-x_m)}{(v^i_\alpha-y_m)(u^i_\alpha-x_m)}\\&\qquad\times\prod_{i=1}^{T}\Bigg(\prod_{\alpha=1}^{k_i}\frac{v^i_\alpha}{u^i_\alpha-v^i_\alpha}\prod_{1\leq\alpha<\beta\leq k_i}\frac{(u^i_\alpha-u^i_\beta)(v^i_\alpha-v^i_\beta)}{(v^i_\alpha-u^i_\beta)(u^i_\alpha-v^i_\beta)}\Bigg)\\&\qquad\prod_{1\leq i<j\leq T}\Bigg(\prod_{\alpha=1}^{k_i}\prod_{\beta=1}^{k_j}\frac{(u^i_\alpha-u^j_\beta)(v^i_\alpha-v^j_\beta)}{(v^i_\alpha-u^j_\beta)(u^i_\alpha-v^j_\beta)}\Bigg),\end{aligned}$$

where Z is given by (8.2)*. If $k_j=0$ for some j, we omit the operator $\Psi(\mathbf{u}^j;\mathbf{v}^j)$ in the left-hand side; and in the right-hand side, the products $\prod_{\alpha=1}^{k_j}$ are equal to 1, by agreement.*

Proof Throughout the proof, conditions (8.4) arise from recording all the required commutations of the operators which are obtained in Sect. 7.3. Moreover, we use Wick's determinant (Proposition 7.14) which leads to the last condition on $\mathbf{u}^i,\mathbf{v}^j$ in (8.4).

Observe that

$$\Psi(y_j)e_{\mathbb{Z}_{\le j-1}} = e_{\mathbb{Z}_{\le j}} \prod_{i=1}^{j-1} \frac{s_i^2(y_i - y_j)}{y_i - s_i^2 y_j},$$

where $j \ge 1$ (note the specific choice of the argument in $\Psi(\cdot)$). Therefore,

$$e_{\mathbb{Z}_{\le T}} = \prod_{1 \le i < j \le T} \frac{y_i - s_j^2 y_j}{s_i^2(y_i - y_j)} \Psi(y_T) \dots \Psi(y_1) e_{\mathbb{Z}_{\le 0}}. \tag{8.5}$$

First, we move each $D^{\mathbb{Z}}(w_i, \theta_i)$ to the left of $\mathbf{\Psi}(\mathbf{u}^i; \mathbf{v}^i)$ and $B^{\mathbb{Z}}(x_i, r_i)$, $j \ge i$. Then we move each $B^{\mathbb{Z}}(x_j, r_j)$ to the right of $\mathbf{\Psi}(\mathbf{u}^i; \mathbf{v}^i)$, $i \le j$. This leads, by Proposition 7.10, to

$$\begin{aligned}
&B^{\mathbb{Z}}(x_T, r_T)\Psi(\mathbf{u}^T; \mathbf{v}^T)D^{\mathbb{Z}}(w_T, \theta_T)\dots \\
&\quad \Psi(\mathbf{u}^2; \mathbf{v}^2)D^{\mathbb{Z}}(w_2, \theta_2)B^{\mathbb{Z}}(x_1, r_1)\Psi(\mathbf{u}^1; \mathbf{v}^1)D^{\mathbb{Z}}(w_1, \theta_1) \\
&= D^{\mathbb{Z}}(w_T, \theta_T)\dots D^{\mathbb{Z}}(w_1, \theta_1)\Psi(\mathbf{u}^T; \mathbf{v}^T)\dots \Psi(\mathbf{u}^1; \mathbf{v}^1)B^{\mathbb{Z}}(x_T, r_T)\dots B^{\mathbb{Z}}(x_1, r_1) \\
&\quad \times \prod_{1 \le i \le j \le T} \left(\frac{\theta_i^{-2} w_i - x_j}{w_i - x_j} \prod_{\alpha=1}^{k_j} \frac{\theta_i^{-2} w_i - v_\alpha^j}{w_i - v_\alpha^j} \frac{w_i - u_\alpha^j}{\theta_i^{-2} w_i - u_\alpha^j} \frac{u_\alpha^j - r_i^{-2} x_i}{u_\alpha^j - x_i} \frac{v_\alpha^j - x_i}{v_\alpha^j - r_i^{-2} x_i} \right).
\end{aligned} \tag{8.6}$$

Now, note that $\langle e_{\mathbb{Z}_{\le 0}}, D^{\mathbb{Z}}(w, \theta) e_{\mathcal{T}} \rangle = \mathbf{1}_{\mathcal{T} = \mathbb{Z}_{\le 0}}$. Thus, we may replace the D operators on the left by any other D operators. So we have, using (8.5),

$$\begin{aligned}
&\big\langle e_{\mathbb{Z}_{\le 0}}, D^{\mathbb{Z}}(w_T, \theta_T)\dots D^{\mathbb{Z}}(w_1, \theta_1)\Psi(\mathbf{u}^T; \mathbf{v}^T)\dots \\
&\quad \Psi(\mathbf{u}^1; \mathbf{v}^1)B^{\mathbb{Z}}(x_T, r_T)\dots B^{\mathbb{Z}}(x_1, r_1)e_{\mathbb{Z}_{\le T}} \big\rangle \\
&= \prod_{1 \le i < j \le T} \frac{y_i - s_i^2 y_j}{s_i^2(y_i - y_j)} \big\langle e_{\mathbb{Z}_{\le 0}}, D(r_T^{-2} x_T, r_T^{-1})\dots \\
&\quad D(r_1^{-2} x_1, r_1^{-1})\Psi(\mathbf{u}^T; \mathbf{v}^T)\dots \Psi(\mathbf{u}^1; \mathbf{v}^1) \\
&\quad \times B^{\mathbb{Z}}(x_T, r_T)\dots B^{\mathbb{Z}}(x_1, r_1)\Psi(y_T)\dots\Psi(y_1)e_{\mathbb{Z}_{\le 0}} \big\rangle,
\end{aligned} \tag{8.7}$$

We chose the new D operators such that together with $B(x_i, r_i)$ they will lead to the operators Ψ^*, cf. (7.21). Now we commute again and have, using (7.23), (7.25), and (7.15):

$$\begin{aligned}
&D(r_T^{-2} x_T, r_T^{-1})\dots D(r_1^{-2} x_1, r_1^{-1})\Psi(\mathbf{u}^T; \mathbf{v}^T)\dots \\
&\quad \Psi(\mathbf{u}^1; \mathbf{v}^1)B^{\mathbb{Z}}(x_T, r_T)\dots B^{\mathbb{Z}}(x_1, r_1) \\
&= \Psi(\mathbf{u}^T; \mathbf{v}^T)\dots \Psi(\mathbf{u}^1; \mathbf{v}^1)D(r_T^{-2} x_T, r_T^{-1})\dots
\end{aligned}$$

$$
\begin{aligned}
& D(r_1^{-2}x_1, r_1^{-1}) B^{\mathbb{Z}}(x_T, r_T) \dots B^{\mathbb{Z}}(x_1, r_1) \\
& \times \prod_{i,j=1}^{T} \prod_{\alpha=1}^{k_j} \frac{u_\alpha^j - x_i}{u_\alpha^j - r_i^{-2}x_i} \frac{v_\alpha^j - r_i^{-2}x_i}{v_\alpha^j - x_i} \\
= {} & \Psi(\mathbf{u}^T; \mathbf{v}^T) \dots \Psi(\mathbf{u}^1; \mathbf{v}^1) \prod_{i=1}^{T} \underbrace{D^{\mathbb{Z}}(r_i^{-2}x_i, r_i^{-1}) B^{\mathbb{Z}}(x_i, r_i)}_{\Psi^*(r_i^{-2}x_i, x_i)} \\
& \times \prod_{i,j=1}^{T} \prod_{\alpha=1}^{k_j} \frac{u_\alpha^j - x_i}{u_\alpha^j - r_i^{-2}x_i} \frac{v_\alpha^j - r_i^{-2}x_i}{v_\alpha^j - x_i} \prod_{1 \le i < j \le T} \frac{r_i^{-2}x_i - x_j}{x_i - x_j}.
\end{aligned}
$$

Now in the matrix element (8.7) we have a total of T operators $\Psi^*(r_i^{-2}x_i, x_i)$ in front of the same number of operators $\Psi(y_j)$. We can commute these operators through each other to form pairs of the operators as $\Psi\Psi^*$. Thanks to (7.26), this only produces the sign $(-1)^{T(T+1)/2}$. Putting this together, for the computation of the matrix element $\langle e_{\mathbb{Z}_{\le 0}}, (\cdots) e_{\mathbb{Z}_{\le 0}} \rangle$, we apply Proposition 7.14 with the variables

$$
\begin{aligned}
\{u_i\} &= \left\{u_\alpha^i : 1 \le i \le T,\ 1 \le \alpha \le k_i\right\} \cup \{y_m : 1 \le m \le T\}, \qquad \zeta_i \equiv 0; \\
\{\kappa_i\} &= \{0 : 1 \le i \le T,\ 1 \le \alpha \le k_i\} \cup \left\{r_i^{-2}x_i : 1 \le i \le T\right\}; \\
\{v_i\} &= \left\{v_\alpha^i : 1 \le i \le T,\ 1 \le \alpha \le k_i\right\} \cup \{x_i : 1 \le i \le T\}.
\end{aligned}
$$

This produces the following expression for the final matrix element $\langle e_{\mathbb{Z}_{\le 0}}, (\cdots) e_{\mathbb{Z}_{\le 0}} \rangle$:

$$
\begin{aligned}
& \prod_{i=1}^{T} v_i x_i (1 - r_i^{-2}) \prod_{i,j=1}^{T} \left(\prod_{\alpha=1}^{k_i} \prod_{\beta=1}^{k_j} \frac{1}{v_\beta^j - u_\alpha^i} \prod_{\alpha=1}^{k_i} \frac{1}{x_j - u_\alpha^i} \right) \\
& \times \prod_{m,i=1}^{T} \left(\frac{1}{x_i - y_m} \prod_{\alpha=1}^{k_i} \frac{1}{v_\alpha^i - y_m} \right) \prod_{1 \le i < j \le T} (y_i - y_j)(x_j - x_i) \\
& \times \prod_{1 \le i < j \le T} \left(\prod_{\alpha=1}^{k_i} \prod_{\beta=1}^{k_j} (u_\alpha^i - u_\beta^j)(v_\beta^j - v_\alpha^i) \right) \prod_{m,i=1}^{T} \prod_{\alpha=1}^{k_i} (u_\alpha^i - y_m) \\
& \times \prod_{i=1}^{T} \left(\prod_{1 \le \alpha < \beta \le k_i} (u_\alpha^i - u_\beta^i)(v_\beta^i - v_\alpha^i) \right) \prod_{i,j=1}^{T} \prod_{\alpha=1}^{k_i} (x_j - v_\alpha^i).
\end{aligned}
$$

Combining this with all the factors resulting from commutations at previous steps of the proof, and with the denominator (8.2), we get the desired identity. □

Remark 8.8 Recall that we assume that for some fixed $\varepsilon > 0$, we have $\varepsilon < y_i < \varepsilon^{-1}$ and $\varepsilon < s_i < 1 - \varepsilon$ for all i. One can check that there exist parameters for which all conditions (8.4) hold and the FG process is well-defined (see Definitions 6.1 and 6.3). For example, we may take the following parameters:

$$\begin{aligned} &y_i \approx 1,\ i \ge 1; \quad y_i \approx 0.9,\ i \le 0; \quad s_i \approx 0.25,\ i \ge 1; \quad s_i \approx 0.95,\ i \le 0; \\ &r_i \approx 0.84; \quad \theta_i \approx 0.84; \quad x_i \approx 0.8; \quad w_i \approx 0.85. \end{aligned} \tag{8.8}$$

Here "≈" means that the parameters are very close to the corresponding values (for all i), but are allowed to be distinct. Given (8.8), one readily sees that u_i, v_j satisfying (8.4) also exist.

8.5 Correlation kernel

We can now compute the correlation kernel for the FG process.

Theorem 8.9 *The point process $\mathcal{S}^{(T)}$ (6.11) corresponding to the FG process (8.1) is determinantal. That is, for any finite $A = \{(t, a^t_\alpha) \colon 1 \le t \le T,\ 1 \le \alpha \le k_t\} \subset \{1, \dots, T\} \times \mathbb{Z}_{\ge 1}$ we have*

$$\mathbb{P}_{\mathcal{P}}\big[A \subset \mathcal{S}^{(T)}\big] = \det\Big[K_{\mathcal{P}}(t, a^t_\alpha; t', a^{t'}_{\alpha'})\Big]. \tag{8.9}$$

Here the determinant is of size $k_1 + \ldots + k_T$ corresponding to $1 \le t, t' \le T$, $1 \le \alpha \le k_t$, $1 \le \alpha' \le k_{t'}$. The kernel $K_{\mathcal{P}}$ has the form

$$\begin{aligned} K_{\mathcal{P}}(t, a; t', a') = \big[\Phi_{a'}(u, 0)\Phi^*_a(0, v)\big] \frac{v}{u - v} \prod_{m=1}^{T} \frac{(u - y_m)(v - x_m)}{(v - y_m)(u - x_m)} \\ \times \prod_{i=1}^{t'} \frac{u - w_i}{u - \theta_i^{-2} w_i} \prod_{i=1}^{t} \frac{v - \theta_i^{-2} w_i}{v - w_i} \prod_{i=1}^{t'-1} \frac{u - x_i}{u - r_i^{-2} x_i} \prod_{i=1}^{t-1} \frac{v - r_i^{-2} x_i}{v - x_i}. \end{aligned} \tag{8.10}$$

Proof From Proposition 8.7 we see that $\mathbb{P}_{\mathcal{P}}\big[A \subset \mathcal{S}^{(T)}\big]$ is the coefficient

$$\begin{aligned} &\left[\prod_{t=1}^{T}\prod_{\alpha=1}^{k_t} \Phi_{\mathsf{a}^t_\alpha}(u^t_\alpha, 0)\Phi^*_{\mathsf{a}^t_\alpha}(0, v^t_\alpha)\right] \prod_{1 \le i \le j \le T} \prod_{\alpha=1}^{k_j} \frac{(v^j_\alpha - \theta_i^{-2} w_i)(u^j_\alpha - w_i)}{(v^j_\alpha - w_i)(u^j_\alpha - \theta_i^{-2} w_i)} \\ &\prod_{1 \le i < j \le T} \frac{(u^j_\alpha - x_i)(v^j_\alpha - r_i^{-2} x_i)}{(u^j_\alpha - r_i^{-2} x_i)(v^j_\alpha - x_i)} \\ &\times \prod_{m,i=1}^{T} \prod_{\alpha=1}^{k_i} \frac{(u^i_\alpha - y_m)(v^i_\alpha - x_m)}{(v^i_\alpha - y_m)(u^i_\alpha - x_m)} \det\left[\frac{v^t_\alpha}{u^{t'}_{\alpha'} - v^t_\alpha}\right], \end{aligned}$$

where we used the Cauchy determinantal formula, and the last determinant is of the same size as in (8.9). The dependence of the remaining expression is of a product form in the u^j_α's and v^j_β's, and we may put this product expression into the determinant. Finally, the operation of extracting the series coefficient may also be placed inside the determinant thanks to Andréief identity (5.12), see also [39].

8.6 Specialization to ascending FG processes

Let us now specialize the results for the general FG process (8.1) obtained in this section to the case of the ascending process (Definition 6.4). Recall that in the latter case the correlation kernel is computed via an Eynard–Mehta type approach (Theorem 6.7 proven in Appendix B). Our aim is to establish the following result whose proof occupies the rest of this subsection:

Theorem 8.10 *Specialize the correlation kernel for the general FG process (given by Theorem 8.9 as a generating series coefficient) to the case of an ascending FG process. Then the series coefficient can be extracted with the help of a double contour integration, which results in the same expression (6.12) for the correlation kernel $K_{\mathcal{AP}}$ as the one obtained using the Eynard–Mehta type approach.*

To match the notation, let us rename the parameter T in the general FG process (8.1) to $N+T$, make the specializations $(x_1; r_1), \ldots, (x_T; r_T)$ empty, rename $(x_{T+1}; r_{T+1}), \ldots (x_{T+N}; r_{T+N})$ to $(x_1; r_1), \ldots (x_N; r_N)$, and set $w_{T+1} = \ldots = w_{T+N-1} = w_{T+N} = 0$. Furthermore, in Proposition 8.7 let us take $k_{T+N} = \ldots = k_{T+1} = 0$. One readily sees as in the proof of Proposition 8.7 that the correlation generating function becomes

$$
\begin{aligned}
&\frac{1}{Z}\big\langle e_{\mathbb{Z}_{\le 0}}, B^{\mathbb{Z}}(x_N, r_N)\ldots B^{\mathbb{Z}}(x_1, r_1)\Psi(\mathbf{u}^T; \mathbf{v}^T)D^{\mathbb{Z}}(w_T, \theta_T) \\
&\quad\times \Psi(\mathbf{u}^{T-1}; \mathbf{v}^{T-1})D^{\mathbb{Z}}(w_{T-1}, \theta_{T-1})\ldots \\
&\quad D^{\mathbb{Z}}(w_2, \theta_2)\Psi(\mathbf{u}^1; \mathbf{v}^1)D^{\mathbb{Z}}(w_1, \theta_1)e_{\mathbb{Z}_{\le N}}\big\rangle \\
&= \det\left[\frac{v^t_\alpha}{u^{t'}_{\alpha'} - v^t_\alpha}\right] \prod_{1\le i\le j\le T}\prod_{\alpha=1}^{k_j} \frac{(v^j_\alpha - \theta_i^{-2}w_i)(u^j_\alpha - w_i)}{(v^j_\alpha - w_i)(u^j_\alpha - \theta_i^{-2}w_i)} \\
&\quad\prod_{i=1}^{T}\prod_{\alpha=1}^{k_i}\prod_{m=1}^{N} \frac{u^i_\alpha - y_m}{v^i_\alpha - y_m}\frac{v^i_\alpha - x_m}{u^i_\alpha - x_m},
\end{aligned}
\tag{8.11}
$$

where Z is now given by (6.9), and the determinant is the same as in Sect. 8.5, that is, of size $k_1 + \ldots + k_T$ such that $1 \le t, t' \le T$, $1 \le \alpha \le k_t$, $1 \le \alpha' \le k_{t'}$.

Identity (8.11) holds under assumptions (8.4) on the parameters which are quite restrictive. In fact, some of these assumptions are artifacts of our proof of Proposition 8.7 and can be removed:

Lemma 8.11 *Identity* (8.11) *holds under the weaker assumptions*

$$\oplus_{x_i;w_j},\ \oplus_{v^j_\alpha;w_i},\ \oplus_{x_j;u^i_\alpha},\ \ominus_{u^j_\alpha;\theta_i^{-2}w_i},\ \begin{cases}\ominus_{u^i_\alpha;v^j_\beta}, & i>j \text{ or } i=j,\alpha\ge\beta;\\ \oplus_{v^j_\beta;u^i_\alpha}, & i<j \text{ or } i=j,\alpha<\beta,\end{cases} \tag{8.12}$$

where we use notation (7.18)*, (7.19).*

Proof Since the right-hand side of (8.11) is rational, it suffices to show that under (8.12) the left-hand side of (8.11) converges. After establishing this, we may drop the unnecessary conditions from (8.4).

Observe that possible infinite summations in the left-hand side of (8.11) may arise in two cases. Either one of the operators $D^{\mathbb{Z}}$ or Ψ adds a vertical arrow at some $L\ge 1$, and then one of the following operators $B^{\mathbb{Z}}$ or Ψ^* removes it; or one of the operators $D^{\mathbb{Z}}$ or Ψ^* removes a vertical arrow at some $L\le 0$, and one of the following operators Ψ adds it back. There are no operators Ψ to the left of $B^{\mathbb{Z}}$, so removals of the arrows at $L\le 0$ by $B^{\mathbb{Z}}$ cannot be compensated and thus do not contribute to the left-hand side of (8.11).

We now use Definition 7.4 and W (2.3) for $B^{\mathbb{Z}}$, $D^{\mathbb{Z}}$ and Theorem 7.11 for Ψ, Ψ^*. We see that at sufficiently large $L\ge 1$, the combination of $D^{\mathbb{Z}}(w,\theta)$ and $B^{\mathbb{Z}}(x,r)$ produces a factor $\prod_{i=m}^{L}\frac{x-s_i^{-2}y_i}{x-y_i}\frac{w-s_i^{-2}y_i}{w-y_i}$, where m is fixed and L grows. This factor is summable over L under $\oplus_{x;w}$. All other pairs of operators are considered similarly. Namely, for $L\ge 1$, pairs of operators lead to conditions as follows:

$$\begin{pmatrix} D^{\mathbb{Z}}(w,\theta),\ \Psi^*(v)\\ \Psi(u),\ B^{\mathbb{Z}}(x,r)\\ \Psi(u),\ \Psi^*(v)\end{pmatrix}\quad \text{leadsto}\quad \begin{pmatrix}\oplus_{v;w}\\ \oplus_{x;u}\\ \oplus_{v;u}\end{pmatrix}$$

And for $L\le 0$, we have

$$\begin{pmatrix} D^{\mathbb{Z}}(w,\theta),\ \Psi(u)\\ \Psi^*(v),\ \Psi(u)\end{pmatrix}\quad \text{leadsto}\quad \begin{pmatrix}\ominus_{u;\theta^{-2}w}\\ \ominus_{u;v}\end{pmatrix}.$$

Finally, note that the two different cases in (8.12) are due to the fact that when Ψ comes before or after Ψ^*, only the case $L\ge 1$ or $L\le 0$, respectively, may lead to infinite sums. □

Fix a finite set $A=\left\{(t,\mathsf{a}^t_\alpha)\colon 1\le t\le T,\ 1\le\alpha\le k_t\right\}$, and denote $k=k_1+\ldots+k_T$ (this is the size of A). Arguing as in the proof of Theorem 8.9, we see that $\mathbb{P}_{\mathcal{AP}}[A\subset\mathcal{S}^{(T)}]$, the correlation function of the ascending FG process, is equal to the coefficient by $\prod_{t=1}^{T}\prod_{\alpha=1}^{k_t}\Phi_{\mathsf{a}^t_\alpha}(u^t_\alpha,0)\Phi^*_{\mathsf{a}^t_\alpha}(0,v^t_\alpha)$ in the expansion of the right-hand side of (8.11). By Lemmas 8.4 and 8.5, this coefficient can be extracted with the help of a $2k$-fold contour integral

$$
\begin{aligned}
&\frac{1}{(2\pi \mathbf{i})^{2k}}\left(\prod_{i=1}^{T}\prod_{\alpha=1}^{k_i}\oint_{\Gamma_{y,*}} du_\alpha^i \oint_{\Gamma_{s^{-2}y,*}} dv_\alpha^i\right)\\
&\prod_{t=1}^{T}\prod_{\alpha=1}^{k_t}\left(\frac{y_{\mathsf{a}_\alpha^t}(s_{\mathsf{a}_\alpha^t}^{-2}-1)}{v-s_{\mathsf{a}_\alpha^t}^{-2}y_{\mathsf{a}_\alpha^t}}\frac{1}{u-y_{\mathsf{a}_\alpha^t}}\prod_{j=1}^{\mathsf{a}_\alpha^t-1}\frac{v-y_j}{v-s_j^{-2}y_j}\frac{u-s_j^{-2}y_j}{u-y_j}\right)\\
&\times \det\left[\frac{1}{u_{\alpha'}^{t'}-v_\alpha^t}\right]\prod_{i=1}^{T}\prod_{\alpha=1}^{k_i}\prod_{m=1}^{N}\frac{u_\alpha^i-y_m}{v_\alpha^i-y_m}\frac{v_\alpha^i-x_m}{u_\alpha^i-x_m}\\
&\prod_{1\le i\le j\le T}\prod_{\alpha=1}^{k_j}\frac{(v_\alpha^j-\theta_i^{-2}w_i)(u_\alpha^j-w_i)}{(v_\alpha^j-w_i)(u_\alpha^j-\theta_i^{-2}w_i)}.
\end{aligned}
\tag{8.13}
$$

Here each contour u_α^i goes around all y_k in the positive direction and leaves all $s_k^{-2}y_k$ outside, while each contour v_β^j encircles all $s_k^{-2}y_k$ and leaves all y_k outside. Moreover, the contours might encircle some of the other poles $u_\alpha^i = v_\beta^j, u_\alpha^i = x_k, u_\alpha^i = \theta_k^{-2}w_k$, or $v_\beta^j = w_k$ of the integrand. These additional residues are not yet specified because Lemmas 8.4 and 8.5 involve series expansions and not actual rational functions. Therefore, we need to determine which of these additional poles the contours in (8.13) encircle. This is done in the next statement.

Proposition 8.12 *The correlation function $\mathbb{P}_{\mathcal{AP}}[A \subset \mathcal{S}^{(T)}]$ is equal to the 2k-fold contour integral (8.13), where:*

- *the integration contour for each u_α^i is positively oriented, encircles all y_k, $\theta_k^{-2}w_k$, and does not encircle any of x_k;*
- *the integration contour for each v_β^j is negatively oriented, encircles all y_k, w_k, and does not encircle any of $s_k^{-2}y_k$;*
- *the contour u_α^i contains the contour v_β^j for $i > j$ or $i = j$, $\alpha \ge \beta$; and the v_β^j contains u_α^i otherwise.*

Proof First, let us take the parameters close to each other as follows:

$$
y_i \approx y,\ i \in \mathbb{Z};\quad s_i \approx s,\ i \in \mathbb{Z};\quad r_i \approx r;\quad \theta_i \approx \theta;\quad x_i \approx x;\quad w_i \approx w, \tag{8.14}
$$

where

$$
\left|\frac{x-s^{-2}y}{x-y}\right| < \left|\frac{w-s^{-2}y}{w-y}\right|,
$$

and the nonnegativity of the specializations (Definition 6.1) holds. In (8.14), "≈" means that the parameters are very close to the corresponding values (for all i), but are all distinct. One can check that such a choice of x, r, w, θ, y, s exists, for example, $y = 1, s = 0.25, x = 0.3, r = 0.5, w = 0.43, \theta = 0.64$.

Under (8.14), conditions $\oplus_{a;b}$ and $\ominus_{a;b}$ are essentially the same since there is no difference between (y_j, s_j) with negative and positive indices. Consider the map (and its inverse)

$$U \mapsto \Xi = \Xi(U) := \frac{U - s^{-2}y}{U - y}, \qquad \Xi \mapsto U = U(\Xi) = \frac{y(s^{-2} - \Xi)}{1 - \Xi}.$$

Clearly, Ξ maps y_j close to ∞, and $s_j^{-2}y_j$ close to 0. We also see that conditions (8.12) are satisfied if the variables u_α^i, v_β^j are chosen so that $|\Xi(x)| < |\Xi(u_\alpha^i)| < |\Xi(w)|$, $|\Xi(x)| < |\Xi(v_\beta^j)| < |\Xi(w)|$, $|\Xi(u_\alpha^i)| < |\Xi(\theta^{-2}w)|$, and

$$\begin{cases} |\Xi(u_\alpha^i)| < |\Xi(v_\beta^j)|, & i > j \text{ or } i = j, \alpha \geq \beta; \\ |\Xi(v_\beta^j)| < |\Xi(u_\alpha^i)|, & i < j \text{ or } i = j, \alpha < \beta. \end{cases}$$

We claim that the integration contours for u_α^i, v_β^j satisfying all the required conditions exist. Indeed, one can simply take $u_\alpha^i = U(c_\alpha^i e^{-\mathbf{i}t})$, $v_\beta^j = U(d_\beta^j e^{\mathbf{i}t})$, where $0 < t < 2\pi$ and

$$\begin{cases} 0 < c_\alpha^i < d_\beta^j, & i > j \text{ or } i = j, \alpha \geq \beta; \\ 0 < d_\beta^j < c_\alpha^i, & i < j \text{ or } i = j, \alpha < \beta. \end{cases}$$

In particular, the radii c_α^i, d_β^j are interlacing in a certain way. Since these radii can be arbitrarily close to each other, this can be achieved. Note the different orientation of the u and the v contours which is due to the fact that Ξ maps y to infinity. This agrees with Lemmas 8.4 and 8.5 in that the u_α^i contours must go around all y_j in the positive direction, and the v_β^j contours must go around all $s_j^{-2}y_j$ in the negative direction. Both families of contours should separate $\{y_k\}$ from $\{s_k^{-2}y_k\}$.

Moreover, inequalities for $|\Xi(u_\alpha^i)|$ and $|\Xi(v_\beta^j)|$ listed above imply that the integration contours are as described in the claim of the proposition. Therefore, by Lemmas 8.4 and 8.5, in the case when all the similarly named parameters are close to each other as in (8.14), we may extract the desired coefficient $\mathbb{P}_{\mathcal{AP}}[A \subset \mathcal{S}^{(T)}]$ from the right-hand side of (8.11) by means of integration over the $2k$ contours described above in the proof.

To complete the proof in the general case, we use analytic continuation. First, a straightforward a priori argument (like in [19, Lemma 8.10]) shows that under $\oplus_{x_i;w_j}$ (for all i, j), any correlation function $\mathbb{P}_{\mathcal{AP}}[A \subset \mathcal{S}^{(T)}]$ of the FG process is a rational function depending on a finite subset of the parameters of the process (the size of the subset depends on A). Second, the $2k$-fold contour integral (8.13) over the contours described above in the proof is also a rational function because it is a finite sum of residues of the integrand. These two rational functions are equal on an open full-dimensional subset in the finite-dimensional space of the parameters that they depend on. Therefore, these functions are equal in general, provided that the correlation function $\mathbb{P}_{\mathcal{A}\mathcal{P}}[A \subset \mathcal{S}^{(T)}]$ is well-defined (i.e., under $\oplus_{x_i;w_j}$) and the $2k$-fold integral is taken over the same residues as before the analytic continuation. □

By applying Andréief identity (5.12) (see also [39]), the $2k$-fold contour integral (8.13) (over the contours described in Proposition 8.12) is rewritten as a determinant

$$\mathbb{P}_{\mathcal{A}\mathcal{P}}\big[A \subset \mathcal{S}^{(T)}\big] = \det\big[K_{\mathcal{A}\mathcal{P}}(t, \mathsf{a}^t_\alpha; t', \mathsf{a}^{t'}_{\alpha'})\big]$$

of the correlation kernel $K_{\mathcal{A}\mathcal{P}}(t, a; t', a')$ given by the double contour integral (6.12). The determinant is of size k, indexed by $1 \le t, t' \le T$ with $1 \le \alpha \le k_t$, $1 \le \alpha' \le k_{t'}$. Let us add two remarks:

- The sign difference in the term $1 - s_a^{-2}$ between (8.13) and the kernel $K_{\mathcal{A}\mathcal{P}}$ (6.12), is due to reversing the direction of the v contour.
- The conditions $\ominus_{u^i_\alpha; v^j_\beta}$ and $\oplus_{v^j_\beta; u^i_\alpha}$ in (8.12) depending on the relative order of the indices (i, α) and (j, β) translate to $\ominus_{u;v}$ for $t' \ge t$ and $\oplus_{v;u}$ for $t' < t$ in the double contour integral kernel $K_{\mathcal{A}\mathcal{P}}(t, a; t', a')$.

Overall, we see that for the ascending FG process, the fermionic operator approach developed in Sects. 7 and 8 and the Eynard–Mehta type approach from Appendix B produce the same correlation kernel $K_{\mathcal{A}\mathcal{P}}$. This completes the proof of Theorem 8.10.

Part III Random Tilings

In this part we represent determinantal point processes from Part II as a certain inhomogeneous dimer model (which can also be viewed as a model of random domino tilings), and study the bulk asymptotic behavior of the model.

9 Dimers and domino tilings

In this section we interpret the ascending FG process defined in Sect. 6 as a nonintersecting path model and a dimer model, and prove Theorem 1.9 from Introduction.

9.1 Layered five vertex model

Let us take six vertex weights $w_{6\mathrm{V}}(i_1, j_1; i_2, j_2)$ as in Fig. 12, top. Assume that they are free fermionic, that is, $a_1a_2 + b_1b_2 = c_1c_2$. Moreover, let $c_1 \ne 0$.

Define two families of *five vertex* weights, $w'_{5\mathrm{V}}$ and $w_{5\mathrm{V}}$, as in Fig. 12, middle and bottom, respectively. One readily sees that these vertex weights also satisfy the free fermion condition.

The six vertex configuration can be replaced by a vertical concatenation of two five vertex configurations [98, Section 4.7]. Let us recall the construction. Consider a stacked two-vertex configuration with the weight $w'_{5\mathrm{V}}$ at the top, and weight $w_{5\mathrm{V}}$

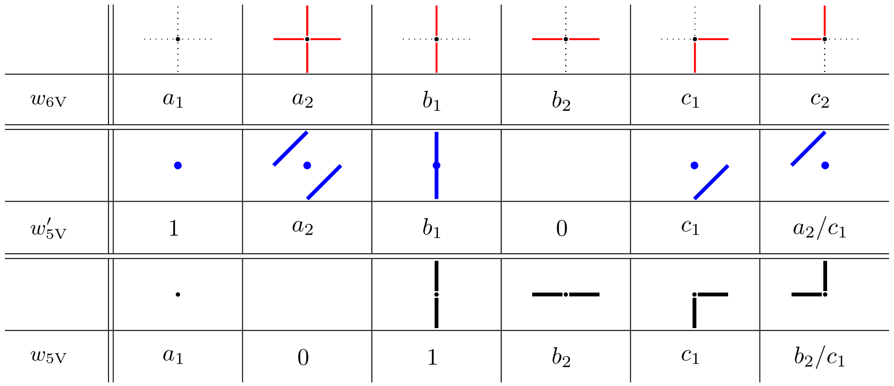


Fig. 12 The six vertex weights and two families of five vertex weights

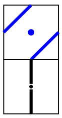
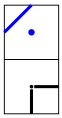
at the bottom. Then we claim that these two five vertex weights produce the same partition function as w_{6V}:

Lemma 9.1 *For any fixed $I_1, I_2, J_1, j_2, j'_2 \in \{0, 1\}$, we have*

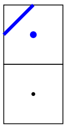
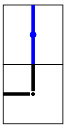
$$\sum_{j_1, j'_1, k \in \{0,1\}} w_{5V}(I_1, j_1; k, j_2)\, w'_{5V}(k, j'_1; I_2, j'_2)\, \mathbf{1}_{j_1 + j'_1 = J_1} = w_{6V}(I_1, J_1; I_2, j_2 + j'_2).$$

In particular, if $j_2 + j'_2$ is greater than 1, then the left-hand side vanishes.

Proof This is done by a straightforward verification. Let us illustrate just two cases. First, for $I_1 = I_2 = J_1 = 1$, we have two configurations to be considered separately (as they correspond to different exit boundary conditions (j_2, j'_2)):

weight of  $= a_2 \cdot 1 = a_2,$ weight of  $= \frac{a_2}{c_1} \cdot c_1 = a_2.$

Second, for $I_1 = 0$, $J_1 = 1$, $I_2 = 1$, we have two configurations to be considered together (as they have the same (j_2, j'_2)):

weight of  + weight of  $= \frac{a_2}{c_1} \cdot a_1 + \frac{b_2}{c_1} \cdot b_1 = c_2.$

All other cases are obtained similarly.

Lemma 9.1 implies that the ascending FG process $\mathcal{AP}(\lambda^{(1)}, \ldots, \lambda^{(T)})$ (6.8) can be realized as a partition function of a path configuration in $\mathbb{Z}_{\geq 1} \times \mathcal{I}_{T,N}$, where

$$\mathcal{I}_{T,N} = \{1, 1', \ldots, T, T', T+1, (T+1)', \ldots, T+N, (T+N)'\}. \tag{9.1}$$

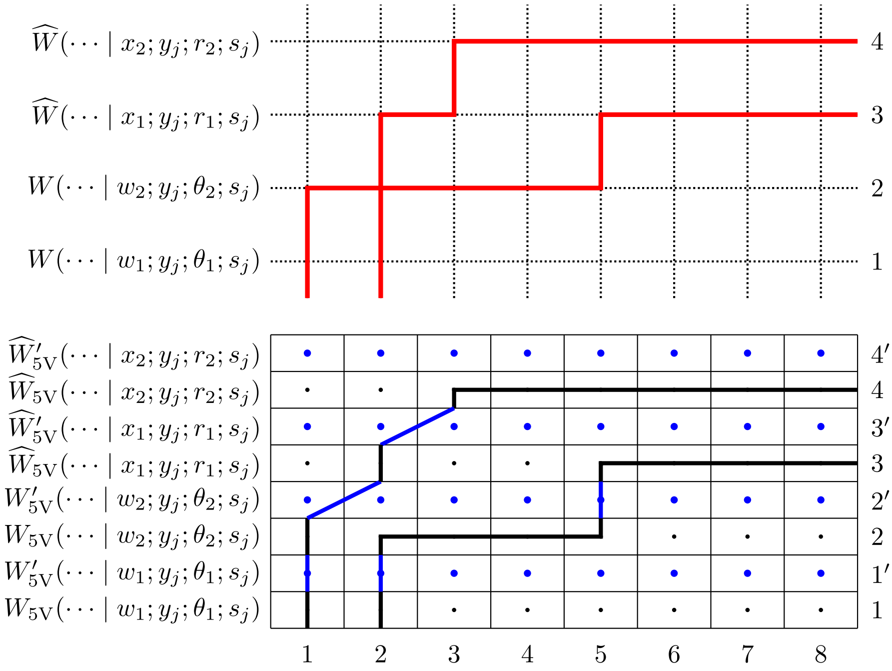


Fig. 13 Top: A path configuration under an FG process with $N = T = 2$, where $\lambda^{(1)} = (0, 0)$ and $\lambda^{(2)} = (3, 1)$. Bottom: One of the possible path configurations under the five vertex realization of the FG process, which corresponds to the particular six vertex configuration given at the top. The index j in y_j, s_j is the horizontal coordinate

Namely, take the vertex weights at the odd horizontals (numbered $1 \le i \le T + N$) to be $W_{5\mathrm{V}}$, $1 \le i \le T$ or $\widehat{W}_{5\mathrm{V}}$, $T + 1 \le i \le T + N$, and the vertex weights at the even horizontals (numbered by i', $1 \le i \le T + N$) to be $W'_{5\mathrm{V}}$ or $\widehat{W}_{5\mathrm{V}}$ in a similar way. Here $W_{5\mathrm{V}}$, W'_{5V} are constructed from the six vertex weights W (2.3) as in Fig. 12, and similarly $\widehat{W}_{5\mathrm{V}}$, $\widehat{W}'_{5\mathrm{V}}$ are constructed from $\widehat{W}$ (2.4). The boundary conditions in $\mathbb{Z}_{\ge 1} \times \mathfrak{I}_{T,N}$, are the same as the boundary conditions for the ascending FG process: there are N paths entering from below, and these N paths exit far to the right through the topmost N odd horizontals. See Fig. 13 for an illustration.

Proposition 9.2 *With the above notation, the joint distribution of the arrow configurations in the layered five vertex model as in Fig.* 13*, bottom, joining horizontals i' and $i + 1$, $1 \le i \le T$, is the same as the joint distribution of $\mathcal{S}(\lambda^{(i)})$, $1 \le i \le T$, under the ascending FG process* (6.8). *(Here we are using notation $\mathcal{S}(\lambda)$ from* (3.1).*)*

Note that the paths in the layered five vertex model are drawn to be nonintersecting (as in Fig. 13, bottom). This allows to identify this model with a dimer model in Sect. 9.2 below.

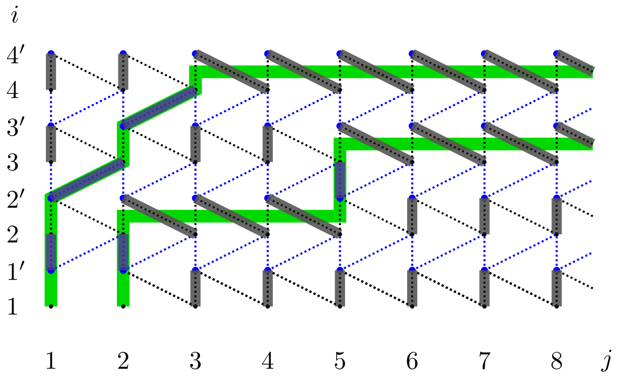


Fig. 14 Graph in $\mathbb{Z}_{\geq 1} \times \mathfrak{I}_{T,N}$ (with $T = N = 2$) and a dimer covering corresponding to the layered five vertex configuration in Fig. 13, bottom. The paths of the layered five-vertex model are shown in green

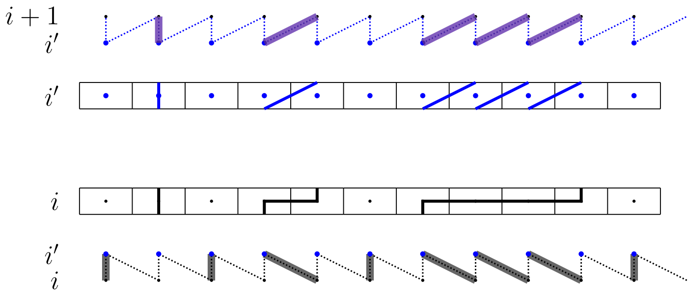


Fig. 15 Illustration of the correspondence between five vertex path configurations and dimer configurations at two different types of layers

9.2 Dimer model

Consider a layered bipartite graph $\mathcal{G}_{T,N}$ with vertices $\mathbb{Z}_{\geq 1} \times \mathfrak{I}_{T,N}$ (cf. (9.1)) in which edges connect the following vertices:

$$(j, i) - (j, i'), \qquad (j, i) - (j - 1, i'),$$
$$(j, i') - (j, i + 1), \qquad (j, i') - (j + 1, i + 1),$$

where $j \geq 1$ and $1 \leq i \leq T + N$. In addition, remove vertices $(1, 1), (2, 1), \ldots, (N, 1)$ from the graph together with all edges incident to these vertices. See Fig. 14 for an illustration. This graph is equivalent to a particular case of a rail-yard graph [7].

Then we construct a one-to-one mapping from the layered five vertex configuration to a dimer covering (i.e., a perfect matching) of the graph $\mathcal{G}_{T,N}$. This is done layer by layer as in Fig. 15. Recall that the probability weight of a particular dimer covering is proportional to the product of the weights of all edges that are covered. We refer to [43, 58, 65] for basics on dimer models on bipartite graphs. Note that due to the behavior of the five vertex paths far to the right, in our dimer covering far to the right we will almost surely see only dimers $(j, i) - (j, i')$ for $1 \leq i \leq T$, and only dimers

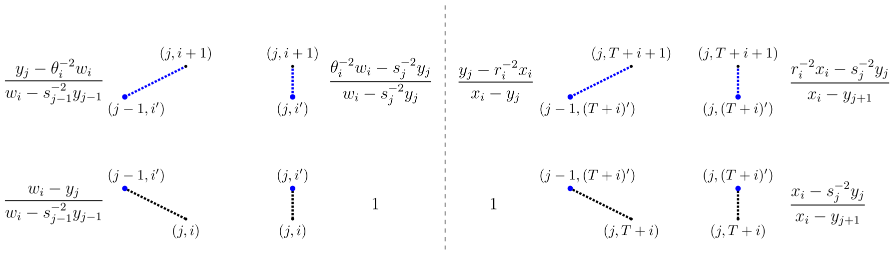


Fig. 16 Edge weights in the dimer model representing the layered five vertex model from Fig. 13, bottom. The left four weights correspond to the lower T rows, so $1 \le i \le T$. The right four weights appear in the upper N rows, and there we have $1 \le i \le N$

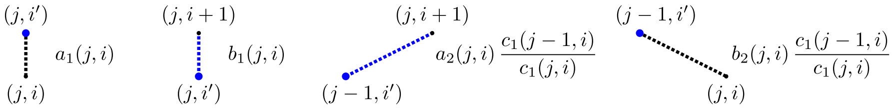


Fig. 17 Dimer weights in the proof of Proposition 9.3

$(j,i)-(j-1,i')$ for $T+1 \le i \le T+N$. This also follows from the form of our edge weights:

Proposition 9.3 *Under the identification between the layered five vertex model and the dimer model as in Figs. 14 and 15, the probability measure (coming from the FG process) is equivalent to the dimer model with the edge weights given in Fig. 16.*

Proof From the identification between the five vertex paths and the dimer covering (see Fig. 15), using the six to five vertex conversion table in Fig. 12, we obtain dimer weights as in Fig. 17.

The concrete values of $a_1, a_2, b_1, b_2, c_1, c_2$ depending on the coordinates (j,i) are equal to W (2.3) or $\widehat{W}$ (2.4), as depicted in Fig. 13.

One readily sees that in the W part, the weights in Fig. 17 produce the left four weights in Fig. 16, as desired.

In the $\widehat{W}$ part (containing the parameters x_j, r_j) let us in addition multiply the dimer weights around each vertex (j,i') by

$$\frac{c_1(j+1,i)}{c_1(j,i)} = \frac{x_i - y_j}{x_i - y_{j+1}}.$$

This does not change the dimer model on finite subgraphs (cf. [58, Section 3.10]), but makes the weight of each edge $(j,i)-(j-1,i')$ to be 1. This agrees with the fact that such dimers appear infinitely often in the full dimer model on our infinite graph. This leads to the right four weights in Fig. 16, and so we are done. □

9.3 Random domino tilings

The representation of the FG process as a dimer model on a bipartite graph described in Sect. 9.2 is useful for asymptotic analysis (performed below in this part), mainly due to the clear dependence of the edge weights on the Cartesian coordinates in the plane, cf. Figure 16. Here we provide its equivalent interpretation as an inhomogeneous domino tiling model.

In the lattice in Fig. 14, shift each row i' to the right by $\frac{1}{2}$. This transforms the lattice into a subset of the square lattice, see Fig. 18, left. Then rotate the whole picture 45° clockwise, and interpret dimers as 1×2 dominoes. In this way, the ascending FG process (6.8) is represented as a random domino tiling of the half-infinite strip with the zigzag boundary and with N additional unit squares removed from the top of the southwest part of the boundary. The corresponding domino tiling is given in Fig. 18, right. The domino weights are inhomogeneous and, moreover, depend on the parity of the coordinates. The weights are also given in Fig. 18. Note that this domino tiling model and the weights are the same as in Figs. 2 and 3 from Introduction, up to a rotation by 45°. Thus, we have completed the proof of Theorem 1.9.

Random domino tilings (in particular, of the Aztec Diamond) is a classical subject in combinatorics and probability [23, 26, 30, 65]. Our model in Fig. 18 is a particular case of a larger family of domino tilings, the *steep tilings* of an infinite strip [10]. While our inhomogeneous domino weights are more general than that in steep tilings, the latter allow for more general boundary conditions which are not yet available in our FG process setup. Moreover, in steep tilings one can prescribe an arbitrary sequence $\in \{+, -\}^{2\ell}$ of asymptotic domino directions at infinity (i.e., the directions of the shaded dominoes in Fig. 18 repeating infinitely many times). We remark that different asymptotic directions of dominoes may be modeled in our setup by passing to the fully general FG processes (Sect. 8.1), but we will not consider this generality in the present work.

Below in this Part III we discuss bulk asymptotics of the inhomogeneous domino tiling model displayed in Fig. 18 coming from ascending FG processes.

Remark 9.4 (Noncolliding lattice walks) When $w_i = y_j$ for all i, j, one of the dimer weights vanishes, see the left part of Fig. 16. Thus, in the bottom T double rows in Fig. 14 we can erase the edges carrying the zero weight. In this way we obtain the hexagonal lattice. Therefore, when $w_i \equiv y_j$, one can interpret the dimer model as a model of N noncolliding lattice walks as in, e.g., [62] (see also [45] for an equivalent lozenge tiling picture). The endpoints of these noncolliding lattice walks are distributed according to the probability weights coming from $F_\lambda(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s})$. On the other hand, the bulk asymptotics of our dimer model would lead to a certain class of random lozenge tilings of the whole plane. We briefly discuss these measures in Sect. 11.

10 Asymptotics in the bulk

10.1 Scaling. Global parameters and local sequences

In this section we perform bulk asymptotic analysis of the correlation kernel $K_{\mathcal{A}\mathcal{P}}(t, a; t', a')$ (6.12). By "bulk asymptotics" we mean the scaling around a global position far from the boundary of the system, such that discrete lattice structure is preserved. More precisely, take $N \to +\infty$ to be an independent parameter going to

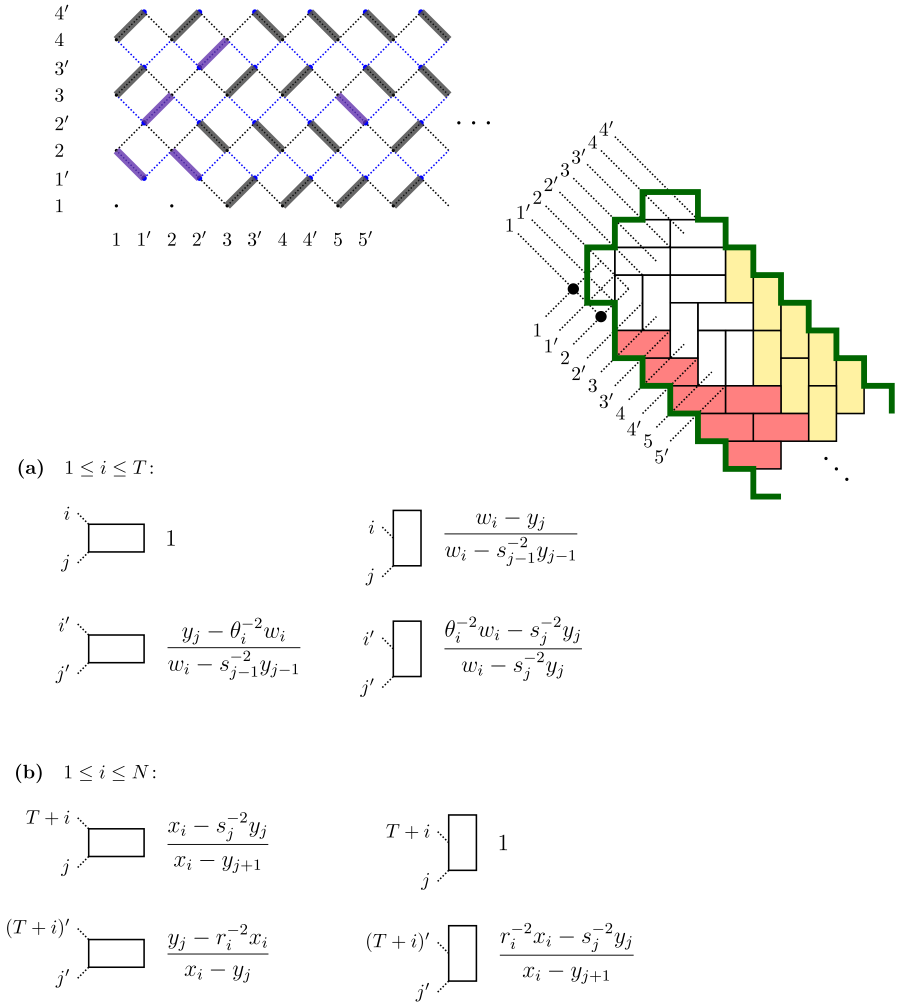


Fig. 18 Mapping from the dimer model in Fig. 14 with weights given in Fig. 16 to domino tilings of a half-strip. The dominoes which are repeated infinitely many times in the down-right direction are shaded

infinity, and set for the variables in the kernel:

$$T \gg N, \qquad t = \lfloor \tau N \rfloor, \quad t' = t + \Delta t, \qquad a = \lfloor \alpha N \rfloor, \quad a' = a + \Delta a, \quad (10.1)$$

where $\tau, \alpha \in \mathbb{R}_{>0}$, $\Delta t, \Delta a \in \mathbb{Z}$ are fixed. Note that since the kernel $K_{\mathcal{A}\mathcal{P}}$ does not depend on T, we just need to take T large enough so that $t, t' \le T$ could grow linearly with N.

Remark 10.1 Along with the bulk limit behavior, dimer models typically possess other interesting scaling limits. In particular, the limit at the edge of the liquid region (cf. Figure 19 below) should bring the Airy kernel or its multiparameter deformations obtained in [18]. We do not anticipate new determinantal kernels at the edge in the case of generic inhomogeneous parameters, but it would be interesting to probe whether special choices of the parameters lead to interesting phase transitions in the edge behavior. We leave this question out of the scope of the present paper.

As we aim to capture a nontrivial lattice limit in the bulk, we may set the parameters $x_i, w_i, \theta_i, y_j, s_j$ of the ascending FG process to be constant outside a finite neighborhood of the global position. Let this neighborhood be of size $L \in \mathbb{Z}_{\ge 1}$, and later (in Sect. 11) we also take the *cutoff* L to infinity. While this restricts the generality of the global limit shape and global fluctuations (such as the Gaussian Free Field fluctuations, cf. [57, 92]), this specialization does not restrict the local lattice behavior. More precisely, set $x_i = x_*$ for all $i = 1, \ldots, N$,

$$w_i = \begin{cases} w_*, & |i - \lfloor \tau N \rfloor| > L; \\ w^{\circ}_{i-\lfloor \tau N \rfloor}, & |i - \lfloor \tau N \rfloor| \le L, \end{cases} \qquad \theta_i = \begin{cases} \theta_*, & |i - \lfloor \tau N \rfloor| > L; \\ \theta^{\circ}_{i-\lfloor \tau N \rfloor}, & |i - \lfloor \tau N \rfloor| \le L, \end{cases} \tag{10.2}$$

and similarly

$$y_j = \begin{cases} y_*, & |j - \lfloor \alpha N \rfloor| > L; \\ y^{\circ}_{j-\lfloor \alpha N \rfloor}, & |j - \lfloor \alpha N \rfloor| \le L, \end{cases} \qquad s_j = \begin{cases} s_*, & |j - \lfloor \alpha N \rfloor| > L; \\ s^{\circ}_{j-\lfloor \alpha N \rfloor}, & |j - \lfloor \alpha N \rfloor| \le L. \end{cases} \tag{10.3}$$

In other words, we have passed to the global parameters $x_*, w_*, \theta_*, y_*, s_*$, and the local sequences $\{w^{\circ}_i\}, \{\theta^{\circ}_i\}, \{y^{\circ}_j\}, \{s^{\circ}_j\}$ with $|i|, |j| \le L$. For convenience, let us also write $w^{\circ}_i = w_*$ for $|i| > L$, and similarly for $y^{\circ}_j, \theta^{\circ}_i, s^{\circ}_j$. Both the global parameters and the local sequences are fixed and do not depend on N.

All the parameters of the ascending FG process must satisfy (6.1), (6.2), and (6.4) in order for the process to be well-defined as a probability distribution (6.8) on sequences of signatures. Under the scaling assumptions (10.2)–(10.3), these conditions read

$$x_* < y_j, \quad w_i < y_j < \theta_i^{-2} w_i < s_j^{-2} y_j, \quad \left| \frac{x_* - s_*^{-2} y_*}{x_* - y_*} \frac{w_* - y_*}{w_* - s_*^{-2} y_*} \right| < 1 - \delta < 1 \tag{10.4}$$

for all i, j. Here we have dropped the assumptions $x_i, w_i, y_j > 0$, cf. Remark 6.2. The ascending FG process does not depend on the parameters r_i, so there are no conditions on the r_i's.

The last inequality in (10.4) is needed for the convergence of the series for the normalizing constant (6.9) in the FG process weights (which in general is guaranteed by (6.4)). Clearly, the convergence of this series is determined only by the global parameters.

By taking x_* sufficiently small (close to $-\infty$), we see that the last inequality in (10.4) follows from $w_* < y_* < s_*^{-2} y_*$ and hence hold automatically. Therefore, we may and will assume that the global parameters satisfy

$$x_* < w_* < y_* < \theta_*^{-2} w_* < s_*^{-2} y_*. \tag{10.5}$$

10.2 Steepest descent

Let us rewrite the correlation kernel $K_{\mathcal{AP}}$ (6.12) in the following form adapted to the scaling regime from Sect. 10.1:

$$\begin{aligned} &K_{\mathcal{AP}}(t, a; t', a') \\ &= \frac{1}{(2\pi \mathbf{i})^2} \oint \oint \frac{\mathcal{E}_N(u)/\mathcal{E}_N(v)\, du\, dv}{u - v} \frac{y_a(1 - s_a^{-2})}{v - y_a} \\ &\quad \frac{1}{u - s_{a'}^{-2} y_{a'}} \frac{\prod_{c=1}^{a'} \frac{u - s_c^{-2} y_c}{u - y_c}}{\prod_{c=1}^{a} \frac{u - s_c^{-2} y_c}{u - y_c}} \frac{\prod_{c=1}^{t'} \frac{u - w_c}{u - \theta_c^{-2} w_c}}{\prod_{c=1}^{t} \frac{u - w_c}{u - \theta_c^{-2} w_c}}, \end{aligned} \tag{10.6}$$

where

$$\mathcal{E}_N(u) := (u - x_*)^{-N} \prod_{k=1}^{N} (u - y_k) \prod_{c=1}^{a} \frac{u - s_c^{-2} y_c}{u - y_c} \prod_{c=1}^{t} \frac{u - w_c}{u - \theta_c^{-2} w_c}.$$

In (10.6) both integration contours are positively oriented simple curves. The u and v contours encircle, respectively, all the points y_*, y_i°, $\theta_*^{-2} w_*$, $(\theta_i^\circ)^{-2} w_i^\circ$ and all the points y_*, y_i°, w_*, w_i°, and no other poles of the integrand except $u = v$. For the latter pole, the u contour is outside the v contour for $\Delta t = t' - t \geq 0$, and inside for $\Delta t < 0$. The integration contours exist thanks to (10.4). The ratios $\prod_{c=1}^{a'} / \prod_{c=1}^{a}$ and $\prod_{c=1}^{t'} / \prod_{c=1}^{t}$ in (10.6) outside of $\mathcal{E}_N$ are finite products (which later will depend only on local sequences of parameters).

Our aim is to perform the steepest descent analysis of $K_{\mathcal{AP}}$ based on critical points of

$$\mathcal{S}_N(u) := \frac{1}{N} \log \mathcal{E}_N(u),$$

following [87, Sections 3.1, 3.2]. Namely, if for all N large enough on the integration contours we have

$$\Re\,(\mathcal{S}_N(u) - \mathcal{S}_N(v)) < 0 \tag{10.7}$$

(here and below $\Re$ and $\Im$ stand for the real and imaginary parts, respectively), then the integral containing $e^{N(\mathcal{S}_N(u) - \mathcal{S}_N(v))}$ goes to zero exponentially with N. We achieve (10.7) by deforming the original u and v integration contours so that they pass through complex conjugate critical points $z, \bar{z}$ of $\mathcal{S}_N(u)$ (first, we need to show that such critical points exist). In the process of deforming the contours to those with (10.7), certain residues will survive and contribute to the limit of $K_{\mathcal{A}\mathcal{P}}$. Observe that this argument works for arbitrary branches of the logarithms in $\log \mathcal{E}_N(u)$.

Since the integration contours in (10.6) can be chosen bounded, we have

$$\begin{aligned}\mathcal{S}_N(u) = &-\log(u - x_*) + (1-\alpha)\log(u - y_*) + \alpha \log(u - y_*/s_*^2) \\ &+ \tau \log(u - w_*) - \tau \log(u - w_*/\theta_*^2) + \text{remainder} =: \mathcal{S}(u) + \text{remainder},\end{aligned} \tag{10.8}$$

where $|\text{remainder}| < C(L)/N$. The constant $C(L)$ independent of N comes from two sources. First, we dropped the integer parts in $a \approx \alpha N$, $t \approx \tau N$ in $\mathcal{E}_N$. Second, in $\mathcal{E}_N$ we have replaced the local parameters $y_{j-\lfloor \alpha N \rfloor}$, $|j| \le L$ (see (10.2)–(10.3)), by y_*, and similarly for $w_i, \theta_i^{-2}, s_j^{-2} y_j$. We see that the critical points of $\mathcal{S}_N(u)$ are close (as $N \to +\infty$) to the critical points of $\mathcal{S}(u)$. One can check that the critical point equation $\mathcal{S}'(u) = 0$ reduces to a cubic polynomial equation in u. Indeed, the terms containing u^4 cancel out because the sum of the coefficients by all the logarithms is zero.

10.3 Moving the contours

Let us fix global parameters satisfying (10.5), and investigate the behavior of the function $\mathcal{S}(u)$ given by (10.8). The cubic polynomial equation for the critical points of $\mathcal{S}(u)$ reads

$$\Big(\alpha y_*(s_*^{-2} - 1) - \tau w_*(\theta_*^{-2} - 1) + y_* - x_*\Big)\, u^3 + c_2 u^2 + c_1 u + c_0 = 0, \tag{10.9}$$

for certain polynomial functions c_0, c_1, c_2 of the global parameters $x_*, w_*, \theta_*, y_*, s_*$ whose explicit expressions we omit for shorter notation.

Definition 10.2 Denote by $\mathcal{L}$ the region in the plane $(\alpha, \tau) \in \mathbb{R}^2_{\ge 0}$ where the discriminant of the cubic (10.9) is negative (see Fig. 19 for an example). In $\mathcal{L}$ the cubic equation has two nonreal complex conjugate roots. Denote by $z = z(\alpha, \tau)$ the root belonging to the upper half plane.

Lemma 10.3 *The map $z\colon \mathcal{L} \to \mathbb{C}$ is a diffeomorphism between $\mathcal{L}$ and the open upper half plane. The region $\mathcal{L}$ is unbounded.*

Proof For each $(\alpha, \tau) \in \mathcal{L}$, there is a unique root $z(\alpha, \tau)$ in the upper half plane, so the map z is injective. Substituting $u = X + iY$ into the cubic equation (10.9) for $\mathcal{S}'(u) = 0$, we may find (α, τ) as rational functions of (X, Y). These rational functions define the map $z^{-1}\colon \mathbb{C} \to \mathcal{L}$.

One can check that the image under z^{-1} of the real line (corresponding to $Y = 0$) is precisely the curve where the discriminant of (10.9) vanishes, i.e., the boundary of $\mathcal{L}$. Moreover, an explicit computation shows that the discriminant of (10.9) has the

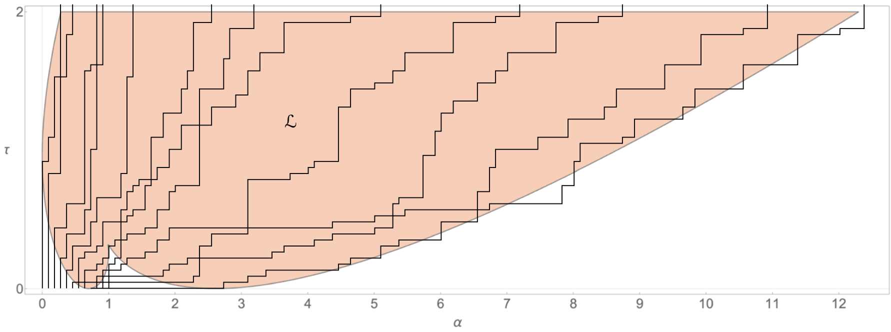


Fig. 19 In the shaded region the discriminant of the cubic equation (10.9) is negative. We also sketch possible lattice path behavior under the FG process. The parameters in the figure are equal to $x_* = \frac{1}{2}$, $w_* = \frac{2}{3}$, $s_* = \frac{1}{2}$, $y_* = \frac{9}{10}$, and $\theta_* = \frac{4}{5}$

form $-(\text{rationalexpression})^2 Y^2$, so it is manifestly negative for all (α, τ) expressed through (X, Y) with $Y \neq 0$. Since Y enters z^{-1} only as Y^2, we see that $z^{-1}: \mathbb{C} \to \mathcal{L}$ is two-to-one and in particular maps the upper half plane to $\mathcal{L}$. As z and z^{-1} are clearly differentiable, z is indeed a diffeomorphism.

To see that the region $\mathcal{L}$ is unbounded, one can check that the boundary point (α, τ) of $\mathcal{L}$ corresponding to $(X, Y) = (x_*, 0)$ is at $\alpha = \tau = +\infty$. □

Fix $(\alpha, \tau) \in \mathcal{L}$. Let us look at the *steepest descent integration contours* $\Im \mathcal{S}(u) = \Im \mathcal{S}(z(\alpha, \tau))$. Since $\mathcal{S}(u)$ involves logarithms, let us now choose their branches to have cuts in the lower half plane, so that $\mathcal{S}(u)$ is holomorphic in the upper half plane and up to the real line except the 5 points (10.5).

Lemma 10.4 *The second derivative $\mathcal{S}''(z(\alpha, \tau))$ at the critical point is nonzero.*

Proof Fix a point $z = X + \mathbf{i}Y$ in the upper half plane, and substitute α and τ as functions of (X, Y) under z^{-1} (see the proof of the previous Lemma 10.3) into $\mathcal{S}''(X + \mathbf{i}Y)$. One can check that the resulting rational expression in $X, Y \in \mathbb{R}$ (with complex coefficients) does not vanish unless $Y = 0$, which is outside the upper half plane. □

By Lemma 10.4, the behavior of $\mathcal{S}(u)$ at the critical point z is exactly quadratic. Therefore, there are four half-contours with $\Im \mathcal{S}(u) = \Im \mathcal{S}(z)$ leaving z. On two of them we have $\Re \mathcal{S}(u) < \Re \mathcal{S}(z)$, $u \neq z$, and on the other two we have $\Re \mathcal{S}(v) > \Re \mathcal{S}(z)$, $v \neq z$. Going around z these half-contours interlace. Let us denote these contours as read in the clockwise direction around z by $\Gamma_1^-, \Gamma_1^+, \Gamma_2^-, \Gamma_2^+$. See Fig. 20 for an illustration.

Lemma 10.5 *We have* $\lim_{R \to +\infty} \Re \mathcal{S}(Re^{\mathbf{i}p}) = 0$, *uniformly in* $p \in [0, \pi]$.

Proof Taylor expanding each logarithm in $\mathcal{S}$ (10.8), we have

$$\Re \log(Re^{\mathbf{i}p} - c) = \log R - \frac{c}{R} \cos p + O(R^{-2}),$$

where $O(R^{-2})$ is uniform in p. Since the coefficients by the logarithms in (10.8) sum to 0, the behavior of $\Re \mathcal{S}(Re^{\mathbf{i}p})$ is not $\log R$ but $O(R^{-1})$, uniformly in p. □

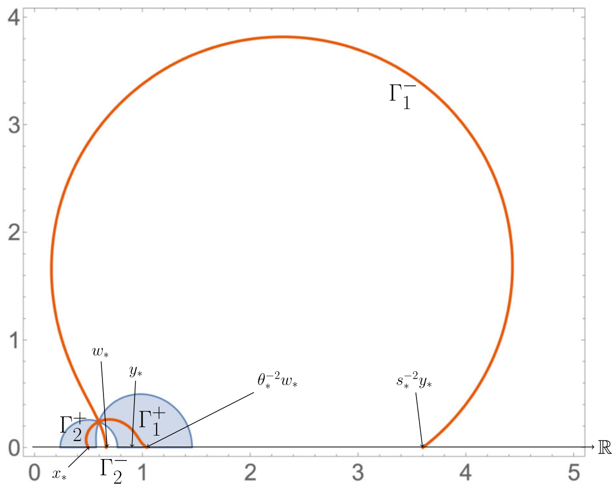


Fig. 20 Steepest descent contours $\Gamma^{\pm}_{1,2}$ in the upper half plane. The four half-contours intersect at the critical point z. In the shaded regions we have $\Re(\mathcal{S}(u) - \mathcal{S}(z)) > 0$. The graph corresponds to $\alpha = \tau = \frac{3}{2}$ and other parameters as in Fig. 19

By Lemma 10.5, the half-contours $\Gamma^{\pm}_{1,2}$ cannot escape to infinity because on them the real part of $\mathcal{S}$ grows to $+\infty$ or decays to $-\infty$. Since the function $\mathcal{S}$ is holomorphic in the upper half plane, these half-contours must end at the real line. More precisely, each of these contours must end at one of the logarithmic singularities (10.5) of $\mathcal{S}$. The signs of $\Re\mathcal{S}$ at these singularities are

$$\Re\mathcal{S}(x_*) = +\infty, \quad \Re\mathcal{S}(w_*) = -\infty, \quad \Re\mathcal{S}(y_*) = (\alpha - 1)\infty,$$
$$\Re\mathcal{S}(\theta_*^{-2}w_*) = +\infty, \quad \Re\mathcal{S}(s_*^{-2}y_*) = -\infty. \tag{10.10}$$

We see that Γ_1^- must end at $s_*^{-2}y_*$; Γ_1^+ must end at y_* if $\alpha > 1$ or at $\theta_*^{-2}w_*$; Γ_2^- must end at y_* if $\alpha < 1$ or at w_*; and Γ_2^+ must end at x_*.

Lemma 10.6 *Assume that the parameters of the ascending FG process are as described in Sect. 10.1. Then*

- *The u contour in (10.6) can be deformed, without picking residues other than at $u = v$, to a positively oriented contour which crosses the real line at $s_*^{-2}y_*$, coincides with Γ_1^- till z, then with Γ_2^- till a small neighborhood of the real line, then crosses $\mathbb{R}$ again between x_* and all y_j. Then the contour extends to the lower half plane symmetrically.*
- *The v contour in (10.6) can be deformed, without picking residues other than at $v = u$, to a positively oriented contour which crosses the real line between $\sup y_j$ and $\inf s_j^{-2}y_j$, in a small neighborhood of $\mathbb{R}$ joins Γ_1^+ and coincides with it till z, then coincides Γ_2^+ till the real line which it crosses again at x_*. Then the contour extends to the lower half plane symmetrically.*

Moreover, on the new contours we have $\Re\,(\mathcal{S}(u) - \mathcal{S}(v)) \le 0$, with equality only for $u = v = z$.

Let us denote the new contours afforded by Lemma 10.6 by Γ_u^{st}, Γ_v^{st}, see Fig. 21 for an example. They are positively oriented simple closed curves in the full complex plane.

Proof of Lemma 10.6 Let the new contours Γ_u^{st}, Γ_v^{st} coincide with the steepest descent ones except in a small neighborhood of $\mathbb{R}$. In the neighborhood of $\mathbb{R}$, let us change the steepest descent contours $\Gamma_{1,2}^{\pm}$ such that $\Re\left(\mathcal{S}(u) - \mathcal{S}(v)\right) \le 0$ still holds on the new contours Γ_u^{st}, Γ_v^{st}. Moreover, we would like the contour deformation from the contours (10.6) to Γ_u^{st}, Γ_v^{st}, to not pick any residues at poles w_i°, y_j°, $(\theta_i^\circ)^{-2}w_i^\circ$, $(s_j^\circ)^{-2}y_j^\circ$. coming from the local parameters (call these the *local residues*). Thanks to (10.10) and the previous statements in this subsection, such a contour deformation does not cross the poles (10.5) coming from the global parameters.

Let us now show the absence of local residues. On the right, the original v contour crossed $\mathbb{R}$ between y_j and $s_j^{-2}y_j$. The steepest descent contour Γ_1^+ crosses $\mathbb{R}$ at y_* or $\theta_*^{-2}w_*$. In the latter case, we let Γ_v^{st} coincide with Γ_1^+. In the former case, we may need to change the contour (in a small neighborhood of $\mathbb{R}$) so that it is still around all the y_j's (this case is illustrated in Fig. 21). As the y_j's are all to the left of $\theta_*^{-2}w_*$ and $\Re\mathcal{S}(\theta_*^{-2}w_*) = +\infty$, the new contour still satisfies $\Re\mathcal{S}(v) > \Re\mathcal{S}(z)$. Since $v = (\theta_i^\circ)^{-2}w_i^\circ$ are not poles of the integrand, this deformation does not pick any local residues. Then the new v contour joins Γ_1^+ and follows it till z, then follows Γ_2^+ till x_* where it crosses the real line. We see that the new v contour is still around all y_j, w_i, and not $s_j^{-2}y_j$, so no local residues are picked.

The argument for the u contour is similar. On the right, the original u contour crossed the real line between $\theta_i^{-2}w_i$ and $s_j^{-2}y_j$, and can be deformed to coincide with Γ_1^- which crosses $\mathbb{R}$ at $s_*^{-2}y_*$ without picking local residues at $(\theta_i^\circ)^{-2}w_i^\circ$ (there are no poles at $u = s_j^{-2}y_j$). On the left, the original u contour crossed $\mathbb{R}$ between x_* and y_j. The steepest descent contour Γ_2^- can cross the real line at w_* or y_*. In the former case, we simply make the new u contour coincide with Γ_2^- and cross $\mathbb{R}$ at w_*. In the latter case, we deform the u contour in a neighborhood of $\mathbb{R}$ so that it still encircles all y_j. As the y_j's are all to the right of w_* and $\Re\mathcal{S}(w_*) = -\infty$, on the new u contour we have $\Re\mathcal{S}(u) < \Re\mathcal{S}(z)$.

We see that the desired contour deformation exists, with $\Re\left(\mathcal{S}(u) - \mathcal{S}(v)\right) \le 0$ on the new contours. ◻

10.4 Asymptotics of the kernel

In the previous Sect. 10.3 we showed how to deform the integration contours for the correlation kernel $K_{\mathcal{AP}}$ (10.6) to the steepest descent contours Γ_u^{st}, Γ_v^{st}. It remains to collect the residues at $u = v$ arising from this deformation. Denote by $\mathcal{I}_{t,a;t',a'}(u,v)$ the integrand in the double contour integral in (10.6).

Lemma 10.7 *The double contour integral in* (10.6) *is equal to*

$$-\frac{1}{2\pi\mathbf{i}}\int_{\bar{z}}^{z} \operatorname*{Res}_{u=v} \mathcal{I}_{t,a;t',a'}(u,v)\,dv + \frac{1}{(2\pi\mathbf{i})^2}\oint_{\Gamma_u^{st}} du \oint_{\Gamma_v^{st}} dv\, \mathcal{I}_{t,a;t',a'}(u,v), \tag{10.11}$$

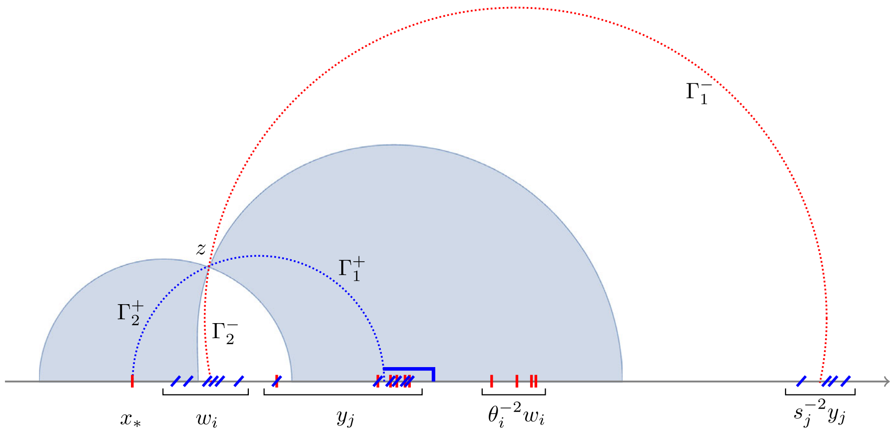


Fig. 21 Deformation of the integration contours in the proof of Lemma 10.6. The shaded region is where $\Re\mathcal{S}(v) > \Re\mathcal{S}(z)$, and the v contour must be inside this region. The u poles on $\mathbb{R}$ are x_*, y_j, and $\theta_i^{-2}w_i$, and the v poles on $\mathbb{R}$ are w_i, y_j, $s_j^{-2}y_j$. We need to modify the new v contour so that in a neighborhood of $\mathbb{R}$ it diverges from Γ_1^+ and encircles all y_j's

where in the single integral the arc from $\bar{z}$ to z is as follows:

- *If $\Delta t = t' - t \geq 0$, the arc crosses the real line to the left of w_* and all w_i°;*
- *If $\Delta t < 0$, the arc crosses the real line between $\theta_*^{-2}w_*, (\theta_i^\circ)^{-2}w_i^\circ$ and $s_*^{-2}y_*, (s_j^\circ)^{-2}y_j^\circ$.*

Proof This follows by considering the contour deformation in two cases. For $\Delta t \geq 0$, the u contour is around the v one, so we pick the residue at $u = v$ and integrate it over the left portion of Γ_v^{st}, which is Γ_2^+ and its symmetric copy in the lower half plane. When $\Delta t < 0$, we integrate minus the residue at $u = v$ over the right portion of the contour Γ_v^{st}. □

Recall that the number of local parameters differing from the global ones participating in (10.11) is at most $2L$, see Sect. 10.1. Therefore, in (10.11) the double contour integral decays to zero. More precisely, this rate of decay is bounded by $C(L) \cdot N^{-\frac{1}{2}}$, where $C(L)$ is independent of N. All of this contribution comes from a small neighborhood of z, and outside of a finite neighborhood of z the decay is exponential in N.

The surviving term in (10.11) given by the single integral is a new determinantal correlation kernel on $\mathbb{Z}^2$:

$$K_{2d}^z(\mathsf{t}, \mathsf{a}; \mathsf{t}', \mathsf{a}') := -\frac{1}{2\pi\mathbf{i}} \int_{\bar{z}}^{z} \operatorname*{Res}_{u=v} \mathcal{I}_{\mathsf{t}+\lfloor\tau N\rfloor,\mathsf{a}+\lfloor\alpha N\rfloor;\mathsf{t}'+\lfloor\tau N\rfloor,\mathsf{a}'+\lfloor\alpha N\rfloor}(u, v)\, dv$$

$$= -\frac{1}{2\pi\mathbf{i}} \int_{\bar{z}}^{z} \frac{y_{\mathsf{a}}^\circ(1 - (s_{\mathsf{a}}^\circ)^{-2})\, dv}{(v - y_{\mathsf{a}}^\circ)(v - (s_{\mathsf{a}'}^\circ)^{-2}y_{\mathsf{a}'}^\circ)} \frac{\prod_{c=-\infty}^{\mathsf{a}'} \frac{v-(s_c^\circ)^{-2}y_c^\circ}{v-y_c^\circ}}{\prod_{c=-\infty}^{\mathsf{a}} \frac{v-(s_c^\circ)^{-2}y_c^\circ}{v-y_c^\circ}} \frac{\prod_{c=-\infty}^{\mathsf{t}'} \frac{v-w_c^\circ}{v-(\theta_c^\circ)^{-2}w_c^\circ}}{\prod_{c=-\infty}^{\mathsf{t}} \frac{v-w_c^\circ}{v-(\theta_c^\circ)^{-2}w_c^\circ}}, \tag{10.12}$$

where $\mathsf{t}, \mathsf{a}, \mathsf{t}', \mathsf{a}' \in \mathbb{Z}$ are fixed and the integration contours are as in Lemma 10.7. Note that the ratios of the products from $-\infty$ in the second line in (10.12) are actually finite.

Recalling the scaling in Sect. 10.1, we see that the kernel K^{z}_{2d} is independent of N and depends on the following data:

- Cutoff $L \in \mathbb{Z}_{\geq 1}$;
- Four local sequences $\{w^\circ_i\}, \{\theta^\circ_i\}, \{y^\circ_j\}, \{s^\circ_j\}, |i|, |j| \leq L$;
- Four global parameters w_*, θ_*, y_*, s_*, where in (10.12) we use the notation $w^\circ_i = w_*$ for $|i| > L$, and similarly for $\theta^\circ_i, y^\circ_j, s^\circ_j$;
- Complex number z in the upper half plane.

We call K^{z}_{2d} the (two-dimensional) *inhomogeneous discrete sine kernel* and discuss it in detail in Sect. 11 below.

Remark 10.8 Note that with the cutoff L, the kernel K^{z}_{2d} defines a determinantal process on the whole plane $\mathbb{Z}^2$, but its parameters $w^\circ_i, \theta^\circ_i, y^\circ_j, s^\circ_j$ only vary in a window of size $2L$. In Sect. 11.1 below we take the limit $L \to +\infty$, and arrive at a fully inhomogeneous kernel with parameters varying in the full plane.

Let us summarize the notation and establish the main result of this section. Fix a cutoff parameter $L \in \mathbb{Z}_{\geq 1}$ and four local sequences $\{w^\circ_i\}, \{\theta^\circ_i\}, \{y^\circ_j\}, \{s^\circ_j\}$, $|i|, |j|$, satisfying $w^\circ_i < y^\circ_j < (\theta^\circ_i)^{-2} w^\circ_i < (s^\circ_j)^{-2} y^\circ_j$ for all i, j. Take global parameters $x_*, w_*, y_*, \theta_*, s_*$ satisfying (10.5), where x_* is sufficiently close to $-\infty$, $w_* \in (\min w_i, \max w_i)$, $y_* \in (\min y_j, \max y_j)$, $\theta_* \in (\min \theta_i, \max \theta_i)$, and $s_* \in (\min s_j, \max s_j)$. This ensures that the ascending FG process (6.8) with the parameters given in (10.2)–(10.3) exists thanks to (10.4). Fix a complex number z in the upper half plane.

Recall from Definition 10.2 the region $\mathcal{L}$ in the plane $(\alpha, \tau) \in \mathbb{R}^2_{\geq 0}$ determined by $x_*, w_*, \theta_*, y_*, s_*$. Let $(\alpha, \tau) = (\alpha(z), \tau(z))$ be the image of our point z under the diffeomorphism from the upper half plane to $\mathcal{L}$ (Lemma 10.3).

We take the scaling (10.1) of T, N, t, t', a, a' determined by (α, τ). Take the ascending FG process and let its parameters $x_i, w_i, \theta_i, y_j, s_j$ behave as defined in (10.2)–(10.3). That is, $x_i = x_*$ are all the same, the parameters w_i, θ_i vary for i in the L-neighborhood of τN, and y_j, s_j vary for j in the L-neighborhood of αN. Outside these neighborhoods the parameters are constant. Adopt the notation $w^\circ_i = w_*$ for $|i| > L$, and similarly for the other three families $\theta^\circ_i, y^\circ_j, s^\circ_j$.

Theorem 10.9 (Theorem 1.11 from Introduction) *Under the scaling and assumptions described before the theorem, the correlation kernel $K_{\mathcal{AP}}$ (given by (6.12) or (10.6)) of the ascending FG process converges to the two-dimensional inhomogeneous discrete sine kernel (10.12):*

$$\lim_{N\to+\infty} K_{\mathcal{AP}}(\mathsf{t} + \lfloor \tau N \rfloor, \mathsf{a} + \lfloor \alpha N \rfloor; \mathsf{t}' + \lfloor \tau N \rfloor, \mathsf{a}' + \lfloor \alpha N \rfloor) = K^{z}_{2d}(\mathsf{t}, \mathsf{a}; \mathsf{t}', \mathsf{a}'),$$

where $\mathsf{t}, \mathsf{a}, \mathsf{t}', \mathsf{a}' \in \mathbb{Z}$ are fixed.

Proof Fix (α, τ) as the image of z under the diffeomorphism from the upper half plane to $\mathcal{L}$. Let us look at the N-dependent kernel $K_{\mathcal{A}\mathcal{P}}$ (10.6) and consider the critical point $z_N(\alpha, \tau)$ (in the upper half plane) of the N-dependent function $\mathcal{S}_N(u)$ (10.8). Compared to $\mathcal{S}(u)$, $\mathcal{S}_N(u)$ may depend on local parameters $w_i^\circ, \theta_i^\circ, y_j^\circ, s_j^\circ$, but recall that the difference is bounded by $C(L)/N$. Therefore, z_N is close to the critical point $z(\alpha, \tau)$ from Definition 10.2.

Let us deform the integration contours in (10.6) to coincide, outside a neighborhood of z_N (which at the same time is a neighborhood of z), with the contours $\Gamma_u^{st}, \Gamma_v^{st}$ described in Lemma 10.6. In this neighborhood, let the contours pass through z_N along the steepest descent directions. Thanks to Lemma 10.6, this deformation of contours does not cross any poles at $x_i, w_i, y_j, \theta_i^{-2} w_i, s_j^{-2} y_j$ which could lead to residues. The only residues which this deformation of contours could produce are at $u = v$, and these residues are accounted for in Lemma 10.7.

After the contour deformation, $K_{\mathcal{A}\mathcal{P}}$ becomes a sum of a single integral from $\bar{z}_N(\alpha, \tau)$ to $z_N(\alpha, \tau)$ and a double integral. In the $N \to +\infty$ limit, the double integral disappears, and the single integral turns into an integral from $\bar{z}$ to z, which is precisely the two-dimensional inhomogeneous sine kernel $K_{2\mathrm{d}}^z$. This completes the proof. □

11 Inhomogeneous discrete sine kernel

In this section we discuss the two-dimensional inhomogeneous discrete sine kernel $K_{2\mathrm{d}}^z$ defined by (10.12), and consider its many degenerations to known correlation kernels. In particular, we prove Theorem 1.12 from Introduction. For simplicity, in this section we drop the "◦" notation from the parameters w_i, θ_i, y_j, s_j of the kernel.

11.1 Definition of the kernel

It is convenient to introduce the following inhomogeneous analogues of power functions to write down the kernel $K_{2\mathrm{d}}^z$:

Definition 11.1 (Inhomogeneous powers) For any two sequences $\mathbf{b} = \{b_i\}_{i\in\mathbb{Z}}$ and $\mathbf{c} = \{c_i\}_{i\in\mathbb{Z}}$, define the following "inhomogeneous powers":

$$\mathcal{P}_{n,n'}(u \mid \mathbf{b}; \mathbf{c}) := \begin{cases} \displaystyle\prod_{j=n+1}^{n'} \frac{u-b_j}{u-c_j}, & n < n'; \\ 1, & n = n'; \\ \displaystyle\prod_{j=n'+1}^{n} \frac{u-c_j}{u-b_j}, & n > n', \end{cases} \qquad n, n' \in \mathbb{Z}. \tag{11.1}$$

We will now define the kernel $K_{2\mathrm{d}}^z$ depending on four sequences of parameters

$$\mathbf{w} = \{w_i\},\ \boldsymbol{\theta} = \{\theta_i\},\ \mathbf{y} = \{y_j\},\ \mathbf{s} = \{s_j\}, \qquad i, j \in \mathbb{Z}, \tag{11.2}$$

and on a point z in the upper half complex plane. Assume that the parameter sequences satisfy

$$\sup_i w_i < \inf_j y_j \le \sup_j y_j < \inf_i \theta_i^{-2} w_i \le \sup_i \theta_i^{-2} w_i < \inf_j s_j^{-2} y_j. \quad (11.3)$$

Definition 11.2 The *two-dimensional inhomogeneous extended sine kernel* is defined as follows:

$$K_{2\mathrm{d}}^{z}(t, a; t', a') = -\frac{1}{2\pi \mathbf{i}} \int_{\bar{z}}^{z} \frac{y_a(1 - s_a^{-2})}{(u - y_a)(u - s_{a'}^{-2} y_{a'})} \mathcal{P}_{a,a'}(u \mid \mathbf{s}^{-2}\mathbf{y}; \mathbf{y})\, \mathcal{P}_{t,t'}(u \mid \mathbf{w}; \boldsymbol{\theta}^{-2}\mathbf{w})\, du, \quad (11.4)$$

where $t, a, t', a' \in \mathbb{Z}$. The integration contour is an arc from $\bar{z}$ to z which crosses the real line

- To the left of all w_i when $\Delta t = t' - t \ge 0$;
- Between $\theta_i^{-2} w_i$ and $s_j^{-2} y_j$ when $\Delta t < 0$.

This integration arc exists thanks to (11.3).

The kernel (11.4) is the same as (10.12), up to changes in notation and the removal of the cutoff parameter $L \in \mathbb{Z}_{\ge 1}$.

Theorem 11.3 (Theorem 1.12 from Introduction) *Under the assumptions* (11.3) *and for any z in the upper half plane with $\Im z > 0$, the kernel $K_{2\mathrm{d}}^{z}$* (11.4) *defines a determinantal random point process on $\mathbb{Z}^2$.*

Proof We show using Theorem 10.9 that the determinantal point process defined by $K_{2\mathrm{d}}^{z}$ arises as a limit of a determinantal random point process coming from an ascending FG process.

Given the data (11.2) satisfying (11.3), pick global parameters $x_* \in (-\infty, \inf w_i)$, $w_* \in (\inf w_i, \sup w_i)$, $y_* \in (\inf y_j, \sup y_j)$, $\theta_* \in (\inf \theta_i, \sup \theta_i)$, and $s_* \in (\inf s_j, \sup s_j)$. For any cutoff $L \in \mathbb{Z}_{\ge 1}$, define truncated local parameter sequences

$$w_i^{(L)} = \begin{cases} w_i, & |i| \le L; \\ w_*, & |i| > L, \end{cases} \quad \theta_i^{(L)} = \begin{cases} \theta_i, & |i| \le L; \\ \theta_*, & |i| > L, \end{cases}$$
$$y_j^{(L)} = \begin{cases} y_j, & |j| \le L; \\ y_*, & |j| > L, \end{cases} \quad s_j^{(L)} = \begin{cases} s_j, & |j| \le L; \\ s_*, & |j| > L. \end{cases} \quad (11.5)$$

Taking x_* smaller if necessary, one can make sure that (10.4) holds. Thus, the ascending FG process (6.8) with global parameters $x_*, w_*, \theta_*, y_*, s_*$ and local sequences (11.5) is well-defined. Take the scaling location (α, τ) corresponding to z as in Lemma 10.3. Applying Theorem 10.9, we see that the kernel $K_{2\mathrm{d}}^{z,(L)}$ with the L-truncated parameter sequences (11.5) is the bulk lattice limit of the kernel $K_{\mathcal{AP}}$ of the FG process. Therefore, in the L-truncated case the kernel $K_{2\mathrm{d}}^{z,(L)}$ indeed defines a stochastic process.

Now observe that for any $t, a, t', a' \in \mathbb{Z}$ and $L > \max\{|t|, |a|, |t'|, |a'|\}$, the matrix element $K^z_{2d}(t, a; t', a')$ (11.4) does not depend on L or the global parameters involved in the truncation (11.5). Therefore, as $L \to +\infty$, the matrix elements of $K^{z,(L)}_{2d}$ stabilize to those of K^z_{2d}. This implies that K^z_{2d} also defines a stochastic process, as desired. □

In fact, our conditions on the parameters w_i, θ_i, y_j, s_j of the inhomogeneous discrete sine kernel are natural in the following sense:

Lemma 11.4 *Conditions (11.3) are equivalent to the fact that the domino weights depending on w_i, θ_i, y_j, s_j (given in Fig. 18,* (a)*) are positive and separated from zero and infinity.*

Proof For fixed i_1, j_1 the positivity of the domino weights depending only on $w_{i_1}, \theta_{i_1}, y_{j_1}, s_{j_1}$ is equivalent to either $w_{i_1} < y_{j_1} < \theta^{-2}_{i_1} w_{i_1} < s^{-2}_{j_1} y_{j_1}$ or $w_{i_1} > y_{j_1} > \theta^{-2}_{i_1} w_{i_1} > s^{-2}_{j_1} y_{j_1}$. Let us pick $i_2 \neq i_1$, $j_2 \neq j_1$, and for $w_{i_2}, \theta_{i_2}, y_{j_2}, s_{j_2}$ we have similarly one of the two strings of inequalities. If, say, $w_{i_1} < y_{j_1} < \theta^{-2}_{i_1} w_{i_1} < s^{-2}_{j_1} y_{j_1}$ but $w_{i_2} > y_{j_2} > \theta^{-2}_{i_2} w_{i_2} > s^{-2}_{j_2} y_{j_2}$, then at $i = i_1$, $j = j_2$ one readily sees that both possibilities

$$w_{i_1} < y_{j_2} < \theta^{-2}_{i_1} w_{i_1} < s^{-2}_{j_2} y_{j_2} \quad \text{or} \quad w_{i_1} > y_{j_2} > \theta^{-2}_{i_1} w_{i_1} > s^{-2}_{j_2} y_{j_2}$$

lead to a contradiction.

Therefore, it must be either $w_i < y_j < \theta^{-2}_i w_i < s^{-2}_j y_j$ or $w_i > y_j > \theta^{-2}_i w_i > s^{-2}_j y_j$ simultaneously for all i, j. If it's the latter, observe that the domino weights are invariant under the simultaneous sign flips $w_i \mapsto -w_i$, $y_j \mapsto -y_j$ for all i, j, which turns the conditions with "$>$" into those with "$<$". Thus, we see that picking the "$<$" sign in all conditions does not restrict the generality. □

Thus, by Theorem 11.3 and Lemma 11.4, the kernel K^z_{2d} defines a *bona fide* stochastic determinantal point process on $\mathbb{Z}^2$ for a maximally generic open family of parameters. Moreover, setting some of the domino weights in Fig. 18, (a) to zero also leads to a stochastic process via a straightforward limit transition. In the rest of this section we compare the kernel K^z_{2d} to similar known kernels, in one and then in two dimensions.

11.2 Discrete sine kernel in one dimension

In one-dimensional slices (corresponding to fixing $t = t' \in \mathbb{Z}$), the process is independent of t, and the kernel (11.4) becomes an inhomogeneous analogue of the discrete sine kernel:

$$K^z_{1d}(a, a') = -\frac{1}{2\pi \mathbf{i}} \int_{\bar{z}}^{z} \frac{y_a(1 - s^{-2}_a)}{(u - y_a)(u - s^{-2}_{a'} y_{a'})} \mathcal{P}_{a,a'}(u \mid \mathbf{s}^{-2}\mathbf{y}; \mathbf{y})\, du. \quad (11.6)$$

The integration contour passes to the left of all y_j.

The kernel K^z_{1d} is clearly not translation invariant, and the density function is given by

$$\rho_a = K^z_{1\mathrm{d}}(a,a) = -\frac{1}{2\pi\mathbf{i}}\int_{\bar z}^{z}\frac{y_a(1-s_a^{-2})\,du}{(u-y_a)(u-s_a^{-2}y_a)} = \frac{1}{2\pi\mathbf{i}}\int_{V_a(\bar z)}^{V_a(z)}\frac{dv}{v} = \frac{\arg V_a(z)}{\pi},$$

where we used the change of variables

$$v = V_a(u) := \frac{u-y_a}{u-s_a^{-2}y_a}. \tag{11.7}$$

This change of variables swaps the lower and upper half planes, hence the minus sign in front of the integral disappears. The integration in v is over an arc crossing the real line to the right of the origin.

In the homogeneous case $y_a = y$, $s_a = s$ for all $a \in \mathbb{Z}$, the change of variables $V_a = V$ does not depend on a, and the density $\rho_a \equiv \rho = \frac{1}{\pi}\arg V(z)$ is constant. We see that (11.6) essentially becomes the usual *discrete sine kernel*:

$$K^z_{1\mathrm{d,\,hom}}(a,a') = \frac{1}{2\pi\mathbf{i}}\int_{V(\bar z)}^{V(z)}\frac{dv}{v^{a'-a+1}} = |V(z)|^{a-a'}\frac{\sin\big(\pi\rho(a'-a)\big)}{\pi(a'-a)}, \quad a,a'\in\mathbb{Z}. \tag{11.8}$$

The factor $|V(z)|^{a-a'}$ is a so-called "gauge transformation", and can be removed from the kernel without changing the determinantal process. The discrete sine kernel in one dimension and the corresponding determinantal point process were obtained in [15] as a bulk limit of Plancherel random partitions. This point process arises from many other discrete determinantal point processes as a lattice (bulk) scaling limit.

11.3 Periodic discrete sine kernel in one dimension

The fully inhomogeneous kernel $K^z_{1\mathrm{d}}(a,a')$ (11.6) on $\mathbb{Z}$ can be specialized to a k-periodic kernel on $\mathbb{Z}$, for any $k \ge 2$ (the case $k=1$ leads to the discrete sine kernel (11.8)). Here let us consider the case with $k=2$, and take a further degeneration. Namely, set

$$y_i = s_i^2 c_i, \qquad c_i = c_0\mathbf{1}_{i\equiv 0 \bmod 2} + c_1\mathbf{1}_{i\equiv 1 \bmod 2},$$

and after that send $s_i \to 0$. This leads to the following kernel:

$$\begin{bmatrix} K_{1\mathrm{d}}(2n,2n') & K_{1\mathrm{d}}(2n,2n'+1) \\ K_{1\mathrm{d}}(2n+1,2n') & K_{1\mathrm{d}}(2n+1,2n'+1)\end{bmatrix} = \frac{1}{2\pi\mathbf{i}}\int_{\bar z}^{z}\begin{bmatrix}\frac{c_0}{1-c_0/u} & c_0 \\ \frac{c_1}{(1-c_0/u)(1-c_1/u)} & \frac{c_1}{1-c_1/u}\end{bmatrix}\Big(\big(1-\frac{c_0}{u}\big)\big(1-\frac{c_1}{u}\big)\Big)^{\Delta n}\frac{du}{u^2}, \tag{11.9}$$

where the integration contour crosses the real line to the left of 0, and $\Delta n = n' - n$. Here the matrix form is just a shorthand for four different integral expressions for the matrix elements of the kernel, depending on the parity. The kernel (11.9) is invariant under translations by $2\mathbb{Z}$, but not by $\mathbb{Z}$ if $c_0 \neq c_1$.

This correlation kernel (11.9) in the $2\mathbb{Z}$ periodic case has appeared (together with its $\mathbb{Z}$-translation invariant extension into the second dimension) in [76, Theorem 3.1] in the study of plane partitions with weights periodic in one direction. See also an extension to the arbitrary $k\mathbb{Z} \times \mathbb{Z}$ periodic case in [77, Theorem 4.2].

Note also that $k\mathbb{Z} \times \mathbb{Z}$ periodic kernels from [77] also arise as particular cases of the determinantal kernels for Gibbs ensembles of nonintersecting paths from [22] or as a bulk limit from the periodic Schur process [16]. Indeed, the setup of [22] allows for fully inhomogeneous parameters in one direction (but requires full $\mathbb{Z}$ translation invariance in the second direction). Therefore, in one-dimensional slices the point processes from [22] produce all our one-dimensional kernels $K^z_{1\text{d}}$ (11.6) with the most general parameters.

11.4 Two-dimensional homogeneous kernel

Let us proceed to discussing two-dimensional kernels. First, let us identify the fully homogeneous case $K^z_{2\text{d}}$ (11.4). That is, let $w_i = w$ for all $i \in \mathbb{Z}$, and similarly for θ, y, s. Recall the change of variables $V_a(u)$ (11.7) which is independent of a in the homogeneous case, and we denote it simply by $V(u)$. We have

$$K^z_{2\text{d, hom}}(t, a; t', a') = \frac{(\text{const})^{\Delta t}}{2\pi \mathbf{i}} \int_{V(\bar{z})}^{V(z)} \left(\frac{v - V(w)}{v - V(\theta^{-2} w)} \right)^{\Delta t} \frac{dv}{v^{\Delta a + 1}}, \quad (11.10)$$

where $\Delta t = t' - t$, $\Delta a = a' - a$, and the integration contour crosses the real line between $V(w)$ and 1 for $\Delta t \geq 0$, and to the left of $V(\theta^{-2} w)$ for $\Delta t < 0$.

The correlation kernel $K^z_{2\text{d, hom}}$ can be identified with the bulk limiting kernel in the liquid phase of the model of random domino tilings of the Aztec diamond, when the dominoes are mapped to a determinantal process on $\mathbb{Z}^2$. The one-dimensional sine kernel for the Aztec diamond model was obtained in [52, Theorem 2.10], and the two-dimensional bulk kernel may be read off from the more general theory of [65], or deduced as a bulk limit from [53, (2.21)].

Moreover, $K^z_{2\text{d, hom}}$ also arises as a particular member of the family of extensions of the one-dimensional discrete sine kernel constructed in [22]. Namely, to get (11.10), one should alternate the "alpha" and the "beta" factors, $(1 - \alpha^+ v)^{-1}$ and $(1 + \beta^+ v)$, in [22, (2)].

Setting $w = y$, that is, $V(w) = 0$, turns the correlation kernel $K^z_{2\text{d, hom}}$ (11.10) (up to a gauge factor which does not change the determinantal process) into the *incomplete beta kernel* introduced in [90]. The incomplete beta kernel is the determinantal kernel of the ergodic translation invariant Gibbs measure on lozenge tilings of the plane (viewed as a determinantal process on $\mathbb{Z}^2$, cf. Remark 9.4), which is unique up to specifying the *slope*. The slope is a two-dimensional real parameter which can be mapped (in our notation) to the point z in the upper half plane. We refer to [65, 95] for further details. Universality of the incomplete beta kernel in the model of uniformly random lozenge tilings in general domains was established recently in [2].

11.5 Other dimer models with periodic weights

We see that our two-dimensional inhomogeneous sine kernel K_{2d}^{z} (Definition 11.2) generalizes the bulk lattice distributions arising in domino and lozenge tilings. Our generalization allows for inhomogeneous parameters in both lattice directions.

By taking the parameters to be periodic (as in Sect. 11.3 but in both directions), one gets doubly periodic determinantal kernels with periods $k\mathbb{Z} \times m\mathbb{Z}$ for arbitrary $k, m \geq 1$. When both $k, m > 1$, the explicit form of the doubly periodic kernels is new. For $m = 1$, the $k\mathbb{Z} \times \mathbb{Z}$ periodic kernels appeared in [16, 22, 76, 77].

While in principle our doubly periodic kernels fall into the general framework of [65], rewriting the general double integral formula for the kernel from [65, Theorem 4.3] in an arc integral form as in K_{2d}^{z} (11.4) is a nontrivial transformation. Moreover, the fully inhomogeneous (non-periodic) kernels in both directions do not immediately follow from the general theory of [65]. It might be possible to obtain non-periodic fully inhomogeneous kernels as limits of the periodic ones from [65], but the double integral form of the latter kernels does not seem well-suited for such a limit transition.

Observe that our two-dimensional inhomogeneous discrete sine kernel K_{2d}^{z} corresponds only to the liquid (also called rough) phase of our path ensembles / domino tilings coming from the ascending FG processes. In the rough phase one expects the variance of the height difference to grow logarithmically with the distance [65]. This behavior is proven in many cases, and the fluctuations are identified with the Gaussian Free Field, cf. [43, 57, 92].

The liquid phase local behavior described by K_{2d}^{z} should be contrasted with that in the gaseous (also called smooth) phase in which the height differences have bounded variance [65]. The gaseous phase is present in doubly periodic (in particular, $2\mathbb{Z} \times 2\mathbb{Z}$ periodic) domino tilings [11, 25, 28], see also, e.g., [24] for a discussion of the case of lozenge tilings. We see that our dimer edge weights (see Fig. 18) do not produce gaseous phases. This is because our weights are *not* fully generic like in, e.g., [25], and instead depend on the parameters in quite a special way. In particular, in the $2\mathbb{Z} \times 2\mathbb{Z}$ periodic case we have verified that the domino weights are gauge equivalent (in the sense of [58, Section 3.10]), in a nontrivial way, to weights periodic in only one direction. This seems to be the reason for not seeing gaseous phases in the bulk of ascending FG processes.

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Part IV Appendix

A Formulas for F_λ and G_λ

Here we employ the row operators (defined in Sect. 2.3) to get explicit formulas for the partition functions F_λ and G_λ of the free fermion six vertex model, and thus prove Theorems 3.9 and 3.10. This Appendix accompanies Sect. 3 and employs algebraic Bethe Ansatz type computations. They follow [19, Section 4.5] (but are more involved in the case of G_λ), see also Part VII and in particular Appendix VII.2 of [56].

A.1 Proof of Theorem 3.9

A.1.1 Recalling the notation

Throughout this subsection we fix a signature $\lambda = (\lambda_1, \ldots, \lambda_N \geq 0)$ with N parts, and sequences

$$\mathbf{x} = (x_1, \ldots, x_N), \qquad \mathbf{y} = (y_1, y_2, \ldots), \qquad \mathbf{r} = (r_1, \ldots, r_N), \qquad \mathbf{s} = (s_1, s_2, \ldots).$$

Recall (Definition 3.3) that the function $F_\lambda(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s})$ is the partition function of the free fermion six vertex model with weights $\widehat{W}$ (2.4) and with boundary conditions determined by λ.

In this subsection we prove Theorem 3.9 stating that F_λ is given by the determinantal expression (3.12) involving the functions $\varphi_k(x)$ (3.11). For convenience, let us explicitly reproduce the desired formula here:

$$F_\lambda(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s}) = \left(\prod_{i=1}^{N} x_i (r_i^{-2} - 1) \prod_{1 \leq i < j \leq N} \frac{r_i^{-2} x_i - x_j}{x_i - x_j} \right) \times \det \left[\frac{1}{y_{\lambda_j + N - j + 1} - x_i} \prod_{m=1}^{\lambda_j + N - j} \frac{y_m - s_m^2 x_i}{s_m^2 (y_m - x_i)} \right]_{i,j=1}^{N}. \tag{A.1}$$

For the proof we will need the row operators $\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}$ defined by (2.22)–(2.23). These operators are built from the weights $\widehat{W}$, depend on two numbers x, r and the sequences $\mathbf{y}$, $\mathbf{s}$, and act (from the right) on tensor products of two-dimensional spaces $V^{(k)} = \mathrm{span}\{e_0^{(k)}, e_1^{(k)}\} \simeq \mathbb{C}^2$. To the signature λ we associate the element $e_{\mathcal{S}(\lambda)}$ in the (formal) infinite tensor product $V^{(1)} \otimes V^{(2)} \otimes \ldots$, where we take $e_1^{(k)}$ in the k-th place if and only if $k \in \mathcal{S}(\lambda)$ and $e_0^{(k)}$ otherwise, see Sect. 3.1. For example, the empty signature $\varnothing$ (which has 0 parts) corresponds to $e_\varnothing = e_0^{(1)} \otimes e_0^{(2)} \otimes \ldots$.

By Proposition 3.4, $F_\lambda(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s})$ is the coefficient of $e_{\mathcal{S}(\lambda)}$ in $e_\varnothing \widehat{B}(x_N, r_N) \ldots \widehat{B}(x_1, r_1)$, and for the proof of Theorem 3.9 we proceed to evaluate this coefficient. One of our main tools is the Yang–Baxter equation stated as a family of commutation relations between the operators (see Proposition 2.5).

A.1.2 Action on a tensor product of two spaces

The crucial part of the argument is to consider the action of $\widehat{B}(x_N, r_N) \dots \widehat{B}(x_1, r_1)$ on a tensor product of two spaces, $V_1 \otimes V_2$. Using the second identity from (2.23), namely, $(v_1 \otimes v_2)\widehat{B} = v_1\widehat{D} \otimes v_2\widehat{B} + v_1\widehat{B} \otimes v_2\widehat{A}$, we see that

$$\widehat{B}(x_N, r_N) \dots \widehat{B}(x_1, r_1) = \sum_{\mathcal{I} \subseteq \{1,\dots,N\}} X_{\mathcal{I}}(\mathbf{x}; \mathbf{r}) \otimes Y_{\mathcal{I}}(\mathbf{x}; \mathbf{r}), \tag{A.2}$$

where

$$X_{\mathcal{I}}(\mathbf{x}; \mathbf{r}) = X_N(\mathcal{I}; x_N, r_N) X_{N-1}(\mathcal{I}; x_{N-1}, r_{N-1}) \dots X_1(\mathcal{I}; x_1, r_1),$$
$$Y_{\mathcal{I}}(\mathbf{x}; \mathbf{r}) = Y_N(\mathcal{I}; x_N, r_N) Y_{N-1}(\mathcal{I}; x_{N-1}, r_{N-1}) \dots Y_1(\mathcal{I}; x_1, r_1),$$
$$X_i(\mathcal{I}; x_i, r_i) = \begin{cases} \widehat{D}(x_i, r_i), & i \in \mathcal{I}; \\ \widehat{B}(x_i, r_i), & i \notin \mathcal{I}, \end{cases} \qquad Y_i(\mathcal{I}; x_i, r_i) = \begin{cases} \widehat{B}(x_i, r_i), & i \in \mathcal{I}; \\ \widehat{A}(x_i, r_i), & i \notin \mathcal{I}. \end{cases}$$

Now, using the commutation relations (2.26)–(2.27) from Proposition 2.5, we move all the operators $\widehat{B}$ to the right in both $X_{\mathcal{I}}$ and $Y_{\mathcal{I}}$, which allows to rewrite (A.2) as

$$\begin{aligned} \sum_{\substack{I \cup J = \{1,\dots,N\} \\ I' \cup J' = \{1,\dots,N\}}} & c_{I;I'}(\mathbf{x}; \mathbf{r})\, \widehat{D}(x_{j_{N-k}}, r_{j_{N-k}}) \dots \widehat{D}(x_{j_1}, r_{j_1}) \widehat{B}(x_{i_k}, r_{i_k}) \dots \widehat{B}(x_{i_1}, r_{i_1}) \\ & \otimes \widehat{A}(x_{j'_{N-m}}, r_{j'_{N-m}}) \dots \widehat{A}(x_{j'_1}, r_{j'_1}) \widehat{B}(x_{i'_m}, r_{i'_m}) \dots \widehat{B}(x_{i'_1}, r_{i'_1}), \end{aligned} \tag{A.3}$$

for some rational functions $c_{I;I'}(\mathbf{x}; \mathbf{r})$, where we have denoted $|I| = k$ and $|I'| = m$, defined $J = \{1, \dots, N\} \setminus I$ and $J' = \{1, \dots, N\} \setminus I'$, and ordered the indices such that $i_\alpha < i_\beta$, $i'_\alpha < i'_\beta$, $j_\alpha < j_\beta$, and $j'_\alpha < j'_\beta$ for all $\alpha < \beta$. Here we also employed the commutativity of $\widehat{A}$ (2.24) and $\widehat{D}$ (2.28). In fact, here one can already see from (2.26)–(2.27) that $m = N - k$, but we will get this relation (and a stronger relation between the sets I, I', J, J') in the next Lemma A.2.

Remark A.1 Let us make an important observation about the coefficients $c_{I;I'}(\mathbf{x}; \mathbf{r})$. Namely, these coefficients are computed using only the commutation relations for the operators $\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}$, and we argue that the $c_{I;I'}(\mathbf{x}; \mathbf{r})$'s do not depend on the order of applying the commutation relations. This property is based on the fact that for generic parameters (x, r), there exists a representation of $\begin{bmatrix} \widehat{A}(x, r) & \widehat{B}(x, r) \\ \widehat{C}(x, r) & \widehat{D}(x, r) \end{bmatrix}$ subject to the same commutation relations, and a highest weight vector (annihilated by $\widehat{C}$ and an eigenfunctions of $\widehat{A}, \widehat{D}$) v_0 in that representation, such that vectors $\Big(\prod_{k \in \mathcal{K}} \widehat{B}(x_k, r_k)\Big)\mathsf{v}_0$, with $\mathcal{K}$ ranging over all subsets of $\{1, 2, \dots, N\}$, are linearly independent. This fact is a corollary of [41, Lemma 14]: our operators are based on the free fermion six vertex weights, and the cited paper deals with more general eight vertex case.

Therefore, if we apply the commutation relations in two ways and get different coefficients $c_{I;I'}(\mathbf{x};\mathbf{r})$ in (A.3), then we can apply these commutation relations in the above highest weight representation, which contradicts the linear independence.

Lemma A.2 *We have $c_{I;I'}(\mathbf{x};\mathbf{r})=0$ if $I\cap I'\neq\varnothing$ or $J\cap J'\neq\varnothing$.*

Proof The two claims with $I\cap I'\neq\varnothing$ and $J\cap J'\neq\varnothing$ are analogous, so we only prove the first one.

Suppose $I\cap I'\neq\varnothing$. Since the operators $\widehat{B}(x,r)$ commute up to a scalar factor (see (2.25)), we may assume that $I\cap I'\ni N$ by permuting terms in the left-hand side of (A.2).

Observe that no summand in (A.2) with $X_N(\mathcal{I};x_N,r_N)=\widehat{D}(x_N,r_N)$ (i.e., $N\in\mathcal{I}$) contributes to a nonzero value of $c_{I;I'}(\mathbf{x},\mathbf{r})$. Indeed, in this case the operator $\widehat{D}(x_N,r_N)$ is the leftmost term in $X_{\mathcal{I}}$, and thus it does not get involved in the commutation relations of the form (2.26), which means that one cannot obtain $\widehat{B}(x_N,r_N)$ from this term. Similarly, no summand in (A.2) with $Y_N(\mathcal{I};x_N,r_N)=\widehat{A}(x_N,r_N)$ (i.e., $N\notin\mathcal{I}$) contributes to a nonzero value of $c_{I;I'}(\mathbf{x};\mathbf{r})$.

However, for any $\mathcal{I}\subset\{0,1\}^N$ we either have $X_N(\mathcal{I};x_N,r_N)=\widehat{D}(x_N,r_N)$ or $Y_N(\mathcal{I};x_N,r_N)=\widehat{A}(x_N,r_N)$, and so we cannot obtain $\widehat{B}(x_N,r_N)$ in both tensor factors. Therefore, terms with $I\cap I'\neq\varnothing$ are zero. □

We see that in (A.3) it must be $I=J'$ and $I'=J$, and we may abbreviate $c_{I;I'}=c_I$. We thus rewrite (A.2)–(A.3) as

$$\begin{aligned}&\widehat{B}(x_N,r_N)\dots\widehat{B}(x_1,r_1)\\&\quad=\sum_{I\cup J=\{1,\dots,N\}}c_I(\mathbf{x};\mathbf{r})\,\widehat{D}(x_{j_{N-k}},r_{j_{N-k}})\dots\widehat{D}(x_{j_1},r_{j_1})\widehat{B}(x_{i_k},r_{i_k})\dots\widehat{B}(x_{i_1},r_{i_1})\\&\qquad\otimes\widehat{A}(x_{i_k},r_{i_k})\dots\widehat{A}(x_{i_1},r_{i_1})\widehat{B}(x_{j_{N-k}},r_{j_{N-k}})\dots\widehat{B}(x_{j_1},r_{j_1}).\end{aligned}\tag{A.4}$$

We will now evaluate the coefficients $c_I(\mathbf{x};\mathbf{r})$. First, set $I=\{N-k+1,N-k+2,\dots,N\}$. Then the operator

$$\begin{aligned}&\widehat{D}(x_{N-k},r_{N-k})\dots\widehat{D}(x_1,r_1)\widehat{B}(x_N,r_N)\dots\widehat{B}(x_{N-k+1},r_{N-k+1})\\&\quad\otimes\widehat{A}(x_N,r_N)\dots\widehat{A}(x_{N-k+1},r_{N-k+1})\widehat{B}(x_{N-k},r_{N-k})\dots\widehat{B}(x_1,r_1)\end{aligned}\tag{A.5}$$

might come from (A.2) only for $\mathcal{I}=\{1,2,\dots,N-k\}$, in which case

$$\begin{aligned}&X_{\mathcal{I}}(\mathbf{x};\mathbf{r})\otimes Y_{\mathcal{I}}(\mathbf{x};\mathbf{r})\\&\quad=\widehat{B}(x_N,r_N)\dots\widehat{B}(x_{N-k+1},r_{N-k+1})\widehat{D}(x_{N-k},r_{N-k})\dots\widehat{D}(x_1,r_1)\\&\quad\otimes\widehat{A}(x_N,r_N)\dots\widehat{A}(x_{N-k+1},r_{N-k+1})\widehat{B}(x_{N-k},r_{N-k})\dots\widehat{B}(x_1,r_1).\end{aligned}$$

In the first term, we use the commutation relation (2.26) to place the $\widehat{D}$ operators on the left and extract the coefficient $c_I(\mathbf{x};\mathbf{r})$ of (A.5). We have thus established:

Lemma A.3 *For $I = I_k := \{N-k+1, N-k+2, \ldots, N\}$, the rational function c_I is equal to*

$$c_{I_k}(\mathbf{x}; \mathbf{r}) = \prod_{i=1}^{N-k} \prod_{j=N-k+1}^{N} \frac{r_i^{-2} x_i - x_j}{x_i - x_j}.$$

We are now in a position to compute $c_I(\mathbf{x}; \mathbf{r})$ for arbitrary $I \subset \{1, \ldots, N\}$ of size k (where k is also arbitrary) by permuting the $\widehat{B}$ operators in the left-hand side of (A.2) thanks to the commutation relation (2.25). For each such I, let σ be a permutation of $\{1, \ldots, N\}$ which is increasing on the intervals $\{1, \ldots, N-k\}$ and $\{N-k+1, \ldots, N\}$, and sends $\{N-k+1, N-k+2, \ldots, N\}$ to I.

Lemma A.4 *With the above notation, we have*

$$c_I(\mathbf{x}; \mathbf{r}) = \operatorname{sgn}(\sigma) \prod_{1 \le i < j \le N} \frac{r_i^{-2} x_i - x_j}{x_i - x_j} \prod_{\substack{i, j \in I \\ i < j}} \left(\frac{r_i^{-2} x_i - x_j}{x_i - x_j} \right)^{-1} \prod_{\substack{i, j \notin I \\ i < j}} \left(\frac{r_i^{-2} x_i - x_j}{x_i - x_j} \right)^{-1}. \quad (A.6)$$

Proof The claim follows from the fact that

$$c_I(\mathbf{x}; \mathbf{r}) = c_{I_k}(\sigma(\mathbf{x}); \sigma(\mathbf{r})) \prod_{\substack{j \le N-k,\, i \ge N-k+1 \\ \sigma(i) < \sigma(j)}} \frac{r_{\sigma(i)}^{-2} x_{\sigma(i)} - x_{\sigma(j)}}{r_{\sigma(j)}^{-2} x_{\sigma(j)} - x_{\sigma(i)}},$$

which in turn holds thanks to (2.25) via induction on the length of the permutation σ (which is the minimal number of elementary transpositions required to represent σ as their product). Then we can further simplify:

$$\begin{aligned} & c_{I_k}(\sigma(\mathbf{x}); \sigma(\mathbf{r})) \prod_{\substack{j \le N-k,\, i \ge N-k+1 \\ \sigma(i) < \sigma(j)}} \frac{r_{\sigma(i)}^{-2} x_{\sigma(i)} - x_{\sigma(j)}}{r_{\sigma(j)}^{-2} x_{\sigma(j)} - x_{\sigma(i)}} \\ &= \prod_{i \in I,\, j \notin I} \frac{x_i - r_j^{-2} x_j}{x_i - x_j} \prod_{i \in I,\, j \notin I\, i < j} \frac{x_j - r_i^{-2} x_i}{x_i - r_j^{-2} x_j} \\ &= \prod_{i \in I,\, j \notin I} (x_i - x_j)^{-1} \prod_{i, j \in I,\, i < j} (x_j - r_i^{-2} x_i)^{-1} \\ &\quad \prod_{i, j \notin I,\, i < j} (x_j - r_i^{-2} x_i)^{-1} \prod_{1 \le i < j \le N} (x_j - r_i^{-2} x_i), \end{aligned}$$

which leads to the desired right-hand side of (A.6). The signature of the permutation σ arises by turning $x_i - x_j$ into $x_j - x_i$ for each pair $i \notin I,\ j \in I$ with $i > j$. ◻

A.1.3 Completing the proof

We are now in a position to prove the determinantal formula (A.1), which finalizes the proof of Theorem 3.9. The goal is to express the coefficient of $e_{\mathcal{S}(\lambda)}$ in $e_\varnothing \widehat{B}(x_N, r_N) \dots \widehat{B}(x_1, r_1)$. We are going to repeatedly apply identity (A.4) (with the coefficients $c_I(\mathbf{x}; \mathbf{r})$ given by (A.6)) to vectors of the form $e_0^{(m)} \otimes v$, $m = 1, 2, \dots$.

Observe that $e_0^{(m)} \widehat{B}(x, r) \widehat{B}(x', r') = 0$. Therefore, any nonzero summand in (A.4) must have $|I| \le 1$. Moreover, when $I = \{i\}$ has one element, any such nonzero contribution to the coefficient of $e_{\mathcal{S}(\lambda)}$ should have $i \in \mathcal{S}(\lambda) = \{\lambda_N + 1, \lambda_{N-1} + 2, \dots, \lambda_1 + N\}$. Therefore, each step of the repeated application of (A.4) for which we choose $|I| = 1$ corresponds to a number from 1 to N (indicating which element of $\mathcal{S}(\lambda)$ is selected), and these numbers must be distinct. We encode this information by a permutation $\tau \in \mathfrak{S}_N$. Using the facts that

$$c_\varnothing(\mathbf{x}; \mathbf{r}) = 1, \qquad c_{\{k\}}(\mathbf{x}; \mathbf{r}) = (-1)^{N-k} \prod_{i=1}^{k-1} \frac{r_i^{-2} x_i - x_k}{x_i - x_k} \prod_{j=k+1}^{N} \frac{r_k^{-2} x_k - x_j}{x_k - x_j},$$

$$e_0^{(m)} \widehat{A}(x, r) = e_0^{(m)}, \qquad e_0^{(m)} \widehat{B}(x, r) = \frac{x(1 - r^2)}{r^2 (y_m - x)}\, e_1^{(m)},$$

$$e_0^{(m)} \widehat{D}(x, r) = \frac{y_m - s_m^2 x}{s_m^2 (y_m - x)}\, e_0^{(m)},$$

we see that the coefficient of $e_{\mathcal{S}(\lambda)}$ in $e_\varnothing \widehat{B}(x_N, r_N) \dots \widehat{B}(x_1, r_1)$ is equal to

$$\prod_{1 \le i < j \le N} \frac{r_i^{-2} x_i - x_j}{x_i - x_j} \sum_{\tau \in \mathfrak{S}_N} \operatorname{sgn}(\tau) \prod_{k=1}^{N} \left(\frac{x_{\tau(k)} (r_{\tau(k)}^{-2} - 1)}{y_{\lambda_k + N - k + 1} - x_{\tau(k)}} \prod_{m=1}^{\lambda_k + N - k} \frac{y_m - s_m^2 x_{\tau(k)}}{s_m^2 (y_m - x_{\tau(k)})} \right).$$

Note that the prefactor $\prod_{1 \le i < j \le N} \frac{r_i^{-2} x_i - x_j}{x_i - x_j}$ arises by taking the product of the $c_{\{k\}}$'s over all $k = 1, \dots, N$, but in this product for each next term the number N of variables decreases by one. Therefore, we end up with a product over $i < j$ instead of over all pairs $i \ne j$. This completes the proof of Theorem 3.9.

A.2 Proof of Theorem 3.10

A.2.1 Recalling the notation

Throughout this subsection we fix $M, N \ge 1$, a signature $\lambda = (\lambda_1 \ge \dots \ge \lambda_N \ge 0)$ with N parts, and sequences of complex parameters

$$\mathbf{x} = (x_1, \dots, x_M), \qquad \mathbf{y} = (y_1, y_2, \dots), \qquad \mathbf{r} = (r_1, \dots, r_M), \qquad \mathbf{s} = (s_1, s_2, \dots).$$

Recall from Definition 3.2 the function $G_\lambda(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s}) = G_{\lambda/0^N}(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s})$ which is the partition function of the free fermion six vertex model with weights W (2.3) and with boundary conditions determined by λ.

Our aim is to prove Theorem 3.10 which gives an explicit formula for G_λ (3.14) in terms of a sum over a pair of permutations. The argument is longer than in the case of F_λ from Appendix A.1 but also involves manipulations with row operators. Namely, we utilize the operators A, B, C, D given by (2.8)–(2.9). They are built from the vertex weights W and depend on x, r and the sequences $\mathbf{y}, \mathbf{s}$. These operators act (from the left) on tensor products of two-dimensional spaces $V^{(k)} = \operatorname{span}\{e_0^{(k)}, e_1^{(k)}\} \simeq \mathbb{C}^2$, where $k \geq 1$. Recall (Sect. 3.1) that to λ we associate the vector $e_{\mathcal{S}(\lambda)}$ in the finitary subspace $\mathcal{V}$ of the infinite tensor product $V^{(1)} \otimes V^{(2)} \otimes \ldots$, where we take $e_1^{(k)}$ in the k-th place if and only if $k \in \mathcal{S}(\lambda)$ and $e_0^{(k)}$ otherwise, see Sect. 3.1. Let us also set

$$e_{[1,N]} = e_1^{(1)} \otimes \ldots \otimes e_1^{(N)} \otimes e_0^{(N+1)} \otimes e_0^{(N+2)} \otimes \ldots . \tag{A.7}$$

Equip all tensor products of the spaces $V^{(k)}$ with the inner product defined by $\langle e_{\mathcal{T}}, e_{\mathcal{T}'}\rangle = \mathbf{1}_{\mathcal{T}=\mathcal{T}'}$ (here we use the notation $e_{\mathcal{T}}$ as in (3.2)). Then by Proposition 3.4 we have

$$G_\lambda(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s}) = \left\langle e_{\mathcal{S}(\lambda)}, D(x_M, r_M)\ldots D(x_2, r_2)D(x_1, r_1)e_{[1,N]}\right\rangle .$$

We will compute the above coefficient of $e_{\mathcal{S}(\lambda)}$ in the action of the product of the D operators using the Yang–Baxter equation stated in Proposition 2.4 as a series of commutation relations between the operators A, B, C, and D.

Remark A.5 Sometimes, to shorten some formulas in the proofs, we will use notation A_i, B_i, C_i, or D_i for $A(x_i, r_i)$, $B(x_i, r_i)$, $C(x_i, r_i)$, and $D(x_i, r_i)$, respectively.

A.2.2 Action of *D* operators on a two-fold tensor product

The next two statements, Lemmas A.6 and A.7, are parallel to the computations with the row operators performed in Appendix A.1.2 in the proof of the formula for F_λ.

Lemma A.6 *Let $\sigma \in \mathfrak{S}_M$ be a permutation. Then*

$$\begin{aligned} &C(x_{\sigma(M)}, r_{\sigma(M)})\ldots C(x_{\sigma(1)}, r_{\sigma(1)}) \\ &\quad = C(x_M, r_M)\ldots C(x_1, r_1) \prod_{\substack{1\le i<j\le M \\ \sigma(j)<\sigma(i)}} \frac{r_{\sigma(j)}^{-2}x_{\sigma(j)} - x_{\sigma(i)}}{r_{\sigma(i)}^{-2}x_{\sigma(i)} - x_{\sigma(j)}}. \end{aligned}$$

Proof This is proven by induction on the length of the permutation σ using the commutation relation (2.11) between the C operators. ◻

Lemma A.7 *As operators on a tensor product of two spaces $V_1 \otimes V_2$, we have*

$$
\begin{aligned}
&D(x_M, r_M)D(x_{M-1}, r_{M-1}) \dots D(x_1, r_1) \\
&\quad = \sum_{\mathcal{I} \subseteq \{1,\dots,M\}} \left(\prod_{i \in \mathcal{I},\, j \notin \mathcal{I}} \frac{r_i^{-2} x_i - x_j}{r_i^{-2} x_i - r_j^{-2} x_j} \right) B(x_{i_k}, r_{i_k}) \dots B(x_{i_1}, r_{i_1}) \\
&\qquad D(x_{j_{M-k}}, r_{j_{M-k}}) \dots D(x_{j_1}, r_{j_1}) \\
&\qquad \otimes D(x_{j_{M-k}}, r_{j_{M-k}}) \dots D(x_{j_1}, r_{j_1}) C(x_{i_k}, r_{i_k}) \dots C(x_{i_1}, r_{i_1}).
\end{aligned}
\tag{A.8}
$$

Here $\mathcal{I} = (i_1 < \dots < i_k)$ and $\mathcal{J} = \{1, \dots, M\} \setminus \mathcal{I} = (j_1 < \dots < j_{M-k})$.

Proof In the proof we use the shorthand notation for the operators from Remark A.5. By the last identity in (2.9), the action of $D_M \dots D_1$ on $V_1 \otimes V_2$ is given by

$$
\sum_{\mathcal{K} \subseteq \{1,\dots,M\}} X_{\mathcal{K}} \otimes Y_{\mathcal{K}}, \tag{A.9}
$$

where $X_{\mathcal{K}} = X_M(\mathcal{K}) \dots X_1(\mathcal{K})$, $Y_{\mathcal{K}} = Y_M(\mathcal{K}) \dots Y_1(\mathcal{K})$, with

$$
X_i(\mathcal{K}) = \begin{cases} B_i, & i \in \mathcal{K}; \\ D_i, & i \notin \mathcal{K}, \end{cases} \qquad Y_i(\mathcal{K}) = \begin{cases} C_i, & i \in \mathcal{K}; \\ D_i, & i \notin \mathcal{K}. \end{cases}
$$

Next, by repeated use of relations (2.15) and (2.17), the sum (A.9) can be expressed in the form

$$
\sum_{I, I' \subseteq \{1,\dots,M\}} h_{I;I'}(\mathbf{x}; \mathbf{r})\, B_{i_k} \dots B_{i_1} D_{j_{M-k}} \dots D_{j_1} \otimes D_{i'_m} \dots D_{i'_1} C_{j'_{M-m}} \dots C_{j'_1}, \tag{A.10}
$$

where $h_{I;I'}(\mathbf{x}; \mathbf{r})$ are rational functions in $\mathbf{x} = (x_1, \dots, x_M)$ and $\mathbf{r} = (r_1, \dots, r_M)$, and the indices are

$$
\begin{aligned}
I &= (i_1 < \dots < i_k), & I' &= (i'_1 < \dots < i'_m), \\
J = I^c &= (j_1 < \dots < j_{M-k}), & J' = (I')^c &= (j'_1 < \dots < j'_{M-m}).
\end{aligned}
$$

By looking at relations (2.15), (2.17) closer, one can already see that $m = M - k$ in (A.10). By Remark A.1, the coefficients $h_{I;I'}(\mathbf{x}; \mathbf{r})$ are independent of the order in which we apply the commutation relations between the operators A, B, C, D to get from (A.9) to (A.10).

By the same argument as in Lemma A.2, one can show that $h_{I;I'}(\mathbf{x}; \mathbf{r}) = 0$ if $I \cap I' \neq \varnothing$ or $J \cap J' \neq \varnothing$. Thus, it must be that $I = J'$ and $J = I'$, and we may rewrite $h_I(\mathbf{x};\mathbf{r}) = h_{I;I'}(\mathbf{x};\mathbf{r})$. This implies that we may write (A.10) as

$$\sum_{I\cup J=\{1,\dots,M\}} h_I(\mathbf{x};\mathbf{r})\, B_{i_k}\dots B_{i_1} D_{j_{M-k}}\dots D_{j_1}\otimes D_{j_{M-k}}\dots D_{j_1} C_{i_k}\dots C_{i_1}. \tag{A.11}$$

It remains to evaluate the coefficients $h_I(\mathbf{x};\mathbf{r})$ in (A.11). This is simpler than for the case of F_λ considered in Appendix A.1.2. First, assume that $I = I_k := \{1, 2, \dots, k\}$. In this case, applying (2.15) and (2.17) to a term $X_{\mathcal{K}} \otimes Y_{\mathcal{K}}$ in (A.9) only gives rise to a nonzero multiple of $B_k \cdots B_1 D_M \cdots D_{k+1} \otimes D_M \cdots D_{k+1} C_k \cdots C_1$ as a summand only if $\mathcal{K} = I_k$. Indeed, otherwise let $k_0 = \min \mathcal{K}^c \le k$. In any expression of $X_{\mathcal{K}} \otimes Y_{\mathcal{K}}$ as a linear combination of $B_{i_k} \cdots B_{i_1} D_{j_{M-k}} \cdots D_{j_1} \otimes D_{j_{M-k}} \cdots D_{j_1} C_{i_k} \cdots C_{i_1}$, one needs to commute D_{k_0} to the right through $X_{k_0-1}, \dots, X_1$, which implies that $j_1 \le k_0 \le k$. Therefore, it must be $\mathcal{K} = I_k$.

For $\mathcal{K} = I_k$, the only way of obtaining $B_k \cdots B_1 D_M \cdots D_{k+1} \otimes D_M \cdots D_{k+1} C_k \cdots C_1$ from $X_{\mathcal{K}} \otimes Y_{\mathcal{K}}$ is through using (2.15) to commute each D_j to the right of each B_i. This produces a factor of $(r_i^{-2}x_i - x_j)/(r_i^{-2}x_i - r_j^{-2}x_j)$ for each such commutation, and so

$$h_{I_k}(\mathbf{x};\mathbf{r}) = \prod_{i\in I_k,\, j\notin I_k} \frac{r_i^{-2}x_i - x_j}{r_i^{-2}x_i - r_j^{-2}x_j}.$$

Finally, to get h_I for general I, observe that the operators D_i commute by (2.13), and therefore $h_{\sigma(I_k)}(\mathbf{x};\mathbf{r}) = h_{I_k}(\sigma(\mathbf{x});\sigma(\mathbf{r}))$, which are precisely the coefficients in the claimed identity in the present lemma, where σ takes I_k to an arbitrary I. This completes the proof. □

For the next proposition, recall the notation $d = d(\lambda) \ge 0$ which is the integer such that $\lambda_d \ge d$ and $\lambda_{d+1} < d + 1$, and $\mu = (\mu_1 < \mu_2 < \dots < \mu_d) = \{1, \dots, N\}\backslash\big(\mathcal{S}(\lambda) \cap \{1, \dots, N\}\big)$. Also consider the N-fold tensor product $V^{(1)} \otimes \dots \otimes V^{(N)}$, and take the following vectors in this space

$$e_{\mathcal{S}_N(\lambda)} := e^{(1)}_{m_1} \otimes e^{(2)}_{m_2} \otimes \dots \otimes e^{(N)}_{m_N}, \qquad e_{[1,N]} = e^{(1)}_1 \otimes e^{(2)}_1 \otimes \dots \otimes e^{(N)}_1, \tag{A.12}$$

where $m_i = \mathbf{1}_{i\in\mathcal{S}(\lambda)}$, and with $e_{[1,N]}$ we are slightly abusing the notation, cf. (A.7).

Proposition A.8 *With the above notation, for any vectors $v_1, v_2 \in V^{(N+1)} \otimes V^{(N+2)} \otimes \ldots$ we have*

$$
\begin{aligned}
&\big\langle e_{\mathcal{S}_N(\lambda)} \otimes v_2, D(x_M, r_M) \ldots D(x_2, r_2) D(x_1, r_1) (e_{[1,N]} \otimes v_1) \big\rangle \\
&= \prod_{j=1}^{M} \prod_{k=1}^{N} \frac{y_k - s_k^2 r_j^{-2} x_j}{y_k - s_k^2 x_j} \\
&\quad \sum_{\substack{\mathcal{I} \subseteq \{1,\ldots,M\} \\ |\mathcal{I}|=d}} \left\langle v_2, \left(\prod_{j \notin \mathcal{I}} D(x_j, r_j) \right) C(x_{i_d}, r_{i_d}) \ldots C(x_{i_1}, r_{i_1}) v_1 \right\rangle \\
&\quad \times \prod_{i \in \mathcal{I},\, j \notin \mathcal{I}} \frac{r_i^{-2} x_i - x_j}{r_i^{-2} x_i - r_j^{-2} x_j} \prod_{i, j \in \mathcal{I},\, i<j} \frac{r_i^{-2} x_i - x_j}{r_i^{-2} x_i - r_j^{-2} x_j} \\
&\quad \times \sum_{\sigma \in \mathfrak{S}_d} \operatorname{sgn}(\sigma) \prod_{j=1}^{d} \left(\frac{s_{\mu_j}^2 x_{i_{\sigma(j)}} \left(r_{i_{\sigma(j)}}^{-2} - 1 \right)}{y_{\mu_j} - s_{\mu_j}^2 r_{i_{\sigma(j)}}^{-2} x_{i_{\sigma(j)}}} \prod_{k=\mu_j+1}^{N} \frac{s_k^2 \left(r_{i_{\sigma(j)}}^{-2} x_{i_{\sigma(j)}} - y_k \right)}{y_k - s_k^2 r_{i_{\sigma(j)}}^{-2} x_{i_{\sigma(j)}}} \right),
\end{aligned}
\tag{A.13}
$$

where $\mathcal{I} = (i_1 < \ldots < i_d)$.

The right-hand side of (A.13) vanishes if $d(\lambda) > M$. Observe that the same is true for the left-hand side. Indeed, a single D operator moves at most one vertical arrow somewhere to the right, and d is the number of gaps (sites with no vertical arrows) among $\{1, \ldots, N\}$ in the configuration encoded by $e_{\mathcal{S}_N(\lambda)}$, so d should not be larger than M.

Proof of Proposition A.8 In this proof we use the shorthand notation for the operators, see Remark A.5. As a first step, we consider how the action of the product of the D and C operators like in the right-hand side of (A.13) acts on tensor products. Fix an integer $n > 0$, a subset $\mathcal{H} = \{h_1, h_2, \ldots, h_k\} \subseteq \{1, \ldots, M\}$, and $u, w \in V^{(n+1)} \otimes V^{(n+2)} \otimes \ldots$. Then we have

$$
\begin{aligned}
&\left\langle e_1^{(n)} \otimes w, \left(\prod_{j \notin \mathcal{H}} D_j \right) C_{h_k} \cdots C_{h_1} \left(e_1^{(n)} \otimes u \right) \right\rangle \\
&= \left\langle e_1^{(n)}, \left(\prod_{j \notin \mathcal{H}} D_j \right) A_{h_k} \cdots A_{h_1} e_1^{(n)} \right\rangle \left\langle w, \left(\prod_{j \notin \mathcal{H}} D_j \right) C_{h_k} \cdots C_{h_1} u \right\rangle.
\end{aligned}
\tag{A.14}
$$

and

$$\Big\langle e_0^{(n)} \otimes w, \Big(\prod_{j\notin\mathcal{H}} D_j\Big) C_{h_k}\cdots C_{h_1}\big(e_1^{(n)}\otimes u\big)\Big\rangle$$
$$= \sum_{i\notin\mathcal{H}} \Big\langle e_0^{(n)}, B_i\Big(\prod_{j\notin\mathcal{H}\cup\{i\}} D_j\Big) A_{h_k}\cdots A_{h_1} e_1^{(n)}\Big\rangle$$
$$\Big\langle w, \Big(\prod_{j\notin\mathcal{H}\cup\{i\}} D_j\Big) C_i C_{h_k}\cdots C_{h_1} u\Big\rangle$$
$$\times \prod_{j\notin\mathcal{H}\cup\{i\}} \frac{r_i^{-2}x_i - x_j}{r_i^{-2}x_i - r_j^{-2}x_j}. \tag{A.15}$$

Indeed, observe that $C(x, r)$ maps $e_1^{(n)}$ to 0, so by the third statement in (2.9) we have

$$C_{h_k}\dots C_{h_1}(e_1^{(n)}\otimes u) = A_{h_k}\dots A_{h_1}e_1^{(n)} \otimes C_{h_k}\dots C_{h_1}u.$$

When applying a product of the D_j's to this vector, a nonzero term with $e_1^{(n)}$ in the first tensor factor may appear only if we act each time by the operators D on both tensor factors, see the fourth statement in (2.9). This (together with the fact that $\langle\cdot,\cdot\rangle$ is multiplicative with respect to the tensor product) leads to (A.14). For (A.15), we use Lemma A.7 expressing the action of a product of the D_j's on a tensor product, and observe that a nonzero term with $e_0^{(n)}$ in the first tensor factor may appear only if $|\mathcal{I}| = 1$ in the right-hand side of (A.8).

The action of all the operators on $e_1^{(n)}$ in the right-hand sides of (A.14)–(A.15) is explicit by (2.8) and (2.3):

$$\Big\langle e_1^{(n)}, \Big(\prod_{j\notin\mathcal{H}} D_j\Big) A_{h_k}\cdots A_{h_1} e_1^{(n)}\Big\rangle = \prod_{k\in\mathcal{H}} \frac{s_n^2(x_k - r_k^2 y_n)}{r_k^2(y_n - s_n^2 x_k)} \prod_{j\notin\mathcal{H}} \frac{y_n - s_n^2 r_j^{-2} x_j}{y_n - s_n^2 x_j};$$

$$\Big\langle e_0^{(n)}, B_i\Big(\prod_{j\notin\mathcal{H}\cup\{i\}} D_j\Big) A_{h_k}\cdots A_{h_1} e_1^{(n)}\Big\rangle$$
$$= \frac{s_n^2 x_i(r_i^{-2}-1)}{y_n - s_n^2 x_i} \prod_{k\in\mathcal{H}} \frac{s_n^2(x_k - r_k^2 y_n)}{r_k^2(y_n - s_n^2 x_k)} \prod_{j\notin\mathcal{H}\cup\{i\}} \frac{y_n - s_n^2 r_j^{-2} x_j}{y_n - s_n^2 x_j}.$$

This means that we can continue our identities as

$$(A.14) = \left\langle w, \Big(\prod_{j\notin\mathcal{H}} D_j\Big) C_{h_k}\cdots C_{h_1} u\right\rangle \prod_{k\in\mathcal{H}} \frac{s_n^2(x_k - r_k^2 y_n)}{r_k^2(y_n - s_n^2 x_k)} \prod_{j\notin\mathcal{H}} \frac{y_n - s_n^2 r_j^{-2} x_j}{y_n - s_n^2 x_j};$$

$$(A.15) = \sum_{i\notin\mathcal{H}} \left\langle w, \Big(\prod_{j\notin\mathcal{H}\cup\{i\}} D_j\Big) C_i C_{h_k}\cdots C_{h_1} u\right\rangle \frac{s_n^2 x_i (r_i^{-2} - 1)}{y_n - s_n^2 x_i}$$
$$\prod_{k\in\mathcal{H}} \frac{s_n^2(x_k - r_k^2 y_n)}{r_k^2(y_n - s_n^2 x_k)}$$
$$\times \prod_{j\notin\mathcal{H}\cup\{i\}} \frac{y_n - s_n^2 r_j^{-2} x_j}{y_n - s_n^2 x_j} \prod_{j\notin\mathcal{H}\cup\{i\}} \frac{r_i^{-2} x_i - x_j}{r_i^{-2} x_i - r_j^{-2} x_j}. \tag{A.16}$$

Now we can evaluate

$$\left\langle e_{\mathcal{S}_N(\lambda)} \otimes v_2, D(x_M, r_M) \dots D(x_2, r_2) D(x_1, r_1)(e_{[1,N]} \otimes v_1)\right\rangle$$

by repeatedly using (A.16). Start with $\mathcal{H} = \varnothing$, and apply the first identity in (A.16) for each $n \notin \mu = \{1, \dots, N\} \setminus \mathcal{S}(\lambda)$, and the second identity in (A.16) for each $n \in \mu$. Each application of the latter involves choosing an index $i \notin \mathcal{H}$. This freedom is encoded by the data $(\mathcal{I}, \sigma)$, where $\mathcal{I} = \{i_1 < i_2 < \dots < i_d\} \subseteq \{1, \dots, M\}$ and $\sigma \in \mathfrak{S}_d$, such that at each step when $n = \mu_k \in \mu$ we remove the index $i_{\sigma(k)}$. For each fixed $(\mathcal{I}, \sigma)$ we have the following factors in the resulting expansion:

- The inner product term $\left\langle v_2, \Big(\prod_{j\notin\mathcal{I}} D_j\Big) C_{i_d} \dots C_{i_1} v_1\right\rangle \prod_{1\le\alpha<\beta\le d:\, \sigma(\beta)<\sigma(\alpha)} \frac{r_{i_{\sigma(\beta)}}^{-2} x_{i_{\sigma(\beta)}} - x_{i_{\sigma(\alpha)}}}{r_{i_{\sigma(\alpha)}}^{-2} x_{i_{\sigma(\alpha)}} - x_{i_{\sigma(\beta)}}}$, where the last factor comes from reordering the C operators thanks to Lemma A.6.
- The factor $\prod_{i\in\mathcal{I},\, j\notin\mathcal{I}} \frac{r_i^{-2} x_i - x_j}{r_i^{-2} x_i - r_j^{-2} x_j} \prod_{1\le\alpha<\beta\le d} \frac{r_{i_{\sigma(\alpha)}}^{-2} x_{i_{\sigma(\alpha)}} - x_{i_{\sigma(\beta)}}}{r_{i_{\sigma(\alpha)}}^{-2} x_{i_{\sigma(\alpha)}} - r_{i_{\sigma(\beta)}}^{-2} x_{i_{\sigma(\beta)}}}$ arises by applying the second identity in (A.16) for each $n \in \mu$. Reordering the denominator in the second factor gives

$$\prod_{1\le\alpha<\beta\le d} \frac{1}{r_{i_{\sigma(\alpha)}}^{-2} x_{i_{\sigma(\alpha)}} - r_{i_{\sigma(\beta)}}^{-2} x_{i_{\sigma(\beta)}}} = \operatorname{sgn}(\sigma) \prod_{i,j\in\mathcal{I},\, i<j} \frac{1}{r_i^{-2} x_i - r_j^{-2} x_j}.$$

- The product $\prod_{j=1}^{d} \frac{s_{\mu_j}^2 x_{i_{\sigma(j)}} (r_{i_{\sigma(j)}}^{-2} - 1)}{y_{\mu_j} - s_{\mu_j}^2 x_{i_{\sigma(j)}}}$ is composed of one factor per each application of the second identity in (A.16) corresponding to $n = \mu_j \in \mu$.

- The product $\prod_{j=1}^{d}\prod_{n=\mu_j+1}^{N}\dfrac{s_n^2(x_{i_{\sigma(j)}}-r_{i_{\sigma(j)}}^2 y_n)}{r_{i_{\sigma(j)}}^2(y_n-s_n^2 x_{i_{\sigma(j)}})}$ arises from both identities in (A.16) which contain the same products over $k\in\mathcal{H}$.
- Finally, the product $\left(\prod_{n=1}^{N}\prod_{j=1}^{M}\dfrac{y_n-s_n^2 r_j^{-2}x_j}{y_n-s_n^2 x_j}\right)\left(\prod_{j=1}^{d}\prod_{n=\mu_j}^{N}\dfrac{y_n-s_n^2 x_{i_{\sigma(j)}}}{y_n-s_n^2 r_{i_{\sigma(j)}}^{-2}x_{i_{\sigma(j)}}}\right)$ arises from the products over $j\notin\mathcal{H}$ or $j\notin\mathcal{H}\cup\{i\}$ in (A.16).

Combining all the terms yields the desired identity.

A.2.3 Commutation of the operators *C* and *D*

In this subsection we establish one of the key formulas concerning the commutation of the operators C and D. We fix $M, N\geq 1$ and sequences of complex numbers

$$\mathbf{x}=(x_1,\ldots,x_N),\qquad \mathbf{r}=(r_1,\ldots,r_N),\qquad \mathbf{w}=(w_1,\ldots,w_M),\qquad \boldsymbol{\theta}=(\theta_1,\ldots,\theta_M).$$

Proposition A.9 *We have*

$$\begin{aligned}
&D(x_N,r_N)\ldots D(x_1,r_1)C(w_M,\theta_M)\ldots C(w_1,\theta_1)\\
&\quad=\sum_{\substack{\mathcal{I}\subseteq\{1,\ldots,N\}\\ \mathcal{H}\subseteq\{1,\ldots,M\}\\ |\mathcal{I}|+|\mathcal{H}|=M}} C(x_{i_k},r_{i_k})\ldots C(x_{i_1},r_{i_1})\,C(w_{h_{M-k}},\theta_{h_{M-k}})\ldots C(w_{h_1},\theta_{h_1})\\
&\qquad\prod_{j\notin\mathcal{H}}D(w_j,\theta_j)\prod_{j\notin\mathcal{I}}D(x_j,r_j)\\
&\qquad\times\prod_{i\in\mathcal{I}}(1-r_i^{-2})x_i\prod_{i\in\mathcal{I},\,j\notin\mathcal{I}}\frac{r_j^{-2}x_j-x_i}{x_j-x_i}\prod_{h\in\mathcal{H},\,j\notin\mathcal{H}}\frac{1}{w_j-w_h}\\
&\qquad\prod_{h\in\mathcal{H},\,j\notin\mathcal{I}}\frac{r_j^{-2}x_j-w_h}{x_j-w_h}\prod_{i\in\mathcal{I},\,j\notin\mathcal{H}}\frac{1}{x_i-w_j}\\
&\qquad\times\prod_{i,j\in\mathcal{I},\,i<j}(r_i^{-2}x_i-x_j)\prod_{i,h\in\mathcal{H},\,h<i}\frac{1}{\theta_i^{-2}w_i-w_h}\prod_{1\leq i<j\leq M}(\theta_j^{-2}w_j-w_i).
\end{aligned}\tag{A.17}$$

Here $\mathcal{I}=(i_1<\ldots<i_k)$ *and* $\mathcal{H}=(h_1<\ldots<h_{M-k})$.

Recall that the operators $D(x_j,r_j)$ commute by (2.13), so we can write their products in any order. This is not the case for the operators $C(w_j,\theta_j)$, which is why their order in (A.17) must be specified explicitly.

The rest of this subsection is devoted to the proof of Proposition A.9. As a first step, let us establish the claim for $M=1$:

Lemma A.10 (Proposition A.9 for $M = 1$) *We have*

$$D(x_N, r_N) \ldots D(x_1, r_1) C(w, \theta) = C(w, \theta) D(x_1, r_1) \ldots D(x_N, r_N) \prod_{j=1}^{N} \frac{r_j^{-2} x_j - w}{x_j - w}$$
$$+ \sum_{i=1}^{N} \Bigl(C(x_i, r_i) D(w, \theta) \prod_{j \neq i} D(x_j, r_j) \Bigr) \frac{(1 - r_i^{-2}) x_i}{x_i - w} \prod_{j \neq i} \frac{r_j^{-2} x_j - x_i}{x_j - x_i}. \tag{A.18}$$

Proof The first term containing $C(w, \theta) D(x_1, r_1) \ldots D(x_N, r_N)$ may only arise if we are picking the first summand in (2.16) for each commutation. This produces the desired product $\prod_{j=1}^{N} \frac{r_j^{-2} x_j - w}{x_j - w}$ as a prefactor.

Now let us explain how to get the summand in the second sum corresponding to $i = 1$. Thanks to the commutativity of the $D(x_j, r_j)$'s, the form of the other summands then would follow. To get the term containing $C(x_1, r_1) D(w, \theta) D(x_2, r_2) \ldots D(x_N, r_N)$, we must pick the second summand in (2.16) once, when moving $C(w, \theta)$ to the left of $D(x_1, r_1)$. This produces $C(x_1, r_1) D(w, \theta) \frac{(1 - r_1^{-2}) x_1}{x_1 - w}$. After that, we move $C(x_1, r_1)$ to the left of all the other $D(x_j, r_j)$'s, always picking the first summand in (2.16). This produces the desired identity.

We now consider the general case $M, N \geq 1$ of (A.17). First, repeatedly using relations (2.11), (2.13), and (2.16), we have

$$\begin{aligned} &D(x_N, r_N) \ldots D(x_1, r_1) C(w_M, \theta_M) \ldots C(w_1, \theta_1) \\ &\quad = \sum_{\mathcal{I}, \mathcal{H}} C(x_{i_k}, r_{i_k}) \ldots C(x_{i_1}, r_{i_1})\, C(w_{h_{M-k}}, \theta_{h_{M-k}}) \ldots C(w_{h_1}, \theta_{h_1}) \\ &\qquad \times \prod_{j \notin \mathcal{H}} D(w_j, \theta_j) \prod_{j \notin \mathcal{I}} D(x_j, r_j) R_{\mathcal{I};\mathcal{H}}(\mathbf{w}; \mathbf{x}; \boldsymbol{\theta}; \mathbf{r}), \end{aligned} \tag{A.19}$$

where the sum is taken over $\mathcal{I} \subseteq \{1, \ldots, N\}$ and $\mathcal{H} \subseteq \{1, \ldots, M\}$, such that $|\mathcal{I}| = k$, $|\mathcal{H}| = M - k$, and k is arbitrary (see (A.17)). Here $R_{\mathcal{I};\mathcal{H}}$ are some rational functions which we will now evaluate.

Lemma A.11 (Evaluation of $R_{\mathcal{I};\mathcal{H}}$ in a special case) *Let* $\mathcal{H} = \{1, 2, \ldots, M-k\}$*, and* $\mathcal{I} = (i_1 < \ldots < i_k) \subseteq \{1, \ldots, N\}$ *with* $|\mathcal{I}| = k$ *be arbitrary. Then*

$$
\begin{aligned}
R_{\mathcal{I};\mathcal{H}}(\mathbf{w}; \mathbf{x}; \boldsymbol{\theta}; \mathbf{r}) = \prod_{i\in\mathcal{I}}(1 - r_i^{-2})x_i \prod_{i\in\mathcal{I},\, j\notin\mathcal{I}} \frac{r_j^{-2}x_j - x_i}{x_j - x_i} \\
\prod_{h\in\mathcal{H},\, j\notin\mathcal{H}} \frac{1}{w_j - w_h} \prod_{h\in\mathcal{H},\, j\notin\mathcal{I}} \frac{r_j^{-2}x_j - w_h}{x_j - w_h} \\
\times \prod_{i\in\mathcal{I},\, j\notin\mathcal{H}} \frac{1}{x_i - w_j} \prod_{i,j\in\mathcal{I},\, i<j} (r_i^{-2}x_i - x_j) \\
\prod_{i,h\in\mathcal{H},\, h<i} \frac{1}{\theta_i^{-2}w_i - w_h} \prod_{1\le i<j\le M} (\theta_j^{-2}w_j - w_i).
\end{aligned}
\tag{A.20}
$$

Proof From the left-hand side of (A.19), we apply (2.16) (together with permutation relations (2.11), (2.13) for the operators C, D) to move all the operators C to the left of all the operators D. The operator

$$
\begin{aligned}
C(x_{i_k}, r_{i_k}) \ldots C(x_{i_1}, r_{i_1}) C(w_{M-k}, \theta_{M-k}) \ldots C(w_1, \theta_1) \\
\prod_{j=M-k+1}^{M} D(w_j, \theta_j) \prod_{j\notin\mathcal{I}} D(x_j, r_j)
\end{aligned}
$$

may arise, after a sequence of applications of Lemma A.10, only if there exists a permutation $\sigma \in \mathfrak{S}_k$ such that the following two conditions are met:

- When moving each $C(w_{M-k+j}, \theta_{M-k+j})$, $1 \le j \le k$, to the left, turn $(w_{M-k+j}, \theta_{M-k+j})$ into $(x_{i_{\sigma(j)}}, r_{i_{\sigma(j)}})$. This corresponds to picking the second summand in (2.16), and this swapping of parameters may happen only once per each C operator.
- When moving each $C(w_j, \theta_j)$, $1 \le j \le M-k$, to the left, we always pick the first summand in (2.16), and the parameters (w_j, θ_j) stay the same throughout the exchanges.

To be able to put all the coefficients together, denote $\sigma_t(\mathcal{I}) = \big(i_{\sigma(t)}, i_{\sigma(t+1)}, \ldots, i_{\sigma(k)}\big)$ for each $1 \le t \le k$. Then, for each integer $1 \le j \le k$, when attempting to commute $C(w_{M-k+j}, \theta_{M-k+j})$ to the left of

$$
\prod_{h\notin\sigma_{j+1}(\mathcal{I})} D(x_h, r_h) \prod_{h=j+1}^{k} D(w_{M-k+h}, \theta_{M-k+h}),
$$

we obtain

$$C\big(x_{i_{\sigma(j)}}, r_{i_{\sigma(j)}}\big) \prod_{h \notin \sigma_j(\mathcal{I})} D(x_h, r_h) \prod_{h=j}^{k} D(w_{M-k+h}, \theta_{M-k+h}).$$

By Lemma A.10, this contributes a factor of

$$\frac{\big(1 - r_{i_{\sigma(j)}}^{-2}\big) x_{i_{\sigma(j)}}}{x_{i_{\sigma(j)}} - w_{M-k+j}} \prod_{h=M-k+j+1}^{M} \frac{\theta_h^{-2} w_h - x_{i_{\sigma(j)}}}{w_h - x_{i_{\sigma(j)}}} \prod_{h \notin \sigma_j(\mathcal{I})} \frac{r_h^{-2} x_h - x_{i_{\sigma(j)}}}{x_h - x_{i_{\sigma(j)}}}. \tag{A.21}$$

This deals with the first case above when we swap the parameters between C and D operators.

In the second case when we do not swap the parameters, each $C(w_j, \theta_j)$ for $1 \le j \le M-k$ must be commuted to the left of $\prod_{h \notin \mathcal{I}} D(x_h, r_h) \prod_{h=M-k+1}^{M} D(w_h, \theta_h)$, which contributes

$$\prod_{h \notin \mathcal{I}} \frac{r_h^{-2} x_h - w_j}{x_h - w_j} \prod_{h=M-k+1}^{M} \frac{\theta_h^{-2} w_h - w_j}{w_h - w_j}. \tag{A.22}$$

Observe that

$$\begin{aligned} \prod_{j=1}^{k} \big(1 - r_{i_{\sigma(j)}}^{-2}\big) x_{i_{\sigma(j)}} &= \prod_{i \in \mathcal{I}} (1 - r_i^{-2}) x_i; \\ \prod_{j=1}^{k} \prod_{h \notin \sigma_j(\mathcal{I})} \frac{r_h^{-2} x_h - x_{i_{\sigma(j)}}}{x_h - x_{i_{\sigma(j)}}} &= \prod_{i \in \mathcal{I},\, h \notin \mathcal{I}} \frac{r_h^{-2} x_h - x_i}{x_h - x_i} \prod_{1 \le h < j \le k} \frac{r_{i_{\sigma(h)}}^{-2} x_{i_{\sigma(h)}} - x_{i_{\sigma(j)}}}{x_{i_{\sigma(h)}} - x_{i_{\sigma(j)}}}, \end{aligned} \tag{A.23}$$

Now, combining the product of (A.21) over $1 \le j \le k$ and (A.22) over $1 \le j \le M-k$, and using (A.23), we see that the desired coefficient depending on $\sigma \in \mathfrak{S}_k$ is equal to

$$\begin{aligned} &\prod_{i \in \mathcal{I}} (1 - r_i^{-2}) x_i \prod_{i \in \mathcal{I},\, h \notin \mathcal{I}} \frac{r_h^{-2} x_h - x_i}{x_h - x_i} \prod_{j=1}^{M-k} \left(\prod_{h \notin \mathcal{I}} \frac{r_h^{-2} x_h - w_j}{x_h - w_j} \prod_{h=M-k+1}^{M} \frac{\theta_h^{-2} w_h - w_j}{w_h - w_j} \right) \\ &\quad \times \prod_{1 \le h < j \le k} \frac{r_{i_{\sigma(h)}}^{-2} x_{i_{\sigma(h)}} - x_{i_{\sigma(j)}}}{x_{i_{\sigma(h)}} - x_{i_{\sigma(j)}}} \prod_{j=1}^{k} \left(\frac{1}{x_{i_{\sigma(j)}} - w_{M-k+j}} \prod_{h=M-k+j+1}^{M} \frac{\theta_h^{-2} w_h - x_{i_{\sigma(j)}}}{w_h - x_{i_{\sigma(j)}}} \right). \end{aligned} \tag{A.24}$$

Note that this is the coefficient of the operator

$$C(x_{i_{\sigma(k)}}, r_{i_{\sigma(k)}}) \ldots C(x_{i_{\sigma(1)}}, r_{i_{\sigma(1)}}) C(w_{M-k}, \theta_{M-k}) \ldots C(w_1, \theta_1)$$

$$\prod_{j=M-k+1}^{M} D(w_j, \theta_j) \prod_{j \notin \mathcal{I}} D(x_j, r_j),$$

and permuting the first k of the C operators to the desired order $C(x_{i_k}, r_{i_k}) \dots C(x_{i_1}, r_{i_1})$ results in an additional factor

$$\prod_{\substack{1 \le \alpha < \beta \le k \\ \sigma(\beta) < \sigma(\alpha)}} \frac{r_{i_{\sigma(\beta)}}^{-2} x_{i_{\sigma(\beta)}} - x_{i_{\sigma(\alpha)}}}{r_{i_{\sigma(\alpha)}}^{-2} x_{i_{\sigma(\alpha)}} - x_{i_{\sigma(\beta)}}}, \tag{A.25}$$

by Lemma A.6.

This implies that the full coefficient $R_{\mathcal{I};\mathcal{H}}(\mathbf{w}; \mathbf{x}; \boldsymbol{\theta}; \mathbf{r})$ equals to the sum of (A.24) times (A.25) over all $\sigma \in \mathfrak{S}_k$. We have

$$\prod_{1 \le h < j \le k} \frac{r_{i_{\sigma(h)}}^{-2} x_{i_{\sigma(h)}} - x_{i_{\sigma(j)}}}{x_{i_{\sigma(h)}} - x_{i_{\sigma(j)}}} \prod_{\substack{1 \le \alpha < \beta \le k \\ \sigma(\beta) < \sigma(\alpha)}} \frac{r_{i_{\sigma(\beta)}}^{-2} x_{i_{\sigma(\beta)}} - x_{i_{\sigma(\alpha)}}}{r_{i_{\sigma(\alpha)}}^{-2} x_{i_{\sigma(\alpha)}} - x_{i_{\sigma(\beta)}}}$$
$$= \operatorname{sgn}(\sigma) \prod_{i,j \in \mathcal{I},\, i<j} \frac{r_i^{-2} x_i - x_j}{x_i - x_j}.$$

Therefore, the summation over σ amounts to computing the determinant:

$$\sum_{\sigma \in \mathcal{I}} \operatorname{sgn}(\sigma) \prod_{j=1}^{k} \left(\frac{1}{x_{i_{\sigma(j)}} - w_{M-k+j}} \prod_{h=M-k+j+1}^{M} \frac{\theta_h^{-2} w_h - x_{i_{\sigma(j)}}}{w_h - x_{i_{\sigma(j)}}} \right)$$
$$= \det \left[\frac{1}{x_{i_\beta} - w_{M-k+\alpha}} \prod_{h=M-k+\alpha+1}^{M} \frac{\theta_h^{-2} w_h - x_{i_\beta}}{w_h - x_{i_\beta}} \right]_{\alpha,\beta=1}^{k}. \tag{A.26}$$

We have already computed this determinant (up to renaming the variables) in (3.9), and so

$$(A.26) = \prod_{i \in \mathcal{I},\, j \notin \mathcal{H}} \frac{1}{x_i - w_j} \prod_{i,j \notin \mathcal{H},\, i<j} (\theta_j^{-2} w_j - w_i) \prod_{i,j \in \mathcal{I},\, i<j} (x_i - x_j),$$

where we recalled that $\mathcal{H} = \{1, 2, \dots, M-k\}$. Combining this with the remainder of (A.24), we arrive at the desired expression (A.20), thus concluding the proof of Lemma A.11. □

Finally, to get $R_{\mathcal{I};\mathcal{H}}$ for general $\mathcal{H}$, we can permute the C operators in the left-hand side of (A.17) thanks to (2.11). More precisely, the two expressions

$$C(w_M,\theta_M)\dots C(w_1,\theta_1)\prod_{1\le i<j\le M}\frac{1}{\theta_j^{-2}w_j-w_i},$$
$$C(w_{h_{M-k}},\theta_{h_{M-k}})\dots C(w_{h_1},\theta_{h_1})\prod_{i,j\in\mathcal{H},\,i<j}\frac{1}{\theta_j^{-2}w_j-w_i}$$

are symmetric in (w_i,θ_i), $1\le i\le M$, and (w_h,θ_h), $h\in\mathcal{H}$, respectively. Defining

$$\widehat{R}_{\mathcal{I};\mathcal{H}}(\mathbf{w};\mathbf{x};\boldsymbol{\theta};\mathbf{r})=R_{\mathcal{I};\mathcal{H}}(\mathbf{w};\mathbf{x};\boldsymbol{\theta};\mathbf{r})\,\frac{\prod_{i,j\in\mathcal{H},\,i<j}(\theta_j^{-2}w_j-w_i)}{\prod_{1\le i<j\le M}(\theta_j^{-2}w_j-w_i)},\tag{A.27}$$

we see that for any permutation $\tau\in\mathfrak{S}_M$ we have $\widehat{R}_{\mathcal{I};\tau(\mathcal{H})}(\tau(\mathbf{w});\mathbf{x};\tau(\boldsymbol{\theta});\mathbf{r})=\widehat{R}_{\mathcal{I};\mathcal{H}}(\mathbf{w};\mathbf{x};\boldsymbol{\theta};\mathbf{r})$. The renormalization in (A.27) cancels out with the two last factors in $R_{\mathcal{I};\{1,\dots,M-k\}}$ in (A.20). This together with the symmetry of (A.27) implies that $R_{\mathcal{I};\mathcal{H}}$ for general $\mathcal{H}$ is given by the same formula. We have thus completed the proof of Proposition A.9.

A.2.4 Action of *C* operators on a two-fold tensor product

In this subsection we perform computations with row operators acting on tensor products which are parallel to those in Appendices A.1.2 and A.2.2, but now involve the C operators.

Lemma A.12 *Let* $\mathbf{x}=(x_1,\dots,x_M)$, $\mathbf{r}=(r_1,\dots,r_M)$. *On any tensor product* $V_1\otimes V_2$ *we have:*

$$\begin{aligned}
&C(x_M,r_M)\dots C(x_1,r_1)\\
&\quad=\sum_{\mathcal{I}\subseteq\{1,\dots,M\}}C(x_{i_k},r_{i_k})\dots C(x_{i_1},r_{i_1})A(x_{j_{M-k}},r_{j_{M-k}})\dots A(x_{j_1},r_{j_1})\\
&\quad\otimes C(x_{j_{M-k}},r_{j_{M-k}})\dots C(x_{j_1},r_{j_1})D(x_{i_k},r_{i_k})\dots D(x_{i_1},r_{i_1})\\
&\quad\times\prod_{i\in\mathcal{I},\,j\in\mathcal{J}}\frac{1}{x_i-x_j}\prod_{1\le i<j\le M}(r_j^{-2}x_j-x_i)\\
&\quad\prod_{i,j\in\mathcal{I},\,i<j}\frac{1}{r_j^{-2}x_j-x_i}\prod_{i,j\in\mathcal{J},\,i<j}\frac{1}{r_j^{-2}x_j-x_i},
\end{aligned}\tag{A.28}$$

where $\mathcal{I}=(i_1<\dots<i_k)$ *and* $\mathcal{J}=\{1,\dots,M\}\setminus\mathcal{I}=(j_1<\dots<j_{M-k})$.

Proof In the proof we use the shorthand notation for the operators from Remark A.5. Due to (2.9), relations in Proposition 2.4, and an argument identical to the beginning of the proof of Lemma A.7, we see that the left-hand side of (A.28) can be written in the form

$$\sum_{\mathcal{I}\subseteq\{1,\ldots,M\}} h_{\mathcal{I}}(\mathbf{x};\mathbf{r}) C_{i_k}\ldots C_{i_1} A_{j_{M-k}}\ldots A_{j_1}\otimes C_{j_{M-k}}\ldots C_{j_1} D_{i_k}\ldots D_{i_1},$$

where the notation $\mathcal{I}$, $\mathcal{J}$ is as in (A.28).

We first evaluate $h_{\mathcal{I}}$ in the special case $\mathcal{I}=\mathcal{I}_k=\{M-k+1,\ldots,M-1,M\}$. The contribution containing the operator $C_M\ldots C_{M-k+1}A_{M-k}\ldots A_1\otimes C_{M-k}\ldots C_1$ $D_M\ldots D_{M-k+1}$ may arise only if we use (2.16) in the second tensor factor to commute all C_j, $j\notin\mathcal{I}_k$, to the left of all D_i, $i\in\mathcal{I}$, without swapping their arguments. Each such commutation gives rise to the factor $\frac{r_i^{-2}x_i-x_j}{x_i-x_j}$. Therefore,

$$\begin{aligned} h_{\mathcal{I}_k}(\mathbf{x};\mathbf{r}) &= \prod_{i=M-k+1}^{M}\prod_{j=1}^{M-k}\frac{r_i^{-2}x_i-x_j}{x_i-x_j}\\ &= \prod_{i\in\mathcal{I}_k,\, j\notin\mathcal{I}_k}\frac{1}{x_i-x_j}\prod_{1\le i<j\le M}(r_j^{-2}x_j-x_i)\\ &\quad \prod_{i,j\in\mathcal{I}_k,\, i<j}\frac{1}{r_j^{-2}x_j-x_i}\prod_{i,j\notin\mathcal{I}_k,\, i<j}\frac{1}{r_j^{-2}x_j-x_i}. \end{aligned} \tag{A.29}$$

Next, thanks to (2.11) the three expressions

$$\frac{C_M\ldots C_1}{\prod_{1\le i<j\le M}(r_j^{-2}x_j-x_i)},\qquad \frac{C_{i_k}\ldots C_{i_1}}{\prod_{i,j\in\mathcal{I},\, i<j}(r_j^{-2}x_j-x_i)},\qquad \frac{C_{j_{M-k}}\ldots C_{j_1}}{\prod_{i,j\notin\mathcal{I},\, i<j}(r_j^{-2}x_j-x_i)}$$

are symmetric in the pairs (x_i, r_i) of variables they depend on (where $1\le i\le M$, $i\in\mathcal{I}$, and $i\notin\mathcal{I}$, respectively). Therefore, the function

$$\widehat{h}_{\mathcal{I}}(\mathbf{x};\mathbf{r}) = h_{\mathcal{I}}(\mathbf{x};\mathbf{r})\,\frac{\prod_{i,j\in\mathcal{I},\, i<j}(r_j^{-2}x_j-x_i)\prod_{i,j\notin\mathcal{I},\, i<j}(r_j^{-2}x_j-x_i)}{\prod_{1\le i<j\le M}(r_j^{-2}x_j-x_i)}$$

satisfies $\widehat{h}_{\tau(\mathcal{I})}(\mathbf{x};\mathbf{r}) = \widehat{h}_{\mathcal{I}}(\tau^{-1}(\mathbf{x});\tau^{-1}(\mathbf{r}))$ for any permutation $\tau\in\mathfrak{S}_M$. Together with (A.29) this shows that for any $\mathcal{I}$ we have $\widehat{h}_{\mathcal{I}}(\mathbf{x};\mathbf{r}) = \prod_{i\in\mathcal{I},\, j\notin\mathcal{I}}(x_i-x_j)^{-1}$, which implies the claim. □

In the next proposition, let $e_0 = e_0^{(i_1)}\otimes e_0^{(i_2)}\otimes\cdots\otimes e_0^{(i_n)}\in V^{(i_1)}\otimes V^{(i_2)}\otimes\cdots\otimes V^{(i_n)}$ for any integers $i_1<i_2<\cdots<i_n$. Moreover, fix $M\ge 1$, $N\ge 0$, and $\mathcal{T}=(t_1<t_2<\ldots<t_M)\subset\mathbb{Z}_{\ge 1}$. Define the vector $e_{\mathcal{T};N} = e_{m_1}^{(N+1)}\otimes e_{m_2}^{(N+2)}\otimes\cdots\in V^{(N+1)}\otimes V^{(N+2)}\otimes\cdots$, where $m_i=1$ if $i\in\mathcal{T}$, and 0 otherwise.

Proposition A.13 *With the above notation we have*

$$\left\langle e_{\mathcal{T};N}, C(x_M, r_M)\cdots C(x_1, r_1)\, e_0\right\rangle = \prod_{1\le i<j\le M} \frac{r_j^{-2}x_j - x_i}{x_i - x_j}$$
$$\times \sum_{\sigma\in\mathfrak{S}_M} \mathrm{sgn}(\sigma) \prod_{j=1}^{M} \left(\frac{y_{t_j+N}\left(1-s_{t_j+N}^2\right)}{y_{t_j+N} - s_{t_j+N}^2 x_{\sigma(j)}} \prod_{i=N+1}^{t_j+N-1} \frac{s_i^2\left(y_i - x_{\sigma(j)}\right)}{y_i - s_i^2 x_{\sigma(j)}} \right),$$

where the inner product is taken in the space $V^{(N+1)}\otimes V^{(N+2)}\otimes\ldots$*.*

Observe that this formula is determinantal, and is in fact equivalent to the determinantal formula for F_λ from Theorem 3.9 proven in Appendix A.1, up to swapping horizontal arrows with empty horizontal edges, and renormalizing. Here, however, we present an independent proof which is more convenient given our previous statements.

Proof of Proposition A.13 In the proof we use the shorthand notation for the operators from Remark A.5. Fix $n > N$ and vectors $v_1, v_2 \in V^{(n+1)}\otimes V^{(n+2)}\otimes\ldots$. By Lemma A.12, we have

$$\left\langle e_0^{(n)}\otimes v_2, C_M C_{M-1}\cdots C_1 e_0\right\rangle = \left\langle e_0^{(n)}, A_M A_{M-1}\cdots A_1 e_0^{(n)}\right\rangle \langle v_2, C_M C_{M-1}\cdots C_1 e_0\rangle;$$
$$\left\langle e_1^{(n)}\otimes v_2, C_M C_{M-1}\cdots C_1 e_0\right\rangle = \sum_{i=1}^{M}\left\langle e_1^{(n)}, C_i A_M\cdots A_{i+1}A_{i-1}\cdots A_1 e_0^{(n)}\right\rangle$$
$$\times \langle v_2, C_M\cdots C_{i+1}C_{i-1}\cdots C_1 D_i e_0\rangle$$
$$\times \prod_{j\ne i}\frac{1}{x_i - x_j}\prod_{j=1}^{i-1}(r_i^{-2}x_i - x_j)\prod_{j=i+1}^{M}(r_j^{-2}x_j - x_i). \tag{A.30}$$

These quantities can be computed as follows:

$$D_i e_0 = e_0;$$
$$\left\langle e_0^{(n)}, A_M A_{M-1}\cdots A_1 e_0^{(n)}\right\rangle = \prod_{j=1}^{M}\frac{s_n^2(y_n - x_j)}{y_n - s_n^2 x_j};$$
$$\left\langle e_1^{(n)}, C_i A_M\cdots A_{i+1}A_{i-1}\cdots A_1 e_0^{(n)}\right\rangle = \frac{y_n(1-s_n^2)}{y_n - s_n^2 x_i}\prod_{j\ne i}\frac{s_n^2(y_n - x_j)}{y_n - s_n^2 x_j},$$

using the definition of the operators (2.8) and formulas for the vertex weights W (2.3). Therefore, (A.30) is continued as

$$\langle e_0^{(n)} \otimes v_2, C_M C_{M-1} \cdots C_1 e_0 \rangle = \langle v_2, C_M C_{M-1} \cdots C_1 e_0 \rangle \prod_{j=1}^{M} \frac{s_n^2 (y_n - x_j)}{y_n - s_n^2 x_j};$$

$$\langle e_1^{(n)} \otimes v_2, C_M C_{M-1} \cdots C_1 e_0 \rangle = \sum_{i=1}^{M} \langle v_2, C_M \cdots C_{i+1} C_{i-1} \cdots C_1 e_0 \rangle$$

$$\times \frac{y_n (1 - s_n^2)}{y_n - s_n^2 x_i} \prod_{j \neq i} \frac{s_n^2 (y_n - x_j)}{y_n - s_n^2 x_j} \prod_{j \neq i} \frac{1}{x_i - x_j} \prod_{j=1}^{i-1} (r_i^{-2} x_i - x_j) \prod_{j=i+1}^{M} (r_j^{-2} x_j - x_i). \tag{A.31}$$

Now we can evaluate $\langle e_{\mathcal{T};N}, C_M \ldots C_1 e_0 \rangle$ by repeatedly applying (A.31). Throughout these applications, we use first or second identity in (A.31), respectively, for each n belonging or not belonging to the set $\{t_1 + N, t_2 + N, \ldots, t_M + N\}$. In the latter case, for $n = N + t_j$, we choose which index $i = i_j \in \{1, \ldots, M\}$ to remove. These choices are encoded by a permutation $\sigma \in \mathfrak{S}_M$ as $i_j = \sigma(j)$. This leads to the desired claim, where, in particular, $\operatorname{sgn}(\sigma)$ arises from reordering the denominators $x_{\sigma(i)} - x_{\sigma(j)}$ to $x_i - x_j$ over all $1 \leq i < j \leq M$. □

A.2.5 Completing the proof

To finalize the proof of Theorem 3.10, let us recall the formula to be established. Fix an arbitrary signature $\lambda = (\lambda_1 \geq \ldots \geq \lambda_N \geq 0)$. Let $d = d(\lambda) \geq 0$ denote the integer such that $\lambda_d \geq d$ and $\lambda_{d+1} < d + 1$. Denote by $\ell_j = \lambda_j + N - j + 1$, $j = 1, \ldots, N$, the elements of the set $\mathcal{S}(\lambda)$. Moreover, we define $\mu = (\mu_1 < \mu_2 < \ldots < \mu_d) = \{1, \ldots, N\} \backslash \big(\mathcal{S}(\lambda) \cap \{1, \ldots, N\}\big)$. Our goal is to show that

$$G_\lambda(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s}) = \prod_{j=1}^{M} \prod_{k=1}^{N} \frac{y_k - s_k^2 r_j^{-2} x_j}{y_k - s_k^2 x_j}$$

$$\sum_{\substack{\mathcal{I}, \mathcal{J} \subseteq \{1, \ldots, M\} \\ |\mathcal{I}| = |\mathcal{J}| = d}} \prod_{\substack{i \in \mathcal{I} \\ 1 \leq j \leq M}} (r_i^{-2} x_i - x_j) \prod_{\substack{i \in \mathcal{I} \\ j \in \mathcal{I}^c}} \frac{1}{r_i^{-2} x_i - r_j^{-2} x_j}$$

$$\times \prod_{\substack{i, j \in \mathcal{I} \\ i < j}} \frac{1}{r_i^{-2} x_i - r_j^{-2} x_j} \prod_{\substack{i \in \mathcal{I}^c \\ j \in \mathcal{J}}} (r_i^{-2} x_i - x_j) \prod_{\substack{i \in \mathcal{J}^c \\ j \in \mathcal{J}}} \frac{1}{x_i - x_j} \prod_{\substack{i, j \in \mathcal{J} \\ i < j}} \frac{1}{x_j - x_i}$$

$$\times \sum_{\sigma, \rho \in \mathfrak{S}_d} \operatorname{sgn}(\sigma \rho) \prod_{h=1}^{d} \left(\frac{y_{\ell_h} \left(1 - s_{\ell_h}^2\right)}{y_{\ell_h} - s_{\ell_h}^2 x_{j_{\rho(h)}}} \prod_{i=N+1}^{\ell_h - 1} \frac{s_i^2 \left(y_i - x_{j_{\rho(h)}}\right)}{y_i - s_i^2 x_{j_{\rho(h)}}} \right)$$

$$\times \prod_{m=1}^{d} \left(\frac{s_{\mu_m}^2}{y_{\mu_m} - s_{\mu_m}^2 r_{i_{\sigma(m)}}^{-2} x_{i_{\sigma(m)}}} \prod_{k=\mu_m+1}^{N} \frac{s_k^2 \left(r_{i_{\sigma(m)}}^{-2} x_{i_{\sigma(m)}} - y_k\right)}{y_k - s_k^2 r_{i_{\sigma(m)}}^{-2} x_{i_{\sigma(m)}}} \right). \tag{A.32}$$

where $\mathcal{I} = (i_1 < i_2 < \ldots < i_d)$ and $\mathcal{J} = (j_1 < j_2 < \ldots < j_d)$.

Recall that

$$G_\lambda(\mathbf{x}; \mathbf{y}; \mathbf{r}; \mathbf{s}) = \left\langle e_{\mathcal{S}_N(\lambda)} \otimes e_{\mathcal{S}_{>N}}(\lambda),\, D(x_M, r_M) \ldots D(x_2, r_2) D(x_1, r_1)(e_{[1,N]} \otimes e_0) \right\rangle,$$

where we have split the vectors into $e_{\mathcal{S}_N(\lambda)}, e_{[1,N]} \in V^{(1)} \otimes \ldots \otimes V^{(N)}$ (cf. (A.12)), and the remaining two vectors belong to $V^{(N+1)} \otimes V^{(N+2)} \otimes \ldots$. Note that the vector $e_{\mathcal{S}_{>N}(\lambda)}$ has exactly d tensor components of the form $e_1^{(k)}$, and the other components are of the form $e_0^{(k)}$. We can use Proposition A.8 to write:

$$\begin{aligned}
&\left\langle e_{\mathcal{S}_N(\lambda)} \otimes e_{\mathcal{S}_{>N}(\lambda)},\, D(x_M, r_M) \ldots D(x_2, r_2) D(x_1, r_1)(e_{[1,N]} \otimes e_0) \right\rangle \\
&= \prod_{j=1}^{M} \prod_{k=1}^{N} \frac{y_k - s_k^2 r_j^{-2} x_j}{y_k - s_k^2 x_j} \sum_{\substack{\mathcal{I} \subseteq \{1,\ldots,M\} \\ |\mathcal{I}| = d}} \\
&\left\langle e_{\mathcal{S}_{>N}(\lambda)}, \left(\prod_{j \notin \mathcal{I}} D(x_j, r_j) \right) C(x_{i_d}, r_{i_d}) \ldots C(x_{i_1}, r_{i_1}) e_0 \right\rangle \\
&\times \prod_{i \in \mathcal{I},\, j \notin \mathcal{I}} \frac{r_i^{-2} x_i - x_j}{r_i^{-2} x_i - r_j^{-2} x_j} \prod_{i, j \in \mathcal{I},\, i<j} \frac{r_i^{-2} x_i - x_j}{r_i^{-2} x_i - r_j^{-2} x_j} \\
&\times \sum_{\sigma \in \mathfrak{S}_d} \operatorname{sgn}(\sigma) \prod_{j=1}^{d} \left(\frac{s_{\mu_j}^2 x_{i_{\sigma(j)}} \left(r_{i_{\sigma(j)}}^{-2} - 1 \right)}{y_{\mu_j} - s_{\mu_j}^2 r_{i_{\sigma(j)}}^{-2} x_{i_{\sigma(j)}}} \prod_{k=\mu_j+1}^{N} \frac{s_k^2 \left(r_{i_{\sigma(j)}}^{-2} x_{i_{\sigma(j)}} - y_k \right)}{y_k - s_k^2 r_{i_{\sigma(j)}}^{-2} x_{i_{\sigma(j)}}} \right).
\end{aligned} \tag{A.33}$$

Let us denote

$$D_{\mathcal{I}^c} := \prod_{i \notin \mathcal{I}} D(x_j, r_j), \qquad C_{\mathcal{I}} := C(x_{i_d}, r_{i_d}) \ldots C(x_{i_1}, r_{i_1}),$$

and use similar notation in what follows. In particular, in all such products of the C operators the indices are decreasing from left to right. Employ Proposition A.9 to write

$$\begin{aligned}
D_{\mathcal{I}^c} C_{\mathcal{I}} = &\sum_{\substack{\mathcal{K} \subseteq \mathcal{I}^c,\, \mathcal{H} \subseteq \mathcal{I} \\ |\mathcal{K}| + |\mathcal{H}| = d}} C_{\mathcal{K}} C_{\mathcal{H}} D_{\mathcal{I} \setminus \mathcal{H}} D_{\mathcal{I}^c \setminus \mathcal{K}} \\
&\times \prod_{k \in \mathcal{K}} (1 - r_k^{-2}) x_k \prod_{i \in \mathcal{K} \cup \mathcal{H},\, j \in \mathcal{I}^c \setminus \mathcal{K}} \frac{r_j^{-2} x_j - x_i}{x_j - x_i} \prod_{h \in \mathcal{H},\, j \in \mathcal{I} \setminus \mathcal{H}} \frac{1}{x_j - x_h} \prod_{i \in \mathcal{K},\, j \in \mathcal{I} \setminus \mathcal{H}} \frac{1}{x_i - x_j} \\
&\times \prod_{i, j \in \mathcal{K},\, i<j} (r_i^{-2} x_i - x_j) \prod_{i, h \in \mathcal{H},\, h<i} \frac{1}{r_i^{-2} x_i - x_h} \prod_{i, j \in \mathcal{I},\, i<j} (r_j^{-2} x_j - x_i).
\end{aligned}$$

Let us insert this into (A.33). Observe that all operators D preserve the vector e_0. Thus, we can continue the computation as

$$
\begin{aligned}
&\left\langle e_{\mathcal{S}_N(\lambda)} \otimes e_{\mathcal{S}_{>N}(\lambda)}, D(x_M, r_M) \dots D(x_2, r_2) D(x_1, r_1)(e_{[1,N]} \otimes e_0)\right\rangle \\
&\quad = \prod_{j=1}^{M} \prod_{k=1}^{N} \frac{y_k - s_k^2 r_j^{-2} x_j}{y_k - s_k^2 x_j} \sum_{\substack{\mathcal{I} \subseteq \{1,\dots,M\} \\ |\mathcal{I}|=d}} \prod_{i,j\in\mathcal{I}} (r_i^{-2} x_i - x_j) \\
&\quad \prod_{\substack{i\in\mathcal{I} \\ j\in\mathcal{I}^c}} \frac{r_i^{-2} x_i - x_j}{r_i^{-2} x_i - r_j^{-2} x_j} \prod_{\substack{i,j\in\mathcal{I} \\ i<j}} \frac{1}{r_i^{-2} x_i - r_j^{-2} x_j} \\
&\quad \times \sum_{\substack{\mathcal{K}\subseteq\mathcal{I}^c,\, \mathcal{H}\subseteq\mathcal{I} \\ |\mathcal{K}|+|\mathcal{H}|=d}} \left\langle e_{\mathcal{S}_{>N}(\lambda)}, C_{\mathcal{K}} C_{\mathcal{H}} e_0 \right\rangle \prod_{k\in\mathcal{K}} (1 - r_k^{-2}) x_k \prod_{i\in\mathcal{K}\cup\mathcal{H},\, j\in\mathcal{I}^c\setminus\mathcal{K}} \frac{r_j^{-2} x_j - x_i}{x_j - x_i} \\
&\quad \times \prod_{h\in\mathcal{H},\, j\in\mathcal{I}\setminus\mathcal{H}} \frac{1}{x_j - x_h} \prod_{i\in\mathcal{K},\, j\in\mathcal{I}\setminus\mathcal{H}} \frac{1}{x_i - x_j} \\
&\quad \prod_{i,j\in\mathcal{K},\, i<j} (r_i^{-2} x_i - x_j) \prod_{i,h\in\mathcal{H},\, h<i} \frac{1}{r_i^{-2} x_i - x_h} \\
&\quad \times \sum_{\sigma\in\mathfrak{S}_d} \operatorname{sgn}(\sigma) \prod_{j=1}^{d} \left(\frac{s_{\mu_j}^2}{y_{\mu_j} - s_{\mu_j}^2 r_{i_{\sigma(j)}}^{-2} x_{i_{\sigma(j)}}} \prod_{k=\mu_j+1}^{N} \frac{s_k^2 \left(r_{i_{\sigma(j)}}^{-2} x_{i_{\sigma(j)}} - y_k\right)}{y_k - s_k^2 r_{i_{\sigma(j)}}^{-2} x_{i_{\sigma(j)}}} \right).
\end{aligned}
$$

Now we are going to apply Proposition A.13 to compute the remaining inner product. Recall that $e_{\mathcal{S}_{>N}(\lambda)}$ has exactly d tensor components equal to $e_1^{(m)}$, for $m \in \{\ell_1, \dots, \ell_d\}$. Denote $(x'_1, \dots, x'_d) = (x_{h_1}, \dots, x_{h_{|\mathcal{H}|}}, x_{k_1}, \dots, x_{k_{|\mathcal{K}|}})$, where $h_1 < \dots < h_{|\mathcal{H}|}, k_1 < \dots < k_{|\mathcal{K}|}$. Then we have

$$
\begin{aligned}
\left\langle e_{\mathcal{S}_{>N}(\lambda)}, C_{\mathcal{K}} C_{\mathcal{H}} e_0 \right\rangle &= (-1)^{\frac{d(d-1)}{2}} \prod_{i,j\in\mathcal{H},\, i<j} \frac{r_j^{-2} x_j - x_i}{x_i - x_j} \prod_{i,j\in\mathcal{K},\, i<j} \frac{r_j^{-2} x_j - x_i}{x_i - x_j} \\
&\times \prod_{i\in\mathcal{H},\, j\in\mathcal{K}} \frac{r_j^{-2} x_j - x_i}{x_i - x_j} \sum_{\rho\in\mathfrak{S}_d} \operatorname{sgn}(\rho) \prod_{j=1}^{d} \left(\frac{y_{\ell_j}\left(1 - s_{\ell_j}^2\right)}{y_{\ell_j} - s_{\ell_j}^2 x'_{\rho(j)}} \prod_{i=N+1}^{\ell_j - 1} \frac{s_i^2 \left(y_i - x'_{\rho(j)}\right)}{y_i - s_i^2 x'_{\rho(j)}} \right).
\end{aligned}
\tag{A.34}
$$

The sign $(-1)^{\frac{d(d-1)}{2}}$ arises from the fact that the t_j's in Proposition A.13 are increasing, while the ℓ_j's in (A.34) are decreasing, so the sign of ρ has to be multiplied by $(-1)^{\frac{d(d-1)}{2}}$. This allows to continue our computation as follows:

$$
\left\langle e_{\mathcal{S}_N(\lambda)} \otimes e_{\mathcal{S}_{>N}(\lambda)}, D(x_M, r_M) \dots D(x_2, r_2) D(x_1, r_1)(e_{[1,N]} \otimes e_0)\right\rangle
$$

$$
\begin{aligned}
&= (-1)^{\frac{d(d-1)}{2}} \prod_{j=1}^{M} \prod_{k=1}^{N} \frac{y_k - s_k^2 r_j^{-2} x_j}{y_k - s_k^2 x_j} \\
&\times \sum_{\substack{\mathcal{I} \subseteq \{1,\dots,M\} \\ |\mathcal{I}|=d}} \prod_{\substack{i \in \mathcal{I} \\ 1 \le j \le M}} (r_i^{-2} x_i - x_j) \prod_{\substack{i \in \mathcal{I} \\ j \in \mathcal{I}^c}} \frac{1}{r_i^{-2} x_i - r_j^{-2} x_j} \prod_{\substack{i,j \in \mathcal{I} \\ i<j}} \frac{1}{r_i^{-2} x_i - r_j^{-2} x_j} \\
&\times \sum_{\substack{\mathcal{K} \subseteq \mathcal{I}^c,\, \mathcal{H} \subseteq \mathcal{I} \\ |\mathcal{K}|+|\mathcal{H}|=d}} \prod_{\substack{i \in \mathcal{I}^c \\ j \in \mathcal{K} \cup \mathcal{H}}} (r_i^{-2} x_i - x_j) \prod_{\substack{i \notin \mathcal{K} \cup \mathcal{H} \\ j \in \mathcal{K} \cup \mathcal{H}}} \frac{1}{x_i - x_j} \prod_{\substack{i,j \in \mathcal{H} \\ i<j}} \frac{1}{x_i - x_j} \\
&\prod_{\substack{i,j \in \mathcal{K} \\ i<j}} \frac{1}{x_i - x_j} \prod_{\substack{i \in \mathcal{H} \\ j \in \mathcal{K}}} \frac{1}{x_i - x_j} \\
&\times \sum_{\sigma,\rho \in \mathfrak{S}_d} \operatorname{sgn}(\sigma\rho) \prod_{j=1}^{d} \left(\frac{y_{\ell_j}\left(1 - s_{\ell_j}^2\right)}{y_{\ell_j} - s_{\ell_j}^2 x'_{\rho(j)}} \prod_{i=N+1}^{\ell_j - 1} \frac{s_i^2 \left(y_i - x'_{\rho(j)}\right)}{y_i - s_i^2 x'_{\rho(j)}} \right) \\
&\times \prod_{j=1}^{d} \left(\frac{s_{\mu_j}^2}{y_{\mu_j} - s_{\mu_j}^2 r_{i_{\sigma(j)}}^{-2} x_{i_{\sigma(j)}}} \prod_{k=\mu_j+1}^{N} \frac{s_k^2 \left(r_{i_{\sigma(j)}}^{-2} x_{i_{\sigma(j)}} - y_k\right)}{y_k - s_k^2 r_{i_{\sigma(j)}}^{-2} x_{i_{\sigma(j)}}} \right).
\end{aligned}
$$

Upon denoting $\mathcal{J} = \mathcal{K} \cup \mathcal{H} = (j_1 < \dots < j_d)$, we arrive at the desired statement (A.32). Note that reordering the indices in $(x'_1, \dots, x'_d)$ in the increasing order leads to an extra $\pm$ sign coming from $\operatorname{sgn}(\rho)$, but this sign is compensated by writing

$$
\prod_{i,j \in \mathcal{H},\, i<j} \frac{1}{x_i - x_j} \prod_{i,j \in \mathcal{K},\, i<j} \frac{1}{x_i - x_j} \prod_{i \in \mathcal{H},\, j \in \mathcal{K}} \frac{1}{x_i - x_j} = \pm \prod_{i,j \in \mathcal{J},\, i<j} \frac{1}{x_i - x_j} \tag{A.35}
$$

(equivalently, one may refer to the symmetry as in the proof of Lemma A.12). Finally, replacing $x_i - x_j$ in (A.35) with $x_j - x_i$ absorbs the sign $(-1)^{\frac{d(d-1)}{2}}$. This completes the proof of Theorem 3.10.

B Correlation kernel via Eynard–Mehta approach

Here we prove Theorem 6.7 on the determinantal structure of the FG measures and processes. We employ an Eynard–Mehta type approach based on [20], see also [31].

B.1 Representation of the ascending FG process in a determinantal form

Recall the notation of the ascending FG process (6.8) from Sect. 6.2. Throughout Appendix B we omit the notation $\mathbf{y}, \mathbf{s}$ in the functions $G_{\mu/\varkappa}(w_i; \mathbf{y}; \theta_i; \mathbf{s})$ and other similar quantities.

Here we use the determinantal formulas for the functions F_λ (Theorem 3.9) and G_μ, $G_{\nu/\lambda}$ to rewrite the probabilities (6.8) in a determinantal form. The formulas for G_μ and $G_{\nu/\lambda}$ are of Jacobi–Trudy type and follow from Cauchy identities and biorthogonality as in Sect. 5.4.

Recall the notation (3.11):

$$\varphi_k(x) = \frac{1}{y_{k+1} - x} \prod_{j=1}^{k} \frac{y_j - s_j^2 x}{s_j^2(y_j - x)}, \qquad k \geq 0.$$

By Theorem 3.9, we have

$$F_\lambda(\rho) = \text{const} \cdot \det\left[\varphi_{\lambda_j+N-j}(x_i)\right]_{i,j=1}^{N}, \tag{B.1}$$

where the constant is independent of λ (we adopt this convention for all such constants throughout Appendix B, and will denote all of them by const).

Next, recall the functions ψ_k (5.1):

$$\psi_k(x) = \frac{y_{k+1}(s_{k+1}^2 - 1)}{y_{k+1} - s_{k+1}^2 x} \prod_{j=1}^{k} \frac{s_j^2(y_j - x)}{y_j - s_j^2 x}, \qquad k \geq 1.$$

For $(\mathbf{w}; \boldsymbol{\theta}) = (w_a, \ldots, w_b; \theta_a, \ldots, \theta_b)$, $a \leq b$, let us define a slight generalization of (5.13):

$$\mathsf{h}_{k,p}(\mathbf{w}; \boldsymbol{\theta}) := \frac{1}{2\pi\mathbf{i}} \oint_{\Gamma_{y,w}} dz \, \frac{\psi_k(z)}{y_p - z} \prod_{j=a}^{b} \frac{z - \theta_j^{-2} w_j}{z - w_j}, \qquad k \geq 0, \quad p \geq 1,$$

where the integration contour $\Gamma_{y,w}$ is positively oriented, surrounds all y_i, w_j, and leaves out all $s_i^{-2} y_i$. The function $G_{\lambda^{(1)}}$ in (6.8) has the following determinantal form (with $a = b = 1$ in $\mathsf{h}_{k,l}$):

$$G_{\lambda^{(1)}}(w_1; \theta_1) = \text{const} \cdot \det\left[\mathsf{h}_{\lambda_i^{(1)}+N-i,\, j}(w_1; \theta_1)\right]_{i,j=1}^{N}. \tag{B.2}$$

Finally, recall the functions $\widetilde{\mathsf{h}}_l$ (5.9) and $\mathsf{g}_{l/k}$ (5.10):

$$\mathsf{g}_{l/k}(\mathbf{w}; \boldsymbol{\theta}) = \widetilde{\mathsf{h}}_{l-k}(\mathbf{w}; \tau_k \mathbf{y}; \boldsymbol{\theta}; \tau_k \mathbf{s}) = \frac{\mathbf{1}_{l \geq k}}{2\pi\mathbf{i}} \oint_{\Gamma_{y,w}} dz \, \varphi_k(z) \psi_l(z) \prod_{j=a}^{b} \frac{z - \theta_j^{-2} w_j}{z - w_j},$$

where the integration contour is around y_j, w_i and not $s_j^{-2} y_j$, and $(\mathbf{w}; \boldsymbol{\theta}) = (w_a, \ldots, w_b; \theta_a, \ldots, \theta_b)$ with $a \leq b$. The skew functions in (6.8) take the following determinantal form:

$$G_{\lambda^{(t)}/\lambda^{(t-1)}}(w_t; \theta_t) = \det\left[\mathsf{g}_{(\lambda_i^{(t)}+N-i)/(\lambda_j^{(t-1)}+N-j)}(w_t; \theta_t)\right]_{i,j=1}^{N}. \tag{B.3}$$

We observe that when evaluated at a single pair of variables $(w;\theta)$, both $\mathsf{h}_{k,j}$ (for $k \ge j$) and $\mathsf{g}_{l/k}$ become explicit:

Lemma B.1 *We have*

$$\mathsf{h}_{k,j}(w;\theta) = \begin{cases} \dfrac{w(1-\theta^{-2})\psi_k(w)}{y_j - w}, & k \ge j; \\ \text{anon}-\text{productexpression}, & k < j, \end{cases}$$

$$\mathsf{g}_{l/k}(w;\theta) = \begin{cases} w(1-\theta^{-2})\varphi_k(w)\psi_l(w), & l > k; \\ \dfrac{\theta^{-2}w - s_{k+1}^{-2}y_{k+1}}{w - s_{k+1}^{-2}y_{k+1}}, & l = k; \\ 0, & l < k. \end{cases}$$

Proof For $\mathsf{h}_{k,j}$ with $k \ge j$, the only pole inside the contour is $z = w$, which leads to the desired formula. The exact form of the functions $\mathsf{h}_{k,j}$ with $k < j$ is not very explicit (apart from the original contour integral expression), but they are not involved in our computations.

For $\mathsf{g}_{l/k}$, in the case $l = k$, the only singularity outside the contour is $z = s_{k+1}^{-2}y_k$, and for $l > k$ the only singularity inside the contour is $z = w$. The respective residues in these two cases lead to the desired formulas. For $l < k$, there are no singularities outside the integration contours, and the integral vanishes. □

Putting together (B.1), (B.2), and (B.3), we get:

Proposition B.2 *The probability weights under the ascending FG process (6.8) have the following product-of-determinants form. For $\ell_j^{(t)} := \lambda_j^{(t)} + N + 1 - j$, we have*

$$\mathcal{AP}(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(T)})$$
$$= \text{const} \cdot \det\left[h_{\ell_i^{(1)}-1,\,j}(w_1;\theta_1)\right] \prod_{t=2}^{T} \det\left[g_{(\ell_i^{(t)}-1)/(\ell_j^{(t-1)}-1)}(w_t;\theta_t)\right] \det\left[\varphi_{\ell_j^{(T)}-1}(x_i)\right],$$

where all determinants are taken with respect to $1 \le i, j \le N$, and const *is a normalizing constant which does not depend on the $\ell^{(j)}$'s.*

B.2 Application of the Eynard–Mehta theorem

The form of the probability weights as in Proposition B.2 puts the ascending FG process into the domain of applicability of the Eynard–Mehta theorem (see, for example, [31], [20, Theorem 1.4]). To express the determinantal correlation kernel of the point process

$$\{(t, \ell_j^{(t)}) : t = 1, \ldots, T,\ j = 1, \ldots, N\} \subset \{1, \ldots, T\} \times \mathbb{Z}_{\ge 1},$$
$$\ell_j^{(t)} = \lambda_j^{(t)} + N + 1 - j, \tag{B.4}$$

one first needs to invert the $N \times N$ "Gram matrix" given by

$$M_{ij} = \sum_{a_1,\dots,a_m \geq 0} \mathsf{h}_{a_1,i}(w_1;\theta_1)\, \mathsf{g}_{a_2/a_1}(w_2;\theta_2) \dots \mathsf{g}_{a_T/a_{T-1}}(w_T;\theta_T)\, \varphi_{a_T}(x_j). \tag{B.5}$$

Note that by Lemma B.1, this series converges absolutely under the condition (6.7).

Proposition B.3 *We have*

$$M_{ij} = \frac{1}{y_i - x_j} \prod_{t=1}^{T} \frac{x_j - \theta_t^{-2} w_t}{x_j - w_t}. \tag{B.6}$$

The proof is based on the following lemma:

Lemma B.4 *Let* $\left| \frac{u - s_j^{-2} y_j}{u - y_j} \frac{v - y_j}{v - s_j^{-2} y_j} \right| < 1 - \delta < 1$ *for all sufficiently large* $j \geq 1$. *Then we have*

$$\sum_{k=0}^{\infty} \varphi_k(u) \psi_k(v) = \frac{1}{u - v}.$$

Proof We have

$$\begin{aligned} \sum_{k=0}^{\infty} \varphi_k(u) \psi_k(v) &= \sum_{k=0}^{\infty} \frac{1}{u - y_{k+1}} \frac{y_{k+1}(1 - s_{k+1}^{-2})}{v - s_{k+1}^{-2} y_{k+1}} \prod_{j=1}^{k} \frac{u - s_j^{-2} y_j}{u - y_j} \frac{v - y_j}{v - s_j^{-2} y_j} \\ &= \sum_{k=0}^{\infty} \frac{1}{u - v} \left(1 - \frac{u - s_{k+1}^{-2} y_{k+1}}{u - y_{k+1}} \frac{v - y_{k+1}}{v - s_{k+1}^{-2} y_{k+1}} \right) \\ &\quad \prod_{j=1}^{k} \frac{u - s_j^{-2} y_j}{u - y_j} \frac{v - y_j}{v - s_j^{-2} y_j}, \end{aligned}$$

and the sum telescopes to $1/(u - v)$ if it converges (which holds under the condition in the hypothesis). □

Proof of Proposition B.3 We represent $\mathsf{h}_{a_1,i}$ as an integral over z_1, and each $\mathsf{g}_{a_t/a_{t-1}}$ as an integral over z_t, $2 \leq t \leq T$. Initially all the integration variables belong to the same contour $\Gamma_{y,w}$. However, in order to apply Lemma B.4 under the integrals, we need to have the following conditions on the contours for all sufficiently large $k \geq 1$:

$$\left| \frac{z_{t+1} - s_k^{-2} y_k}{z_{t+1} - y_k} \frac{z_t - y_k}{z_t - s_k^{-2} y_k} \right| < 1 - \delta < 1, \qquad \left| \frac{x_j - s_k^{-2} y_k}{x_j - y_k} \frac{z_T - y_k}{z_T - s_k^{-2} y_k} \right| < 1 - \delta < 1,$$

where $t = 1, \dots, T - 1$, $j = 1, \dots, N$. Clearly, under certain restrictions on the parameters, such contours exist. Moreover, we may also choose them to be nested: z_1

around all y_k and w_t, z_B around z_A if $B > A$, and all contours must leave outside all the points $s_k^{-2} y_k$. On these contours, we have by Lemma B.4:

$$M_{ij} = \frac{1}{(2\pi \mathbf{i})^T} \oint \ldots \oint \frac{1}{y_i - z_1} \frac{dz_1 \ldots dz_T}{(x_j - z_T)(z_T - z_{T-1}) \ldots (z_2 - z_1)} \prod_{t=1}^{T} \frac{z_t - \theta_t^{-2} w_t}{z_t - w_t}.$$

This integral is computed as follows. First, for z_T there is a single pole $z_T = x_j$ outside the contour (and the integrand has the zero residue at infinity). Taking the residue clears the denominator $x_j - z_T$ and substitutes $z_T = x_j$. After that, we repeat the procedure for $z_{T-1}, \ldots, z_1$, which leads to the desired formula.

Finally, the restrictions on the parameters under which the contours exist are lifted by an analytic continuation, since Lemmas B.1 and B.4 imply that the summation in (B.5) produces an a priori rational function. □

The matrix $M = [M_{ij}]_{i,j=1}^N$ is readily inverted:

Lemma B.5 *We have, for $i, j = 1, \ldots, N$,*

$$\begin{aligned} M_{ij}^{-1} &= \frac{1}{x_i - y_j} \frac{\prod_{k=1}^N (x_i - y_k)(y_j - x_k)}{\prod_{k \ne i} (x_i - x_k) \prod_{k \ne j} (y_j - y_k)} \prod_{t=1}^T \frac{x_i - w_t}{x_i - \theta_t^{-2} w_t} \\ &= \frac{1}{(2\pi \mathbf{i})^2} \oint_{\Gamma_{x_i}} d\xi \oint_{\Gamma_{y_j}} d\eta \frac{1}{\xi - \eta} \prod_{k=1}^N \frac{(\xi - y_k)(\eta - x_k)}{(\xi - x_k)(\eta - y_k)} \prod_{t=1}^T \frac{\xi - w_t}{\xi - \theta_t^{-2} w_t}, \end{aligned} \tag{B.7}$$

where the contours for ξ and η are small nonintersecting positively oriented circles around x_i and y_j, respectively, which do not include any other poles of the integrand.

Proof The first expression for M_{ij}^{-1} is obtained using the Cauchy determinant, since all minors (and hence all cofactors) of M are determinants of similar form. The contour integral expression corresponds to taking residues at the simple poles $\xi = x_i$ and $\eta = y_j$.

By the Eynard–Mehta theorem as in [20, Theorem 1.4], the correlation kernel of the determinantal point process (B.4) on $\{1, \ldots, T\} \times \mathbb{Z}_{\ge 1}$ takes the form (the shifts $a + 1, a' + 1$ correspond to the shifts in the determinantal representation in Proposition B.2):

$$\begin{aligned} &K_{\mathcal{A}\mathcal{P}}(t, a+1; t', a'+1) \\ &= -\mathbf{1}_{t > t'} \sum_{\alpha_{t'+1}, \ldots, \alpha_{t-1} \ge 0} \mathsf{g}_{\alpha_{t'+1}/a'}(w_{t'+1}; \theta_{t'+1}) \ldots \mathsf{g}_{\alpha_{t-1}/\alpha_{t-2}}(w_{t-1}; \theta_{t-1}) \mathsf{g}_{a/\alpha_{t-1}}(w_t; \theta_t) \\ &\quad + \sum_{i,j=1}^N M_{ji}^{-1} \sum_{\alpha_1, \ldots, \alpha_{t-1} \ge 0} \mathsf{h}_{\alpha_1, i}(w_1; \theta_1) \mathsf{g}_{\alpha_2/\alpha_1}(w_2; \theta_2) \ldots \mathsf{g}_{a/\alpha_{t-1}}(w_t; \theta_t) \\ &\quad \times \sum_{\beta_{t'+1}, \ldots, \beta_T \ge 0} \mathsf{g}_{\beta_{t'+1}/a'}(w_{t'+1}; \theta_{t'+1}) \ldots \mathsf{g}_{\beta_T/\beta_{T-1}}(w_T; \theta_T) \varphi_{\beta_T}(x_j). \end{aligned} \tag{B.8}$$

The iterated sums over the α_j's in the first and the second terms are finite and thus converge, and the sum over the β_j's is infinite but converges under (6.7), see Lemma B.1.

B.3 Computation of the kernel

Let us now compute all the sums in (B.8), and arrive at the resulting formula for the correlation kernel.

For the first summand arising when $t > t'$, we pass to the nested contours (z_B around z_A if $B > A$) as in the proof of Proposition B.3. We obtain

$$\sum_{\alpha_{t'+1},\dots,\alpha_{t-1}\ge 0} \mathsf{g}_{\alpha_{t'+1}/a'}(w_{t'+1};\theta_{t'+1})\dots \mathsf{g}_{\alpha_{t-1}/\alpha_{t-2}}(w_{t-1};\theta_{t-1})\mathsf{g}_{a/\alpha_{t-1}}(w_t;\theta_t)$$
$$= \frac{1}{(2\pi \mathbf{i})^{t-t'}} \oint \dots \oint \varphi_{a'}(z_{t'+1})\psi_a(z_t) \frac{dz_{t'+1}\dots dz_t}{(z_{t'+2}-z_{t'+1})\dots(z_{t-1}-z_{t-2})(z_t-z_{t-1})}$$
$$\prod_{i=t'+1}^{t} \frac{z_i-\theta_i^{-2}w_i}{z_i-w_i}, \tag{B.9}$$

where we extended the sum over the α_j's to all $\alpha_j \ge 0$ under the integral, and the infinite sums under the integral are computed using Lemma B.4. Next, deforming the contours $z_{t-1}, z_{t-2}, \dots, z_{t'+1}$ (in this order) to infinity, each integration in z_i picks up a residue at a single pole outside the integration contour at $z_i = z_t$. This leaves a single integral:

$$(B.9) = \frac{1}{2\pi \mathbf{i}} \oint_{\Gamma_{y,w}} dz\, \varphi_{a'}(z)\psi_a(z) \prod_{i=t'+1}^{t} \frac{z-\theta_i^{-2}w_i}{z-w_i}. \tag{B.10}$$

Arguing in a similar manner, we can compute

$$\sum_{\alpha_1,\dots,\alpha_{t-1}\ge 0} \mathsf{h}_{\alpha_1,i}(w_1;\theta_1)\mathsf{g}_{\alpha_2/\alpha_1}(w_2;\theta_2)\dots \mathsf{g}_{a/\alpha_{t-1}}(w_t;\theta_t)$$
$$= \frac{1}{(2\pi \mathbf{i})^{t}} \oint \dots \oint \frac{\psi_a(z_t)}{y_i-z_1} \frac{dz_1\dots dz_t}{(z_2-z_1)(z_3-z_2)\dots(z_t-z_{t-1})} \prod_{d=1}^{t} \frac{z_d-\theta_d^{-2}w_d}{z_d-w_d}$$
$$= \frac{1}{2\pi \mathbf{i}} \oint_{\Gamma_{y,w}} \frac{\psi_a(z)\,dz}{y_i-z} \prod_{d=1}^{t} \frac{z-\theta_d^{-2}w_d}{z-w_d},$$

and

$$\sum_{\beta_{t'+1},\dots,\beta_T\geq 0} \mathsf{g}_{\beta_{t'+1}/a'}(w_{t'+1};\theta_{t'+1})\dots\mathsf{g}_{\beta_T/\beta_{T-1}}(w_T;\theta_T)\varphi_{\beta_T}(x_j)$$
$$= \frac{1}{(2\pi\mathbf{i})^{T-t'}}\oint\dots\oint \frac{\varphi_{a'}(z_{t'+1})}{x_j-z_T}\frac{dz_{t'+1}\dots dz_T}{(z_{t'+2}-z_{t'+1})\dots(z_{T-1}-z_{T-2})(z_T-z_{T-1})}$$
$$\prod_{c=t'+1}^{T}\frac{z_c-\theta_c^{-2}w_c}{z_c-w_c}$$
$$= \varphi_{a'}(x_j)\prod_{c=t'+1}^{T}\frac{x_j-\theta_c^{-2}w_c}{x_j-w_c}.$$

In the latter computation we pick the residues at $z_T = x_j, \dots, z_{t'+1} = x_j$ (in this order), which is the only pole outside the corresponding integration contour. Finally, we take the last two quantities, multiply by M_{ji}^{-1}, and sum as in (B.8). Using (B.7), we have

$$\sum_{i,j=1}^{N}\frac{1}{(2\pi\mathbf{i})^2}\oint_{\Gamma_{x_j}}d\xi\oint_{\Gamma_{y_i}}d\eta\,\frac{1}{\xi-\eta}\prod_{k=1}^{N}\frac{(\xi-y_k)(\eta-x_k)}{(\xi-x_k)(\eta-y_k)}\prod_{t=1}^{T}\frac{\xi-w_t}{\xi-\theta_t^{-2}w_t}$$
$$\times\frac{1}{2\pi\mathbf{i}}\oint_{\Gamma_{y,w}}\frac{\psi_a(z)\,dz}{y_i-z}\prod_{d=1}^{t}\frac{z-\theta_d^{-2}w_d}{z-w_d}\,\varphi_{a'}(x_j)\prod_{c=t'+1}^{T}\frac{x_j-\theta_c^{-2}w_c}{x_j-w_c}$$
$$=\frac{1}{(2\pi\mathbf{i})^3}\oint_{\Gamma_x}d\xi\oint_{\Gamma_y}d\eta\oint_{\Gamma_{y,w}}dz\,\frac{1}{\eta-\xi}\frac{1}{\eta-z}\prod_{k=1}^{N}\frac{(\xi-y_k)(\eta-x_k)}{(\xi-x_k)(\eta-y_k)}$$
$$\times\frac{y_{a+1}(1-s_{a+1}^{-2})}{z-s_{a+1}^{-2}y_{a+1}}\frac{1}{y_{a'+1}-\xi}\prod_{j=1}^{a}\frac{z-y_j}{z-s_j^{-2}y_j}\prod_{j=1}^{a'}\frac{\xi-s_j^{-2}y_j}{\xi-y_j}$$
$$\prod_{d=1}^{t}\frac{z-\theta_d^{-2}w_d}{z-w_d}\prod_{c=1}^{t'}\frac{\xi-w_c}{\xi-\theta_c^{-2}w_c}.$$

To obtain the latter expression we substituted $x_j = \xi, y_i = \eta$, and changed the contours Γ_x, Γ_y for these variables to encircle all x_k's or all y_k's, respectively, while leaving all other poles outside. Observe now that the only pole in η outside the integration contour which produces a nonzero residue is at $\eta = z$. Indeed, the residue at $\eta = \xi$ eliminates all poles inside the ξ contour, and thus vanishes. Therefore, we may continue the above computation as follows:

$$=\frac{1}{(2\pi\mathbf{i})^2}\oint_{\Gamma_x}d\xi\oint_{\Gamma_{y,w}}dz\,\frac{1}{z-\xi}\prod_{k=1}^{N}\frac{(\xi-y_k)(z-x_k)}{(\xi-x_k)(z-y_k)}$$
$$\times\frac{y_{a+1}(1-s_{a+1}^{-2})}{z-s_{a+1}^{-2}y_{a+1}}\frac{1}{\xi-y_{a'+1}}\prod_{j=1}^{a}\frac{z-y_j}{z-s_j^{-2}y_j}\prod_{j=1}^{a'}\frac{\xi-s_j^{-2}y_j}{\xi-y_j}$$

$$\prod_{d=1}^{t} \frac{z-\theta_d^{-2} w_d}{z-w_d} \prod_{c=1}^{t'} \frac{\xi - w_c}{\xi - \theta_c^{-2} w_c}.$$

Let us drag the ξ contour through infinity, so that now it encircles the z contour $\Gamma_{y,w}$, and also all the points $\theta_i^{-2} w_i$. This leads to an extra minus sign.

Finally, we need to add the additional summand (B.10) if $t > t'$. In this case, observe that dragging the z contour so that it is outside of the ξ contour produces the same expression as (B.10), but with the opposite sign. Moreover, we need to undo the shifts $a+1, a'+1$ corresponding to the determinantal representation in Proposition B.2. Renaming the integration variables as $\xi = u$, $z = v$ leads to the final expression for the correlation kernel of the ascending FG process:

$$\begin{aligned} K_{\mathcal{AP}}(t,a;t',a') =& \frac{1}{(2\pi \mathbf{i})^2} \oint_{\Gamma_{y,w,\theta^{-2}w}} du \oint_{\Gamma_{y,w}} dv \, \frac{1}{u-v} \prod_{k=1}^{N} \frac{(u-y_k)(v-x_k)}{(u-x_k)(v-y_k)} \\ & \times \frac{y_a(1-s_a^{-2})}{v-s_a^{-2} y_a} \frac{1}{u-y_{a'}} \prod_{j=1}^{a-1} \frac{v-y_j}{v-s_j^{-2} y_j} \prod_{j=1}^{a'-1} \frac{u-s_j^{-2} y_j}{u-y_j} \\ & \prod_{d=1}^{t} \frac{v-\theta_d^{-2} w_d}{v-w_d} \prod_{c=1}^{t'} \frac{u-w_c}{u-\theta_c^{-2} w_c}. \end{aligned}$$

where the u contour is outside for $t \le t'$, and the v contour is outside for $t > t'$. This completes the proof of Theorem 6.7 in the ascending FG process case.

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