Research Article

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Systemic risk models for disjoint and overlapping groups with equilibrium strategies

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Abstract: We analyze the systemic risk for disjoint and overlapping groups of financial institutions by proposing new models with realistic game features. Specifically, we generalize the systemic risk measure proposed in [F. Biagini, J.-P. Fouque, M. Frittelli and T. Meyer-Brandis, On fairness of systemic risk measures, *Finance Stoch.* **24** (2020), no. 2, 513–564] by allowing individual banks to choose their preferred groups instead of being assigned to certain groups. We introduce the concept of Nash equilibrium for these new models, and analyze the optimal solution under Gaussian distribution of the risk factor. We also provide an explicit solution for the risk allocation of the individual banks and study the existence and uniqueness of Nash equilibrium both theoretically and numerically. The developed numerical algorithm can simulate scenarios of equilibrium, and we apply it to study the banking structure with real data and show the validity of the proposed model.

Keywords: Systemic risk, CCPs, grouping, equilibrium

MSC 2010: 60A99, 91A06, 91B50, 91G99

1 Introduction

Financial institutions are increasingly and tightly connected together at an unprecedented scale, and the complex dynamics of the inter-connectedness aggregate their idiosyncratic risks within the financial system. Consequently, failures of individual institutions due to excessive risk-taking may quickly propagate throughout the entire financial network and systemically cause cascading disasters. Such financial crises (e.g., [7, 24, 30]) have dramatically demonstrated the importance of understanding the nature of systemic risk and designing models and methods to capture and analyze it. In the literature, various aspects related to systemic risk have been studied, such as modeling and measurements [10, 14, 17, 29], the analysis of the contagion [2, 12, 22], the spread of a potential exogenous shock into the system [23], and many others [13, 15]. Recently, the framework has also been extended to include effects such as multi-maturities [26], bankruptcy costs [18], and exchanging contingent convertible debt obligations [19]. We refer interested readers to the book [25] for a comprehensive review.

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For a given financial network and a given random shock, one then determines the "cascade" mechanism, which generates many defaults. This mechanism often requires a detailed description of the balance sheet of each institution; assumptions on the interbank network and exposures, on the recovery rate at default, on the liquidation policy; the analysis of direct liabilities, bankruptcy costs, cross-holdings, leverage structures, fire sales, and liquidity freezes. Meanwhile, central clearing counterparties (CCPs) are introduced to the financial markets to mitigate the cascade defaults. They require default funds from their members to absorb the cost of clearing member's defaults, and several mechanisms of default funds have been designed [1]. While some literature [4, 11] concentrated on single aspects of central counterparty clearing, the papers [3, 5] treated the CCP as a complex package: it offers netting services, requires guarantee fund contributions, changes the profile of liquidations across the network, and comes with its own capital requirements [27]. From an individual bank's perspective, [28] studied how it assesses its risks based on the partial information available to it provided by CCP. In our analysis, we consider only guarantee fund contributions in a CCP to reduce the systemic risk.

In the previous works [8, 9], one of the authors (J.-P. Fouque) and his collaborators introduced a general class of systemic risk measures that allow for random allocations to individual banks before aggregation of their risks. They also proved the dual representation of a particular subclass of such systemic risk measures and the existence and uniqueness of the optimal allocation. They interpreted the systemic risk measures as the minimal amount of cash that secures the aggregated system by allocating capital to the single institutions before aggregating the individual risks, which allows for a possible ranking of the institutions in terms of systemic risk measured by the optimal allocations. The papers [6, 20] developed systemic risk measures in a similar spirit, covering allocation first followed by aggregation frameworks.

In this paper, we generalize the systemic risk measure under exponential utility functions proposed in [8, 9] by allowing individual banks to choose their allocations of the risks into different groups instead of being assigned to specific groups. This brings game features into modeling, making it more realistic and providing baselines for a CCP to design its default fund mechanism from rational banks. To solve the new models with game features, we first define the concept of Nash equilibrium based on banks' fair systemic risk allocation, a concept introduced in [9], and discuss the existence and uniqueness of equilibrium strategies. Then some explicit formulas are derived under the Gaussian distribution of the risk factor. In the overlapping group case, still focusing on the risk measure produced by exponential utility functions, we first provide explicit expressions for the systemic risk measure and fair risk allocation of each bank under general risk factor. Sensitivity and monotonicity properties are also established. The concept of Nash equilibrium is then extended to the overlapping group case, namely, its existence and uniqueness are discussed theoretically and numerically under Gaussian assumptions and with two groups. In particular, we propose a numerical algorithm based on fictitious play to identify the Nash equilibrium, and use it to study synthetic examples and the bank-CCP structure with real data, showing the validity of the proposed model. It is worth mentioning that the complex mechanism of CCP is by no means fully captured by our modeling on group choice and systemic risk measure. In practice, the CCP requires multiple layers of capital, margin fund, etc., and we hope our stylized model could provide CCPs guidelines for further mechanism design. To bridge the gap, for instance, two of the authors and their collaborator design deep learning algorithms [21] to compute the optimal strategy of capital allocations and improve it to a more realistic situation of scenario-dependent cash distribution without cross-subsidization.

The rest of the paper is organized as follows. In Section 2, we first describe the systemic risk models with game features for disjoint groups. Then we introduce the concept of group formation and Nash equilibrium, and analyze the optimal solution under Gaussian distribution of the risk factor. In Section 3, we generalize the model and concept of Nash equilibrium to the overlapping group case. We propose numerical methods for computing Nash equilibrium and give several examples in Section 4. We make conclusive remarks in Section 5.

2 Fair systemic risk measure on disjoint groups

2.1 Review on systemic risk measure

A concept of systemic risk measure was proposed in [8, 9], where the authors considered the following problem of risk allocations of N individuals labeled as $\{1, \dots, N\}$. Given a joint distribution of an N-dimensional, real-valued random vector $\mathbf{X} := (X^1, \dots, X^N)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the risk sensitivity vector $\boldsymbol{\alpha} := (\alpha^1, \dots, \alpha^N) \in (0, \infty)^N$, the risk tolerance value $B < 0$ and the partition set

$$\{I_m := \{n_{m-1} + 1, \dots, n_m\},\ m = 1, \dots, h\}$$

(indexed by a vector $\boldsymbol{n} := (n_1, \dots, n_h)$ with $0 = n_0 < n_1 < \cdots < n_h = N$ for some $h \geq 1$) of the N elements $\{1, \dots, N\}$, one defines the aggregate risk

$$\boldsymbol{\rho}(\mathbf{X}) := \inf\left\{\sum_{n=1}^{N} Y^n : \mathbf{Y} = (Y^1, \dots, Y^N) \in \mathcal{C}_0^{(\mathbf{n})},\ \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n)\right] = B\right\}, \tag{2.1}$$

where we take $u_n(x) = -\frac{1}{\alpha_n} e^{-\alpha_n x}$ as exponential utility functions, α_n could be interpreted as the risk aversion of individual n, and the random allocation $\mathcal{C}_0^{(\mathbf{n})}$ of partition index vector $\boldsymbol{n}$ and the associated partition $\{I_m,\ m = 1, \dots, h\}$ are given by

$$\begin{aligned}\mathcal{C}_0^{(\mathbf{n})} := \Big\{&\mathbf{Y} \in \mathbb{L}^0(\mathbb{R}^N) : \text{there exists a real vector } \mathbf{d} := (d_1, \dots, d_h) \in \mathbb{R}^h \\ &\text{such that } \sum_{i \in I_m} Y^i = d_m \text{ for every } m = 1, \dots, h\Big\}.\end{aligned}$$

The partition set $\{I_m,\ m = 1, \dots, h\}$ represents the grouping among the individuals and is determined by the vector $\boldsymbol{n}$. Here $\mathcal{C}_0^{(\boldsymbol{n})}$ is a subfamily of random vectors $\mathbb{L}^0(\mathbb{R}^N)$ associated with $\boldsymbol{n}$ such that all the partial sums of elements divided by the partition are deterministic real numbers.

Theorem 6.2 in [9] shows that the infimum of (2.1) is attained by

$$\begin{aligned}&Y_{\mathbf{X}}^i := -X^i + \frac{S_m + d_m}{\alpha_i \beta_m}, \quad S_m := \sum_{k \in I_m} X^k, \quad \beta_m := \sum_{k \in I_m} \frac{1}{\alpha_k}, \\ &d_m := \beta_m \log\Big(-\frac{\beta}{B} \mathbb{E}[e^{-S_m/\beta_m}]\Big), \quad \beta := \sum_{n=1}^{N} \frac{1}{\alpha_n} = \sum_{m=1}^{h} \beta_m\end{aligned} \tag{2.2}$$

for $m = 1, \dots, h$ and $i \in I_m$, and

$$\boldsymbol{\rho}(\mathbf{X}) = \sum_{n=1}^{N} Y_{\mathbf{X}}^n = \sum_{m=1}^{h} d_m.$$

Moreover, the systemic risk allocation $\rho^{i,(\boldsymbol{n})}(\mathbf{X})$ of individual i is given by

$$\rho^{i,(\boldsymbol{n})}(\mathbf{X}) := \mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^m}[Y_{\mathbf{X}}^i] = (\mathbb{E}[e^{-S_m/\beta_m}])^{-1} \mathbb{E}[Y_{\mathbf{X}}^i e^{-S_m/\beta_m}], \quad i \in I_m, \tag{2.3}$$

for $m = 1, \dots, h$, where $\mathbb{Q}_{\mathbf{X}}^m$ is a tilted probability measure, absolutely continuous with respect to $\mathbb{P}$, determined by the Radon–Nikodym derivative

$$\frac{\mathrm{d}\mathbb{Q}_{\mathbf{X}}^m}{\mathrm{d}\mathbb{P}} := (\mathbb{E}[e^{-S_m/\beta_m}])^{-1} e^{-S_m/\beta_m}, \quad m = 1, \dots, h. \tag{2.4}$$

By the construction, one has

$$\boldsymbol{\rho}(\mathbf{X}) = \sum_{m=1}^{h} d_m = \sum_{m=1}^{h} \sum_{i \in I_m} \rho^{i,(\boldsymbol{n})}(\mathbf{X}) = \sum_{n=1}^{N} \rho^{n,(\boldsymbol{n})}(\mathbf{X}).$$

2.2 Groups formation and Nash equilibrium

In this section, we generalize the systemic risk measure to a game setup. For a game with N individuals, we assume there are N buckets $(B_1, \dots, B_N)$ for each individual to choose which one she belongs to. The choice of individual n, $n = 1, \dots, N$, is denoted by a_n, and $a_n = j$ means individual n chooses the bucket j. We call $\mathscr{A}$ the set of all strategies. A set of strategies $\boldsymbol{a} := (a_1, \dots, a_N) \in \mathscr{A}$ generates m groups by considering only the non-empty buckets for some $1 \le m \le N$. Different sets of strategies may generate the same groups denoted by $\mathcal{C}(\boldsymbol{a})$. We say two strategies $\boldsymbol{a}^{(1)}$ and $\boldsymbol{a}^{(2)}$ are equivalent if the partitions $\mathcal{C}(\boldsymbol{a}^{(1)})$ and $\mathcal{C}(\boldsymbol{a}^{(2)})$ are equivalent. With the individual systemic risk allocation in (2.3), the objective function of individual n under the partition $\mathcal{C}(\boldsymbol{a})$ is defined by

$$\rho^n(\mathcal{C}(\boldsymbol{a})) = \rho^n(\mathbf{X}; \mathcal{C}(\boldsymbol{a})) = \mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^m}[Y_{\mathbf{X}}^n],$$

where $n \in I_m$ for some $m \in \{1, \dots, h\}$, $Y_{\mathbf{X}}^n$ and $\mathbb{Q}_{\mathbf{X}}^m$ depend on the partition $\mathcal{C}(\boldsymbol{a})$.

Let $\hat{\boldsymbol{a}} = (\hat{a}_1, \dots, \hat{a}_N) \in \mathscr{A}$ and $(\hat{\boldsymbol{a}}^{-n}, a^n) = (\hat{a}_1, \dots, \hat{a}_{n-1}, a^n, \hat{a}_{n+1}, \dots, \hat{a}_N) \in \mathscr{A}$.

Definition 2.1. With the systemic risk allocation map $\mathcal{C} \mapsto \rho^{\cdot}(\mathbf{X}; \mathcal{C})$ in (2.3) and the above definitions, the configuration $\mathcal{C}(\hat{\boldsymbol{a}})$ is a Nash equilibrium if, for every $n = 1, \dots, N$ and a^n,

$$\rho^n(\mathbf{X}; \mathcal{C}(\hat{\boldsymbol{a}})) \le \rho^n(\mathbf{X}; \mathcal{C}(\hat{\boldsymbol{a}}^{-n}, a^n)), \tag{2.5}$$

i.e., the systemic risk allocation of individual n is minimized under grouping $\mathcal{C}(\hat{\boldsymbol{a}})$, given other individuals' choices are $\hat{\boldsymbol{a}}^{-n}$. If there are multiple Nash equilibrium strategies satisfying (2.5) and all the partitions associated with these Nash equilibrium strategies are equivalent, we say the Nash equilibrium strategy is unique up to equivalence relation.

In this paper, we shall consider the following questions:

- Does a Nash equilibrium exist?
- If it exists, is it unique?

We view the equilibrium as a network of risk-sharing. It is easy to show that a single group with all the individuals, called full risk-sharing, is a Nash equilibrium. We call this the trivial Nash equilibrium. This follows from the fact that a configuration with a group having only one individual is never a Nash equilibrium; see [9, Section 6.2 "Monotonicity"]. For simplicity, we take Gaussian distribution for the risk factors and discuss the proposed model in detail with some examples.

2.3 Extreme examples

Here we discuss some extreme cases. Let $|I_m|$ (≥ 1) be the number of elements in I_m for $m \ge 1$. Under the exchangeability assumption on the joint distribution of $\mathbf{X}$ and the identical exponential utility functions with $\alpha_1 = \cdots = \alpha_N = \alpha > 0$, the marginal distributions of $X^i e^{-S_m/\beta_m}$, $i \in I_m$, are identical with the same expectation, that is,

$$\mathbb{E}[-X^i e^{-S_m/\beta_m}] = \frac{-\beta_m}{|I_m|}\mathbb{E}\Big[\frac{S_m}{\beta_m} e^{-S_m/\beta_m}\Big], \quad i \in I_m, \tag{2.6}$$

and hence, with $\beta_m = \alpha^{-1}|I_m|$, $\beta = \alpha^{-1}N$ for $m = 1, \dots, h$,

$$\mathbb{E}[Y_{\mathbf{X}}^i e^{-S_m/\beta_m}] = \mathbb{E}\Big[\Big(-X^i + \frac{S_m}{\alpha_k \beta_m} + \frac{d_m}{\alpha_i \beta_m}\Big) e^{-S_m/\beta_m}\Big] = \frac{d_m}{|I_m|}\mathbb{E}[e^{-S_m/\beta_m}].$$

Then substituting it into (2.3) gives the systemic risk allocation of individual i,

$$\rho^i(\mathbf{X}; \{I_\cdot\}) = \frac{d_m}{|I_m|} = \frac{1}{\alpha}\log\Big(\frac{-N}{\alpha \cdot (-B)}\mathbb{E}[e^{-\alpha S_m/|I_m|}]\Big), \quad i \in I_m.$$

Under the i.i.d. Gaussian distribution assumptions for $\mathbf{X}$, i.e., X^i, $i = 1, \dots, N$, are independent, identically distributed Gaussian random variables with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$, $S_m = \sum_{i \in I_m} X^i$ is distributed normally with mean $\mu|I_m|$ and variance $\sigma^2|I_m|$ for $m \ge 1$. Direct calculation yields the systemic risk

allocation of individual i,

$$\rho^i(\mathbf{X};\{I_\cdot\}) = \frac{1}{\alpha}\log\Big(\frac{N}{\alpha\cdot(-B)}\Big) - \mu + \frac{\alpha\sigma^2}{2|I_m|}, \quad i \in I_m, \tag{2.7}$$

which is a decreasing function of the size $|I_m|$ of the group I_m that individual i belongs to.

Example 2.1 (I.i.d. Gaussian with the same exponential utility function). The strategy $\hat{\boldsymbol{a}} := (1,\dots,1)$ that everyone chooses the same group is a unique Nash equilibrium. In this case, $\mathcal{C}(\hat{\boldsymbol{a}}) = \{I_1 = \{1,\dots,N\}\}$ with $|I_1| = N$, $h = 1$. Every individual k belongs to the same group I_1, and by (2.7),

$$\rho^k(\mathbf{X};\mathcal{C}(\hat{\boldsymbol{a}})) = \frac{1}{\alpha}\log\Big(\frac{N}{\alpha\cdot(-B)}\Big) - \mu + \frac{\alpha\sigma^2}{2N} \le \rho^k(\mathbb{X};\mathcal{C}(\hat{\boldsymbol{a}}^{-k}, a_k)), \quad \boldsymbol{a} \in \mathscr{A}.$$

To see its uniqueness, if $\boldsymbol{a}^*$ were a Nash equilibrium strategy with $\mathcal{C}(\boldsymbol{a}^*) = \{I_m^*,\ m = 1,\dots,h^*\}$, which is not equivalent to $\mathcal{C}(\hat{\boldsymbol{a}})$, then $h^* \ge 2$ and all the sets I_m^*, $m = 1,\dots,h^*$, would satisfy $1 \le |I_m^*| \le N-1$. Take the group number $\ell_0 := \arg\min_{1\le i\le h}|I_i^*|$ of its smallest size. For a fixed individual $k \in I_{\ell_0}^*$, there exists $\boldsymbol{a} \in \mathscr{A}$ with $a_k = j_0$ such that, in the new partition $\mathcal{C}(\boldsymbol{a}^{*-k}, a_k) = \{I_m\}$, the individual k belongs to another group I_{m_0} with $|I_{m_0}| > |I_{\ell_0}|$, and hence, by (2.7),

$$\rho^k(\mathbf{X};\mathcal{C}(\boldsymbol{a}^*)) = \rho^k(\mathbf{X};\{I_m^*,\ m = 1,\dots,h^*\}) > \rho^k(\mathbf{X};\mathcal{C}(\boldsymbol{a}^{*-k}, j_0)).$$

This contradicts with the definition of Nash equilibrium. Thus $\hat{\boldsymbol{a}}$ is a unique Nash equilibrium up to equivalence relation.

Example 2.2 (Non-random, equal outcomes with the same exponential utility function). Instead, if $\mathbf{X}$ is a deterministic constant vector of μ's ($\mu \in \mathbb{R}$) with $\sigma^2 \equiv 0$, that is, $X^i = \mu$ for every $i = 1,\dots,N$, then there is no contribution from I_m in the systemic risk allocation

$$\rho^i(\mathbf{X};\{I_\cdot\}) = \frac{1}{\alpha}\log\Big(\frac{N}{\alpha\cdot(-B)}\Big) - \mu, \quad i \in I_m,$$

and hence the risk sharing is arbitrary and undetermined.

Next we shall relax the condition on α's. We still assume X^i are i.i.d. Gaussians with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. In this case, $\frac{S_m}{\beta_m}$ is normally distributed with mean $\frac{\mu|I_m|}{\beta_m}$ and variance $\frac{|I_m|\sigma^2}{\beta_m^2}$. Then direct calculations produce

$$\begin{aligned} \frac{d_m}{\beta_m} &= \log\Big(\frac{\beta}{-B}\Big) + \log\mathbb{E}[e^{-S_m/\beta_m}] = \log\Big(\frac{\beta}{-B}\Big) - \frac{\mu|I_m|}{\beta_m} + \frac{|I_m|\sigma^2}{2\beta_m^2}, \\ \mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^m}\Big[\frac{S_m}{\beta_m}\Big] &= (\mathbb{E}[e^{-S_m/\beta_m}])^{-1}\mathbb{E}\Big[\frac{S_m}{\beta_m}e^{-S_m/\beta_m}\Big] = \frac{\mu|I_m|}{\beta_m} - \frac{|I_m|\sigma^2}{\beta_m^2}, \end{aligned} \tag{2.8}$$

where $\mathbb{Q}_{\mathbf{X}}^m$ is the tilted measure defined by (2.4). Also, by (2.6),

$$\mathbb{E}\Big[-X^k e^{-S_m/\beta_m} + \frac{S_m}{\alpha_k\beta_m}e^{-S_m/\beta_m}\Big] = \Big(\frac{-\beta_m}{|I_m|} + \frac{1}{\alpha_k}\Big)\mathbb{E}\Big[\frac{S_m}{\beta_m}e^{-S_m/\beta_m}\Big] \tag{2.9}$$

for $k \in I_m$. Hence, substituting (2.8)–(2.9) into (2.3) brings, for $k \in I_m$,

$$\rho^k(\mathbf{X};\{I_\cdot\}) = \Big(\frac{-\beta_m}{|I_m|} + \frac{1}{\alpha_k}\Big)\mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^m}\Big[\frac{S_m}{\beta_m}\Big] + \frac{d_m}{\alpha_k\beta_m} = -\mu + \frac{1}{\alpha_k}\log\Big(\frac{\beta}{(-B)}\Big) + \frac{\sigma^2}{\beta_m}\Big(1 - \frac{|I_m|}{2\alpha_k\beta_m}\Big).$$

In order to find a Nash equilibrium, we evaluate the quantity

$$\eta^k(\{I_\cdot\}) := \frac{1}{\beta_m}\Big(1 - \frac{|I_m|}{2\alpha_k\beta_m}\Big) = \alpha_k\frac{\prod_{\{j\in I_m: j\ne k\}}\alpha_j}{\sum_{j\in I_m}\alpha_j}\Big(1 - \frac{|I_m|\prod_{\{j\in I_m: j\ne k\}}\alpha_j}{2\sum_{j\in I_m}\alpha_j}\Big), \quad k \in I_m. \tag{2.10}$$

Note that $\eta^k(\{I_\cdot\})$ is an increasing function of α_k, and thus, if $\arg\max_{k\in I_m}\alpha_k = k_0$, then

$$\eta^{k_0}(\{I_\cdot\}) \ge \eta^k(\{I_\cdot\}) \quad \text{for every} k \in I_m. \tag{2.11}$$

However, it is not necessarily true that

$$\rho^{k_0}(\mathbf{X};\{I_\cdot\}) \geq \rho^k(\mathbf{X};\{I_\cdot\}) \quad \text{for every } k \geq 0$$

because of the term $\frac{1}{\alpha_k}\log(\frac{\beta}{-B})$. Note that $\eta^k(\{I_\cdot\})$ is a decreasing function of $\frac{|I_m|}{\beta_m}$ and is a decreasing function of β_m.

If $N = 2$ and $0 < \alpha_1 \leq \alpha_2$, then $\hat{\boldsymbol{a}} := (1, 1)$ is a unique Nash equilibrium. This is directly verified by the inequalities

$$\eta^k(\mathcal{C}(\hat{\boldsymbol{a}})) = \alpha_k \cdot \frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2} \leq \frac{\alpha_k}{4} < \frac{\alpha_k}{2} = \eta^k(\{\{1\},\{2\}\}), \quad k = 1, 2, \tag{2.12}$$

and hence $\rho^k(\mathbf{X};\mathcal{C}(\hat{\boldsymbol{a}})) < \rho^k(\mathbf{X};\mathcal{C}(\hat{\boldsymbol{a}}^{-k},\mathbf{a}))$ for every $k = 1, 2$, and $\boldsymbol{a} \in \mathscr{A}$ which is not equivalent to $\hat{\boldsymbol{a}}$.

If $N \geq 2$ and $0 < \alpha_1 \leq \alpha_2 \cdots \leq \alpha_N$, then $\hat{\boldsymbol{a}} := (1, \ldots, 1)$ is a Nash equilibrium for a similar calculation to (2.12),

$$\begin{aligned}\eta^k(\mathcal{C}(\hat{\boldsymbol{a}})) &= \alpha_k \cdot \frac{\prod_{i\neq k}\alpha_i}{\alpha_1+\cdots+\alpha_k}\Big(1 - \frac{N\prod_{i\neq k}\alpha_i}{2(\alpha_1+\cdots+\alpha_N)}\Big) < \alpha_k \cdot \frac{\prod_{i\neq k}\alpha_i}{\alpha_1+\cdots+\alpha_k}\Big(1 - \frac{\prod_{i\neq k}\alpha_i}{(\alpha_1+\cdots+\alpha_N)}\Big)\\ &\leq \frac{\alpha_k}{4} < \frac{\alpha_k}{2} = \eta^k(\{\{k\},\{1,\ldots,N\}\setminus\{k\}\}), \quad k = 1,\ldots,N.\end{aligned}$$

There is no reason to move out of the alliance of $\{1, \ldots, N\}$ and to become an outcast. This observation can be generalized: under this setup, there is no reason to move out of the alliance of a group I_m of size greater than or equal to 2, i.e., $|I_m| \geq 2$ and to become an outcast.

We conjecture that $\hat{\boldsymbol{a}}$ is a unique Nash equilibrium under a wide range of configurations of α's. For each $\mathcal{C}(\boldsymbol{a}) = \{I_m,\ m = 1, \ldots, h\}$, let us consider the group heads k_m, $m = 1, \ldots, h$, and the head of the group heads k^* defined by

$$k_m := \operatorname*{arg\,max}_{\ell\in I_m} \alpha_\ell, \quad m = 1,\ldots,h, \qquad k^* := \operatorname*{arg\,max}_{k\in\{k_1,\ldots,k_m\}} \eta^k(\mathcal{C}(\mathbf{a})). \tag{2.13}$$

Let us denote by m^* the group name of k^*, i.e., $k^* \in I_{m^*}$. By setting the group heads and the head of the group heads, we see from the observation made in (2.11) that

$$\begin{aligned}&\eta^{k_m}(\mathcal{C}(\boldsymbol{a})) \geq \eta^{\ell}(\mathcal{C}(\boldsymbol{a})) \quad \text{for every } \ell \in I_m,\ m = 1,\ldots,h,\\ &\eta^{k^*}(\mathcal{C}(\boldsymbol{a})) \geq \eta^{k_m}(\mathcal{C}(\boldsymbol{a})) \quad \text{for every } m = 1,\ldots,h.\end{aligned} \tag{2.14}$$

Lemma 2.1. *If there is a group head $k_* \in \{k_1, \ldots, k_m\} \setminus \{k^*\}$ in a group m_*, i.e.,*

$$k_* \in I_{m_*} \quad \text{from } \mathcal{C}(\boldsymbol{a}) = \{I_m,\ m = 1,\ldots,h\}$$

such that

$$\frac{\alpha_{k^*}}{\alpha_{k_*}} \leq 1 + \frac{1}{|I_{m_*}|}, \tag{2.15}$$

then $\eta^{k^*}(\mathcal{C}(\boldsymbol{a})) \geq \eta^{k^*}(\mathcal{C}(\tilde{\boldsymbol{a}}))$, where $\mathcal{C}(\tilde{\boldsymbol{a}}) = \{\tilde{I}_m,\ m = 1,\ldots,h\}$ is obtained only by removing k^* from group m^* and adding k^* into group m_*, that is,

$$\tilde{I}_{m^*} := I_{m^*} \setminus \{k^*\}, \quad \tilde{I}_{m_*} := I_{m_*} \cup \{k^*\}.$$

In addition, either if the inequality in (2.15) *is strict or if the strict inequality $\eta^{k^*}(\mathcal{C}(\boldsymbol{a})) > \eta^{k_*}(\mathcal{C}(\boldsymbol{a}))$ holds, then $\mathcal{C}(\boldsymbol{a})$ is not a Nash equilibrium.*

Proof. We rewrite $\eta^k(\{I_\cdot\}) = f(\frac{1}{\beta_m}; \frac{|I_m|}{2\alpha_k})$ in (2.10) with a quadratic function $f(x; a) := x(1 - ax)$, $x > 0$. Note that $f(0) = 0 = f(\frac{1}{a})$ and $f(x; a)$ is increasing in the interval $(0, \frac{1}{2a})$. For each group head $k_1, \ldots, k_h$, we have

$$\frac{|I_m|}{\alpha_{k_m}} \leq \sum_{k\in I_m}\frac{1}{\alpha_k} = \beta_m \quad \text{or} \quad \frac{1}{\beta_m} \leq \frac{\alpha_{k_m}}{|I_m|}, \quad m = 1,\ldots,h.$$

Thus, with $a = \frac{|I_m|}{2\alpha_{k_m}}$, we have $\frac{1}{\beta_m} \leq \frac{1}{2a}$, and hence $x \mapsto f(x; \frac{|I_m|}{2\alpha_k})$ is increasing in the interval $(0, \frac{1}{\beta_m})$ for $m = 1, \ldots, h$.

By the definition of k^* in (2.13) and (2.14), we have

$$\eta^{k^*}(\mathcal{C}(\boldsymbol{a})) = \frac{1}{\beta_{m^*}}\Big(1 - \frac{|I_{m^*}|}{2\alpha_{k^*}\beta_{m^*}}\Big) \geq \frac{1}{\beta_{m_*}}\Big(1 - \frac{|I_{m_*}|}{2\alpha_{k_*}\beta_{m_*}}\Big) = f\Big(\frac{1}{\beta_{m_*}}; \frac{|I_{m_*}|}{2\alpha_{k_*}}\Big) = \eta^{k_*}(\mathcal{C}(\boldsymbol{a})).$$

Then, by the monotonicity of $x \mapsto f(x; \frac{|I_{m_*}|}{2\alpha_{k_*}})$ in the interval $(0, \frac{1}{\beta_{m_*}})$, we have

$$\begin{aligned}\eta^{k^*}(\mathcal{C}(\boldsymbol{a})) &\geq f\Big(\frac{1}{\beta_{m_*}}; \frac{|I_{m_*}|}{2\alpha_{k_*}}\Big) \geq f\Big(\frac{1}{\beta_{m_*} + \frac{1}{\alpha_{k^*}}}; \frac{|I_{m_*}|}{2\alpha_{k_*}}\Big) \\ &= \frac{1}{\beta_{m_*} + (\frac{1}{\alpha_{k^*}})}\Big(1 - \frac{|I_{m_*}|}{2\alpha_{k_*}(\beta_{m_*} + \frac{1}{\alpha_{k^*}})}\Big) \\ &\geq \frac{1}{\beta_{m_*} + (\frac{1}{\alpha_{k^*}})}\Big(1 - \frac{|I_{m_*}| + 1}{2\alpha_{k^*}(\beta_{m_*} + \frac{1}{\alpha_{k^*}})}\Big) = \eta^{k^*}(\mathcal{C}(\tilde{\boldsymbol{a}})),\end{aligned}$$

where we used (2.15) in the last inequality and $\mathcal{C}(\tilde{\boldsymbol{a}})$ is obtained only by removing k^* from the group m^* and adding k^* into the group m_*. Thus, for k^*, it is better to move from m^* to m_*, and hence $\mathcal{C}(\boldsymbol{a})$ is not a Nash equilibrium. □

Example 2.3. Suppose that we have $\mathcal{C}(\boldsymbol{a}) = \{I_m\}_{m=1,2,3}$, $I_1 = \{1, 2\}$, $I_2 = \{3, 4, 5\}$, $I_3 = \{6, 7, 8, 9, 10\}$ with

$$\alpha_1 = \alpha_2 = 2, \quad \alpha_3 = \alpha_4 = \alpha_5 = 3, \quad \alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = 4, \quad \alpha_{10} = 5.$$

Condition (2.15) holds with a strictly inequality, and $\mathcal{C}(\boldsymbol{a})$ is not a Nash equilibrium.

Example 2.4. Suppose that we have $\mathcal{C}(\boldsymbol{a}) = \{I_m\}_{m=1,2}$, $I_1 = \{1, 2, 3\}$, $I_2 = \{4, 5, 6\}$ with

$$\alpha_1 = \alpha_2 = 2, \quad \alpha_3 = 4, \quad \alpha_4 = \alpha_5 = \alpha_6 = 3.$$

Condition (2.15) does not hold; however, $\mathcal{C}(\boldsymbol{a})$ is not a Nash equilibrium.

2.4 Case discussion: Correlated Gaussian distribution

In a system with N individuals, we assume the joint distribution of the column vector $\mathbf{X} = (X^i,\ i = 1, \dots, N)^T$ follows a multivariate Gaussian distribution, that is, $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu} \in \mathbb{R}^N$ and $\boldsymbol{\Sigma} \in \mathbb{R}^{N\times N}$ is positive semi-definite. The exponential utility functions for N individuals have positive parameters $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N)$.

For every partition set I_m, $m = 1, \dots, h$, define a group vector $A_m \in \mathbb{R}^{1\times N}$ which consists of only 0's and 1's. For all the $j \in I_m$, the j-th element in A_m is 1, otherwise 0. For example, in a 4-player system, if individuals 1 and 4 are in group 1, and individuals 2 and 3 in group 2, the corresponding vectors for the two groups are $A_1 = (1, 0, 0, 1)$ and $A_2 = (0, 1, 1, 0)$. Then, following (2.2), we have

$$S_m = \sum_{i\in I_m} X^i = A_m\mathbf{X} \sim N(A_m\boldsymbol{\mu}, A_m\boldsymbol{\Sigma}A_m^T) \overset{\text{def}}{=} N(\mu_m^s, (\sigma_m^s)^2). \tag{2.16}$$

The results in Appendix A.10 produce that, for $m = 1, \dots, h$ and for $k \in I_m$,

$$d_m = \beta_m \log\Big(-\frac{\beta}{B}\mathbb{E}[e^{-S_m/\beta_m}]\Big) = \beta_m \log\Big(\frac{\beta}{-B}\Big) - \mu_m^s + \frac{(\sigma_m^s)^2}{2\beta_m},$$

and the systemic risk allocation of individual k is given by

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^m}[Y_{\mathbf{X}}^k] = \mathbb{E}\Big[Y_{\mathbf{X}}^k \cdot \frac{\mathrm{d}\mathbb{Q}_{\mathbf{X}}^m}{\mathrm{d}\mathbb{P}}\Big] &= \frac{\mathbb{E}[(-X^k + \frac{1}{\alpha_k\beta_m}S_m + \frac{1}{\alpha_k\beta_m}d_m)\cdot e^{-S_m/\beta_m}]}{\mathbb{E}(e^{-S_m/\beta_m})} \\ &= -\mu_k + \frac{1}{\alpha_k}\log\Big(\frac{\beta}{-B}\Big) + \frac{1}{\beta_m}A_m\boldsymbol{\Sigma}_{[,k]} - \frac{(\sigma_m^s)^2}{2\beta_m^2\alpha_k}.\end{aligned} \tag{2.17}$$

Remark 2.1 (Effect of mean). From the above formula of the systemic risk allocation for individual k, we find that the mean of individual k has no effect on her risk allocation no matter which group she belongs to. Thus, in the following discussion, without loss of generality, we take all means to be the constant zero.

Remark 2.2 (Comparison between trivial grouping and multi-groups). The total risk allocation for multiple groups ($h \geq 2$) is always greater than the total risk allocation for the trivial grouping ($h = 1$). The proof of this statement will be given in Appendix A.1, and we refer interested readers to read "Monotonicity" in [9] for general proof free of the distribution of risk factors.

In the following, we present three concrete examples to help better understand on banks' rational choices under this fair risk allocation.

Claim 2.1. *If* $\mathbf{X} = (X^i,\ i = 1, \ldots, N)^T$ *has the same standard deviation* $\sigma > 0$ *and correlation coefficient* $\rho \in [-1, 1)$*, and the utility parameters are identical, denoted by* α *(*> 0*), then there is only one trivial Nash equilibrium, that is, all individuals are in the same group.*

Claim 2.2. *In the case of* $N = 4$*, we assume all the utility parameter* α_i *are the same and equal to* 1*, and all individuals have the same standard deviation denoted by* $\sigma > 0$*. If the correlation matrix is a block matrix with uniform correlation coefficient, i.e., the correlation matrix is given by*

$$\begin{pmatrix} 1 & \rho & 0 & 0 \\ \rho & 1 & 0 & 0 \\ 0 & 0 & 1 & \rho \\ 0 & 0 & \rho & 1 \end{pmatrix},$$

we have the following conclusion about Nash equilibrium.

- *If* $\rho \in [-\frac{3}{13}, \frac{3}{8}]$*, there is no nontrivial Nash equilibrium.*
- *If* $\rho \in [-1, -\frac{3}{13})$*, grouping "*$\{1, 2\}$-$\{3, 4\}$*", i.e., the first two and the second two individuals are in two different groups, is a nontrivial Nash equilibrium.*
- *If* $\rho \in (\frac{3}{8}, 1]$*, groupings "*$\{1, 3\}$-$\{2, 4\}$*" and "*$\{1, 4\}$-$\{2, 3\}$*" are both nontrivial Nash equilibria for the system.*

According to the claim, when the correlation is not strong, all individuals tend to be together to form a trivial Nash equilibrium. Otherwise, negatively correlated individuals tend to be in the same group, while positively correlated individuals tend to be separate.

Claim 2.3. *In the case of* $N = 5$*, we assume the standard deviation are uniform and the* α_i*'s are* 1*. Let the covariance matrix be*

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma^2 & \rho\sigma^2 & 0 & 0 & 0 \\ \rho\sigma^2 & \sigma^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma^2 & \rho\sigma^2 & \rho\sigma^2 \\ 0 & 0 & \rho\sigma^2 & \sigma^2 & \rho\sigma^2 \\ 0 & 0 & \rho\sigma^2 & \rho\sigma^2 & \sigma^2 \end{pmatrix},$$

which is of block form with positive standard deviation σ *and the correlation coefficient* $\rho \in (-1, 1)$*.*

- *If* $\rho \in (-1, -\frac{2}{7}]$*, grouping "*$\{1, 2\}$-$\{3, 4, 5\}$*" is a Nash equilibrium.*
- *If* $\rho \in (-\frac{2}{7}, 1)$*, there is no nontrivial Nash equilibrium.*
- *Grouping "*$\{1, 2, 3\}$-$\{4, 5\}$*" cannot be a nontrivial Nash equilibrium for any value of* ρ*.*

We give derivations of Claims 2.1, 2.2, and 2.3 in Appendix A.2, A.3, and A.4, respectively. The result further shows that individuals tend to stay with highly negatively correlated individuals to minimize the systemic risk if they exist. It is impossible for individuals to stay with correlated and uncorrelated individuals at the same time except for the trivial case when all individuals are together.

3 Systemic risk measure on overlapping groups

To further study the systemic risk measure under exponential utility functions, we generalize the systemic risk allocation for N individuals on disjoint groups in Section 2 to the risk allocation for them on overlapping ,groups where they can choose multiple groups to allocate their risks. Assuming there are at most h

groups, the weighted risk factors for the n-th individual assigned to multiple groups are labeled as $w_{n,k}X^n$, $k = 1, \dots, h$, with $\sum_{k=1}^{h} w_{n,k} = 1$. In the weight $w_{n,k}$, the index k refers to the group number the individual n joins, and the weight can be of any value between 0 and 1. If $w_{n,k} = 0$ for some k, then we say the individual is not in the k-th group. Therefore, we can extend the systemic risk measure given by (2.1) to a general measure ρ defined by

$$\boldsymbol{\rho}(\mathbf{X}) := \inf\left\{\sum_{n=1}^{N}\sum_{k=1}^{h} Y^{n,k} : \mathbf{Y} \in \mathcal{C}_0^{\text{new}},\ \mathbb{E}\left[\sum_{n=1}^{N}\sum_{k=1}^{h} u_n(w_{n,k}X^n + Y^{n,k})\right] = B\right\}, \tag{3.1}$$

where we take $u_n(x) = -\frac{1}{\alpha_n}e^{-\alpha_n x}$ as exponential utility functions, h is the maximum number of groups individuals can contribute to in total and it is a finite integer, and the random allocation $\mathcal{C}_0^{\text{new}}$ is given by

$$\mathcal{C}_0^{\text{new}} = \Bigg\{\mathbf{Y} = (Y^{i,j}, 1 \le i \le N, 1 \le j \le h) \in L^0(\mathbb{R}^{N\times h}) : \text{there exists } d = (d_1, \dots, d_h) \in \mathbb{R}^h \text{ such that } \sum_{i=1}^{N} Y^{i,j} = d_j \text{ for } j = 1, \dots, h\Bigg\}. \tag{3.2}$$

Remark 3.1. Here h is an integer fixed a priori, to eliminate the situation that an individual wants to split the risk X^i into infinitely many groups. An alternative way is to impose a minimum value requirement for nonzero weights to avoid too many groups for an individual to participate in, denoted by $w^{\min}$. Then, naturally, $h = \lfloor \frac{1}{w^{\min}} \rfloor \cdot N$.

Remark 3.2. The generalized system (3.1)–(3.2) still meets the assumptions made in [9] because the measure on overlapping groups can be seen as the measure (2.1) on disjoint groups with more individuals with weighted risk factors. Thus the existence and uniqueness of optimal allocation solution $\mathbf{Y_X}$ of the primal problem (3.1) is guaranteed, according to the discussion in [9, Section 4].

Given the grouping for all individuals, we define the family of sets

$$\{I_j := \{i \in \mathbb{N} : w_{i,j} > 0,\ i = 1, \dots, N\},\ j = 1, \dots, h\}.$$

Theorem 3.1. *The optimal value of $\boldsymbol{\rho}(\mathbf{X})$ in (3.1) is attained by*

$$d_j = \beta_j \log\left(-\frac{\beta}{B}\mathbb{E}[e^{-S_j/\beta_j}]\right), \quad Y^{i,j}_{\mathbf{X}} = \left[-w_{i,j}X^i + \frac{1}{\alpha_i\beta_j}(S_j + d_j)\right]\mathbf{1}_{w_{i,j}>0},$$

where

$$S_j = \sum_{i=1}^{N} w_{i,j}X^i = \sum_{i\in I_j} w_{i,j}X^i, \quad \beta_j = \sum_{i=1}^{N}\frac{1}{\alpha_i}\mathbf{1}_{w_{i,j}>0} = \sum_{i\in I_j}\frac{1}{\alpha_i} \quad \text{for } j = 1, \dots, h \text{ and } i = 1, \dots, N,$$

$$\beta = \sum_{j=1}^{h}\beta_j = \sum_{j=1}^{h}\sum_{i=1}^{N}\frac{1}{\alpha_i}\mathbf{1}_{w_{i,j}>0}, \quad \text{and} \quad \boldsymbol{\rho}(\mathbf{X}) = \sum_{j=1}^{h}\sum_{i=1}^{N} Y^{i,j}_{\mathbf{X}} = \sum_{j=1}^{h} d_j.$$

The systemic risk allocation for individual i is $\sum_{j=1}^{h}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}[Y^{i,j}_{\mathbf{X}}]$ with the density

$$\frac{d\mathbb{Q}^j_{\mathbf{X}}}{d\mathbb{P}} := \frac{e^{-S_j/\beta_j}}{\mathbb{E}[e^{-S_j/\beta_j}]}, \quad j = 1, \dots, h.$$

The proof of Theorem 3.1 is left to Appendix A.5. According to the theorem, we define

$$\rho^i(\mathbf{X}) := \mathbb{E}_{\mathbb{Q}_{\mathbf{X}}}[Y^i_{\mathbf{X}}] = \sum_{j=1}^{h}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}[Y^{i,j}_{\mathbf{X}}] = \sum_{j=1}^{h}\mathbb{E}\left[Y^{i,j}_{\mathbf{X}}\cdot\frac{d\mathbb{Q}^j_{\mathbf{X}}}{d\mathbb{P}}\right] \tag{3.3}$$

as the total fair systemic risk allocation for individual i.

Remark 3.3. Compared with the disjoint group case well discussed by Biagini et al. [9], the model here can be seen as an extended disjoint group case, where we consider that one individual can be divided into several sub-individuals and join different groups. Thus, for the measures $\{Q^{i,j}_{\mathbf{X}},\ 1 \le i \le N,\ 1 \le j \le h\}$, we have $Q^{i,j}_{\mathbf{X}} = Q^{l,k}_{\mathbf{X}} := Q^m_{\mathbf{X}}$ if $j, k \in I_m$ for group $m = 1, \dots, h$.

3.1 Sensitivity analysis

Based on the main theorem, Theorem 3.1, we perform a sensitivity analysis by adding a perturbation on the risk factors. Consider the risk factors are given by $\mathbf{X} + \varepsilon\mathbf{Z}$, where $\varepsilon \in \mathbb{R}$ and $\mathbf{X} := (X^1, \dots, X^N)$, $\mathbf{Z} := (Z^1, \dots, Z^N)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then we have, for $w_{i,j} > 0$,

$$Y^{i,j}_{\mathbf{X}+\varepsilon\mathbf{Z}} = -w_{i,j}(X^i + \varepsilon Z^i) + \frac{1}{\alpha_i\beta_j}(S_j + \varepsilon S^Z_j) + \frac{1}{\alpha_i\beta_j} d_j^{\mathbf{X}+\varepsilon\mathbf{Z}},$$
$$d_j^{\mathbf{X}+\varepsilon\mathbf{Z}} = \beta_j \log\Big(-\frac{\beta}{B}\mathbb{E}[e^{-(S_j+\varepsilon S^Z_j)/\beta_j}]\Big),$$

where $S_j = \sum_{i\in I_j} w_{i,j}X^i$, $S^{\mathbf{Z}}_j = \sum_{i\in I_j} w_{i,j}Z^i$, and $S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j = S_j + \varepsilon S^{\mathbf{Z}}_j$.

Proposition 3.1. *Let ρ be the systemic risk measure in* (3.1).

- *Marginal risk contribution of group j:*

$$\frac{\partial}{\partial\varepsilon} d_j^{\mathbf{X}+\varepsilon\mathbf{Z}}\Big|_{\varepsilon=0} = \mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}[-S^{\mathbf{Z}}_j], \quad j = 1, \dots, h.$$

- *Local causal responsibility for individual i in group j:*

$$\frac{\partial}{\partial\varepsilon}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}[Y^{i,j}_{\mathbf{X}+\varepsilon\mathbf{Z}}]\Big|_{\varepsilon=0} = \mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}[-w_{i,j}Z^i], \quad i \in I_j.$$

- *Marginal risk allocation for individual i in group j:*

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}[Y^{i,j}_{\mathbf{X}+\varepsilon\mathbf{Z}}]\Big|_{\varepsilon=0} &= \mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}[-w_{i,j}Z^i] - \frac{1}{\beta_j}\operatorname{Cov}_{\mathbb{Q}^j_{\mathbf{X}}}(Y^{i,j}_{\mathbf{X}}, S^{\mathbf{Z}}_j)\\ &= \mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}[-w_{i,j}Z^i] + \frac{w_{i,j}}{\beta_j}\operatorname{Cov}_{\mathbb{Q}^j_{\mathbf{X}}}(X^i, S^{\mathbf{Z}}_j) - \frac{1}{\alpha_i\beta_j^2}\operatorname{Cov}_{\mathbb{Q}^j_{\mathbf{X}}}(S_j, S^{\mathbf{Z}}_j). \qquad (3.4)\end{aligned}$$

We leave the proof of Proposition 3.1 to Appendix A.6. Note that if we replace $\mathbb{Q}^j_{\mathbf{X}}$ with $\mathbb{P}$, none of the results above hold. To interpret these formulas, first we look at the first term in (3.4), $\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}[-w_{i,j}Z^i]$. This term contains only the increment Z^i in individual i and thus is not a systemic contribution. Summing this term over all individuals in group j gives

$$\sum_{i\in I_j}\frac{\partial}{\partial\varepsilon}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}[Y^{i,j}_{\mathbf{X}+\varepsilon\mathbf{Z}}]\Big|_{\varepsilon=0} = \mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}[-S^{\mathbf{Z}}_j] = \frac{\partial}{\partial\varepsilon} d_j^{\mathbf{X}+\varepsilon\mathbf{Z}}\Big|_{\varepsilon=0}. \qquad (3.5)$$

This shows the first term contributes to the marginal risk allocation of individual i without any systemic influence. When Z^i is positive, which means an increment is added, this results in a risk deduction, regardless of the relation to other individuals. When $\mathbf{Z}$ is deterministic, we can see, in (3.4), the marginal risk allocation to individual i in group j is $\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}[-w_{i,j}Z^i] = -w_{i,j}Z^i$ and the covariance terms do not exist anymore.

To better study the effect of other terms in (3.4), we take $\mathbf{Z} = Z^k\mathbf{e}_k$, where $k \neq i$. Then, from (3.4), we obtain

$$\frac{\partial}{\partial\varepsilon}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon Z^k\mathbf{e}_k}}[Y^{i,j}_{\mathbf{X}+\varepsilon Z^k\mathbf{e}_k}]\Big|_{\varepsilon=0} = \frac{w_{i,j}}{\beta_j}\operatorname{Cov}_{\mathbb{Q}^j_{\mathbf{X}}}(X^i, Z^k) - \frac{1}{\alpha_i\beta_j^2}\operatorname{Cov}_{\mathbb{Q}^j_{\mathbf{X}}}(S_j, Z^k). \qquad (3.6)$$

Supposing that $\frac{w_{i,j}}{\beta_j}\operatorname{Cov}_{\mathbb{Q}^j_{\mathbf{X}}}(X^i, Z^k) < 0$, we look at the first term which relates to the covariance between (X^i, Z^k). When they have a negative correlation under the systemic risk probability $\mathbb{Q}^j_{\mathbf{X}}$, the increase in individual k will result in a decrease of the risk allocation for individual i. That means, individual i takes advantage of the decrease of others. Since the overall marginal risk allocation of group j does not change according to (3.5), some other individuals in the group would pay for this advantage. This is related to the last term.

The last term in (3.4) or (3.6) contains both the systemic contribution $-\frac{1}{\beta_j^2}\operatorname{Cov}_{\mathbb{Q}^j_{\mathbf{X}}}(S_j, Z^k)$ which only depends on the group S_j, and the systemic relevance part $\frac{1}{\alpha_i}$ of individual i. The systemic component is distributed among the individuals based on $\frac{1}{\alpha_i}$. In addition, this term compensates for possible risk decrease in the second term of (3.4) since the overall marginal risk allocation of group j is fixed.

Proposition 3.2 (Sensitivity with respect to weights). *For any i, j such that $w_{i,j} > 0$,*

$$\begin{aligned}\frac{\partial \mathbb{E}_{\mathbb{Q}_{\mathbf{x}}^j}[Y_{\mathbf{X}}^{i,j}]}{\partial w_{i,j}} &= -\mathbb{E}_{\mathbb{Q}_{\mathbf{x}}^j}[X^i] - \frac{1}{\alpha_i \beta_j^2}\operatorname{Cov}_{\mathbb{Q}_{\mathbf{x}}^j}(X^i, S_j) + \frac{w_{i,j}}{\beta_j}\operatorname{Var}_{\mathbb{Q}_{\mathbf{x}}^j}(X^i)\\ &= -\mathbb{E}_{\mathbb{Q}_{\mathbf{x}}^j}[X^i] - \frac{1}{\beta_j}\operatorname{Cov}_{\mathbb{Q}_{\mathbf{x}}^j}\Big(X^i, \frac{1}{\alpha_i \beta_j} S_j - w_{i,j} X^i\Big).\end{aligned}$$

We give the proof in Appendix A.7.

3.2 Monotonicity

In a grouping set sequence $\{I_1, \dots, I_h\}$, for some set I_m, assume there is a non-empty subset $I_{m'}$ of I_m, and for every $k \in I_{m'}$, assume the weight for risk factor X^k is $w_{k,m'} \in (0, w_{k,m}]$. Then define

$$I_{m''} = \{k \in I_m : w_{k,m} - w_{k,m'} > 0\},$$

and the corresponding weights are $w_{k,m''} = w_{k,m} - w_{k,m'}$ for all $k \in I_{m''}$. Then there will be $h + 1$ groups, and the new grouping set sequence is $\{I_1, \dots, I_{m'}, I_{m''}, \dots, I_h\}$, while the weights structure is the same as before except those of groups $I_{m'}$ and $I_{m''}$. The optimal risk allocations under the new grouping of the primal problem coincide with $Y^{k,r}$, $k \in I_r$, for $r \neq m$. For $r = m$, $k \in I_{m'}$ or $I_{m''}$, we first know $w_{k,m'} \leq w_{k,m}$, $w_{k,m''} \leq w_{k,m}$, and we have the following.

Proposition 3.3. *Under the above setup, define $Y^{k,m}$, $k \in I_m$, the optimal allocation of group m to the primal problem given h groups. Define $Y^{k,m'}$, $k \in I_{m'}$, and $Y^{k,m''}$, $k \in I_{m''}$, the optimal allocations of groups m' and m'' to the primal problem given $h + 1$ groups, where $I_{m'} \in I_m$ and $I_{m''} \in I_m$. Then*

$$\mathbb{E}_{\mathbb{Q}_{\mathbf{x}}^m}\Big[\sum_{k \in I_{m'}} \frac{w_{k,m'}}{w_{k,m}} Y^{k,m}\Big] \leq \eta'_m \log\Big\{-\frac{\beta'}{B}\mathbb{E}\Big[\exp\Big(-\frac{1}{\eta'_m}\sum_{k \in I_{m'}} w_{k,m'} X^k\Big)\Big]\Big\}, \tag{3.7}$$

where

$$\eta'_m = \sum_{k \in I_{m'}} \frac{w_{k,m'}}{w_{k,m}} \frac{1}{\alpha_k}.$$

Particularly, if both $\sum_{k \in I_{m'}} w_{k,m'} X^k$ and $\sum_{k \in I_{m''}} w_{k,m''} X^k$ are nonnegative, it holds that

$$\mathbb{E}_{\mathbb{Q}_{\mathbf{x}}^m}\Big[\sum_{k \in I_{m'}} \frac{w_{k,m'}}{w_{k,m}} Y^{k,m}\Big] \leq d_{m'}, \quad \mathbb{E}_{\mathbb{Q}_{\mathbf{x}}^m}\Big[\sum_{k \in I_{m''}} \frac{w_{k,m''}}{w_{k,m}} Y^{k,m}\Big] \leq d_{m''},$$

and thus

$$\sum_{k \in I_{m'}} \frac{w_{k,m'}}{w_{k,m}} Y^{k,m} + \sum_{k \in I_{m''}} \frac{w_{k,m''}}{w_{k,m}} Y^{k,m} = \sum_{k \in I_m} Y^{k,m} = d_m \leq d_{m'} + d_{m''},$$

where

$$d_{m'} = \beta'_m \log\Big\{-\frac{\beta'}{B}\mathbb{E}\Big[\exp\Big(-\frac{1}{\beta'_m}\sum_{k \in I_{m'}} w_{k,m'} X^k\Big)\Big]\Big\}$$

and $d_{m''}$ are the total risks of groups m' and m''. We point out that each individual profits from this decrease (of groups) by avoiding being (group) alone ($I_m \sim I_{m'} \cup I_{m''}$).

Also, when the system allows only disjoint grouping, i.e., each individual joins groups with weight 1*, inequality* (3.7) *implies the monotonicity result in* [9].

The proof is left to Appendix A.8.

3.3 Generalized group formation and Nash equilibrium

For a game with N individuals and h groups, similar to before, we assume there are h buckets for each individual to choose which ones she belongs to and how much she puts. This induces a corresponding weight

matrix for all individuals

$$\boldsymbol{W} = (w_{i,j}) = \begin{pmatrix} w_1 \\ \vdots \\ w_i \\ \vdots \\ w_N \end{pmatrix} \in \mathbb{R}^{N\times h},$$

defined as their strategies to distribute their risks, in order to minimize the individual total risk allocation. Each vector w_i contains values of weights showing which groups individual i belongs to and how much she wants to distribute the risk. So there is a natural constraint: $\sum_{j=1}^{h} w_{i,j} = 1$ for every $i = 1, \dots, N$. Recall that the weight of individual i in group j is denoted by $w_{i,j} \in [0, 1]$, and $w_{i,j} = 0$ means individual i is not in group j, $w_{i,j} = 1$ means individual i only joins group j. The case $w_{i,j} \in (0, 1)$ means, besides group j, individual i joins some other groups at the same time. Different sets of strategies may generate the same groups denoted by $\mathcal{C}(\boldsymbol{W})$.

The objective function of individual i is defined by

$$\rho^i(\mathcal{C}(\boldsymbol{W})) := \mathbb{E}_{\mathbb{Q}_{\mathbf{X}}}[Y^i_{\mathbf{X}}] = \sum_{j=1}^{h} \mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}[Y^{i,j}_{\mathbf{X}}],$$

where $i = 1, \dots, N$ and ρ^i is the total fair systemic risk allocation for individual i defined in (3.3).

Let

$$\hat{\boldsymbol{W}} = (\hat{w}_{i,j}) = \begin{pmatrix} \hat{w}_1 \\ \vdots \\ \hat{w}_i \\ \vdots \\ \hat{w}_N \end{pmatrix} \quad \text{and} \quad (\hat{\boldsymbol{W}}^{-i}, w^i) = \begin{pmatrix} \hat{w}_1 \\ \vdots \\ w^i \\ \vdots \\ \hat{w}_N \end{pmatrix}$$

be the weight matrix $\hat{\boldsymbol{W}}$ with the weight vector for individual i, i.e., the i-th row, is replaced by a new vector w^i whose elements sum up to 1.

Definition 3.1. The grouping $\mathcal{C}(\hat{\boldsymbol{W}})$ defined by the weight matrix $\hat{\boldsymbol{W}}$ is a Nash equilibrium if, for every i and any w^i,

$$\rho^i(\mathcal{C}(\mathbf{X}; \hat{\boldsymbol{W}})) \le \rho^i(\mathcal{C}(\mathbf{X}; \hat{\boldsymbol{W}}^{-i}, w^i)),$$

i.e., the systemic risk allocation of individual i is minimized under grouping $\mathcal{C}(\hat{\boldsymbol{W}})$, given other individuals' strategies are $\hat{\boldsymbol{W}}^{-i}$.

According the definition of Nash equilibrium, it is to be determined when the grouping, determined by the matrix $\boldsymbol{W}$, is optimized and how individuals distribute their risks under Nash equilibrium.

Remark 3.4. One cannot claim that, in this overlapping group case, it is still true that a single group with all the individuals is a (trivial) Nash equilibrium. This follows from the proof in Appendix A.9. When B, the minimal level of expected utility, is small, this means the system has a high tolerance with respect to risks. Then individuals tend to split into different groups, so there is no trivial Nash equilibrium. This can help explain why banks tend to join multiple central clearing counterparties (CCPs) to allocate their risks.

3.4 Case discussion: Correlated Gaussian distribution

In this section, we take a Gaussian distribution for the risk factors for simplicity and discuss in detail. Similar to Section 2.4, assume the joint distribution of $\mathbf{X} = (X^i,\ i = 1, \dots, N)^T$ follows a multivariate Gaussian distribution, that is, $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu} \in \mathbb{R}^N$ and $\boldsymbol{\Sigma} \in \mathbb{R}^{N\times N}$ is positive semi-definite. And define the column vector of $\boldsymbol{W}$ as a group vector given by

$$A_j = \begin{cases} 0, & i \notin I_j, \\ w_{i,j}, & i \in I_j,\ i = 1, \dots, N, \end{cases} \qquad A_j \in \mathbb{R}^{1\times N} \quad \text{for } j = 1, \dots, h.$$

Then the group sum

$$S_j = \sum_{i\in I_j} w_{i,j}X^i = A_j\mathbf{X} \sim N(A_j\boldsymbol{\mu}, A_j\boldsymbol{\Sigma}A_j^T) \overset{\text{def}}{=} N(\mu_j^s, (\sigma_j^s)^2)$$

follows, where

$$\mu_j^s = \sum_{k\in I_j} w_{k,j}\mu_k, \quad (\sigma_j^s)^2 = \sum_{m,k\in I_j} w_{m,j}w_{k,j}\sigma_{km}.$$

Using the results in Appendix A.10, we have, for $j = 1, \dots, h$ and for $i \in I_j$,

$$d_j = \beta_j \log\Big(-\frac{\beta}{B}\mathbb{E}[e^{-S_j/\beta_j}]\Big) = \beta_j \log\Big(\frac{\beta}{-B}\Big) - \mu_j^s + \frac{(\sigma_j^s)^2}{2\beta_j},$$

and the systemic risk allocation of individual i in group j is given by

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_\mathbf{x}^j}[Y_\mathbf{X}^{i,j}] &= \mathbb{E}\Big[Y_\mathbf{X}^{i,j}\cdot\frac{\mathrm{d}\mathbb{Q}_\mathbf{x}^j}{\mathrm{d}\mathbb{P}}\Big] = \frac{\mathbb{E}[(-w_{i,j}X^i + \frac{1}{\alpha_i\beta_j}S_j + \frac{1}{\alpha_i\beta_j}d_j)\cdot e^{-S_j/\beta_j}]}{\mathbb{E}(e^{-S_j/\beta_j})} \\
&= -w_{i,j}\Big(\mu_i - \frac{1}{\beta_j}A_j\boldsymbol{\Sigma}_{[,i]}\Big) + \frac{1}{\alpha_i\beta_j}\Big(\mu_j^s - \frac{(\sigma_j^s)^2}{\beta_j}\Big) + \frac{1}{\alpha_i\beta_j}\Big(\beta_j\log\Big(\frac{\beta}{-B}\Big) - \mu_j^s + \frac{(\sigma_j^s)^2}{2\beta_j}\Big) \\
&= \frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) - w_{i,j}\mu_i + \frac{w_{i,j}}{\beta_j}A_j\boldsymbol{\Sigma}_{[,i]} - \frac{(\sigma_j^s)^2}{2\beta_j^2\alpha_i},
\end{aligned} \tag{3.8}$$

where $A_j\boldsymbol{\Sigma}_{[,i]} = \sum_{k\in I_j} w_{k,j}\sigma_{ki}$.

3.4.1 Optimal weights in a general system

Continuing with the Gaussian distribution assumption, we use formula (3.8) to first find the optimal weight vector w^* for a given individual assuming the weight structure for other individuals is known. Then we search for a Nash equilibrium numerically using an algorithm in Section 4. That is, we minimize the total fair systemic risk allocation of individual i defined in (3.3) over the weight distributions $w_i = (w_{i,j},\ j = 1, \dots, N)$,

$$\min_{w_i}\mathbb{E}_{\mathbb{Q}_\mathbf{x}}[Y^i] = \min_{w_i}\sum_{j=1}^h \mathbb{E}_{\mathbb{Q}_\mathbf{x}^j}[Y^{i,j}] = \min_{w_{i,j}}\sum_{j=1}^h \mathbf{1}_{w_{i,j}>0}\Big[\frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) - w_{i,j}\mu_i + \frac{w_{i,j}}{\beta_j}A_j\boldsymbol{\Sigma}_{[,i]} - \frac{(\sigma_j^s)^2}{2\beta_j^2\alpha_i}\Big],$$

subject to

$$\sum_{j=1}^h w_{i,j}\mathbf{1}_{w_{i,j}>0} = 1.$$

For simplicity, we consider the problem when individual i joins at most two groups to discuss the optimal weights.

3.4.2 Risk allocation under at most two groups

When individual i joins at most 2 groups, there are three cases to discuss about the weights. Without loss of generality, we assume the weights are denoted by $(w_{i,1}, w_{i,2})$ for individual i, the weights for others are fixed, and there is at least one other individual in group 1 and 2. Given that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_\mathbf{x}}[Y^i] = {} & \mathbf{1}_{w_{i,1}>0}\Big[\frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) - w_{i,1}\mu_i + \frac{w_{i,1}}{\beta_1}A_1\boldsymbol{\Sigma}_{[,i]} - \frac{(\sigma_1^s)^2}{2\beta_1^2\alpha_i}\Big] \\
& + \mathbf{1}_{w_{i,2}>0}\Big[\frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) - w_{i,2}\mu_i + \frac{w_{i,2}}{\beta_2}A_2\boldsymbol{\Sigma}_{[,i]} - \frac{(\sigma_1^s)^2}{2\beta_2^2\alpha_i}\Big],
\end{aligned}$$

where $\beta_j = \sum_{i\in\{k:w_{k,j}>0\}}\frac{1}{\alpha_i}$ and $\beta = \sum_{j=1}^h \beta_j$, we discuss the following two boundary cases and one non-boundary case.

- Boundary case 1: $(w_{i,1}, w_{i,2}) = (1, 0)$. Then

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_X}[Y^i] &= \frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) - \mu_i + \frac{1}{\beta_1}A_1\mathbf{\Sigma}_{[,i]} - \frac{(\sigma_1^s)^2}{2\beta_1^2\alpha_i} \\ &= \frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) - \mu_i + \frac{1}{\beta_1}\Big(\sum_{k=1,\,k\neq i}^{N} w_{k,1}\sigma_{ki} + \sigma_{ii}\Big) \\ &\quad - \frac{1}{2\beta_1^2\alpha_i}\Big(\sum_{m,k=1,\,m,k\neq i}^{N} w_{k,1}w_{m,1}\sigma_{km} + 2\sum_{k=1,\,k\neq i}^{N} w_{k,1}\sigma_{ki} + \sigma_{ii}\Big). \end{aligned} \tag{3.9}$$

- Boundary case 2: $(w_{i,1}, w_{i,2}) = (0, 1)$. Then

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_X}[Y^i] &= \frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) - \mu_i + \frac{1}{\beta_2}A_2\mathbf{\Sigma}_{[,i]} - \frac{(\sigma_2^s)^2}{2\beta_2^2\alpha_i} \\ &= \frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) - \mu_i + \frac{1}{\beta_2}\Big(\sum_{k=1,\,k\neq i}^{N} w_{k,2}\sigma_{ki} + \sigma_{ii}\Big) \\ &\quad - \frac{1}{2\beta_2^2\alpha_i}\Big(\sum_{m,k=1,\,m,k\neq i}^{N} w_{k,2}w_{m,2}\sigma_{km} + 2\sum_{k=1,\,k\neq i}^{N} w_{k,2}\sigma_{ki} + \sigma_{ii}\Big). \end{aligned} \tag{3.10}$$

  In the above formulas,

$$\beta_1 = \sum_{k\neq i, w_{k,1}>0}\frac{1}{\alpha_k} + \frac{1}{\alpha_i}, \quad \beta_2 = \sum_{k\neq i, w_{k,2}>0}\frac{1}{\alpha_k} + \frac{1}{\alpha_i}, \quad \beta = \sum_{k\neq i, w_{k,1}>0}\frac{1}{\alpha_k} + \sum_{k\neq i, w_{k,2}>0}\frac{1}{\alpha_k} + \frac{1}{\alpha_i}.$$

- Non-boundary case: $(w_{i,1}, w_{i,2}) = (w, 1-w)$, while $0 < w < 1$. Then

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_X}[Y^i] &= \frac{2}{\alpha_i}\log\Big(\frac{\beta'}{-B}\Big) - \mu_i + \frac{w}{\beta_1}A_1'\mathbf{\Sigma}_{[,i]} + \frac{1-w}{\beta_2}A_2'\mathbf{\Sigma}_{[,i]} - \frac{(\sigma_1')^2}{2\beta_1^2\alpha_i} - \frac{(\sigma_2')^2}{2\beta_2^2\alpha_i} \\ &= \frac{2}{\alpha_i}\log\Big(\frac{\beta'}{-B}\Big) - \mu_i + \frac{w}{\beta_1}\Big(\sum_{k=1,\,k\neq i}^{N} w_{k,1}\sigma_{ki} + w\sigma_{ii}\Big) + \frac{1-w}{\beta_2}\Big(\sum_{k=1,\,k\neq i}^{N} w_{k,2}\sigma_{ki} + (1-w)\sigma_{ii}\Big) \\ &\quad - \frac{1}{2\beta_1^2\alpha_i}\Big(\sum_{m,k=1,\,m,k\neq i}^{N} w_{k,1}w_{m,1}\sigma_{km} + 2w\sum_{k=1,\,k\neq i}^{N} w_{k,1}\sigma_{ki} + w^2\sigma_{ii}\Big) \\ &\quad - \frac{1}{2\beta_2^2\alpha_i}\Big(\sum_{m,k=1,\,m,k\neq i}^{N} w_{k,2}w_{m,2}\sigma_{km} + 2(1-w)\sum_{k=1,\,k\neq i}^{N} w_{k,2}\sigma_{ki} + (1-w)^2\sigma_{ii}\Big), \end{aligned} \tag{3.11}$$

  where β_1, β_2 are the same as before but $\beta' = \beta_1 + \beta_2 = \beta + \frac{1}{\alpha_i}$.

3.4.3 Risk allocation comparison between boundary and non-boundary cases

Here we compare the minimal risk allocation of non-boundary case (3.11) with the risks of boundary cases (3.9) and (3.10). First we investigate the non-boundary case and prove $\mathbb{E}_{\mathbb{Q}_X}[Y^i]$ in (3.11) is a quadratic function of w and the minimal point is w^* given by (3.12). Taking partial derivative of (3.11) gives

$$\begin{aligned}\frac{\partial\mathbb{E}_{\mathbb{Q}_X}[Y^i]}{\partial w} &= \frac{1}{\beta_1}\sum_{k=1,\,k\neq i}^{N} w_{k,1}\sigma_{ki} + \frac{2w}{\beta_1}\sigma_{ii} - \frac{1}{\beta_2}\sum_{k=1,\,k\neq i}^{N} w_{k,2}\sigma_{ki} - \frac{2(1-w)}{\beta_2}\sigma_{ii} \\ &\quad - \frac{1}{\beta_1^2\alpha_i}\Big(\sum_{k=1,\,k\neq i}^{N} w_{k,1}\sigma_{ki} + w\sigma_{ii}\Big) + \frac{1}{\beta_2^2\alpha_i}\Big(\sum_{k=1,\,k\neq i}^{N} w_{k,2}\sigma_{ki} + (1-w)\sigma_{ii}\Big) \\ &= \sum_{k=1,\,k\neq i}^{N}\Big(\frac{w_{k,1}}{\beta_1} - \frac{w_{k,2}}{\beta_2}\Big)\sigma_{ki} - \sum_{k=1,\,k\neq i}^{N}\Big(\frac{w_{k,1}}{\beta_1^2\alpha_i} - \frac{w_{k,2}}{\beta_2^2\alpha_i}\Big)\sigma_{ki} \\ &\quad + w\Big[\Big(\frac{2}{\beta_1} + \frac{2}{\beta_2}\Big) - \Big(\frac{1}{\beta_1^2\alpha_i} + \frac{1}{\beta_2^2\alpha_i}\Big)\Big]\sigma_{ii} - \Big(\frac{2}{\beta_2} - \frac{1}{\beta_2^2\alpha_i}\Big)\sigma_{ii} = 0, \end{aligned}$$

and hence

$$w^* = \frac{\sum_{k=1,\,k\neq i}^{N}\left(\frac{w_{k,1}}{\beta_1^2\alpha_i} - \frac{w_{k,2}}{\beta_2^2\alpha_i}\right)\sigma_{ki} - \sum_{k=1,\,k\neq i}^{N}\left(\frac{w_{k,1}}{\beta_1} - \frac{w_{k,2}}{\beta_2}\right)\sigma_{ki} + \left(\frac{2}{\beta_2} - \frac{1}{\beta_2^2\alpha_i}\right)\sigma_{ii}}{\left[\left(\frac{2}{\beta_1} + \frac{2}{\beta_2}\right) - \left(\frac{1}{\beta_1^2\alpha_i} + \frac{1}{\beta_2^2\alpha_i}\right)\right]\sigma_{ii}} \tag{3.12}$$

is a minimizer since

$$\frac{\partial^2 \mathbb{E}_{\mathbb{Q}_X}[Y^i]}{\partial w^2} = \left[\left(\frac{2}{\beta_1} + \frac{2}{\beta_2}\right) - \left(\frac{1}{\beta_1^2\alpha_i} + \frac{1}{\beta_2^2\alpha_i}\right)\right]\sigma_{ii} > 0.$$

Here note that

$$\frac{2}{\beta_1} - \frac{1}{\beta_1^2\alpha_i} > 0, \quad \frac{2}{\beta_2} - \frac{1}{\beta_2^2\alpha_i} > 0.$$

It is clear that the denominator of (3.12) is positive. If

$$\sum_{k=1,\,k\neq i}^{N}\left(\frac{w_{k,1}}{\beta_1^2\alpha_i} - \frac{w_{k,2}}{\beta_2^2\alpha_i}\right)\sigma_{ki} - \sum_{k=1,\,k\neq i}^{N}\left(\frac{w_{k,1}}{\beta_1} - \frac{w_{k,2}}{\beta_2}\right)\sigma_{ki} > -\left(\frac{2}{\beta_2} - \frac{1}{\beta_2^2\alpha_i}\right)\sigma_{ii},$$

we have $w^* > 0$, and if

$$\sum_{k=1,\,k\neq i}^{N}\left(\frac{w_{k,1}}{\beta_1^2\alpha_i} - \frac{w_{k,2}}{\beta_2^2\alpha_i}\right)\sigma_{ki} - \sum_{k=1,\,k\neq i}^{N}\left(\frac{w_{k,1}}{\beta_1} - \frac{w_{k,2}}{\beta_2}\right)\sigma_{ki} < \left(\frac{2}{\beta_1} - \frac{1}{\beta_1^2\alpha_i}\right)\sigma_{ii},$$

we have $w^* < 1$.

In conclusion, let

$$A := \sum_{k=1,\,k\neq i}^{N}\left(\frac{w_{k,1}}{\beta_1^2\alpha_i} - \frac{w_{k,2}}{\beta_2^2\alpha_i}\right)\sigma_{ki} - \sum_{k=1,\,k\neq i}^{N}\left(\frac{w_{k,1}}{\beta_1} - \frac{w_{k,2}}{\beta_2}\right)\sigma_{ki},$$
$$B_1 := \left(\frac{2}{\beta_1} - \frac{1}{\beta_1^2\alpha_i}\right)\sigma_{ii}, \quad B_2 := \left(\frac{2}{\beta_2} - \frac{1}{\beta_2^2\alpha_i}\right)\sigma_{ii}.$$

In the non-boundary case, the local optimal weights for individual i joining two groups are non-zero, i.e.,

$$w^* = \frac{A + B_2}{B_1 + B_2} \in (0, 1) \quad \text{and} \quad 1 - w^* \in (0, 1)$$

if and only if $-B_2 < A < B_1$, i.e.,

$$-\left(\frac{2}{\beta_2} - \frac{1}{\beta_2^2\alpha_i}\right)\sigma_{ii} < \sum_{k=1,\,k\neq i}^{N}\left(\frac{w_{k,1}}{\beta_1^2\alpha_i} - \frac{w_{k,2}}{\beta_2^2\alpha_i}\right)\sigma_{ki} - \sum_{k=1,\,k\neq i}^{N}\left(\frac{w_{k,1}}{\beta_1} - \frac{w_{k,2}}{\beta_2}\right)\sigma_{ki} < \left(\frac{2}{\beta_1} - \frac{1}{\beta_1^2\alpha_i}\right)\sigma_{ii}. \tag{3.13}$$

In Appendix A.11, we investigate the condition further by reducing it to a simplified sufficient condition.

Then, given w^*, the minimal risk of (3.11) is

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_X}[Y^i]|_{w=w^*} = {} & \frac{2}{\alpha_i}\log\left(\frac{\beta'}{-B}\right) - \mu_i + w^* \sum_{k=1,\,k\neq i}^{N}\left[\left(\frac{w_{k,1}}{\beta_1} - \frac{w_{k,2}}{\beta_2}\right) - \left(\frac{w_{k,1}}{\beta_1^2\alpha_i} - \frac{w_{k,2}}{\beta_2^2\alpha_i}\right)\right]\sigma_{ki} \\ & + \sum_{k=1,\,k\neq i}^{N}\left(\frac{w_{k,2}}{\beta_2} - \frac{w_{k,2}}{\beta_2^2\alpha_i}\right)\sigma_{ki} + \left(\frac{(w^*)^2}{\beta_1} - \frac{(w^*)^2}{2\beta_1^2\alpha_i}\right)\sigma_{ii} + \left(\frac{(1-w^*)^2}{\beta_2} - \frac{(1-w^*)^2}{2\beta_2^2\alpha_i}\right)\sigma_{ii} \\ & - \frac{1}{2\beta_1^2\alpha_i}\sum_{m,k=1,\,m,k\neq i}^{N} w_{k,1}w_{m,1}\sigma_{km} - \frac{1}{2\beta_2^2\alpha_i}\sum_{m,k=1,\,m,k\neq i}^{N} w_{k,2}w_{m,2}\sigma_{km}.\end{aligned} \tag{3.14}$$

- When condition (3.13) holds, i.e., $w^* \in (0, 1)$, we compare the minimal risk of non-boundary case (3.11) with that of boundary cases (3.9) and (3.10) in Appendix A.12. We conclude that, when it holds that

$$\begin{cases}\frac{2}{\alpha_i}\log\left(\frac{\beta'}{-B}\right) - \frac{1}{\alpha_i}\log\left(\frac{\beta}{-B}\right) < \frac{(A-B_1)^2}{2(B_1+B_2)} + \frac{1}{2\beta_2^2\alpha_i}\sum_{m,k=1,\,m,k\neq i}^{N} w_{k,2}w_{m,2}\sigma_{km}, \\ \frac{2}{\alpha_i}\log\left(\frac{\beta'}{-B}\right) - \frac{1}{\alpha_i}\log\left(\frac{\beta}{-B}\right) < \frac{(A+B_2)^2}{2(B_1+B_2)} + \frac{1}{2\beta_1^2\alpha_i}\sum_{m,k=1,\,m,k\neq i}^{N} w_{k,1}w_{m,1}\sigma_{km},\end{cases} \tag{3.15}$$

the minimal risk for individual i is achieved at $(w_{i1}, w_{i2}) = (w^*, 1 - w^*)$ with non-zero weights. This is a necessary and sufficient condition to determine which one is superior when condition (3.13) is true.

- When condition (3.13) does not hold, if $w^* \le 0$ and

$$\frac{2}{\alpha_i}\log\Big(\frac{\beta'}{-B}\Big) - \frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) \ge \frac{1}{2\beta_1^2\alpha_i}\sum_{m,k=1,\,m,k\neq i}^{N} w_{k,1}w_{m,1}\sigma_{km},$$

the minimal risk is achieved at the boundary case $(w_{i1}, w_{i2}) = (0, 1)$. If $w^* \ge 1$ and

$$\frac{2}{\alpha_i}\log\Big(\frac{\beta'}{-B}\Big) - \frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) \ge \frac{1}{2\beta_2^2\alpha_i}\sum_{m,k=1,\,m,k\neq i}^{N} w_{k,2}w_{m,2}\sigma_{km},$$

the minimal risk is achieved at $(w_{i1}, w_{i2}) = (1, 0)$.

4 Numerical algorithm for Nash equilibrium

For a game of (large) N individuals, it is hard to find the risk allocation of each individual associated with the grouping system. Instead, using the discussion introduced in Section 2 and Section 3.4.1, under the assumption of Gaussian distribution for risk factors, we can do numerical analysis of some examples via Python to search for a Nash equilibrium for the system such that no individual could achieve a smaller fair risk allocation by changing grouping or weights under the equilibrium. We conclude that, for the disjoint group case, a nontrivial Nash equilibrium does not always exist and neither does the overlapping group case. If we apply the overlapping group setup to the real world, that is, interpreting $\sum_{n=1}^{N} Y^{n,j}$ as the default fund of the CCP j that is liable for any participating institution/bank, the numerical results indicate that big banks tend to join multiple CCPs, while small banks tend to choose one.

4.1 Numerical algorithm

In this section, we introduce numerical algorithms based on fictitious play. For the disjoint group case discussed in Section 2, we have the following algorithm.

(1) Let N individuals be in N different groups, i.e., $a_n = n$ for $1 \le n \le N$, as the initial state.
(2) At each stage, one individual is randomly picked with equal probability, and it chooses to join the group which gives the minimal risk allocation.
(3) Step (2) is repeated until the grouping is stabilized, and no individual has the incentive to move anymore.

For the overlapping group case in Section 3, the algorithm is similar. We take the number of groups h fixed, and the initial weights for every individual in groups are randomly generated. The optimal weights at each stage are determined based on the discussion in Section 3.4.1.

4.2 Numerical examples

Example 4.1 (Nearly block correlation matrix with positive, uniform ρ). In the case of $N = 4$, assume that the means and standard deviations are the same and α_i's are 1, i.e., $\alpha = [1, 1, 1, 1]$, $\mu = [10, 10, 10, 10]$, $\sigma_i \equiv \sigma$ for all i. The correlation matrix is

$$\begin{pmatrix} 1 & 0.4 & 0 & 0 \\ 0.4 & 1 & 0.05 & 0 \\ 0 & 0.05 & 1 & 0.4 \\ 0 & 0 & 0.4 & 1 \end{pmatrix}.$$

Then, for the disjoint group case, there exists one nontrivial Nash equilibrium "{1, 3}-{2, 4}".

Example 4.2 ($N = 10$). We take $\rho_{ij} = 0.8$ for all $i \neq j$, $i, j = 1, \ldots, N$, except $\rho_{19} = -0.3$. The values for other parameters are listed below:

$$\begin{aligned}
&\mu = [1, 1, 1, 2, 2, 3, 6, 6, 6, 7],\\
&\sigma = [4, 2.8, 1.6, 1, 3.8, 2.8, 0.9, 1.1, 4.2, 1.8],\\
&\alpha = [\underline{0.4}, 1.2, 1.8, 2.2, \underline{0.4}, 0.9, 2.8, 2.2, \underline{0.4}, 1.9],\\
&B = -8, \quad \text{the initial weights } (w_{i,1}, w_{i,2}) = (0.3, 0.7) \text{ for all } i.
\end{aligned}$$

By the algorithm presented in Section 4.1, we find the optimal weights for each individual one by one, and it turns out there exists a nontrivial Nash equilibrium in the system

$$\begin{pmatrix} w_{1,1} & w_{1,2} \\ \vdots & \vdots \\ w_{i,1} & w_{i,2} \\ \vdots & \vdots \\ w_{10,1} & w_{10,2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0.51 & 0.49 \\ 0.48 & 0.52 \\ 0.44 & 0.56 \\ 0 & 1 \\ 0.49 & 0.51 \\ 0.44 & 0.56 \\ 0.45 & 0.55 \\ 1 & 0 \\ 0.49 & 0.51 \end{pmatrix}.$$

We can see for some individuals which seek risks, i.e., with extremely small risk aversion parameters, they prefer being alone instead of separated.

As mentioned at the beginning of the section, the setup of systemic risks in the individual-group structure can be applied to the bank-CCP structure in real life, where individuals are banks and groups are CCPs. Then an individual with a large utility parameter alpha represents a core bank which is very risk-averse.

Example 4.3 ($N = 10$, less risk-averse individuals). In this example, the utility parameters are modified to compare with the previous example, and we interpret the results using the "bank-CCP" language. Assuming there are two core banks (4, 7) and eight peripheral banks, the correlation matrix is given in Figure 1.

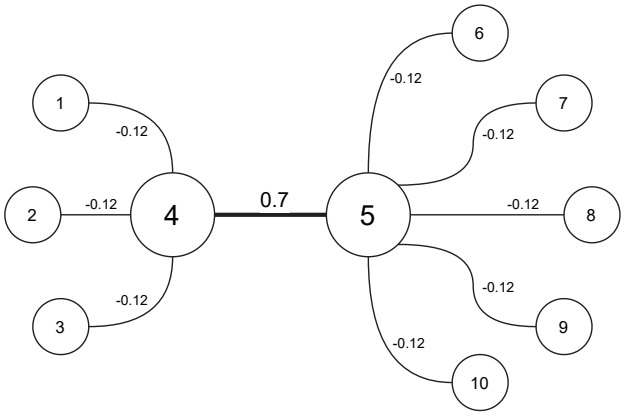
The values for other parameters are listed below:

$$\begin{aligned}
&\mu = [1, 1, 2, 2, 3, 4, 5, 5, 6, 7],\\
&\sigma = [4, 2.8, 2.2, 1.7, 1.4, 3.2, 3.8, 1.9, 4.2, 2.5],\\
&\alpha = [0.4, 1, 1.1, \underline{2.2}, \underline{2.8}, 0.9, 0.8, 1.4, 0.6, 1.3],\\
&B = -8,
\end{aligned}$$

and the initial weights are randomly generated based on uniform distribution between 0 and 1 for every i. There exists a nontrivial Nash equilibrium in the system

$$\begin{pmatrix} w_{1,1} & w_{1,2} \\ \vdots & \vdots \\ w_{i,1} & w_{i,2} \\ \vdots & \vdots \\ w_{10,1} & w_{10,2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0.46 & 0.54 \\ 0.32 & 0.68 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

We can see banks tend to stay with negatively correlated banks to mitigate the systemic risks. And risk-averse banks prefer splitting their risks by joining more CCPs.



	1 2 3	4	5	6 7 8 9 10
1 2 3	$\rho_{pp} = -0.25$	$\rho_{pc} = -0.12$	$\rho_{pc'} = -0.09$	$\rho_{pp'} = 0.05$
4	$\rho_{pc} = -0.12$		$\rho_{cc'} = 0.7$	$\rho_{cp'} = -0.09$
5	$\rho_{pc'} = -0.09$	$\rho_{cc'} = 0.7$		$\rho_{c'p'} = -0.12$
6 7 8 9 10	$\rho_{pp'} = 0.05$	$\rho_{cp'} = -0.09$	$\rho_{c'p'} = -0.12$	$\rho_{p'p'} = -0.25$

Figure 1: Correlation structure in Example 4.3: The top diagram shows partial correlations for 10 banks, where 4 and 5 are core banks and coefficients are labeled for each bank pair. The correlation table on the bottom shows the correlations matrix, where diagonals are all 1 and $\rho_{\cdot,\cdot}$ represents the correlation between two distinct banks. Subscripts c, c' stand for core bank 4 and core bank 5 respectively; p stands for peripheral banks 1, 2, 3, and p' stands for peripheral banks 6, 7, 8, 9, 10.

The analysis of the systemic risk and grouping formation can be applied to the reality, and it turns out our numerical results are consistent with the choices of CCPs for banks and financial institutions. One example is shown below using real data.

Example 4.4 (Real-life example). We take two CCPs who are clearing the same products but in a different region. One is the Chicago Mercantile Exchange Inc. (CME), which operates two separate clearing services, one for commodity and financial futures and options, and one for interest rate swaps and swaptions. The other one is the European Commodity Clearing (ECC), which is a central clearing house in Europe specializing in energy and commodity products. We select six clearing members and list them in order: J.P. Morgan (JPM), Goldman Sachs (GS), BNP Paribas (BNP), StoneX Group (SNEX), Banco Santander (SAN), and Interactive Brokers Group (IBKR). Among these firms, JPM, GS, SNEX, and BNP are members of both CCPs. SAN is only in ECC, while IBKR is only in CME.

	JPM	GS	BNP	SNEX	SAN	IBKR
	1	0.82	0.61	0.87	−0.27	0.86
	0.82	1	0.86	0.83	0.04	0.79
	0.61	0.86	1	0.65	0.25	0.60
	0.87	0.83	0.65	1	−0.35	0.89
	−0.27	0.04	0.25	−0.35	1	−0.24
	0.86	0.79	0.60	0.89	−0.24	1
σ	0.262	0.245	0.235	0.264	0.236	0.233

Table 1: Correlation matrix and standard deviation for 6 banks in Example 4.4.

We estimate the bank correlation matrix and standard deviation σ from banks' stock prices and list them in Table 1. Without loss of generality, we assume the expected values of their risks are all 0 since they have no effect on Nash equilibria according to formula (3.8). The value for B is the same as in the previous example. The risk-aversion parameters are chosen according to clearing members' "sizes",

$$\alpha = [2, 1.8, 1.7, 1.9, 1.2, 0.85],$$

and we list them in the following order: [JPM, GS, BNP, SNEX, SAN, IBKR].

The numerical results show that there exists a nontrivial Nash equilibrium

$$\begin{pmatrix} w_{1,1} & w_{1,2} \\ \vdots & \vdots \\ \vdots & \vdots \\ w_{6,1} & w_{6,2} \end{pmatrix} = \begin{pmatrix} 0.73 & 0.27 \\ 0.61 & 0.39 \\ 0.56 & 0.44 \\ 0.54 & 0.46 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is consistent with the fact that the first four firms JPM, GS, BNP, SNEX are spitted and join in both CCP groups, while SAN and IBKR belong to different CCPs. However, the distribution of weights cannot be verified here since related data of banks are not revealed in CCP documents.

5 Conclusion

In this paper, we generalize the systemic risk measure proposed in [8, 9] by allowing individual banks to choose their preferred groups instead of being assigned to certain groups. This introduces realistic game features in the proposed models and allows us to analyze the systemic risk for disjoint and overlapping groups (e.g., central clearing counterparties (CCP)). We introduce the concept of Nash equilibrium for these new models and analyze the optimal solution under the Gaussian distribution of the risk factor. We also provide an explicit solution for the individual banks' risk allocation and study the existence and uniqueness of Nash equilibrium both theoretically and numerically. The developed numerical algorithm can simulate scenarios of equilibrium, and we apply it to study the bank-CCP structure with real data and show the validity of the proposed model. While our formulation of constructing equilibrium groupings is based on minimizing the capital as a form of cross-absorption of losses, other factors such as netting agreements at the group level could also impact the change in the liability structure. However, this is beyond the scope of this paper and can be studied as further research. Further research also includes obtaining more actual data on bank balances and bank interconnections to conduct more in-depth research and analysis. The participation percentage of financial institutions is left to be validated and explained with more data. To better study the real CCP, it is also valuable to include more CCP mechanisms in the measure model, such as waterfall mechanisms, the existence of multiple layers of capital and CCP clearing fee charge, as in [16], and their effects on the equilibrium.

A Appendix

A.1 Comparison between trivial grouping and multi-groups

We first look at the trivial grouping, i.e., $m = h = 1$ and all $k \in I_1 = \{1, 2, \ldots, N\}$. The group parameter and group vectors are

$$\beta_m = \beta = \sum_{i=1}^{N} \frac{1}{\alpha_i}, \quad A_m = (1, 1, \ldots, 1).$$

Then, following (2.17), the systemic risk allocation of individual i is

$$\mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^1}[Y_{\mathbf{X}}^i] = -\mu_i + \frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) + \frac{1}{\beta}\sum_{j=1}^{N}\sigma_{ji} - \frac{1}{2\beta^2\alpha_i}\sum_{k,j=1}^{N}\sigma_{jk}.$$

The total systemic risk allocation for the system is

$$\sum_{i=1}^{N}\mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^1}[Y_{\mathbf{X}}^i] = -\sum_{i=1}^{N}\mu_i + \beta\log\Big(\frac{\beta}{-B}\Big) + \frac{1}{2\beta}\sum_{i,j=1}^{N}\sigma_{ji}. \tag{A.1}$$

Then, for the multi-group case, assuming $m = 1, \dots, h$ ($h \geq 2$) and for $k_m \in I_m \neq \emptyset$, the systemic risk allocation of individual k_m is

$$\mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^m}[Y_{\mathbf{X}}^{k_m}] = -\mu_{k_m} + \frac{1}{\alpha_{k_m}}\log\Big(\frac{\beta}{-B}\Big) + \frac{1}{\beta_m}\sum_{j\in I_m}\sigma_{jk_m} - \frac{1}{2\beta_m^2\alpha_{k_m}}\sum_{j,l\in I_m}\sigma_{jl}.$$

The total risk allocation is

$$\sum_{m=1}^{h}\sum_{k_m\in I_m}\mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^m}[Y_{\mathbf{X}}^{k_m}] = -\sum_{i=1}^{N}\mu_i + \beta\log\Big(\frac{\beta}{-B}\Big) + \frac{1}{2}\sum_{m=1}^{h}\frac{1}{\beta_m}\sum_{j,k\in I_m}\sigma_{jk}. \tag{A.2}$$

We need to compare the total risk of trivial grouping (A.1) and the total risk of nontrivial grouping (A.2). For simplicity, we take $h = 2$ in the nontrivial grouping case and compare.

Assume an N-individual system is divided into two subgroups with sizes $N_1 = |I_1|$, $N_2 = |I_2|$, respectively. Given all risk factors, define

$$S = \sum_{i=1}^{N}X^i, \quad S_1 = \sum_{i\in I_1}X^i, \quad S_2 = \sum_{i\in I_2}X^i.$$

Note that $S = S_1 + S_2$. Therefore,

$$\operatorname{Var}(S) = \sum_{i,j=1}^{N}\sigma_{ij}, \quad \operatorname{Var}(S_1) = \sum_{i,j\in I_1}\sigma_{ij}, \quad \operatorname{Var}(S_2) = \sum_{i,j\in I_2}\sigma_{ij}.$$

Then we compare the last term in (A.1), (A.2),

$$\text{(A.1):} \qquad \frac{1}{2\beta}\sum_{i,j=1}^{N}\sigma_{i,j} = \frac{1}{2\beta}\operatorname{Var}(S), \tag{A.3}$$

$$\text{(A.2):} \quad \frac{1}{2}\Big(\frac{1}{\beta_1}\sum_{j,k\in I_1}\sigma_{j,k} + \frac{1}{\beta_2}\sum_{j,k\in I_2}\sigma_{j,k}\Big) = \frac{1}{2}\Big(\frac{1}{\beta_1}\operatorname{Var}(S_1) + \frac{1}{\beta_2}\operatorname{Var}(S_2)\Big). \tag{A.4}$$

Since

$$\begin{aligned}
\operatorname{Var}(S) &= \operatorname{Var}(S_1) + \operatorname{Var}(S_2) + 2\operatorname{Cov}(S_1, S_2),\\
2\beta_1\beta_2\beta\cdot((\text{A.4}) - (\text{A.3})) &= \beta_1\beta_2\operatorname{Var}(S_1) + \beta_2^2\operatorname{Var}(S_1) + \beta_1^2\operatorname{Var}(S_2) + \beta_1\beta_2\operatorname{Var}(S_2)\\
&\quad - \beta_1\beta_2\big(\operatorname{Var}(S_1) + \operatorname{Var}(S_2) + 2\operatorname{Cov}(S_1, S_2)\big)\\
&= \beta_2^2\operatorname{Var}(S_1) + \beta_1^2\operatorname{Var}(S_2) - 2\beta_1\beta_2\operatorname{Cov}(S_1, S_2)\\
&= \operatorname{Var}(\beta_2 S_1 - \beta_1 S_2) \geq 0.
\end{aligned}$$

Thus (A.3) ≤ (A.4), which is equivalent to (A.1) ≤ (A.2). (The equality holds only if $S_1 = S_2 = S$, which we can exclude.)

We can conclude the trivial grouping has a smaller total systemic risk allocation for the N-player system compared with two-group case. It can be extended to the general grouping case and shows the advantage of trivial grouping or fewer groups in terms of the total risk. And it is consistent with the monotonicity property proved in [9].

A.2 Proof of Claim 2.1

Under the assumption $\beta_m = |I_m|\frac{1}{\alpha}$, equation (2.17) becomes

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^m}[Y_{\mathbf{X}}^k] &= -\mu_k + \frac{1}{\alpha}\log\Big(\frac{\beta}{-B}\Big) + \frac{\alpha}{|I_m|}(\sigma^2 + (|I_m|-1)\rho\sigma^2) \\ &\quad - \frac{1}{2\alpha}\Big(\frac{\alpha}{|I_m|}\Big)^2\Big(|I_m|\sigma^2 + \binom{|I_m|}{2}2\rho\sigma^2\Big) \\ &= -\mu_k + \frac{1}{\alpha}\log\Big(\frac{\beta}{-B}\Big) + \frac{\alpha}{2|I_m|}\sigma^2 + \Big(\alpha\frac{|I_m|-1}{|I_m|} - \alpha\frac{|I_m|-1}{2|I_m|}\Big)\rho\sigma^2 \\ &= -\mu_k + \frac{1}{\alpha}\log\Big(\frac{\beta}{-B}\Big) + \alpha\frac{1}{2|I_m|}\sigma^2 + \alpha\frac{|I_m|-1}{2|I_m|}\rho\sigma^2 \\ &= -\mu_k + \frac{1}{\alpha}\log\Big(\frac{\beta}{-B}\Big) + \frac{\alpha}{2}\Big((1-\rho)\frac{1}{|I_m|} + \rho\Big)\sigma^2.\end{aligned}$$

First, when $\rho \neq 1$, the function is monotonically decreasing in $|I_m|$. So the risk allocation for every individual/individual is maximized when $|I_m| = n$, i.e., all individuals/individuals are in the same group.

If we assume individuals are separated in several groups which makes it a Nash equilibrium for all. For some individual i in the second largest group, moving to the largest group makes it achieve a smaller risk allocation, which is better. If there are only two equal size groups, i.e., "$\frac{n}{2}$-$\frac{n}{2}$", one individual i in group 1 joining the other one gives "$(\frac{n}{2}-1)$-$(\frac{n}{2}+1)$" and $\mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^1}[Y_{\mathbf{X}}^i] > \mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^2}[Y_{\mathbf{X}}^i]$. So "$(\frac{n}{2}-1)$-$(\frac{n}{2}+1)$" cannot be a Nash equilibrium neither.

In conclusion, nontrivial grouping strategy cannot be a Nash equilibrium and only $|I_m| = n$ is a Nash equilibrium under the case that all standard deviation and utility parameters are the same, and the correlation coefficient is $\rho \in [-1, 1)$. When $\rho = 1$, the risk for every individual is constant and grouping has no effect.

A.3 Proof of Claim 2.2

First we look at the grouping "{1, 2}-{3, 4}" and find the equivalent condition to make it a Nash equilibrium. We compare risks for individual 1 under two cases, according to (2.17),

$$\begin{aligned}&\text{under "\{1, 2\}-\{3, 4\}":} \quad \mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^1}[Y_{\mathbf{X}}^1] = -\mu_1 + \log\Big(\frac{\beta}{-B}\Big) + \frac{1}{2}(\sigma^2 + \rho\sigma^2) - \frac{1}{8}(2\sigma^2 + 2\rho\sigma^2), \\ &\text{under "\{2\}-\{1, 3, 4\}":} \quad \mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^2}[Y_{\mathbf{X}}^1] = -\mu_1 + \log\Big(\frac{\beta}{-B}\Big) + \frac{1}{3}\sigma^2 - \frac{1}{18}(3\sigma^2 + 2\rho\sigma^2).\end{aligned}$$

For individual 1, to make the risk allocation under "{1, 2}-{3, 4}" at most that under "{2}-{1, 3, 4}",

$$\frac{1}{3}\sigma^2 - \frac{1}{18}(3\sigma^2 + 2\rho\sigma^2) \geq \frac{1}{2}(\sigma^2 + \rho\sigma^2) - \frac{1}{8}(2\sigma^2 + 2\rho\sigma^2) \iff \rho \leq -\frac{3}{13}.$$

For individuals 2, 3, 4, we repeat a similar discussion, and the condition is the same, $\rho \leq -\frac{3}{13}$. So, in conclusion, if $\rho \leq -\frac{3}{13}$, grouping "{1, 2}-{3, 4}" is a Nash equilibrium.

In the grouping "{1, 3}-{2, 4}", we follow the same discussion about comparing risk allocations for all individuals. For example, for individual 1,

$$\begin{aligned}&\text{under "\{1, 3\}-\{2, 4\}":} \quad \mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^1}[Y_{\mathbf{X}}^1] = -\mu_1 + \log\Big(\frac{\beta}{-B}\Big) + \frac{1}{2}\sigma^2 - \frac{1}{8}2\sigma^2, \\ &\text{under "\{3\}-\{1, 2, 4\}":} \quad \mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^2}[Y_{\mathbf{X}}^1] = -\mu_1 + \log\Big(\frac{\beta}{-B}\Big) + \frac{1}{3}(\sigma^2 + \rho\sigma^2) - \frac{1}{18}(3\sigma^2 + 2\rho\sigma^2).\end{aligned}$$

For individual 1, to make the risk allocation under "{1, 3}-{2, 4}" at most that under "{3}-{1, 2, 4}",

$$\frac{1}{3}(\sigma^2 + \rho\sigma^2) - \frac{1}{18}(3\sigma^2 + 2\rho\sigma^2) \geq \frac{1}{2}\sigma^2 - \frac{1}{8}2\sigma^2 \iff \rho \geq \frac{3}{8}.$$

For individuals 2, 3, 4, we repeat a similar discussion, and the condition is the same, $\rho \geq \frac{3}{8}$. As a result, grouping "{1, 3}-{2, 4}" is a Nash equilibrium if $\rho \geq \frac{3}{8}$. It is also the equivalent condition for grouping "{1, 4}-{2, 3}" to be a Nash equilibrium

A.4 Proof of Claim 2.3

Similar to the proof of Claim 2.2, we look at the grouping "{1, 2}-{3, 4, 5}" and find the equivalent condition to make it a Nash equilibrium. Under the assumptions, we first compare risks for individual 1 under two cases, according to (2.17),

$$\text{under "\{1, 2\}-\{3, 4, 5\}":}\quad \mathbb{E}_{\mathbb{Q}^1_{\mathbf{X}}}[Y^1_{\mathbf{X}}] = -\mu_1 + \log\Big(\frac{\beta}{-B}\Big) + \frac{1}{2}(\sigma^2 + \rho\sigma^2) - \frac{1}{8}(2\sigma^2 + 2\rho\sigma^2),$$
$$\text{under "\{2\}-\{1, 3, 4, 5\}":}\quad \mathbb{E}_{\mathbb{Q}^2_{\mathbf{X}}}[Y^1_{\mathbf{X}}] = -\mu_1 + \log\Big(\frac{\beta}{-B}\Big) + \frac{1}{4}\sigma^2 - \frac{1}{32}(4\sigma^2 + 6\rho\sigma^2).$$

For individual 1, to make the risk allocation under "{1, 2}-{3, 4, 5}" at most that under "{2}-{1, 3, 4, 5}",

$$\frac{1}{4}\sigma^2 - \frac{1}{32}(4\sigma^2 + 6\rho\sigma^2) \geq \frac{1}{2}(\sigma^2 + \rho\sigma^2) - \frac{1}{8}(2\sigma^2 + 2\rho\sigma^2) \iff \rho \leq -\frac{2}{7}.$$

For individual 2, this is the same condition to have a smaller risk allocation. We then compare risks for individual 3 under two cases,

$$\text{under "\{1, 2\}-\{3, 4, 5\}":}\quad \mathbb{E}_{\mathbb{Q}^2_{\mathbf{X}}}[Y^3_{\mathbf{X}}] = -\mu_3 + \log\Big(\frac{\beta}{-B}\Big) + \frac{1}{3}(\sigma^2 + 2\rho\sigma^2) - \frac{1}{18}(3\sigma^2 + 6\rho\sigma^2),$$
$$\text{under "\{1, 2, 3\}-\{4, 5\}":}\quad \mathbb{E}_{\mathbb{Q}^1_{\mathbf{X}}}[Y^3_{\mathbf{X}}] = -\mu_3 + \log\Big(\frac{\beta}{-B}\Big) + \frac{1}{3}\sigma^2 - \frac{1}{18}(3\sigma^2 + 2\rho\sigma^2).$$

To make the risk allocation for individual 3 under "{1, 2}-{3, 4, 5}" at most that under "{1, 2, 3}-{4, 5}", we get the equivalent condition $\rho \leq 0$. This is true for individuals 4, 5 as well. In conclusion, when $\rho \leq -\frac{2}{7}$, "{1, 2}-{3, 4, 5}" is a Nash equilibrium for all individuals.

In the grouping "{1, 3}-{2, 4, 5}", we follow the same discussion about comparing risk allocations for all individuals. Individual 1 will stay with individual 3 instead of joining the other group if $\rho = 1$. Individual 2 will stay if $\rho \geq 0$. Individual 3 will stay with individual 1 if $\rho \geq \frac{2}{5}$, and individuals 4, 5 will not move for any ρ. As a result, grouping "{1, 3}-{2, 4, 5}" is not a Nash equilibrium for any $\rho \in (-1, 1)$.

For the grouping "{1, 2, 3}-{4, 5}", after the discussion about the condition for every individual not moving, we find a contradiction which proves, for any value of ρ, this grouping cannot be a Nash equilibrium.

A.5 Proof of Theorem 3.1

We can rewrite the systemic risk measure ρ as

$$\boldsymbol{\rho}(\mathbf{X}) = \inf\Bigg\{\sum_{m=1}^{h} d_m : d = (d_1, \dots, d_h) \in \mathbb{R}^h,\ \mathbf{Y} = (Y^{i,j}, i \in I_j, 1 \leq j \leq h) \in L^0(\mathbb{R}^{\sum_{m=1}^{h}|I_m|*h}),$$
$$\mathbb{E}\Big[\sum_{m=1}^{h}\sum_{k\in I_m} -\frac{1}{\alpha_k}\exp[-\alpha_k(w_{k,m}X^k + Y^{k,m})]\Big] = B,\ \sum_{i\in I_j} Y^{i,j} = d_j \text{ for } j = 1, 2, \dots, h\Bigg\},$$

where $(I_1, \dots, I_h)$ are the group index sets given. For any group m, we define the smallest element as $m^0 \in I_m$ and fix another element $m^* \in I_m$ and $m^* \neq m^0$. (When there is only one element in the group, $m^* = m^0$, and the discussion will be similar.) In the following proof, we assume $|I_m| \geq 2$ for all $m = 1, \dots, h$. We also assume Y is defined on a finite space, $Y^{k,m} \in \{y_1^{k,m}, \dots, y_M^{k,m}\}$ for all k, m and $y_j^{k,m} \in \mathbb{R}$. Then we have

$$y_j^{m^*,m} = d_m - \sum_{k\in I_m,\, k\neq m^*} y_j^{k,m} \quad \text{for } m = 1, \dots, h.$$

We show the results with the Lagrange method with the function defined by

$$\mathcal{L}(\mathbf{d}, \mathbf{Y}, \lambda) = \sum_{m=1}^{h} d_m + \lambda\Bigg\{\sum_{j=1}^{M} p_j \sum_{m=1}^{h}\Big[\sum_{k\in I_m,\, k\neq m^*} \frac{-1}{\alpha_k}\exp(-\alpha_k(w_{k,m}X^k(\omega_j) + y_j^{k,m}))$$
$$+ \frac{-1}{\alpha_{m^*}}\exp\Big(-\alpha_{m^*}\Big(w_{m^*,m}X^{m^*}(\omega_j) + d_m - \sum_{k\in I_m,\, k\neq m^*} y_j^{k,m}\Big)\Big)\Big] - B\Bigg\}.$$

We compute partial derivatives of $\mathcal{L}$ with respect to all variables and get all equivalent conditions in the following.

(1) Given m, for $j = 1, \dots, M$, $k \in I_m$ and $k \neq m^*$, we have $\frac{\partial \mathcal{L}}{\partial y_j^{k,m}} = 0$ if and only if, for every fixed j,

$$0 = \frac{\partial \mathcal{L}}{\partial y_j^{k,m}} = \lambda p_j \Big[\exp(-\alpha_k (w_{k,m} X^k(\omega_j) + y_j^{k,m})) - \exp\Big(-\alpha_{m^*} \Big(w_{m^*,m} X^{m^*}(\omega_j) + d_m - \sum_{k \in I_m,\, k \neq m^*} y_j^{k,m} \Big) \Big) \Big]$$

or, equivalently, we have

$$\exp(-\alpha_k (w_{k,m} X^k(\omega_j) + y_j^{k,m})) = \exp\Big(-\alpha_{m^*} \Big(w_{m^*,m} X^{m^*}(\omega_j) + d_m - \sum_{k \in I_m,\, k \neq m^*} y_j^{k,m} \Big) \Big). \tag{A.5}$$

This implies

$$\exp(-\alpha_k (w_{k,m} X^k(\omega_j) + y_j^{k,m})) = \exp(-\alpha_{m^0} (w_{m^0,m} X^{m^0}(\omega_j) + y_j^{m^0,m})).$$

Then, for $k \neq m^*$,

$$y_j^{k,m} = \frac{1}{\alpha_k} (\alpha_{m^0} w_{m^0,m} X^{m^0}(\omega_j) - \alpha_k w_{k,m} X^k(\omega_j) + \alpha_{m^0} y_j^{m^0,m}), \tag{A.6}$$

and by (A.5) and (A.6), we obtain (details are shown below)

$$\begin{aligned} y_j^{m^0,m} &= \frac{1}{\alpha_{m^0} \beta_m} \Big(\sum_{k \in I_m} w_{k,m} X^k(\omega_j) \Big) - w_{m^0,m} X^{m^0}(\omega_j) + \frac{1}{\alpha_{m^0} \beta_m} d_m \\ &= \frac{1}{\alpha_{m^0} \beta_m} S_m(\omega_j) - w_{m^0,m} X^{m^0}(\omega_j) + \frac{1}{\alpha_{m^0} \beta_m} d_m, \end{aligned} \tag{A.7}$$

where $\beta_m = \sum_{k \in I_m} \frac{1}{\alpha_k}$, $S_m = \sum_{k \in I_m} w_{k,m} X^k$.

Proof. By (A.6),

$$\begin{aligned} \sum_{k \in I_m,\, k \neq m^*} y_j^{k,m} &= \sum_{k \in I_m,\, k \neq m^*} \frac{1}{\alpha_k} \alpha_{m^0} (w_{m^0,m} X^{m^0}(\omega_j) + y_j^{m^0,m}) - \sum_{k \in I_m,\, k \neq m^*} w_{k,m} X^k(\omega_j) \\ &= \Big(\beta_m - \frac{1}{\alpha_{m^*}} \Big) \alpha_{m^0} (w_{m^0,m} X^{m^0}(\omega_j) + y_j^{m^0,m}) - \sum_{k \in I_m,\, k \neq m^*} w_{k,m} X^k(\omega_j), \end{aligned}$$

$$w_{m^*,m} X^{m^*}(\omega_j) + d_m - \sum_{k \in I_m,\, k \neq m^*} y_j^{k,m} = d_m - \Big(\beta_m - \frac{1}{\alpha_{m^*}} \Big) \alpha_{m^0} (w_{m^0,m} X^{m^0}(\omega_j) + y_j^{m^0,m}) + S_m(\omega_j).$$

Thus, using (A.5), we obtain

$$\begin{aligned} & -\alpha_{m^*} \Big(w_{m^*,m} X^{m^*}(\omega_j) + d_m - \sum_{k \in I_m,\, k \neq m^*} y_j^{k,m} \Big) \\ &\quad = -\alpha_{m^*} d_m + (\beta_m \alpha_{m^*} - 1) \alpha_{m^0} (w_{m^0,m} X^{m^0}(\omega_j) + y_j^{m^0,m}) - \alpha_{m^*} S_m(\omega_j) \\ &\quad = -\alpha_{m^0} (w_{m^0,m} X^{m^0}(\omega_j) + y_j^{m^0,m}) \quad \text{by (A.5)} \\ &\implies \alpha_{m^*} \beta_m \alpha_{m^0} (w_{m^0,m} X^{m^0}(\omega_j) + y_j^{m^0,m}) = \alpha_{m^*} d_m + \alpha_{m^*} S_m(\omega_j) \\ &\implies \text{(A.7)}. \end{aligned}$$

□

(2) For $m = 1, \dots, h$, the derivative with respect to d_m, $\frac{\partial \mathcal{L}}{\partial d_m} = 0$ if and only if

$$0 = \frac{\partial \mathcal{L}}{\partial d_m} = 1 + \lambda \sum_{j=1}^{M} p_j \exp\Big(-\alpha_{m^*} \Big(w_{m^*,m} X^{m^*}(\omega_j) + d_m - \sum_{k \in I_m,\, k \neq m^*} y_j^{k,m} \Big) \Big). \tag{A.8}$$

By (A.5),

$$0 = 1 + \lambda \sum_{j=1}^{M} p_j \exp(-\alpha_k (w_{k,m} X^k(\omega_j) + y_j^{k,m})) \quad \text{for all } k \in I_m \text{ and } k \neq m^*,$$

i.e.,

$$\sum_{j=1}^{M} p_j \exp(-\alpha_k (w_{k,m} X^k(\omega_j) + y_j^{k,m})) = \frac{-1}{\lambda}. \tag{A.9}$$

(3) The derivative with respect to λ, $\frac{\partial \mathcal{L}}{\partial \lambda} = 0$ if and only if

$$
\begin{aligned}
B &= \mathbb{E}\Big[\sum_{m=1}^{h}\sum_{k\in I_m}\frac{-1}{\alpha_k}\exp[-\alpha_k(w_{k,m}X^k + Y^{k,m})]\Big] \\
&= \sum_{j=1}^{M} p_j \cdot \sum_{m=1}^{h}\Big[\sum_{k\in I_m,\, k\neq m^*}\frac{-1}{\alpha_k}\exp(-\alpha_k(w_{k,m}X^k(\omega_j) + y_j^{k,m})) \\
&\qquad + \frac{-1}{\alpha_{m^*}}\exp\Big(-\alpha_{m^*}\Big(w_{m^*,m}X^{m^*}(\omega_j) + d_m - \sum_{k\in I_m,\, k\neq m^*} y_j^{k,m}\Big)\Big)\Big] \\
&= \sum_{j=1}^{M}\sum_{m=1}^{h}\Big[\sum_{k\in I_m,\, k\neq m^*}\frac{-1}{\alpha_k}\cdot p_j\exp(-\alpha_k(w_{k,m}X^k(\omega_j) + y_j^{k,m})) \\
&\qquad + \frac{-1}{\alpha_{m^*}}\cdot p_j\exp\Big(-\alpha_{m^*}\Big(w_{m^*,m}X^{m^*}(\omega_j) + d_m - \sum_{k\in I_m,\, k\neq m^*} y_j^{k,m}\Big)\Big)\Big] \\
&= \sum_{m=1}^{h}\Big[\sum_{k\in I_m,\, k\neq m^*}\frac{-1}{\alpha_k}\frac{-1}{\lambda} + \frac{-1}{-\alpha_{m^*}}\frac{-1}{\lambda}\Big] \quad \text{by (A.8), (A.9)} \\
&= \frac{1}{\lambda}\sum_{m=1}^{h}\sum_{k\in I_m}\frac{1}{\alpha_k} = \frac{1}{\lambda}\beta.
\end{aligned}
$$

Hence

$$\lambda = \frac{\beta}{B}. \tag{A.10}$$

We then compute d_m by inserting (A.10) and (A.7) in (A.9) for $k = m^0 \in I_m$, and $m^0 \neq m^*$,

$$
\begin{aligned}
-\frac{B}{\beta} &= \sum_{j=1}^{M} p_j\exp(-\alpha_{m^0}(w_{m^0,m}X^{m^0}(\omega_j) + y_j^{m^0,m})) = \sum_{j=1}^{M} p_j\exp\Big(-\frac{1}{\beta_m}(S_m(\omega_j) + d_m)\Big) \\
&= e^{-d_m/\beta_m}\sum_{j=1}^{M} p_j\exp\Big(-\frac{1}{\beta_m}S_m(\omega_j)\Big) = e^{-d_m/\beta_m}\mathbb{E}(e^{-S_m/\beta_m}).
\end{aligned}
$$

So

$$d_m = \beta_m\log\Big(\frac{\beta}{-B}\mathbb{E}(e^{-S_m/d_m})\Big).$$

Then, back to (A.6) and (A.7), for $k, m^0, m^* \in I_m$ and $k \neq m^0 \neq m^*$,

$$
\begin{aligned}
y_j^{m^0,m} &= \frac{1}{\alpha_{m^0}\beta_m}S_m(\omega_j) - w_{m^0,m}X^{m^0}(\omega_j) + \frac{1}{\alpha_{m^0}\beta_m}d_m, \\
y_j^{k,m} &= \frac{\alpha_{m^0}}{\alpha_k}w_{m^0,m}X^{m^0}(\omega_j) - w_{k,m}X^k(\omega_j) \\
&\qquad + \frac{1}{\alpha_k}\Big[\frac{1}{\beta_m}S_m(\omega_j) - \alpha_{m^0}w_{m^0,m}X^{m^0}(\omega_j) + \frac{1}{\beta_m}d_m\Big] \\
&= -w_{k,m}X^k(\omega_j) + \frac{1}{\alpha_k\beta_m}(S_m(\omega_j) + d_m), \\
y_j^{m^*,m} &= d_m - \sum_{k\in I_m,\, k\neq m^*} y_j^{k,m} \\
&= d_m + \sum_{k\in I_m,\, k\neq m^*} w_{k,m}X^k(\omega_j) - \sum_{k\in I_m,\, k\neq m^*}\frac{1}{\alpha_k\beta_m}(S_m(\omega_j) + d_m) \\
&= d_m + (S_m(\omega_j) - w_{m^*,m}X^{m^*}(\omega_j)) - \Big(\beta_m - \frac{1}{\alpha_{m^*}}\Big)\frac{1}{\beta_m}(S_m(\omega_j) + d_m) \\
&= -w_{m^*,m}X^{m^*}(\omega_j) + \frac{1}{\alpha_{m^*}\beta_m}(S_m(\omega_j) + d_m).
\end{aligned}
$$

So, for given m, for all $k \in I_m$, we have

$$Y^{k,m} = -w_{k,m}X^k + \frac{1}{\alpha_k\beta_m}(S_m + d_m), \quad \text{where} \quad d_m = \beta_m\log\Big(\frac{\beta}{-B}\mathbb{E}(e^{-S_m/d_m})\Big).$$

In addition, the systemic risk measure is given by

$$\boldsymbol{\rho}(\mathbf{X}) = \sum_{m=1}^{h} d_m = \sum_{m=1}^{h} \beta_m \log\Big(\frac{\beta}{-B}\mathbb{E}(e^{-S_m/d_m})\Big) = \beta\log\Big(\frac{\beta}{-B}\Big) + \sum_{m=1}^{h}\beta_m \log(\mathbb{E}(e^{-S_m/d_m})).$$

A.6 Proof of Proposition 3.1

First we prove the marginal risk allocation for individual i in group j. By Theorem 3.1, for $i \in I_j$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}[Y^{i,j}_{\mathbf{X}+\varepsilon\mathbf{Z}}] &= \mathbb{E}\Big[Y^{i,j}_{\mathbf{X}+\varepsilon\mathbf{Z}}\cdot\frac{\mathrm{d}\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}{\mathrm{d}\mathbb{P}}\Big] \\
&= \mathbb{E}\Big[\Big(-w_{i,j}(X^i+\varepsilon Z^i) + \frac{1}{\alpha_i\beta_j}(S_j+\varepsilon S^{\mathbf{Z}}_j) + \frac{1}{\alpha_i\beta_j}d^{\mathbf{X}+\varepsilon\mathbf{Z}}_j\Big)\cdot\frac{e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j}}{\mathbb{E}[e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j}]}\Big] \\
&= -w_{i,j}\frac{\mathbb{E}((X^i+\varepsilon Z^i)e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j})}{\mathbb{E}(e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j})} + \frac{1}{\alpha_i\beta_j}\frac{\mathbb{E}((S_j+\varepsilon S^{\mathbf{Z}}_j)e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j})}{\mathbb{E}(e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j})} \\
&\quad + \frac{1}{\alpha_i}\Big(\log\Big(\frac{\beta}{-B}\Big) + \log(\mathbb{E}(e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j}))\Big) \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Since $S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j = S_j + \varepsilon S^{\mathbf{Z}}_j$, and assuming everything is well-defined so that we can use Leibniz integral rule, then we have the following results:

$$\frac{\partial}{\partial\varepsilon}\mathbb{E}(e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j}) = \mathbb{E}\Big(-\frac{S^{\mathbf{Z}}_j}{\beta_j}e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j}\Big),$$

$$\frac{\partial}{\partial\varepsilon}\mathbb{E}((S_j+\varepsilon S^{\mathbf{Z}}_j)e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j}) = \mathbb{E}\Big[S^{\mathbf{Z}}_j e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j} - (S_j+\varepsilon S^{\mathbf{Z}}_j)\frac{S^{\mathbf{Z}}_j}{\beta_j}e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j}\Big].$$

Then we compute the derivatives

$$\begin{aligned}
\frac{\partial\mathrm{I}}{\partial\varepsilon} &= -w_{i,j}\frac{\partial}{\partial\varepsilon}\Big(\frac{\mathbb{E}((X^i+\varepsilon Z^i)e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j})}{\mathbb{E}(e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j})}\Big) \\
&= -w_{i,j}\frac{1}{(\mathbb{E}e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j})^2}\Big[\mathbb{E}(Z^i e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j})\mathbb{E}(e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j}) + \mathbb{E}\Big((X^i+\varepsilon Z^i)\Big(-\frac{S^{\mathbf{Z}}_j}{\beta_j}\Big)e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j}\Big)\mathbb{E}(e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j}) \\
&\qquad - \mathbb{E}((X^i+\varepsilon Z^i)e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j})\mathbb{E}\Big(-\frac{S^{\mathbf{Z}}_j}{\beta_j}e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j}\Big)\Big] \\
&= -w_{i,j}\Big[\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}[Z^i] - \frac{1}{\beta_j}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}[(X^i+\varepsilon Z^i)S^{\mathbf{Z}}_j] + \frac{1}{\beta_j}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}[X^i+\varepsilon Z^i]\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}[S^{\mathbf{Z}}_j]\Big] \\
&= \mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}[-w_{i,j}Z^i] + \frac{w_{i,j}}{\beta_j}\operatorname{Cov}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}(X^i+\varepsilon Z^i, S^{\mathbf{Z}}_j), \\
\frac{\partial\mathrm{II}}{\partial\varepsilon} &= \frac{1}{\alpha_i\beta_j}\frac{\partial}{\partial\varepsilon}\Big(\frac{\mathbb{E}((S_j+\varepsilon S^{\mathbf{Z}}_j)e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j})}{\mathbb{E}(e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j})}\Big) \\
&= \frac{1}{\alpha_i\beta_j}\frac{1}{(\mathbb{E}e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j})^2}\Big[\Big(\mathbb{E}(S^{\mathbf{Z}}_j e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j}) + \mathbb{E}\Big((S_j+\varepsilon S^{\mathbf{Z}}_j)\Big(-\frac{S^{\mathbf{Z}}_j}{\beta_j}\Big)e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j}\Big)\Big)\mathbb{E}(e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j}) \\
&\qquad - \mathbb{E}((S_j+\varepsilon S^{\mathbf{Z}}_j)e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j})\mathbb{E}\Big(-\frac{S^{\mathbf{Z}}_j}{\beta_j}e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j}\Big)\Big] \\
&= \frac{1}{\alpha_i\beta_j}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}[S^{\mathbf{Z}}_j] - \frac{1}{\alpha_i\beta_j^2}\operatorname{Cov}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}(S_j+\varepsilon S^{\mathbf{Z}}_j, S^{\mathbf{Z}}_j), \\
\frac{\partial\mathrm{III}}{\partial\varepsilon} &= \frac{1}{\alpha_i}\frac{1}{\mathbb{E}(e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j})}\cdot\mathbb{E}\Big(-\frac{S^{\mathbf{Z}}_j}{\beta_j}e^{-S^{\mathbf{X}+\varepsilon\mathbf{Z}}_j/\beta_j}\Big) = -\frac{1}{\alpha_i\beta_j}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}[S^{\mathbf{Z}}_j].
\end{aligned}$$

As a result, we obtain

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}[Y^{i,j}_{\mathbf{X}+\varepsilon\mathbf{Z}}] &= \frac{\partial}{\partial\varepsilon}(\mathrm{I}+\mathrm{II}+\mathrm{III})\\ &= -w_{i,j}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}[Z^i] + \frac{w_{i,j}}{\beta_j}\operatorname{Cov}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}(X^i+\varepsilon Z^i, S^{\mathbf{Z}}_j) + \frac{1}{\alpha_i\beta_j}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}[S^{\mathbf{Z}}_j]\\ &\quad - \frac{1}{\alpha_i\beta_j^2}\operatorname{Cov}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}(S_j+\varepsilon S^{\mathbf{Z}}_j, S^{\mathbf{Z}}_j) - \frac{1}{\alpha_i\beta_j}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}[S^{\mathbf{Z}}_j]\\ &= \mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}[-w_{i,j}Z^i] + \frac{w_{i,j}}{\beta_j}\operatorname{Cov}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}(X^i+\varepsilon Z^i, S^{\mathbf{Z}}_j) - \frac{1}{\alpha_i\beta_j^2}\operatorname{Cov}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}(S_j+\varepsilon S^{\mathbf{Z}}_j, S^{\mathbf{Z}}_j)\\ &= \mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}[-w_{i,j}Z^i] - \frac{1}{\beta_j}\operatorname{Cov}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}(Y^{i,j}_{\mathbf{X}+\varepsilon\mathbf{Z}}, S^{\mathbf{Z}}_j).\end{aligned}$$

Then, when $\varepsilon = 0$, we have the formula for marginal risk allocation for individual i in group j in Proposition 3.1.

The marginal risk contribution of group j is trivial, and for the conclusion on local causal responsibility, we have

$$\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}[Y^{i,j}_{\mathbf{X}+\varepsilon\mathbf{Z}}] = \mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}\Big[\Big(-w_{i,j}(X^i+\varepsilon Z^i) + \frac{1}{\alpha_i\beta_j}(S_j+\varepsilon S^{\mathbf{Z}}_j) + \frac{1}{\alpha_i\beta_j}d_j^{\mathbf{X}+\varepsilon\mathbf{Z}}\Big)\Big].$$

Then, according to the previous proof,

$$\frac{\partial}{\partial\varepsilon}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}[Y^{i,j}_{\mathbf{X}+\varepsilon\mathbf{Z}}] = \mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}\Big(-w_{i,j}Z^i + \frac{1}{\alpha_i\beta_j}S^{\mathbf{Z}}_j\Big) - \frac{1}{\alpha_i\beta_j}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}+\varepsilon\mathbf{Z}}}(S^{\mathbf{Z}}_j),$$

and thus

$$\frac{\partial}{\partial\varepsilon}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}[Y^{i,j}_{\mathbf{X}+\varepsilon\mathbf{Z}}]\Big|_{\varepsilon=0} = \mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}[-w_{i,j}Z^i].$$

A.7 Proof of Proposition 3.2

By Theorem 3.1, for $i \in I_j$,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}[Y^{i,j}] &= \mathbb{E}\Big[Y^{i,j}\cdot\frac{\mathrm{d}\mathbb{Q}^j_{\mathbf{X}}}{\mathrm{d}\mathbb{P}}\Big] = \mathbb{E}\Big[\Big(-w_{i,j}X^i + \frac{1}{\alpha_i\beta_j}(S_j+d_j)\Big)\cdot\frac{e^{-S_j/\beta_j}}{\mathbb{E}[e^{-S_j/\beta_j}]}\Big]\\ &= -w_{i,j}\frac{\mathbb{E}(X^ie^{-S_j/\beta_j})}{\mathbb{E}(e^{-S_j/\beta_j})} + \frac{1}{\alpha_i\beta_j}\frac{\mathbb{E}(S_je^{-S_j/\beta_j})}{\mathbb{E}(e^{-S_j/\beta_j})} + \frac{1}{\alpha_i}\Big(\log\Big(\frac{\beta}{-B}\Big) + \log(\mathbb{E}(e^{-S_j/\beta_j}))\Big)\\ &= \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

Assuming everything is well-defined so that we can use the Leibniz integral rule, then we have the following results:

$$\frac{\partial}{\partial w_{i,j}}\mathbb{E}(e^{-S_j/\beta_j}) = \mathbb{E}\Big(-\frac{X^i}{\beta_j}e^{-S_j/\beta_j}\Big), \quad \frac{\partial}{\partial w_{i,j}}\mathbb{E}(S_je^{-S_j/\beta_j}) = \mathbb{E}\Big[X^ie^{-S_j/\beta_j} - S_j\frac{X^i}{\beta_j}e^{-S_j/\beta_j}\Big].$$

Compute the derivatives

$$\begin{aligned}\frac{\partial\mathrm{I}}{\partial w_{i,j}} &= \frac{\partial}{\partial w_{i,j}}\Big(-w_{i,j}\frac{\mathbb{E}(X^ie^{-S_j/\beta_j})}{\mathbb{E}(e^{-S_j/\beta_j})}\Big)\\ &= -\frac{\mathbb{E}(X^ie^{-S_j/\beta_j})}{\mathbb{E}(e^{-S_j/\beta_j})} - w_{i,j}\frac{\mathbb{E}\big(-\frac{(X^i)^2}{\beta_j}e^{-S_j/\beta_j}\big)\cdot\mathbb{E}(e^{-S_j/\beta_j}) - \mathbb{E}(X^ie^{-S_j/\beta_j})\cdot\mathbb{E}\big(-\frac{X^i}{\beta_j}e^{-S_j/\beta_j}\big)}{\big(\mathbb{E}(e^{-S_j/\beta_j})\big)^2}\\ &= -\frac{\mathbb{E}(X^ie^{-S_j/\beta_j})}{\mathbb{E}(e^{-S_j/\beta_j})} + \frac{w_{i,j}}{\beta_j}\frac{\mathbb{E}\big((X^i)^2e^{-S_j/\beta_j}\big)}{\mathbb{E}(e^{-S_j/\beta_j})} - \frac{w_{i,j}}{\beta_j}\Big(\frac{\mathbb{E}(X^ie^{-S_j/\beta_j})}{\mathbb{E}(e^{-S_j/\beta_j})}\Big)^2\\ &= -\mathbb{E}_{\mathbb{Q}^j_{\mathbf{X}}}[X^i] + \frac{w_{i,j}}{\beta_j}\operatorname{Var}_{\mathbb{Q}^j_{\mathbf{X}}}(X^i),\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathrm{II}}{\partial w_{i,j}} &= \frac{1}{\alpha_i\beta_j}\frac{\partial}{\partial w_{i,j}}\Big(\frac{\mathbb{E}(S_j e^{-S_j/\beta_j})}{\mathbb{E}(e^{-S_j/\beta_j})}\Big)\\
&= \frac{1}{\alpha_i\beta_j}\frac{1}{(\mathbb{E}(e^{-S_j/\beta_j}))^2}\Big(\mathbb{E}\Big[X^i e^{-S_j/\beta_j} - S_j\frac{X^i}{\beta_j}e^{-S_j/\beta_j}\Big]\cdot\mathbb{E}(e^{-S_j/\beta_j}) - \mathbb{E}(S_j e^{-S_j/\beta_j})\cdot\mathbb{E}\Big(-\frac{X^i}{\beta_j}e^{-S_j/\beta_j}\Big)\Big)\\
&= \frac{1}{\alpha_i\beta_j}\Big(\frac{\mathbb{E}(X^i e^{-S_j/\beta_j})}{\mathbb{E}(e^{-S_j/\beta_j})} - \frac{1}{\beta_j}\mathbb{E}\Big(X^i S_j\frac{e^{-S_j/\beta_j}}{\mathbb{E}(e^{-S_j/\beta_j})}\Big) + \frac{1}{\beta_j}\mathbb{E}\Big(X^i\frac{e^{-S_j/\beta_j}}{\mathbb{E}(e^{-S_j/\beta_j})}\Big)\cdot\mathbb{E}\Big(S_j\frac{e^{-S_j/\beta_j}}{\mathbb{E}(e^{-S_j/\beta_j})}\Big)\Big)\\
&= \frac{1}{\alpha_i\beta_j}\mathbb{E}_{\mathbb{Q}_{\mathbf{x}}^j}[X^i] - \frac{1}{\alpha_i\beta_j^2}\operatorname{Cov}_{\mathbb{Q}_{\mathbf{x}}^j}(X^i, S_j),\\
\frac{\partial \mathrm{III}}{\partial w_{i,j}} &= \frac{\partial}{\partial w_{i,j}}\Big(\frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) + \frac{1}{\alpha_i}\log(\mathbb{E}(e^{-S_j/\beta_j}))\Big) = \frac{1}{\alpha_i}\frac{1}{\mathbb{E}(e^{-S_j/\beta_j})}\cdot\mathbb{E}\Big(-\frac{X^i}{\beta_j}e^{-S_j/\beta_j}\Big)\\
&= -\frac{1}{\alpha_i\beta_j}\mathbb{E}_{\mathbb{Q}_{\mathbf{x}}^j}[X^i].
\end{aligned}$$

As a result,

$$\begin{aligned}
\frac{\partial \mathbb{E}_{\mathbb{Q}_{\mathbf{x}}^j}[Y^{i,j}]}{\partial w_{i,j}} &= \frac{\partial}{\partial w_{i,j}}(\mathrm{I}+\mathrm{II}+\mathrm{III})\\
&= -\mathbb{E}_{\mathbb{Q}_{\mathbf{x}}^j}[X^i] + \frac{w_{i,j}}{\beta_j}\operatorname{Var}_{\mathbb{Q}_{\mathbf{x}}^j}(X^i) + \frac{1}{\alpha_i\beta_j}\mathbb{E}_{\mathbb{Q}_{\mathbf{x}}^j}[X^i] - \frac{1}{\alpha_i\beta_j^2}\operatorname{Cov}_{\mathbb{Q}_{\mathbf{x}}^j}(X^i, S_j) - \frac{1}{\alpha_i\beta_j}\mathbb{E}_{\mathbb{Q}_{\mathbf{x}}^j}[X^i]\\
&= -\mathbb{E}_{\mathbb{Q}_{\mathbf{x}}^j}[X^i] - \frac{1}{\alpha_i\beta_j^2}\operatorname{Cov}_{\mathbb{Q}_{\mathbf{x}}^j}(X^i, S_j) + \frac{w_{i,j}}{\beta_j}\operatorname{Var}_{\mathbb{Q}_{\mathbf{x}}^j}(X^i), \quad i\in I_j.
\end{aligned}$$

A.8 Proof of Proposition 3.3

Define

$$\eta'_m = \sum_{k\in I_{m'}}\frac{w_{k,m'}}{w_{k,m}}\frac{1}{\alpha_k}.$$

By Theorem 3.1, for $k\in I_m$,

$$\sum_{k\in I_{m'}}\frac{w_{k,m'}}{w_{k,m}}Y^{k,m} = \frac{S_m+d_m}{\beta_m}\sum_{k\in I_{m'}}\frac{w_{k,m'}}{w_{k,m}}\frac{1}{\alpha_k} - \sum_{k\in I_{m'}}\frac{w_{k,m'}}{w_{k,m}}w_{k,m}X^k = \frac{S_m+d_m}{\beta_m}\eta'_m - \sum_{k\in I_{m'}}w_{k,m'}X^k.$$

Then

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_{\mathbf{x}}^m}\Big[\sum_{k\in I_{m'}}\frac{w_{k,m'}}{w_{k,m}}Y^{k,m}\Big]\\
&\quad= \mathbb{E}_{\mathbb{Q}_{\mathbf{x}}^m}\Big(\frac{\eta'_m}{\beta_m}S_m - \sum_{k\in I_{m'}}w_{k,m'}X^k\Big) + \frac{\eta'_m}{\beta_m}\beta_m\log\Big\{-\frac{\beta}{B}\mathbb{E}[e^{-S_m/\beta_m}]\Big\}\\
&\quad= \eta'_m\log\Big\{\exp\Big[\frac{1}{\eta'_m}\mathbb{E}_{\mathbb{Q}_{\mathbf{x}}^m}\Big(\frac{\eta'_m}{\beta_m}S_m - \sum_{k\in I_{m'}}w_{k,m'}X^k\Big)\Big]\Big\} + \eta'_m\log\Big\{-\frac{\beta}{B}\mathbb{E}[e^{-S_m/\beta_m}]\Big\}\\
&\quad\le \eta'_m\log\Big\{\mathbb{E}_{\mathbb{Q}_{\mathbf{x}}^m}\Big[\exp\Big(\frac{1}{\beta_m}S_m - \frac{1}{\eta'_m}\sum_{k\in I_{m'}}w_{k,m'}X^k\Big)\Big]\Big\} + \eta'_m\log\Big\{-\frac{\beta}{B}\mathbb{E}[e^{-S_m/\beta_m}]\Big\}\\
&\quad= \eta'_m\log\Big\{\mathbb{E}\Big[\frac{e^{S_m/\beta_m}e^{-S_m/\beta_m}\exp(-\frac{1}{\eta'_m}\sum_{k\in I_{m'}}w_{k,m'}X^k)}{\mathbb{E}[e^{-S_m/\beta_m}]}\Big]\Big\} + \eta'_m\log\Big\{-\frac{\beta}{B}\mathbb{E}[e^{-S_m/\beta_m}]\Big\}\\
&\quad= \eta'_m\log\Big\{-\frac{\beta}{B}\mathbb{E}\Big[\exp\Big(-\frac{1}{\eta'_m}\sum_{k\in I_{m'}}w_{k,m'}X^k\Big)\Big]\Big\}\\
&\quad< \beta'_m\log\Big\{-\frac{\beta'}{B}\mathbb{E}\Big[\exp\Big(-\frac{1}{\beta'_m}\sum_{k\in I_{m'}}w_{k,m'}X^k\Big)\Big]\Big\} \quad \text{if } \sum_{k\in I_{m'}}w_{k,m'}X^k \text{ is nonnegative}\\
&\quad:= d_{m'}.
\end{aligned} \tag{A.11}$$

In conclusion, if both $\sum_{k\in I_{m'}} w_{k,m'}X^k$ and $\sum_{k\in I_{m''}} w_{k,m''}X^k$ are nonnegative, the inequality holds for the risk allocations of subgroup $I_{m'}$, as well as $I_{m''}$. Otherwise, we have the inequality given by (A.11).

A.9 Necessary and sufficient condition for B in Remark 3.4

Here we show a necessary and sufficient condition for B to have a trivial Nash equilibrium through an example. We assume all risk factors are i.i.d. Gaussian random variables, where $\sigma_{ij} = 0$, $i \neq j$ and $\sigma_{ii} = \sigma$ for all i, j.

When all banks are in one group, i.e., $h = 1$, $\beta = \sum_{i=1}^{N} \frac{1}{\alpha_i}$,

$$\mathbb{E}_{\mathbb{Q}_{\mathbf{X}}}[Y_{\mathbf{X}}^1] = \mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^1}[Y_{\mathbf{X}}^{1,1}] = \frac{1}{\alpha_1}\log\Big(\frac{\beta}{-B}\Big) - \mu_1 + \frac{1}{\beta_1}\sigma - \frac{1}{2\beta_1^2\alpha_1}N\sigma. \tag{A.12}$$

When bank 1 decides to split and put some weights in another group, e.g. there exist $w_{1,1}, w_{1,2} > 0$ and $w_{1,1} + w_{1,2} = 1$, then $\beta' = \beta + \frac{1}{\alpha_1}$ and $\beta_1 = \beta$, $\beta_2 = \frac{1}{\alpha_1}$,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_{\mathbf{X}}}[Y_{\mathbf{X}}^1] &= \mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^1}[Y_{\mathbf{X}}^{1,1}] + \mathbb{E}_{\mathbb{Q}_{\mathbf{X}}^2}[Y_{\mathbf{X}}^{1,2}] \\ &= \frac{2}{\alpha_1}\log\Big(\frac{\beta'}{-B}\Big) - \mu_1 + \frac{w_{1,1}^2}{\beta_1}\sigma + \frac{w_{1,2}^2}{\beta_2}\sigma - \frac{1}{2\beta_1^2\alpha_1}((N-1)\sigma + w_{1,1}^2\sigma) - \frac{1}{2\beta_2^2\alpha_1}w_{1,2}^2\sigma. \end{aligned}\tag{A.13}$$

To have a trivial Nash equilibrium, for bank 1, it should hold that (A.13) ≥ (A.12), which gives

$$\frac{1}{\alpha_1}\log(-B) \le \frac{1}{\alpha_1}\log\Big(\frac{(\beta')^2}{\beta}\Big) - \Big[\frac{1}{\beta_1} - \Big(\frac{w_{1,1}^2}{\beta_1} + \frac{w_{1,2}^2}{\beta_2}\Big)\Big]\sigma - \frac{1}{2\alpha_1}\Big[\Big(\frac{w_{1,1}^2}{\beta_1^2} + \frac{w_{1,2}^2}{\beta_2^2}\Big) - \frac{1}{\beta_1^2}\Big]\sigma. \tag{A.14}$$

Then, by extending (A.14) to all banks, we can get the necessary and sufficient condition on B to have a trivial Nash equilibrium: for all $i = 1, \dots, N$, B satisfies

$$\frac{1}{\alpha_i}\log(-B) \le \frac{1}{\alpha_i}\log\Big(\frac{(\beta')^2}{\beta}\Big) - \Big[\frac{1}{\beta_1} - \Big(\frac{w_{i,1}^2}{\beta_1} + \frac{w_{i,2}^2}{\beta_2}\Big)\Big]\sigma - \frac{1}{2\alpha_i}\Big[\Big(\frac{w_{i,1}^2}{\beta_1^2} + \frac{w_{i,2}^2}{\beta_2^2}\Big) - \frac{1}{\beta_1^2}\Big]\sigma,$$

where $\beta_2 = \frac{1}{\alpha_i}$, $\beta_1 = \sum_{i=1}^{N}\frac{1}{\alpha_i}$ and $\beta' = \beta_1 + \beta_2$.

Recall that B is negative and stands for the minimal level of expected utility. Intuitively, when B is small, $\log(-B)$ is large, then some of the inequalities tend to be violated so that there would be no trivial Nash equilibrium in the system. On the other hand, when B is large (close to 0), $\log(-B)$ will be extremely small so that a trivial Nash equilibrium may exist in the system.

A.10 Remark for S_m in equation (2.16)

Remark A.1. Some results about S_m are the following:

$$\begin{aligned}\mathbb{E}(e^{-S_m/\beta_m}) &= \exp\Big(-\frac{1}{\beta_m}\mu_m^s + \frac{1}{2\beta_m^2}(\sigma_m^s)^2\Big), \\ \mathbb{E}(X^i e^{-S_m/\beta_m}) &= B\Big(\mu_i - \frac{1}{\beta_m}A_m\mathbf{\Sigma}_{[,i]}\Big)\exp\Big(-\frac{1}{\beta_m}\mu_m^s + \frac{1}{2\beta_m^2}(\sigma_m^s)^2\Big) \quad \text{for } i \in I_m \text{ (see below for a proof)}, \\ \mathbb{E}(S_m e^{-S_m/\beta_m}) &= \Big(\mu_m^s - \frac{1}{\beta_m}(\sigma_m^s)^2\Big)\exp\Big(-\frac{1}{\beta_m}\mu_m^s + \frac{1}{2\beta_m^2}(\sigma_m^s)^2\Big).\end{aligned}$$

Proof. Define $\mathbf{t}^T = (t_1, \dots, t_N)$. Then

$$\mathbb{E}(X^i e^{\mathbf{t}^T\mathbf{X}}) = \frac{\partial M_{\mathbf{X}}(\mathbf{t})}{\partial t_i} = \frac{\partial}{\partial t_i}\exp\Big(\boldsymbol{\mu}^T\mathbf{t} + \frac{1}{2}\mathbf{t}^T\mathbf{\Sigma}\mathbf{t}\Big) = (\mu_i + \mathbf{t}^T\mathbf{\Sigma}_{[,i]})\exp\Big(\boldsymbol{\mu}^T\mathbf{t} + \frac{1}{2}\mathbf{t}^T\mathbf{\Sigma}\mathbf{t}\Big).$$

Thus we obtain

$$\mathbb{E}(X^i e^{-S_m/\beta_m}) = \mathbb{E}(X^i e^{-(A_m/\beta_m)\mathbf{X}}) = \Big(\mu_i - \frac{1}{\beta_m}A_m\mathbf{\Sigma}_{[,i]}\Big)\exp\Big(-\frac{1}{\beta_m}\mu_m^s + \frac{1}{2\beta_m^2}(\sigma_m^s)^2\Big). \qquad \square$$

A.11 Sufficient condition for local optimal weights

To investigate condition (3.13) further, we make some reasonable assumptions on estimates and introduce some situations when they hold.

Assuming all $w_{k,1}, w_{k,2} \neq 0$ for all $k \neq i$, then $\beta_1 = \beta_2 = \sum \frac{1}{\alpha_k}$. Then (3.13) is equivalent to

$$-\Big(\frac{2}{\beta_1} - \frac{1}{\beta_1^2\alpha_i}\Big)\sigma_{ii} < \Big(\frac{1}{\beta_1^2\alpha_i} - \frac{1}{\beta_1}\Big)\sum_{k=1,\,k\neq i}^{N}(w_{k,1} - w_{k,2})\sigma_{ki} < \Big(\frac{2}{\beta_1} - \frac{1}{\beta_1^2\alpha_i}\Big)\sigma_{ii}.$$

Since

$$\frac{1}{\beta_1^2\alpha_i} - \frac{1}{\beta_1} < 0, \quad |(w_{k,1} - w_{k,2})\sigma_{ki}| \le |\sigma_{ki}| \quad \text{for all } k, i,$$

we can deduce a sufficient condition for $w^*_{(i)} \in (0, 1)$ for individual i,

$$\Big(1 - \frac{1}{\beta_1\alpha_i}\Big)\sum_{k=1,\,k\neq i}^{N}|\sigma_{ki}| < \Big(2 - \frac{1}{\beta_1\alpha_i}\Big)\sigma_{ii}. \tag{A.15}$$

Remark A.2. From above, we can have a rough estimation: if $\sigma_i \sim \sigma$, $\rho > 0$ and $\alpha = [1, 1, 1, 1]$ (i.e., no extremely large σ and no extremely small α), the inequality is true when $\rho < \frac{7}{9}$, according to

$$\Big(1 - \frac{1}{4}\Big)\cdot(4-1)\rho\sigma^2 < \Big(2 - \frac{1}{4}\Big)\sigma^2.$$

This explains why in numerical experiments, when we apply reasonable values of parameters, the optimal weights are often located between $(0, 1)$. And when $(w_{k,1}, w_{k,2})$ are close for most $k \neq i$, the weights for individual i are around 0.5 because of small $A \sim 0$ and $B_1 \sim B_2$ in this case.

If, for individual i, σ_i is small and $\sigma_i \ll \sigma_{i^0}$ for some i^0, then the sufficient condition does not hold, and by numerical results, we found $w^*_{(i)}$ is not in $(0, 1)$ anymore.

Assume for some k('s), $w_{k,1}, w_{k,2}$ can be 0 or 1. Then $\beta_1 \neq \beta_2$, but it is still true that $-1 \le w_{k,1} - w_{k,2} \le 1$ for all k. Then (3.13) can be rewritten as

$$-\Big(\frac{2}{\beta_2} - \frac{1}{\beta_2^2\alpha_i}\Big)\sigma_{ii} < \sum_{k=1,\,k\neq i}^{N}\Big(w_{k,1}\Big(\frac{1}{\beta_1^2\alpha_i} - \frac{1}{\beta_1}\Big) - w_{k,2}\Big(\frac{1}{\beta_2^2\alpha_i} - \frac{1}{\beta_2}\Big)\Big)\sigma_{ki} < \Big(\frac{2}{\beta_1} - \frac{1}{\beta_1^2\alpha_i}\Big)\sigma_{ii}.$$

A sufficient condition for $w^*_{(i)} \in (0, 1)$ for individual i is

$$\begin{cases} \Big(\frac{1}{\beta_1} - \frac{1}{\beta_1^2\alpha_i}\Big)\sum_{k=1,\,k\neq i}^{N}|\sigma_{ki}| < \Big(\frac{2}{\beta_2} - \frac{1}{\beta_2^2\alpha_i}\Big)\sigma_{ii}, \\ \Big(\frac{1}{\beta_2} - \frac{1}{\beta_2^2\alpha_i}\Big)\sum_{k=1,\,k\neq i}^{N}|\sigma_{ki}| < \Big(\frac{2}{\beta_1} - \frac{1}{\beta_1^2\alpha_i}\Big)\sigma_{ii}. \end{cases}$$

This is a generalization of (A.15).

A.12 Necessary and sufficient condition for optimal weights

First we compare the minimal risk over non-zero weights (3.14) with the corner case $(w_{i1}, w_{i2}) = (0, 1)$,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_{\mathbf{x}}}[Y^i]|_{w=w^*} - (3.10) = {} & \frac{2}{\alpha_i}\log\Big(\frac{\beta'}{-B}\Big) - \frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) \\ & + w^*\sum_{k=1,\,k\neq i}^{N}\Big[\Big(\frac{w_{k,1}}{\beta_1} - \frac{w_{k,2}}{\beta_2}\Big) - \Big(\frac{w_{k,1}}{\beta_1^2\alpha_i} - \frac{w_{k,2}}{\beta_2^2\alpha_i}\Big)\Big]\sigma_{ki} \\ & + \Big(\frac{(w^*)^2}{\beta_1} - \frac{(w^*)^2}{2\beta_1^2\alpha_i}\Big)\sigma_{ii} + ((1-w^*)^2 - 1)\Big(\frac{1}{\beta_2} - \frac{1}{2\beta_2^2\alpha_i}\Big)\sigma_{ii} \\ & - \frac{1}{2\beta_1^2\alpha_i}\sum_{m,k=1,\,m,k\neq i}^{N} w_{k,1}w_{m,1}\sigma_{km} \end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\alpha_i}\log\Big(\frac{\beta'}{-B}\Big) - \frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) + w^*\cdot(-A) + \frac{(w^*)^2}{2}B_1 + \frac{w^*(w^*-2)}{2}B_2 \\
&\qquad - \frac{1}{2\beta_1^2\alpha_i}\sum_{m,k=1,\,m,k\neq i}^{N} w_{k,1}w_{m,1}\sigma_{km} \quad \text{(use notation } A, B_1, B_2\text{)} \\
&= \frac{2}{\alpha_i}\log\Big(\frac{\beta'}{-B}\Big) - \frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) - w^*\cdot\underbrace{\Big(A - \frac{w^*}{2}B_1 - \frac{w^*-2}{2}B_2\Big)}_{=\Delta} \\
&\qquad - \frac{1}{2\beta_1^2\alpha_i}\sum_{m,k=1,\,m,k\neq i}^{N} w_{k,1}w_{m,1}\sigma_{km} \\
&= \frac{2}{\alpha_i}\log\Big(\frac{\beta'}{-B}\Big) - \frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) - \frac{(A+B_2)^2}{2(B_1+B_2)} - \frac{1}{2\beta_1^2\alpha_i}\sum_{m,k=1,\,m,k\neq i}^{N} w_{k,1}w_{m,1}\sigma_{km}.
\end{aligned}
\tag{A.16}
$$

Since $w^* = \frac{A+B_2}{B_1+B_2}$, we get $\Delta = A - \frac{w^*}{2}(B_1+B_2) + B_2 = \frac{A+B_2}{2}$.

We compare the minimal risk of non-boundary case (3.14) with the boundary case $(w_{i1}, w_{i2}) = (1, 0)$,

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}_X}[Y^i]|_{w=w^*} - (3.9) &= \frac{2}{\alpha_i}\log\Big(\frac{\beta'}{-B}\Big) - \frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) \\
&\qquad + (w^*-1)\sum_{k=1,\,k\neq i}^{N}\Big[\Big(\frac{w_{k,1}}{\beta_1} - \frac{w_{k,2}}{\beta_2}\Big) - \Big(\frac{w_{k,1}}{\beta_1^2\alpha_i} - \frac{w_{k,2}}{\beta_2^2\alpha_i}\Big)\Big]\sigma_{ki} \\
&\qquad + ((w^*)^2-1)\Big(\frac{1}{\beta_1} - \frac{1}{2\beta_1^2\alpha_i}\Big)\sigma_{ii} + (1-w^*)^2\Big(\frac{1}{\beta_2} - \frac{1}{2\beta_2^2\alpha_i}\Big)\sigma_{ii} \\
&\qquad - \frac{1}{2\beta_2^2\alpha_i}\sum_{m,k=1,\,m,k\neq i}^{N} w_{k,2}w_{m,2}\sigma_{km} \\
&= \frac{2}{\alpha_i}\log\Big(\frac{\beta'}{-B}\Big) - \frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) - (w^*-1)\Big(A - \frac{w^*+1}{2}B_1 - \frac{w^*-1}{2}B_2\Big) \\
&\qquad - \frac{1}{2\beta_2^2\alpha_i}\sum_{m,k=1,\,m,k\neq i}^{N} w_{k,2}w_{m,2}\sigma_{km} \\
&= \frac{2}{\alpha_i}\log\Big(\frac{\beta'}{-B}\Big) - \frac{1}{\alpha_i}\log\Big(\frac{\beta}{-B}\Big) - \frac{(A-B_1)^2}{2(B_1+B_2)} - \frac{1}{2\beta_2^2\alpha_i}\sum_{m,k=1,\,m,k\neq i}^{N} w_{k,2}w_{m,2}\sigma_{km}.
\end{aligned}
\tag{A.17}
$$

If both (A.17) and (A.16) are less than 0, we get conditions (3.15), which are the necessary and sufficient conditions to conclude that non-zero weights $(w^*, 1-w^*)$ are the optimal weights to minimize the total risk.

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References

[1] C. Albanese, Y. Armenti and S. Crépey, XVA metrics for CCP optimization, *Stat. Risk Model.* **37** (2020), no. 1–2, 25–53.

[2] H. Amini, R. Cont and A. Minca, Resilience to contagion in financial networks, *Math. Finance* **26** (2016), no. 2, 329–365.

[3] H. Amini, D. Filipović and A. Minca, Systemic risk and central clearing counterparty design, Swiss Finance Inst. Res. Paper 13-34, 2015.

[4] H. Amini, D. Filipović and A. Minca, To fully net or not to net: Adverse effects of partial multilateral netting, *Oper. Res.* **64** (2016), no. 5, 1135–1142.

[5] H. Amini, D. Filipović and A. Minca, Systemic risk in networks with a central node, *SIAM J. Financial Math.* **11** (2020), no. 1, 60–98.
[6] Y. Armenti, S. Crépey, S. Drapeau and A. Papapantoleon, Multivariate shortfall risk allocation and systemic risk, *SIAM J. Financial Math.* **9** (2018), no. 1, 90–126.
[7] M. Asmild and M. Zhu, Controlling for the use of extreme weights in bank efficiency assessments during the financial crisis, *European J. Oper. Res.* **251** (2016), no. 3, 999–1015.
[8] F. Biagini, J.-P. Fouque, M. Frittelli and T. Meyer-Brandis, A unified approach to systemic risk measures via acceptance sets, *Math. Finance* **29** (2019), no. 1, 329–367.
[9] F. Biagini, J.-P. Fouque, M. Frittelli and T. Meyer-Brandis, On fairness of systemic risk measures, *Finance Stoch.* **24** (2020), no. 2, 513–564.
[10] M. K. Brunnermeier and P. Cheridito, Measuring and allocating systemic risk, *Risks* **7** (2019), no. 2, Paper No. 46.
[11] A. Capponi, W. A. Cheng and S. Rajan, Systemic risk: The dynamics under central clearing, Working Paper, Office of Financial Research, 2015.
[12] R. Cifuentes, G. Ferrucci and H. S. Shin, Liquidity risk and contagion, *J. European Econ. Assoc.* **3** (2005), no. 2–3, 556–566.
[13] R. Cont and A. Minca, Credit default swaps and systemic risk, *Ann. Oper. Res.* **247** (2016), no. 2, 523–547.
[14] R. Cont, A. Moussa and E. B. Santos, Network structure and systemic risk in banking systems, in: *Handbook on Systemic Risk*, Cambridge University, Cambridge (2013), 327–368.
[15] B. Craig and G. von Peter, Interbank tiering and money center banks, *J. Financial Intermediation* **23** (2014), no. 3, 322–347.
[16] Z. Cui, Q. Feng, R. Hu and B. Zou, Systemic risk and optimal fee for central clearing counterparty under partial netting, *Oper. Res. Lett.* **46** (2018), no. 3, 306–311.
[17] L. Eisenberg and T. H. Noe, Systemic risk in financial systems, *Manag. Sci.* **47** (2001), no. 2, 236–249.
[18] M. Elliott, B. Golub and M. O. Jackson, Financial networks and contagion, *Amer. Econ. Rev.* **104** (2014), no. 10, 3115–53.
[19] Z. Feinstein and T. R. Hurd, Contingent convertible obligations and financial stability, preprint (2020), https://arxiv.org/abs/2006.01037.
[20] Z. Feinstein, B. Rudloff and S. Weber, Measures of systemic risk, *SIAM J. Financial Math.* **8** (2017), no. 1, 672–708.
[21] Y. Feng, M. Min and J.-P. Fouque, Deep learning for systemic risk measures, preprint (2022), https://arxiv.org/abs/2207.00739.
[22] P. Gai and S. Kapadia, Contagion in financial networks, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **466** (2010), no. 2120, 2401–2423.
[23] P. Gai and S. Kapadia, Liquidity hoarding, network externalities, and interbank market collapse, *Proc. Roy. Soc. A* **466** (2010), no. 2401–2423, Paper No. 439.
[24] A. G. Haldane and R. M. May, Systemic risk in banking ecosystems, *Nature* **469** (2011), no. 7330, 351–355.
[25] T. R. Hurd, *Contagion! Systemic Risk in Financial Networks*, Springer Briefs Quant. Finance, Springer, Cham, 2016.
[26] M. Kusnetsov and L. A. M. Veraart, Interbank clearing in financial networks with multiple maturities, *SIAM J. Financial Math.* **10** (2019), no. 1, 37–67.
[27] M. L. Lin and J. Surti, Capital Requirements for over-the-counter derivatives central counterparties, International Monetary Fund, 2013.
[28] A. Lipton, Systemic risks in central counterparty clearing house networks, Margin in Derivatives Trading, Risk. net, 2018.
[29] S. Weber and K. Weske, The joint impact of bankruptcy costs, fire sales an cross-holdings on systemic risk in financial networks, *Probab. Uncertain. Quant. Risk* **2** (2017), Paper No. 9.
[30] Financial Crisis Inquiry Commission, The financial crisis inquiry report: The final report of the National Commission on the causes of the financial and economic crisis in the United States including dissenting views, Cosimo, 2011.