Particle Motion under the Conservative Piece of the Self-Force is Hamiltonian

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We consider the motion of a point particle in a stationary spacetime under the influence of a scalar, electromagnetic, or gravitational self-force. We show that the conservative piece of the first-order self-force gives rise to Hamiltonian dynamics, and we derive an explicit expression for the Hamiltonian on phase space. Specialized to the Kerr spacetime, our result generalizes the Hamiltonian function previously obtained by Fujita *et al.* [Classical Quantum Gravity **34**, 134001 (2017).], which is valid only for nonresonant orbits.

Introduction.—The two-body problem in general relativity has been the focus of intense observational and theoretical interest in recent years. On the observational side, LIGO and VIRGO have detected several dozen coalescences of binary systems containing black holes and neutron stars [1–3] via the gravitational waves that they emit. The near future should bring many more detections from upgraded instruments, from the next generation ground based detectors Cosmic Explorer [4] and Einstein Telescope [5], from the space based detector LISA [6], and potentially from pulsar timing arrays [7]. On the theoretical side a wide variety of approaches valid in different regimes have been used to understand the dynamics of black hole binaries with ever greater precision: numerical relativity [8], the post-Newtonian approximation [9–12], the post-Minkowskian approximation [13] for which amplitude methods from quantum field theory have been fruitfully brought to bear [14], the small mass ratio approximation [15,16], and the effective one-body framework which synthesizes information from the other approaches [17,18].

An issue that arises in this field is whether one can define dissipative and conservative sectors of the dynamics for which the conservative sector admits a Hamiltonian description. While this is not possible in the fully nonlinear, dynamical regime, it has been achieved in the post-Newtonian and post-Minkowskian approximations to various orders, and it is a foundational assumption of the effective one-body framework. Its status within the small mass ratio regime, however, has been an open question beyond the leading order of geodesic motion. In that regime the small body is treated as a point particle, and the leading order self-force acting on that body is computed by taking a gradient of a suitably regularized version of the body's self-field [15,16], computed as a perturbation of the large black hole spacetime. That force can be split into time-even conservative and time-odd dissipative pieces. Hamiltonian descriptions of the conservative motion have been derived in special cases (orbits in the Schwarzschild spacetime [19] and nonresonant orbits in Kerr [20]). General orbits in Kerr however have been an open question.

In this Letter we show that the leading order conservative self-forced motion of a nonspinning body in any stationary spacetime admits a Hamiltonian description, and derive an explicit expression for the Hamiltonian. We then discuss a number of applications in the context of binary black holes: implications for our understanding of the integrability of the motion, and the identification of a new class of gauge invariant observables that may be useful for comparing different computational methods.

General result in Hamiltonian dynamics.—We start by deriving a general result in the theory of Hamiltonian systems. We define a *pseudo-Hamiltonian* dynamical system to consist of a phase space Γ, a closed, nondegenerate two form Ω_{AB}, and a smooth pseudo-Hamiltonian function $\mathcal{H}:\Gamma \times \Gamma \to \mathbf{R}$, for which the dynamics are given by integral curves of the vector field

$$v^A = \Omega^{AB} \frac{\partial}{\partial Q^B} \mathcal{H}(Q, Q')|_{Q'=Q}, \qquad (1)$$

where $\Omega^{AB}\Omega_{BC} = \delta_C^A$ and Q^A are coordinates on Γ. Pseudo-Hamiltonian systems need not be Hamiltonian, and can be used to describe dissipation [21].

We now specialize to a pseudo-Hamiltonian system which is a perturbation of a Hamiltonian system, with symplectic form and pseudo-Hamiltonian

$$\Omega_{AB} = \Omega_{0AB}, \qquad (2a)$$

$$\mathcal{H}(Q, Q') = H_0(Q) + \varepsilon \mathcal{H}_1(Q, Q') + O(\varepsilon^2). \quad (2b)$$

Here ε is a formal expansion parameter. We denote by $Q \to \varphi_\tau(Q)$ the zeroth order Hamiltonian flow, defined by the condition

$$\frac{d}{d\tau}\bigg|_{\tau=0}\varphi_\tau^A(Q) = \Omega_0^{AB}\partial_B H_0, \qquad (3)$$

and the group composition law

$$\varphi_\tau[\varphi_{\tau'}(Q)] = \varphi_{\tau+\tau'}(Q). \qquad (4)$$

The pseudo-Hamiltonian perturbation $\mathcal{H}_1$ is defined in terms of a function $G:\Gamma\times\Gamma\to\mathbf{R}$ via

$$\mathcal{H}_1(Q,Q') = \int_{-\infty}^{\infty} d\tau'\tilde{G}(0,Q,\tau',Q'), \qquad (5)$$

where we have defined

$$\tilde{G}(\tau,Q,\tau',Q') = G[\varphi_\tau(Q),\varphi_{\tau'}(Q')]. \qquad (6)$$

The function G is assumed to satisfy the conditions

$$G(Q,Q') = G(Q',Q), \qquad (7a)$$

$$\tilde{G}(\tau,Q,\tau',Q') \to 0 \quad \text{as } \tau \quad \text{or} \quad \tau'\to\pm\infty. \qquad (7b)$$

We now show that with these assumptions, the pseudo-Hamiltonian system (2) is Hamiltonian to linear order in ε. To do so we need to find a perturbed Hamiltonian $\tilde{H} = H_0 + \varepsilon\tilde{H}_1 + O(\varepsilon^2)$, and a perturbed symplectic form $\tilde{\Omega}_{AB} = \Omega_{0AB} + \varepsilon\tilde{\Omega}_{1AB} + O(\varepsilon^2)$, for which the equation of motion $dQ^A/d\tau = \tilde{\Omega}^{AB}\partial_B\tilde{H}$ coincides with that given by Eqs. (1) and (2) to $O(\varepsilon)$. This yields the requirement

$$\partial_B\tilde{H}_1 - \tilde{\Omega}_{1BC}\Omega_0^{CD}\partial_D H_0 = \frac{\partial}{\partial Q^B}\mathcal{H}_1(Q,Q')|_{Q'=Q}. \qquad (8)$$

We choose the perturbation to the symplectic form to be

$$\tilde{\Omega}_{1BC} = \left[\frac{\partial}{\partial Q^B}\frac{\partial}{\partial Q^C}\int d\tau\int d\tau'\chi(\tau,\tau')\tilde{G}(\tau,Q,\tau',Q')\right]_{Q'=Q} \qquad (9)$$

where

$$\chi(\tau,\tau') = \frac{1}{2}\text{sgn}(\tau) - \frac{1}{2}\text{sgn}(\tau'). \qquad (10)$$

Because of the antisymmetry property $\chi(\tau',\tau) = -\chi(\tau,\tau')$ and the symmetry property (7a) of G, the expression (9) defines a closed two form on phase space. Using the symplectic form perturbation (9) and the pseudo-Hamiltonian perturbation (5) we find that the requirement (8) reduces to

$$\partial_B\tilde{H}_1 = \left[\frac{\partial}{\partial Q^B}\int d\tau'\tilde{G}(0,Q,\tau',Q')\right]_{Q'=Q}$$
$$+ \Omega_0^{CD}\partial_D H_0\left[\frac{\partial}{\partial Q^B}\frac{\partial}{\partial Q^C}\int d\tau\int d\tau'\chi\tilde{G}(\tau,Q,\tau',Q')\right]_{Q'=Q}. \qquad (11)$$

We now proceed to simplify the second term in Eq. (11), in several stages. First, we bring the factor $\Omega_0^{CD}\partial_D H_0$ inside the square brackets and replace it with the tensor $\Omega_0^{C'D'}\partial'_D H_0$ at Q'. This replacement is valid because of the subsequent evaluation at $Q' = Q$. Second, we can replace the differential operator $\Omega_0^{C'D'}\partial'_D H_0\partial_{C'}$ using the zeroth order Hamiltonian flow (3). The second term becomes

$$\left\{\frac{\partial}{\partial Q^B}\frac{d}{d\Delta\tau'}\bigg|_{\Delta\tau'=0}\int d\tau\int d\tau'\chi\tilde{G}[\tau,Q,\tau',\varphi_{\Delta\tau'}(Q')]\right\}_{Q'=Q}. \qquad (12)$$

Third, using the definition (6) of $\tilde{G}$ together with the group property (4) of the Hamiltonian flow we have

$$\tilde{G}[\tau,Q,\tau',\varphi_{\Delta\tau'}(Q')] = \tilde{G}(\tau,Q,\tau'+\Delta\tau',Q'). \qquad (13)$$

Hence the term (12) can be rewritten as

$$\left[\frac{\partial}{\partial Q^B}\int d\tau\int d\tau'\chi\frac{d}{d\tau'}\tilde{G}(\tau,Q,\tau',Q')\right]_{Q'=Q}. \qquad (14)$$

Fourth, we integrate by parts with respect to τ' and make use of the condition (7b) to eliminate the boundary terms. The derivative of the expression (10) for the function χ gives a delta function, $d\chi/d\tau' = -\delta(\tau')$. The final result is

$$\left[\frac{\partial}{\partial Q^B}\int d\tau\tilde{G}(\tau,Q,0,Q')\right]_{Q'=Q}. \qquad (15)$$

Using the definition (6), the symmetry property (7a) and relabeling $\tau\to\tau'$ this can be written as

$$\left[\frac{\partial}{\partial Q^{B'}}\int d\tau'\tilde{G}(0,Q,\tau',Q')\right]_{Q'=Q}. \qquad (16)$$

Finally inserting this expression as a replacement for the second term in the condition (11), we see that the right hand side is now a total derivative, as desired, and the resulting expression for the perturbation to the Hamiltonian is

$$\tilde{H}_1(Q) = \int d\tau' \tilde{G}(0, Q, \tau', Q). \tag{17}$$

This completes the proof that the system (2) is Hamiltonian.

We can obtain a more convenient representation of this Hamiltonian system by making a linearized phase space diffeomorphism parametrized by the vector field $\varepsilon\xi^A$, under which we have

$$\tilde{H}_1 \to H_1 = \tilde{H}_1 + \mathcal{L}_\xi H_0, \tag{18a}$$

$$\tilde{\Omega}_{1AB} \to \Omega_{1AB} = \tilde{\Omega}_{1AB} + (\mathcal{L}_\xi \Omega_0)_{AB}. \tag{18b}$$

If we choose $\xi^A = \Omega_0^{AB}\eta_B$ then we find $\Omega_{1AB} = \tilde{\Omega}_{1AB} - \partial_A\eta_B + \partial_B\eta_A$. We now choose

$$\eta_A = \frac{1}{2}\left[\frac{\partial}{\partial Q^A}\int d\tau \int d\tau' \chi \tilde{G}(\tau, Q, \tau', Q')\right]_{Q'=Q}, \tag{19}$$

which yields from Eq. (9) that $\Omega_{1AB} = 0$. Hence the new symplectic form coincides with the unperturbed symplectic form:

$$\Omega_{AB} = \Omega_{0AB} + O(\varepsilon^2). \tag{20}$$

Similarly by inserting Eq. (19) into Eq. (18a) and simplifying using the same techniques as for Eq. (11) yields

$$H_1(Q) = \frac{1}{2}\int d\tau' \tilde{G}(0, Q, \tau', Q), \tag{21}$$

which differs from the original result (17) by a factor of 2.

Application to motion under the conservative self-force.—We now explain how the motion of a particle under the action of its conservative first-order gravitational self-force in a stationary spacetime (M, g_{ab}) can be cast as a pseudo-Hamiltonian system of the form (2), by slightly modifying the pseudo-Hamiltonian construction of Fujita *et al.* [20]. This will allow us to apply our Hamiltonian result [Eqs. (20) and (21)].

For the zeroth order geodesic motion we use phase space coordinates (x^μ, p_μ) with symplectic form $\Omega_0 = dp_\mu \wedge dx^\mu$ and Hamiltonian

$$H_0 = -\sqrt{-g^{\mu\nu}(x)p_\mu p_\nu}. \tag{22}$$

The time parameter τ associated with this Hamiltonian is then proper time normalized with respect to g_{ab}, while the conserved value of $-H_0$ is the mass of the particle.

For the first-order motion, consider a particle at location $x^{\mu'}$ with initial 4-momentum $p_{\mu'}$. Writing $Q' = (x', p')$, we

denote by $\varphi_{\tau'}(Q') = [x^{\bar\mu}(\tau'), p_{\bar\mu}(\tau')]$ the geodesic with initial data Q'. From this geodesic we can compute the Lorenz gauge metric perturbation

$$h^{\mu\nu}(x; Q') = \frac{1}{\sqrt{-g^{\mu'\nu'}p_{\mu'}p_{\nu'}}}\int d\tau' G^{\mu\nu\bar\mu\bar\nu}[x, x'(\tau')]p_{\bar\mu}(\tau')p_{\bar\nu}(\tau').$$

Here the symmetric Green's function $G^{\mu\nu\bar\mu\bar\nu}$ is the average of the retarded and advanced Green's functions, regularized according to the Detweiler-Whiting prescription [15,22]. The conservative forced motion of the particle is then equivalent at linear order to geodesic motion in the metric $g_{\mu\nu} + h_{\mu\nu}$, where Q' is held fixed when evaluating the geodesic equation and then evaluated at $Q' = Q$ [16,22].

We can therefore obtain a pseudo-Hamiltonian description of the dynamics by replacing the metric $g_{\mu\nu}(x)$ in Eq. (22) with $g_{\mu\nu}(x) + h_{\mu\nu}(x, Q')$. Expanding to linear order in $h_{\mu\nu}$, comparing with Eqs. (2b), (5), and (6), and setting to unity the formal expansion parameter ε we can read off the function $G(Q, Q')$ on phase space to be

$$G(Q, Q') = -\frac{G^{\mu\nu\mu'\nu'}(x, x')p_\mu p_\nu p_{\mu'}p_{\nu'}}{2\sqrt{-g^{\lambda\sigma}p_\lambda p_\sigma}\sqrt{-g^{\lambda'\sigma'}p_{\lambda'}p_{\sigma'}}}. \tag{23}$$

Similar constructions work for scalar and electromagnetic self-forces. For a particle endowed with a scalar charge q and electromagnetic charge e we replace the initial Hamiltonian expression (22) with $-\sqrt{-g^{\mu\nu}(p_\mu - eA_\mu)(p_\nu - eA_\nu)} - q\Phi$. The expression (23) gets replaced by $-q^2 G_{\rm sc}(x, x')$ in the scalar case, where $G_{\rm sc}$ is the scalar Green's function, and with

$$-\frac{e^2 G^{\mu\mu'}(x, x')p_\mu p_{\mu'}}{\sqrt{-g^{\lambda\sigma}p_\lambda p_\sigma}\sqrt{-g^{\lambda'\sigma'}p_{\lambda'}p_{\sigma'}}}$$

in the electromagnetic case, where $G^{\mu\mu'}$ is the Lorenz gauge electromagnetic Green's function.

The function satisfies (23) the symmetry property (7a). It will also satisfy the decay property (7b) if the retarded Green's function falls off at late times at fixed spatial position. This is known to be true for scalar fields in a class of stationary spacetimes [23], while for black holes it is a lore of the field that perturbations decay at late times as a power law [24]. This decay was shown for the Weyl scalars in black hole spacetimes by Barack [25], and it is also generally believed to be true for tensor perturbations, although it has not yet been established rigorously; see Refs. [26,27] for recent developments.

From this pseudo-Hamiltonian formulation it follows that the motion under the conservative self-force is described by the Hamiltonian (21), in any stationary spacetime for which the retarded Green's function goes to zero at late times.

Specialization to motion near a black hole.—Specialize now to the motion of a particle orbiting a Kerr black hole. In this context it is useful to derive an explicit form for the Hamiltonian in action angle variables.

We use the variables $(q^\alpha, j_\alpha) = (q^t, q^r, q^\theta, q^\phi, j_t, j_r, j_\theta, j_\phi)$ defined in Refs. [28,29], deformed via Eq. (19). In these variables the symplectic form is $\Omega = dj_\alpha \wedge dq^\alpha$, and the full Hamiltonian from Eqs. (21) and (22) is

$$H = H_0(j_\alpha) + H_1(q^\alpha, j_\alpha). \quad (24)$$

The zeroth order geodesic motion is given by $q^\alpha(\tau) = q_0^\alpha + \Omega_0^\alpha(j)\tau$, $j_\alpha = \text{const}$, where $\Omega_0^\alpha = \partial H_0/\partial j_\alpha$ are the zeroth order frequencies.

We now fix a value m of the conserved quantity $-H$, which is the mass of the particle to leading order. For describing motion on the mass shell $H = -m$ it will be convenient to define rescaled versions of the symplectic form and Hamiltonian,

$$\hat{\Omega}_{AB} = \Omega_{AB}/m, \qquad \hat{H} = H/m. \quad (25)$$

This rescaling preserves Hamilton's equations. Using the fact that under the transformation $(x^\mu, p_\mu) \to (x^\mu, sp_\mu)$ with $s > 0$ we have $(q^\alpha, j_\alpha) \to (q^\alpha, sj_\alpha)$ [29], $H_0 \to sH_0$, and $H_1 \to s^2 H_1$ [cf. Eq. (23)], the dynamical system can be written as

$$\hat{\Omega} = dJ_\alpha \wedge dq^\alpha, \qquad \hat{H} = \hat{H}_0(J) + m\hat{H}_1(q, J), \quad (26)$$

where $J_\alpha = j_\alpha/m$.

Next, the Hamiltonian (26) is independent of q^t because of the symmetries of the Kerr background. This allows a description in terms of a six-dimensional Hamiltonian system instead of the eight-dimensional system (26), following the method of Sec. 45 of Ref. [30]. The new system has symplectic form $\Omega = dJ_i \wedge dq^i$, time parameter q^t, and Hamiltonian $h = h(q^i, J_i)$ defined by

$$\hat{H}[q^i, -h(q^i, J_i), J_i] = -1 \quad (27)$$

where $q^i = (q^r, q^\theta, q^\phi)$. From Eqs. (25) and (27) we obtain

$$h(q^i, J_i) = h_0(J_i) + mh_1(q^i, J_i) + O(m^2) \quad (28)$$

where h_0 and h_1 are given by $\hat{H}_0(-h_0, J_i) = -1$ and $h_1 = \hat{H}_1(q^i, -h_0, J_i)/\Omega_0^t$. The zeroth order frequencies are now $\omega_0^i = \partial h_0/\partial J_i = \Omega_0^i/\Omega_0^t$.

The Hamiltonian perturbation h_1 is independent of q^ϕ due to the symmetry of the Kerr background, and can be expanded in Fourier modes on the torus parametrized by $\mathbf{q} = (q^r, q^\theta)$:

$$h_1(\mathbf{q}, J_i) = \sum_{k_r=-\infty}^{\infty} \sum_{k_\theta=-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{q}} h_{1\,\mathbf{k}}(J_i). \quad (29)$$

It is possible to obtain an explicit formula for the coefficients $h_{1\,\mathbf{k}}$ starting from a Fourier expansion of the function (23) in action angle variables

$$G(q, J, q', J')$$
$$= \int d\omega \sum_{m,\mathbf{k},\mathbf{k}'} e^{-i\omega(q^t - q^{t'}) - im(q^\phi - q^{\phi'})} e^{i\mathbf{k}\cdot\mathbf{q} + i\mathbf{k}'\cdot\mathbf{q}'} G_{\omega m \mathbf{k}\mathbf{k}'}(J, J').$$

Combining this with Eqs. (21), (27), (28), and (29) gives

$$h_{1\,\mathbf{k}} = \frac{\pi}{(\Omega_0^t)^2} \sum_{m,\mathbf{l}} G_{\omega m(\mathbf{k}/2 + \mathbf{l}/2)(\mathbf{k}/2 - \mathbf{l}/2)}(J_t, J_i, J_t, J_i), \quad (30)$$

where we sum over all pairs of integers $\mathbf{l} = (l_r, l_\theta)$ for which $k_r + l_r$ and $k_\theta + l_\theta$ are even, and we evaluate at $J_t = -h_0(J_i)$ and at $\omega = m\omega_0^\phi + (\mathbf{k} - \mathbf{l}) \cdot \omega_0/2$.

Application: Integrability of dynamics.—We now turn to discussing some applications. Since the motion is Hamiltonian one can ask whether it is also integrable. It will be integrable to linear order if and only if all the resonant mode amplitudes vanish, that is,

$$h_{1\,\mathbf{k}}(J_i) = 0 \quad \text{whenever } \mathbf{k} \cdot \omega_0(J_i) = 0, \qquad \mathbf{k} \neq \mathbf{0}. \quad (31)$$

This condition is easy to prove: under a linearized canonical transformation with generating function $G(\mathbf{q}, J_i) = \sum_{\mathbf{k}} \exp[i\mathbf{k}\cdot\mathbf{q}] G_{\mathbf{k}}(J_i)$ we have $h_{1\,\mathbf{k}} \to h_{1\,\mathbf{k}} + i(\mathbf{k} \cdot \omega_0)G_{\mathbf{k}}$. Thus choosing $G_{\mathbf{k}}(J_i) = -ih_{1\,\mathbf{k}}(J_i)/\mathbf{k}\cdot\omega_0$ yields $h_{1\,\mathbf{k}} = 0$ for all nonzero $\mathbf{k}$ and thus an integrable system (The resulting Hamiltonian coincides with that found by Ref. [20], who excluded resonances), and this choice is possible without divergences only when the condition (31) is satisfied. Conversely, if the system is integrable there must exist perturbed versions $J_i + m\delta J_i$ of the action variables which have vanishing Poisson brackets with the Hamiltonian $h_0 + mh_1$, which yields at linear order the relation

$$k_i h_{1\,\mathbf{k}} = (\mathbf{k} \cdot \omega_0)\delta J_{i\,\mathbf{k}} \quad (32)$$

between Fourier components, enforcing the condition (31).

An alternative version of the integrability condition (31) is that the average of the conservative time derivative of the Carter constant $Q(J_i)$ over any orbit on any resonant torus should vanish. Computing a time derivative using Eqs. (28) and (29) gives $dQ/d\tau = \Omega_0^t(\partial Q/\partial J_i)dJ_i/dq^t = -i\Omega_0^t(\partial Q/\partial J_i)\sum_{\mathbf{k}} k_i h_{1\,\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{q}}$. Now using $\mathbf{q}(\tau) = \mathbf{q}_0 + \Omega_0\tau$, writing the resonant vectors as $\mathbf{k} = N\mathbf{k}_0 = N(n, -p, 0)$ for integers N and taking an orbit average gives [31]

$$\left\langle \frac{dQ}{d\tau} \right\rangle = -i\Omega_0^t \left(n\frac{\partial Q}{\partial J_r} - p\frac{\partial Q}{\partial J_\theta} \right) \sum_{N=-\infty}^{\infty} N h_{1N\mathbf{k}_0} e^{iN q_{\text{res}}}, \quad (33)$$

where $q_{\text{res}} = \mathbf{k}_0 \cdot \mathbf{q}_0 = nq_0^r - pq_0^\theta$ is the resonant combination of the phases. The left hand side vanishing for all q_{res} is equivalent to all the resonant amplitudes $h_{1N\mathbf{k}_0}$ vanishing.

One of us conjectured in Ref. [32] that the linear integrability condition (31) is satisfied in Kerr, based on the fact that enhanced symmetries present in the post-Newtonian limit enforce this condition. However, this was a weak argument, since it is possible for symmetries to be present only near the boundary of phase space that corresponds to the post-Newtonian limit, and not in the interior (just as for asymptotic spacetime symmetries). Indeed, recently Nasipak and Evans have shown numerically that $\langle dQ/d\tau \rangle = 0$ fails for conservative scalar self-forces in Kerr on resonances [33,34]. The gravitational self-force case is presumably similar, although this will need to be confirmed numerically (see Ref. [35]).

If the gravitational case is indeed nonintegrable, the qualitative consequences for the conservative dynamics are well understood in general contexts from the theory of weakly perturbed Hamiltonian systems [30,36]. They have been explored in the contexts of tidal and other perturbations to extreme mass ratio inspirals in Refs. [37–40]. Suppose we focus attention on one resonant torus $J_i = J_i^*$ and neglect the effect of other resonances. First, away from this torus the invariant tori $J_i = $ constant are deformed [cf. Eq. (32)] but preserved (as predicted by the Kolmogorov–Arnold–Moser (KAM) theorem [30]). Second, within a shell of width $J_i - J_i^* \sim \sqrt{m}$ the dynamics is altered: in the $m \to 0$ limit the resonant torus is destroyed and replaced by a number of islands of size $\sim\sqrt{m}$ in phase space within which the motion is integrable [40] [This can be seen explicitly in the description of the near-resonance dynamics derived by van de Meent, Eq. (18) of Ref. [41], dropping the dissipative terms (the first term on the right hand side and half of the oscillatory terms); the solutions consist of rotational or librational (islands) motions, depending on the energy.]. One can define action angle variables within each island, but they do not join continuously onto the global action angle variables. At finite m chaotic regions develop within the shell. Third, motion that starts within the shell is confined to remain within it by the surrounding surviving invariant tori, since the system is effectively two dimensional (J_ϕ is conserved) [36]. There are no large excursions to $J_i - J_i^* \sim O(1)$, unlike in higher dimensions.

When one considers the full $O(m)$ dynamics with the dissipative component of the self-force included, the nonintegrable mode coefficients $h_{1\mathbf{k}}$ can drive transient resonances which give $O(\sqrt{m})$ kicks to the action variables J_i [32], and also sustained resonances in which the orbit evolves along a nonadiabatic path in the space of parameters J_i maintaining the condition $\mathbf{k} \cdot \omega_0(J_i) = 0$ [41]. However neither of these are smoking gun signatures of the breakdown of integrability, since both can be produced when $h_{1\mathbf{k}} = 0$ by the oscillatory dissipative components of the self-force [41].

Application: Gauge invariant observables.—Gauge invariant observables such as invariant redshifts, frequencies of innermost stable circular orbits, etc. have proven enormously useful for cross-checks between different computational methods [16]. The simple form (21) of our Hamiltonian may be helpful for computing such observables, since one expects the complicated phase space coordinate transformation (19) not to be relevant for gauge invariant observables.

For generic orbits, nonintegrability of the dynamics would impede the definition of such observables. For example one can no longer label orbits by their three fundamental frequencies of motion. However new gauge invariant observables do arise in this context, the resonant amplitudes $h_{1\mathbf{k}}$ themselves [See Eq. (30) for a general expression for these amplitudes in terms of the Green's function, and Eq. (33) for a prescription for resonant amplitudes in terms of the self-force.], for which the action-angle variables are defined geometrically at zeroth order [29] and which at first order are invariant under linearized phase space coordinate transformations. These observables are not accessible from within post-Newtonian or post-Minkowski theory, but could be useful for comparisons between self-force theory and numerical relativity. They could for example be extracted from the kicks in the conserved quantities obtained from numerical waveforms.

Conclusion.—We have shown that the conservative dynamics of the two-body problem in general relativity in the small mass ratio limit is Hamiltonian to the first subleading order, when the small body is nonspinning. It would be interesting to extend this result to include the spin of the small body, and to second-order conservative self-forces.

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