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Decomposition in Chern–Simons theories in three dimensions

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In this paper, we discuss decomposition in the context of three-dimensional Chern–Simons theories. Specifically, we argue that a Chern–Simons theory with a gauged noneffectively-acting one-form symmetry is equivalent to a disjoint union of Chern–Simons theories, with discrete theta angles coupling to the image under a Bockstein homomorphism of a canonical degree-two characteristic class. On three-manifolds with boundary, we show that the bulk discrete theta angles (coupling to bundle characteristic classes) are mapped to choices of discrete torsion in boundary orbifolds. We use this to verify that the bulk three-dimensional Chern–Simons decomposition reduces on the boundary to known decompositions of two-dimensional (WZW) orbifolds, providing a strong consistency test of our proposal.

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1. Introduction

Decomposition is the observation that some local quantum field theories are equivalent to disjoint unions of other local quantum field theories, essentially a counterexample to old lore linking locality and cluster decomposition. It was first[a] observed

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[a] For purposes of historical language translation, before the term "one-form symmetry" was coined, theories with one-form symmetries were sometimes called "gerby" theories, in reference to the fact that a gerbe is a fiber bundle whose fibers are higher groups.

in Ref. 1 in two-dimensional (2D) gauge theories and orbifolds with trivially-acting subgroups (nonminimally-charged matter),[2–4] and since then has been developed in many other references, see e.g. Refs. 5–30 and 31–35 for reviews.

Decomposition is not limited to two dimensions, and indeed four-dimensional versions of decomposition have been described in Refs. 16 and 18. The common thread linking these different examples involves what is now called a higher-form symmetry: a quantum field theory in d space–time dimensions decomposes if it has a global $(d-1)$-form symmetry (possibly realized noninvertibly).[16,18]

In this paper, following up,[30] we turn to decomposition in three-dimensional (3D) Chern–Simons theories with gauged noneffectively-acting one-form symmetries. Briefly, we find that

$$[\mathrm{Chern}-\mathrm{Simons}(H)/BA] = \coprod_{\theta\in\hat{K}} \mathrm{Chern}-\mathrm{Simons}(G)_{\theta}, \tag{1.1}$$

where $G = H/(A/K)$, $K \subset A$ defines the trivially-acting subgroup, and θ indicates a discrete theta angle coupling to an appropriate characteristic class of G bundles, On the boundary, this reduces to decomposition in noneffectively-acting orbifolds of 2D WZW models. A key role is played by the fact that the bulk discrete theta angles (coupling to bundle characteristic classes) become discrete torsion on the boundary, a result we explain in detail. The fact that the bulk decomposition correctly implies a known decomposition of the 2D boundary theory provides a strong consistency check on our proposal.

In two dimensions, decomposition has had a variety of applications, for example in giving nonperturbative constructions of geometries in phases of some gauged linear sigma models (GLSMs),[5,36–48] in Gromov–Witten theory,[6–11] in computing elliptic genera to check claims about IR limits of pure supersymmetric gauge theories,[17] and recently in understanding Wang–Wen–Witten anomaly resolution.[27–29,49]

Chern–Simons theories are the starting point for many physics questions, and so we anticipate that the results of this paper should have a variety of applications. For example, as is well known, 3D AdS gravity can be understood as a Chern–Simons theory,[50] making Chern–Simons theories a natural playground for addressing questions in 3D gravity, an approach used in e.g. Ref. 51 to address Marolf–Maxfield factorization questions.[52] We anticipate that this work may have analogous uses.

Similarly, one of the original applications of 2D decomposition was to understand phases of certain gauged linear sigma models, where decomposition was used locally (ala Born–Oppenheimer) to understand IR limits of certain theories as nonperturbatively-realized branched covers of spaces.[5] We expect that similar ideas could be used to understand the IR limits of certain Chern–Simons-matter theories.

We begin in Sec. 2 with a review of decomposition in 2D WZW orbifolds, which not only serves as a review of decomposition, but also describes the decomposition pertinent to boundaries in the 3D Chern–Simons theories we discuss.

In Sec. 3, we describe the primary proposal of this paper, namely, decomposition in Chern–Simons theories with gauged one-form symmetry groups, which takes the form (1.1). All Chern–Simons theories are assumed to have levels such that the theories exist on the three-manifolds over which they are defined. We describe how this bulk decomposition maps to boundary WZW models, and reproduces standard results on decomposition in 2D noneffective orbifolds, which serves as a strong consistency test of our claims. We also observe that in all these examples, the boundary discrete theta angles (choices of discrete torsion in boundary WZW models) are trivial, which is often reflected in the bulk discrete theta angles.

In Sec. 4, we discuss the spectra of these theories. We begin with an explanation and review of monopole operators, local operators (analogues of twist fields in 2D orbifolds) which can be used to construct projection operators. We then discuss line operators. When gauging ordinary one-form symmetries, the standard technology of anyon condensation can be used to describe the line operators. However, to describe noneffectively-acting one-form symmetries (in which a subgroup acts trivially), as relevant for this paper, requires a minor extension, which we propose and utilize.

In Sec. 5, we walk through the details of bulk and boundary decomposition, spectrum computations, and consistency tests such as level-rank duality in a variety of concrete examples.

Finally in Sec. 6, we briefly discuss the related case of boundary G/G models. These 2D theories decompose, and we briefly discuss their corresponding bulk theories.

In App. A, we summarize some results on line operators that are used in the main text. In App. B, we give a brief overview of crossed modules, to make this paper self-contained, as they are used in the description of 3D decomposition. In App. C, we describe gauging effectively-acting one-form symmetries without appealing to line operators.

2. Warm-Up: Decomposition in WZW Orbifolds

As a warm-up exercise, let us briefly review decomposition in two dimensions, and apply it towards orbifolds of WZW models.

Consider an orbifold $[X/\Gamma]$ where a central subgroup $K \subset \Gamma$ acts trivially on X. As has been discussed previously (see e.g. Ref. 1), for an ordinary (orientation-preserving) orbifold

$$\mathrm{QFT}([X/\Gamma]) = \coprod_{\theta \in \hat{K}} \mathrm{QFT}([X/G]_{\theta(\omega)}), \tag{2.1}$$

where $\theta(\omega)$ is a choice of discrete torsion, given as the image of the extension class $[\omega] \in H^2(G, K)$ corresponding to

$$1 \to K \to \Gamma \to G \to 1 \tag{2.2}$$

under the map $\theta : K \to U(1)$, yielding $\theta(\omega) \in H^2(K, U(1))$.

Consider a Γ orbifold of a WZW model for a group H, with $K \subset \Gamma$ acting trivially, and $G = \Gamma/K$ a subset of the center of H, acting freely on H. Then, as a special case of the decomposition above, we have that

$$[\mathrm{WZW}(H)/\Gamma] = \coprod_{\theta \in \hat{K}} \mathrm{WZW}(H/G)_{\theta(\omega)}, \tag{2.3}$$

with both sides at the same level. That said, (ordinary) discrete torsion vanishes for cyclic subgroups, so the only occasion on which $\theta(\omega)$ can be nontrivial will be if $H = \mathrm{Spin}(4n)$ and $\Gamma/K = \mathbb{Z}_2 \times \mathbb{Z}_2$. (We will discuss that case in Subsec. 5.7.)

For example, consider a $\mathbb{Z}_4$ orbifold of an $SU(2)$ WZW model, where a $\mathbb{Z}_2 \subset \mathbb{Z}_4$ acts trivially, and the $\mathbb{Z}_2$ coset is the freely-acting center of $SU(2)$. For an ordinary (orientation-preserving) orbifold, since there is no discrete torsion in a $\mathbb{Z}_2$ orbifold, we have that

$$[\mathrm{WZW}(SU(2))/\mathbb{Z}_4] = \coprod_2 \mathrm{WZW}(SO(3)) \tag{2.4}$$

(with all WZW models at the same level).

Although we will not utilize orientifolds in this paper, in principle one can also consider orientation-reversing orbifolds (orientifolds) of WZW models, see e.g. Refs. 53–58. See Ref. 59 and references therein for discussions of discrete torsion in orientifolds.

So far we have discussed discrete torsion weighting different universes. In principle, WZW models can also be weighted by analogues of discrete theta angles. Although these are better known in the case of gauge theories,[b] the point is that if a group manifold G has a torsion characteristic class, some $w \in H^2(G, F)$ for some coefficient module F, then there exists a discrete theta angle $\theta \in \hat{F}$ that weights maps into G, via a term in the action of the form

$$\int_\Sigma \langle \theta, \phi^* w \rangle, \tag{2.5}$$

where Σ is the worldsheet and $\phi : \Sigma \to G$ any map in the path integral. If $G = \tilde{G}/Z$ for some finite group Z, these discrete theta angles can also, for appropriate w, correspond to choices of discrete torsion in a Z orbifold of a WZW model on $\tilde{G}$.

In Subsec. 3.3, we shall see that the choices of discrete theta angle above that arise in the WZW orbifolds appearing on boundaries of decompositions of one-form-gauged Chern–Simons theories, are the same as choices of discrete torsion.

3. Decomposition in Noneffective One-Form Symmetry Gaugings

In general terms, one expects a decomposition in a d-dimensional quantum field theory whenever it has a global $(d-1)$-form symmetry.[16,18]

A typical example of a decomposition in two dimensions involves gauging a noneffective group action: a group action in which a subgroup acts trivially on the theory being gauged, in the sense that its generator commutes with the operators of that theory: $[J, \mathcal{O}] = 0$. Gauging a trivially-acting group results in a global one-form symmetry, which is responsible for a decomposition.

In principle, an analogous phenomenon exists in three dimensions, involving the gauging of "trivially-acting" one-form symmetries. Here, for a one-form action to be

[b] Discrete theta angles in gauge theories in unrelated contexts have a long history, see e.g. Ref. 60 (Sec. 6 in Ref. 61 and Sec. 4 in Ref. 36) for 2D examples and Refs. 62–64 for four-dimensional examples.

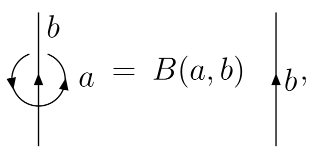
trivial means that it commutes with the line operators in the theory, as we shall elaborate below.

In this section, after a short overview of the notion of noneffective one-form symmetries, we make a precise prediction for decomposition.

3.1. *Noneffective one-form symmetry group actions*

We define a "trivially-acting" one-form symmetry in terms of the fusion algebra of the corresponding lines, and a "non-effective" one-form symmetry is one in which a subset of the lines acts trivially.

First, let us recall some basics of gauging one-form symmetries, which in three dimensions we will be described by the fusion algebra of line operators (see e.g. Subsec. 3.1 in Refs. 65–67 and Sec. 2 in Ref. 68) and references therein for a detailed discussion, with gauging as in e.g. Ref. 69. Anomalies in such a gauging are discussed in e.g. (Subsec. 2.3 in Ref. 51 and Sec. 2.1 in Refs. 70–73). In order to be gaugeable, its 't Hooft anomaly must vanish, which requires that the lines be mutually transparent, meaning that they have trivial mutual braiding. In particular, a one-form symmetry necessarily has abelian lines, for which the braiding is completely characterized by their spins (see e.g. Eq. (2.28) in Ref. 51 and Sec. 2 in Ref. 70), schematically



(3.1)

where

$$B(a,b) = \exp(2\pi i(h(a \times b) - h(a) - h(b))), \tag{3.2}$$

where a, b denote lines, and $h(a)$ mod 1 is the spin of the line a. Note that if the spins are integers, then $B = 1$ and there is no obstruction. Conversely, if $B = 1$, then spins are integers or half integers.

We take[c] a "trivially-acting BK" to be described by a set of lines $\{g\}$ such that all other lines b both

1. have trivial monodromy under g, meaning $B(g, b) = 1$, and also are
2. invariant under fusion with g, $g \times b = b$,

for all g. (In effect, there are two conditions in three dimensions, whereas invariance in two dimensions really boils down to a single constraint of the form $[J, \mathcal{O}] = 0$).

[c] We are using "B" to mean several different things in this section. We use BK to denote a one-form symmetry, a standard notation in mathematics, going back decades. (In physics, the notation $K^{[1]}$ is sometimes used instead). Later, we will use BG to denote a classifying space. In this section, we also use $B(a, b)$ to denote line monodromies.

To be clear, this notion can be somewhat counterintuitive. Consider for example $SU(2)$ Chern–Simons theory. This theory has a $B\mathbb{Z}_2$ one-form symmetry defined by the center of $SU(2)$. However, although the classical action is invariant under the center, the Wilson lines are not invariant, as the $B\mathbb{Z}_2$ action multiplies Wilson lines by phases (corresponding to the n-ality of the corresponding representation with respect to the center). In particular, the $B\mathbb{Z}_2$ action on $SU(2)$ Chern–Simons theory defined by the center of $SU(2)$ is not trivial.

3.2. *Basic decomposition prediction*

In Ref. 30, it was argued that in a quotient by a 2-group Γ of the form

$$1 \to BK \to \Gamma \to G \to 1, \tag{3.3}$$

where the BK acts trivially, the path integral sums over both K gerbes and a subset of G bundles, specifically G bundles satisfying a constraint.

In general, if one has a group H and an abelian group A with a map $d : A \to H$ whose image is in the center of H, then the crossed module[d] $\Gamma_. = \{A \to H\}$ defines a 2-group we shall label Γ. So long as we are interested in flat bundles, we can apply the same analysis as Ref. 30, and argue that Γ bundles on a three-manifold M map to $G = H/\text{im } A$ bundles satisfying a condition. This 2-group fits into an exact sequence

$$1 \to K \to A \xrightarrow{d} H \to G \to 1, \tag{3.4}$$

where $K = \text{Ker } d$. (Physically, d just encodes the A action, by projecting it to a subgroup of the center of H). This exact sequence defines an element

$$\omega \in H^3_{\text{group}}(G, K) = H^3_{\text{sing}}(BG, K), \tag{3.5}$$

which we will give explicitly in (3.9), and the condition that G bundles must satisfy to be in the image of Γ bundles is that

$$\phi^* \omega = 0, \tag{3.6}$$

for $\phi : M \to B\Gamma$ the map defining the Γ bundle on M, for the same reasons discussed in Ref. 30.

Next, we describe the element ω corresponding to the extension (3.4), appearing in the constraint (3.6) above. Let $Z = \text{im} d \subset Z(H)$, the center of H, and w_G the Z-valued degree-two characteristic class for G correspond to a generator of $H^2_{\text{sing}}(BG, Z)$. (For example, for $G = SO(n)$, w_G is the second Stiefel–Whitney class w_2). Let $\alpha \in H^2_{\text{group}}(Z, K)$ be the class of the extension

$$1 \to K \to A \to Z \to 1, \tag{3.7}$$

and let

$$\beta_\alpha : H^2_{\text{sing}}(BG, Z) \to H^3_{\text{sing}}(BG, K) \tag{3.8}$$

[d] See App. B for an introduction to crossed modules, or alternatively (App. A) in Ref. 74 and Sec. 2 in Ref. 75.

be the Bockstein homomorphism in the long exact sequence associated to the extension (3.7). Then

$$\omega = \beta_\alpha(w_G) \in H^3_{\rm sing}(BG, K). \tag{3.9}$$

When discussing boundary WZW models, it will be useful to describe ω differently. To that end, we use the fact that

$$H^n_{\rm sing}(BG, Z) = {\rm Map}(BG, K(Z, n)), \tag{3.10}$$

to write w_G and α as maps

$$w_G : BG \to K(Z, 2), \ \alpha : BZ(= K(Z, 1)) \to K(K, 2). \tag{3.11}$$

Since Eilenberg–Maclane spaces are in the stable category, where B exists as a functor, we can define

$$B\alpha : K(Z, 2) \to K(K, 3), \tag{3.12}$$

hence

$$B\alpha \circ w_G : BG \to K(K, 3), \tag{3.13}$$

and so defines an element of $H^3_{\rm sing}(BG, K)$. Furthermore, $B\alpha$ is just the Bockstein homomorphism β_α, hence

$$\omega = B\alpha \circ w_G = \beta_\alpha(w_G), \tag{3.14}$$

and so we recover the description of ω above.

So far, we have argued that on general principles, our Γ gauge theory should be described by a G gauge theory such that the G bundles satisfy the constraint (3.6). Just as in Refs. 1 and 30, such a restriction on instantons can be implemented by a sum over universes. The constraint (3.6), namely, $\phi^*\omega = 0$, is implemented by summing over G Chern–Simons theories with discrete theta angles coupling to ω, formally

$$[{\rm Chern-Simons}(H)/BA] = \coprod_{\theta \in \hat{K}} {\rm Chern-Simons}(G)_\theta, \tag{3.15}$$

where θ is the 3D discrete theta angle coupling to $\phi^*\omega$, for levels and underlying three-manifolds for which these theories are defined.[e] This is our prediction for decomposition in 3D Chern–Simons theories.

The G Chern–Simons theory is defined to be the $B({\rm im}A)$ gauging of the H Chern–Simons theory, at the same level as the H Chern–Simons theory. This is

[e] As has been noted in e.g. Ref. 69 (App. C in Ref. 76 and App. A in Ref. 77–80), not every Chern–Simons theory with every level is well-defined on every three-manifold. The basic issue is that Chern–Simons actions are not precisely gauge-invariant, but under gauge transformations shift by an amount proportional to 2π. Depending upon the gauge group and the three-manifold, the proportionality factor may or may not be integral. If k times that proportionality factor is integral, then the exponential of the action is gauge-invariant, and the theory is well-defined; if that product is not integral, then the path integral is not gauge-invariant and so not defined. Even if it is defined, it may depend upon subtle choices. For example, App. A in Ref. 77 argues that the (ordinary, bosonic) $U(1)_1$ Chern–Simons theory is well-defined only on spin three-manifolds, and furthermore that the choices of values of the action, the Chern–Simons invariants in the sense of Refs. 81 and 82, are in one-to-one correspondence with the spin structures. More generally, gauging one-form symmetries can create issues of this form, precisely because one twists gauge fields by gerbes, which results in "twisted" bundles and connections not present in the original theory, of fractional instanton numbers.

important to distinguish because sometimes gauging one-form symmetries can shift levels. For example, Sec. C.1 in Ref. 76 argues that, schematically, $U(1)_{4m}/B\mathbb{Z}_2 = U(1)_m$, and not $U(1)_{4m}$, despite the fact that as groups, $U(1)/\mathbb{Z}_2 = U(1)$.

The reader should note that the decomposition statement above correctly reproduces ordinary one-form gaugings. Consider the case that $K = 1$, so that the map $d : A \to H$ is one-to-one into the center of H. Then, decomposition (3.15) correctly predicts that

$$[\mathrm{Chern}-\mathrm{Simons}(H)/BA] = \mathrm{Chern}-\mathrm{Simons}(G), \tag{3.16}$$

which is a standard result (see e.g. Ref. 69). Decomposition becomes interesting in cases in which $K \neq 1$.

In Sec. 5, we will check this statement in several examples, outlining how it both reproduces known results as well as explains new cases.

3.3. *Boundary WZW models*

Let us now turn to Chern–Simons theories on manifolds with boundary, and the corresponding theories on the boundaries. We will see that the bulk Chern–Simons decomposition of the previous section correctly predicts a decomposition of boundary WZW models, which matches existing results on decomposition in 2D orbifolds. This matching involves a rather interesting relation between characteristic classes of bundles on three-manifolds and choices of discrete torsion in 2D orbifolds. In particular, the fact that the 3D decomposition correctly reproduces 2D decomposition on the boundary is an important consistency test of our proposal.

Briefly, as has been discussed elsewhere (see e.g. Refs. 83–87, Subsec. 4.2 in Ref. 88, Subsec. 5.2 in Ref. 89, and in related contexts[90,91]), on a three-manifold with boundary, a bulk Chern–Simons theory for gauge group G naturally couples to a (chiral) WZW model for the group G on the boundary. If the Chern–Simons theory has level k, then (see e.g. Subsec. 4.2 in Ref. 88) the boundary WZW model has level $\tau(k)$, where

$$\tau : H^n_{\mathrm{sing}}(BG, F) \to H^{n-1}_{\mathrm{sing}}(G, F) \tag{3.17}$$

is the loop space map[f] for any abelian group F, and we take Chern–Simons levels[g] $k \in H^4_{\mathrm{sing}}(BG, \mathbb{Z})$, and WZW levels $\tau(k) \in H^3_{\mathrm{sing}}(G, \mathbb{Z})$. Similarly, if the Chern–Simons theory has a discrete theta angle coupling to some characteristic class defined by an element of $\omega \in H^3(BG, F)$, then the boundary WZW model couples[h]

[f] This is the natural map

$$\begin{aligned} &H^n_{\mathrm{sing}}(BG, F) = \mathrm{Map}(BG, K(F, n)) \\ &\to \mathrm{Map}(\Omega(BG), \Omega(K(F, n))) = \mathrm{Map}(G, K(F, n-1)) = H^{n-1}_{\mathrm{sing}}(G, F). \end{aligned} \tag{3.18}$$

which sends any $f \in \mathrm{Map}(BG, K(F, n))$ to $\Omega(f)$. For later use, to construct explicit maps, one needs concrete choices of e.g. $X \mapsto \Omega BX$, for which we refer the reader to e.g. Refs. 92–95. As such choices do not alter cohomology classes, we will not discuss them explicitly in this paper.

[g] As before, levels are assumed to be such that the theory exists.

[h] We would like to thank Y. Tachikawa for a discussion of discrete theta angles in this context.

to a discrete theta angle defined by $\tau(\omega) \in H^2(G, F)$. Such discrete theta angles in 2D WZW models are reviewed in Sec. 2.

Given that standard bulk Chern–Simons/boundary WZW model relationship reviewed above, the 3D decomposition prediction 3.15 implies that in the associated boundary RCFT, an A orbifold of a WZW model for H is equivalent to a disjoint union of WZW models for G

$$[\mathrm{WZW}(H)/A] = \coprod_{\theta \in \hat{K}} \mathrm{WZW}(G)_\theta, \tag{3.19}$$

with levels and discrete theta angles related to those of the bulk theory by the map τ. We will see later in this section that although the WZW discrete theta angles θ are derived from characteristic classes in the Chern–Simons theory, they nevertheless correspond to choices of discrete torsion in the boundary orbifolds.

As a consistency check, let us show that τ commutes with gauging BA, so that the levels on the left-and right-hand sides of (3.19) match, just as they did[i] in the bulk prediction (3.15). First, for G as any topological group, there is a natural homotopy equivalence between the loop space $\Omega(BG)$ and G (meaning that BG is a delooping of G). Also, for any abelian group F, the Eilenberg–Maclane space $K(F, n-1)$ is homotopy equivalent to loop space $\Omega(K(F, n))$. Since

$$H^n_{\mathrm{sing}}(BG, F) = \mathrm{Map}(BG, K(F, n)) \tag{3.20}$$

and since Ω is a functor, for any continuous homomorphism $f : G_1 \to G_2$ between topological groups G_1, G_2, there is a continuous map $Bf : BG_1 \to BG_2$ and natural maps

$$\mathrm{Map}(BG_2, K(F, n)) \to \mathrm{Map}(BG_1, K(F, n)), \tag{3.21}$$

$$a \mapsto B(f \circ a) \tag{3.22}$$

and

$$\mathrm{Map}(G_2, K(F, n-1)) \to \mathrm{Map}(G_1, K(F, n-1)), \tag{3.23}$$

$$b \mapsto f \circ b. \tag{3.24}$$

Combining these maps, one finds that for any Lie group G with K as a subgroup of the center, the following diagram commutes:

$$\begin{array}{ccc} H^3_{\mathrm{sing}}(B(G/K), F) & \longrightarrow & H^2_{\mathrm{sing}}(G/K, F) \\ \downarrow & & \downarrow \\ H^3_{\mathrm{sing}}(BG, F) & \longrightarrow & H^2_{\mathrm{sing}}(G, F). \end{array} \tag{3.25}$$

This tells us that the levels appearing on either side of the boundary WZW relation (3.19) match, as expected, consistent with the prediction (3.15) of the bulk Chern–Simons theory.

[i] Modulo subtleties discussed there in special cases, such as those arising from the fact that $U(1)/\mathbb{Z}_k = U(1)$ as a group, but the corresponding Chern–Simons theories have different levels.

Now, we will argue that the WZW model discrete theta angles, arising as τ of characteristic classes in the Chern–Simons theory, are the same as choices of discrete torsion in the boundary theory. This will be important in understanding how the 3D Chern–Simons decomposition compares to 2D decompositions as reviewed in Sec. 2. For simplicity, we will assume that H is the universal covering, so that $Z = \pi_1(G)$. (Similar results exist in more general cases).

To that end, since τ is the loop space functor, we can write

$$\tau(\beta_\alpha(w_G)) = \tau(B\alpha \circ w_G) = \Omega(B\alpha \circ w_G) = \Omega(B\alpha) \circ \Omega(w_G). \tag{3.26}$$

Now, $\Omega(B\alpha) = \alpha$, and

$$\Omega(w_G) \in \mathrm{Map}(\Omega(BG), \Omega(K(Z,2))) = \mathrm{Map}(G, K(Z,1)), \tag{3.27}$$

so $\Omega(w_G)$ is a map $G \to BZ$. Now, we claim that $\Omega(w_G)$ is also the cell attachment map p of the Postnikov tower, $p : G \to B\pi_1(G) = BZ$, where $Z = \pi_1(G)$.

To make this clear, recall that the Postnikov tower map is the classifying map for the universal cover. In other words, if $\tilde{G}$ is the universal covering group of G, then $p^* EZ = \tilde{G}$. Now, on the other hand, $B\tilde{G} \to BG$ is a principal $K(Z,1)$ bundle on BG, which corresponds to a map $BG \to B(K(Z,1)) = K(Z,2)$, which is w_G. Applying the loop space functor gives the map $\Omega(w_G) : G \to K(Z,1) = BZ$, which is then more or less tautologically p.

In particular, we see that

$$\tau(B\alpha \circ w_G) = \alpha \circ \Omega(w_G) = \alpha \circ p. \tag{3.28}$$

The expression above relates the Chern–Simons discrete theta angles (coupling to bundle characteristic classes) to discrete torsion on the boundary. We can see this as follows. If $\phi : \Sigma \to G$ is any map from the worldsheet Σ into the target G, then $p \circ \phi : \Sigma \to BZ$ defines a Z-twisted sector over Σ. In particular, the discrete theta angle phase

$$\langle \theta, \phi^*(\alpha \circ p) \rangle, \tag{3.29}$$

for $\theta : K \to U(1)$ any character of K, corresponds to discrete torsion in the Z-twisted sector defined by $p \circ \phi$, specifically discrete torsion given by $\theta(\alpha) \in H^2_{\mathrm{group}}(Z, U(1))$, for $\alpha \in H^2_{\mathrm{group}}(Z, K)$. Thus, we see that τ relates discrete theta angles coupling to bundle characteristic classes on three-manifolds, to discrete torsion in 2D orbifolds on boundaries.

In passing, this phenomenon that 3D bulk discrete theta angles become discrete torsion in boundary 2D orbifolds is also visible in the case that the bulk theory is a finite 2-group orbifold, see Subsec. 3.2 in Ref. 30.

Now, let us compare the decomposition (3.19) in boundary WZW models, implied by bulk Chern–Simons decomposition, to standard results[1] on decomposition in 2D orbifolds, as reviewed earlier in Sec. 2.

Certainly, the form of the boundary decomposition (3.19) is identical to that arising in 2D orbifolds with trivially-acting central subgroups, possibly modulo the form of the discrete theta angles. We have just argued that the discrete theta angles arising on the boundary correspond to choices of discrete torsion, and in fact, the discrete torsion phases arising in the boundary case match those in the ordinary 2D case.

We can relate these two pictures of boundary discrete theta angles as follows. Recall $\alpha \in H^2_{\rm group}(Z, K)$ is the class of the extension

$$1 \to K \to A \to Z \to 1. \tag{3.30}$$

In 2D decomposition in A orbifolds with trivially-acting central subgroups K, the discrete torsion phase factors on a universe associated with $\theta \in \hat{K}$ are precisely the image of α under θ:

$$H^2_{\rm group}(Z, K) \to H^2_{\rm group}(Z, U(1)), \tag{3.31}$$

$$\alpha \mapsto \theta \circ \alpha. \tag{3.32}$$

These are the same as the discrete torsion phases arising in the boundary WZW decomposition (3.19), as we have just discussed, and we will confirm explicitly in examples in Sec. 5 that the decomposition above in the boundary theory precisely coincides with the decomposition of WZW orbifolds given in (2.3). This matching is an important consistency test of our proposal.

3.4. *Nontriviality of discrete theta angles*

In the boundary WZW models appearing in these decompositions, the discrete torsion on each universe appearing in a decomposition is trivial. For most single group factors, this is because the center is usually a cyclic group, and cyclic group orbifolds have no discrete torsion. The exceptions are the groups Spin$(4n)$, which have center $\mathbb{Z}_2 \times \mathbb{Z}_2$. That finite group admits discrete torsion; however, to generate the discrete torsion in a decomposition of a string orbifold, the orbifold group must be nonabelian, and so cannot arise as the boundary of a 3D theory, as we will discuss in greater detail in Subsec. 5.7.

In at least some examples, not only are the boundary discrete theta angles (discrete torsions) trivial, but the bulk discrete theta angles are also trivial. For example,[j] in bulk theories, for cases in which $K = \mathbb{Z}_2$, $Z = \mathbb{Z}_2$, and $A = \mathbb{Z}_4$, so that the extension α is

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1, \tag{3.33}$$

the bulk discrete theta angle couples to the Bockstein β_α of a distinguished element $w_G \in H^2(M_3, \mathbb{Z}_2)$. Now, for this α

$$\beta_\alpha(w_G) = {\rm Sq}^1(w_G), \tag{3.34}$$

and as we will argue in Subsec. 5.3

$${\rm Sq}^1(w_G) = w_1(TM_3) \cup w_G, \tag{3.35}$$

hence it can only be nonzero on nonorientable spaces. However, we only define Chern–Simons theories on oriented three-manifolds, so for all cases we consider, these bulk discrete theta angles vanish.

[j] We would like to thank Y. Tachikawa for making this observation.

Similarly, if the three-manifold is T^3, the pertinent Bockstein homomorphism will vanish, and one cannot get a nonzero bulk discrete theta angle. Briefly, for any short exact sequence

$$1 \to K \to A \to Z \to 1, \tag{3.36}$$

for K, A, Z abelian, the induced map

$$H^2(T^3, A) \to H^2(T^3, Z) \tag{3.37}$$

is surjective (since each of those cohomology groups is just Hom from a free abelian group into the coefficients), which implies that in the long exact sequence

$$H^2(T^3, K) \to H^2(T^3, A) \to H^2(T^3, Z) \stackrel{\beta}{\to} H^3(T^3, K), \tag{3.38}$$

the Bockstein $\beta = 0$, and so the bulk discrete theta angles are trivial in the corresponding cases.

For another example, consider Lens spaces. From Example 3E.2 in Ref. 96, for the Bockstein associated to the short exact sequence

$$1 \to \mathbb{Z}_m \to \mathbb{Z}_{m^2} \to \mathbb{Z}_m \to 1, \tag{3.39}$$

the associated Bockstein maps generators of $H^1(L, \mathbb{Z}_m)$ to generators of $H^2(L, \mathbb{Z}_m)$, for L as a Lens space, but $\beta^2 = 0$, hence the associated Bockstein map

$$\beta : H^2(L, \mathbb{Z}_m) \to H^3(L, \mathbb{Z}_m) \tag{3.40}$$

necessarily vanishes, and so the bulk discrete theta angles are trivial in corresponding cases.

More generally, whether the bulk discrete theta angles are always trivial is a reflection of the map $\tau : H^3_{\rm sing}(BG, K) \to H^2_{\rm sing}(BG, K)$. For example, if τ is injective, then triviality of the boundary discrete theta angles implies triviality of the bulk discrete theta angles. We leave general questions about the injectivity of τ for future work.

In passing, note that in the bulk, orientability plays a key role. At least abstractly, it is tempting to speculate about more general cases involving, e.g. orientifolds of boundary WZW models, as might arise if the three-manifold descends to a solid Klein bottle (a three-manifold whose boundary is the 2D Klein bottle). On such a non-orientable space, at least sometimes the discrete theta angles would be nontrivial. Furthermore, in orientifolds, discrete torsion is counted by $H^2_{\rm group}(Z, U(1))$ with a nontrivial action on the coefficients (see e.g. Refs. 53, 54, 58 and 59), so that for example $H^2_{\rm group}(\mathbb{Z}_2, U(1))$ can be nonzero, which again would result in boundary WZW models with nonzero discrete theta angle contributions.

4. Spectra

In this section, we briefly describe the spectra of monopole operators and line operators in a theory with a gauged trivially-acting one-form symmetry, and argue that the results are consistent with decomposition (3.15).

4.1. *Monopole operators*

In 2D theories, when one gauges a noneffectively-acting group, one gets twist fields and Gukov–Witten operators corresponding to conjugacy classes in the trivially-acting subgroup. In 3D theories, instead of twist fields, one has monopole operators (see e.g. Refs. 97 and 98), which play the same role. In this section, we will outline their properties.

In two dimensions, twist fields generate branch cuts, which in the language of topological defect lines are real codimension-one walls that implement the gauging of the zero-form symmetry. In three dimensions, when gauging a one-form symmetry, from thinking about topological defect ones, one sees the theory has codimension-two lines, which end in monopole operators, in the same way that in two dimensions, the orbifold branch cuts terminate in twist fields.

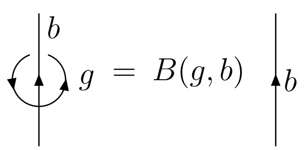
We can think of the monopole operators in three dimensions as local disorder operators: on a sphere surrounding the monopole operator associated to a BG symmetry, one has a nontrivial G gerbe, corresponding to an element of $H^2(S^2, G)$ (for G assumed finite), just as on a circle surrounding a twist field in two dimensions one has a nontrivial bundle.

In two dimensions, the twist fields associated to trivially-acting gauged zero-form symmetries are local dimension-zero operators, which can be used to form projectors onto the universes of decomposition. In three dimensions, the monopole operators associated to trivially-acting gauged one-form symmetries are closely analogous, and can again be used to form projection operators, in exactly the same fashion. In Subsec. 4.1.4 in Ref. 30, projection operators are explicitly constructed from monopole operators in 3D theories, and we encourage the reader to consult that reference for further details.

4.2. *Line operator spectrum*

Given a "gaugable" one-form symmetry, described by a subset of the lines in the theory, there is a standard procedure for computing the spectrum of lines in the gauged theory, given as follows (see e.g. Sec. 2 in Ref. 69 and Subsec. 2.5 in Ref. 99).[51] For $B\mathbb{Z}_n$, let g denote a line generating the others, and then

- Exclude from the spectrum all lines a which have monodromy[k] $B(g, a) \neq 1$, under a generator g, where the g action on a line b is determined by the process



(4.1)

- Identify any two lines b, $g \times b$ that differ by fusion with g, and finally

[k] In terms of the S-matrix, $B(a, b) = S_{ab}/S_{0b}$, see e.g. Eq. (40) in Ref. 68.

- Lines b that are invariant under fusion (meaning $a = g \times a$) become n lines in the spectrum of the gauged theory.

This is closely analogous to 2D orbifolds, in which one omits noninvariant operators, and fixed points lead to twist fields.

This is a special case of a more general procedure, sometimes known as anyon condensation, which is also applicable to noninvertible symmetries, unlike the basic algorithm above. (See e.g. Subsec. 4.1. in Ref. 51),[25,99–102] for further details.

Now, in our case, the (noneffectively-acting) one-form symmetry we wish to gauge is not described by a set of lines within the original theory. Sometimes, in some special cases, we can describe it by adding lines to the theory, such as in the case that the entire gauged one-form symmetry acts trivially. In general, however, that procedure is not well-defined. Consider for example the case of $SU(2)_4$ Chern–Simons theory, whose spectrum of lines $\{(0),(1),(2),(3),(4)\}$ is as described in App. A. Let us consider gauging a $B\mathbb{Z}_4$. Now, $SU(2)_4$ has a $B\mathbb{Z}_2$ symmetry, corresponding to the lines (0), (1), so one could imagine extending it to $B\mathbb{Z}_4$ by replacing $\{(0),(1)\}$ with $\{(0),\ell_1,\ell_2,\ell_3\}$ which obey

$$\ell_i \times \ell_j = \ell_{i+j \bmod 4}, \tag{4.2}$$

with $\ell_0 = (0)$. In order for the image of $\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_4$ to act trivially, we require

$$\ell_2 \times (2,3,4) = (2,3,4), \tag{4.3}$$

and for this to descend to the ordinary $SU(2)_4$, we also require

$$\ell_1 \times (2,3,4) = \ell_3 \times (2,3,4), \tag{4.4}$$

which must match $(1) \times (2,3,4)$ in the $SU(2)_4$ fusion algebra given in App. A

$$\ell_{1,3} \times (2) = (3), \quad \ell_{1,3} \times (3) = (2), \quad \ell_{1,3} \times (4) = (4). \tag{4.5}$$

The new lines ℓ_i are then defined to have trivial monodromy with all other lines

$$B(\ell_i, x) = 1 \tag{4.6}$$

for all lines x. This much is uniquely specified by the statement of the extension.

Now, we have not completely specified the extension of $SU(2)_4$; for example, the product $(2) \times (3)$ in $SU(2)_4$ contains a (1), so we would still need to decide whether to replace (1) with ℓ_1 or ℓ_3, for example. However, in making such choices, we find an internal contradiction with the structure we have already described, namely, a failure of associativity. For example, in $SU(2)_4$, as described in App. A

$$(2) \times (3) = (1) + (4), \quad (3) \times (3) = (0) + (4). \tag{4.7}$$

We could replace the (1) above with either ℓ_1 or ℓ_3. Suppose we take

$$(2) \times (3) = \ell_1 + (4). \tag{4.8}$$

Then,

$$\ell_1 \times ((2) \times (3)) = \ell_1 \times (\ell_1 + (4)) = \ell_2 + (4), \tag{4.9}$$

$$(\ell_1 \times (2)) \times (3) = (3) \times (3) = (0) + (4). \tag{4.10}$$

However, $\ell_2 \neq (0)$, so we see that

$$\ell_1 \times ((2) \times (3)) \neq (\ell_1 \times (2)) \times (3), \tag{4.11}$$

and so associativity is broken. We encounter a similar problem if we choose

$$(2) \times (3) = \ell_3 + (4) \tag{4.12}$$

instead and consider fusion with ℓ_3. Put simply, we cannot enlarge the $B\mathbb{Z}_2$ inside $SU(2)_4$ to a noneffective $B\mathbb{Z}_4$, without breaking associativity.

With this in mind, we outline here [1] a minor extension of the prescription of Sec. 2 in Ref. 69 and Subsec. 2.5 in Ref. 51, 99 for counting line operators in 3D theories with gauged one-form symmetries. (As we will not be using noninvertible symmetries, we will not attempt to describe the analogous construction for condensation algebra objects here).

Our approach is motivated by the action of a group G on a set M: we distinguish G and M, G is not a subset of M in general, though we can still define an action of G on M that enables us to make sense of the quotient M/G. Let G be a finite abelian group, so that BG is a group of one-form symmetries, and associate lines to elements of G. Consider a set of simple lines $\mathcal{C}$ (objects in a braided tensor category, which we will gauge).

An action of BG on $\mathcal{C}$ is then described by giving, for each $g \in G$ and line $L \in \mathcal{C}$,

- a monodromy $B(g, L)$, such that

$$B(g_1, L)B(g_2, L) = B(g_1 g_2, L), \tag{4.13}$$

  and
- a fusion $g \times L \in \mathbb{Z}[\mathcal{C}]$, meaning $g \times L = \sum_c N_{gL}^c c$ for $c \in \mathcal{C}$ and $N_{gL}^c \in \mathbb{Z}$, with the property that

$$g_1 \times (g_2 \times L) = (g_1 g_2) \times L. \tag{4.14}$$

We say that a line in BG corresponding to $g \in G$ acts trivially if, for all $L \in \mathcal{C}$,

$$B(g, L) = 1 \quad \text{and} \quad g \times L = L, \tag{4.15}$$

and then to say BG acts noneffectively, as in Subsec. 3.1, means that some for some $g \neq 1$ in G, the line corresponding to G acts trivially.

[1] Although we are only interested in isomorphism classes of objects, presumably the full categorical description is in terms of module categories, or as a minor variation on the group actions described in Sec. III. B in Ref. 68. As we are only interested here in counting (isomorphism classes of) objects, and this is merely a minor variation on existing methods, we will be very schematic.

Now, given an action of BG on $\mathcal{C}$, we propose to construct the lines of the quotient $\mathcal{C}/BG$ as follows, by close analogy with (Subsec. 2 in Ref. 69 and Subsec. 2.5 in Ref. 99).[51]

- Exclude any L such that for some $g \in G$, $B(g, L) \neq 1$,
- Identify $L \sim g \times L$ for each $g \in G$,
- For each $g \in G$ such that $g \times L = L$, we get a copy of L in $\mathcal{C}/BG$.

It is straightforward to check that in the special case the lines in BG are a subset of those in $\mathcal{C}$, this reduces to the prescription reviewed earlier and in Sec. 2 in Ref. 69 and Subsec. 2.5 in Ref. 99.[51]

As another special case, note that if all of BG acts trivially, then in the quotient $\mathcal{C}/BG$,

- No lines in $\mathcal{C}$ are excluded, since $B(g, L) = 1$ for all L,
- No lines are identified, since $g \times L = L$ for all L, so fusion does not relate different lines,
- Since $g \times L = L$ for each $g \in G$ and each $L \in \mathcal{C}$, we get $|G|$ copies of the lines in $\mathcal{C}$.

This is consistent with the expectations of decomposition in this case: if we gauge a BG that acts completely trivially on a theory, in the sense above, one expects to get $|G|$ copies of the theory.

We will apply this computation in specific examples in Chern–Simons theories later in this paper, but for the moment, we give two toy examples, to illustrate the idea.

First, consider $B\mathbb{Z}_2/B\mathbb{Z}_2$. Let the lines of $\mathcal{C} = B\mathbb{Z}_2$ be generated over $\mathbb{Z}$ by $\{(0), (1)\}$, where

$$(0) \times (0) = (0), (0) \times (1) = (1), (1) \times (1) = (0), \tag{4.16}$$

and $B\mathbb{Z}_2$ acts as

$$g \times (0) = (1), g \times (1) = (0), \tag{4.17}$$

and we take all monodromies $B = 1$. Then, applying the procedure above, to get the lines of $\mathcal{C}/B\mathbb{Z}_2$,

- Since $B(g, L) = 1$ for all $L \in \mathcal{C}$, no lines are excluded,
- Since $g \times (0) = (1)$, $(0) \sim (1)$,
- No lines are invariant.

Hence, the quotient is generated by one single line, as one would expect.

Next, consider $B\mathbb{Z}_2/B\mathbb{Z}_4$, where the $B\mathbb{Z}_4$ acts noneffectively. Let the lines of $\mathcal{C} = B\mathbb{Z}_2$ be as above, and the generator g of $B\mathbb{Z}_4$ acts as

$$g \times (0) = (1), g \times (1) = (0). \tag{4.18}$$

As before, we take all monodromies $B = 1$. Applying the procedure above

- Since $B(g, L) = 1$ for all $L \in \mathcal{C}$, no lines are excluded.
- Since $g \times (0) = (1)$, $(0) \sim (1)$,
- Since $g^2 \times (0) = (0)$ and $g^2 \times (1) = (1)$, $(0) \sim (1)$ appears twice in the quotient.

Thus, the quotient $\mathcal{C}/B\mathbb{Z}_4$ is generated over $\mathbb{Z}$ by two lines, as expected since a $\mathbb{Z}_2 \subset \mathbb{Z}_4$ describes trivially-acting lines.

We should also briefly observe that the theories we are describing, which decompose, have the property that they violate the axiom of remote detectability in a topological order, see e.g. Refs. 103–105. This axiom says that there are no invisible lines in the bulk theory (technically, that the category of lines has trivial center). Violation of remote detectability signals multiple vacua and therefore a decomposition, much as cluster decomposition in other contexts.[1]

4.3. *Bulk-boundary map*

Let us now consider the bulk-boundary map between lines in the 3D bulk and on the 2D boundary. Let $\mathcal{C}$ be the category of lines which act trivially in the bulk. Suppose we have a line in $\mathcal{C}$ which ends on the boundary, defining an object in the 2D vertex operator algebra V. We can describe this bulk-boundary relation by a functor

$$F : \mathcal{C} \to \mathrm{Rep}(V), \tag{4.19}$$

(for $\mathrm{Rep}(V)$ the category of representations of V) which takes a line to the vector space of ways and the line can end on the boundary, giving point operators.

As observed in Subsec. 3.5 in Ref. 106, a one-form symmetry that acts trivially in the bulk might act nontrivially on the boundary, and the theory can still decompose, much as with Chan–Paton factors and D-branes in 2D theories. Broadly speaking, the different line operators in the 3D bulk end on the various 2D sectors of the boundary theory.

A 3D theory may have surface operators which are not totally determined by the line operators. In the case where the 3D theory has only a local vacuum, all the surfaces can be built as condensations, i.e. networks of lines. However, when there are multiple vacua, as in the cases we are interested in, then this fails to be true. The surfaces which are not built as a network of lines will end on a line on the boundary. These lines define the "action" of a trivially acting zero-form symmetry, In two dimensions, if one gauges a trivially-acting zero-form symmetry, then one obtains an emergent global one-form symmetry (and hence a decomposition).

From the decomposition conjecture (3.15), the different universes and hence the different ground states are labeled by elements of the Pontryagin dual of the one-form symmetry group. On the other hand, the surfaces in the bulk which enact a 2-form symmetry, come from gauging a trivially acting one-form symmetry. So while the lines that the surface ends on has trivial action on the boundary, the surface itself is not necessarily trivial in the bulk. This is summarized in the following diagram, where F is the functor that makes objects to the boundary:

$$
\begin{array}{ccc}
\underline{\text{Bulk symmetry}} & & \underline{\text{Boundary symmetry}} \\
\\
\text{trivially-acting one-form} & \xrightarrow{F} & \text{trivially-acting zero-form} \\
\big\downarrow \text{gauge} & & \big\downarrow \text{gauge} \\
\text{global two-form} & \xrightarrow{F} & \text{global one-form}
\end{array}
\tag{4.20}
$$

5. Examples

In the following several subsections, we will walk through examples of the decomposition proposed in Sec. 3. Where possible, we will apply level-rank duality to perform self-consistency tests. In all cases, we will compare to the decomposition of the boundary WZW model. In particular, as reviewed in Sec. 2, decomposition is reasonably well-understood in 2D theories, and so we get solid consistency tests by checking that the boundary WZW decomposition implied by the bulk Chern–Simons decomposition matches existing 2D results.

In each case, we will assume that levels are chosen so that the theories are well-defined, but will not list those conditions explicitly.

5.1. *Chern–Simons*$(SU(2))/B\mathbb{Z}_2$, $K = 1$

In this section, we will reproduce a well-known result as a special case of the decomposition prediction (3.15).

Specifically, we consider gauging the $B\mathbb{Z}_2$ central one-form symmetry in $SU(2)$ Chern–Simons theory.

Here, this $B\mathbb{Z}_2$ is not trivially-acting, and so no decomposition is expected. In particular, this gauging is known (see e.g. Ref. 69) to be equivalent to the $SO(3)$ Chern–Simons theory at the same level. At the level of the path integral for the gauge theory, this is discussed in App. C.

We can understand this as a special case of the decomposition prediction (3.15). In the language of that statement, we identify $A = \mathbb{Z}_2$, $H = SU(2)$, and $d : A \to H$ is the inclusion map of the center, $\mathbb{Z}_2 \hookrightarrow SU(2)$. Then, the kernel of d vanishes, so $K = 1$, and $G = H/A = SO(3)$. This corresponds to the exact sequence

$$1 \to 1 \to \mathbb{Z}_2 \xrightarrow{d} SU(2) \to SO(3) \to 1. \tag{5.1}$$

Furthermore, in the case, since $K = 1$, the extension class $[\omega] \in H^3(G, K)$ is trivial, $\omega = 1$, so $\phi^*\omega = 1$ and there is no discrete theta angle.

Putting this together, we see that the decomposition prediction (3.15) in this case is

$$[\text{Chern}-\text{Simons}(SU(2))/B\mathbb{Z}_2] = \text{Chern}-\text{Simons}(SO(3)), \tag{5.2}$$

which reproduces known results.

Let us also compute the line operator spectrum in this example. This is a standard computation, but we will quickly outline it using the tools of Subsec. 4.2, with an eye towards later, more obscure, versions. There are five line operators in $SU(2)_4$ Chern–Simons, as listed in App. A, which we denote

$$(0), (1), (2), (3), (4). \tag{5.3}$$

We gauge a $B\mathbb{Z}_2$, with lines $\{\ell_0, \ell_1\}$, where

$$\ell_i \times \ell_j = \ell_{i+j \text{ mod } 2}, \tag{5.4}$$

and which act on the $SU(2)_4$ lines as

$$\ell_0 \times L = L, \ell_1 \times L = (1) \times L, \tag{5.5}$$

and with

$$B(\ell_0, L) = +1,\ B(\ell_1, 0) = B(\ell_1, 1) = B(\ell_1, 4) = +1, B(\ell_1, 2) = B(\ell_1, 3) = -1. \tag{5.6}$$

(Clearly, we can identify the action of this $B\mathbb{Z}_2$ with the action of the lines (0), (1) in $SU(2)_4$). It is straightforward to check that this gives a well-defined action in the sense of Subsec. 4.2. Applying the procedure there, to get the lines of $SU(2)_4/B\mathbb{Z}_2$,

- the lines (2) and (3) are not invariant under monodromies and so should be excluded,
- from $(1) \times (1) = (0)$, the lines (0) and (1) should be identified in the quotient, and
- from $(1) \times (4) = (4)$, the line (4) is duplicated,

so that the $SU(2)_4/B\mathbb{Z}_2$ spectrum consists of the vacuum line and two copies of (4), which is the standard result for $SO(3)_4$.

Now, let us turn to the boundary theory. On the boundary, this reduces to the statement

$$[\text{WZW}(SU(2))/\mathbb{Z}_2] = \text{WZW}(SO(3)), \tag{5.7}$$

which is standard.

5.2. *Chern–Simons*$(\boldsymbol{SU(2)}) \times$ [point$/\boldsymbol{B\mathbb{Z}_2}$], $\boldsymbol{K} = \mathbb{Z}_2$

Now, let us apply the decomposition prediction (3.15) to a different case, namely, one in which we gauge a trivially-acting $B\mathbb{Z}_2$ "acting" on an $SU(2)$ Chern–Simons theory, uncoupled from the center one-form symmetry of the $SU(2)$ theory.

This is perhaps the cleanest example of a $B\mathbb{Z}_2$ gauging that acts trivially: we gauge a $B\mathbb{Z}_2$ in bulk that does nothing at all to the $SU(2)$.

Let us apply the decomposition prediction (3.15) to this case. Here, in the notation of (3.15), we take $H = SU(2)$ and $A = \mathbb{Z}_2$; however, the map $d : A \to H$ maps all of $\mathbb{Z}_2$ to 1. In this case, the kernel of d, K, is all of $\mathbb{Z}_2$, and $G = H = SU(2)$. The decomposition prediction for this case is that

$$[\mathrm{Chern}-\mathrm{Simons}(SU(2))/B\mathbb{Z}_2] = \coprod_{\theta \in \hat{K}} \mathrm{Chern}-\mathrm{Simons}(SU(2)), \tag{5.8}$$

two copies of the $SU(2)$ Chern–Simons theory. Furthermore, in this case there are no nontrivial discrete theta angles, hence the decomposition prediction can be written more simply as

$$[\mathrm{Chern}-\mathrm{Simons}(SU(2))/B\mathbb{Z}_2] = \coprod_{2} \mathrm{Chern}-\mathrm{Simons}(SU(2)). \tag{5.9}$$

Let us briefly consider the spectrum of line operators, following the procedure discussed in Subsec. 4.2. We describe the trivially-acting $B\mathbb{Z}_2$ in terms of two lines $\{\ell_0, \ell_1\}$, where

$$\ell_i \times \ell_j = \ell_{i+j \bmod 2}, \tag{5.10}$$

and with an action on the lines of $SU(2)_4$ given by

$$B(\ell_i, L) = +1, \ell_i \times L = L. \tag{5.11}$$

It is straightforward to check that this gives a well-defined action in the sense of Subsec. 4.2.

Next, we compute the spectrum of $SU(2)_4/B\mathbb{Z}_2$, for this trivially-acting $B\mathbb{Z}_2$. From the rules in Subsec. 4.2,

- None of the original lines of the $SU(2)$ Chern–Simons theory is omitted, as they all have trivial monodromy under the generator (a),
- Since $(a) \times (a) = (0)$, we see that in the gauged theory, (a) and (0) are identified with one another,
- Since all of the original lines are invariant under fusion $((a) \times (x) = (x))$, they are all duplicated.

As a result, the line operator spectrum of the gauged theory is two copies of the line operator spectrum of the original $SU(2)$ Chern–Simons theory, consistent with decomposition. This result could also be obtained by adding one new line a to the lines of $SU(2)_4$, which interacts trivially with all other lines, and then condensing $\{(0), a\}$ in the ordinary fashion, though as we discussed in Subsec. 4.2, it will not always be possible to do that.

Next, we turn to the boundary theory. In the boundary WZW model, bulk decomposition becomes the statement that

$$[\mathrm{WZW}(SU(2))/\mathbb{Z}_2] = \coprod_{2} \mathrm{WZW}(SU(2)). \tag{5.12}$$

In the $\mathbb{Z}_2$ orbifold on the left, the $\mathbb{Z}_2$ acts trivially on the $SU(2)$ WZW model, for which case ordinary 2D decomposition predicts exactly the statement above, that the completely-trivially-acting $\mathbb{Z}_2$ orbifold of a WZW model is just two copies of the same WZW model. Thus, the boundary theory matches results from 2D decomposition, as expected.

5.3. *Chern–Simons*$(SU(2))/B\mathbb{Z}_4$, $K = \mathbb{Z}_2$

Consider a $SU(2)$ Chern–Simons theory in three dimensions, and gauge a $B\mathbb{Z}_4$ that acts via projecting to a $B\mathbb{Z}_2$ which acts as the center symmetry. In this case, there is a trivially-acting $B\mathbb{Z}_2$, so in broad brushstrokes one expects two copies of a $B\mathbb{Z}_2$-gauged $SU(2)$ Chern–Simons theory.

Let us walk through the prediction of the decomposition prediction (3.15) in this case. Here, we have $H = SU(2)$ and $A = \mathbb{Z}_2$, with the map $d : A \to SU(2)$ mapping the $\mathbb{Z}_4$ onto the center $\mathbb{Z}_2$ of $SU(2)$. Thus, the map d is surjective, but not injective: its kernel $K = \mathbb{Z}_2$. Similarly,

$$G = H/\mathrm{im}\ d = SU(2)/\mathbb{Z}_2 = SO(3). \tag{5.13}$$

Putting this together, we see in this case that the decomposition prediction (3.15) is

$$\begin{aligned}&[\mathrm{Chern-Simons}(SU(2))/B\mathbb{Z}_4]\\ &\quad = \mathrm{Chern-Simons}(SO(3))_+ \coprod \mathrm{Chern-Simons}(SO(3))_-,\end{aligned} \tag{5.14}$$

where the $\pm$ denotes the two values of the discrete theta angle coupling to the characteristic class defined by $\beta_\alpha(w_G = w_{SO(3)})$, for α the class of the extension

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1, \tag{5.15}$$

and where here, $w_{SO(3)} = w_2$, the second Stiefel–Whitney class.

Next, we will argue[m] that the characteristic class $\beta_\alpha(w_2)$ is the third Stiefel–Whitney class w_3. From the Wu formula (Problem 8-A in Ref. 107) for Steenrod squares, which map $\mathrm{Sq}^k : H^\bullet(X, \mathbb{Z}_2) \to H^{\bullet+k}(X, \mathbb{Z}_2)$, $k \geq 0$:

$$\begin{aligned}\mathrm{Sq}^k(w_m(\xi)) &= \sum_{t=0}^{k} \binom{k-m}{t} w_{k-t}(\xi) \cup w_{m+t}(\xi)\\ &= \sum_{t=0}^{k} \binom{m-k+t-1}{t} w_{k-t}(\xi) \cup w_{m+t}(\xi),\end{aligned} \tag{5.16}$$

[m] E.S. would like to thank Y. Tachikawa for observing the pertinent properties of w_3.

where each $w_j = w_j(\xi)$ for a real vector bundle ξ, and in the equality, we have used the fact that

$$\begin{aligned}\binom{k-m}{t} &= \frac{(k-m)(k-m-1)\cdots(k-m-t+1)}{t!},\\ &= (\pm)\frac{(m-k)(m-k+1)\cdots(m-k+t-1)}{t!}\\ &\equiv \binom{m-k+t-1}{t} \bmod 2.\end{aligned} \tag{5.17}$$

(See e.g. Ref. 108 for this and related observations). As a result, for any real vector bundle

$$\mathrm{Sq}^1(w_2) = w_1 \cup w_2 + w_0 \cup w_3 = w_1 \cup w_2 + w_3, \tag{5.18}$$

so if $w_1 = 0$, as is the case for $SO(3)$ bundles, then $w_3 = \mathrm{Sq}^1(w_2)$. (In principle, this is one explanation of why all $SO(3)$ bundles can be constructed by twisting $SU(2)$ bundles by $\mathbb{Z}_2$ gerbes: the gerbe characteristic class determines not only the second Stiefel–Whitney class w_2 of the $SO(3)$ bundles, but also w_3 via Sq^1, as above).

Furthermore, the action of Sq^1 is the Bockstein homomorphism β associated to the extension

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1, \tag{5.19}$$

(see e.g. Sec. 4.L in Ref. 96,) meaning

$$\mathrm{Sq}^1(x) = \beta(x) \tag{5.20}$$

for any x. The extension (5.19) above coincides with α in this case, so we see that in this example, the discrete theta angle couples to

$$\beta_\alpha(w_2) = \mathrm{Sq}^1(w_2), \tag{5.21}$$

using (3.14). We also see that in this example, this class can be described even more simply as w_3, the third Stiefel–Whitney class, as $w_3 = \mathrm{Sq}^1(w_2)$.

Now, on a three-manifold M, we can write $\mathrm{Sq}^1(x)$ for any x in terms of the Wu class $\nu_1 \in H^1(M, \mathbb{Z}_2)$ as (Chap. 11 in Ref. 107)

$$\mathrm{Sq}^1(x) = \nu_1 \cup x. \tag{5.22}$$

Furthermore (Theorem 11.14 in Ref. 107)

$$\nu_1 = w_1(TM), \tag{5.23}$$

so assembling these pieces, we have that

$$w_3(\xi) = \mathrm{Sq}^1(w_2(\xi)) = w_1(M) \cup w_2(\xi). \tag{5.24}$$

As a result, the third Stiefel–Whitney class w_3 will only be nontrivial on a nonorientable three-manifold M. However, Chern–Simons theories are not defined on nonorientable spaces.

In Subsec. 5.9, we will use level-rank duality to perform a self-consistency check of decomposition in this case.

Now, let us check this prediction by computing the line spectrum in this gauged Chern–Simons theory. First, following Subsec. 4.2, we define a $B\mathbb{Z}_4$ by lines $\{\ell_0, \ell_1, \ell_2, \ell_3\}$ such that

$$\ell_i \times \ell_j = \ell_{i+j \text{ mod } 4}, \tag{5.25}$$

and which acts on the lines of $SU(2)_4$ (described in App. A) as follows:

$$\begin{aligned} B(\ell_{0,2}, L) = +1, B(\ell_{1,3}, 0) = B(\ell_{1,3}, 1) = B(\ell_{1,3}, 4) = +1, \\ B(\ell_{1,3}, 2) = B(\ell_{1,3}, 3) = -1, \end{aligned} \tag{5.26}$$

$$\ell_0 \times L = \ell_2 \times L = L, \ell_1 \times L = \ell_3 \times L = (1) \times L. \tag{5.27}$$

It is straightforward to check that this action of $B\mathbb{Z}_4$ on the lines of $SU(2)_4$ is well-defined in the sense of Subsec. 4.2. As ℓ_2 acts trivially, this is also a noneffective action, in the sense of Subsec. 3.1.

Next, we follow the procedure outlined in Subsec. 4.2 to get the lines of $SU(2)_4/B\mathbb{Z}_4$:

- Lines (2) and (3) have $B(\ell_{1,3}, L) \neq +1$, and so are omitted.
- Since $\ell_{1,3} \times (1) = (0)$, we identify the lines $(0) \sim (1)$.
- Since $\ell_i \times (4) = (4)$ for all i, we get four copies of (4) in the spectrum of $SU(2)_4/B\mathbb{Z}_4$, and since $\ell_{0,2} \times (1) = (1)$, $\ell_{0,2} \times (0) = (0)$, we get two copies of $(0) \sim (1)$.

Thus, we see that we get two copies of the lines of $SO(3)_4$, consistent with the expectations from the decomposition.

Before going on, let us compute the lines in one more example, specifically $SU(2)_4/B\mathbb{Z}_{2p}$, where the $\mathbb{Z}_{2p}$ projects to the $\mathbb{Z}_2$ center of $SU(2)_4$, with kernel $\mathbb{Z}_4$. The lines of $B\mathbb{Z}_{2p}$ are $\{\ell_0, \ldots, \ell_{2p-1}\}$, where

$$\ell_i \times \ell_j = \ell_{i+j \text{ mod } 2p}, \tag{5.28}$$

and their action on $SU(2)_4$ is given by

$$B(\ell_{\text{even}}, L) = +1, B(\ell_{\text{odd}}, 0) = +1 = B(\ell_{\text{odd}}, 1) = B(\ell_{\text{odd}}, 4), \tag{5.29}$$

$$B(\ell_{\text{odd}}, 2) = -1 = B(\ell_{\text{odd}}, 3), \tag{5.30}$$

$$\ell_{\text{even}} \times L = L, \ell_{\text{odd}} \times L = (1) \times L. \tag{5.31}$$

As before, it is straightforward to check that this action of $B\mathbb{Z}_{2p}$ is well-defined in the sense of Subsec. 4.2, and since $\{\ell_{\text{even}}\}$ acts trivially, it is a noneffective action, in the sense of Subsec. 3.1.

Next, we follow the procedure outlined in Subsec. 4.2 to get the lines of $SU(2)_4/B\mathbb{Z}_{2p}$:

- Lines (2) and (3) have $B(\ell_{\text{odd}}, L) \neq +1$, and so are omitted.
- Since $\ell_{\text{odd}} \times (1) = (0)$, we identify the lines $(0) \sim (1)$.
- Since $\ell_i \times (4) = (4)$ for all i, we get $2p$ copies of (4), and since $\ell_{\text{even}} \times (1) = (1)$, $\ell_{\text{even}} \times (0) = (0)$, we get p copies of $(0) \sim (1)$.

Altogether, we find p copies of the lines of $SO(3)_4$, consistent with expectations from decomposition, since $B\mathbb{Z}_p$ acts trivially.

Before going on, let us briefly discuss the boundary theory. The Chern–Simons decomposition (5.14) becomes a decomposition of WZW models, formally

$$[\text{WZW}(SU(2))/\mathbb{Z}_4] = \text{WZW}(SO(3))_+ \coprod \text{WZW}(SO(3))_-. \qquad (5.32)$$

Here the $\mathbb{Z}_2$ discrete theta angle couples to the image of the element of $H^3(BSO(3), \mathbb{Z}_2)$ (corresponding to third Stiefel–Whitney classes) in $H^2(SO(3), \mathbb{Z}_2) = \mathbb{Z}_2$. However, the generator of this group is $\text{Sq}^1(a)$, where a generates $H^1(SO(3), \mathbb{Z}_2)$ and for reasons discussed previously, $\text{Sq}^1(a) = w_1(TM) \cup a$, hence is nonzero only if the 2D space is nonorientable.

We will consider various generalizations of this example, returning to this example for special levels to utilize level-rank duality consistency checks in Subsec. 5.9.

5.4. *Chern–Simons*$(SU(n))/B\mathbb{Z}_{np}$, $K = \mathbb{Z}_p$

Next, we will consider gauging the action of $B\mathbb{Z}_{np}$ on $SU(n)$ Chern–Simons, where the $\mathbb{Z}_{np}$ acts by projecting to the center $\mathbb{Z}_n$ of $SU(n)$, and study the discrete theta angles for special values of n and p beyond those discussed already.

In terms of the decomposition prediction (3.15), we take $A = \mathbb{Z}_{np}$, $H = SU(n)$, and $d : A \to H$ acts by projecting to $Z = \mathbb{Z}_n \subset Z(H)$. Then, the kernel $K = \mathbb{Z}_p$, $G = SU(n)/\mathbb{Z}_n$, and we have the long exact sequence

$$1 \to \mathbb{Z}_\ell \to \mathbb{Z}_{n\ell} \to SU(n) \to SU(n)/\mathbb{Z}_n \to 1. \qquad (5.33)$$

In general terms, decomposition (3.15) then predicts that

$$[\text{Chern}-\text{Simons}(SU(n))/BA] = \coprod_{\theta \in \hat{K}} \text{Chern}-\text{Simons}(SU(n)/\mathbb{Z}_n)_{\theta(\omega)}, \qquad (5.34)$$

where the $\theta(\omega)$ are discrete theta angles coupling to the characteristic class defined by $\beta_\alpha(w_{SU(n)/\mathbb{Z}_n})$, where $w_{SU(n)/\mathbb{Z}_n} \in H^2_{\text{sing}}(BSU(n)/\mathbb{Z}_n, \mathbb{Z}_n)$ is a generalization of the second Stiefel–Whitney class to $n \geq 2$, and β_α is the Bockstein map in the long exact sequence associated to the extension

$$1 \to K(=\mathbb{Z}_p) \to A(=\mathbb{Z}_{np}) \to Z(=\mathbb{Z}_n) \to 1, \qquad (5.35)$$

with extension class $\alpha \in H^2_{\text{group}}(Z, K)$.

We will evaluate this expression for some special cases in which we will simplify the expression for discrete theta angles. We will use,[109] which provides the cohomology of $SU(n)/\mathbb{Z}_n$, which (modulo a degree shift) is essentially the same. (See also Refs. 110–115.)

First, consider the case that p is a prime number that does not divide n. Then, from Sec. 7 in Ref. 109

$$H^{\bullet}_{\rm sing}(BSU(n)/\mathbb{Z}_n, \mathbb{Z}_p) = H^{\bullet}_{\rm sing}(BSU(n), \mathbb{Z}_p), \tag{5.36}$$

and so there is no $\mathbb{Z}_p$-valued characteristic class in degree three, hence no discrete theta angle. In this case, the decomposition above can be written more simply as

$$[\text{Chern}-\text{Simons}(SU(n))/BA] = \coprod_p \text{Chern}-\text{Simons}(SU(n)/\mathbb{Z}_n). \tag{5.37}$$

Next, suppose that $p = 2$, and $n = 2m$ for m odd. From Corollary 4.2 in Ref. 109, the group $H^3_{\rm sing}(BSU(n)/\mathbb{Z}_n, \mathbb{Z}_2) \neq 0$, and so for $w_{SU(n)/\mathbb{Z}_n} \in H^2_{\rm sing}(BSU(n)/\mathbb{Z}_n, \mathbb{Z}_n)$, we get a discrete theta angle coupling to $\beta_\alpha(w_{SU(n)/\mathbb{Z}_n})$, the image of $w_{SU(n)/\mathbb{Z}_n}$ under the Bockstein map associated to the extension

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_{pn} \to \mathbb{Z}_n \to 1, \tag{5.38}$$

with extension class $\alpha \in H^2_{\rm group}(\mathbb{Z}_n, \mathbb{Z}_p)$. Since $p = 2$, we can write $\beta_\alpha(w_{SU(n)/\mathbb{Z}_n}) = \text{Sq}^1(w_{SU(n)/\mathbb{Z}_n})$, as before, and also just as before, it is only nonzero on nonoriented spaces, as we saw for the case of $SU(2)$ and $SO(3)$ theories in Subsec. 5.3.

Now, let us consider the corresponding boundary WZW model. The bulk decomposition above predicts

$$\left[\text{WZW}(SU(n))/\mathbb{Z}_{np}\right] = \coprod_{\theta \in \hat{K}} \text{WZW}(SU(n)/\mathbb{Z}_n)_{\theta(\omega)}. \tag{5.39}$$

Now, from ordinary 2D decomposition, since there is no discrete torsion in $\mathbb{Z}_n$,

$$[\text{WZW}(SU(n))/\mathbb{Z}_{np}] = \coprod_{\ell} \text{WZW}(SU(n)/\mathbb{Z}_n). \tag{5.40}$$

This is certainly consistent with the special cases computed above, in which the bulk discrete theta angle vanishes.

5.5. *Chern–Simons*(Spin(n))/$B\mathbb{Z}_{2p}$, $K = \mathbb{Z}_p$

Next, we consider a simple generalization of the example above, in which we gauge a $B\mathbb{Z}_{2p}$ action on Spin(n) Chern–Simons, in which the $B\mathbb{Z}_{2p}$ acts by first projecting to $B\mathbb{Z}_2$ which acts through (a subgroup of) the center. We begin by discussing the case that the $\mathbb{Z}_2$ is such that $\text{Spin}(n)/\mathbb{Z}_2 = SO(n)$. In the case that n is divisible by four, there is a second choice of $\mathbb{Z}_2$ subgroup, for which the quotient $\text{Spin}(n)/\mathbb{Z}_2 \neq SO(n)$. We will discuss the second case at the end of this section.

In terms of the decomposition prediction (3.15), we take $A = \mathbb{Z}_{2p}$, $H = \mathrm{Spin}(n)$, and $d : A \to H$ is the map that projects $\mathbb{Z}_{2p}$ onto the $\mathbb{Z}_2$ in the center of $\mathrm{Spin}(n)$ such that $\mathrm{Spin}(n)/\mathbb{Z}_2 = SO(n)$. Then, the kernel of d is $K = \mathbb{Z}_p$, $G = H/A = SO(n)$, and we have the exact sequence

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_{2p} \to \mathrm{Spin}(n) \to SO(n) \to 1. \tag{5.41}$$

This extension is nontrivial, and defines a discrete theta angle coupling to $\beta_\alpha(w_{SO(n)})$, with $w_{SO(n)} = w_2$, the second Stiefel–Whitney class, as before, and the Bockstein homomorphism β_α is associated to the extension

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_{2p} \to \mathbb{Z}_2 \to 1 \tag{5.42}$$

of extension class $\alpha \in H^2_{\mathrm{group}}(\mathbb{Z}_2, \mathbb{Z}_p)$.

Decomposition then predicts (3.15)

$$[\mathrm{Chern-Simons}(\mathrm{Spin}(n))/B\mathbb{Z}_{2p}] = \coprod_{\theta \in \hat{\mathbb{Z}}_p} \mathrm{Chern-Simons}(SO(n))_\theta, \tag{5.43}$$

where θ denotes the discrete theta angle coupling.

In the case that $p = 2$, for the same reasons as discussed in Subsec. 5.3, we can identify $\beta_\alpha(w_2)$ with w_3, the third Stiefel–Whitney class. However, by the same reasoning as described in Subsec. 5.3, the third Stiefel–Whitney class will only be nontrivial on nonorientable three-manifolds. Therefore, on orientable three-manifolds, for $p = 2$, the statement of decomposition reduces to

$$[\mathrm{Chern-Simons}(\mathrm{Spin}(n))/B\mathbb{Z}_4] = \coprod_2 \mathrm{Chern-Simons}(SO(n)). \tag{5.44}$$

Next, let us briefly compare to the boundary WZW model. On the boundary, from the decomposition (5.43), we have

$$[\mathrm{WZW}(\mathrm{Spin}(n))/\mathbb{Z}_{2p}] = \coprod_{\theta \in \hat{\mathbb{Z}}_p} \mathrm{WZW}(SO(n))_\theta, \tag{5.45}$$

For the case $p = 2$, for the same reasons as noted in Subsec. 5.3, for oriented spaces, the discrete theta angles are trivial, as the characteristic class they couple to vanishes. As a result, on oriented spaces, for $p = 2$ we can equivalently write

$$[\mathrm{WZW}(\mathrm{Spin}(n))/\mathbb{Z}_4] = \coprod_2 \mathrm{WZW}(SO(n)). \tag{5.46}$$

This is consistent with the prediction of decomposition in two dimensions in this case. As reviewed in Sec. 2, essentially because there is no discrete torsion in a $\mathbb{Z}_2$ orbifold, in a 2D WZW orbifold by $\mathbb{Z}_{2p}$ with trivially-acting $\mathbb{Z}_p$, we have

$$[\mathrm{WZW}(\mathrm{Spin}(n))/\mathbb{Z}_4] = \coprod_p \mathrm{WZW}(SO(n)). \tag{5.47}$$

For $p = 2$, this is certainly consistent with the bulk description.

So far we have discussed the case that the $\mathbb{Z}_{2p}$ maps to $\mathbb{Z}_2 \subset \text{Spin}(n)$ such that

$$\text{Spin}(n)/\mathbb{Z}_2 = SO(n). \tag{5.48}$$

In the case that n is divisible by four, there is another choice of $\mathbb{Z}_2$ subgroup of the center of $\text{Spin}(n)$, which leads to a quotient

$$\text{Spin}(n)/\mathbb{Z}_2 \neq SO(n), \tag{5.49}$$

which for example projects out the vector representation. (See e.g. Ref. 116 for a discussion in a different context.) This second quotient group is sometimes denoted Semi-spin(n), abbreviated $Ss(n)$ (see e.g. Sec. 11 in Ref. 115). Relevant material on the cohomology of $Ss(n)$ can be found in e.g. Sec. 9 in Ref. 109.

5.6. *Chern–Simons*$(\mathbf{Spin}(4n+2))/B\mathbb{Z}_{4p}$, $K = \mathbb{Z}_p$

Let us consider the case of a Chern–Simons theory with gauge group $\text{Spin}(4n+2)$ and a gauged $B\mathbb{Z}_{4p}$, where the $\mathbb{Z}_{4p}$ maps to the center $(\mathbb{Z}_4)$ of $\text{Spin}(4n+2)$, with kernel $K = \mathbb{Z}_p$.

In terms of the decomposition prediction (3.15), we take $A = \mathbb{Z}_{4p}$, $H = \text{Spin}(4n+2)$, and $d : A \to H$ projects $\mathbb{Z}_{4p}$ onto the central $\mathbb{Z}_4 \subset \text{Spin}(4n+2)$. The kernel of d is $K = \mathbb{Z}_p$, $G = H/A = SO(4n+2)/\mathbb{Z}_2$, and we have the exact sequence

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_{4p} \to \text{Spin}(4n+2) \to SO(4n+2)/\mathbb{Z}_2 \to 1. \tag{5.50}$$

Decomposition then predicts (3.15)

$$\begin{aligned} &[\text{Chern}-\text{Simons}(\text{Spin}(4n+2))/B\mathbb{Z}_{4p}] \\ &= \coprod_{\theta \in \hat{\mathbb{Z}}_p} \text{Chern}-\text{Simons}(SO(4n+2)/\mathbb{Z}_2)_{\theta(\omega)}, \end{aligned} \tag{5.51}$$

where the discrete theta angle couples to a characteristic class $\beta_\alpha(w_{\text{Spin}(4n+2)/\mathbb{Z}_4})$ for β_α the Bockstein map associated to the short exact sequence

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_{4p} \to \mathbb{Z}_4 \to 1 \tag{5.52}$$

of extension class $\alpha \in H^2_{\text{group}}(\mathbb{Z}_4, \mathbb{Z}_p)$.

Consider for example the case $p = 2$. From Lemma 8.1 in Ref. 109, $SO(4n+2)/\mathbb{Z}_2$ has one characteristic class in $H^3(BSO(4n+2)/\mathbb{Z}_2, \mathbb{Z}_2)$, related to w_3 of a covering $SO(4n+2)$ bundle.

In the boundary WZW model, the decomposition (5.51) predicts

$$[\text{WZW}(\text{Spin}(4n+2))/\mathbb{Z}_{4p}] = \coprod_{\theta \in \hat{\mathbb{Z}}_p} \text{WZW}(SO(4n+2)/\mathbb{Z}_2)_\theta. \tag{5.53}$$

Ordinary 2D decomposition predicts in this case that

$$[\text{WZW}(\text{Spin}(4n+2))/\mathbb{Z}_{4p}] = \coprod_p \text{WZW}(SO(4n+2)/\mathbb{Z}_2), \tag{5.54}$$

essentially because there is no discrete torsion in a $\mathbb{Z}_4$ orbifold.

5.7. *Chern–Simons*(Spin(4*n*))/$B(\mathbb{Z}_2 \times \mathbb{Z}_{2p})$, $K = \mathbb{Z}_p$

Next, we consider the case of a $B(\mathbb{Z}_2 \times \mathbb{Z}_{2p})$ action on a Spin$(4n)$ Chern–Simons theory. Here, Spin$(4n)$ has center $\mathbb{Z}_2 \times \mathbb{Z}_2$, and the $\mathbb{Z}_2 \times \mathbb{Z}_{2p}$ acts by first mapping to the center.

In terms of the decomposition prediction (3.15), we take $A = \mathbb{Z}_2 \times \mathbb{Z}_{2p}$, $H = \text{Spin}(4n)$, $d : A \to H$ maps A onto the center, $K = \text{Ker } d = \mathbb{Z}_p$, hence we predict

$$[\text{Chern}-\text{Simons}(\text{Spin}(4n))/B(\mathbb{Z}_2 \times \mathbb{Z}_{2p})] = \coprod_{\theta \in \hat{\mathbb{Z}}_p} \text{Chern}-\text{Simons}(SO(4n)/\mathbb{Z}_2)_\theta, \tag{5.55}$$

where the discrete theta angle couples to $\beta_\alpha(w_{\text{Spin}(4n)/\mathbb{Z}_2 \times \mathbb{Z}_2})$, for β_α the Bockstein map associated to the short exact sequence

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_2 \times \mathbb{Z}_{2p} \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1 \tag{5.56}$$

of extension class $\alpha \in H^2_{\text{group}}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_p)$.

Consider for example $p = 2$. From Lemma 8.1 in Ref. 109, $SO(4n)/\mathbb{Z}_2$ has one characteristic class in $H^3(BSO(4n)/\mathbb{Z}_2, \mathbb{Z}_2)$, related to w_3 of a covering $SO(4n)$ bundle.

Now, let us consider this in the boundary WZW model. The bulk decomposition (5.55) predicts that

$$[\text{WZW}(\text{Spin}(4n))/(\mathbb{Z}_2 \times \mathbb{Z}_{2p})] = \coprod_{\theta \in \hat{\mathbb{Z}}_p} \text{WZW}(SO(4n)/\mathbb{Z}_2)_\theta, \tag{5.57}$$

whereas discussed in Subsec. 3.3, the boundary discrete theta angles θ correspond to choices of discrete torsion, here in a $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold.

We can understand those boundary discrete theta angles more precisely by comparing them to the predictions of 2D decomposition. We have a $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_{2p}$ orbifold, with trivially-acting $K = \mathbb{Z}_p$, and $G = \Gamma/K = \mathbb{Z}_2 \times \mathbb{Z}_2$. In principle, G can contain discrete torsion, since $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$, so we should compute to see if we get nontrivial discrete torsion in any factors. Any such discrete torsion is the image of the extension class in $H^2(G, K)$ corresponding to

$$1 \to K \to \Gamma \to G \to 1 \tag{5.58}$$

under the map $K \to U(1)$ defined by the representation of K corresponding to that universe, and the extension class is nontrivial; nevertheless, as discussed in Subsec. 6.1 in Ref. 26, its image in $H^2(G, U(1))$ is trivial for both irreducible representations of K. As a result, 2D decomposition predicts

$$[\text{WZW}(\text{Spin}(4n))/(\mathbb{Z}_2 \times \mathbb{Z}_{2p})] = \coprod_{p} \text{WZW}(SO(4n)/\mathbb{Z}_2). \tag{5.59}$$

In particular, the boundary discrete theta angles vanish.

In passing, we should observe that this is a nontrivial constraint. The two choices of discrete torsion in the WZW model for $\mathrm{Spin}(4n)/\mathbb{Z}_2 \times \mathbb{Z}_2$ correspond to two distinct quantum theories, each of which can be described as the WZW model for $SO(4n)$, see e.g. Refs. 117–121. Furthermore, in two dimensions, certainly there exist examples in which both choices of discrete torsion appear. For example, only slightly generalizing results in[1]

$$[\mathrm{WZW}(\mathrm{Spin}(4n))/D_4] = \mathrm{WZW}(SO(4n)/\mathbb{Z}_2)_+ \coprod \mathrm{WZW}(SO(4n)/\mathbb{Z}_2)_-, \tag{5.60}$$

$$[\mathrm{WZW}(\mathrm{Spin}(4n))/\mathbb{H}] = \mathrm{WZW}(SO(4n)/\mathbb{Z}_2)_+ \coprod \mathrm{WZW}(SO(4n)/\mathbb{Z}_2)_-, \tag{5.61}$$

where in both D_4 and $\mathbb{H}$ the $\mathbb{Z}_2$ center is taken to act trivially, and the $\pm$ indicates the two choices of discrete torsion.

However, because both the dihedral group D_4 and the group of unit quaternions $\mathbb{H}$ are nonabelian, there is no Chern–Simons version of the decompositions above. That is fortuitous, as of the two $SO(4n)/\mathbb{Z}_2$ WZW models, the one with nonzero discrete torsion also does not have a Chern–Simons dual.[121,122]

More generally, in order to get a 2D decomposition of $[\mathrm{WZW}(\mathrm{Spin}(4n))/\Gamma]$ to copies of $\mathrm{WZW}(SO(4n)/\mathbb{Z}_2)$ with nontrivial discrete torsion, it is straightforward to check that Γ must be nonabelian, and so does not admit a Chern–Simons description.

5.8. *Chern–Simons*$(Sp(n))/B\mathbb{Z}_{2p}$, $K = \mathbb{Z}_p$

Next, consider the case of a Chern–Simons theory with gauge group $Sp(n)$ and a gauged $B\mathbb{Z}_{2p}$, where the $\mathbb{Z}_{2p}$ maps to the center ($\mathbb{Z}_2$) of $Sp(n)$.

In terms of the decomposition prediction (3.15), we take $A = \mathbb{Z}_{2p}$, $H = Sp(n)$, and $d : A \to H$ projects $\mathbb{Z}_{2p}$ onto the central $\mathbb{Z}_2 \subset Sp(n)$, with $K = \mathrm{Ker}\, d = \mathbb{Z}_p$. Decomposition then predicts (3.15)

$$[\mathrm{Chern-Simons}(Sp(n))/B\mathbb{Z}_{2p}] = \coprod_{\theta \in \hat{\mathbb{Z}}_p} \mathrm{Chern-Simons}(Sp(n)/\mathbb{Z}_2)_\theta, \tag{5.62}$$

where the discrete theta angle couples to a characteristic class $\beta_\alpha(w_{Sp(n)/\mathbb{Z}_2})$ for β_α the Bockstein map associated to the short exact sequence

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_{2p} \to \mathbb{Z}_2 \to 1 \tag{5.63}$$

of extension class $\alpha \in H^2_{\mathrm{group}}(\mathbb{Z}_2, \mathbb{Z}_p)$. See e.g. Sec. 8 in Ref. 109 for results on pertinent characteristic classes.

In the boundary WZW model, the bulk decomposition (5.62) predicts

$$[\mathrm{WZW}(Sp(n))/\mathbb{Z}_2] = \coprod_{\theta \in \hat{\mathbb{Z}}_p} \mathrm{WZW}(Sp(n)/\mathbb{Z}_2)_\theta. \tag{5.64}$$

Because there is no discrete torsion in a $\mathbb{Z}_2$ orbifold, 2D decomposition predicts in this case that

$$[\mathrm{WZW}(Sp(n))/\mathbb{Z}_2] = \coprod_p \mathrm{WZW}(Sp(n)/\mathbb{Z}_2). \tag{5.65}$$

5.9. Chern–Simons$(U(1))_k/B\mathbb{Z}_\ell p$, $K = \mathbb{Z}_p$

Consider a $U(1)_k$ Chern–Simons theory in three dimensions. This theory has a global $B\mathbb{Z}_k$ symmetry which can be gauged (see e.g. Refs. 123 and 124, App. C in Ref. 70). It has slightly different properties depending upon whether k is even or odd (see e.g. Subsec. 2.2 in Ref. 51):

- When k is even, this theory has k line operators, labeled by elements of $\mathbb{Z}_k$. If k is 0 mod 8, then the $B\mathbb{Z}_k$ one-form symmetry generator has integer spin. If k is 2 mod 8, then the one form generator has spin 1/4 and if k is 4 mod 8, then the one-form symmetry generator is spin 1/2.
- When k is odd, the theory has $2k$ lines labeled by elements of $\mathbb{Z}_{2k}$ and is moreover a spin TQFT. The line with the label k is the transparent fermion.

Now, consider gauging a $B\mathbb{Z}_{\ell p}$, where ℓ divides n, where the $\mathbb{Z}_{\ell p}$ projects to $\mathbb{Z}_\ell \subset \mathbb{Z}_k$, for that $B\mathbb{Z}_k$ above, with kernel $B\mathbb{Z}_p$. Let us apply the decomposition prediction (3.15) to this case.

In the language of (3.15), $A = \mathbb{Z}_{\ell p}$ and $H = U(1)$. Here, the map $d : A \to H$ is given by projecting $A = \mathbb{Z}_{\ell p}$ to a $\mathbb{Z}_\ell \subset \mathbb{Z}_k \subset U(1)$, and so it has kernel $K = \mathbb{Z}_p$. Furthermore

$$G = H/\mathrm{im} d = U(1)/\mathbb{Z}_\ell = U(1). \tag{5.66}$$

In this case, $BU(1) = \mathbb{CP}^\infty$ has no odd degree cohomology, so there cannot be any discrete theta angle. Thus, the decomposition prediction (3.15) for this case is that

$$[\mathrm{Chern-Simons}(U(1)_k)/B\mathbb{Z}_{\ell p}] = \coprod_p [\mathrm{Chern-Simons}(U(1)_k)/B\mathbb{Z}_\ell], \tag{5.67}$$

a sum of p theories (consistent with a trivially-acting $B\mathbb{Z}_p$) with no discrete theta angles.

In particular, note that the right-hand side is a sum of $U(1)_k/B\mathbb{Z}_\ell$ Chern–Simons theories, which is not necessarily the same as a union of $U(1)_k$ Chern–Simons theories. Although as groups $U(1)/\mathbb{Z}_k = U(1)$, gauging a Chern–Simons theory by a one-form symmetry is a bit different. For example, $U(1)_{4m}/B\mathbb{Z}_2 = U(1)_m$, from Sec. C.1 in Ref. 76. (On the boundary, one has a $U(1)$ WZW model, meaning a sigma model on S^1, with radius determined by the level. Gauging the $B\mathbb{Z}_k$ in bulk becomes gauging a $\mathbb{Z}_k$ rotation in the boundary theory, which changes the radius and hence the level).

We can use level-rank duality to perform a consistency test. Beginning with the decomposition described in Subsec. 5.3 at level 1, namely

$$\begin{aligned}&[\mathrm{Chern-Simons}(SU(2)_1)/B\mathbb{Z}_4]\\ &\quad = \mathrm{Chern-Simons}(SO(3)_1)_+ \coprod \mathrm{Chern-Simons}(SO(3)_1)_-. \end{aligned}\tag{5.68}$$

Here we have kept track of the discrete theta angle; we only consider Chern–Simons theories on orientable manifolds, so no discrete theta angle is visible, so the prediction of Subsec. 5.3 in this case is more simply

$$[\text{Chern}-\text{Simons}(SU(2)_1)/B\mathbb{Z}_4] = \coprod_2 \text{Chern}-\text{Simons}(SO(3)_1). \tag{5.69}$$

From level-rank duality, we know Subsecs. 3.1 and 3.2 in Ref. 125

$$U(1)_2 = U(1)_{-2} \leftrightarrow SU(2)_1, \tag{5.70}$$

so we have that

$$\begin{aligned}[\text{Chern}-\text{Simons}(U(1)_2)/B\mathbb{Z}_2] &= [SU(2)_1/B\mathbb{Z}_2]\\ &= \text{Chern}-\text{Simons}(SO(3)_1).\end{aligned} \tag{5.71}$$

Thus, we see from level-rank duality that our decomposition in Subsec. 5.3 implies

$$[\text{Chern}-\text{Simons}(U(1)_2)/B\mathbb{Z}_4] = \coprod_2 [\text{Chern}-\text{Simons}(U(1)_2)/B\mathbb{Z}_2], \tag{5.72}$$

which is a special case of the result (5.67), confirming in this case that the decomposition prediction (3.15) is giving results compatible with this example of level-rank duality.

Next, we compute the spectrum of line operators in $U(1)_8/B\mathbb{Z}_{2p}$, using the methods of Subsec. 4.2, where in the gauging, the $\mathbb{Z}_{2p}$ projects to $\mathbb{Z}_2$ with trivially acting $\mathbb{Z}_p$. We describe the $\mathbb{Z}_{2p}$ by a set of lines $\{\ell_i\}$, $i \in \{0, \dots, 2p-1\}$, where

$$\ell_i \times \ell_j = \ell_{i+j \bmod 8}. \tag{5.73}$$

$U(1)_8$ has eight lines, labeled

$$(0), (1), (2), (3), (4), (5), (6), (7) \tag{5.74}$$

whose properties are listed in App. A, and for which $\{(0), (4)\}$ encode a $B\mathbb{Z}_2$. The action of $B\mathbb{Z}_{2p}$ on the lines of $U(1)_8$ is given as follows:

$$B(\ell_{\text{even}}, L) = +1, B(\ell_{\text{odd}}, L) = B(4, L), \tag{5.75}$$

$$\ell_{\text{even}} \times L = L, \ell_{\text{odd}} \times L = (4) \times L, \tag{5.76}$$

using the monodromies and fusion algebra described in App. A. It is straightforward that this gives a well-defined action in the sense of Subsec. 4.2.

Next, we compute the spectrum of lines in $U(1)_8/B\mathbb{Z}_8$, following the procedure of Subsec. 4.2.

- The lines (1), (3), (5), (7) have $B(\ell_{\text{odd}}, L) = -1 \neq +1$, and so are excluded.
- $\ell_1 \times (0) = (4)$, $\ell_1 \times (2) = (6)$, so we identify $(0) \sim (4)$, $(2) \sim (6)$.
- $\ell_{\text{even}} \times L = L$, so we get p copies of $(0) \sim (4)$ and $(2) \sim (6)$.

Thus, the resulting spectrum is p copies of $\{(0) \sim (4), (2) \sim (6)\}$, which is the same as p copies of the line operator spectrum of $U(1)_8/B\mathbb{Z}_2$, as expected from decomposition, since there is a trivially-acting $\mathbb{Z}_p$.

Next, let us compare to boundary WZW models. A (boundary) WZW model for the group $U(1)$ is the same as a $c = 1$ free scalar, of radius determined by the level. (See e.g. App. C.1 in Ref. 76 for discussions of the RCFTs arising at particular values of the level). Gauging the bulk one-form symmetry corresponds to orbifolding the boundary $c = 1$ theory, which just changes the radius of the target-space circle in that boundary $c = 1$ theory.

In a 2D sigma model with target S^1, if we orbifold by a $\mathbb{Z}_{kp}$ where $\mathbb{Z}_p \subset \mathbb{Z}_{kp}$ acts trivially, then from 2D decomposition, the resulting theory is equivalent to p copies of the effectively-acting $\mathbb{Z}_k$ orbifold, precisely matching (5.67), as expected.

5.10. *Exceptional groups*

So far we have discussed quotients of Chern–Simons theories for the gauge groups $SU(n)$, $\mathrm{Spin}(n)$, and $Sp(n)$. We can also consider cases with exceptional gauge groups. Although G_2, F_4, and E_8 have no center, the group E_6 has center $\mathbb{Z}_3$, and E_7 has center $\mathbb{Z}_2$ (see e.g. App. A in Ref. 126).

For example, applying decomposition (3.15), for a $\mathbb{Z}_{3p}$ that acts on E_6 by projecting to the $\mathbb{Z}_3$ center with kernel $\mathbb{Z}_p$,

$$[\text{Chern}-\text{Simons}(E_6)/B\mathbb{Z}_{3p}] = \coprod_{\theta \in \hat{\mathbb{Z}}_p} \text{Chern}-\text{Simons}(E_6/\mathbb{Z}_3)_\theta, \tag{5.77}$$

where the discrete theta angle couples to $\beta_\alpha(w_{E_6/\mathbb{Z}_3})$, for β_α the Bockstein map associated to the short exact sequence

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_{3p} \to \mathbb{Z}_3 \to 1 \tag{5.78}$$

of extension class $\alpha \in H^2_{\text{group}}(\mathbb{Z}_3, \mathbb{Z}_p)$.

Similarly, from decomposition (3.15), for a $\mathbb{Z}_{2p}$ that acts on E_7 by projecting to the $\mathbb{Z}_2$ center with kernel $\mathbb{Z}_p$

$$[\text{Chern}-\text{Simons}(E_7)/B\mathbb{Z}_{2p}] = \coprod_{\theta \in \hat{\mathbb{Z}}_p} \text{Chern}-\text{Simons}(E_7/\mathbb{Z}_2)_\theta, \tag{5.79}$$

where the discrete theta angle couples to $\beta_\alpha(w_{E_7/\mathbb{Z}_2})$, for β_α the Bockstein map associated to the short exact sequence

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_{2p} \to \mathbb{Z}_3 \to 1 \tag{5.80}$$

of extension class $\alpha \in H^2_{\text{group}}(\mathbb{Z}_2, \mathbb{Z}_p)$.

In both cases, in the boundary WZW model, this reduces to 2D decomposition of a WZW orbifold, with the discrete theta angles becoming choices of discrete torsion. In both cases, as the orbifolds involve cyclic groups, discrete torsion is trivial, so the boundary decomposition yields just a disjoint union of copies of the same WZW orbifold.

5.11. *Chern–Simons*$(H_1 \times H_2)/BA$

For completeness, let us also briefly discuss decomposition in gauged Chern–Simons theories whose gauge groups are a product of Lie groups. Specifically, consider the gauge of a gauged BA action, for A finite and abelian, on a Chern–Simons theory for $H_1 \times H_2$ (at various levels, such that the gauge theory is well defined on the given three-manifold). Bulk decomposition takes the same form as (3.15)

$$[\mathrm{Chern-Simons}(H_1 \times H_2)/BA] = \coprod_{\theta \in \hat{K}} \mathrm{Chern-Simons}(G)_\theta, \tag{5.81}$$

where

$$1 \to K \to A \stackrel{d}{\to} H_1 \times H_2 \to G \to 1, \tag{5.82}$$

and the discrete theta angle couples to $\beta_\alpha(w_G)$, for β_α the Bockstein homomorphism associated to

$$1 \to K \to A \to Z \to 1, \tag{5.83}$$

classified by $\alpha \in H^2_{\mathrm{group}}(Z, K)$, where Z is a subgroup of the product of the centers of $H_{1,2}$, given by the image of d.

On the boundary, as before, this reduces to decomposition in the 2D theory, here

$$[\mathrm{WZW}(H_1 \times H_2)/A] = \coprod_{\theta \in \hat{K}} \mathrm{WZW}(G)_\theta, \tag{5.84}$$

where the discrete theta angles θ now correspond to choices of discrete torsion in a

$$[\mathrm{WZW}(H_1 \times H_2)/Z] \tag{5.85}$$

orbifold. Essentially, because A is abelian, for ultimately the same reasons as in Subsec. 5.7, the discrete torsion is trivial on each universe.

5.12. *Finite 2-group orbifolds*

So far, we have focused on Chern–Simons theories in three dimensions, but the same ideas apply to the finite 2-group orbifolds discussed in Ref. 30. There, orbifolds by 2-groups Γ were described, where Γ is an extension

$$1 \to BK \to \Gamma \to G \to 1, \tag{5.86}$$

where G, K are both finite and K is abelian, determined by $[\omega] \in H^3_{\mathrm{group}}(G, K)$. Now, Γ can also be described by a crossed module $\{d : A \to H\}$, corresponding to a four-term exact sequence of ordinary groups

$$1 \to K \to A \stackrel{d}{\to} H \to G \to 1, \tag{5.87}$$

also determined (up to equivalences) by $[\omega] \in H^3_{\rm group}(G, K)$ (see e.g. Sec. IV.9 in Ref. 127 for related observations).

In this language, we can write the 2-group orbifold $[X/\Gamma]$ in terms of the crossed module as

$$[X/\Gamma] = [[X/H]/BA], \tag{5.88}$$

at least for a presentation in which A is abelian.

For this slightly different physical realization in terms of finite groups, the statement of decomposition (3.15) is modified, but only slightly

$$[X/\Gamma] = [[X/H]/BA] = \oplus_{\theta \in \hat{K}} [X/G]_{\omega(\theta)}, \tag{5.89}$$

where the discrete torsion (formerly discrete theta angle) $\omega(\theta)$ is defined by $\phi^* \omega$. In this sense, the decomposition described in this paper is simply a variation on the 2-group orbifold decomposition described in Ref. 30. The fact that bulk discrete theta angles (here, C-field analogues of discrete torsion) become (ordinary) discrete torsion in the boundary theory was also observed in Subsec. 3.2 in Ref. 30.

In passing, we should also observe that results in finite 2-group orbifolds have a qualitatively different form. For example, Subsec. 4.4 in Ref. 30 described an orbifold by a 2-group extension

$$1 \to B\mathbb{Z}_2 \to \Gamma \to (\mathbb{Z}_2)^3 \to 1. \tag{5.90}$$

In this case, $[X/\Gamma]$ is equivalent to a pair of copies of $[X/(\mathbb{Z}_2)^3]$ orbifolds, each with a different C field discrete torsion in $H^3_{\rm group}((\mathbb{Z}_2)^3, U(1))$, which is nontrivial even on T^3. One could imagine an analogous theory here, such as a quotient of $SU(2)^3$ Chern–Simons by BA (for A a finite abelian group, with $K = \mathbb{Z}_2$ kernel, say) that leads to a disjoint union of $SO(3)^3$ Chern–Simons theories. Here, however, in the case of Chern–Simons theories, no analogue of C field discrete torsion is present for T^3, partly because (as noted in Subsec. 3.4) the pertinent Bockstein homomorphism vanishes. Part of the difference between these two theories is that in the Chern–Simons case, the pertinent exact sequence of finite groups has the form

$$1 \to \mathbb{Z}_2 \to A \to (\mathbb{Z}_2)^3 \to 1, \tag{5.91}$$

whereas by contrast the analogous sequence in Ref. 30, namely (5.90), can be alternately encoded as a four-term sequence

$$1 \to \mathbb{Z}_2 \to P' \to Q' \to (\mathbb{Z}_2)^3 \to 1, \tag{5.92}$$

which realizes an element of $H^3_{\rm group}((\mathbb{Z}_2)^3, \mathbb{Z}_2)$. By contrast, the short exact sequence (5.91) realizes an element of $H^2_{\rm group}((\mathbb{Z}_2)^3, \mathbb{Z}_2)$, cohomology of different degrees; the crossed module construction realizes a 2-group, but involves different groups.

6. Boundary G/G Models

For completeness, in this section we include a different example of a decomposition.

Consider gauged WZW models G/H at level k, on the boundary of a 3D theory. Because the H action being gauged is an adjoint action,[128] if the center $Z(H)$ of H is nonzero, it acts trivially, and in two dimensions, the resulting gauged WZW model decomposes into universes indexed by irreducible representations of $Z(H)$.

Now, let us compare to the bulk theory. From Sec. 3 in Ref. 69, for the gauged WZW model G/H at level k, the bulk 3D theory is a $(G \times H)/Z$ gauge theory, with Z as the common center of G and H, with action

$$k\ell S_{\mathrm{CS}}(G) - kS_{\mathrm{CS}}(H), \tag{6.1}$$

where ℓ is the index of the embedding $H \hookrightarrow G$,

Consider the special case of the 2D G/G model, on the boundary of a 3D theory. The G/G model decomposes into universes indexed by the integrable representations. (In principle, this is because it is a unitary topological field theory[129,130]; the specific relation to decomposition is via noninvertible symmetries, as discussed in Refs. 20 and 21.) From the discussion above, the bulk dual to the boundary G/G model appears to have an identically-zero action (6.1). Since the boundary theory is a topological field theory, this would be trivially consistent.

For more general boundary G/H-gauged WZW models, the bulk action (6.1) does not vanish identically. Decomposition of the boundary suggests that the bulk may also decompose, in which case the bulk theory should admit a global two-form symmetry. We leave elucidating that symmetry for future work.

7. Conclusions

In this paper, we have discussed decomposition in 3D Chern–Simons theories with gauged noneffectively-acting one-form symmetries. In the bulk decomposition, the different universes of the decomposition have discrete theta angles coupling to bundle characteristic classes, specifically, images under Bockstein maps of canonical degree-two characteristic classes. On the boundary, those map to choices of discrete torsion, and the bulk decomposition becomes a standard orbifold decomposition, involving WZW models, which serves as a strong consistency test.

There are many directions this work could be taken. One example would be to consider decomposition in gauged Chern–Simons theories in which the original theory has a discrete theta angle, analogous to decomposition in 2D orbifolds with discrete torsion.[26] Another example would be to consider decomposition in Chern–Simons-matter theories, rather than pure Chern–Simons. Similarly, it would be interesting to consider decomposition in holomorphic Chern–Simons,[131] or deformations of Chern–Simons theories, that arise when studying disk instanton corrections in string compactifications.

It would also be interesting to understand dimensional reduction of decomposition to two dimensions. The dimensional reduction of pure Chern–Simons is the 2D G/G model (which as a unitary TFT already admits a decomposition[20,21,129,130]), and the BK symmetry in three dimensions should become a $K \times BK$ symmetry in the two-dimensional theory.

In condensed matter physics, there exists a realization of Chern–Simons theories known as the Levin–Wen model,[134] and it would be interesting to consider this story in that setting.

In a different direction, Chern–Simons theories can also arise on boundaries of four-dimensional theories, and it would be interesting to study decomposition in that context, perhaps relating it to the decomposition arising after instanton restriction in Ref. 16. There, the instanton restriction resulted in a disjoint union of 4D Yang–Mills theories with theta angle terms of the form

$$\frac{1}{8\pi^2} \frac{2\pi m}{k} \int \mathrm{Tr} F \wedge F, \tag{7.1}$$

for $m \in \{0, 1, \ldots, k-1\}$, which implements the restriction on instantons. On a boundary, that would become a disjoint union of theories, whose actions have Chern–Simons terms of the form

$$\frac{1}{8\pi^2} \frac{2\pi m}{k} \int \omega_{\mathrm{CS}}, \tag{7.2}$$

clearly related to the disjoint unions of Chern–Simons theories we discuss in this paper. We leave such considerations for future work.

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Appendix A. Line Operators

In this section, we briefly review some basics of line operators in Chern–Simons theories and their quantum numbers, to make this paper self-contained.

In general, the line operators in a Chern–Simons theory at level k correspond to integrable representations, which for a model at level k, are the representations of highest weight λ satisfying the unitarity bound (Eq. (9.30) in Ref. 132)

$$2\frac{\psi \cdot \lambda}{\psi^2} \leq k, \tag{A.1}$$

for ψ as the highest weight of the adjoint representation. (For example, for $SU(n)$ the integrable representations at any level are classified by Young diagrams of width bounded by the level.) Similarly, for a given WZW primarily associated to an integrable representation of highest weight λ, the L_0 eigenvalue is (Eq. (15.87) in Ref. 133)

$$h = \frac{(\lambda, \lambda + 2\rho)}{2(k+g)}, \tag{A.2}$$

where g is the dual Coxeter number and ρ the Weyl vector (half-sum of positive roots). In passing, a representation is integrable if and only if its dual is integrable, and its dual defines WZW primaries of the same conformal weight, see e.g. Subsec. 8.3 in Ref. 126. Similarly, the quantum dimension is given by (Eq. (16.66) in Ref. 133)

$$\prod_{\alpha>0} \frac{\sin\left(\frac{\pi(\lambda+\rho,\alpha)}{k+g}\right)}{\sin\left(\frac{\pi(\rho,\alpha)}{k+g}\right)}. \tag{A.3}$$

For use in examples in the text, the line operators of $SU(2)_4$ are[n]

$SU(2)_4$	Integrable rep.	$\tilde{\lambda}$	h	q-dim
(0)	**1**	[0,4]	0	1
(1)	□□□□	[4,0]	1	1
(2)	□	[1,3]	1/8	$\sqrt{3}$
(3)	□□□	[3,1]	5/8	$\sqrt{3}$
(4)	□□	[2,2]	1/3	2,

where $\tilde{\lambda}$ denotes the Dynkin label of each line, h is the conformal weight of the corresponding boundary chiral primary as above, and q-dim denotes the quantum dimension.

The fusion algebra of $SU(2)_4$ lines can be computed with the program Kac,[135] and that algebra is given as follows:

$$\begin{aligned}
&(0)\times(0) = (0),\ (2)\times(2) = (0) + (4),\\
&(0)\times(1) = (1),\ (2)\times(3) = (1) + (4),\\
&(0)\times(2) = (2),\ (2)\times(4) = (2) + (3),\\
&(0)\times(3) = (3),\ (3)\times(3) = (0) + (4),\\
&(0)\times(4) = (4),\ (3)\times(4) = (2) + (3),\\
&(1)\times(1) = (0),\ (4)\times(4) = (0) + (1) + (4).
\end{aligned}$$

[n] We would like to thank M. Yu for providing the results for line operators of $SU(2)_4$ and $U(1)_8$ listed in this appendix.

$$(1) \times (2) = (3),$$
$$(1) \times (3) = (2),$$
$$(1) \times (4) = (4),$$

We see that the lines (0), (1) are mutually transparent, and their fusion products have the structure of the group $\mathbb{Z}_2$.

From the table above, it is straightforward to compute the monodromies of the line (1) about other lines, using

$$B(a, b) = \exp(2\pi i(h(a \times b) - h(a) - h(b))), \tag{A.4}$$

and one finds

$$B(1, 1) = +1, \tag{A.5}$$
$$B(1, 2) = -1, \tag{A.6}$$
$$B(1, 3) = -1, \tag{A.7}$$
$$B(1, 4) = +1, \tag{A.8}$$

so that all monodromies are in $\{\pm 1\}$, as expected for a $B\mathbb{Z}_2$, and also consistent with the fact that (2) and (3) correspond to Wilson lines for an odd number of copies of the $\not\models$ representation.

Similarly, it will be useful later to write down the fusion algebra for $U(1)_8$. Here, there are eight lines, labeled (0) through (7), with conformal weights and quantum dimensions

$U(1)_8$	h	q-dim
(0)	0	1
(1)	1/16	1
(2)	1/4	1
(3)	9/16	1
(4)	1	1
(5)	9/16	1
(6)	1/4	1
(7)	1/16	1

and the fusion algebra acts by addition, as

$$(a) \times (b) = (a + b \bmod 8). \tag{A.9}$$

From the table of lines above, it is clear that there is a $B\mathbb{Z}_2$ corresponding to the lines $\{(0), (4)\}$. For use in Subsec. 5.9, we list here pertinent monodromies:

$$B((0), L) = +1, B(4, 0) = B(4, 2) = B(4, 4) = B(4, 6) = +1, \tag{A.10}$$

$$B(4, 1) = B(4, 3) = B(4, 5) = B(4, 7) = -1. \tag{A.11}$$

Appendix B. Overview of Crossed Modules

In this paper, we have described 2-groups using crossed modules. As they play an important role in the decomposition statement in 3D Chern–Simons theories, to make this paper self-contained, we include a brief overview here.

Briefly, a crossed module consists of the following data:

- a pair of groups G_0, G_1,
- a group homomorphism $d : G_1 \to G_0$,
- a group homomorphism $\alpha : G_0 \to \mathrm{Aut}(G_1)$,

such that

1. The composition

$$G_1 \xrightarrow{d} G_1 \xrightarrow{\alpha} \mathrm{Aut}(G_1) \tag{B.1}$$

is the conjugation action of G_1 on itself, meaning

$$\alpha(d(g_1))(h) = g_1 h g_1^{-1}, \tag{B.2}$$

for $g_1, h \in G_1$, or equivalently that

$$\begin{array}{ccc} G_1 \times G_1 & \xrightarrow{d\times \mathrm{Id}} & G_0 \times G_1 \\ & {}_{\mathrm{Ad}}\searrow \quad \swarrow_{\alpha} & \\ & G_1 & \end{array} \tag{B.3}$$

commutes

2. d is equivariant for the G_0 action on the source and target, meaning

$$d(\alpha(g_0)(h)) = g_0 d(h) g_0^{-1} \tag{B.4}$$

for $g_0, h \in G_0$, or equivalently that

$$\begin{array}{ccc} G_0 \times G_1 & \xrightarrow{\alpha} & G_1 \\ {\scriptstyle \mathrm{Id}\times d}\downarrow & & \downarrow{\scriptstyle d} \\ G_0 \times G_0 & \xrightarrow{\mathrm{Ad}} & G_0, \end{array} \tag{B.5}$$

commutes.

In the description above, $\mathrm{Ad} : G \to \mathrm{Aut}(G)$ denotes the adjoint action of G to itself, namely, $\mathrm{Ad}(g)(x) = gxg^{-1}$.

Some examples of crossed modules include the following:

- For G_1 any group, let $G_0 = \mathrm{Aut}(G_1)$, with $d : G_1 \to \mathrm{Aut}(G_1)$ the natural inclusion (meaning $d(g) = \mathrm{Ad}(g)$) and $\alpha : \mathrm{Aut}(G_1) \to \mathrm{Aut}(G_1)$ the identity.
- Let G_0 be any group and G_1 a normal subgroup of G_0, with $d : G_1 \to G_0$ inclusion, and $\alpha : G_0 \to \mathrm{Aut}(G_1)$ by conjugation.

A crossed module can be encoded in a four-term exact sequence:

$$1 \to \mathrm{Ker}\ d \to G_1 \stackrel{d}{\to} G_0 \to \mathrm{Coker}\ d \to 1. \tag{B.6}$$

In the case that Ker d is abelian, this is sometimes alternatively expressed as the extension

$$1 \to B(\mathrm{Ker}\ d) \to \Gamma \to \mathrm{Coker}\ d \to 1, \tag{B.7}$$

for Γ the 2-group corresponding to the crossed module.

Physically, in this paper, the map d encodes the action of the noneffectively-acting BA, by mapping A to a subset of the center of the Chern–Simons gauge group, which acts nontrivially.

For more information on crossed modules, see for example[136] for further mathematics background, or App. A in Ref. 74 and Sec. 2 in Ref. 75 in physics.

Appendix C. Generalities on Gauging Effectively-Acting One-Form Symmetries

For most of this paper, we have discussed gauging one-form symmetries in terms of line operators, but it is worth observing that this operation can also be understood in terms of local actions, which we will briefly review in this section.

Suppose in general terms we have a G gauge theory, and we gauge the action of a one-form symmetry BK, where BK acts nontrivially on the line operators of the theory. (For example, this is the case if K is a subset of the center of G).

In general terms, when gauging the BK on a G gauge theory,

- the path integral sums over K gerbes, and
- for each K gerbe, the path integral sums over gerbe-twisted G bundles, defined by transition functions which close on triple overlaps only up to a cocycle representing the gerbe characteristic class.

Consider for example gauging an effectively-acting $B\mathbb{Z}_n$ in an $SU(n)$ gauge theory. The twisted $SU(n)$ gauge fields above are all the same as ordinary $SU(n)/\mathbb{Z}_n$

gauge fields, and the gerbe characteristic classes correspond to (some) characteristic classes of $SU(n)/\mathbb{Z}_n$ bundles. Let us look at this in more detail:

1. The transition functions g_{ij} of a twisted bundle no longer close on triple overlaps, but rather obey
$$g_{ij}g_{jk}g_{ki} = h_{ijk} \tag{C.1}$$
for a cocycle h_{ijk} representing an element of $H^2(Y, \mathbb{Z}_n)$ corresponding to the gerbe characteristic class, and
2. Across overlaps, the gauge field A obeys
$$A_i = g_{ij}A_j g_{ij}^{-1} + g_{ij}^{-1}dg_{ij} - I\Lambda_{ij}, \tag{C.2}$$
where I is the identity and Λ_{ij} is a locally-defined one-form field, with the property that if the gerbe were to admit a connection B, then on the same overlaps
$$B_i = B_j + d\Lambda_{ij}. \tag{C.3}$$

Now, this procedure should generate all G/K bundles. One example of this involves the relation between $SU(2)$ and $SO(3)$ bundles in 3D theories. As is well known
$$\text{Chern}-\text{Simons}(SU(2))/B\mathbb{Z}_2 = \text{Chern}-\text{Simons}(SO(3)), \tag{C.4}$$
for the $B\mathbb{Z}_2$ corresponding to the center one-form symmetry. Viewed as a $B\mathbb{Z}_2$ quotient of an $SU(2)$ gauge theory, the path integral

- sums over $\mathbb{Z}_2$ gerbes, whose characteristic class is $w \in H^2(M, \mathbb{Z}_2)$, and
- sums over w-twisted $SU(2)$ bundles, meaning that the $SU(2)$ transition functions close on triple overlaps only up to w, and that gauge transformations across patches only have to match up to a $\mathbb{Z}_2$ shift.

Interpreted in terms of $SO(3)$ bundles, the characteristic class $w \in H^2(M, \mathbb{Z}_2)$ is the second Stiefel–Whitney class of an $SO(3)$ bundle. (The other possibly nonzero characteristic class, the third Stiefel–Whitney class $w_3 \in H^3(M, \mathbb{Z}_2)$, is determined by w_2 as $w_3 = \mathrm{Sq}^1(w_2)$, see Subsec. 5.3). The fact that gauge transformations only respect $SU(2)$ up to $\mathbb{Z}_2$ shifts, and that $SU(2)$ transition functions only close up to w, are indicative of general aspects of $SO(3)$ bundles.

Thus, we see that the $B\mathbb{Z}_2$-gauged $SU(2)$ theory really does recover all $SO(3)$ bundles, even those with nonzero w_3, as expected.

If we instead gauged a BA action on a G Chern–Simons theory with a trivially-acting subgroup BK, then, for reasons detailed in Ref. 30, we would recover

$G/(A/K)$ gauge theory, with a restriction on $G/(A/K)$ bundles. One role of decomposition is to implement that restriction.

References

1. S. Hellerman, A. Henriques, T. Pantev, E. Sharpe and M. Ando, *Adv. Theor. Math. Phys.* **11**, 751 (2007), arXiv:hep-th/0606034.
2. T. Pantev and E. Sharpe, arXiv:hep-th/0502027.
3. T. Pantev and E. Sharpe, *Nucl. Phys. B* **733**, 233 (2006), arXiv:hep-th/0502044.
4. T. Pantev and E. Sharpe, *Adv. Theor. Math. Phys.* **10**, 77 (2006), arXiv:hep-th/0502053.
5. A. Căldăraru, J. Distler, S. Hellerman, T. Pantev and E. Sharpe, *Commun. Math. Phys.* **294**, 605 (2010), arXiv:0709.3855.
6. E. Andreini, Y. Jiang and H.-H. Tseng, arXiv:0812.4477.
7. E. Andreini, Y. Jiang and H.-H. Tseng, *Commun. Anal. Geom.* **24**, 223 (2016), arXiv:0905.2258.
8. E. Andreini, Y. Jiang and H.-H. Tseng, *J. Diff. Geom.* **99**, 1 (2015), arXiv:0907.2087.
9. H.-H. Tseng, *Int. Math. Res. Not.* **2011**, 2444 (2011), arXiv:0912.3580.
10. A. Gholampour and H.-H. Tseng, *Proc. Amer. Math. Soc.* **141**, 191 (2013), arXiv:1001.0435.
11. X. Tang and H.-H. Tseng, *Adv. Math.* **250**, 496 (2014), arXiv:1004.1376.
12. S. Hellerman and E. Sharpe, *Adv. Theor. Math. Phys.* **15**, 1141 (2011), arXiv:1012.5999.
13. L. B. Anderson, B. Jia, R. Manion, B. Ovrut and E. Sharpe, *Adv. Theor. Math. Phys.* **19**, 531 (2015), arXiv:1307.2269.
14. E. Sharpe, *Phys. Rev. D* **90**, 25030 (2014), arXiv:1404.3986.
15. E. Sharpe, *Int. J. Mod. Phys. A* **34**, 1950233 (2020), arXiv:1911.05080.
16. Y. Tanizaki and M. Ünsal, *J. High Energy Phys.* **3**, 123 (2020), arXiv:1912.01033.
17. R. Eager and E. Sharpe, arXiv:2009.03907.
18. A. Cherman and T. Jacobson, *Phys. Rev. D* **103**, 105012 (2021), arXiv:2012.10555.
19. A. Cherman, T. Jacobson and M. Neuzil, arXiv:2111.00078.
20. Z. Komargodski, K. Ohmori, K. Roumpedakis and S. Seifnashri, *J. High Energy Phys.* **3**, 103 (2021), arXiv:2008.07567.
21. T. C. Huang, Y. H. Lin and S. Seifnashri, *J. High Energy Phys.* **12**, 28 (2021), arXiv:2110.02958.
22. M. Nguyen, Y. Tanizaki and M. Ünsal, *J. High Energy Phys.* **3**, 238 (2021), arXiv:2101.02227.
23. M. Nguyen, Y. Tanizaki and M. Ünsal, *Phys. Rev. D* **104**, 65003 (2021), arXiv:2104.01824.
24. M. Honda, E. Itou, Y. Kikuchi and Y. Tanizaki, arXiv:2110.14105.
25. M. Yu, arXiv:2111.13697.
26. D. Robbins, E. Sharpe and T. Vandermeulen, *J. High Energy Phys.* **21**, 134 (2020), arXiv:2101.11619.
27. D. G. Robbins, E. Sharpe and T. Vandermeulen, *J. High Energy Phys.* **2**, 108 (2022), arXiv:2107.12386.
28. D. G. Robbins, E. Sharpe and T. Vandermeulen, *Phys. Rev. D* **104**, 85009 (2021), arXiv:2106.00693.
29. D. G. Robbins, E. Sharpe and T. Vandermeulen, *Int. J. Mod. Phys. A* **36**, 2150220 (2021), arXiv:2107.13552.
30. T. Pantev, D. Robbins, E. Sharpe and T. Vandermeulen, arXiv:2204.13708.

31. E. Sharpe, Derived categories and stacks in physics, in *Homological Mirror Symmetry: New Developments and Perspectives*, eds. A. Kapustin, M. Kreuzer and K.-G. Schlesinger, *Lecture Notes in Physics*, Vol. 757 (Springer, Berlin, 2009), arXiv:hep-th/0608056.
32. E. Sharpe, Landau-Ginzburg models, gerbes, and Kuznetsov's homological projective duality, in *Superstrings, Geometry, Topology, and* $\mathbb{C}^*$ *Algebras*, *Proceedings of Symposia in Pure Mathematics*, Vol. 81 (American Mathematical Society, Providence, RI, 2010).
33. E. Sharpe, *J. Phys. Conf. Ser.* **462**, 12047 (2013), arXiv:1004.5388.
34. E. Sharpe, *Fort. Phys.* **67**, 1910019 (2019), arXiv:1903.02880.
35. E. Sharpe, arXiv:2204.09117.
36. K. Hori, *J. High Energy Phys.* **10**, 121 (2013), arXiv:1104.2853.
37. N. M. Addington, E. P. Segal and E. Sharpe, *Adv. Theor. Math. Phys.* **18**, 1369 (2014), arXiv:1211.2446.
38. E. Sharpe, *J. Geom. Phys.* **74**, 256 (2013), arXiv:1212.5322.
39. J. Halverson, V. Kumar and D. R. Morrison, *J. High Energy Phys.* **9**, 143 (2013), arXiv:1305.3278.
40. M. Ballard, D. Deliu, D. Favero, M. U. Isik and L. Katzarkov, arXiv:1306.3957.
41. E. Sharpe, *Phys. Lett. B* **726**, 390 (2013), arXiv:1306.5440.
42. K. Hori and J. Knapp, *J. High Energy Phys.* **11**, 70 (2013), arXiv:1308.6265.
43. K. Hori and J. Knapp, arXiv:1612.06214.
44. K. Wong, *J. High Energy Phys.* **3**, 132 (2017), arXiv:1702.00730.
45. M. Kapustka and M. Rampazzo, *Commun. Numer. Theor. Phys.* **13**, 725 (2019), arXiv:1711.10231.
46. H. Parsian, E. Sharpe and H. Zou, *Int. J. Mod. Phys. A* **33**, 1850113 (2018), arXiv:1803.00286.
47. Z. Chen, T. Pantev and E. Sharpe, *J. Geom. Phys.* **137**, 204 (2019), arXiv:1806.01283.
48. J. Guo and M. Romo, arXiv:2111.00025.
49. J. Wang, X. G. Wen and E. Witten, *Phys. Rev. X* **8**, 31048 (2018), arXiv:1705.06728.
50. E. Witten, *Nucl. Phys. B* **311**, 46 (1988).
51. F. Benini, C. Copetti and L. Di Pietro, arXiv:2203.09537.
52. D. Marolf and H. Maxfield, *J. High Energy Phys.* **8**, 44 (2020), arXiv:2002.08950.
53. K. Gawedzki, R. R. Suszek and K. Waldorf, *Commun. Math. Phys.* **284**, 1 (2008), arXiv:hep-th/0701071.
54. I. Brunner, *J. High Energy Phys.* **1**, 7 (2002), arXiv:hep-th/0110219.
55. G. Pradisi, A. Sagnotti and Y. S. Stanev, *Phys. Lett. B* **354**, 279 (1995), arXiv:hep-th/9503207.
56. G. Pradisi, A. Sagnotti and Y. S. Stanev, *Phys. Lett. B* **356**, 230 (1995), arXiv:hep-th/9506014.
57. U. Schreiber, C. Schweigert and K. Waldorf, *Commun. Math. Phys.* **274**, 31 (2007), arXiv:hep-th/0512283.
58. C. Bachas, N. Couchoud and P. Windey, *J. High Energy Phys.* **12**, 3 (2001), arXiv:hep-th/0111002.
59. E. Sharpe, *J. Geom. Phys.* **61**, 1017 (2011), arXiv:0908.0087.
60. K. Hori, arXiv:hep-th/9402019.
61. K. Hori, *Commun. Math. Phys.* **182**, 1 (1996), arXiv:hep-th/9411134.
62. D. Gaiotto, G. W. Moore and A. Neitzke, *Adv. Theor. Math. Phys.* **17**, 241 (2013), arXiv:1006.0146.
63. O. Aharony, N. Seiberg and Y. Tachikawa, *J. High Energy Phys.* **8**, 115 (2013), arXiv:1305.0318.

64. J. P. Ang, K. Roumpedakis and S. Seifnashri, *J. High Energy Phys.* **4**, 87 (2020), arXiv:1911.00589.
65. K. Roumpedakis, S. Seifnashri and S. H. Shao, arXiv:2204.02407.
66. A. Joyal, R. Street, *Macquarie Math. Rep.* (1986) 860081. http://maths.mq.edu.au/˜street/JS1.pdf.
67. C. Schommer-Pries, arXiv:1112.1000.
68. M. Barkeshli, P. Bonderson, M. Cheng and Z. Wang, *Phys. Rev. B* **100**, 115147 (2019), arXiv:1410.4540.
69. G. W. Moore and N. Seiberg, *Phys. Lett. B* **220**, 422 (1989).
70. P. S. Hsin, H. T. Lam and N. Seiberg, *SciPost Phys.* **6**, 39 (2019), arXiv:1812.04716.
71. G. W. Moore and N. Seiberg, *Commun. Math. Phys.* **123**, 177 (1989).
72. B. Bakalov and A. Kirillov, *Lectures on Tensor Categories and Modular Functors*, *University Lecture Series*, Vol. 21 (American Mathematical Society, Providence, Rhode Island, 2001).
73. A. Kitaev, *Ann. Phys.* **321**, 2 (2006), arXiv:cond-mat/0506438.
74. Y. Lee, K. Ohmori and Y. Tachikawa, *J. High Energy Phys.* **10**, 114 (2021), arXiv:2108.05369.
75. L. Bhardwaj, *SciPost Phys.* **12**, 152 (2022), arXiv:2107.06816.
76. N. Seiberg and E. Witten, *Prog. Theor. Exp. Phys.* **2016**, 12C101 (2016), arXiv:1602.04251.
77. N. Seiberg, T. Senthil, C. Wang and E. Witten, *Ann. Phys.* **374**, 395 (2016), arXiv:1606.01989.
78. D. Belov and G. W. Moore, arXiv:hep-th/0505235.
79. D. S. Freed, *Adv. Math.* **113**, 237 (1995), arXiv:hep-th/9206021.
80. D. S. Freed, https://web.ma.utexas.edu/users/dafr/cs2.pdf.
81. A. Borel, R. Friedman and J. W. Morgan, arXiv:math/9907007.
82. J. de Boer, R. Dijkgraaf, K. Hori, A. Keurentjes, J. Morgan, D. R. Morrison and S. Sethi, *Adv. Theor. Math. Phys.* **4**, 995 (2002), arXiv:hep-th/0103170.
83. S. Elitzur, G. W. Moore, A. Schwimmer and N. Seiberg, *Nucl. Phys. B* **326**, 108 (1989).
84. M. Bos and V. P. Nair, *Int. J. Mod. Phys. A* **5**, 959 (1990).
85. S. Axelrod, S. Della Pietra and E. Witten, *J. Diff. Geom.* **33**, 787 (1991).
86. O. Coussaert, M. Henneaux and P. van Driel, *Class. Quantum Grav.* **12**, 2961 (1995), arXiv:gr-qc/9506019.
87. https://classes.golem.ph.utexas.edu/category/2006/10/wzw_as_transition_1gerbe_of_cs.html.
88. R. Dijkgraaf and E. Witten, *Commun. Math. Phys.* **129**, 393 (1990).
89. K. Gawedzki, arXiv:hep-th/9904145.
90. D. Fiorenza, H. Sati and U. Schreiber, *J. Geom. Phys.* **74**, 130 (2013), arXiv:1207.5449.
91. D. Fiorenza, C. Rogers and U. Schreiber, arXiv:1304.0236.
92. G. Segal, *Topology* **13**, 293 (1974).
93. G. Dunn, *J. Pure Appl. Algebra* **113**, 159 (1996).
94. P. May, *The Geometry of Iterated Loop Spaces*, *Lecture Notes in Mathematics*, Vol. 271 (Springer-Verlag, Berlin, 1972).
95. P. May and R. Thomason, *Topology* **17**, 205 (1978).
96. A. Hatcher, *Algebraic Topology* (Cambridge University Press, 2002).
97. V. Borokhov, A. Kapustin and X. K. Wu, *J. High Energy Phys.* **11**, 49 (2002), arXiv:hep-th/0206054.
98. V. Borokhov, A. Kapustin and X. K. Wu, *J. High Energy Phys.* **12**, 44 (2002), arXiv:hep-th/0207074.
99. C. Cordova, P. S. Hsin and N. Seiberg, *SciPost Phys.* **4**, 21 (2018), arXiv:1711.10008.

100. L. Kong, *Nucl. Phys. B* **886**, 436 (2014), arXiv:1307.8244.
101. J. Fuchs, I. Runkel and C. Schweigert, *Nucl. Phys. B* **646**, 353 (2002), arXiv:hep-th/0204148.
102. D. Gaiotto and T. Johnson-Freyd, arXiv:1905.09566.
103. M. Levin, *Phys. Rev. X* **3**, 21009 (2013), arXiv:1301.7355.
104. L. Kong and X. G. Wen, arXiv:1405.5858.
105. T. Johnson-Freyd, arXiv:2003.06663.
106. M. Yu, *J. High Energy Phys.* **8**, 61 (2021), arXiv:2010.01136.
107. J. Milnor and J. Stasheff, *Characteristic Classes*, *Annals of Mathematics Studies*, Vol. 76 (Princeton University Press, Princeton, New Jersey, 1974).
108. https://math.stackexchange.com/questions/800804/calculate-the-wu-class-from-the-stiefel-whitney-class.
109. P. Baum and W. Browder, *Topology* **3**, 305 (1965).
110. X. Gu, *J. Topol. Anal.* **13**, 535 (2021), arXiv:1612.00506.
111. H. Duan, arXiv:1710.09222.
112. V. Kac, *Invent. Math.* **80**, 69 (1985).
113. D. Notbohm, Classifying spaces of compact Lie groups and finite loop spaces, in *Handbook of Algebraic Topology* (North-Holland, Amsterdam, 1995), pp. 1049–1094.
114. A. Kono and M. Mimura, *Publ. RIMS Kyoto Univ.* **10**, 691 (1975).
115. A. Borel, *Bull. Amer. Math. Soc.* **61**, 397 (1955).
116. E. Witten, *J. High Energy Phys.* **2**, 6 (1998), arXiv:hep-th/9712028.
117. G. Felder, K. Gawedzki and A. Kupiainen, *Commun. Math. Phys.* **117**, 127 (1988).
118. K. Gawedzki and N. Reis, *J. Geom. Phys.* **50**, 28 (2004), arXiv:math/0307010.
119. M. R. Gaberdiel, *Nucl. Phys. B* **460**, 181 (1996), arXiv:hep-th/9508105.
120. I. Runkel and R. R. Suszek, *Adv. Theor. Math. Phys.* **13**, 1137 (2009), arXiv:0808.1419.
121. K. Gawedzki and K. Waldorf, *J. High Energy Phys.* **9**, 73 (2009), arXiv:0908.1130.
122. K. Waldorf, private communication.
123. M. Kreuzer and A. N. Schellekens, *Nucl. Phys. B* **411**, 97 (1994), arXiv:hep-th/9306145.
124. J. Fuchs, A. N. Schellekens and C. Schweigert, *Nucl. Phys. B* **473**, 323 (1996), arXiv:hep-th/9601078.
125. P. S. Hsin and N. Seiberg, *J. High Energy Phys.* **9**, 95 (2016), arXiv:1607.07457.
126. J. Distler and E. Sharpe, *Adv. Theor. Math. Phys.* **14**, 335 (2010), arXiv:hep-th/0701244.
127. P. J. Hilton and U. Stammbach, *A Course in Homological Algebra*, *Graduate Texts in Mathematics*, Vol. 4 (Springer-Verlag, New York, 1971).
128. E. Witten, *Nucl. Phys. B* **371**, 191 (1992).
129. B. Durhuus and T. Jonsson, *J. Math. Phys.* **35**, 5306 (1994), arXiv:hep-th/9308043.
130. G. W. Moore and G. Segal, arXiv:hep-th/0609042.
131. R. Thomas, Gauge theory on Calabi–Yau manifolds, Ph.D. thesis, University of Oxford (1997).
132. P. H. Ginsparg, Applied conformal field theory, in *Fields, Strings, and Critical Phenomena (Les Houches 1988 proceedings)*, eds. E. Brézin and J. Zinn-Justin (Elsevier, 1989), pp. 1–68, arXiv:hep-th/9108028.
133. P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory* (Springer-Verlag, New York, 1997).
134. M. A. Levin and X. G. Wen, *Phys. Rev. B* **71**, 45110 (2005), arXiv:cond-mat/0404617.
135. https://www.nikhef.nl/t58/Site/Kac.html.
136. R. Brown, P. Higgins and R. Sivera, *Nonabelian Algebraic Topology: Filtered Spaces, Crossed Complexes, Cubical Homotopy Groupoids* (European Mathematical Society, Zurich, Switzerland, 2011).