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A general framework for gravitational charges and holographic renormalization

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We develop a general framework for constructing charges associated with diffeomorphisms in gravitational theories using covariant phase space techniques. This framework encompasses both localized charges associated with space–time subregions, as well as global conserved charges of the full space–time. Expressions for the charges include contributions from the boundary and corner terms in the subregion action, and are rendered unambiguous by appealing to the variational principle for the subregion, which selects a preferred form of the symplectic flux through the boundaries. The Poisson brackets of the charges on the subregion phase space are shown to reproduce the bracket of Barnich and Troessaert for open subsystems, thereby giving a novel derivation of this bracket from first principles. In the context of asymptotic boundaries, we show that the procedure of holographic renormalization can be always applied to obtain finite charges and fluxes once suitable counterterms have been found to ensure a finite action. This enables the study of larger asymptotic symmetry groups by loosening the boundary

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conditions imposed at infinity. We further present an algorithm for explicitly computing the counterterms that renormalize the action and symplectic potential, and, as an application of our framework, demonstrate that it reproduces known expressions for the charges of the generalized Bondi–Metzner–Sachs algebra.

Keywords: Gravitational charges; asymptotic symmetries; holographic renormalization; covariant phase space.

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1. Introduction and Summary

Canonical methods in general relativity and other gravitational theories provide an important tool for understanding the theory's observables and degrees of freedom. These methods are particularly well-suited for characterizing the subtle role played by diffeomorphisms, which serve as the gauge symmetries of these theories. The gauge nature of diffeomorphisms is captured by the fact that, in the absence of boundaries, they generate transformations on the gravitational phase space corresponding to degenerate directions of the presymplectic form; equivalently, the Hamiltonians generating these diffeomorphisms vanish on-shell. Introducing boundaries, either at infinity or finite locations in space–time, partially breaks the full diffeomorphism invariance of theory, and results in nontrivial charges associated with the broken gauge symmetries. The nonzero contribution to the charges comes purely from an integral over the boundary of the space–time region, which is a manifestation of the familiar fact that the on-shell Hamiltonian is a pure boundary term in diffeomorphism-invariant theories.

An important technical tool for investigating properties of diffeomorphism invariance is the covariant phase space formalism.[1–7] Its advantage over other constructions of gravitational phase spaces is the fact that covariance is maintained throughout. This allows the consequences of diffeomorphism invariance to be easily discerned, the most important of which is the localization of diffeomorphism charges to contributions from the boundary. These boundary charges find applications in a number of questions in gravitational physics, including black hole entropy,[6–13] asymptotic symmetries,[14,15] entanglement and edge modes,[16–19] and holography.[20–22] Given the breadth of scenarios in which boundary charges find use, it is important to have a well-defined framework that constructs these charges in an unambiguous manner. Unfortunately, there are a number of complications that arise related to ambiguities in the formalism, renormalization at asymptotic boundaries, and equivocal definitions of charges, that have led to differing results and conclusions regarding boundary charges in various contexts. The goal of this work is to develop a general framework that addresses these complications and sharply characterizes the choices that must be made to resolve the various ambiguities.

One major motivation for having such a framework is its applications to holography in asymptotically flat space–time, an arena in which the Hamiltonian formulation can provide important insights.[23] One can approach holography in a bottom-up manner, wherein one uses knowledge of the symmetries and charges of the theory at asymptotic boundaries to extrapolate properties of a putative dual theory. The classic example of this is the discovery by Brown and Henneaux that the asymptotic charge algebra of AdS_3 gravity coincides with the Virasoro algebra of CFT_2.[24] In a similar manner, systematically understanding the symmetries and charges at null infinity could help characterize the structure of the boundary theory. In particular, motivated by the UV/IR correspondence of the standard AdS/CFT dictionary, one might hope that a prescription for IR renormalization of classical observables using the Hamiltonian formalism leads to insights on universal properties of the putative boundary theory in the UV. This procedure is known as holographic renormalization,[21,25–28,a] and its use has been expanded to asymptotically flat applications; for example, it is needed in order to obtain finite charges associated with the generalized BMS group.[29–31]

When describing boundary charges, it is often useful to distinguish between *global* charges and *localized* charges.[32] Given a set of boundary conditions that define a phase space, global charges are given by integrals over a complete Cauchy surface. They include contributions from all the degrees of freedom of the theory, and generate the corresponding symmetry on the global phase space.[32] They are integrals over the codimension-2 boundaries of the Cauchy surface, which will typically be a sum over cross-sections of all the codimension-1 boundaries in the space–time that the Cauchy slice intersects.

Localized charges instead arise when defining a phase space associated with a subsystem of the full theory, such as when considering a subregion of space–time. Standard examples include: the interior of a timelike tube in space–time, as it occurs in the Brown–York quasilocal charge construction;[33] the exterior region of a finite null hypersurface;[13,32] and the domain of dependence of a partial Cauchy surface ending on a cut of $\mathscr{I}^+$ in asymptotically flat space–times.[34] Such subsystems are fundamentally open Hamiltonian systems, which interact through their boundary with degrees of freedom of the complementary region. Because of this interaction, the subregion symplectic form is not conserved under evolution along the boundary, and hence there is no integrable charge generating the diffeomorphism associated with this evolution on the subsystem phase space. Instead, localized charges are defined as a best approximation for the generator of the diffeomorphism on the subregion.

A procedure for defining localized charges in the covariant phase space was put forward by Wald and Zoupas,[34] and subsequently developed in Refs. 13 and 32. These charges satisfy a modification of Hamilton's equation in which the symplectic form evaluated on a diffeomorphism variation yields the variation of the charge, plus an additional term representing the flux. In order to produce unambiguous results, a criterion must be given for separating the charge from the flux in this equation,

[a]The name holographic renormalization arose because the formalism originated in the context of holographic dualities between bulk and boundary theories. However the formalism itself as used here does not require any such dualities and can be defined in purely classical contexts.

determining this criterion is the main challenge in obtaining well-defined localized charges. An additional set of independent ambiguities, known as Jacobson–Kang–Myers (JKM) ambiguities,[35] arises in the definitions of the theory's Lagrangian and symplectic potential, and naively affects both localized and global charges. One would like to have a coherent framework in which all the ambiguities are resolved through a single unified principle.

We will show that the crucial ingredient is the choice of action for the subregion, including boundary and corner terms. Equivalently, this can be viewed as a preferred choice for the symplectic flux at each of the boundaries, which appear as the boundary terms in the variational principle for the chosen action. The idea to use the action principle to resolve ambiguities in the covariant phase was first proposed in Ref. 36, motivated by holographic considerations in asymptotically AdS space–times.[21,26–28] This principle is also partially inspired by Euclidean gravity, wherein one takes the action to be the fundamental object from which all other observables are computed. Furthermore, it ties in with the Brown–York construction of quasilocal charges,[33] in which the subregion action plays a central role, and one can show that these quasilocal charges agree with the canonical charges constructed when utilizing the action principle to fix their ambiguities.[37] This perspective based on the full subregion action will allow us to resolve both sets of localized charge ambiguities in one fell swoop. Moreover, it will enable us to give a simple general argument that holographic renormalization can always be performed to obtain finite charges and fluxes, without imposing *any* boundary conditions on the field variations beyond those contained in the equations of motion. Indeed, as explained in Ref. 38, such generality is one of the main novelties that the holographic approach brings to the study of gravitational charges. Thus, our framework unifies many different aspects of gravitational charges in diffeomorphism-invariant theories.

In what follows, we give a detailed summary of each of our main results.

1.1. *Extended summary of results*

We begin in Sec. 2 by presenting the general framework for utilizing the covariant phase space in constructing gravitational charges. While much of the material in this section is a review, we present a number of results for handling background structures in the theory, which modify a number of formulas by noncovariant contributions.[b] The reasons for allowing noncovariances are twofold. First, as was shown in Ref. 13, central extensions in gravitational charge algebras arise due to noncovariant boundary terms in the action, and such extensions often contain critical information about properties of the theory. Second, allowing for noncovariance extends the applicability of the covariant phase space to noncovariant formulations of the theory, such as the ADM formulation,[40] facilitating a straightforward comparison between the formulations.

[b]Noncovariant corrections to covariant phase space quantities have also been explored in Ref. 39, which contains some overlap with the results of Sec. 2.

The main objective of Sec. 2 is to arrive at unambiguous expressions for the gravitational charges. Ambiguities can arise in two related but conceptually distinct ways. The first are the JKM ambiguities,[35] which occur in the formulation of the covariant phase space by Wald and collaborators[5–7,34] due to the fact that various quantities, such as the Lagrangian or the symplectic current, are defined only up to addition of exact differential forms. We demonstrate in Subsec. 2.4 that the gravitational charges can be defined in such a way as to be completely invariant under the JKM transformations, including transformations involving noncovariant quantities. This provides a powerful link between covariant and noncovariant formulations of the theory, since any two formulations can be viewed as being related by a JKM transformation. This then demonstrates that the charges are not sensitive to the specific choices made in setting up the canonical framework.

The second set of ambiguities occurs for localized charges constructed via the Wald–Zoupas procedure.[34] These charges depend on the form of the flux through the boundary of the subregion, and a prescription is needed to fix the expression for the flux. Wald and Zoupas gave a proposal called the *stationarity requirement* for fixing the ambiguity, which requires that the decomposition of the symplectic potential be chosen such that the flux vanishes identically in stationary space–times. This condition, along with a requirement on the covariance properties of the flux, was shown to yield unambiguous localized charges for BMS generators in 4D asymptotically flat space–time.[34] On the other hand, there has been much recent interest in extended symmetry algebras at null infinity,[29–31,41–45] which were missed in older analyses due to imposition of diffeomorphism-freedom conditions at the boundary that do not correspond to degeneracy directions of the symplectic form and are thus not true gauge degrees of freedom. One can demonstrate that the stationarity and covariance requirements do not produce finite charges associated with these extended symmetries,[46] and for sufficiently permissive boundary conditions, the stationarity requirement may either fail, or not fully fix all possible ambiguities in the flux. This motivates finding an alternative for fixing the flux ambiguities.

We therefore focus in this work on a different resolution that is more closely tied to the variational principle associated with the subregion. This resolution was first proposed by Compère and Marolf[36] (see also Refs. 47 and 48), motivated by the covariant Peierls bracket construction that far predates the more modern treatments of the covariant phase space.[20,49–52] These ideas were subsequently expanded upon and formalized in the work of Harlow and Wu[53] and the extension of this construction to Wald–Zoupas localized charges was recently described by two of us.[13] It has also been employed in applications of extended symmetries of asymptotically AdS spaces and their flat space limits in Refs. 54 and 55.[c] The variational

[c]A related approach described in Refs. 18 and 39 absorbs all boundary terms in the action into a bulk Lagrangian. Often, this produces results consistent with the action variational principle, but it lacks some of the flexibility of the present formulation, requires arbitrary choices in how to extend the boundary term into the bulk, and cannot handle the corner improvements described in Subsec. 2.5.

principle pertains to the full action for the subsystem, involving an integral of the Lagrangian in the bulk plus additional boundary terms, which are chosen to ensure the action is stationary for a given choice of boundary conditions. For a closed system, the boundary conditions are essential in determining the dynamics of the theory. Localized subregions instead behave like open systems due to the presence of symplectic flux through the boundary, and in this case boundary conditions should not be imposed, as they would unnecessarily constrain the dynamics. Nevertheless, the boundary contribution in the variation of the action is used to describe the flux through the boundary, and hence the form of the flux is largely determined by the choice of boundary condition one would have to impose if viewing the subregion as a closed system.

From the viewpoint of the variational principle, resolving the ambiguities in the covariant phase space formalism thus amounts to finding a preferred form for the flux, or, equivalently, to a preferred boundary condition one would impose if treating the system as closed. A particularly natural choice is to require that the flux be of Dirichlet form, meaning it depends algebraically on variations of the intrinsic variables on the boundary. For example, at a timelike boundary in theories where the only dynamical field is the metric, the Dirichlet condition implies that the flux takes the form $\mathcal{E} = \pi^{ij}\delta h_{ij}$, where h_{ij} is the induced metric and π^{ij} can involve both intrinsic and extrinsic quantities. Similarly, on a null surface, the Dirichlet form of the flux is $\mathcal{E} = \pi^{ij}\delta q_{ij} + \pi_i \delta n^i$, where q_{ij} is the degenerate induced metric and n^i is the null generator. Arguments in favor of the Dirichlet form of the flux were presented in Ref. 13, and include the connection to junction conditions at a surface, the semiclassical description of the path integral when gluing subregions, and a straightforward relation to the Brown–York and holographic constructions. For most of this work, we focus on the Dirichlet form of the flux, but emphasize that most of the formal constructions work for other choices corresponding to different boundary conditions, although these other choices yield different values of the charges and can affect their algebra. The dependence of gravitational charges on the choice of boundary conditions was recently verified in Ref. 56, which explored the effect of imposing Neumann and York conformal boundary conditions as opposed to Dirichlet.

The demonstration in Subsec. 2.4 that the action, symplectic form, and localized charges are all insensitive to ambiguities is then performed by working out how the individual contributions to each of these quantities change under JKM transformations once the expression for the flux has been fixed. We also introduce a class of *boundary canonical transformations*, which resemble the JKM transformations, but act nontrivially on the form of the flux, and hence change expressions for the charges. Because these boundary canonical transformations change the subregion action, this emphasizes that different choices of action generically produce different charges. A careful treatment of the definition of all quantities involved in constructing the localized charges reveals an additional set of corner ambiguities in the charges described in Subsecs. 2.1 and 2.5, that naively affect the values of the charge. We further demonstrate in Subsec. 2.5 that a corner improvement term in the localized charges fixes this ambiguity as well.

Having obtained ambiguity-free expressions for the charges, we proceed in Sec. 3 to determine the algebra they satisfy. This algebra can be defined by way of the bracket introduced by Barnich and Troessaert in Ref. 57 (henceforward referred to as the BT bracket), where it was postulated as a sensible choice that reproduces the algebra satisfied by the vector fields generating the diffeomorphisms on space–time, up to extensions. We present a new result deriving this bracket from first principles by identifying it as the Poisson bracket of the localized charges on the subregion phase space. This derivation relies on the flux being of Dirichlet form, but the arguments continue to hold for a class of alternative forms of the flux, subject to certain conditions. The bracket of the localized charges in general does not close, but instead produces additional generators $K_{\xi,\zeta}$ that yield an extension of the algebra satisfied by the space–time vector fields. Explicit expressions for the extension terms are given in Eqs. (3.11) and (3.19), which are consistent with the expressions originally derived in Ref. 13, suitably generalized to allow noncovariances in the bulk Lagrangian. We further show that the brackets between the new generators $K_{\xi,\zeta}$ and the localized charges H_ξ coincides with the bracket postulated by Barnich and Troessaert, as long as the generators $K_{\xi,\zeta}$ depend only on intrinsic variables at the surface when employing the Dirichlet flux condition. This requirement is nontrivially satisfied for charges constructed at null surfaces in general relativity, which serves as a consistency check on the use of the BT bracket.

The final sections of this paper are devoted to charges constructed at asymptotic boundaries. In Sec. 4, as a segue into holographic renormalization, we review a number of asymptotic symmetry algebras that have been proposed for 4D asymptotically flat space. Our presentation focuses on the different universal structures each algebra preserves, and we specifically analyze the cases of the standard BMS group, the generalized BMS group,[29,30] and the recently proposed Weyl BMS group,[45] which in fact coincides with the symmetry group obtained in Ref. 32 for finite null boundaries. Detailed derivations of these universal structures and their associated symmetry groups are given in App. E.

We then turn to an analysis of the holographic renormalization procedure that is needed to obtain finite results for asymptotic charges and their fluxes. This procedure can be viewed as finding a boundary canonical transformation that renders the action finite, after which all JKM-invariant quantities are finite as well. We further show that a JKM transformation can be performed to make each individual term in the expressions for the charges finite as well. It has often been remarked that one reason for imposing boundary conditions on fields at asymptotic boundaries is to ensure that the charges and fluxes have a finite limit to the boundary. The framework of holographic renormalization instead provides a different perspective:[38] one should allow for the most general asymptotic expansion of the dynamical fields that are consistent with the equations of motion, and handle any divergences using the counterterms that renormalize the action. It was first demonstrated by Compère and Marolf that in asymptotically AdS space, the resulting symplectic structure obtained via the holographic renormalization procedure is finite for all fluctuations of the dynamical fields, which further implies the charges and fluxes are finite as well, consistent with previous results on holographic asymptotic charges.[21,26–28] In Subsec. 5.2, we show that this argument applies quite generally to any asymptotic boundary, and give a general argument that the fluxes and charges are finite once a set of boundary terms that renormalize the action have been found. In Subsec. 5.3 we show that holographic renormalization can always be successfully carried out, by giving an algorithm for computing the terms that one must add to the symplectic potential and Lagrangian to obtain finite renormalized quantities. It is impossible to simultaneously maintain covariance and achieve finiteness, so our renormalized quantities break covariance through dependence on a choice of background structure. This is entirely analogous to the situation in AdS/CFT, where renormalized asymptotic charges necessarily depend on the choice of radial cutoff surface, which translates into the appearance of the Weyl anomaly on the boundary.[26,27,58] Finally, in Sec. 6, we apply the formalism described in Subsec. 5.3 to explicitly compute the renormalized symplectic potential and the localized charges associated with the generalized BMS group in vacuum general relativity in 4D asymptotically flat space–times.

We conclude in Sec. 7 with several points of discussion and avenues for future work.

1.2. *Notation*

Unless otherwise stated, we will work in $d+1$ space–time dimensions with metric signature $(-,+,+,\cdots)$. We will use the indices a,b,c for $(d+1)$-dimensional tensors in space–time and i,j,k for d-dimensional tensors intrinsic to a surface embedded in space–time. The conformal factor in our notation will be denoted by Φ (instead of the more commonly used symbol, Ω, which we will reserve for the symplectic form). We will use $\mathscr{I}$ to denote null infinity in asymptotically flat space–times, $\mathscr{I}^+$ where we specialize to future null infinity, and $\hat{=}$ to denote equality on $\mathscr{I}^+$ (or more generally on a null surface). The null normal to a null surface will be denoted by n_a, and the auxiliary null vector on a null surface will be denoted by l^a. κ is used to denote the inaffinity associated with a null vector and is defined by $n^a\nabla_a n^b \hat{=} \kappa n^b$. Often an index free notation will be used to denote differential forms, although the indices will be made explicit where convenient. For example, $\eta \equiv \eta_{i_1 i_2\cdots i_d}$ and $\mu \equiv \mu_{i_1 i_2\cdots i_{d-1}}$ will denote the volume forms on codimension-1 and codimension-2 surfaces, respectively. We will use $i_v\eta$ to denote the inner product of a vector field, v^a, with a differential form (in this case η). On occasion, the contracted indices will be displayed while the uncontracted indices will be left implicit. In other places, where convenient, all of the indices will be made explicit. In summary, we will freely use any of the expressions $i_v\eta$, $v^i\eta_i$, $v^i\eta_{ij_2\cdots j_d}$ to denote the contraction of v^i into the form η.

Table 1. A summary of the various differential forms that are defined in our covariant phase space formalism, showing their space–time degrees and phase space ($\mathscr{S}$) degrees, along with the equations where they are first introduced. We generally employ a convention where Greek or calligraphic letters denote forms with phase space degree greater than zero, and Latin letters denote forms of phase space degree zero. See the paragraph above (2.5) for the meaning of the prime notation, the paragraph below (2.4) for the meaning of the ^{c} superscript, and footnote g for the meaning of the ^{vc} superscript.

$\mathscr{S}$ degree \ Space–time degree	$(d+1)$	d	$(d-1)$	$(d-2)$
0	L' (2.5), $\overset{c}{L}$ (2.8a)	b' (2.7a), $\overset{c}{r}$ (2.9a), ℓ' (2.12), J'_ξ (2.21), $\overset{vc}{J}_\xi$ (2.23), a (2.36a), B (2.38a)	e (2.9a), $\overset{vc}{Q}_\xi$ (2.25), Q'_ξ (2.26), h_ξ (2.30), f (2.40a), c' (2.43), $\tilde{h}_\xi$ (2.46)	
1		θ (2.5), $\overset{c}{\theta}$ (2.8b), $\mathcal{E}$ (2.12)	λ' (2.7b), $\overset{c}{\rho}$ (2.9b), ν (2.10), β' (2.12), Λ (2.38a), ε (2.43)	χ (2.9b), γ' (2.43), μ_ξ (5.20b), ζ (2.51)
2		ω' (2.11)		

Pullbacks to surfaces will be denoted using underlines, i.e. the pullback of θ to a surface will be denoted by $\underline{\theta}$. When working with the covariant phase space, $\mathscr{F}$ will be used to denote the field configuration space of a theory, while $\mathscr{S}$ will represent the space of field configurations that satisfy the equations of motion. Operations on it including $L_{\hat{\xi}}$, δ, $I_{\hat{\xi}}$, and $\Delta_{\hat{\xi}}$ will be defined in Subsec. 2.1, and capitalized calligraphic letters $\mathscr{A}$, $\mathscr{B}$, ... will be used as abstract indices on $\mathscr{S}$. Note also that for simplicity, we will not distinguish between "pre-symplectic" and "symplectic" for quantities defined on the pre-phase space and the true phase space (see the second paragraph of Subsec. 2.1 for details). Finally, Table 1 lists various differential forms used in this paper along with their degrees on phase space and on space–time, and the equations where they first appear.

Finally, when dealing with subregions, it is important to keep track of the orientations of the various components of its boundary, for which we follow the conventions of Ref. 53. Beginning with the codimension-0 subregion $\mathcal{U}$ with $\mathcal{N}$ a null or timelike component of the boundary, we choose the orientation of $\mathcal{N}$ to be that induced as part of $\partial\mathcal{U}$. The orientation of a spatial surface Σ inside of $\mathcal{U}$ whose boundary intersects $\mathcal{N}$ will be oriented as part of the boundary of its past, and the codimension-2 surface $\partial\Sigma$ defining a cut of $\mathcal{N}$ will inherit the induced orientation as a boundary of Σ. Note that this means that $\partial\Sigma$ has the opposite orientation as that induced as part of the boundary of its past in $\mathcal{N}$. We define the volume form η on $\mathcal{N}$ to be one consistent with this choice of orientation, and similarly define μ on $\partial\Sigma$ to be consistent with its orientation. See App. C for the details of these volume forms when $\mathcal{N}$ is a null surface.

2. Gravitational Charges at Finite Boundaries

In any gravitational theory defined on a space–time region with boundary, there are nonzero charges associated with diffeomorphisms that act near the boundary. Depending on the context, one can distinguish between two related notions of charges, namely, *global* charges and *localized* charges. Global charges are defined when the space–time region under consideration can be viewed as a closed system, which occurs when considering the entire space–time, or else working with a subregion of space–time on which boundary conditions are imposed to prevent any interaction with the complementary region. These charges generate the symmetry transformation of their associated diffeomorphism on phase space via Hamilton's equation, and are conserved under time evolution. On the other hand, localized charges are defined for a subregion of space–time, which is not assumed to be isolated from its complement. Such charges need not be conserved due to the presence of nonzero fluxes through the boundary, and in general will not faithfully generate the transformation associated with the diffeomorphism. Nevertheless, these localized charges provide useful notions of quasilocal energy and momentum for subregions in phase space, and, as we will discuss, satisfy an algebra that closely resembles the diffeomorphism algebra of their corresponding vector fields.

Despite the distinctions, the two notions of charges are not entirely independent of each other. Instead, a global charge can be viewed as a special case of a localized charge, in which the space–time region is specialized to a closed system and the fluxes of the charge vanish. For this reason, we will focus in this work on the more general construction of localized charges, and simply mention at various points how the construction can be specialized to global charges.

This section reviews the construction of localized gravitational charges using covariant phase space techniques. The procedure was initially developed by Wald and Zoupas,[34] and in this work we specifically focus on a number of recent developments on the handling of boundaries in the covariant phase space that have led to resolutions of the various ambiguities that can appear in the formalism.[13,36,47,48,53] The resolution comes from demanding that the symplectic potential $\mathcal{E}$ describing the flux through the subregion's boundary be of Dirichlet form. We will demonstrate explicitly that this fixes both the standard JKM ambiguities present in the covariant phase space formalism,[7,35] as well as the additional ambiguity in identifying the flux when employing the Wald–Zoupas procedure. In fact, we will see that the formalism is invariant under generically *noncovariant* JKM transformations, which, in particular, allows for formulations involving a bulk Lagrangian that is not space–time covariant, such as in the ADM formulation of the theory.[40] This provides maximal flexibility in identifying charges, allowing one to switch between a covariant or noncovariant formulation depending on the application; invariance under JKM transformations ensures that the final result for the charges will not depend on this intermediate choice. We also describe in Subsec. 2.5 a resolution of an additional set of ambiguities involving corner contributions to the action, leading to an improved set of localized charges. These corner-improved charges generalize the proposal of Ref. 13 to allow for a noncovariant bulk Lagrangian and symplectic potential.

Throughout this section, we assume that boundaries are at finite locations in space–time, and that all quantities have finite limits to the boundaries. This assumption excludes asymptotic boundaries such as spatial infinity or future null infinity in asymptotically flat space–times, which can be brought to a finite location in space–time via conformal compactification, at the expense of having some of the dynamical fields diverge on the boundary. Later in Sec. 5, we will discuss the modifications and generalizations of the formalism that are necessary to handle asymptotic boundaries, based on the technique of holographic renormalization.

2.1. *Covariant phase space*

We begin with a brief review of the covariant phase space construction[1–7] in order to establish notation, which largely coincides with that used in Ref. 13, and to point to places where we generalize the standard treatments. For recent reviews and more in-depth discussions of the covariant phase space, see Refs. 32 and 53.

The idea behind the covariant phase space is to provide a canonical description of a field theory defined on a manifold $\mathcal{M}$ without breaking covariance by singling out a foliation of constant-time slices, as it is done in more standard phase space constructions. This is achieved by working with the space $\mathscr{S}$ of all field configurations satisfying the equations of motion, viewed as a subspace of the space $\mathscr{F}$ of all field configurations. In a globally hyperbolic space–time, each solution in $\mathscr{S}$ can be identified, up to gauge transformations, with its initial data defined on a Cauchy slice Σ, and since this initial data comprises the usual phase space of the theory, we see that there is a canonical identification between $\mathscr{S}$ modulo gauge transformations and the standard noncovariant phase space.[d] Since the phase space arises as a quotient of $\mathscr{S}$ by the action of the gauge group, we will find that $\mathscr{S}$ has the structure of a pre-phase space, on which we will construct a pre-symplectic form that has degenerate directions. Most calculations will be done on $\mathscr{S}$, bearing in mind that eventually the quotient must be taken to arrive at expressions for the true phase space. Throughout this work, we will drop the "pre" label for objects defined on $\mathscr{S}$, and simply point out where it is important to distinguish between the pre-phase space and true phase space.

The spaces $\mathscr{F}$ and $\mathscr{S}$ are infinite-dimensional manifolds, on which certain standard differential geometry concepts are well-defined. The dynamical fields ϕ (which will later be taken to consist of the metric and any matter fields) define a collection of functions on field space, and the gradients of these functions are denoted $\delta\phi$. Differential forms of higher degree on field space can then be constructed by taking

[d] We will later consider subregions of space–time which are not globally hyperbolic, so this identification will not hold in those cases, but the construction nevertheless will allow us to define a sensible notion of phase space for the subregion.

wedge products, and we will employ the notation where the product $\alpha\beta$ of two field-space differential forms is always assumed to be a field-space wedge product, and hence satisfies $\alpha\beta = (-1)^{ab}\beta\alpha$, where a and b are the respective form degrees of α and β. The operator δ then defines an exterior derivative on the space of field-space differential forms in the usual way. Vector fields are defined by infinitesimal variations of the field configuration, and since vectors tangent to solution space $\mathscr{S}$ must preserve the equations of motion, they are parametrized by solutions of the linearized field equations. Given a vector field V on $\mathscr{S}$, we denote the operation of contraction with a differential form by I_V, so that in particular $I_V\delta\phi$ gives a phase space function that returns the linearized solution corresponding to V around each background solution. We can also take Lie derivatives along a given vector field V in field space, which we denote L_V, and its action on differential forms can be computed via Cartan's magic formula,

$$L_V = I_V\delta + \delta I_V \,. \tag{2.1}$$

Our main focus in this work will be diffeomorphism-invariant theories. Infinitesimal diffeomorphisms are generated by vector fields ξ^a on space–time, and they act on fields via the space–time Lie derivative $\pounds_\xi\phi$. Diffeomorphism invariance implies that $\pounds_\xi\phi$ is a solution to the linearized field equations, and hence defines a vector field on $\mathscr{S}$, denoted $\hat{\xi}$, through the equation $I_{\hat{\xi}}\delta\phi = \pounds_\xi\phi$. The vector field ξ^a can itself be viewed as a function on field space, and often it is taken to be a constant, meaning $\delta\xi^a = 0$. However, in many applications it is useful to consider transformations generated by field-dependent diffeomorphisms, for which $\delta\xi^a \neq 0$. The Lie bracket $[\hat{\xi},\hat{\zeta}]_{\mathscr{F}}$ on field space of the vectors $\hat{\xi}$ associated with field-dependent ξ^a is given by (see App. A)

$$[\hat{\xi},\hat{\zeta}]_{\mathscr{F}} = -\widehat{[\![\xi,\zeta]\!]}\,, \tag{2.2}$$

$$[\![\xi,\zeta]\!]^a = [\xi,\zeta]^a - I_{\hat{\xi}}\delta\zeta^a + I_{\hat{\zeta}}\delta\xi^a \,. \tag{2.3}$$

This expression employs the modified Lie bracket $[\![\cdot\,,\cdot]\!]$ introduced in Ref. 59, and its relation to the field space Lie bracket was noted in Ref. 60. Since the vectors $\hat{\xi}$ are tangent to the solution space submanifold $\mathscr{S}$ in $\mathscr{F}$, the bracket $[\hat{\xi},\hat{\zeta}]_{\mathscr{S}}$ is also given by (2.2).

We will be interested in objects defined on field space that may not transform covariantly under diffeomorphisms. Noncovariances arise in objects that depend on a background structure such as a nondynamical field. Being nondynamical means that such a field is constant in field space, and hence $L_{\hat{\xi}}$ acts trivially on it. In order to track the lack of covariance of a field space differential form, it is useful to define the anomaly operator $\Delta_{\hat{\xi}}$, first introduced in Ref. 61, which acts on field space differential forms constructed from local fields as[e]

$$\Delta_{\hat{\xi}} = L_{\hat{\xi}} - \pounds_\xi - I_{\widehat{\delta\xi}} \,. \tag{2.4}$$

[e]The operator $I_{\widehat{\delta\xi}}$ acts on the local field variations as $I_{\widehat{\delta\xi}}\delta\phi = \pounds_{\delta\xi}\phi$ (see App. A for additional details).

This operator provides a means for replacing field space Lie derivatives $L_{\hat{\xi}}$ with space–time Lie derivatives $\pounds_\xi$, keeping track of the anomalous transformation of an object when doing so. A covariant object is one that satisfies $\Delta_{\hat{\xi}}\alpha = 0$, so, for example, since the dynamical fields are covariant, the statement $\Delta_{\hat{\xi}}\phi = 0$ is equivalent to the oft-used identity $L_{\hat{\xi}}\phi = \pounds_\xi\phi$. On the other hand, a nondynamical field ψ satisfies $L_{\hat{\xi}}\psi = 0$ even though the space–time Lie derivative is generically nonzero. In this case, the anomaly is given by $\Delta_{\hat{\xi}}\psi = -\pounds_\xi\psi$. When it is important to emphasize that a certain object is fully covariant, we will denote it with an overset c, as in $\overset{c}{\alpha}$; hence, for any such quantity, one may always assume $\Delta_{\hat{\xi}}\overset{c}{\alpha} = 0$.

The dynamics of the theory is specified in terms of its Lagrangian L', taken to be a top form on space–time, so that the action is given by $\int_\mathcal{M} L'$ up to boundary terms. As we will discuss shortly, various quantities that we will consider depend on ambiguities in the definition of the Lagrangian and related quantities, and we employ the notation that quantities that depend on these ambiguities are indicated with a prime, as in L'. Any primed quantity should be assumed to be noncovariant in general. Varying the Lagrangian yields the field equations and symplectic potential θ' for the theory according to

$$\delta L' = E \cdot \delta\phi + d\theta' \,. \tag{2.5}$$

The solution space $\mathscr{S}$ which will serve as the pre-phase space for the theory consists of all field configurations satisfying the field equations $E = 0$. Our main focus will be theories whose field equations are diffeomorphism-invariant, meaning $\Delta_{\hat{\xi}}(E \cdot \delta\phi) = 0$. A condition that guarantees diffeomorphism invariance is that the Lagrangian be covariant up to an exact term, $\Delta_{\hat{\xi}} L' = da'_\xi$. We will further restrict attention to theories in which the anomalous term a'_ξ can be written as the anomalous transformation of some other quantity defined on the boundary, $a'_\xi = \Delta_{\hat{\xi}} b'$. This implies that there exists a choice of Lagrangian that differs from L' by an exact term, $\overset{c}{L} = L' - db'$, and is fully covariant, $\Delta_{\hat{\xi}}\overset{c}{L} = 0$.[f] Iyer and Wald have shown that whenever there is a covariant Lagrangian, one can find a symplectic potential $\overset{c}{\theta}$ that is covariant as well, $\Delta_{\hat{\xi}}\overset{c}{\theta} = 0$.[7] The covariant symplectic potential can differ from θ' by the addition of an exact term and a total variation, and hence

[f] This assumption precludes theories such as topologically massive gravity[62,63] whose Lagrangians are not covariant for any choice of boundary term due to the presence of Chern–Simons-like terms, but nevertheless yield diffeomorphism invariant field equations. The most general definition of a diffeomorphism-invariant theory would be one whose equations of motion satisfy $\Delta_{\hat{\xi}}(E \cdot \delta\phi) = 0$, which, in light of Eq. (2.5), implies the anomaly of the Lagrangian need only satisfy

$$\Delta_{\hat{\xi}}\delta L' = d\Delta_{\hat{\xi}}\theta' \,. \tag{2.6}$$

Given that the formalism is invariant under addition of noncovariant boundary terms, as discussed in Subsec. 2.4, it seems likely that most of the results described in this work can be extended to this more general class of diffeomorphism-invariant theories. It would be interesting to analyze such generalizations in more detail, for example, as explored in Ref. 39.

there must exist quantities b' and λ' satisfying the equations

$$\Delta_{\hat{\xi}} L' = d\Delta_{\hat{\xi}} b' \,, \tag{2.7a}$$

$$\Delta_{\hat{\xi}} \theta' = \Delta_{\hat{\xi}} \delta b' + d\Delta_{\hat{\xi}} \lambda' \,. \tag{2.7b}$$

For a given Lagrangian L' and symplectic potential θ', Eqs. (2.7a) and (2.7b) will be taken as the definitions of b' and λ'. Once b' and λ' satisfying these equations have been found, the associated covariant Lagrangian and symplectic potential are defined to be

$$\overset{c}{L} = L' - db' \,, \tag{2.8a}$$

$$\overset{c}{\theta} = \theta' - \delta b' - d\lambda' \,. \tag{2.8b}$$

Equations (2.7a) and (2.7b) fix b' and λ' in terms of L' and θ' up to shifts of the form

$$b' \to b' + \overset{c}{r} + de \,, \tag{2.9a}$$

$$\lambda' \to \lambda' - \delta e + \overset{c}{\rho} + d\chi \tag{2.9b}$$

with $\overset{c}{r}$ and $\overset{c}{\rho}$ covariant and e and χ generically noncovariant. However, we will see below that the localized charges and other relevant quantities do not depend on the freedom to shift by the covariant quantities $\overset{c}{r}$, $\overset{c}{\rho}$, nor on the shift in λ' by $d\chi$. In principle, the charges *are* sensitive to the shift by e if $\Delta_{\hat{\xi}} e \neq 0$, but this can be resolved using a more refined treatment of corner terms, as explained in Subsec. 2.5.

Finally, we mention that the standard ambiguities that appear when working with L' and θ' arise from the fact that any other Lagrangian that differs from L' by an exact term, $L' + da'$, yields the same equation of motion, and hence is an equally valid choice for defining the bulk dynamics. For such a shifted Lagrangian, any shifted symplectic potential of the form

$$\theta' + \delta a' + d\nu' \tag{2.10}$$

will satisfy the relation (2.5), and hence defines a valid symplectic potential. These freedoms to shift L' and θ' are often presented as ambiguities in the covariant phase space formalism;[7,35] however, it has recently been understood that such ambiguities may be resolved by specifying the form of the boundary condition one would impose to ensure vanishing symplectic flux through the boundary of the subregion.[13,36,47,48,53] This resolution is explored in detail in Subsec. 2.4, where it is shown that the charges, fluxes, and subregion action all involve combinations of the various objects that are manifestly invariant under these shifts.

2.2. *Symplectic form*

Before constructing localized charges associated with a subregion, we must first restrict the solution space to the subregion, and equip it with a symplectic structure. To this end, we let $\mathcal{U}$ denote the open set in $\mathcal{M}$ defining the subregion of interest, whose boundary includes a timelike or null component $\mathcal{N}$. There may be additional boundaries to the future and past of $\mathcal{U}$, and, although these do not play a major role in the construction of charges in this work, these additional boundaries will become important when considering more detailed resolutions of corner ambiguities, as discussed in Subsec. 2.5. We will restrict attention to the space of solutions within the subregion $\mathcal{U}$, with no boundary conditions imposed at $\mathcal{N}$. We denote this restricted solution space by $\mathscr{S}_{\mathcal{U}}$.

We now consider spatial slices Σ in $\mathcal{U}$ whose boundaries $\partial\Sigma$ lie in $\mathcal{N}$. We will define a symplectic form Ω associated with $\partial\Sigma$ as an integral over Σ and $\partial\Sigma$. The resulting localized phase spaces $(\mathscr{S}_{\mathcal{U}}, \Omega)$ will serve as the starting point for constructing localized charges, and it is important to remember that they depend on both the subregion solution space $\mathscr{S}_{\mathcal{U}}$ as well as a choice of cut of the boundary.

Two specific examples that illustrate this general framework are as follows. First, we take $\mathcal{U}$ to be a globally hyperbolic, asymptotically flat space–time, $\mathcal{N}$ to be future null infinity $\mathscr{I}^+$, and Σ to be an asymptotically null slice which intersects $\mathscr{I}^+$ in some cut $\partial\Sigma$.[34] Second, we take $\mathcal{U}$ to be a timelike tube in space–time, $\mathcal{N}$ to be the timelike boundary $\partial\mathcal{U}$ of the tube, and Σ to be a spatial slice whose boundary $\partial\Sigma$ lies in $\mathcal{N}$. This second example is the context for the Brown–York quasilocal charge construction.[33] Note that in both of these examples, the subregion solution space $\mathscr{S}_{\mathcal{U}}$ is not in one-to-one correspondence with the space of initial data on Σ. This is a general feature of the framework, since Σ is generally not a Cauchy surface for the subregion. In the timelike tube example this arises because we have not imposed any boundary conditions on $\partial\mathcal{U}$.

The symplectic form will be constructed as a sum of two terms, one capturing the bulk contribution and one involving a boundary contribution. The bulk term is constructed as the integral over a spatial slice Σ through $\mathcal{U}$ of the symplectic current,

$$\omega' = \delta\theta' \, . \tag{2.11}$$

To determine the boundary contribution, we first consider the pullback $\underline{\theta}'$ of the symplectic potential to $\mathcal{N}$, and decompose it into three terms

$$\underline{\theta}' \mathrel{\widehat{=}} -\delta\ell' + d\beta' + \mathcal{E} \, , \tag{2.12}$$

where we refer to ℓ' as the *boundary term*, β' as the *corner term*, and $\mathcal{E}$ as the *flux term*. The reason for this terminology relates to the variational principle for the subregion. Neglecting contributions from past and future boundaries, the action for the subregion $\mathcal{U}$ is defined to be

$$S = \int_{\mathcal{U}} L' + \int_{\mathcal{N}} \ell' \, . \tag{2.13}$$

Varying this action and applying Eqs. (2.5) and (2.12), we find

$$\delta S = \int_{\mathcal{U}} E \cdot \delta\phi + \int_{\mathcal{N}} \mathcal{E} + \int_{\partial\mathcal{N}} \beta' \tag{2.14}$$

and hence it is stationary both with the bulk field equations hold, $E \cdot \delta\phi = 0$ and when the flux through the boundary vanishes, $\mathcal{E} = 0$. The corner term β' localizes to the past and future boundaries of $\mathcal{N}$, and in a complete treatment, additional corner contributions to the action should be added at the codimension-2 boundaries of $\mathcal{N}$ and the past and future boundaries, as described in e.g. Refs. 64–66. Although not crucial to the remaining discussion of this paper, these corner contributions to the action can produce some modifications to the formalism, as described in Subsec. 2.5.

Without specifying the form of the flux term $\mathcal{E}$, Eq. (2.12) is ambiguous, since we can always shift it by exact terms and total variations $\mathcal{E} \to \mathcal{E} + \delta B - d\Lambda$ by making compensating changes to ℓ' and β'. These changes affect the subregion action (2.13), as well as the definitions of the charges, and hence to avoid such ambiguities, it is paramount to specify a criterion for selecting a preferred choice for $\mathcal{E}$. In making such a choice, it is important to realize that the form of $\mathcal{E}$ determines the boundary condition one would impose in a variational principle for the subregion by the above discussion. While different choices are available for these boundary conditions, we mention that it is often most useful to choose those in which $\mathcal{E}$ takes a Dirichlet form, meaning only variations of intrinsic quantities on the surface without derivatives appear in $\mathcal{E}$. For a timelike surface, this means

$$\mathcal{E} = \pi^{ij}\delta h_{ij} \,, \tag{2.15}$$

where h_{ij} is the induced metric, while for a null surface it means[13,37]

$$\mathcal{E} = \pi^{ij}\delta q_{ij} + \pi_i \delta n^i \,, \tag{2.16}$$

where q_{ij} is the degenerate induced metric, and n^i is the null generator. A number of arguments in favor of the Dirichlet form of the flux were presented in Ref. 13, such as the relation to junction conditions across $\mathcal{N}$ and the semiclassicality of the gravitational path integral when gluing subregions. We will also utilize this condition in Sec. 3 when deriving the algebra satisfied by the localized charges, but we argue that other forms of the flux also allow the derivation to go through. In writing Eqs. (2.15) and (2.16), we have restricted attention to theories such as general relativity that admit a Dirichlet variational principle (or equivalently, possesses second-order equations of motion), and have neglected any contributions from matter fields to the symplectic potential. Note that the conjugate momenta π^{ij}, π_i can involve objects constructed from both the extrinsic and intrinsic geometry of the surface. The Dirichlet requirement fixes the form of $\mathcal{E}$ up to addition of boundary and corner terms constructed entirely from intrinsic quantities, and in Sec. 5 we will discuss how these purely intrinsic ambiguities are used in the context of holographic renormalization.

We can further interpret how to view $\mathcal{E}$ by taking a variation of Eq. (2.12) and rearranging terms, which yields

$$\delta\mathcal{E} = \underline{\omega}' - d\delta\beta' \,. \tag{2.17}$$

This shows that $\mathcal{E}$ serves as a symplectic potential for the pullback of the symplectic form $\underline{\omega}' - d\delta\beta'$. Here, the term $d\delta\beta'$ is precisely of the form of the ambiguity in the symplectic potential described in Eq. (2.10), and as described in Ref. 53, by considering an extension of β' away from the surface $\mathcal{N}$, we can view $\theta' - d\beta'$ as the symplectic potential everywhere in the bulk. The associated symplectic current is then $\omega' - d\delta\beta'$, and integrating this over a spatial slice Σ yields a symplectic form that is the sum of a bulk and boundary term,

$$\Omega = \int_\Sigma \omega' - \int_{\partial\Sigma} \delta\beta' \,. \tag{2.18}$$

We remark that we will require the quantities L', θ', b' and λ' to be continuous everywhere on the space–time subregion and in particular everywhere on its boundary. This condition is necessary for passing from $(d+1)$-dimensional bulk integrals to d-dimensional boundary integrals using Stokes' theorem. By contrast, the quantities ℓ', β' and $\mathcal{E}$ associated with the decomposition of the pullback of θ' to a boundary component will not be required to be continuous across a corner joining two boundary components. This greater generality for these quantities goes hand in hand with the use of corner terms in the formalism in Subsec. 2.5 and Sec. 5.

2.3. *Localized charges*

Having identified a symplectic structure for the subregion $\mathcal{U}$, we can proceed to construct gravitational charges associated with diffeomorphisms that act near the boundary $\mathcal{N}$. Diffeomorphism invariance of the field equations implies the existence of a conserved Noether current J'_ξ associated with each diffeomorphism generated by a given vector ξ^a. It follows from Eq. (2.7a) that under the action of diffeomorphisms on phase space, the Lagrangian L' transforms as

$$I_{\hat\xi}\delta L' = \pounds_\xi L' + \Delta_{\hat\xi} L' = d i_\xi L' + d\Delta_{\hat\xi} b' \,. \tag{2.19}$$

However, from the definition (2.5), the left-hand side can be written as

$$I_{\hat\xi}\delta L' = E\cdot I_{\hat\xi}\delta\phi + dI_{\hat\xi}\theta' \,, \tag{2.20}$$

and so defining the Noether current to be

$$J'_\xi = I_{\hat\xi}\theta' - i_\xi L' - \Delta_{\hat\xi} b' \,, \tag{2.21}$$

we see that $dJ'_\xi = -E\cdot I_{\hat\xi}\delta\phi$, which vanishes on shell. Here, we find a correction to the usual definition of the Noether current involving the noncovariance of the boundary term, $\Delta_{\hat\xi} b'$, which was identified previously in Ref. 53. We can relate the Noether current (2.21) to the Noether current $\overset{vc}{J}_\xi$ constructed from the covariant Lagrangian and symplectic potential[g]

$$\overset{vc}{J}_\xi = I_{\hat\xi}\overset{c}{\theta} - i_\xi \overset{c}{L} \tag{2.23}$$

[g] The superscript "vc" in this expression stands for "vector covariant," and is used to indicate that the only noncovariance in $\overset{vc}{J}_\xi$ arises from its dependence on the noncovariant vector field ξ^a. This notation will be used to indicate any quantity such as $\overset{vc}{J}_\xi$ depending linearly on ξ^a and its derivatives whose noncovariance is given by

$$\Delta_{\hat\zeta}\overset{vc}{J}_\xi = \overset{vc}{J}_{(\Delta_{\hat\zeta}\xi)} = -\overset{vc}{J}_{(\llbracket \zeta,\xi\rrbracket - I_{\hat\xi}\delta\zeta)} \,. \tag{2.22}$$

using (2.8a) and (2.8b), which produces

$$J'_\xi = I_{\hat\xi}\overset{c}{\theta} + I_{\hat\xi}\delta b' + dI_{\hat\xi}\lambda' - i_\xi \overset{c}{L} - i_\xi db' - I_{\hat\xi}\delta b' + \pounds_\xi b' = \overset{vc}{J}_\xi + d\left(i_\xi b' + I_{\hat\xi}\lambda'\right), \tag{2.24}$$

showing that $\overset{vc}{J}_\xi$ and J'_ξ differ by an exact term. Furthermore, since $\overset{vc}{J}_\xi$ is identically closed on shell and covariantly constructed for any vector ξ^a, it can be expressed as the exterior derivative of a potential,

$$\overset{vc}{J}_\xi = d\overset{vc}{Q}_\xi\,, \tag{2.25}$$

that is covariantly constructed from ξ^a and the dynamical fields.[67] The relation (2.24) then implies that J'_ξ is also expressible as the exterior derivative $J'_\xi = dQ'_\xi$ of a potential Q'_ξ, given by

$$Q'_\xi = \overset{vc}{Q}_\xi + i_\xi b' + I_{\hat\xi}\lambda'\,. \tag{2.26}$$

The localized charges H_ξ are now constructed by evaluating the contraction of the field space vector field $\hat\xi$ into the symplectic form. Using the identity (see App. B)

$$-I_{\hat\xi}\omega' = d\left(\delta Q'_\xi - Q'_{\delta\xi} - i_\xi\theta' - \Delta_{\hat\xi}\lambda'\right), \tag{2.27}$$

and the definition (2.18) for the subregion symplectic form, we find that the contraction of $\hat\xi$ into Ω is given by

$$-I_{\hat\xi}\Omega = \int_{\partial\Sigma}\left(\delta Q'_\xi - Q'_{\delta\xi} - \Delta_{\hat\xi}\lambda' - i_\xi\theta' + I_{\hat\xi}\delta\beta'\right). \tag{2.28}$$

Note that because this contraction localizes to a pure boundary integral, any diffeomorphism supported purely in the interior of Σ is a degeneracy of Ω, reflecting that such transformations are pure gauge. If ξ^a generated a genuine, global symmetry of the subregion phase space, the right-hand side of (2.28) would have to be the total variation δH_ξ of a quantity H_ξ that would be identified as the charge generating the symmetry. In this case, Eq. (2.28) simply becomes the statement of Hamilton's equation, $-I_{\hat\xi}\Omega = \delta H_\xi$. However, it is clear from inspection that the terms $Q'_{\delta\xi} + \Delta_{\hat\xi}\lambda' + i_\xi\theta' + I_{\hat\xi}\delta\beta'$ generically do not take the form of a total variation upon integration over $\partial\Sigma$, absent the imposition of boundary conditions. While such boundary conditions arise naturally for global charges for closed subsystems, in the more general context of an open, localized phase space, such boundary conditions unnecessarily constrain the dynamics and eliminate dynamical degrees of freedom associated with fluxes of radiation modes. In this case, we seek to define a set of localized charges, which satisfy a modification of Hamilton's equation involving a term representing the flux of degrees of freedom escaping the subregion.

Using the decomposition (2.12) of θ', we find that Eq. (2.28) can be reorganized into the form (see App. B)

$$-I_{\hat\xi}\Omega = \int_{\partial\Sigma}\delta h_\xi - \int_{\partial\Sigma}\left(i_\xi\mathcal{E} - \Delta_{\hat\xi}(\beta' - \lambda') + h_{\delta\xi}\right), \tag{2.29}$$

where we have defined the localized charge density h_ξ to be

$$h_\xi = Q'_\xi + i_\xi \ell' - I_{\hat{\xi}}\beta' \,. \tag{2.30}$$

This formula takes the same form as the expression derived by Harlow and Wu,[53] applied now in a context where boundary conditions are not imposed on the phase space, as in Ref. 13. The first term in (2.29) is a total variation, and we are led to identify this with the localized charge associated with ξ^a,

$$H_\xi = \int_{\partial\Sigma} h_\xi \,. \tag{2.31}$$

The remaining terms in (2.29) represent the loss of symplectic flux through the boundary $\mathcal{N}$ under a flow generated by ξ^a that moves $\partial\Sigma$ along this boundary. The modified Hamilton's equation involving nontrivial fluxes for the localized charge then takes the form

$$\delta H_\xi = -I_{\hat{\xi}}\Omega + \mathcal{F}_{\hat{\xi}} \,, \tag{2.32}$$

where

$$\mathcal{F}_{\hat{\xi}} \equiv \int_{\partial\Sigma} \Big(i_\xi \mathcal{E} - \Delta_{\hat{\xi}}(\beta' - \lambda') + h_{\delta\xi} \Big) \,. \tag{2.33}$$

We denote this flux by $\mathcal{F}_{\hat{\xi}}$ rather than simply $\mathcal{F}_\xi$ to emphasize that it can depend nontrivially on the field-dependence of the generator of the diffeomorphism.[h]

The charges constructed via Eq. (2.31) obey a nontrivial conservation equation, which can be obtained by computing the exterior derivative of the charge density h_ξ. This yields the identity (derived in App. B)

$$dh_\xi = I_{\hat{\xi}}\mathcal{E} - \Delta_{\hat{\xi}}(\ell' + b') - i_\xi(L' + d\ell') \,, \tag{2.34}$$

and integrating this between two cuts S_1 and S_2 of the boundary $\mathcal{N}$ produces the anomalous continuity equation,

$$H_\xi(S_2) - H_\xi(S_1) = -\int_{\mathcal{N}_1^2} \Big(I_{\hat{\xi}}\mathcal{E} - \Delta_{\hat{\xi}}(\ell' + b') \Big) \,, \tag{2.35}$$

where the last term in (2.34) does not contribute since ξ^a is taken to be tangent to $\mathcal{N}$. The minus sign in this equation appears due to the choice of orientations of $\mathcal{N}$ and $\partial\Sigma$, discussed in Subsec. 1.2. The first term on the right of this equation is interpreted as the symplectic flux out of the subregion, while the second term involving $\Delta_{\hat{\xi}}(\ell' + b')$ is an anomalous violation of the conservation equation.[13]

[h] We have separated the entire contribution coming from $\delta\xi^a$ into the flux term, although for cases where $\delta\xi^a$ takes a specific form, it may be possible to separate off a total variation from $h_{\delta\xi}$ to include as a correction to the charge. Such field dependence is used in Refs. 68–73, for example, to cancel some terms appearing in the flux, to arrive at integrable generators in the absence of gravitational waves.

2.4. *Ambiguities and boundary canonical transformations*

At this point, it is worth commenting that various objects introduced above, such as Ω (2.18) and h_ξ (2.30), have been defined without the prime notation. This is because these objects are in fact insensitive to the two ambiguities we have mentioned to this point, corresponding to shifting $L' \to L' + da$ and $\theta' \to \theta' + \delta a + d\nu$, where a and ν are allowed to be noncovariant in general. We refer to these shifts of L' and θ' as JKM transformations, having first been identified in the work of Jacobson, Kang and Myers.[35] Under such a transformation, we require that the flux $\mathcal{E}$ remains invariant, since we are taking the form of the flux to be a physical input defining the dynamics of the subregion. In order to keep $\mathcal{E}$ invariant even while θ' changes under the transformation, we must also shift the quantities ℓ' and β' appearing in the decomposition (2.12). The JKM transformations of the basic quantities defining the phase space are then given by

$$L' \to L' + da\,, \tag{2.36a}$$

$$\theta' \to \theta' + \delta a + d\nu\,, \tag{2.36b}$$

$$\mathcal{E} \to \mathcal{E}\,, \tag{2.36c}$$

which then imply

$$\omega' \to \omega' + d\delta\nu\,, \tag{2.37a}$$

$$b' \to b' + a\,, \tag{2.37b}$$

$$\lambda' \to \lambda' + \nu\,, \tag{2.37c}$$

$$\ell' \to \ell' - a\,, \tag{2.37d}$$

$$\beta' \to \beta' + \nu\,, \tag{2.37e}$$

$$J'_\xi \to J'_\xi + d(i_\xi a + I_{\hat{\xi}}\nu)\,, \tag{2.37f}$$

$$Q'_\xi \to Q'_\xi + i_\xi a + I_{\hat{\xi}}\nu\,. \tag{2.37g}$$

Given these transformations, it is immediate to check from the definitions (2.18), (2.30), (2.33) and (2.13) that the symplectic form Ω, the charge density h_ξ, the symplectic flux $\mathcal{F}_{\hat{\xi}}$, and the subregion action S are all invariant.

Note in particular that if we start with a noncovariant Lagrangian and symplectic potential L' and θ', and perform a JKM transformation with $a = -b'$, $\nu = -\lambda'$, we obtain a covariant Lagrangian and symplectic potential, $\overset{c}{L}$ and $\overset{c}{\theta}$, which are often used in standard treatments of the covariant phase space. In this case, the expressions for the charges and symplectic form reduce to those constructed in Ref. 13, which utilized the covariant Lagrangian and symplectic potential. Invariance of the charges under JKM transformations then implies that these expressions will agree with charges constructed using a Lagrangian and symplectic form that differs by the addition of noncovariant boundary and corner terms. The main message here is that the only choices that affect the charges are the field equations $E \cdot \delta\phi = 0$ and the form of the flux $\mathcal{E}$. The quantity $\mathcal{E}$ can be viewed as a boundary equation of motion, analogous to the bulk expression $E \cdot \delta\phi$, and, although we do not impose this field equation for generic localized subregions, the equation $\mathcal{E} = 0$ would be the boundary condition one would have to impose to have a well-defined variational principle for the subregion. Note that it is not necessary to absorb all the boundary terms in the action principle into total derivative terms in the definition of the Lagrangian L'. Instead, when writing a variational principle for the subregion, one should view L' as the bulk part of the action, and ℓ' as the boundary contribution, as in Eq. (2.13), and together they produce an action that is independent of JKM transformations. The choice of a particular L' is then largely a matter of convenience. When discussing consequences of diffeomorphism invariance, such as the first law of black hole mechanics,[6,7] it is usually most transparent to work with covariant $\overset{c}{L}$ and $\overset{c}{\theta}$. However, it can also be advantageous to work with a noncovariant Lagrangian, such as when using the ADM formalism,[40] or in the context of holographic renormalization in order to obtain finite space–time integrals, as expanded upon in Sec. 5. The results of this section indicate that charges obtained in either formulation coincide, and the transformations (2.36) provide a means for translating between different choices.

Since the resolution of JKM ambiguities employed in this work relies on fixing a preferred choice of the flux $\mathcal{E}$, it is worth commenting on the different choices that are available for $\mathcal{E}$. The different possible choices are obtained from transformations that alter the decomposition (2.12) of the presymplectic potential, while leaving L' and θ' invariant. Such transformations induce changes in the flux, corner and boundary terms of the form[i]

$$\mathcal{E} \to \mathcal{E} + \delta B - d\Lambda\,, \tag{2.38a}$$

$$\ell' \to \ell' + B\,, \tag{2.38b}$$

$$\beta' \to \beta' + \Lambda\,. \tag{2.38c}$$

We refer to such transformations as a *boundary canonical transformations*, since they affect the division of the canonical pairs that appear in $\mathcal{E}$ into coordinates and momenta.[j] For example, starting with a Dirichlet flux on a timelike boundary $\mathcal{E} = \pi^{ij}\delta h_{ij}$, the boundary canonical transformation with $B = -\pi^{ij}h_{ij}$ yields a Neumann form of the flux, $\mathcal{E}_N = \mathcal{E} - \delta(\pi^{ij}h_{ij}) = -h_{ij}\delta\pi^{ij}$. Quantities that

[i] By composing with a JKM transformation (2.36) one can obtain an alternative form of boundary canonical transformations in which $L' \to L' + dB$ and $\theta' \to \theta' + \delta B - d\Lambda$ while ℓ' and β' are invariant. However the form (2.38) is more general since it naturally accommodates transformation parameters B and Λ that are discontinuous from one boundary component to the other, while L', θ' and the JKM parameters a and ν are required to be continuous (see the discussion after Eq. (2.18) above).

[j] See Ref. 74 for a related discussion interpreting such transformations as a canonical transformation in the context of holographic renormalization.

were invariant under the JKM transformations considered above transform nontrivially under these boundary canonical transformations; in particular, the action, symplectic form, and charge density change according to

$$S \to S + \int_{\mathcal{N}} B\,, \tag{2.39a}$$

$$\Omega \to \Omega - \int_{\partial\Sigma} \delta\Lambda\,, \tag{2.39b}$$

$$h_\xi \to h_\xi + i_\xi B - I_{\hat{\xi}}\Lambda\,. \tag{2.39c}$$

There are a number of situations where boundary canonical transformations are relevant. The most important example for this work is in considerations of holographic renormalization, where the naive Dirichlet form for $\mathcal{E}$ does not admit a finite limit to the asymptotic boundary. In this case, one seeks to find a counterterm $B = \ell_{\text{ct}}$ constructed from intrinsic quantities on the boundary such that the resulting renormalized action is finite as the boundary is taken to infinity. We show in Subsec. 5.2 that this ensures that the renormalized flux $\mathcal{E}_{\text{ren}}$ also has a finite limit, and hence is sufficient to construct finite asymptotic charges. Another example in which such boundary canonical transformations appear is in AdS/CFT when considering the alternative quantization of low mass bulk fields.[75]

2.5. *Corner improvements*

While specifying the form of the flux $\mathcal{E}$ resolves the standard JKM ambiguities in the covariant phase space formalism, there is an additional ambiguity that remains even after fixing $\mathcal{E}$. This ambiguity occurs because the decomposition (2.12) determines ℓ' and β' only up to shifts of the form

$$\ell' \to \ell' + df\,, \tag{2.40a}$$

$$\beta' \to \beta' + \delta f\,, \tag{2.40b}$$

with f generically noncovariant. Under such a shift, the charge density h_ξ is not invariant, instead transforming as

$$h_\xi \to h_\xi - \Delta_{\hat{\xi}} f - d i_\xi f\,, \tag{2.41}$$

and the term $\Delta_{\hat{\xi}} f$ will affect the value of the integrated charge H_ξ. A similar shift occurs in h_ξ under the transformations of b' and λ' described in Eqs. (2.9a) and (2.9b) by a noncovariant quantity e, leading to a shift in the charge density

$$h_\xi \to h_\xi - \Delta_{\hat{\xi}} e - d i_\xi e\,. \tag{2.42}$$

In order to handle these additional ambiguities, a correction must be added to the charges that cancels the dependence on these shifts. This improvement term in the charges was described in App. C of Ref. 13 when working with covariant L' and θ' (so that b' and λ' are set to zero), and here we will describe the generalization of this procedure to generically noncovariant L' and θ'.

The resolution comes from noting that we must not only fix the form of the flux $\mathcal{E}$ on the bounding hypersurface $\mathcal{N}$, but also must fix a preferred corner flux on the codimension-2 surface $\partial\Sigma$ on which the charge is being evaluated. In this case, the quantity $\beta' - \lambda'$ serves as a higher codimension symplectic potential, and hence to resolve the ambiguities, we decompose it in a similar manner as θ' from Eq. (2.12):

$$\beta' - \lambda' = -\delta c' + d\gamma' + \varepsilon\,, \tag{2.43}$$

where ε is the corner flux. We will obtain unambiguous charges by specifying a preferred form of ε, which could be determined by a Dirichlet variational principle for a subregion of space–time that includes corners, as discussed, for example in Ref. 66. In this case, c' is related to the corner terms one adds to the action, although the full action must include terms coming from both hypersurfaces intersecting at the corner.[k] The quantity γ' can be viewed as a codimension-2 symplectic potential, and in principle we could further consider decomposing it in a similar manner to θ' and $\beta' - \lambda'$. Doing so would yield quantities associated with contributions to the action and flux associated with codimension-3 defects in the shape of the subregion. Such features would arise at caustics of a null hypersurface, and also when considering singular diffeomorphisms such as superrotations that produce defects on a codimension-2 surface.[42,59,76–79] We note, however, in the absence of such codimension-3 features, the quantity γ' drops out of any expression for the charges, and hence we will not consider it further in this work, although a careful analysis of this type of term would be an interesting future direction.

The quantity $\Delta_{\hat{\xi}}(\beta' - \lambda')$ appears in the identity (2.29), and the decomposition (2.43) motivates including the c' term in the localized charge as opposed to the

[k] In slightly more detail, we consider a region $\mathcal{U}$ bounded by two hypersurfaces $\mathcal{N}^+$ and $\mathcal{N}^-$ intersecting at a codimension-2 corner $\mathcal{C}$, oriented such that $\partial\mathcal{U} \supset \mathcal{N}^+ - \mathcal{N}^-$, $\mathcal{N}^+ \supset \mathcal{C}$ and $\mathcal{N}^- \supset \mathcal{C}$, where the signs indicate the relative orientations. The action including contributions from only these boundaries is given by

$$S = \int_{\mathcal{U}} L' + \int_{\mathcal{N}^+} \ell'_+ - \int_{\mathcal{N}^-} \ell'_- + \int_{\mathcal{C}} (c'_+ - c'_-)\,. \tag{2.44}$$

Here, $\ell'_\pm$ and $c'_\pm$ arise from independent decompositions on $\mathcal{N}^\pm$, and these quantities, along with $\beta'_\pm$, need not be continuous when moving from $\mathcal{N}^-$ to $\mathcal{N}^+$ through $\mathcal{C}$. On the other hand, b' and λ' should be continuous across $\mathcal{C}$, since they arise from L' and θ' which are continuous throughout $\mathcal{U}$. Invariance under the standard JKM transformations follows as before, and we can also check invariance under the e and f ambiguities described in Eqs. (2.9a), (2.9b), (2.40a) and (2.40b). For these ambiguities, the quantity e is required to be continuous through $\mathcal{C}$, but f can take on separate values $f_\pm$ at $\mathcal{C}$. This then implies that the action is invariant under these transformations,

$$S \to S + \int_{\mathcal{N}^+} df_+ - \int_{\mathcal{N}^-} df_- + \int_{\mathcal{C}} (-e - f_+ + e + f_-) = S\,. \tag{2.45}$$

The corner improvement in the present section only considers contributions from the single hypersurface $\mathcal{N}^+$ ending on $\mathcal{C}$, but it would be interesting to extend this analysis to account for the contributions from $\mathcal{N}^-$.

flux. The improved charge density is then defined to be

$$\tilde{h}_\xi = h_\xi - \Delta_{\hat{\xi}} c' \tag{2.46}$$

$$\overset{vc}{=} Q_\xi + i_\xi(\ell' + b' + dc') - I_{\hat{\xi}}\varepsilon + d\left(i_\xi c' - I_{\hat{\xi}}\gamma'\right), \tag{2.47}$$

where the expression in the second line follows from applying the definitions (2.30), (2.26) and (2.43). The improved flux that generalizes Eq. (2.33) is given by

$$\tilde{\mathcal{F}}_{\hat{\xi}} = \int_{\partial\Sigma} \left(i_\xi \mathcal{E} - \Delta_{\hat{\xi}}\varepsilon + \tilde{h}_{\delta\xi}\right). \tag{2.48}$$

Defining the improved localized charge $\tilde{H}_\xi$ as the integral over $\partial\Sigma$ of $\tilde{h}_\xi$, we find that improved charges and fluxes still satisfy the modified Hamilton's equation,

$$\delta\tilde{H}_\xi = -I_{\hat{\xi}}\Omega + \tilde{\mathcal{F}}_{\hat{\xi}}. \tag{2.49}$$

Once a preferred form for the corner flux ε is chosen, the shifts in β' and λ' described in (2.40b) and (2.9b) require that c' transform according to

$$c' \to c' - f - e. \tag{2.50}$$

It follows immediately that the improved charge density (2.46) shifts only by exact terms under the transformation, and hence the integrated improved charge $\tilde{H}_\xi$ is invariant.[1]

The corner flux ε in Eq. (2.43) can be shifted by exact terms and total variations, leaving the left-hand side $\beta' - \lambda'$ fixed. The transformations that achieve this are analogous to the boundary canonical transformations (2.38), but arise in the codimension-2 context rather than in codimension-1. We call these transformations *corner canonical transformations*, given that they change the form of ε. One type of corner canonical transformation is an adjustment of the decomposition (2.43) by $\gamma' \to \gamma' + \zeta$, $\varepsilon \to \varepsilon - d\zeta$, leaving all other quantities fixed. A second type is a transformation (2.40) with parameter $f = F$, followed by a boundary canonical transformation (2.38) with parameters $B = -dF$ and $\Lambda = -\delta F$. Under the combined transformations we have

$$c' \to c' - F, \quad \gamma' \to \gamma' + \zeta, \quad \varepsilon \to \varepsilon - \delta F - d\zeta, \tag{2.51}$$

while ℓ' and β' are invariant and $\tilde{h}_\xi \to \tilde{h}_\xi + \Delta_{\hat{\xi}} F$. This combination of transformations is designed to leave θ' invariant. We will make use of these corner canonical transformations in our discussion of holographic renormalization in Sec. 5.

We emphasize that in our formalism the charges are uniquely determined by a choice of subregion action principle. The various canonical transformations considered here coincide with a change in action principle and a corresponding change in the charges. A small subtlety related to this point occurs in regard to the effect of the boundary canonical transformation $\mathcal{E} \to \mathcal{E} - d\Lambda$. For consistency,

[1] A slightly different proposal for a corner-improved charge was recently considered in Ref. 56, which amounts to defining the improved charge density to be $\bar{h}_\xi = h_\xi + \pounds_\xi c' = \tilde{h}_\xi + I_{\hat{\xi}}\delta c'$. We note that this alternative proposal does not have the same invariance properties under the ambiguities as does $\tilde{h}_\xi$, which serves as an argument in favor of (2.46).

Table 2. A summary of the various transformations in the covariant phase space formalism and how they act on the differential forms. The first row lists the eight different independent transformations, the second row their names, and the third the equation numbers where the transformations are defined. The name acronyms are Jacobson–Kang–Myers (JKM) transformation, boundary canonical transformation (BCT), and corner canonical transformation (CCT). The remaining rows list the various quantities that occur in the formalism, their names, defining equations, and how they transform under the transformations. The first eight rows list the fundamental independent quantities, while the remaining quantities are derived from the first eight. Quantities indicated with a † are required to be continuous from one component of the boundary to another, while those without this symbol may be discontinuous at these transitions. These discontinuities are associated with the appearance of corner terms in the integrated action (5.2). The five transformations in the columns for a, ν, e, χ and ζ do not change the integrated action S, symplectic form Ω, or integrated (improved) localized charges, and thus are analogous to gauge freedom in the formalism. By contrast, the three transformations in the columns for B, Λ and F do alter these quantities, reflecting the fact that the boundary flux $\mathcal{E}$ and corner flux ε must be specified (for example via a complete action principle) in order to determine unique localized charges.

	Transformation		a†	ν†	e†	χ†	B	Λ	F	ζ
	Name		JKM	JKM			BCT	BCT	CCT	CCT
	Equation		(2.36)	(2.36)	(2.9)	(2.9)	(2.38)	(2.38)	(2.51)	(2.51)
	Quantity	Eqs.								
L'†	Bulk Lagrangian	(2.5)	da							
θ'†	Symplectic pot.	(2.5)	δa	$d\nu$						
b'†	L' noncovariance	(2.7)	a		de					
λ'†	θ' noncovariance	(2.7)		ν	$-\delta e$	$d\chi$				
ℓ'	Boundary action	(2.12)	$-a$				B			
c'	Corner action	(2.43)			$-e$				$-F$	
γ'		(2.43)				$-\chi$				ζ
ε	Corner flux	(2.43)						Λ	$-\delta F$	$-d\zeta$
$\mathcal{E}$	Boundary flux	(2.12)					δB	$-d\Lambda$		
S	Action	(2.13)					$\int_{\mathcal{N}} B$		$-\int_{\partial N} F$	
Ω	Sympl. form	(2.18)						$-\int_{\partial\Sigma} \delta\Lambda$		
β'	Corner term	(2.12)		ν				Λ		
J'_ξ	Noether current	(2.21)	$di_\xi a$	$dI_{\hat{\xi}}\nu$	$-\Delta_{\hat{\xi}} de$					
Q'_ξ	Noether charge	(2.26)	$i_\xi a$	$I_{\hat{\xi}}\nu$	$-\Delta_{\hat{\xi}} e$	$dI_{\hat{\xi}}\chi$				
h_ξ	Localized charge	(2.30)			$-di_\xi e$ $-\Delta_{\hat{\xi}} e$	$dI_{\hat{\xi}}\chi$	$i_\xi B$	$-I_{\hat{\xi}}\Lambda$		
$\tilde{h}_\xi$	Improved charge	(2.46)			$-di_\xi e$ $-di_\xi e$	$dI_{\hat{\xi}}\chi$	$i_\xi B$	$-I_{\hat{\xi}}\Lambda$	$\Delta_{\hat{\xi}} F$	

this transformation must shift the corner flux according to $\varepsilon \to \varepsilon + \Lambda$ (see Table 2), and these combined transformations have the effect of leaving the variation δS of the subregion action invariant. One might be tempted to conclude that the corner-improved charges should then be invariant under this transformation since the subregion action is meant to uniquely determine the charges; however, according to Table 2, this transformation in fact shifts the charge density $\tilde{h}_\xi$ nontrivially. This suggests that one must not only specify the form of the subregion action, but also the full form of the fluxes $\mathcal{E}$ and ε in order to obtain unique expressions for the charges. In actuality, both $\mathcal{E}$ and ε can be uniquely extracted from the action provided one specifies on which quantities the on-shell action functionally depends. In particular, for a Dirichlet action principle, the on-shell action is a functional of the boundary induced metric h_{ij} and the corner induced metric q_{AB}. This uniquely determines the momenta $\pi^{ij} = \frac{\delta S}{\delta h_{ij}}$ and $\pi^{AB} = \frac{\delta S}{\delta q_{AB}}$, and hence the fluxes by the relation $\mathcal{E} = \pi^{ij}\delta h_{ij}$ and $\varepsilon = \pi^{AB}\delta q_{AB}$. Hence, even though the charges depend on the precise forms of the fluxes, we see that these are ultimately determined in terms of the subregion action principle.

A summary of the various transformations we have defined in this section is given in Table 2.

3. Brackets of Localized Charges

With the definition (2.30) of the localized charges in hand, we would next like to compute the algebra they satisfy. Since these charges arise from the action of diffeomorphisms on a subregion of a space–time manifold, we should expect the charge algebra to be closely related to the algebra satisfied by the corresponding vector fields ξ^a on space–time under the Lie bracket. Diffeomorphisms of space–time induce an action on the solution space $\mathscr{S}$ which serves as a phase space of our theory, leading to a related Lie bracket of the vector fields $\hat{\xi}$ associated with the space–time vector fields ξ^a. As mentioned in Eqs. (2.2) and (2.3), the field space bracket is simply minus the space–time Lie bracket for field-independent generators, and for field-dependent generators, it is given by minus the modified Lie bracket $[\![\cdot, \cdot]\!]$.

Normally when dealing with Hamiltonian charges for a symplectic manifold, the Poisson bracket of the charges can be obtained by contracting the vector fields generating the symmetry into the symplectic form. However, the localized charges do not satisfy Hamilton's equation due to the term involving $\mathcal{F}_{\hat{\xi}}$ in (2.32). This means that the charges H_ξ do not generate the same flow as the vector $\hat{\xi}$ on the phase space. Nevertheless, the charges H_ξ are functions on phase space, and hence possess a well-defined Poisson bracket. We will find in this section that this Poisson bracket on the subregion phase space reproduces the bracket introduced by Barnich and Troessaert in Ref. 57, providing a novel derivation of this bracket and justifying its use in defining the algebra of localized charges. Note that there is no contradiction in the fact that these charges have a well-defined Poisson bracket despite the presence of fluxes in Hamilton's equation because the flow generated by H_ξ differs from the flow generated by $\hat{\xi}$, and it is only the latter that does not preserve the symplectic form when fluxes are present. Due to the difference between the flows generated by H_ξ and $\hat{\xi}$, we will find that the charge algebra differs from the algebra of vector fields under the modified bracket by the extension terms $K_{\xi,\zeta}$ appearing in Eqs. (3.10) and (3.11).

To compute the Poisson bracket, it is first helpful to introduce an abstract index notation for tensors on field space. We let $\mathscr{A}, \mathscr{B}, \ldots$ denote tensor indices on $\mathscr{S}_{\mathcal{U}}$, so that, for example, the symplectic form on phase space can be written $\Omega_{\mathscr{A}\mathscr{B}}$. We then define $\Omega^{\mathscr{A}\mathscr{B}}$ to be an inverse of $\Omega_{\mathscr{B}\mathscr{C}}$. The meaning of this statement is somewhat subtle because $\Omega_{\mathscr{B}\mathscr{C}}$ is degenerate on $\mathscr{S}_{\mathcal{U}}$, and so inverting it requires some gauge-fixing procedure to define the subspace of $\mathscr{S}_{\mathcal{U}}$ on which we are constructing the inverse. This gauge fixing will yield a tensor $\Omega^{\mathscr{A}\mathscr{B}}$ satisfying

$$\Omega_{\mathscr{A}\mathscr{B}}\Omega^{\mathscr{B}\mathscr{C}}\Omega_{\mathscr{C}\mathscr{D}} = \Omega_{\mathscr{A}\mathscr{D}} \,. \tag{3.1}$$

Note that the true subregion phase space $\mathscr{P}_{\mathcal{U}}$ is obtained from $\mathscr{S}_{\mathcal{U}}$ by quotienting out the degenerate directions, and on this quotient space we expect $\Omega^{\mathscr{A}\mathscr{B}}$ to descend to a well-defined inverse that is independent of the gauge-fixing procedure. We also assume that the vector fields $\hat{\xi}^{\mathscr{A}}$ have been constructed to be tangent to the gauge-fixed submanifold, so that $\Omega^{\mathscr{A}\mathscr{B}}\Omega_{\mathscr{B}\mathscr{C}}\hat{\xi}^{\mathscr{C}} = \hat{\xi}^{\mathscr{A}}$. Often, this requirement introduces field dependence into the vector ξ^a, which is one of the reasons for considering field-dependent symmetry generators.

The Poisson bracket of the localized charges is defined to be[m]

$$\{H_\xi, H_\zeta\} = \Omega^{\mathscr{A}\mathscr{B}}(\delta H_\xi)_{\mathscr{A}}(\delta H_\zeta)_{\mathscr{B}} \,. \tag{3.2}$$

Then since the variation of the localized charges satisfies (2.32), we find for the Poisson bracket

$$\{H_\xi, H_\zeta\} = \Omega^{\mathscr{A}\mathscr{B}}\left(\Omega_{\mathscr{A}\mathscr{C}}\hat{\xi}^{\mathscr{C}} + (\mathcal{F}_{\hat{\xi}})_{\mathscr{A}}\right)\left(\Omega_{\mathscr{B}\mathscr{D}}\hat{\zeta}^{\mathscr{D}} + (\mathcal{F}_{\hat{\zeta}})_{\mathscr{B}}\right) \tag{3.3}$$

$$= -\hat{\xi}^{\mathscr{B}}\hat{\zeta}^{\mathscr{D}}\Omega_{\mathscr{B}\mathscr{D}} - \hat{\xi}^{\mathscr{B}}(\mathcal{F}_{\hat{\zeta}})_{\mathscr{B}} + \hat{\zeta}^{\mathscr{A}}(\mathcal{F}_{\hat{\xi}})_{\mathscr{A}} + \Omega^{\mathscr{A}\mathscr{B}}(\mathcal{F}_{\hat{\xi}})_{\mathscr{A}}(\mathcal{F}_{\hat{\zeta}})_{\mathscr{B}} \tag{3.4}$$

$$= \{H_\xi, H_\zeta\}_{\mathrm{BT}} + \Omega^{\mathscr{A}\mathscr{B}}(\mathcal{F}_{\hat{\xi}})_{\mathscr{A}}(\mathcal{F}_{\hat{\xi}})_{\mathscr{B}} \,, \tag{3.5}$$

where in the last line we have introduced the Barnich–Troessaert (BT) bracket

$$\{H_\xi, H_\zeta\}_{\mathrm{BT}} = I_{\hat{\xi}} I_{\hat{\zeta}}\Omega - I_{\hat{\xi}}\mathcal{F}_{\hat{\zeta}} + I_{\hat{\zeta}}\mathcal{F}_{\hat{\xi}} = -I_{\hat{\xi}}\delta H_\zeta + I_{\hat{\zeta}}\mathcal{F}_{\hat{\xi}} \,. \tag{3.6}$$

This bracket was proposed in Ref. 57 as a means for defining an algebra for localized charges that only satisfies the modified version of Hamilton's equation (2.32), but the interpretation of it as a Poisson bracket on a phase space was left as an open problem. From Eq. (3.5), we see that the BT bracket in fact coincides with the ordinary Poisson bracket of the charges H_ξ, H_ζ, provided we can argue that

[m]The sign in the definition of the Poisson bracket here is the more common choice, which is opposite to the sign employed in Ref. 13.

the final term quadratic in the fluxes $\mathcal{F}_{\hat{\xi}}$, $\mathcal{F}_{\hat{\zeta}}$ vanishes. To see how this occurs, we first assume that the corner term β' that appears in (2.12) is covariant, $\Delta_{\hat{\xi}}\beta' = 0$, that the generators are field-independent $\delta\xi^a = 0$, and that the flux has been put into Dirichlet form, as in (2.15) for a timelike surface or (2.16) for a null surface. In this case, the symplectic flux $\mathcal{F}_{\hat{\xi}}$ for a timelike surface simply reduces to

$$\mathcal{F}_{\hat{\xi}} = \int_{\partial\Sigma} i_\xi \pi^{ij} \delta h_{ij} \,. \tag{3.7}$$

In this form, we see that the final term in (3.5) involves the contraction of $\Omega^{\mathscr{A}\mathscr{B}}$ into two metric variations $\delta h_{ij}(x)\delta h_{kl}(x')$ at each pair of points x, x' on the spatial codimension-2 surface $\partial\Sigma$. Hence, it is determined entirely in terms of the Poisson bracket of the intrinsic metric on the surface $\{h_{ij}(x), h_{kl}(x')\}$:

$$\int dx \int dx' \, i_\xi \pi^{ij}(x) i_\zeta \pi^{kl}(x') \{h_{ij}(x), h_{kl}(x')\} \,. \tag{3.8}$$

However, this bracket should vanish on general grounds, since it involves values of the induced metric (without time derivatives) at causally separated points on $\partial\Sigma$. Additionally, at coincident points $x = x'$, no delta functions should appear in the Poisson bracket due to the absence of time derivatives. Because of this, we conclude that the final term (3.5) vanishes, and hence the Poisson bracket of the localized charges agrees with the BT bracket, at least in the case that the symplectic flux has been reduced to the form (3.7).[n] The story is entirely analogous for a null surface, and similarly relies on the vanishing of the brackets between intrinsic quantities at the cut $\partial\Sigma$, $\{q_{ij}, q_{kl}\} = \{n^i, q_{jk}\} = \{n^i, n^j\} = 0$. For example, for the components δq_{AB} of the induced metric perturbation on the future horizon of a Schwarzschild black hole, Ref. 81 derives the commutators

$$\begin{aligned} &\{q_{AB}(\underline{\theta}, v), q_{CD}(\underline{\theta}', v')\} \\ &\qquad \propto (q_{AC}q_{BD} + q_{AD}q_{BC} - q_{AB}q_{CD})\delta^2(\underline{\theta}, \underline{\theta}') \left[\Theta(v - v') - \frac{1}{2}\right], \end{aligned} \tag{3.9}$$

where $(\underline{\theta}, v) = (\theta, \phi, v)$ are coordinates on the horizon. This commutator vanishes[o] at $v = v'$.

We can now ask whether any of the conditions leading to this conclusion can be relaxed. We can allow for the additional terms $\Delta_{\hat{\xi}}(\beta' - \lambda')$ and $h_{\delta\xi}$ that appear in (2.33), provided that these also can be put into Dirichlet form. For the corner term $\beta' - \lambda'$, this can be done using the corner improvement procedure described in Subsec. 2.5 by selecting a Dirichlet form for the corner flux ε appearing in (2.43).

[n]Interestingly, Ref. 80 found that commutativity of the intrinsic metric on a codimension-2 slice of the boundary is violated in the presence of a nonzero Immirzi parameter when utilizing the first-order formulation of general relativity. This suggests that the naive BT bracket would be modified in this case, and it would be interesting to investigate these corrections in more detail.
[o]That $\Theta(v - v') - 1/2$ should be interpreted as 0 for $v = v'$ can be seen by integrating Eq. (3.9) against $w^{AB}(\underline{\theta})w^{CD}(\underline{\theta}')$ for some w^{AB} which yields at $v = v'$ the commutator of an operator with itself.

For the $h_{\delta\xi}$ term, this likely leads to some restrictions on the allowed field dependence in ξ^a.[p] Finally, we can even allow for choices of the flux term $\mathcal{E}$ that are not Dirichlet form, for example, we could instead require Neumann form $\mathcal{E} = -h_{ij}\delta\pi^{ij}$. As long as the flux $\mathcal{E}$ is of the form where the equation $\mathcal{E} = 0$ defines a valid boundary condition for the variational principle,[q] and the condition of vanishing symplectic flux $\mathcal{F}_{\hat{\xi}} = 0$ imposes no further restrictions, one will still be able to argue that the flux terms in (3.5) vanish.

The interpretation given here of the BT bracket as an ordinary Poisson bracket on the subregion phase space is a novel proposal of this work, and can be compared to previous arguments for arriving at this definition of the bracket for localized charges H_ξ. In Ref. 13, two of us suggested a heuristic derivation of the bracket, in which the bracket represented the Poisson bracket on a larger phase space consisting of the subregion and a complementary phase space that collects the flux, yielding a closed global phase space. This interpretation is similar to the one presented in this work, but differs in that our present proposal shows that no auxiliary system is needed to interpret the bracket as a Poisson bracket. It is likely the two proposals are consistent with each other, after employing a gluing construction as discussed in Subsec. 7.2. Another proposal by Troessaert[82] suggested an interpretation in which the boundary fields on which Dirichlet conditions would be imposed in the variational principle are interpreted as classical sources for the subregion phase space, motivated by similar interpretations appearing in AdS/CFT. This interpretation appears to be close in spirit to the proposal in this paper; however, Troessaert's construction involves an explicit decomposition of the bracket into an ordinary bulk Poisson bracket and corrections involving variations of the boundary sources. This makes comparison to the present interpretation difficult, although it would be interesting to further investigate whether the two proposals are consistent with each other. Finally, we mention some recent work by Wieland[83] in which the bracket arose as a Dirac bracket after constraining the phase space to remove all radiative modes from the theory. By contrast, the bracket in this work imposes no such constraint, and hence disagrees with Wieland's proposal. However, it may be that the two proposals agree for a specific choice of transformations and charges that are "purely Coulombic," as might be expected for charges associated with diffeomorphisms acting near the boundary.

Having argued that the BT bracket coincides with the Poisson bracket of the localized charges, we can use it to compute the canonical algebra satisfied by these functions on the local phase space. A straightforward calculation (see App. B) using the bracket definition (3.6) and the expressions (2.30) and (2.33) for the charge

[p]Once the field-dependence of ξ^a has been fixed, one could decompose $h_{\delta\xi} = -\delta a + d\tau + \epsilon$, and include the a contribution in the charge and ϵ in the flux. Such a decomposition will lead to additional modifications of the brackets of the charges. This kind of decomposition has been investigated recently in Refs. 68–73.

[q]This can equivalently be phrased as finding a Lagrangian submanifold for the boundary phase space involving the symplectic pairs (π^{ij}, h_{ij}).

density and symplectic flux yields the following charge representation theorem:

$$\{H_\xi, H_\zeta\} = \left(H_{[\![\xi,\zeta]\!]} + K_{\xi,\zeta}\right), \tag{3.10}$$

$$K_{\xi,\zeta} = \int_{\partial\Sigma} \left(i_\xi \Delta_{\hat{\zeta}}(\ell' + b') - i_\zeta \Delta_{\hat{\xi}}(\ell' + b')\right). \tag{3.11}$$

Hence, we see that the bracket reproduces the algebra of the vector fields ξ^a under the modified bracket (2.3), up to an extension parametrized by a new set of generators $K_{\xi,\zeta}$. The combination $\ell' + b'$ that appears in the formula for the extension is a JKM-invariant quantity, and reduces to the expression for the extension in Ref. 13 when utilizing a covariant Lagrangian L'. Note that Eq. (B.9) indicates that the extension would involve an additional contribution $i_\xi i_\zeta(L' + d\ell')$, except for the fact that we have assumed that ξ^a and ζ^a are both parallel to the same hypersurface, causing this term to pull back to zero. In fact, this term was first derived in Ref. 17 assuming boundary conditions to make such transformations integrable, and was also recently explored in Ref. 39. Assuming charges associated with the two independent normal directions to $\partial\Sigma$ can be consistently defined,[17,39,84] this suggests the full formula for the extension is given by the sum of (3.11) and the integral of $i_\xi i_\zeta(L' + d\ell')$.[r]

The generators $K_{\xi,\zeta}$ are local functionals of the vector fields ξ^a, ζ^a and the geometric quantities defined on the boundary, and we can therefore compute their brackets with the original localized charges using a similar set of arguments as above:

$$\begin{aligned} \{H_\xi, K_{\zeta,\psi}\} &= \Omega^{\mathscr{A}\mathscr{B}}\left(\delta H_\xi\right)_{\mathscr{A}}\left(\delta K_{\zeta,\psi}\right)_{\mathscr{B}} \\ &= -\hat{\xi}^{\mathscr{B}}\left(\delta K_{\zeta,\psi}\right)_{\mathscr{B}} + \Omega^{\mathscr{A}\mathscr{B}}\left(\mathcal{F}_\xi\right)_{\mathscr{A}}\left(\delta K_{\zeta,\psi}\right)_{\mathscr{B}}. \end{aligned} \tag{3.12}$$

To simplify this further, we postulate that $K_{\xi,\zeta}$ is a functional of only intrinsic variables on the surface, $K_{\xi,\zeta} = K_{\xi,\zeta}[h_{ij}]$ (including any field-dependence in the vectors ξ^a, ζ^a). It can be checked that this condition is satisfied in general relativity with a finite null boundary,[13] and we expect it to hold more generally for theories that admit a Dirichlet variational principle. Its variation can therefore be written as

$$\delta K_{\xi,\zeta} = \int_{\partial\Sigma} \left(k^{ij}_{\xi,\zeta}\delta h_{ij} + d\sigma_\xi\right) \tag{3.13}$$

and the exact piece $d\sigma_\xi$ integrates to zero on $\partial\Sigma$. On a null surface, we similarly require that $\delta K_{\xi,\zeta}$ involve only the variations δq_{ij} and δn^i. As before, using

[r] The bracket defined in Ref. 39 further differs from the BT bracket since it is defined to subtract the extension term $K_{\xi,\zeta}$, so that it tautologically leads to a representation of the vector field algebra. It was later shown in Ref. 85 that this bracket arises as a Poisson bracket of integrable Hamiltonian charges in the extended phase space constructed by Ciambelli, Leigh and Pai,[86] and the generalization to ambiguity-free charges analogous to those constructed in this work was given in Ref. 87. The need to extend the phase space to arrive at the bracket considered in these works shows that it is closely related but essentially different from the BT bracket.

the assumption that $\mathcal{F}_{\hat{\xi}}$ is in Dirichlet form and that the intrinsic variables commute on $\partial\Sigma$, we see that the second term in (3.12) drops out, and we derive the relation

$$\{H_\xi, K_{\zeta,\psi}\} = -I_{\hat{\xi}}\delta K_{\zeta,\psi} \,. \tag{3.14}$$

Finally, the assumption that $K_{\xi,\zeta}$ is a functional of only intrinsic quantities leads by the same arguments to the result that the $K_{\xi,\zeta}$ generators commute among themselves,

$$\{K_{\xi,\zeta}, K_{\psi,\chi}\} = 0 \,. \tag{3.15}$$

As before, if we are instead working with a flux $\mathcal{E}$ that is not of Dirichlet form, the commutation relations (3.14) and (3.15) will remain valid as long as $K_{\xi,\zeta}$ is a functional only of quantities that would be fixed by the variational principle associated with the chosen form of the flux.

The relations (3.10), (3.14) and (3.15) fully define the algebra satisfied by the canonical charges H_ξ and the extension charges $K_{\xi,\zeta}$. In the case that $I_{\hat{\xi}}\delta K_{\zeta,\psi}$ can be expressed only in terms of the generators $K_{\chi,\rho}$, the charges $(H_\xi, K_{\zeta,\psi})$ yield a representation of an Abelian extension of the algebra satisfied by the vector fields under the bracket $[\![\xi, \zeta]\!]$. This condition was confirmed, for example, for a class of vector fields satisfying a Witt algebra acting on Killing horizons in Ref. 13, where it was further demonstrated that only a single independent generator $K_{\xi,\zeta}$ arises, yielding a central extension, the Virasoro algebra. In the most general case, however, one would expect $I_{\hat{\xi}}\delta K_{\zeta,\psi}$ to be expressed as a sum of H_ξ and $K_{\chi,\rho}$, producing a more complicated algebra, presumably related to Diff($\mathcal{N}$) or Diff($\mathcal{U}$), in which $K_{\xi,\zeta}$ generates an Abelian subalgebra. It would be interesting to explore these more complicated algebras in future work.

The requirement that $K_{\xi,\zeta}$ be a functional of intrinsic data on the boundary is a nontrivial consistency requirement in order to conclude the algebraic relation (3.14). To further motivate it, we remark that this requirement can be related to a generalized notion of symmetry for the subregion.[s] Normally, symmetries are defined as transformations that leave the subregion action (2.13) invariant. However, we can also consider transformations that change the action only by a boundary term that is constructed entirely from intrinsic data,

$$I_{\hat{\xi}}\delta S = \int_{\mathcal{N}} \mathcal{A}_\xi \,. \tag{3.16}$$

In the Dirichlet variational principle where the intrinsic data is fixed by a boundary condition, requiring $\mathcal{A}_\xi$ to depend only on intrinsic quantities then says that the action is invariant up to a constant. In such a situation, one can still derive a Noether charge that generates the symmetry, and it is conserved up to quantities constructed from the intrinsic geometry, which commute with all observables. The

[s]We thank Don Marolf for discussions on this point.

quantity $I_{\hat{\xi}}\delta S$ can be reexpressed using the anomaly operator as

$$I_{\hat{\xi}}\delta S = \int_{\mathcal{N}} \Delta_{\hat{\xi}}(\ell' + b') + i_\xi(L' + d\ell'), \tag{3.17}$$

where we find both of the contributions that arise in the formula for the extension, as discussed below (3.11).[t] Restricting to ξ^a that is tangent to $\mathcal{N}$, we see that $\mathcal{A}_\xi$ and $\Delta_{\hat{\xi}}(\ell' + b')$ coincide. Hence, $\Delta_{\hat{\xi}}(\ell' + b')$ must be expressible purely in terms of intrinsic quantities to be consistent with this generalized symmetry principle.

The charge algebra (3.10), (3.11) was derived for localized charges H_ξ constructed without employing the corner improvement described in Subsec. 2.5. When working with the improved charges $\tilde{H}_\xi$ defined as integrals of the improved charge density (2.46), the Poisson bracket of the charges is again given by the BT bracket (3.6) after replacing the flux terms involving $\mathcal{F}_{\hat{\xi}}$ to the modified fluxes $\tilde{\mathcal{F}}_{\hat{\xi}}$, defined in (2.48). As before, the charge algebra reproduces the modified bracket algebra of the vector fields up to an extension (derived in App. B),

$$\{\tilde{H}_\xi, \tilde{H}_\zeta\} = \tilde{H}_{[\![\xi,\zeta]\!]} + \tilde{K}_{\xi,\zeta}, \tag{3.18}$$

$$\tilde{K}_{\xi,\zeta} = \int_{\partial\Sigma} \left(i_\xi \Delta_{\hat{\zeta}}(\ell' + b' + dc') - i_\zeta \Delta_{\hat{\xi}}(\ell' + b' + dc')\right). \tag{3.19}$$

An important property of the BT bracket is that it reduces to a Dirac bracket for the generators H_ξ when boundary conditions are imposed to make them integrable, meaning they satisfy Hamilton's equation (2.28) with no fluxes. For a flux in Dirichlet form, this boundary condition is just that the intrinsic quantities on the surface are fixed. In this case, since we also require that $\Delta_{\hat{\xi}}(\ell' + b')$ is constructed purely from intrinsic quantities, the boundary condition also imposes that $\Delta_{\hat{\xi}}(\ell' + b')$ is constant on the constrained phase space, and hence $\delta K_{\xi,\zeta} = 0$. In this case, the vector fields generating the symmetry must be chosen to preserve the boundary condition, and we find that the generators $K_{\xi,\zeta}$ yield a central extension of the vector field algebra, as required by general arguments[32,88,89] on the properties of group actions on a symplectic manifold.

The more general setup considered in this work does not impose such a boundary condition, and this allows for Abelian extensions or more general forms of the algebra. The new generators $K_{\xi,\zeta}$ appearing in the extension are functionals of the intrinsic geometry evaluated on a slice of the boundary $\mathcal{N}$. It is helpful to view the collection of all such intrinsic functionals as forming an Abelian algebra of boundary observables localized on the cut $\partial\Sigma$. The charges H_ξ then act on any such functional $f[h_{ij}]$, generating its evolution along the vector ξ^a just as in Eq. (3.14),

$$\{H_\xi, f[h_{ij}]\} = -I_{\hat{\xi}}\delta f[h_{ij}]. \tag{3.20}$$

[t] As a side consequence, this demonstrates that extensions appear in the bracket of canonical charges only for transformations that do not leave the subregion action invariant.

Hence, even when ℓ is covariant so that the extension terms $K_{\xi,\zeta}$ in (3.10) vanish, we can still construct an extended algebra by allowing the generators H_ξ to act on the intrinsic functionals $f[h_{ij}]$. These intrinsic functionals are reminiscent of the edge modes arising in Ref. 90 when accounting for the Hayward term in the gravitational subregion action. It would be interesting to further explore this connection between edge mode degrees of freedom and the action of the localized charges H_ξ on functionals of intrinsic data.

4. Vacuum General Relativity at Null Infinity

In the formalism developed in the past two sections, we have implicitly assumed that the boundaries $\mathcal{N}$ are at finite locations in space–time, and that all quantities are finite on those boundaries. In Sec. 5, we will extend the formalism to handle the case of asymptotic boundaries, treated in a conformal completion framework to bring them to finite locations. In this context, the Lagrangian and symplectic potential can diverge at the boundaries and must be suitably renormalized using the techniques of holographic renormalization.[21,27,28,31,91] In this section, we take a detour to provide a motivating example for our treatment of holographic renormalization of Sec. 5: an analysis of various asymptotic symmetry groups for vacuum general relativity in asymptotically flat space–times.

As discussed in the introduction, in recent years a number of different field configuration spaces have been suggested that modify the boundary conditions imposed at $\mathscr{I}^+$, and that give rise to extensions of the Bondi–Metzner–Sachs (BMS) group of asymptotic symmetries. The BMS group arises when one defines a field configuration space by fixing some of the diffeomorphism freedom on the boundary. However, some of the relevant linearized diffeomorphisms are not degeneracy directions of the symplectic form, and thus do not correspond to gauge degrees of freedom. Hence it is a nontrivial restriction on the theory to impose these conditions. Lifting some of the boundary conditions leads to an enlarged symmetry group called the generalized BMS group.[29–31,46]

In this section, we first review the field configuration space definitions which lead to the BMS group and generalized BMS groups, using the language of Ref. 32. We then further relax the boundary conditions at null infinity so as to uncover an even bigger symmetry group. This procedure allows us to uncover new boundary degrees of freedom, or edge modes.[16,17,92,93] The enlarged symmetry group coincides with the symmetry group on finite null surfaces derived in Ref. 32. Following Ref. 45 we will call this group the *Weyl BMS* group. Some of the details of the analysis are relegated to App. E.

4.1. *Field configuration spaces*

We describe asymptotically flat space–times using the conformal completion framework, reviewed in App. D. Some of the relevant fields on space–time are the physical metric $\tilde{g}_{ab}$, the conformal factor Φ, the unphysical metric $g_{ab} = \Phi^2 \tilde{g}_{ab}$ and normal $n_a = \nabla_a \Phi$, while some of the fields on $\mathscr{I}^+$ are the null generator n^i, inaffinity κ, and volume forms η_{ijk} and μ_{ij}. We restrict attention throughout to $3+1$ dimensions. Higher-dimensional space–time does possess supertranslation symmetries[94] and associated charges,[96,u] and it would be interesting to investigate analogous extensions of those symmetry groups.

We now review a number of field configuration spaces corresponding to different boundary conditions at $\mathscr{I}^+$. To start, we fix $\mathcal{M}$ and $\mathscr{I}^+$ and consider the set of all asymptotically flat space–times $(\mathcal{M}, g_{ab}, \Phi)$. Since everything should be invariant under the conformal transformation $(g_{ab}, \Phi) \to (e^{-2\sigma} g_{ab}, e^{-\sigma}\Phi)$, it is convenient to fix the conformal freedom by fixing a choice of conformal factor Φ_0 on a neighborhood $\mathcal{D}$ of $\mathscr{I}^+$, with $\Phi_0 = 0$ and $\nabla_a \Phi_0 \neq 0$ on $\mathscr{I}^+$. We now define the large configuration phase of all unphysical metrics with that conformal factor:

$$\Gamma_0 = \{(\mathcal{M}, g_{ab}, \Phi)\,|\,\Phi = \Phi_0 \text{ on } \mathcal{D},\ \tilde{G}_{ab} = 0\}\,. \tag{4.1}$$

This is the most general configuration space consistent with the equations of motion. All of the spaces we will consider will correspond to subspaces of Γ_0 obtained by imposing specific boundary conditions.

Consider first the BMS field configuration space.[34,46] We fix a conformal factor Φ_0 on a neighborhood $\mathcal{D}$ of $\mathscr{I}^+$, fix a particular unphysical metric $g_{0\,ab}$ on $\mathscr{I}^+$, and define

$$\Gamma_{\text{BMS}} = \{(\mathcal{M}, g_{ab}, \Phi)\,|\,g_{ab|\mathscr{I}^+} = g_{0\,ab|\mathscr{I}^+},\ \Phi = \Phi_0 \text{ on } \mathcal{D},\ \nabla_a\nabla_b\Phi|_{\mathscr{I}^+} = 0\}\,. \tag{4.2}$$

Here we have used the conformal freedom (D.3) to fix the conformal factor, imposed the Bondi condition (D.9) to set $\nabla_a\nabla_b\Phi$ to zero on $\mathscr{I}^+$, and fixed the unphysical metric on $\mathscr{I}^+$. The original justification for imposing these conditions was that the entire space Γ_0 can be obtained by taking the orbit of Γ_{BMS} under diffeomorphisms and conformal transformations.[34] However, not all of these diffeomorphisms are gauged in the sense of corresponding to degeneracy directions of the symplectic form (see Sec. 6 for more details), which has led to the recent consideration of enlarged configuration spaces and weaker boundary conditions.[14] The enlargement leads to new degrees of freedom on the boundary, known as boundary gravitons or edge modes.[16,17,92,93]

We next consider the generalized BMS[v] configuration space of Campiglia and Laddha.[29–31] Here we fix a conformal factor Φ_0 on a neighborhood $\mathcal{D}$ of $\mathscr{I}^+$, fix a volume form $\bar{\eta}_{0\,ijk}$ and null generator $\bar{n}_0^i$ on $\mathscr{I}^+$ that satisfy the identity (D.7c) with $\kappa = 0$, and define[46]

$$\Gamma_{\text{GBMS}} = \{(\mathcal{M}, g_{ab}, \Phi)\,|\,n^i = \bar{n}_0^i,\ \eta_{ijk} = \bar{\eta}_{0\,ijk},\ \Phi = \Phi_0 \text{ on } \mathcal{D},\ \nabla_a\nabla_b\Phi|_{\mathscr{I}^+} = 0\}\,. \tag{4.3}$$

[u]Reference 95 argued for imposing boundary conditions that remove these supertranslation symmetries, because of the divergence of the associated symplectic flux. However, as argued in Sec. 5 of this paper, such divergences can generically be addressed using holographic renormalization and so should not be used as a criterion for determining which boundary conditions to impose.
[v]The terminology "extended BMS group" was used in Ref. 46, but "generalized BMS group" seems to be more common.

Here the normal n^i and volume form η_{ijk} are understood to be computed from g_{ab} and Φ as described in App. D. Compared to the BMS configuration space (4.2), we have replaced the unphysical metric evaluated on $\mathscr{I}^+$ with the volume form and the intrinsic normal.

The field configuration space can be further expanded by dropping the volume form. This yields the Weyl BMS field configuration space[45]

$$\Gamma_{\text{WBMS}} = \{(\mathcal{M}, g_{ab}, \Phi)\,|\, n^i = \bar{n}_0^i,\ \Phi = \Phi_0 \text{ on } \mathcal{D},\ \nabla_a\nabla_b\Phi|_{\mathscr{I}^+} = 0\}\,. \tag{4.4}$$

Note that the three different configuration spaces we have defined are related by

$$\Gamma_{\text{BMS}} \subset \Gamma_{\text{GBMS}} \subset \Gamma_{\text{WBMS}}\,, \tag{4.5}$$

when $\bar{n}_0^i$ and $\bar{\eta}_{0\,ijk}$ are those computed from $g_{0\,ab}$ on $\mathscr{I}$.

4.2. *Symmetry groups*

We now turn to describing the symmetry groups and algebras of the three field configuration spaces (4.2)–(4.4). The derivations of these groups are given in App. E, where we use the universal structure approach of Ashtekar[97] and the techniques of Ref. 32.

We start by picking a convenient class of coordinate systems on $\mathscr{I}^+$. Choose a cross-section $\mathcal{C}$ of $\mathscr{I}^+$ and choose coordinates $\theta^A = (\theta^1, \theta^2)$ on $\mathcal{C}$. Extend the definition of θ^A to all of $\mathscr{I}^+$ by demanding that θ^A be constant along integral curves of $\bar{n}_0^i$. Here for the spaces Γ_{GBMS} and Γ_{WBMS}, $\bar{n}_0^i$ is the intrinsic normal that appears explicitly in the definitions, while for the BMS case (4.2), $\bar{n}_0^i$ is computed from the metric $g_{0\,ab}$ on $\mathscr{I}^+$ and from Φ_0. Along each integral curve we define a parameter u by setting $u = 0$ on $\mathcal{C}$ and demanding that

$$\bar{\mathbf{n}}_0 = \partial/\partial u\,. \tag{4.6}$$

This construction yields a coordinate system $y^i = (u, \theta^A)$ on $\mathscr{I}^+$.

In this class of coordinate systems, the diffeomorphisms $\varphi : \mathscr{I}^+ \to \mathscr{I}^+$ have the following form for all three groups:

$$\hat{u} = e^{\alpha(\theta^A)}\left[u + \gamma(\theta^A)\right], \tag{4.7a}$$

$$\hat{\theta}^A = \chi^A(\theta^B)\,, \tag{4.7b}$$

where for a point $\mathcal{P}$ on $\mathscr{I}^+$ we have defined $y^i = y^i(\mathcal{P})$ and $\hat{y}^i = y^i(\varphi(\mathcal{P}))$. Equation (4.7b) defines a mapping χ from $\mathcal{C}$ to itself, or equivalently from the space of generators of $\mathscr{I}^+$ to itself. This set of maps is isomorphic to the set $\text{Diff}(S^2)$ of diffeomorphisms of the two-sphere. Writing the map (4.7) as (α, γ, χ), the group composition law is

$$(\alpha_2, \gamma_2, \chi_2) \circ (\alpha_1, \gamma_1, \chi_1) = (\alpha_3, \gamma_3, \chi_3)\,, \tag{4.8}$$

where

$$\alpha_3 = \alpha_1 + \alpha_2 \circ \chi_1 \,, \tag{4.9a}$$

$$\gamma_3 = \gamma_1 + e^{-\alpha_1}\gamma_2 \circ \chi_1 \,, \tag{4.9b}$$

$$\chi_3 = \chi_2 \circ \chi_1 \,. \tag{4.9c}$$

The linearized version of the mapping (4.7) is $\hat{y}^i = y^i + \xi^i$, where the generator $\boldsymbol{\xi}$ is

$$\boldsymbol{\xi} = \left[\gamma(\theta^A) + \alpha(\theta^A)u\right]\partial_u + \xi^A(\theta^B)\partial_A \,. \tag{4.10}$$

Using the notation (4.7), the structure of the three different groups can be summarized as follows:

- For the BMS group, as is well known, the function γ parametrizes supertranslations and can be freely specified. The map χ and the function α are constrained by

$$\chi^* \bar{q}_{AB} = e^{2\alpha}\bar{q}_{AB} \,, \tag{4.11}$$

  where χ^* is the pullback operator and q_{AB} is the spatial metric. It follows that χ is a global conformal isometry of the sphere, of which there is a six-parameter family isomorphic to the Lorentz group, and α is determined by χ. The group structure is the semidirect product

$$SO(1,3) \ltimes \mathcal{S} \,, \tag{4.12}$$

  where $\mathcal{S}$ is the normal subgroup of supertranslations given by $\alpha = 0$, $\chi =$ identity, and the subgroup $\gamma = 0$ is isomorphic to the Lorentz group $SO(1,3)$. Note that the semidirect product in Eq. (4.12) has the property that the supertranslation γ transforms under the conformal isometries of the two-sphere as a scalar density of weight 1/2, $\gamma \to e^{-\alpha}\chi^*\gamma$, from Eqs. (4.9) and (4.11).[98]
- For the generalized BMS group, the only change is that the six-parameter group of conformal isometries is replaced by the infinite-dimensional group Diff(S^2) of diffeomorphisms of the two-sphere. Thus, the supertranslation function γ can be chosen freely as before, the diffeomorphism χ can be freely chosen, and the function α is determined as a function of χ by

$$\chi^* \bar{\mu}_{AB} = e^{2\alpha}\bar{\mu}_{AB} \,, \tag{4.13}$$

  where $\bar{\mu}_{ij} = -\bar{\eta}_{ijk}\bar{n}^k$. The group structure is the semidirect product

$$\text{Diff}(S^2) \ltimes \mathcal{S} \,. \tag{4.14}$$

  The semidirect product here has the property that the supertranslation γ still transforms as a scalar density of weight 1/2,

$$\gamma \to e^{-\alpha}\chi^*\gamma \,, \tag{4.15}$$

  but now under all diffeomorphisms of the two-sphere instead of just the conformal isometries. This follows from Eqs. (4.9) and (4.13).

- Finally, for the Weyl BMS group, the constraint (4.13) that determines α as a function of χ is lifted, and α can now be freely chosen. This group is isomorphic to the group of symmetries on any null surface at a finite location defined in Ref. 32, as can be seen by comparing Eqs. (4.7) here with Eqs. (4.7) of that paper. The group has the following structure. We define the subgroup $\mathcal{T}$ of all supertranslations to be given by $\chi =$ identity, which is parametrized by α and γ. This is a normal subgroup and the total group has the structure

$$\text{Diff}(S^2) \ltimes \mathcal{T}. \tag{4.16}$$

The supertranslation group $\mathcal{T}$ contains two subgroups. First, there is the subgroup $\mathcal{S}$ given by $\alpha = 0$, parametrized by γ. These were called *affine supertranslations* in Ref. 32 since they correspond to displacements in affine parameter. This is a normal subgroup of both $\mathcal{T}$ and of the full group. Second, there is the non-normal subgroup $\mathcal{W}$ given by $\gamma = 0$, parametrized by α. These were called *Killing supertranslations* in Ref. 32 since they correspond to displacements in Killing parameter when there is a Killing vector field. They were also called Weyl rescalings in Ref. 45, as mentioned earlier. The supertranslation group has the structure

$$\mathcal{T} = \mathcal{W} \ltimes \mathcal{S}, \tag{4.17}$$

so the full symmetry group can be written as

$$\text{Diff}(S^2) \ltimes (\mathcal{W} \ltimes \mathcal{S}). \tag{4.18}$$

Here the first semidirect product is such that the supertranslation functions α and γ transform as scalars under diffeomorphisms of the two-sphere, not as scalar densities,[w,x] from Eqs. (4.9).

5. Gravitational Charges at Asymptotic Boundaries: Holographic Renormalization

5.1. *Introduction and overview*

We now return to the context of general theories and general space–time dimensions. The general formalism for gravitational boundary symmetries and charges

[w]However, if we change from the $\text{Diff}(S^2)$ subgroup $(\alpha, \gamma, \chi) = (0, 0, \chi)$ to the alternative $\text{Diff}(S^2)$ subgroup given by $(\alpha, \gamma, \chi) = [\alpha(\chi), 0, \chi]$, with $\alpha(\chi)$ the function of χ given by imposing Eq. (4.11), then in the semidirect product (4.18) the Killing supertranslations $\mathcal{W}$ transform as scalars but the affine supertranslations $\mathcal{S}$ transform as scalar densities as in Eq. (4.15), from Eqs. (4.9). This alternative $\text{Diff}(S^2)$ subgroup is the one that arises naturally within the generalized BMS subgroup $(\alpha, \gamma, \chi) = [\alpha(\chi), \gamma, \chi]$, which is why affine supertranslations transform as scalar densities in the GBMS and BMS cases.[98]

[x]One can also express the Weyl BMS group as the semidirect product $(\text{Diff}(S^2) \ltimes \mathcal{W}) \ltimes \mathcal{S}$. In this case the action of the second semidirect product endows the affine supertranslations with a certain weight under the Weyl rescalings $\mathcal{W}$, as well as an independent density weight under the diffeomorphisms, which again can be adjusted at will by choosing the $\text{Diff}(S^2)$ subgroup appropriately.

developed in Sec. 2 assumed that the boundaries are at finite locations in space–time (for example future event horizons), and that all the quantities defined are finite on those boundaries. In this section we will extend the formalism to handle cases of asymptotic boundaries, treated in a conformal completion framework to bring them to finite locations.[y] In general the Lagrangian and symplectic potential can then diverge at the boundaries. Some previous general treatments of the covariant phase space framework either did not treat asymptotic boundaries in general,[53] or used the finiteness of certain quantities on the boundary as a criterion to determine which boundary conditions to impose and which field configuration space to use,[34] which is in general unnecessarily restrictive.

The key idea of the extended formalism is holographic renormalization,[21,26–31,36,100] which exploits the boundary canonical transformations and JKM transformations discussed in Subsec. 2.4 to make the integrated action and symplectic potential θ' finite on the boundary. Once one has a renormalized symplectic potential, the formalism of Sec. 2 then yields finite gravitational global and localized charges.

In Sec. 2 we discussed the fact that it is sometimes necessary to introduce a background structure which violates covariance in order to obtain gravitational charges. For example this occurs with certain boundary conditions on finite null surfaces.[13] Similarly, here it is sometimes the case that the transformations that are necessary to renormalize the Lagrangian and symplectic form require the introduction of some background structures. This is the case, for example, for generalized BMS symmetries (Subsec. 4.1) in vacuum general relativity at null infinity, where it was shown in Ref. 46 that a completely covariant renormalization of the symplectic potential is impossible. A similar situation arises in asymptotically AdS space–times, where it is necessary to introduce a space–time foliation near the boundary as a background structure when renormalizing the action. Dependence of the renormalized quantities on this foliation signals the appearance of Weyl anomalies in the dual CFT description.[21,26,27,36,58]

The various possible background structures that we will consider are (i) a foliation of space–time near the boundary; (ii) a choice of conformal factor Φ near the boundary; and (iii) a choice of vector field $\mathbf{v}$ near the boundary which satisfies $v^a \nabla_a \Phi = 1$, which we will call a *rigging vector field*. We expect that in many situations only the foliation will be necessary to affect holographic renormalization. However, later in this section will make use of the more restrictive assumption of the existence of a rigging vector field to argue that holographic renormalization can always be successfully carried out.

We start by defining our notation and conventions for the covariant phase space framework with asymptotic boundaries. As discussed in Subsec. 2.2, our phase

[y] Although we assume a conformal completion, our general framework would also be applicable to situations like odd-dimensional asymptotically flat space–times where the conformal framework does not apply,[99] by making use of a radial foliation.

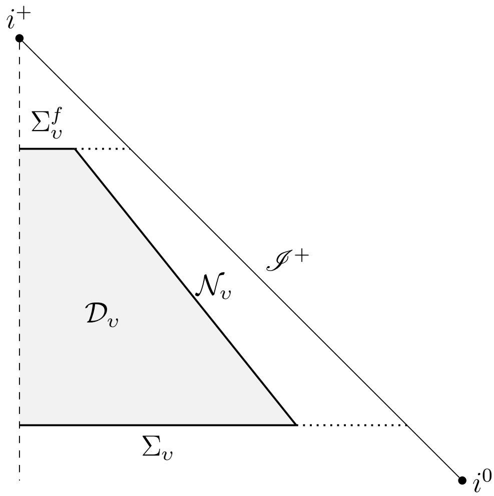

Fig. 1. Standard setup for holographic renormalization in asymptotically flat space–times. The subregion under consideration is associated with a segment of $\mathscr{I}^+$ to the future of a spatial surface Σ. The cutoff subregion $\mathcal{D}_v$ is depicted in gray, and its boundary components $\mathcal{N}_{j,v}$ consist of Σ_v, $\mathcal{N}_v$ and Σ_v^f.

subspaces are defined by a space–time subregion $\mathcal{U}$, and a spatial slice Σ whose boundary $\partial\Sigma$ lies in $\partial\mathcal{U}$. We define $\mathcal{D} = \mathcal{U} \cap I^+(\Sigma)$, the intersection of the subregion with the chronological future of the spatial slice, i.e. the set of all points to the future of Σ in $\mathcal{U}$. The region $\mathcal{D}$ will be the setting for the action principle for the phase subspace. We will denote by $\mathcal{N}_j$ the various components of the boundary of $\mathcal{D}$, one of whom will be the portion of the boundary $\mathcal{N}$ discussed in Subsec. 2.2 in the chronological future of Σ, and one of whom will be the spatial slice Σ. For asymptotic boundaries we use the conformal completion framework and work with conformally rescaled fields which are finite on the boundary. The conformal factor Φ is chosen to be vanishing on asymptotic boundaries and to have a nonzero gradient there, and to be positive on $\mathcal{D}$.

In order to discuss renormalized actions, we define a cutoff manifold $\mathcal{D}_v$ to be the set of points in $\mathcal{D}$ with $\Phi > v$, which excludes a neighborhood of the asymptotic boundaries. We will assume that the boundary of the truncated manifold can be decomposed into a number of components in the same way as the full manifold:

$$\partial\mathcal{D}_v = \bigcup_j \mathcal{N}_{j,v} \,. \tag{5.1}$$

The standard example we will have in mind is depicted in Fig. 1. Here, the region $\mathcal{D}_v$ has a single timelike boundary $\mathcal{N}_v$ that limits to a segment of $\mathscr{I}^+$, and Σ_v comprises the past boundary of $\mathcal{D}_v$. We also include a future boundary Σ_v^f so that the cutoff region is bounded in space–time, making all cutoff integrals manifestly finite.

On each boundary we define a boundary action ℓ'_j, and on the boundaries $\partial\mathcal{N}_j$ we define corner actions c'_j. Then the total action is defined to be

$$S_\upsilon = \int_{\mathcal{D}_\upsilon} L' + \sum_j \int_{\mathcal{N}_{j,\upsilon}} \ell'_j + \sum_j \int_{\partial\mathcal{N}_{j,\upsilon}} c'_j \,. \tag{5.2}$$

A successful renormalization of the action consists of finding definitions of ℓ'_j and c'_j so that S_υ has a finite limit as $\upsilon \to 0$. We will generally drop the subscript j in the remainder of this section.

In the action (5.2) we allow the boundary term ℓ'_j to have different forms on different boundaries $\mathcal{N}_j$, and be in effect discontinuous across the interfaces $\partial\mathcal{N}_j$ between two different boundaries. This is the reason for including the corner terms, which otherwise would give a vanishing contribution if $\ell'_j + dc'_j$ were a continuous function on $\partial\mathcal{D}$. An additional reason for separating out corner terms is as follows. One might imagine eliminating such terms by replacing $\ell'_j \to \ell'_j + dc'_j$. However such a replacement may violate corner covariance; it can arise that there exist definitions of c'_j that are covariant with respect to corner-preserving diffeomorphisms, but no extensions of these definitions to objects that are fully covariant with respect to diffeomorphisms of the entire boundary.

The general scheme for holographic renormalization can be described in terms of a number of stages, starting with conventional stages to renormalize the action, and then subsequent stages to renormalize the symplectic potential and to adjust corner terms. We now give an overview of the various stages. Although in practice the charges and other JKM-invariant quantities would only be computed at the end of the process, we list in the overview which of these quantities would change at each stage in order to clarify the logical structure of the process. In later subsections we will show that the steps described here can be carried out successfully for general theories using a rigging vector field and a boundary vector field as background structures (although we expect that a bulk foliation will in general be sufficient).

(1) We start with a divergent Lagrangian L' and symplectic potential θ'. We choose the decomposition (2.12) to make $\ell' = \beta' = 0$. We assume that the initial quantities are covariant, so that $b' = \lambda' = 0$, and it follows from Eq. (2.43) that $c' = \gamma' = \varepsilon = 0$. The action S, symplectic form Ω, gravitational charges $\tilde{H}_\xi$ and flux $\mathcal{E}$ are all divergent.

(2) We perform a boundary canonical transformation (2.38) with B and Λ chosen to put the flux $\mathcal{E}$ into Dirichlet form. The modifies $\mathcal{E}$, ℓ', β', S and $\tilde{H}_\xi$, and in particular involves adding a boundary term to ℓ'. For example, in vacuum general relativity with a timelike boundary the corresponding boundary term B is the Gibbons–Hawking–York term.

(3) We perform a second boundary canonical transformation (2.38) to make $L' + d\ell'$ finite and also to make the flux $\mathcal{E}$ finite, while preserving the Dirichlet form of the flux. The parameter B of the transformation is a boundary counterterm that is added to ℓ' to make $L' + d\ell'$ finite on the boundary, which is a functional of the intrinsic data on the boundary.[26,27,91] The parameter Λ is computed

from B in the same way that the symplectic potential θ' is computed from the bulk Lagrangian, by varying with respect to the intrinsic data, see for example Ref. 55. Schematically

$$\delta B = \frac{\delta B}{\delta(\text{intrinsic})}\delta(\text{intrinsic}) + d\Lambda\,,$$

which guarantees that the modification $\mathcal{E} \to \mathcal{E} + \delta B - d\Lambda$ to the flux preserves Dirichlet form. The quantities which change in this step are $\mathcal{E}$, ℓ', β', Ω, S and $\tilde{H}_\xi$. In Subsec. 5.3 we show explicitly that it is always possible to find a boundary canonical transformation (B, Λ) that makes both $L' + d\ell'$ and $\mathcal{E}$ finite. (It should be possible to always specialize the transformation to preserve Dirichlet form although we do not show this here.)

(4) We next repeat these steps in one lower dimension to adjust corner terms. We perform a corner canonical transformation (2.51) parametrized by forms F and ζ in order to put the corner flux ε into Dirichlet form. This involves identifying appropriate intrinsic degrees of freedom on the corners $\partial\mathcal{N}$. Additional quantities that change are S and $\tilde{H}_\xi$.

(5) We then perform a second corner canonical transformation (2.51) to make the integrated action S and corner flux ε finite, while maintaining the Dirichlet form of the corner flux. The parameter $-F$ of the transformation is a corner counterterm that is added to c' to make the integrated action finite, which is a functional of the intrinsic data on the corner. The parameter ζ is computed from F as described in step (3) above, by varying with respect to the intrinsic data, to ensure that the modification (2.51) to the corner flux ε preserves Dirichlet form. The gravitational charges $\tilde{H}_\xi$ as well as the symplectic form Ω are now finite, if the corner terms can be chosen so that $\Delta_{\hat{\xi}}(\ell' + dc')$ is finite [from Subsec. 2.5 the charges $\tilde{H}_\xi$ will be finite if $\mathcal{E}$, $L' + d\ell'$ and $\Delta_{\hat{\xi}}(b' + \ell' + dc')$ are all finite, and $b' = 0$ here]. In Subsec. 5.4 we show explicitly that this is the case: it is always possible to find a corner canonical transformation (f, ζ) of this kind that makes the integrated action S, corner flux ε and $\ell' + dc'$ finite. The general proof requires the introduction of additional background structure on the boundary.

(6) We use a JKM transformation (2.36) with $a = \ell'$ and $\nu = -\beta'$. This step is not necessary to obtain finite charges, but it is convenient in order to make other quantities in the formalism finite. It makes finite the Lagrangian L' and symplectic potential θ', and also modifies b', λ' and ℓ', but does not affect the charges, action or symplectic form.

(7) Finally we use a transformation of the form (2.9) with $e = c'$, $\chi = \gamma'$ and $\overset{c}{r} = \overset{c}{\rho} = 0$. This step is also optional, since it does not change the charges, but it is convenient as it makes c' and γ' vanish and λ' finite. The other quantity that is modified is b', which can still be divergent. However $\Delta_{\hat{\xi}} b' = \Delta_{\hat{\xi}}(b' + \ell' + dc')$ will be finite (since this quantity was finite at step (5) and is not modified by steps (6) or (7)), which is sufficient for finiteness of the charges.

Table 3. A summary of the steps in the general holographic renormalization procedure of this paper. The first row lists the numbered steps described in Subsec. 5.1, and the second row the transformations used in getting to each step from the previous step (detailed in Table 2). The remaining rows list the various quantities that occur in the formalism, their names, defining equations, and how they transform under the steps. The symbol $\to$ means that the corresponding quantity is altered by the transformation of that step. The symbols ∞, f and 0 mean that the quantity is diverging, finite and vanishing, respectively, at the end of the step.

	Step		1	2	3	4	5	6	7
	Transformations			B, Λ	B, Λ	F, ζ	F, ζ	a, ν	e, χ
	Quantity	Eqs.							
S	Action	(2.13)	∞	$\to\infty$	$\to\infty$	$\to\infty$	$\to$ f		
Ω	Sympl. form	(2.18)	∞	$\to\infty$	$\to$ f				
$\int_{\partial\Sigma}\tilde{h}_\xi$	Integ. charge	(2.46)	∞	$\to\infty$	$\to\infty$	$\to\infty$	$\to$ f		
L'	Bulk Lagrangian	(2.5)	∞	∞	∞	∞	∞	$\to$ f	
θ'	Symplectic pot.	(2.5)	∞	∞	∞	∞	∞	$\to$ f	
$\mathcal{E}$	Boundary flux	(2.12)	∞	$\to\infty$	$\to$ f				
b'	Pot., noncovar.	(2.7)	0	0	0	0	0	$\to\infty$	$\to\infty$
λ'	Pot., noncovar.	(2.7)	0	0	0	0	0	$\to\infty$	$\to$ f
ℓ'	Boundary action	(2.12)	0	$\to\infty$	$\to\infty$	∞	∞	$\to 0$	
β'	Corner term	(2.12)	0	$\to\infty$	$\to\infty$	∞	∞	$\to 0$	
c'	Corner action	(2.43)	0	0	0	$\to\infty$	$\to\infty$	∞	$\to 0$
γ'		(2.43)	0	0	0	$\to\infty$	$\to\infty$	∞	$\to 0$
ε	Corner flux	(2.43)	0	$\to\infty$	$\to\infty$	$\to\infty$	$\to$ f		

The algorithm described here is summarized in Table 3, which shows which quantities change at each step, and when quantities diverge or become finite.

Although we show that this procedure can always be carried out in order to obtain finite renormalized charges, there are a number of subtleties that arise in the asymptotically flat case that are not present in asymptotically dS or AdS spaces.[z] These subtleties relate to the form of the counterterm for the action ℓ_{ct} and their dependence on the free data associated with the cutoff surface $\mathcal{N}$. In asymptotically (A)dS spaces, this free data can be chosen to be the induced metric on the cutoff surface, which is fully unconstrained by the equations of motion in the limit that the cutoff is taken to the boundary. The counterterms needed to renormalize the action are covariantly constructed from the boundary metric, and hence they are given by local expressions in terms of the free data. Locality and covariance of the counterterms are important in the holographic correspondence, as they allow the on-shell action to be interpreted as the generating functional of correlation functions in a local CFT dual.[25,101,102]

[z]We thank Kostas Skenderis and Ioannis Papadimitriou for discussions on this point.

For asymptotically flat space–times, one finds that the induced metric on the cutoff surface is not freely specifiable, but instead satisfies a number of differential constraints in the limit that the cutoff surface is taken to infinity.[103] Even though counterterms that are local and covariant with respect to the induced metric on the cutoff surface can be constructed,[91] since this induced metric is not freely specifiable, the variation of the on-shell action with respect to this induced metric is necessarily subject to constraints. This generically leads to nonlocal dependence of the counterterms on the free data on the cutoff surface, and further complicates a simple holographic interpretation for asymptotically flat spaces.[74,103,104] While this does not affect the main results of this work on obtaining finite gravitational charges, addressing these subtleties is an important question for further developing the holographic correspondence in flat space.

The remainder of this section is organized as follows. In Subsec. 5.2 we provide a very general argument which shows that the existence of a renormalized total action (including bulk, boundary and corner terms) is sufficient to show that it is possible to renormalize the symplectic potential. The renormalized symplectic potential and charges will in general depend on the choice of foliation used to renormalize the action, as discussed above. In Subsec. 5.3 we present a similar and complementary result. Given any covariant theory in which the Lagrangian and symplectic potential diverge near the boundary, we demonstrate the existence of a renormalized Lagrangian and a renormalized symplectic potential, by explicitly computing the counterterms that one needs to subtract off in order to obtain finite quantities. These counterterms again depend on the choice of conformal factor, and in addition depend on a choice of rigging vector field.

As an application of our holographic renormalization formalism, in Sec. 6 we specialize to vacuum general relativity in asymptotically flat $(3 + 1)$-dimensional space–times, specialize to the generalized BMS field configuration space, and compute the renormalized symplectic potential and the associated localized charges at future null infinity.

5.2. *Existence of renormalized symplectic potential*: *General argument assuming a finite action*

In this subsection we show that a renormalized action functional is sufficient to provide a renormalized symplectic potential. From this one can obtain a complete set of IR finite observables which act on the boundary phase space.

We emphasize from the outset that noncovariances and background structures play a crucial role in this renormalization procedure, which thus avoids the no-go theorem of Ref. 46. In particular, the general argument relies on the introduction of a background foliation near the boundary, on which the renormalized action depends [cf. Eq. (5.2)]. Thus, any renormalization of the boundary action can itself involve noncovariant counterterms. This is entirely analogous to the way holographic renormalization works in AdS/CFT, where a radial foliation is introduced near the AdS boundary, and different choices of foliations lead to different values of the boundary action.[21,26–28,36,58] On the boundary, this is simply the statement that UV regulators break conformal invariance. However, we do demand that the renormalization procedure respects boundary covariance, a point which was also emphasized in Ref. 53. Thus, we do not introduce any background structures beyond the foliation.

We start by taking a variation of the integrated action (5.2) and using the equation of motion (2.5) to eliminate the bulk contribution. Applying the decomposition (2.12) of the symplectic potential results in[aa]

$$\delta S_\upsilon = \sum_j \int_{\mathcal{N}_{j,\upsilon}} \mathcal{E}_j + \sum_j \int_{\partial\mathcal{N}_{j,\upsilon}} (\beta'_j + \delta c_j)\,. \tag{5.3}$$

We next use the decomposition (2.43) of β'_j to give

$$\delta S_\upsilon = \sum_j \int_{\mathcal{N}_{j,\upsilon}} \mathcal{E}_j + \sum_j \int_{\partial\mathcal{N}_{j,\upsilon}} (\lambda' + \varepsilon_j)\,. \tag{5.4}$$

Finally, we note that λ' is assumed to be continuous on $\partial\mathcal{D}_\upsilon$, implying continuity at the interfaces between different boundaries $\mathcal{N}_j$. Since the contributions from λ' always occur in pairs with opposite signs coming from the two $\mathcal{N}_{j,\upsilon}$ intersecting at each corner, the overall contribution from λ' vanishes, and so

$$\delta S_\upsilon = \sum_j \int_{\mathcal{N}_{j,\upsilon}} \mathcal{E}_j + \sum_j \int_{\partial\mathcal{N}_{j,\upsilon}} \varepsilon_j\,. \tag{5.5}$$

By assumption, the left-hand side has a finite limit as $\upsilon \to 0$, implying that the sum of all the boundary and corner fluxes on the right must also be finite in the limit. Hence, if any individual contribution in these sums diverges, it must be canceled by a divergence appearing in a different term. In this case, one expects to be able to choose each of the corner fluxes ε_j such that $\mathcal{E}_j + d\varepsilon_j$ has a finite limit.

To see how this is borne out in more detail, we can focus on the standard example given in Fig. 1 in which the codimension-1 boundaries consist of Σ_υ, Σ^f_υ and $\mathcal{N}_\upsilon$, and the corners are $\partial\Sigma_\upsilon$ and $\partial\Sigma^f_\upsilon$. The spatial surfaces Σ_υ and Σ^f_υ intersect the Φ foliation transversally, and so any divergence coming from an integral of the respective fluxes over these surfaces must be localized in the $\Phi \to 0$ regions of these surfaces, which are just their boundaries. Since the boundaries of Σ_υ and Σ^f_υ coincide with the boundaries of $\mathcal{N}_\upsilon$, it follows that any remaining divergence in the flux on $\mathcal{N}_\upsilon$ must cancel against divergences at its boundary. Hence, it is possible to shift the flux $\mathcal{E}_\mathcal{N}$ by an exact term to cancel its divergence. This allows us to conclude that we can arrange for the ε_j to be chosen such that $\mathcal{E}_j + d\varepsilon_j$ is finite on each boundary.

[aa]Note that some of the boundaries $\partial\mathcal{N}_{j,\upsilon}$ may have multiple components, in which case the quantities c'_j and ε_j are understood to be independently specifiable on each component.

Finally, from the decomposition (2.43), it follows that the symplectic potential θ' is finite up to exact terms and total variations. Hence we can find a JKM transformation (2.36) that makes the symplectic potential at any given asymptotic boundary finite. This renormalized symplectic potential will in general depend on the choice of foliation.

5.3. *Explicit renormalization using background structures*

In this subsection we provide an explicit algorithm for holographic renormalization in general contexts, based on allowing the counterterms to depend on a rigging vector field in addition to a foliation. The intent is to provide an existence proof for background-dependent counterterms. However we expect that in applications it will be generally possible to find counterterms that depend only on a foliation. Note that the dependence of the counterterms on the additional background structure provided by the rigging vector is possibly related to the nonlocality of these terms relative to the free data on the asymptotic boundary, as discussed in Refs. 103 and 104.

As discussed in Subsec. 5.1, the setup is that we have a region $\mathcal{D}$ in a $(d+1)$-dimensional space–time, and a portion $\mathcal{N}$ of the boundary of $\mathcal{D}$, where we are using conformal compactification to treat the asymptotic boundary $\mathcal{N}$ as a finite boundary. We assume that the Lagrangian L' and symplectic potential θ' are smooth in the interior of $\mathcal{D}$ but can diverge on $\mathcal{N}$, and assume a smooth conformal factor Φ with $\Phi = 0$ on $\mathcal{N}$. We fix a rigging vector field $\mathbf{v}$ which is defined on a neighborhood of $\mathcal{N}$, is nowhere vanishing, and satisfies $v^a \nabla_a \Phi = 1$.[105,106] Note that Φ is determined in terms of $\mathbf{v}$ from this condition and the condition $\Phi = 0$ on $\mathcal{N}$.

Consider now a boundary canonical transformation (2.38) parametrized by B and Λ. If we combine this with a JKM transformation (2.36) with $a = B$ and $\nu = -\Lambda$, the Lagrangian and symplectic potential transform as

$$L' \to L'_{\text{ren}} = L' + dB\,, \tag{5.6a}$$

$$\theta' \to \theta'_{\text{ren}} = \theta' + \delta B - d\Lambda\,, \tag{5.6b}$$

The two main results of this section are:

(1) There exists a transformation of this kind for which the renormalized Lagrangian and symplectic potential (5.6) (and not just their pullbacks to surfaces of constant Φ) have finite limits to the boundary $\mathcal{N}$. We will construct this transformation explicitly. This is sufficient to make the charges $\tilde{H}_\xi$ finite, assuming the property of corner terms described in step (3) of Subsec. 5.1.

(2) The anomaly

$$\Delta_\xi \underline{\theta}'_{\text{ren}} \tag{5.7}$$

in the pullback to the boundary of the renormalized symplectic potential is the sum of an exact term and a total variation, both of which are finite on the boundary.

The derivation of these results is given in App. F. Here we summarize some of the details.

5.3.1. *Renormalized symplectic potential and Lagrangian*

We start by defining some notations. We define for $\upsilon > 0$ a map

$$\pi_\upsilon : \mathcal{N} \to \mathcal{D}\,, \tag{5.8}$$

which moves any point υ units along the integral curve of $\mathbf{v}$ passing through that point. The image of $\mathcal{N}$ under this mapping is the surface $\Phi = \upsilon$ which we will denote by $\mathcal{N}_\upsilon$. Any differential form Λ on a neighborhood of $\mathcal{N}$ is uniquely determined by specifying (i) $i_v\Lambda$, and (ii) the pullbacks $\pi_\upsilon^*\Lambda$ for the values of υ that cover the neighborhood, via $\Lambda = d\Phi \wedge i_v\Lambda + \Lambda_h$ with $i_v\Lambda_h = 0$ and $\pi_\upsilon^*\Lambda_h = \pi_\upsilon^*\Lambda$.

The quantities B and Λ that define the boundary canonical transformation (5.6) are given by

$$i_v B = 0\,, \tag{5.9a}$$

$$\pi_\upsilon^* B = \int_\upsilon^{\upsilon_0} d\bar{\upsilon}\, \pi_{\bar{\upsilon}}^* i_v L'\,, \tag{5.9b}$$

$$i_v \Lambda = 0\,, \tag{5.9c}$$

$$\pi_\upsilon^* \Lambda = -\int_\upsilon^{\upsilon_0} d\bar{\upsilon}\, \pi_{\bar{\upsilon}}^* i_v \theta'\,, \tag{5.9d}$$

where we have chosen a fixed $\upsilon_0 > 0$. The renormalized Lagrangian and symplectic potential are given by

$$i_v L'_{\text{ren}} = 0\,, \tag{5.10a}$$

$$i_v \theta'_{\text{ren}} = 0\,, \tag{5.10b}$$

$$\pi_\upsilon^* \theta'_{\text{ren}} = \pi_{\upsilon_0}^* \theta'\,. \tag{5.10c}$$

The expressions (5.9) and (5.10) become more transparent when expressed in a suitable class of coordinate systems. We choose a coordinate system $(x^0, x^1, \ldots, x^d)$ for which $x^0 = \Phi$ and $\mathbf{v} = \partial/\partial x^0$. We define for convenience the basis d-forms, $(d-1)$-forms and $(d-2)$-forms

$$\varpi = dx^1 \wedge \cdots \wedge dx^d\,, \tag{5.11a}$$

$$\varpi_i = -i_{\partial_i}\varpi = (-1)^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^d\,, \tag{5.11b}$$

$$\varpi_{ij} = i_{\partial_i}\varpi_j\,, \tag{5.11c}$$

where the hat on a basis one-form means that one-form is omitted in the wedge product, and i and j run over $1 \cdots d$. We expand the symplectic potential θ', Lagrangian L', and boundary canonical transformation forms B and Λ as

$$L' = \mathcal{L}\, dx^0 \wedge \varpi\,, \tag{5.12a}$$

$$\theta' = \theta'^{\,0}\varpi + \theta'^{\,i}\, dx^0 \wedge \varpi_i\,, \tag{5.12b}$$

$$B = B^0 \varpi + B^i \, dx^0 \wedge \varpi_i \,, \tag{5.12c}$$

$$\Lambda = \Lambda^i \varpi_i + \Lambda^{ij} \, dx^0 \wedge \varpi_{ij} \,, \tag{5.12d}$$

together with a similar notation for L'_{ren} and θ'_{ren}. Using these notations, Eqs. (5.9) reduce to

$$B^i = 0 \,, \tag{5.13a}$$

$$B^0(\upsilon, x^j) = \int_{\upsilon}^{\upsilon_0} d\bar{\upsilon} \, \mathcal{L}(\bar{\upsilon}, x^j) \,, \tag{5.13b}$$

$$\Lambda^i(\upsilon, x^j) = -\int_{\upsilon}^{\upsilon_0} d\bar{\upsilon} \, \theta'^{\,i}(\bar{\upsilon}, x^j) \,, \tag{5.13c}$$

$$\Lambda^{ij}(\upsilon, x^j) = 0 \,, \tag{5.13d}$$

where we have written $(x^0, x^1, \ldots, x^d) = (\upsilon, x^1, \ldots, x^d) = (\upsilon, x^j)$. Similarly Eqs. (5.10) reduce to

$$\mathcal{L}_{\text{ren}} = 0 \,, \tag{5.14a}$$

$$\theta'^{\,i}_{\text{ren}} = 0 \,, \tag{5.14b}$$

$$\theta'^{\,0}_{\text{ren}}(\upsilon, x^j) = \theta'^{\,0}_{\text{ren}}(0, x^j) = \theta'^{\,0}(\upsilon_0, x^j) \,. \tag{5.14c}$$

We now turn to some examples of applications of this formalism. In many cases we can split the Lagrangian and symplectic potential into diverging and finite pieces, $L' = L'_{\text{div}} + L'_{\text{finite}}$, $\theta' = \theta'_{\text{div}} + \theta'_{\text{finite}}$, such that the diverging pieces obey the identity (2.5) on shell, $\delta L'_{\text{div}} = d\theta'_{\text{div}}$. It is convenient then to compute the boundary canonical transformation using just the diverging pieces, which is sufficient to make L'_{ren} and θ'_{ren} finite. In this case the result (5.14) for the renormalized Lagrangian and symplectic potential on the boundary becomes

$$\mathcal{L}_{\text{ren}}(0, x^j) = \mathcal{L}_{\text{finite}}(0, x^j) \,, \tag{5.15a}$$

$$\theta'^{\,i}_{\text{ren}}(0, x^j) = \theta'^{\,i}_{\text{finite}}(0, x^j) \,, \tag{5.15b}$$

$$\theta'^{\,0}_{\text{ren}}(0, x^j) = \theta'^{\,0}_{\text{finite}}(0, x^j) + \theta'^{\,0}_{\text{div}}(\upsilon_0, x^j) \,. \tag{5.15c}$$

If we can further choose υ_0 to make the second term in Eq. (5.15c) vanish, then we obtain

$$\theta'^{\,0}_{\text{ren}}(0, x^j) = \theta'^{\,0}_{\text{finite}}(0, x^j) \,, \tag{5.16}$$

so the pullback of the renormalized symplectic potential is just the pullback of its finite piece.

A class of examples which includes vacuum general relativity at null infinity (see Sec. 6) is when L'_{div} and θ'_{div} are finite polynomials in Φ^{-1}. In this case the second term in Eq. (5.15c) can be made to vanish by choosing $\upsilon_0 = \infty$. The choice $\upsilon_0 = x^0 = \infty$ seems at first glance to be problematic, since the coordinates need only be defined for a finite range of values of x^0. However, we can regard the choice of value of v_0 as a specification of a prescription for finding antiderivatives of the specific functions encountered in Eqs. (5.13), so the inconsistency can be finessed.

A more general class of examples is obtained by allowing log terms, which arise for example in asymptotically AdS space–times. Suppose that there exists some integer t for which $\Phi^t L'$ and $\Phi^t \theta'$ are smooth functions of Φ and $\Phi \log \Phi$. Then we can expand L' and θ' as

$$\theta'^{\,0} = \sum_{p=-t}^{-1} \sum_{q=0}^{\infty} \theta^{0(p,q)} \Phi^p (\Phi \log \Phi)^q + \theta'^{\,0}_{\text{finite}} , \tag{5.17a}$$

$$\theta'^{\,i} = \sum_{p=-t}^{-1} \sum_{q=0}^{\infty} \theta^{i(p,q)} \Phi^p (\Phi \log \Phi)^q + \theta'^{\,i}_{\text{finite}} , \tag{5.17b}$$

$$\mathcal{L} = \sum_{p=-t}^{-1} \sum_{q=0}^{\infty} \mathcal{L}^{(p,q)} \Phi^p (\Phi \log \Phi)^q + \mathcal{L}_{\text{finite}} , \tag{5.17c}$$

where the coefficients depend only on x^i and not on $x^0 = \Phi$. The integrals in Eqs. (5.13) can conveniently be evaluated by assuming similar expansions for the integrals and equating coefficients of $\Phi^p(\Phi \log \Phi)^q$, which yields recursion relations that can be solved. This yields

$$\Lambda^i = \sum_{\substack{k=-t+1 \\ k\neq 0}}^{\infty} \sum_{p=-t+1}^{\min(k,-1)} \sum_{j=-t}^{p-1} \frac{1}{k} \left\{ \prod_{l=j+1}^{p-1} \left(\frac{l}{k} - 1 \right) \right\} \theta^{i(j,k-j-1)} \Phi^p (\Phi \log \Phi)^{k-p}$$
$$- \sum_{p=-t}^{-1} \frac{1}{p} \theta^{i\,(p,-p-1)} \Phi^p (\Phi \log \Phi)^{-p} \tag{5.18}$$

and

$$B^0 = - \sum_{\substack{k=-t+1 \\ k\neq 0}}^{\infty} \sum_{p=-t+1}^{\min(k,-1)} \sum_{j=-t}^{p-1} \frac{1}{k} \left\{ \prod_{l=j+1}^{p-1} \left(\frac{l}{k} - 1 \right) \right\} \mathcal{L}^{(j,k-j-1)} \Phi^p (\Phi \log \Phi)^{k-p}$$
$$+ \sum_{p=-t}^{-1} \frac{1}{p} \mathcal{L}^{(p,-p-1)} \Phi^p (\Phi \log \Phi)^{-p} . \tag{5.19}$$

Here we have effectively chosen $v_0 = \infty$ for terms in the integrands with $p+q < -1$, $v_0 = 1$ for $p + q = -1$, and $v_0 = 0$ for $p + q > -1$. These choices again make the second term in Eq. (5.15c) effectively vanish,[bb] so we again recover the result (5.16).

[bb]Note that the condition $\delta L'_{\text{div}} = d\theta'_{\text{div}}$ is not satisfied in this case due to mixing with the finite terms, but one can directly check that Eq. (5.16) is still valid.

5.3.2. *Anomalies of renormalized symplectic form and Lagrangian*

Although the renormalized Lagrangian L'_{ren} and renormalized symplectic potential θ'_{ren} are finite, they are no longer covariant, assuming one which starts with covariant quantities. In this section we will discuss the corresponding anomalies, which are computed explicitly in App. F. This will yield the second result described above, that the dependence of the pullback of the renormalized symplectic potential on the background structures arises only through corner and boundary terms.

We will show that the anomalies in B and Λ are of the form

$$\Delta_{\hat{\xi}} B = (\Delta_{\hat{\xi}} B)_{\text{finite}} + d\kappa_{\xi} \,, \tag{5.20a}$$

$$\Delta_{\hat{\xi}} \Lambda = (\Delta_{\hat{\xi}} \Lambda)_{\text{finite}} + \delta\kappa_{\xi} - d\mu_{\xi} - I_{\delta\hat{\xi}} \Lambda \,, \tag{5.20b}$$

where the indicated quantities are finite and κ_ξ and μ_ξ are quantities that in general can diverge on the boundary. Inserting these expressions into Eqs. (5.6) for the renormalized Lagrangian and symplectic potential, and acting with the anomaly operator yields for field-independent symmetries

$$\Delta_{\hat{\xi}} L'_{\text{ren}} = d(\Delta_{\hat{\xi}} B)_{\text{finite}} \,, \tag{5.21a}$$

$$\Delta_{\hat{\xi}} \theta'_{\text{ren}} = \delta(\Delta_{\hat{\xi}} B)_{\text{finite}} - d(\Delta_{\hat{\xi}} \Lambda)_{\text{finite}} \,. \tag{5.21b}$$

If we now take a pullback to the boundary, and use the fact that the pullback operator commutes with the anomaly operator and the space–time and phase space exterior derivatives d and δ, we obtain

$$\Delta_{\hat{\xi}} \underline{\theta}'_{\text{ren}} = \delta {\pi_0}^* \big(\Delta_{\hat{\xi}} B\big)_{\text{finite}} - d {\pi_0}^* \big(\Delta_{\hat{\xi}} \Lambda\big)_{\text{finite}} \,. \tag{5.22}$$

Here on the right-hand side we have denoted the pullback by $\pi_{0\,*}$ instead of using our usual boldface notation.

The explicit expressions for the quantities appearing in the anomalies (5.20) are as follows. The finite pieces are given by

$$i_v (\Delta_{\hat{\xi}} B)_{\text{finite}} = 0 \,, \tag{5.23a}$$

$$\pi_v^* (\Delta_{\hat{\xi}} B)_{\text{finite}} = \pi_{v_0}^* i_\xi L' \,, \tag{5.23b}$$

$$i_v (\Delta_{\hat{\xi}} \Lambda)_{\text{finite}} = 0 \,, \tag{5.23c}$$

$$\pi_v^* (\Delta_{\hat{\xi}} \Lambda)_{\text{finite}} = -\pi_{v_0}^* \left[i_v \theta' \pounds_\xi \Phi \right] . \tag{5.23d}$$

In coordinate notation these relations are

$$(\Delta_{\hat{\xi}} B)^i_{\text{finite}} = 0 \,, \tag{5.24a}$$

$$(\Delta_{\hat{\xi}} B)^0_{\text{finite}}(v, x^j) = (\xi^0 \mathcal{L})(v_0, x^j) \,, \tag{5.24b}$$

$$(\Delta_{\hat{\xi}} \Lambda)^{ij}_{\text{finite}} = 0 \,, \tag{5.24c}$$

$$(\Delta_{\hat{\xi}} \Lambda)^i_{\text{finite}}(v, x^j) = -(\xi^0 \theta'^{\,i})(v_0, x^j) \,, \tag{5.24d}$$

where we have decomposed the symmetry generator as $\boldsymbol{\xi} = \xi^0 \partial_0 + \xi^i \partial_i$. The quantities κ_ξ and μ_ξ are given by

$$i_v \kappa_\xi = 0\,, \tag{5.25a}$$

$$\pi_v^* \kappa_\xi = \int_v^{v_0} d\bar{v}\, \pi_{\bar{v}}^* i_\xi i_v L' - i_{\pi_v^* \xi} \int_v^{v_0} d\bar{v}\, \pi_{\bar{v}}^* i_v L'\,, \tag{5.25b}$$

$$i_v \mu_\xi = 0\,, \tag{5.25c}$$

$$\pi_v^* \mu_\xi = \int_v^{v_0} d\bar{v}\, \pi_{\bar{v}}^* i_\xi i_v \theta' - i_{\pi_v^* \xi} \int_v^{v_0} d\bar{v}\, \pi_{\bar{v}}^* i_v \theta'\,. \tag{5.25d}$$

In coordinate notation these relations are

$$\kappa_\xi = \left(\xi^i \int_v^{v_0} \mathcal{L}\, d\bar{v} - \int_v^{v_0} \xi^i \mathcal{L}\, d\bar{v} \right) \varpi_i\,, \tag{5.26a}$$

$$\mu_\xi = -\frac{1}{2} \left[\xi^i \int_v^{v_0} \theta'^{\,j}\, d\bar{v} - \xi^j \int_v^{v_0} \theta'^{\,i}\, d\bar{v} - \int_v^{v_0} (\xi^i \theta'^{\,j} - \xi^j \theta'^{\,i}) d\bar{v} \right] \varpi_{ij}\,, \tag{5.26b}$$

where the integrands are evaluated at $v = \bar{\varepsilon}$.

5.4. *Renormalization of corner terms*

In this subsection we show that it is always possible to find a corner canonical transformation (2.51) that makes the integrated action S, corner flux ε as well as $\ell' + dc'$ finite.

The basic idea is the trivial integral identity, for any function $\mathcal{L}(u, v)$ of two variables u, v:

$$\int_{u_0}^{\infty} d\bar{u} \int_{v_0}^{\infty} d\bar{v}\, \mathcal{L}(\bar{u}, \bar{v}) = \int_u^{\infty} d\bar{u} \int_v^{\infty} d\bar{v}\, \mathcal{L}(\bar{u}, \bar{v}) - \int_u^{u_0} d\bar{u} \int_v^{\infty} d\bar{v}\, \mathcal{L}(\bar{u}, \bar{v}) - \int_u^{\infty} d\bar{u} \int_v^{v_0} d\bar{v}\, \mathcal{L}(\bar{u}, \bar{v}) + \int_u^{u_0} d\bar{u} \int_v^{v_0} d\bar{v}\, \mathcal{L}(\bar{u}, \bar{v})\,. \tag{5.27}$$

Here the first term on the right-hand side will be the bulk action, which can diverge as $v \to 0$ or $u \to 0$ (we assume that the integrals are finite at large u and v). The second and third terms on the right-hand side are boundary terms that are added at the boundaries $u =$ constant and $v =$ constant (Eq. (5.13)). Finally the fourth term is the corner term that when added makes the total integral manifestly finite in the limit $u \to 0$, $v \to 0$, as can be seen from the left-hand side.

We now translate this idea into a covariant notation and add the additional dimensions which were suppressed in the above argument. Consider two boundaries $\mathcal{N}$ and $\tilde{\mathcal{N}}$ which intersect in a $(d-1)$-surface $\mathcal{C}$. In Subsec. 5.3 we introduced a vector field $\mathbf{v}$ and a diffeomorphism π_v which moves points v units along integral curves of $\mathbf{v}$. We also introduced a coordinate v which vanishes on $\mathcal{N}$ and for which $v^a \nabla_a v = 1$.

We now introduce the additional background structure of a nowhere vanishing vector field $\mathbf{u}$ on the boundary $\mathcal{N}$. We can then introduce a coordinate u on $\mathcal{N}$ by $u = 0$ on $\mathcal{C}$ and $u^a \nabla_a u = 1$ on $\mathcal{N}$. We can then extend the definitions of $\mathbf{u}$ and u off $\mathcal{N}$ by Lie transporting with respect to $\mathbf{v}$. Finally we define the diffeomorphism $\tilde{\pi}_u$ to be the map that moves points u units along integral curves of $\mathbf{u}$.

Using this notation, the boundary term B associated with the boundary $\mathcal{N}$ is given by Eqs. (5.9) and (5.13). There is an analogous boundary term $\tilde{B}$ for the boundary $\tilde{\mathcal{N}}$, given by

$$i_u \tilde{B} = 0\,, \tag{5.28a}$$

$$\tilde{\pi}_u^* \tilde{B} = \int_u^{u_0} d\bar{u}\, \tilde{\pi}_{\bar{u}}^* i_u L'\,. \tag{5.28b}$$

Finally the corner term c on $\mathcal{C}$ required to make the total action finite is given by

$$i_u c = 0\,, \tag{5.29a}$$

$$i_v c = 0\,, \tag{5.29b}$$

$$\tilde{\pi}_u^* \pi_v^* c = \int_u^{u_0} d\bar{u} \int_v^{v_0} d\bar{v}\, \tilde{\pi}_{\bar{u}}^* \pi_{\bar{v}}^* i_u i_v L'\,. \tag{5.29c}$$

6. Vacuum General Relativity at Future Null Infinity: Gravitational Charges

In this section, to explicitly demonstrate the holographic renormalization procedure described in Subsec. 5.3, we specialize to the case of future null infinity in vacuum general relativity in four-dimensional asymptotically flat space–times, and to the generalized BMS (GBMS) field configuration space (4.3). We then compute the renormalized symplectic potential and the localized charge using the results laid out in Subsec. 5.3 and Sec. 2, with the conformal factor and rigging vector field taken to be those associated with Bondi–Sachs coordinates. The results obtained in this section agree with expressions for the charges obtained previously in, e.g. Refs. 31 and 107, explicitly demonstrating the utility of the holographic renormalization algorithm described in Subsec. 5.3. Although we do not consider it here, the holographic renormalization algorithm should generalize to the larger WBMS group described in Subsec. 4.2, for which finite renormalized charges were obtained in Ref. 45.

We identify the coordinate system discussed in Subsec. 5.3 with the Bondi–Sachs coordinates according to $(x^0, x^1, x^2, x^3) = (\Phi, u, \theta, \phi) \equiv (\Phi, u, x^A)$. The physical metric in these coordinates corresponds to the line element[31,107,cc]

$$d\tilde{s}^2 = -U e^{2F}\, du^2 + 2e^{2F}\Phi^{-2}\, du\, d\Phi + \Phi^{-2} h_{AB} \left(dx^A - U^A du\right)\left(dx^B - U^B du\right). \tag{6.1}$$

[cc]We use the symbol "F" here in place of the symbol β that is used in Refs. 31 and 107 to avoid confusion with our symbol for the corner term in decomposition of the symplectic potential [as in (2.12)].

Here we have used $\Phi = 1/r$ in place of the radial coordinate r used in Refs. 31 and 107 (see, e.g. the discussion in Ref. 108 for more details on the construction of the Bondi–Sachs coordinate system in the physical space–time). In these coordinates, $\Phi = 0$ corresponds to $\mathscr{I}^+$, and the null generator of $\mathscr{I}^+$ is given by $n^i \mathrel{\widehat{=}} (\partial_u)^i$. Moreover, in these coordinates, we have a foliation of $\mathscr{I}^+$ by cross-sections of constant u. The 1-form on $\mathscr{I}^+$ normal to this foliation is given by $l_i := -\nabla_i u$. The functions appearing in (6.1) are smooth in their dependence on (Φ, u, x^A) and their expansions in powers of Φ, after imposing the Einstein equations, are given by[31]

$$U = \frac{1}{2}\mathcal{R} - 2\Phi M + O\left(\Phi^2\right), \tag{6.2a}$$

$$F = -\frac{1}{32}\Phi^2 C^{AB} C_{AB} + O\left(\Phi^3\right), \tag{6.2b}$$

$$U^A = -\frac{1}{2}\Phi^2 \mathscr{D}_B C^{AB} + 2\Phi^3 L^A + O\left(\Phi^4\right), \tag{6.2c}$$

$$h_{AB} = q_{AB} + \Phi C_{AB} + \frac{1}{4}\Phi^2 q_{AB} C^{CD} C_{CD} + O\left(\Phi^3\right), \tag{6.2d}$$

where $\mathcal{R}$ denotes the Ricci scalar of the leading order sphere metric q_{AB}, and $\sqrt{q}$ denotes the square root of its determinant. In addition, $\mathscr{D}_A$ is the derivative operator compatible with q_{AB}. Moreover, C_{AB} is (-2 times) the shear associated with the auxiliary normal $l^a := -\Phi^2 \tilde{g}^{ab} \nabla_a u$ and satisfies $q^{AB} C_{AB} = 0$ as well as $\delta(q^{AB} C_{AB})$ for all perturbations. Furthermore, M denotes the Bondi mass aspect and L^A is related to the angular momentum aspect.[dd] Note also that it follows from (4.3) that $\delta n^i = \delta\sqrt{q} = 0$. Finally, capital Roman indices are raised and lowered using q_{AB} throughout this section.

Next, we compute the symplectic potential θ', which we take to be the space–time covariant one given in Eq. (39) of Ref. 34 and so here $\theta' = \overset{c}{\theta}$. We will also take L' to be the covariant Einstein–Hilbert Lagrangian, so $L' = \overset{c}{L}$. We make use of the Einstein equations for the background metric, the linearized Einstein equations for the perturbations and the Bondi condition (D.9). We find that the symplectic potential diverges as Φ^{-2} and that there are no logarithmic divergences. In particular, the divergent pieces are given in the notation of (5.17) by[ee] (setting $16\pi G = 1$)

$$\begin{gathered}\theta_0^{(-2,0)} = 0\,, \quad \theta_0^{(-1,0)} = \sqrt{q}\left(-\delta\mathcal{R} - \frac{1}{2} N_{AB}\delta q^{AB}\right), \\ \theta_1^{(-2,0)} = -\frac{\sqrt{q}}{2} C_{AB}\delta q^{AB}\,, \quad \theta_1^{(-1,0)} = 0\,.\end{gathered} \tag{6.3}$$

[dd]The angular momentum aspect N_A, defined in Ref. 107, is related to L_A by $N_A = -3L_A + \frac{3}{32}\mathscr{D}_A(C_{BC}C^{BC}) + \frac{3}{4}C_A{}^B \mathscr{D}^C C_{BC}$.

[ee]We omit writing the explicit expressions for $\theta_A^{(-2,0)}$ and $\theta_A^{(-1,0)}$ since they will not be needed for the explicit charge calculation later in this section.

Note that here δq^{AB} is the variation of the inverse metric, q^{AB}, and not $q^{AC}q^{BD}\delta q_{CD}$. In addition, we have

$$\theta_0^{(0,0)} = \sqrt{q}\left[2\delta M + 2\partial_u \delta Z + \delta(\mathscr{D}_A \mathscr{U}^A) + \frac{1}{2}N_{AB}\delta C^{AB} - \frac{1}{4}\mathcal{R}C_{AB}\delta q^{AB} - \mathscr{D}_A \mathscr{U}_B \delta q^{AB}\right]. \tag{6.4}$$

Note also that here

$$N_{AB} := \partial_u C_{AB}\,, \quad Z := -\frac{1}{32}C_{AB}C^{AB}\,, \quad \mathscr{U}^A := -\frac{1}{2}\mathscr{D}_B C^{AB}\,. \tag{6.5}$$

We now consider the renormalization of the symplectic potential. Using the expression (5.18) specialized to $t = 2$, $d = 3$ with the coefficients of the logarithmic terms taken to vanish, we have that

$$\begin{aligned}\Lambda = &- \left[\log \Phi \theta_1^{(-1,0)} - \Phi^{-1}\theta_1^{(-2,0)}\right] dx^2 \wedge dx^3 \\ &- \left[\log \Phi\, \theta_3^{(-1,0)} - \Phi^{-1}\theta_3^{(-2,0)}\right] dx^1 \wedge dx^2 \\ &- \left[\log \Phi \theta_2^{(-1,0)} - \Phi^{-1}\theta_2^{(-2,0)}\right] dx^3 \wedge dx^1\,.\end{aligned} \tag{6.6}$$

We follow the procedure described in Sec. 5 which instructs us to find a boundary canonical transformation that yields finite renormalized quantities $L'_{\text{ren}} = L' + dB$ and $\theta'_{\text{ren}} = \theta' - d\Lambda$ [recall (5.6a) and (5.6b)]. Since we have taken $L' = \overset{c}{L}$ and $\theta' = \overset{c}{\theta}$, we pick $\Lambda = -\lambda'$ and $B = b'$ [recall (2.8a) and (2.8b)] to parametrize our boundary canonical transformation. Note however that in vacuum general relativity with zero cosmological constant, the Lagrangian vanishes on shell. For that reason, it does not need to be renormalized, and so we take $b' = B = 0$. Using the explicit expressions for the (unrenormalized) symplectic potential along with the linearized Einstein equations to compute $\theta'_{\text{ren}} = \theta' - d\Lambda$ [(5.6b) with $B = 0$], we find that the effect of adding $d\Lambda$ is to cancel the diverging pieces in each component of θ' while leaving the finite pieces unchanged. Moreover, the pullback of the renormalized symplectic potential to $\mathscr{I}^+$ is given by (6.4).

Note that our expression for the symplectic potential and the subsequent renormalization procedure, when implemented in Bondi coordinates, coincide with those in Ref. 31. Note also that even though we have demonstrated our renormalization procedure for conditions that correspond to the generalized BMS configuration space, subject to the Bondi condition, the procedure itself is completely general and can be applied to any of the extensions of the BMS algebra discussed in Sec. 4, with or without the Bondi condition. It is guaranteed to work in any of these cases using the general algorithm described in Subsec. 5.3.

Having obtained an expression for the pullback of the (renormalized) symplectic potential, we now seek to obtain a decomposition of it into a boundary term, a corner term, and a flux term in keeping with (2.12), that is, a decomposition of the form

$$\underline{\theta}'_{\text{ren}} \mathrel{\hat{=}} -\delta\ell' + d\beta' + \mathcal{E}\,. \tag{6.7}$$

Comparing this with (6.4) suggests the following choice for the flux term $\mathcal{E}$

$$\mathcal{E} = -\eta\left[\frac{1}{2}N_{AB}\delta C^{AB} - \frac{1}{4}\mathcal{R}C_{AB}\delta q^{AB} - \mathscr{D}_A\mathscr{U}_B\delta q^{AB}\right], \tag{6.8}$$

where η is the volume element on $\mathscr{I}^+$ given by $\eta = -\sqrt{q}\, du \wedge d\theta \wedge d\phi$. Moreover, we can read off that

$$\ell' = \eta\left[2M + 2\partial_u Z + \mathscr{D}_A\mathscr{U}^A\right], \quad \beta' = 0\,, \tag{6.9}$$

where we have used the fact that $\partial_u\delta Z = \pounds_n\delta Z = \delta\partial_u Z$ since $\delta n^i = 0$.

Note that while C_{AB} is not an intrinsic quantity on $\mathscr{I}^+$, its variation still occurs in our expression for the flux in (6.8). This appears to be at odds with the Dirichlet form of the flux advocated for in this work, since C_{AB} is related to the extrinsic geometry of $\mathscr{I}^+$ with respect to the auxiliary null direction l^a. In asymptotically dS or AdS spaces, the equations of motion allow one to solve for C_{AB} in terms of the leading metric q_{AB} at $\mathscr{I}^+$,[109–111] and hence flux terms involving δC_{AB} are still consistent with Dirichlet form. This is no longer the case in asymptotically flat space–times, in which C_{AB} represents free data on $\mathscr{I}^+$. Nevertheless, from (6.2d) we see that C_{AB} is a subleading component of the spherical part of the metric, h_{AB}, which is an intrinsic quantity on each $\Phi = \text{const}$ surface, which limit to $\mathscr{I}^+$. It is therefore not entirely surprising that C_{AB} appears as a configuration variable in the expression for the flux. Furthermore, the news tensor N_{AB} that appears conjugate to C^{AB} in the expression of the flux is given by the u-derivative of C_{AB} according to (6.5), as one would expect of a momentum variable, lending additional support to interpreting (6.8) as the appropriate analog of the Dirichlet form of the flux. An interesting question for future work would be to understand better the principle for selecting a preferred form of the flux for asymptotically null surfaces, rather than postulating the form as is done in this section.

We now proceed to calculate

$$H_\xi = \lim_{S' \to S}\int_{S'} h_\xi\,, \tag{6.10}$$

where S' here denotes $u = \text{const}$ cross-sections of a one-parameter family of $\Phi = \text{const}$ surfaces that limit to $\mathscr{I}^+$ in the unphysical space–time. As denoted above, to define the charge, H_ξ, on a cross-section, S, of $\mathscr{I}^+$, we will perform this integral and then take the limit $S' \to S$. We calculate (2.30), where ξ^a for a generalized BMS vector field is given by[15]

$$\begin{aligned}\xi^a = f\partial_u + \left[Y^A - \Phi\mathscr{D}^A f + \frac{1}{2}\Phi^2 C^{AB}\mathscr{D}_B f + O(\Phi^3)\right]\partial_A \\ + \Phi^2\left[\frac{1}{2}\Phi^{-1}\mathscr{D}_A Y^A - \frac{1}{2}\mathscr{D}^2 f - \frac{1}{2}\Phi\mathscr{U}^A\mathscr{D}_A f + \frac{1}{4}\Phi\mathscr{D}_A\left(\mathscr{D}_B f C^{AB}\right) + O(\Phi^2)\right]\partial_\Phi\,.\end{aligned} \tag{6.11}$$

Here $f(u, x^A) = \gamma(x^A) + \frac{1}{2} u \mathscr{D}_B Y^B(x^A)$ where $\gamma(x^A)$ is the supertranslation function and Y^A is the generator of arbitrary smooth diffeomorphisms on S^2. Moreover, Q'_ξ is given by (2.26) and $\overset{vc}{Q}_\xi$ in vacuum general relativity is given by (where recall that we have set $\frac{1}{16\pi G} = 1$)

$$\overset{vc}{Q}_{\xi_{ab}} = -\tilde{\epsilon}_{cdab} \tilde{\nabla}^c \xi^d \,. \tag{6.12}$$

Using (6.3), (6.6), (6.11) and the fact that

$$I_{\hat{\xi}} \delta q^{AB} = -\mathscr{D}^A Y^B - \mathscr{D}^B Y^A + q^{AB} \mathscr{D}_C Y^C \,, \tag{6.13}$$

we find that

$$I_{\hat{\xi}} \lambda' = I_{\hat{\xi}} \Lambda = -\Phi^{-1} \mu C_{AB} \mathscr{D}^A Y^B + \cdots \,, \tag{6.14}$$

where $\cdots$ denotes terms that vanish upon pullback to S' and are hence not relevant for the calculation of the charge. Note also that $\mu = -i_n \eta = \sqrt{q}\, d\theta \wedge d\phi$. Then, explicitly calculating $\overset{vc}{Q}_\xi$, one finds that its pullback to S' has a piece that diverges as $\Phi \to 0$ that is given by $\Phi^{-2} \mu\, \mathscr{D}_A Y^A - \Phi^{-1} \mu\, Y^A \mathscr{D}^B C_{AB}$. The first term is a total derivative which drops out of the integral over S'. Moreover, the second term cancels with (6.14) up to a total derivative term. We therefore see that upon integrating over S', the diverging piece drops out of $\int_{S'} Q'_\xi$. Taking the limit $S' \to S$ we then obtain

$$\begin{aligned} \int_S Q'_\xi = -\int_S \mu \Big[& 2f \partial_u Z - 2fM - \mathscr{U}^A \mathscr{D}_A f \\ & - 2Y^A \Big\{ N_A - \frac{1}{4} C_A{}^B \mathscr{D}^C C_{BC} - \frac{1}{16} \mathscr{D}_A (C_{BC} C^{BC}) \Big\} \Big] \,. \end{aligned} \tag{6.15}$$

Using this in addition to (6.9) and (6.11) to compute (2.30) and dropping total derivative terms, we find that the final expression for H_ξ is given by

$$H_\xi = \int_S \mu \Big[4fM + 2Y^A \Big\{ N_A - \frac{1}{4} C_A{}^B \mathscr{D}^C C_{BC} - \frac{1}{16} \mathscr{D}_A \big(C_{BC} C^{BC} \big) \Big\} \Big] \,. \tag{6.16}$$

This expression is the same as the one derived for the (usual) BMS charge in, for example Refs. 107, 108 and 57, and is also consistent with the expression for the charge given in (9.21) of Ref. 45. It was pointed out in Ref. 31 that this expression diverges in limits to the end-points of $\mathscr{I}^+$ (i.e as $u \to \pm\infty$) when one allows for the most general fall-offs in u of C_{AB}: $C_{AB} = O(u)$, that are compatible with the action of the GBMS group on the boundary fields. To cure these "corner" divergences, one would have to implement an additional renormalization step, similar in spirit to the one discussed in Subsec. 5.3. Presumably, one would have to add to the expression for Λ in (6.6) terms that are finite as $\Phi \to 0$ but which diverge as $u \to \pm\infty$. This would modify the expression for β' after which one would have to pick an expression for υ (see (6.20) and the discussion around it) which would lead to a different expression for the charge, H_ξ. However, addressing this issue is beyond the scope of this paper and we leave it to future work. Indeed, it would be interesting to carry out these steps to attempt to derive the GBMS charge expression in (5.49) of Ref. 31 which does not have the aforementioned divergences.

We note that the decomposition we picked in (6.9) was not unique even after having picked the expression for the flux, $\mathcal{E}$, which we take to be given by (6.8). Instead of the choice made in (6.9), one could instead have picked

$$\ell' = \eta\big[2M + \mathscr{D}_A\mathscr{U}^A\big]\,, \quad \beta' = 2\delta Z\mu\,. \tag{6.17}$$

Also, because of the Bondi condition, $(\mathscr{D}_A\mathscr{U}^A)\eta = d(i_{\mathscr{U}}\eta)$, and therefore one could also consider a decomposition of (6.4) in which

$$\ell' = \eta[2M + 2\partial_u Z]\,, \quad \beta' = -\delta(i_{\mathscr{U}}\eta)\,, \tag{6.18}$$

where we have defined a vector $\mathscr{U}^i$ on $\mathscr{I}^+$ such that $\mathscr{U}^i l_i = 0$ and $\mathscr{U}^A = \mathscr{U}^i e_i{}^A$ where $e_i{}^A$ is a projector onto angular directions. Finally, one could also consider

$$\ell' = 2\eta M\,, \quad \beta' = -\delta(i_{\mathscr{U}}\eta - 2Z\mu)\,. \tag{6.19}$$

To resolve the ambiguity between these choices, as described in Subsec. 2.5, one needs to implement a corner improvement where one looks for a decomposition of $\beta' - \lambda'$ of the form [see (2.43)]

$$\beta' - \lambda' = -\delta c' + d\gamma' + \varepsilon\,. \tag{6.20}$$

As described in (2.46), the improved expression for the charge density is given by $\tilde{h}_\xi = h_\xi - \Delta_{\hat{\xi}} c'$. To obtain unambiguous charges,[ff] one needs to fix an expression for ε in addition to the expression for $\mathcal{E}$ which we fixed to be given by (6.8). Here, we pick $\varepsilon = \Lambda$,[gg,hh] with the specific choice of Λ given by (6.6). Since we have computed $\lambda' = -\Lambda$, we see from Eq. (6.20) that this choice amounts to always setting $\delta c' = -\beta'$. It is then easy to check that calculating the charge in the same way as before but with h_ξ replaced with $\tilde{h}_\xi$ for each of the three cases given in Eqs. (6.17)–(6.19), the final charge expression remains unchanged and in each case is just given by (6.16), even though the boundary Lagrangian ℓ' is different in each case. This demonstrates that, as highlighted earlier in the paper, fixing an expression for the flux terms in the problem, on the boundary as well as the corners ($\mathcal{E}$ and ε in this case), gives us an unambiguous expression for the charge.

[ff] The γ' term in (6.20) only contributes an exact piece to the charge density and therefore, in the present context, its choice does not affect the charge. We therefore pick $\gamma' = 0$ here for convenience.
[gg] Note from (6.6) that Λ is actually divergent on $\mathscr{I}^+$, and so really this decomposition is done on a cutoff surface *near* $\mathscr{I}^+$ after extending β' arbitrarily away from $\mathscr{I}^+$. Obtaining a finite corner flux *on* $\mathscr{I}^+$ would entail a more careful analysis of corner terms in the action of vacuum general relativity which we leave to future work.
[hh] Note also that to ensure finiteness of the charge in the $u \to \pm\infty$ limits described earlier (an issue we have chosen to ignore here), one would need to pick a different expression for ε. Presumably, this would follow from a more careful analysis of the corner terms in the action and the resulting charge expression *will* indeed be modified in that case.

7. Discussion

We conclude with a discussion of a number of future directions and applications of this work.

7.1. *More general asymptotic symmetries*

The holographic renormalization argument presented in Sec. 5 demonstrates that all asymptotic charges can be rendered finite once appropriate counterterms have been found to produce a finite renormalized gravitational action. This holds without imposing asymptotic boundary conditions on the dynamical fields, and hence motivates exploring formulations of the theory in which the standard boundary conditions are relaxed. Indeed, the arguments of Sec. 5 were inspired by similar considerations for asymptotically anti-de Sitter space–times[36] in which the standard Dirichlet boundary condition was relaxed. This produces an enlarged asymptotic symmetry algebra for these space–times, which have been further explored in the works on the Λ-BMS group.[54,110] In the past, finiteness of the action and charges has been suggested as a reason for selecting boundary conditions for the theory, but the analysis of this work suggests that this is unnecessary, since finiteness can instead be achieved through holographic renormalization. The only reason for imposing boundary conditions should be to obtain a well-defined variational principle, or, equivalently, to ensure the phase space describes a closed system that does not lose symplectic flux through its boundary.

Relaxing the standard boundary conditions of four-dimensional asymptotically flat space–times leads to the enlarged symmetry algebras discussed in Sec. 4. Each of the symmetry groups described there still fixes some structure at null infinity, but since holographic renormalization applies in the absence of any such boundary condition, it is tempting to propose an even more general set of symmetries. These would be obtained by relaxing the final condition leading to the Weyl BMS configuration space (4.4), namely, not imposing n^i be fixed. We would expect to obtain in this manner all diffeomorphisms of $\mathscr{I}^+$ as asymptotic symmetries, and it would be interesting to compute expressions for the associated charges.[ii] The enlarged algebra may also be related to the extended symmetries of finite null surfaces explored in Ref. 113.

Another context in which extended symmetries can arise is in higher-dimensional asymptotically flat space–times. In higher than four space–time dimensions, there exist consistent boundary conditions that eliminate the supertranslations as asymptotic symmetries. However, in light of the relation between supertranslations and the Weinberg soft graviton theorems,[114–116] which hold in all dimensions, it is desirable to find a phase space in higher dimensions that admits a nontrivial action of supertranslations. Such relaxed boundary conditions have been explored in Refs. 94, 117 and 118, and the general holographic renormalization argument suggests that

[ii]The appearance of Diff($\mathscr{I}$) has also been suggested to appear in the context of asymptotically de Sitter and anti-de Sitter spaces in Ref. 112.

a phase space can be constructed in which these transformations produce finite charges.[jj] It would be interesting to carry out the holographic renormalization procedure in these higher-dimensional cases and to construct the phase space on which the renormalized BMS charges are defined, as well as to obtain charges associated with higher-dimensional versions of the symmetry algebras described in Sec. 4. Some ideas in this direction have been explored in Refs. 55, 119–121.

A final application would be to investigate the recently proposed $w_{1+\infty}$ symmetry of 4D asymptotically flat gravity, which was derived at the level of celestial amplitudes.[122] An interesting question to address is whether the charge generators of this algebra arise from asymptotic diffeomorphisms, to give a spacetime interpretation of the symmetry transformations. The holographic renormalization procedure in this work provides an ideal framework for investigating this question.

7.2. *Gluing and quantization*

One of the main motivations for considering localized charges is to understand the embedding of the localized phase spaces and their observables into the global phase space of the theory. In the classical context, understanding this embedding can help give meaning to quasilocal notions of energy, which are relevant in practice since astrophysical processes are usefully described using local descriptions of objects' locations and momenta, despite the fact that local observables are nonperturbatively ill-defined in a diffeomorphism-invariant theory. There is a natural construction known as Marsden–Weinstein symplectic reduction[123] by which localized phase spaces can be assembled into a global phase space, ensuring in the process that the localized charges become trivial, as would be expected for charges associated with a gauge symmetry. This application of symplectic reduction to the gluing of local phase spaces was discussed in the work of Donnelly and Freidel.[16] The idea is to take two adjacent localized phase spaces $\mathscr{P}_1$ and $\mathscr{P}_2$, each containing a set of charges H^i_ξ, $i = 1, 2$, associated with diffeomorphisms that act at their common boundary. One then constructs the product phase space $\mathscr{P}_{12} = \mathscr{P}_1 \times \mathscr{P}_2$, which also admits an action of the boundary symmetry, generated by the sum of the individual charges, $H^{\mathrm{tot}}_\xi = H^1_\xi + H^2_\xi$. The reduced phase space is obtained by then restricting to the submanifold of zero total charge $H^{\mathrm{tot}}_\xi = 0$, and further quotienting by the flow generated by the charges within this submanifold. This two-step process results in a new phase space $\mathscr{P}_{\mathrm{red}} = \mathscr{P}_{12}//G$, with G the group of boundary symmetries. The fact that the boundary symmetries should act trivially on the global phase space is now encapsulated by the restriction to the zero charge submanifold and further quotienting by the group action. This process thus gives a way of

[jj] These relaxed boundary conditions have been questioned in Ref. 95 on the grounds of not leading to finite fluxes through $\mathscr{I}^+$, but such divergences can always be handled by the procedure of holographic renormalization, at the expense of introducing some dependence on a background structure (see, for example Ref. 96).

realizing the individual phase spaces $\mathscr{P}_i$ within the global phase space, although they are not symplectic submanifolds due to the quotient procedure needed in the construction.

The work of Donnelly and Freidel focused on symmetries in general relativity that preserve the codimension-2 boundary of a subregion Cauchy surface.[16] Such symmetries are simpler to handle since the flux term in Hamilton's equations (2.29) identically vanishes (assuming covariant $\beta' - \lambda'$ and field-independent generators), and the Wald–Zoupas procedure is not needed in order to construct localized charges. The more general localized charges considered in this work are defined even when there are nonzero fluxes, and in Sec. 3 we showed that their Poisson brackets on the localized phase space are given by the BT bracket, which reproduces the diffeomorphism algebra of the vector fields (or a suitable modification when generators are field-dependent) whenever the extension term $K_{\xi,\zeta}$ can be shown to vanish. This is enough to apply the Marsden–Weinstein reduction procedure, since the localized charges generate an action of the boundary symmetry group on the localized phase space, even though this action does not generically act like a diffeomorphism on all observables, due to the failure of such a transformation to satisfy Hamilton's equation. It would be very interesting to carry out this procedure in more detail in order to better understand the relevance of localized charges within the full global phase space.

An even more interesting question is to understand how to apply the reduction in the case of nonvanishing extension terms in the algebra of localized charges, as in Eq. (3.10). The extensions $K_{\xi,\zeta}$ represent additional independent charges, and together with the H_ξ generators they produce an algebra that is larger than the original set of boundary symmetries. There is a question of how to interpret these additional charges, and how to interpret the reduction with respect to the additional generators. The mathematical machinery for handling such situations is called symplectic reduction by stages,[124] and it would be worth investigating whether the reduced phase space obtained using this procedure reproduces the expected global phase space.

Another major motivation for carrying out this reduction procedure is in the applications to the quantum theory of subregions in a gravitational theory. There is an analogous procedure to Marsden–Weinstein reduction whereby the physical Hilbert space $\mathcal{H}_{\text{phys}}$ is realized as a subspace of the tensor product $\mathcal{H}^1 \otimes \mathcal{H}^2$, where $\mathcal{H}^i$ are the Hilbert spaces constructed via quantization of the localized phase spaces $\mathscr{P}^i$.[16,125] This subspace is defined as the zero charge eigenspace associated with the boundary symmetries in the localized phase spaces, and restricting to this physical Hilbert space has the interpretation of imposing the constraints associated with diffeomorphism invariance. There are a number of results beginning with the works of Guillemin and Sternberg that show in certain situations that the process of quantization commutes with symplectic reduction.[126] Hence, we should expect that the localized phase spaces $\mathscr{P}^i$ provide useful semiclassical descriptions of the local Hilbert spaces $\mathcal{H}^i$.

These local phase spaces are important when addressing questions regarding entanglement entropy for subregions in gravitational theories and the entropy associated with black hole horizons. It has long been appreciated that black holes possess an entropy proportional to their area,[127–129] and in a variety of contexts, this entropy can be usefully interpreted as entanglement entropy.[130–133] Even more generic subregions in gravity are expected to possess a finite entropy;[134–136] for example, in holography, subregions bounded by extremal codimension-2 surfaces have an entropy given by the Ryu–Takayanagi formula, which is interpreted as the entanglement entropy of a subregion of the boundary conformal field theory.[137] The construction of localized Hilbert spaces as described above is then crucial for giving a bulk Hilbert space interpretation of this entropy. The larger Hilbert space $\mathcal{H}^1 \otimes \mathcal{H}^2$ in which the physical Hilbert space is embedded is known as the extended Hilbert space, and contains additional edge mode degrees of freedom that contribute to the entanglement entropy.[125] These edge modes can be viewed as objects charged under the boundary symmetries considered in this work, and hence the localized charges play a central role in characterizing edge mode degrees of freedom. In some cases, considerations of boundary symmetries can in fact be shown to determine the entropy given some reasonable assumptions on the quantization of the localized phase space. The best examples of this often involve a set of Virasoro symmetries or a related centrally extended algebra acting on a Killing horizon.[8,9,11–13] In this case, the quantization is conjectured to involve a CFT, and the Cardy formula for such a theory then is able to reproduce the Bekenstein–Hawking entropy of the horizon. It is interesting that the central extension in these examples seems to play an important role in determining the entropy, and this may be related to interesting properties of the reduction procedure for algebras involving nonzero extensions.

7.3. *Corner improvements*

In Subsec. 2.5, we described an additional correction that must be added to the localized charges to arrive at an expression that is fully invariant under the extra ambiguities mentioned in that section. This correction was first described in App. C of Ref. 13, and this work generalizes the proposal to allow for noncovariances in L' and θ'. As mentioned in the text, the correction to the charge density involves a quantity c' which appears as a contribution to the subregion action from codimension-2 corners. Note there are additional questions involving the precise relation between the full corner contribution to the action and the c' appearing in the charge, since, as discussed in footnote k, there are independent contributions to the corner action coming from the boundary of each hypersurface $\mathcal{N}^{\pm}$ ending at the corner. Spelling out the precise relation between these contributions to the action and the localized charges would be an interesting future direction to explore.

Ambiguities of the type described in Subsec. 2.5 arose in the construction of GBMS charges in Sec. 6, where there could have been other possible choices for the form of the corner flux than the one we picked. It would be interesting to relate the choice made there to a more careful analysis of boundary terms needed to obtain a finite variational principle for subregions bounded by $\mathscr{I}^+$, and to carefully derive these terms from a corner Dirichlet principle as well as a corner-improved holographic renormalization procedure, as described in Subsec. 5.1. A possible result of such an analysis would be to obtain GBMS charges that are finite in the limit to either end of $\mathscr{I}^+$. This would allow comparison to the expression obtained by Compère, Fiorucci and Ruzziconi in Eq. (5.49) of Ref. 15, which does satisfy this finiteness property but was derived somewhat indirectly by using input from soft theorems.

Finally, we mention that localized charges constructed via the Brown–York procedure, as described recently in Ref. 37, also enjoy the property of being free of the ambiguities discussed in Subsec. 2.5, since these charges only depend on the form of the codimension-1 flux $\mathcal{E}$. On the other hand, these charges can differ from the canonical charges for transformations that act anomalously on the boundary structures, and hence may yield different expressions than the corner-improved charges. It would be useful to carry out this comparison in detail.

7.4. *Alternative resolutions of the ambiguity*

In this work, we have emphasized that resolving the ambiguities in the covariant phase space construction amounts to choosing a preferred form of the flux. Following Ref. 13, we advocated for the use of a Dirichlet form of the flux, given the close connection to standard holographic constructions, junction conditions, and the Brown–York formulation of localized charges recently explored in Ref. 37. Additional intrinsic counterterms preserving the Dirichlet form of the flux are necessary for asymptotic symmetries, where they are needed to ensure a finite flux through the boundary, and were related to the holographic renormalization of the action in Subsec. 5.2. Previously, there have been other proposals for resolving the ambiguities, and we take a moment to briefly comment on these alternative approaches.

The approach initially advocated by Wald and Zoupas,[34] and employed in subsequent work, for example Ref. 32, fixes some ambiguities using a stationarity condition, although for sufficiently permissive boundary conditions, this requirement either does not yield a unique result, or else fails to hold. A different approach is to focus on the properties of a given Lagrangian, and to extract a preferred symplectic potential using homotopy operators of the variational bicomplex.[18,138,139] While this certainly yields an unambiguous result, there is still a degree of arbitrariness in the fact that homotopy operators for a given complex in general are not unique. In fact, the original formulas by Iyer and Wald[7] for the symplectic potential are completely unambiguous. The ambiguity instead arises in addressing why one particular formula for the symplectic potential is preferred over another. In this regard, we find that resolving the ambiguity by focusing on properties of the flux yields a clearer explanation of what choices have been made in finding the resolution. It would still be interesting to carefully relate the resolutions we explore in this work to those involving the variational bicomplex, and understand the extent to which these two approaches can be made equivalent. Finally, we mention the work of Kirklin,[140] who uses a construction based on the path integral for a subregion, and extracts a manifestly unambiguous symplectic potential using ideas closely related to the Peierls bracket construction.[49] This procedure has a number of advantages beyond being manifestly unambiguous, including making a more direct connection to the quantum description of the subregion, and being completely covariant with respect to the codimension-2 corner of the subregion; i.e. it does not require a preferred codimension-1 hypersurface $\mathcal{N}$ bounding the subregion. Unfortunately, the construction is sufficiently different from the standard covariant phase space that it is not immediately clear what the specific form of the corner contribution to the symplectic potential is in Kirklin's construction. It would be very interesting to make this comparison, and determine whether his construction is related to the Dirichlet form of the flux that was the focus of this work.

7.5. *Casimir energy of vacuum AdS*

A byproduct of the localized charge construction in Subsec. 2.3 is that the resulting charges are largely free from the usual ambiguity to be shifted by phase space constants. The reason for this is that there are fewer quantities that qualify as true constants when no boundary condition is imposed on the intrinsic boundary data. The requirement that the charges satisfy Eq. (2.32) is therefore a stronger condition than the one occurring in standard canonical frameworks in which boundary conditions are imposed to ensure the flux $\mathcal{F}_{\hat{\xi}}$ vanishes. The additional content in Eq. (2.32) is that the charge H_ξ must satisfy this equation even for variations that violate the boundary conditions. For example, when taking $\mathcal{E}$ to be in Dirichlet form, and choosing ξ^a such that $\Delta_{\hat{\xi}}(\beta' - \lambda') + h_{\delta\xi}$ vanishes, one would find that imposing a Dirichlet boundary condition causes the entire flux $\mathcal{F}_{\hat{\xi}}$ to vanish, and H_ξ is then the charge that integrates Hamilton's equation for the transformation. However, any other quantity H'_ξ that differs from H_ξ by a functional of the intrinsic quantities on the boundary would also satisfy Hamilton's equation, since such intrinsic functionals are phase space constants once the Dirichlet boundary condition is imposed. On the other hand, these intrinsic functionals have a nontrivial variation for fluctuations that do not hold the intrinsic data fixed, in which case H'_ξ will fail to satisfy (2.32) in the larger phase space considered in this work where such variations are permitted. This allows us to conclude that the charge H_ξ is unique up to an overall constant that is independent of the bulk and boundary geometry. The expression (2.31) represents a valid choice for fixing this constant, and allows for meaningful comparison of the values of the charges in different space–times.

An important context in which such a comparison arises is in odd-dimensional asymptotically AdS spaces, where, depending on the choice of boundary conformal frame, the charges in vacuum AdS can take on nonzero values. In particular, for asymptotic time translations, the nonzero charge is interpreted as the Casimir energy for the dual CFT.[27] This result crucially relies on the ability to compare the charges in different conformal frames, and for the alternative definition of canonical charges proposed by Ashtekar, Magnon and Das (AMD),[141,142] the energy vanishes for vacuum AdS, regardless of the choice of conformal frame. The resolution of this discrepancy lies in the fact that the AMD charges differ from the charges constructed from a holographic stress tensor by an intrinsic functional of the boundary geometry.[21,22] This intrinsic functional has the effect of subtracting off the value of the charge of vacuum AdS in the appropriate conformal frame, so that the AMD charges always vanish in vacuum AdS.

This raises the question as to which definition of charge coincides with the expression (2.31) in the context of asymptotically AdS space–times. The answer can be inferred from the results of Ref. 37 (see also Refs. 20, 21 and 53), which showed that when the flux is chosen to be of Dirichlet form, H_ξ agrees with the Brown–York charges constructed from the boundary stress tensor obtained by varying the subregion action with respect to the intrinsic boundary variables.[33,kk] Since the Casimir energy is obtained from holographic charges constructed using the Brown–York method, it is immediately apparent that the charges H_ξ considered here will reproduce the Casimir energy of asymptotically AdS space–times, and therefore differ from the AMD charges. It is important to emphasize that, like the holographic charges, any shifts in the localized charges H_ξ are derived from a corresponding change in the subregion action, since the action principle completely determines the expression for the charges. This property is not shared by the AMD charges, and there does not appear to be any action principle that would yield the AMD formula for the charges via the method of Subsec. 2.3. Our construction thus provides a novel means of obtaining this Casimir energy from canonical methods that does not suffer from ambiguities associated with shifting the charges by intrinsic functionals.

7.6. *Implications for holography*

There are a number of potential applications of this work to various aspects of holography. The arguments of Subsec. 5.1 on holographic renormalization of the symplectic potential are largely motivated by well-known constructions that originated in AdS/CFT.[21,25–28] Although Dirichlet boundary conditions were initially thought to be necessary in order to obtain a finite symplectic form, it was pointed

[kk]More precisely, the equivalence between the Brown–York and canonical definitions of charges was shown to hold for transformations that act covariantly on the intrinsic geometry of the boundary. In the case of a null boundary, we showed in Ref. 37 that for transformations that act anomalously on the null generator n^i, in the sense $\Delta_{\hat{\xi}} n^i = w_\xi n^i$ for some function w_ξ, the two definitions of charges differ by an intrinsic functional constructed from w_ξ. In the asymptotically AdS context, a similar anomaly should arise for asymptotic symmetries associated with conformal isometries of the boundary metric with nontrivial conformal factors. In these cases, the holographic charges and canonical charges H_ξ likely differ, and it would be interesting to investigate whether this difference has any physical interpretation. Note that this subtlety does not affect the discussion of the Casimir energy, since that involves charges associated with time translation, which is a boundary isometry with vanishing conformal factor.

out in the work of Compère and Marolf that in fact the holographic renormalization procedure also yields a finite boundary symplectic form, after taking into account the appropriate corner contributions.[36] This then motivates definitions of a wide class of charges associated with all boundary diffeomorphisms, instead of focusing only on the subalgebra of conformal Killing vectors of the boundary metric. For example, in the context of asymptotically de Sitter or anti-de Sitter spaces, such considerations led to the identifications of the Λ-BMS symmetry algebras, which are useful in obtaining the BMS symmetries upon taking a flat space limit.[54,55,110] The general proof in Subsec. 5.2 that such renormalization is always possible, independent of the details of the space–time asymptotics, suggests that the associated generalized charges are always present, and hence should have an interpretation in the dual holographic description.

One puzzling aspect of interpreting these charges holographically is that the symmetry algebras constructed in this way are much larger than the algebras typically encountered in standard examples of AdS/CFT. For example, in asymptotically AdS spaces, the dual quantum theory is a conformal field theory, where the only conserved diffeomorphism charges are those associated with conformal isometries. On the other hand, the charges considered in this work are generically not conserved, due to the presence of nonzero fluxes through the boundary, and hence there is no immediate contradiction with standard holographic considerations. The existence of these charges appears to be most closely tied to the ability to define a local stress tensor operator in the dual theory. As recently reviewed in Ref. 37, the entire set of localized charges can be constructed using the Brown–York stress tensor on the subregion boundary. Although each individual charge may not be conserved, the stress tensor itself satisfies a covariant conservation equation as a consequence of the gravitational constraints. In a holographic dual picture, the dictionary relates the Brown–York stress tensor to the local stress tensor of the dual field theory. Because the continuity equation relating the nonconservation of the charges to the flux is intimately related to the covariant conservation equation of the stress tensor, one could speculate that the diffeomorphism charges become important when characterizing the theory in a hydrodynamical regime, which gives a coarse-grained, effective description of the quantum theory in which the important degrees of freedom are those associated with conserved quantities, such as the stress tensor. This connection between gravity and hydrodynamics has been noted in holography in the fluid-gravity correspondence,[143,144] and has also appeared in various other contexts including the membrane paradigm of black holes[145,146] and considerations of the Einstein equation of state.[147]

There are a number of other possible holographic applications of this work. The considerations of localized charges are well-adapted to describing gravitational theories in local subregions, and in some cases these subregions can be given a holographic interpretation in terms of a CFT deformed by an irrelevant $T\bar{T}$ or T^2 deformation.[148,149] Some ideas relating the $T\bar{T}$ deformation to covariant phase space constructions were recently considered in Ref. 150.

Another area of interest to which the localized charges may be relevant is in the recent models of black hole evaporation that reproduce the Page curve,[151–153] where outgoing Hawking radiation in an asymptotically flat AdS black hole is collected in a nongravitational theory on flat space, in order to induce evaporation. This gluing construction is similar in spirit to the reduction procedure described in Subsec. 7.2 for combining subregions, and hence it may be worthwhile to understand the evaporation models from that perspective. Furthermore, the gluing construction should in principle be possible in setups where both subregions are gravitational, and hence may yield a useful way of understanding black hole evaporation models without restricting one of the subregions to be nongravitational. This may help address recent criticisms of applicability of the evaporation models to genuine asymptotically flat gravitational systems raised in Refs. 154 and 155.

Finally, the considerations of null surfaces and holographic renormalization is particularly well-adapted to applications in celestial holography, which seeks to find a dual of asymptotically flat space in terms of a celestial CFT.[23,156,157] In particular, it would be worthwhile to understand the covariant counterterms needed to renormalize the action and the associated null Brown–York stress tensor recently considered in Ref. 37, without explicitly employing the auxiliary rigging vector used in Subsec. 5.3.

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Appendix A. Field Space Calculations

Here we collect some identities satisfied by various operators on field space. Given a vector field V on $\mathscr{F}$, its action on differential forms via the Lie derivative is given by Cartan's magic formula

$$L_V = I_V \delta + \delta I_V \,. \tag{A.1}$$

More generally, if ν is a vector-valued one-form on $\mathscr{F}$, we can define a derivation of the exterior algebra of degree 0 denoted I_ν which is given by contraction on the vector index and then antisymmetrization of the remaining covariant indices; on a p-form α, this is given by[158]

$$(I_\nu \alpha)_{\mathscr{A}_1\mathscr{A}_2\cdots\mathscr{A}_p} = \nu^{\mathscr{B}}{}_{\underline{\mathscr{A}_1}} \alpha_{\mathscr{B}\underline{\mathscr{A}_2\cdots\mathscr{A}_p}}\,, \tag{A.2}$$

where the underline denotes antisymmetrization of the indices. The graded commutator of I_ν with the exterior derivative δ defines a new derivation of degree 1 denoted L_ν,

$$L_\nu = I_\nu \delta - \delta I_\nu\,. \tag{A.3}$$

In particular, a field-dependent vector field ξ^a has nontrivial variation $\delta\xi^a$ which is a one form on field space. The map $\xi^a \mapsto \hat{\xi}$ extends to $\delta\xi^a$, producing a vector-valued one form on $\mathscr{F}$ denoted $\widehat{\delta\xi}$. This object then defines derivations $I_{\widehat{\delta\xi}}$ and $L_{\widehat{\delta\xi}}$ by the above definitions. A vector-valued differential form ρ of higher degree defines derivations I_ρ and L_ρ in a similar manner.

Lemma A.1. *The various derivations defined above satisfy*

$$[L_{\hat{\xi}}, \pounds_\zeta] = \pounds_{(I_{\hat{\xi}}\delta\zeta)}\,, \tag{A.4}$$

$$[\pounds_\xi, I_{\widehat{\delta\zeta}}] = 0\,, \tag{A.5}$$

$$[I_{\hat{\xi}}, I_{\widehat{\delta\zeta}}] = I_{\widehat{I_{\hat{\xi}}\delta\zeta}}\,, \tag{A.6}$$

$$[I_{\widehat{\delta\xi}}, I_{\widehat{\delta\zeta}}] = I_{\hat{\sigma}}\,, \quad \sigma^a = I_{\widehat{\delta\xi}}\delta\zeta^a - I_{\widehat{\delta\zeta}}\delta\xi^a\,, \tag{A.7}$$

$$[L_{\hat{\xi}}, I_{\widehat{\delta\zeta}}] = I_{\hat{\tau}}\,, \quad \tau^a = [\delta\zeta, \xi]^a + \delta I_{\hat{\xi}}\delta\zeta^a - I_{\widehat{\delta\zeta}}\delta\xi^a\,, \tag{A.8}$$

$$[L_{\hat{\xi}}, L_{\hat{\zeta}}] = -L_{\widehat{[\![\xi,\zeta]\!]}}\,, \quad [\![\xi,\zeta]\!]^a = [\xi,\zeta]^a - I_{\hat{\xi}}\delta\zeta^a + I_{\hat{\zeta}}\delta\xi^a\,. \tag{A.9}$$

In particular, (A.9) *implies that the field space Lie bracket is given by*

$$[\hat{\xi}, \hat{\zeta}]_{\mathscr{F}} = -\widehat{[\![\xi,\zeta]\!]}\,. \tag{A.10}$$

Proof. For (A.4), we compute

$$\begin{aligned}
[L_{\hat{\xi}}, \pounds_\zeta] &= I_{\hat{\xi}}\delta\pounds_\zeta + \delta I_{\hat{\xi}}\pounds_\zeta - \pounds_\zeta I_{\hat{\xi}}\delta - \pounds_\zeta \delta I_{\hat{\xi}} \\
&= I_{\hat{\xi}}\pounds_{\delta\zeta} + \pounds_\zeta I_{\hat{\xi}}\delta + \pounds_{\delta\zeta} I_{\hat{\xi}} - \pounds_\zeta I_{\hat{\xi}}\delta - \pounds_\zeta\delta I_{\hat{\xi}} \\
&= \pounds_{I_{\hat{\xi}}\delta\zeta} - \pounds_{\delta\zeta} I_{\hat{\xi}} + \pounds_{\delta\zeta} I_{\hat{\xi}}
\end{aligned} \tag{A.11}$$

yielding the identity. Equation (A.5) is identically true from the definition of how $\pounds_\xi$ and $I_{\widehat{\delta\zeta}}$ act on field space differential forms. Equations (A.6) and (A.7) follow from the Nijenhuis–Richardson bracket for two algebraic derivations.[158]

For Eq. (A.8), we know from the general structure of brackets of derivations that $[L_{\hat{\xi}}, I_{\widehat{\delta\zeta}}]$ must be an algebraic derivation, and hence is determined by its action on a basis of one-forms $\delta\phi$. This then produces

$$\begin{aligned}
[L_{\hat{\xi}}, I_{\widehat{\delta\zeta}}]\delta\phi &= I_{\hat{\xi}}\delta\pounds_{\delta\zeta}\phi + \delta\pounds_{I_{\hat{\xi}}\delta\zeta}\phi - I_{\widehat{\delta\zeta}}\delta\pounds_{\xi}\phi \\
&= -\pounds_{I_{\hat{\xi}}\delta\zeta}\delta\phi + \pounds_{\delta\zeta}\pounds_{\xi}\phi + \pounds_{\delta(I_{\hat{\xi}}\delta\zeta)}\phi \\
&\quad + \pounds_{I_{\hat{\xi}}\delta\zeta}\delta\phi - \pounds_{I_{\widehat{\delta\zeta}}\delta\xi}\phi - \pounds_{\xi}\pounds_{\delta\zeta}\phi \\
&= \pounds_{\left([\delta\zeta,\xi]+\delta I_{\hat{\xi}}\delta\zeta - I_{\widehat{\delta\zeta}}\delta\xi\right)}\phi\,,
\end{aligned} \tag{A.12}$$

which then reproduces the right-hand side of (A.8).

Finally, for the commutator $[L_{\hat{\xi}}, L_{\hat{\zeta}}]$, we know that the resulting derivation will be a Lie derivative, and hence it is determined by its action on the scalars ϕ. We can therefore compute

$$\begin{aligned}
[L_{\hat{\xi}}, L_{\hat{\zeta}}]\phi &= L_{\hat{\xi}}\pounds_{\zeta}\phi - L_{\hat{\zeta}}\pounds_{\xi}\phi \\
&= \pounds_{I_{\hat{\xi}}\delta\zeta}\phi + \pounds_{\zeta}\pounds_{\xi}\phi - \pounds_{I_{\hat{\zeta}}\delta\xi}\phi - \pounds_{\xi}\pounds_{\zeta}\phi \\
&= -\pounds_{[\![\xi,\zeta]\!]}\phi\,.
\end{aligned} \tag{A.13}$$

□

Lemma A.2. *The operator* $\Delta_{\hat{\xi}}$ *satisfies the following identities*:

$$[\delta, \Delta_{\hat{\xi}}] = \Delta_{\widehat{\delta\xi}}\,, \tag{A.14}$$

$$[\Delta_{\hat{\xi}}, \Delta_{\hat{\zeta}}] = \Delta_{[\hat{\xi},\hat{\zeta}]_{\mathscr{F}}} = -\Delta_{\widehat{[\![\xi,\zeta]\!]}}\,, \tag{A.15}$$

$$[\Delta_{\hat{\xi}}, I_{\hat{\zeta}}] = I_{\widehat{\Delta_{\hat{\xi}}\zeta}} = -I_{\widehat{[\![\xi,\zeta]\!]}} + I_{\widehat{I_{\hat{\zeta}}\delta\xi}}\,. \tag{A.16}$$

Proof. Equation (A.14) follows from

$$\begin{aligned}
\delta\Delta_{\hat{\xi}} &= \delta(L_{\hat{\xi}} - \pounds_{\xi} - I_{\delta\xi}) \\
&= L_{\hat{\xi}}\delta - \pounds_{\delta\xi} - \pounds_{\xi}\delta + L_{\widehat{\delta\xi}} - I_{\widehat{\delta\xi}}\delta \\
&= \Delta_{\hat{\xi}}\delta + \Delta_{\widehat{\delta\xi}}
\end{aligned} \tag{A.17}$$

since $I_{\widehat{\delta\delta\xi}} = 0$.

To derive Eq. (A.15), we can use the identities in Lemma A.1 to derive

$$\begin{aligned}
[\Delta_{\hat{\xi}}, \Delta_{\hat{\zeta}}] &= [(L_{\hat{\xi}} - \pounds_{\xi} - I_{\widehat{\delta\xi}}), (L_{\hat{\zeta}} - \pounds_{\zeta} - I_{\widehat{\delta\zeta}})] \\
&= -L_{\widehat{[\![\xi,\zeta]\!]}} + \pounds_{[\xi,\zeta]} - \pounds_{I_{\hat{\xi}}\delta\zeta} + \pounds_{I_{\hat{\zeta}}\delta\xi} \\
&\quad - [L_{\hat{\xi}}, I_{\widehat{\delta\zeta}}] - [I_{\widehat{\delta\xi}}, L_{\hat{\zeta}}] + [I_{\widehat{\delta\xi}}, I_{\widehat{\delta\zeta}}]\,.
\end{aligned} \tag{A.18}$$

The last three commutators all combine into a single contraction $I_{\hat{\alpha}}$, and using (A.7) and (A.8) we find

$$\alpha^a = -[\delta\zeta, \xi]^a - \delta I_{\hat{\xi}}\delta\zeta^a + I_{\widehat{\delta\zeta}}\delta\xi^a + [\delta\xi, \zeta]^a + \delta I_{\hat{\zeta}}\delta\xi^a - I_{\widehat{\delta\xi}}\delta\zeta^a + I_{\widehat{\delta\xi}}\delta\zeta^a + I_{\widehat{\delta\zeta}}\delta\xi^a = \delta[\![\xi, \zeta]\!]^a\,. \tag{A.19}$$

Hence, Eq. (A.18) becomes

$$[\Delta_{\hat{\xi}}, \Delta_{\hat{\zeta}}] = -L_{\widehat{[\![\xi,\zeta]\!]}} + \pounds_{[\![\xi,\zeta]\!]} + I_{\widehat{\delta[\![\xi,\zeta]\!]}} = -\Delta_{\widehat{[\![\xi,\zeta]\!]}}\,. \tag{A.20}$$

Finally, for Eq. (A.16), we apply Eqs. (A.6) and (A.10) to compute

$$[\Delta_{\hat{\xi}}, I_{\hat{\zeta}}] = [L_{\hat{\xi}} - \pounds_{\xi} - I_{\widehat{\delta\xi}}, I_{\hat{\zeta}}] \tag{A.21}$$

$$= I_{[\hat{\xi},\hat{\zeta}]_{\mathscr{F}}} + I_{\widehat{I_{\hat{\zeta}}\delta\xi}} \tag{A.22}$$

$$= -I_{\widehat{[\![\xi,\zeta]\!]}} + I_{\widehat{I_{\hat{\zeta}}\delta\xi}} = I_{\widehat{\Delta_{\hat{\xi}}\zeta}}\,. \tag{A.23}$$

□

Appendix B. Phase Space Calculations

The standard Iyer–Wald identity[6,7] for computing the contraction of a vector field into the symplectic current receives modifications when θ' contains noncovariances. Making generous use of Cartan's magic formula in addition to Eqs. (2.4), (2.7b), (2.21), (2.24), (A.14), as well as the fact that on-shell, $\delta L' = d\theta'$, we find that

$$\begin{aligned}-I_{\hat{\xi}}\omega' &= -L_{\hat{\xi}}\theta' + \delta I_{\hat{\xi}}\theta' \\ &= -\pounds_{\xi}\theta' - \Delta_{\hat{\xi}}\theta' - I_{\widehat{\delta\xi}}\theta' + \delta(J'_{\xi} + i_{\xi}L' + \Delta_{\hat{\xi}}b') \\ &= -i_{\xi}d\theta' - di_{\xi}\theta' - \Delta_{\hat{\xi}}\delta b' - d\Delta_{\hat{\xi}}\lambda' \\ &\quad - J'_{\delta\xi} - \Delta_{\widehat{\delta\xi}}b' + d\delta Q'_{\xi} + i_{\xi}\delta L' + \delta\Delta_{\hat{\xi}}b' \\ &= d(\delta Q'_{\xi} - Q'_{\delta\xi} - i_{\xi}\theta' - \Delta_{\hat{\xi}}\lambda')\,,\end{aligned} \tag{B.1}$$

where we used $\delta i_{\xi}L' = i_{\delta\xi}L' + i_{\xi}\delta L'$ in the third line. This is then used in determining the charges and fluxes that appear upon contracting $-I_{\hat{\xi}}$ into the symplectic form. Taking into account the additional boundary contribution to Ω [Eq. (2.18)], the result localizes to a boundary integral, whose integrand, using Eqs. (2.4) and (2.12), is given by

$$\begin{aligned}&\delta Q'_{\xi} - Q'_{\delta\xi} - i_{\xi}\underline{\theta}' - \Delta_{\hat{\xi}}\lambda' + I_{\hat{\xi}}\delta\beta' \\ &\quad = \delta Q'_{\xi} - Q'_{\delta\xi} + i_{\xi}\delta\ell' - \pounds_{\xi}\beta' + di_{\xi}\beta' - i_{\xi}\mathcal{E} - \Delta_{\hat{\xi}}\lambda' + L_{\hat{\xi}}\beta' - \delta I_{\hat{\xi}}\beta' \\ &\quad = \delta(Q'_{\xi} + i_{\xi}\ell' - I_{\hat{\xi}}\beta') - Q'_{\delta\xi} - i_{\delta\xi}\ell' + \Delta_{\hat{\xi}}(\beta' - \lambda') + I_{\widehat{\delta\xi}}\beta' - i_{\xi}\mathcal{E} + di_{\xi}\beta' \\ &\quad = \delta h_{\xi} - h_{\delta\xi} - i_{\xi}\mathcal{E} + \Delta_{\hat{\xi}}(\beta' - \lambda') + di_{\xi}\beta'\,,\end{aligned} \tag{B.2}$$

where we recall the definition of the charge density

$$h_\xi = Q'_\xi + i_\xi \ell' - I_{\hat\xi}\beta' \,. \tag{B.3}$$

Integrating this expression over the boundary of a Cauchy surface then yields Eq. (2.29).

The exterior derivative of h_ξ can then be explicitly computed, using Eqs. (2.4) and (2.12),

$$\begin{aligned} dh_\xi &= J'_\xi + \pounds_\xi \ell' - i_\xi d\ell' - I_{\hat\xi} d\beta' \\ &= I_{\hat\xi}\underline{\theta}' - i_\xi L' - \Delta_{\hat\xi} b' + I_{\hat\xi}\delta\ell' - \Delta_{\hat\xi}\ell' - i_\xi d\ell' - I_{\hat\xi} d\beta' \\ &= I_{\hat\xi}\mathcal{E} - \Delta_{\hat\xi}(\ell' + b') - i_\xi(L' + d\ell') \,, \end{aligned} \tag{B.4}$$

which verifies Eq. (2.34).

When computing the bracket between the localized charges, it is helpful to have an expression for the anomaly of the charge density. First, we note using the expression (2.26) for Q'_ζ, the transformation property (2.22) satisfied by the covariant part $\overset{vc}{Q}_\zeta$, and the identity (A.16), that the anomaly of Q'_ζ is given by

$$\Delta_{\hat\xi} Q'_\zeta = -Q'_{[\![\xi,\zeta]\!]} + Q'_{I_{\hat\zeta}\delta\xi} + i_\zeta \Delta_{\hat\xi}\ell' - I_{\hat\zeta}\Delta_{\hat\xi}\beta' \,, \tag{B.5}$$

and similarly it follows that the anomaly of the charge density is

$$\Delta_{\hat\xi} h_\zeta = -h_{[\![\xi,\zeta]\!]} + h_{I_{\hat\zeta}\delta\xi} + i_\zeta \Delta_{\hat\xi}(\ell' + b') - I_{\hat\zeta}\Delta_{\hat\xi}(\beta' - \lambda') \,. \tag{B.6}$$

The bracket (3.6) of the charges is then given by

$$\{H_\xi, H_\zeta\} = -I_{\hat\xi}\delta H_\zeta + I_{\hat\zeta}\mathcal{F}_{\hat\xi} = \int_{\partial\Sigma} m_{\xi,\zeta} \,, \tag{B.7}$$

and by applying the definition (2.33) of $\mathcal{F}_{\hat\xi}$ and using (B.4) and (B.6), the integrand can evaluate to

$$m_{\xi,\zeta} = -\pounds_\xi h_\zeta - \Delta_{\hat\xi} h_\zeta + I_{\hat\zeta}\big(i_\xi \mathcal{E} - \Delta_{\hat\xi}(\beta' - \lambda') + h_{\delta\xi}\big) \tag{B.8}$$

$$= h_{[\![\xi,\zeta]\!]} - i_\zeta \Delta_{\hat\xi}(\ell' + b') + i_\xi \Delta_{\hat\zeta}(\ell' + b') + i_\xi i_\zeta(L' + d\ell') - d i_\xi h_\zeta \,. \tag{B.9}$$

Integrating this over the surface $\partial\Sigma$ yields the charge representation theorem quoted in Eq. (3.10), using that ξ^a and ζ^a are both tangent to $\mathcal{N}$ which causes the term $i_\xi i_\zeta(L' + d\ell')$ to pull back to zero.

A similar computation yields the bracket for the corner-improved charges constructed in Subsec. 2.5. Working with an improved charge density $\tilde h_\xi$ defined by dropping the final exact term in Eq. (2.47) which integrates to zero in the charge,

$$\tilde h_\xi = \overset{vc}{Q}_\xi + i_\xi(\ell' + b' + dc') - I_{\hat\xi}\varepsilon \,, \tag{B.10}$$

we find that its exterior derivative is given by

$$d\tilde{h}_\zeta = I_{\hat{\zeta}}\mathcal{E} - \Delta_{\hat{\zeta}}(\ell' + b' + dc') - i_\zeta(L' + d\ell'), \tag{B.11}$$

and its anomaly by

$$\Delta_{\hat{\xi}}\tilde{h}_\zeta = \tilde{h}_{\Delta_{\hat{\xi}}\zeta} + i_\zeta\Delta_{\hat{\xi}}(\ell' + b' + dc') - I_{\hat{\zeta}}\Delta_{\hat{\xi}}\varepsilon\,. \tag{B.12}$$

The bracket of the charges is

$$\{\tilde{H}_\xi, \tilde{H}_\zeta\} = -I_{\hat{\xi}}\delta\tilde{H}_\zeta + I_{\hat{\zeta}}\tilde{\mathcal{F}}_{\hat{\xi}} = \int_{\partial\Sigma}\tilde{m}_{\xi,\zeta} \tag{B.13}$$

with $\tilde{\mathcal{F}}_{\hat{\xi}}$ defined in (2.48). Then applying (B.11) and (B.12), the integrand evaluates to

$$\tilde{m}_{\xi,\zeta} = -\pounds_\xi\tilde{h}_\zeta - \Delta_{\hat{\xi}}\tilde{h}_\zeta + I_{\hat{\zeta}}\big(i_\xi\mathcal{E} - \Delta_{\hat{\xi}}\varepsilon + \tilde{h}_{\delta\xi}\big) \tag{B.14}$$

$$= \tilde{h}_{[\![\xi,\zeta]\!]} + i_\xi\Delta_{\hat{\zeta}}(\ell' + b' + dc')$$

$$- i_\zeta\Delta_{\hat{\xi}}(\ell' + b' + dc') + i_\xi i_\zeta(L' + d\ell') - di_\xi\tilde{h}_\zeta\,, \tag{B.15}$$

which by the same arguments as above yields the corner-improved charge representation theorem, Eqs. (3.18) and (3.19).

Appendix C. Scaling Transformations on a Null Surface

Consider a space–time $(\mathcal{M}, g_{ab})$ containing a null surface $\mathcal{N}$. In this appendix we review the various geometric quantities that are naturally defined on $\mathcal{N}$ (see for example Sec. 3 of Ref. 32 for more details), and how they transform under rescalings of the null normal and under conformal transformations of the metric. We restrict to four-dimensional space–times in this section.

We pick a smooth future-directed normal covector n_a on $\mathcal{N}$, and define the inaffinity κ, a function on $\mathcal{N}$, by[11]

$$n^a\nabla_a n^b \mathrel{\hat{=}} \kappa n^b\,, \tag{C.1}$$

where we are using $\hat{=}$ to mean equality when evaluated on $\mathcal{N}$. The contravariant normal $n^a = g^{ab}n_b$, when evaluated on $\mathscr{I}^+$, can be viewed as an intrinsic vector n^i, since $n^a n_a = 0$. We denote by q_{ij} the degenerate induced metric, and by η_{ijk} the 3-volume form on $\mathcal{N}$ given by taking the pullback of η_{abc} where η_{abc} is any three form with $4\eta_{[abc}n_{d]} = \varepsilon_{abcd}$. Finally we define a 2-volume form by

$$\mu_{ij} = -\eta_{ijk}n^k\,. \tag{C.2}$$

[11]If the extension of n_a away from $\mathcal{N}$ is chosen to satisfy $\nabla_{[a}n_{b]} = 0$, the quantity κ is equivalently given by the relation $\nabla_a(n_b n^b) \mathrel{\hat{=}} 2\kappa n_a$ which is the usual definition of surface gravity for a horizon when n_a is a Killing vector field. Thus the inaffinity is sometimes called surface gravity for general normals n_a, although in the most general case where $\nabla_{[a}n_{b]} \neq 0$, these two definitions of κ will not agree.

Next, we take the pullback on the index a of $\nabla_a n^b$, which is then orthogonal to n_b on the index b. This quantity therefore defines an intrinsic tensor $W_i{}^j$ called the Weingarten map.[159] The second fundamental form or shape tensor is $K_{ij} = W_i{}^k q_{kj}$, which can be decomposed as

$$K_{ij} = \frac{1}{2}\Theta q_{ij} + \sigma_{ij} \tag{C.3}$$

in terms of an expansion[mm] Θ and a symmetric traceless shear tensor σ_{ij}.

These fields on a null surface obey the relations[32,159]

$$q_{ij} n^j = 0\,, \tag{C.4a}$$

$$K_{ij} n^j = 0\,, \tag{C.4b}$$

$$W_i{}^j n^i = \kappa n^j\,, \tag{C.4c}$$

$$(\pounds_n - \Theta) q_{ij} = 2\sigma_{ij}\,, \tag{C.4d}$$

$$(\pounds_n - \Theta) \eta_{ijk} = 0\,, \tag{C.4e}$$

$$(\pounds_n - \Theta) \mu_{ij} = 0\,, \tag{C.4f}$$

$$(\pounds_n - \kappa)\Theta = -\frac{1}{2}\Theta^2 - \sigma_{ij}\sigma_{kl} q^{ik} q^{kl} - R_{ab} n^a n^b\,, \tag{C.4g}$$

where q^{ij} is any tensor that satisfies $q_{ij} q^{jk} q_{kl} = q_{il}$.

Consider now rescaling the normal according to

$$n^i \to e^{\sigma} n^i\,, \tag{C.5}$$

where σ is a smooth function on $\mathcal{N}$. We can also perform a conformal transformation on the metric,

$$g_{ab} \to e^{2\Upsilon} g_{ab}\,. \tag{C.6}$$

Here Υ is a smooth function on a neighborhood of $\mathcal{N}$, but we will be interested only in Υ restricted to $\mathcal{N}$. Under the combined effect of these transformations the various fields transform as

$$n_a \to e^{\sigma + 2\Upsilon} n_a\,, \tag{C.7a}$$

$$q_{ij} \to e^{2\Upsilon} q_{ij}\,, \tag{C.7b}$$

$$\mu_{ij} \to e^{2\Upsilon} \mu_{ij}\,, \tag{C.7c}$$

$$\eta_{ijk} \to e^{2\Upsilon - \sigma} \eta_{ijk}\,, \tag{C.7d}$$

$$\kappa \to e^{\sigma}(\kappa + \pounds_n \sigma + 2\pounds_n \Upsilon)\,, \tag{C.7e}$$

[mm] The relation of the expansion Θ to the divergence $\nabla_a n^a$ of the normal depends on how one extends the definition of n^a off the null surface. If that extension satisfies $n_a n^a = 0$, then $\Theta = \nabla_a n^a - \kappa$. If that extension satisfies $\nabla_{[a} n_{b]} = 0$, then we have instead $\Theta = \nabla_a n^a - 2\kappa$.[160]

$$\Theta \to e^{\sigma}(\Theta + 2\pounds_n \Upsilon)\,, \tag{C.7f}$$

$$K_{ij} \to e^{\sigma+2\Upsilon}(K_{ij} + q_{ij}\pounds_n \Upsilon)\,, \tag{C.7g}$$

$$W_i{}^j \to e^{\sigma}\big[W_i{}^j + D_i(\sigma+\Upsilon)n^j + \pounds_n \Upsilon \delta_i^j\big]\,, \tag{C.7h}$$

where D_i is any derivative operator on $\mathcal{N}$. These transformation laws preserve the relations (C.4).

In applying this framework to null surfaces $\mathcal{N}$ at a finite location in space–time,[32] the metric g_{ab} is the physical metric. Hence there is no freedom to conformally rescale the metric, and we must take $\Upsilon = 0$. In this case the scaling laws (C.7) reduce to the scaling laws[nn] given in Eq. (3.3) of Ref. 32. By contrast, in applying the framework to future null infinity $\mathcal{N} = \mathscr{I}^+$, the metric g_{ab} is the unphysical metric and is subject to the conformal rescaling freedom (D.3), which also includes a rescaling of the normal. In this case we must take $\Upsilon = -\sigma$, and with this specialization the scaling laws (C.7) reduce to the laws (D.8) given in App. D.

Appendix D. Asymptotically Flat Space−times: Notations and Conventions

In this appendix we review the definition of asymptotically flat space–times in $3+1$ dimensions, and define the notations we use for the conformal completion framework used to describe them.

Consider vacuum space–times that are asymptotically flat at null infinity, $\mathscr{I}$, in the sense of Ref. 161. This means that we have a manifold $\mathcal{M}$ with boundary $\mathscr{I}$ which is topologically $\mathbb{R} \times S^2$, and an unphysical metric g_{ab} which is smooth on $\mathcal{M}$ for which $\mathscr{I}$ is null. We also have a smooth conformal factor Φ on $\mathcal{M}$ which satisfies $\Phi = 0$ on $\mathscr{I}$ and for which

$$n_a = \nabla_a \Phi \tag{D.1}$$

vanishes nowhere on $\mathscr{I}$. Finally the physical metric

$$\tilde{g}_{ab} = \Phi^{-2} g_{ab} \tag{D.2}$$

satisfies the vacuum Einstein equation $\tilde{G}_{ab} = 0$ on $\mathcal{M} \setminus \mathscr{I}$. The conformal transformation

$$(g_{ab}, \Phi) \to \left(e^{-2\sigma} g_{ab}, e^{-\sigma}\Phi\right)\,, \tag{D.3}$$

where σ is a smooth function on M, preserves the physical metric. Although normally one would expect the theory to be invariant under this conformal freedom, it is possible in general contexts for the definitions of gravitational charges to depend on background structures like the choice of conformal frame (as it does in AdS),

[nn]Note that the quantities denoted here by $W_i{}^j$, Θ, q_{ij}, μ_{ij}, η_{ijk} and n^i were denoted there $\mathcal{K}_i{}^j$, θ, h_{ij}, ε_{ij}, ε_{ijk} and ℓ^i, respectively.

as we argued in Sec. 2. The following discussion can be easily adapted to past null infinity but we will focus on future null infinity just for simplicity.

As for any null surface, the metric g_{ab} and normal n_a determine a number of geometric quantities on $\mathscr{I}^+$, reviewed in App. C. These include the inaffinity κ, the expansion Θ, the shear tensor σ_{ij}, the induced metric q_{ij}, the 3-volume form η_{ijk}, the 2-volume form μ_{ij}, the second fundamental form or shape tensor K_{ij}, and the Weingarten map $W_i{}^j$. For general null surfaces these quantities obey a number of identities given in Eqs. (C.4). We now review properties of these quantities that are specific to $\mathscr{I}^+$.

First, the normal n_a is a pure gradient from Eq. (D.1), and so $\nabla_{[a}n_{b]} = 0$. Since $\mathscr{I}^+$ is null we have $n_a n^a = \Phi g + O(\Phi^2)$ for some function g on $\mathscr{I}^+$. Taking a gradient, evaluating at $\Phi = 0$, using the symmetry of $\nabla_a n_b$ and using the definition (C.1) of the inaffinity κ now yields that

$$g^{ab} n_a n_b = 2\kappa\Phi + O(\Phi^2)\,. \tag{D.4}$$

Second, it follows from the vacuum Einstein equation satisfied by the physical metric that

$$\nabla_{(a} n_{b)} \,\widehat{=}\, f g_{ab} \tag{D.5}$$

for some function f on $\mathscr{I}^+$; see, e.g. Eq. (2.6) of Ref. 46. As a reminder we are using $\widehat{=}$ to mean equality when evaluated on $\mathscr{I}^+$. Combining this with Eq. (D.1) yields $\nabla_a n_b \,\widehat{=}\, f g_{ab}$, from which we obtain $f = \kappa$ and

$$\nabla_a \nabla_b \Phi \,\widehat{=}\, \kappa g_{ab}\,, \tag{D.6a}$$

$$\Theta = 2\kappa\,, \tag{D.6b}$$

$$\sigma_{ij} = 0\,, \tag{D.6c}$$

$$W_i{}^j = \kappa \delta_i^j\,. \tag{D.6d}$$

Inserting Eqs. (D.6b) and (D.6c) into the general identities (C.4) for any null surface yields the relations

$$q_{ij} n^j = 0\,, \tag{D.7a}$$

$$(\pounds_n - 2\kappa) q_{ij} = 0\,, \tag{D.7b}$$

$$(\pounds_n - 2\kappa) \eta_{ijk} = 0\,, \tag{D.7c}$$

$$(\pounds_n - 2\kappa) \mu_{ij} = 0\,. \tag{D.7d}$$

Under the conformal transformation (D.3) the transformation laws for the various fields on $\mathscr{I}^+$ are given by the special case $\Upsilon = -\sigma$ of the transformation laws (C.7) discussed in App. C, and are given by

$$n^i \to e^{\sigma} n^i\,, \tag{D.8a}$$

$$n_a \to e^{-\sigma} n_a\,, \tag{D.8b}$$

$$q_{ij} \to e^{-2\sigma} q_{ij} \,, \tag{D.8c}$$

$$\mu_{ij} \to e^{-2\sigma} \mu_{ij} \,, \tag{D.8d}$$

$$\eta_{ijk} \to e^{-3\sigma} \eta_{ijk} \,, \tag{D.8e}$$

$$\kappa \to e^{\sigma} (\kappa - \pounds_n \sigma) \,. \tag{D.8f}$$

These transformation laws preserve the relations (D.6) and (D.7). Using the freedom (D.8f) one can enforce the Bondi condition

$$\kappa = 0 \,. \tag{D.9}$$

However in most of our analysis in this paper we will not make this specialization and will allow κ to be nonzero.

Appendix E. Symmetry Groups at Future Null Infinity in Vacuum General Relativity

In this appendix we derive the symmetry groups that correspond to the three different field configuration spaces defined in Sec. 4 in the body of the paper. Rather than proceeding directly, it will be more convenient to proceed in three stages, following the universal intrinsic structure approach of Ashtekar[97] and the techniques of Ref. 32:

- We define universal intrinsic structures in each of the three cases, and derive the corresponding group of diffeomorphisms of $\mathscr{I}^+$ that preserve these structures.
- We define boundary structures on $\mathscr{I}^+$ in each of the three cases, and define associated field configuration spaces. These configuration spaces are related to those given in Sec. 4 by taking orbits under the conformal transformations.
- Finally, we show that the symmetry groups of the intrinsic structures coincide with those of the field configuration spaces associated with the boundary structures, and with the symmetry groups of the spaces of Sec. 4.

We first explain these steps in detail in the BMS context, and then outline the extensions to the generalized BMS and Weyl BMS contexts.

E.1. *Bondi–Metzner–Sachs case*

E.1.1. *Definition of intrinsic structure*

Consider triplets of tensor fields (n^i, q_{ij}, κ) defined on $\mathscr{I}^+$ that satisfy the relations (D.7a) and (D.7b) for which the vector field n^i is complete. We define any two such triplets to be equivalent if they are related by a rescaling of the form given by Eqs. (D.8a), (D.8c) and (D.8f):

$$(n^i, q_{ij}, \kappa) \sim (e^{\sigma} n^i, e^{-2\sigma} q_{ij}, e^{\sigma}\kappa - e^{\sigma} \pounds_n \sigma) \,. \tag{E.1}$$

We denote the equivalence class associated with a given triple as

$$\mathfrak{u}_{21} = [n^i, q_{ij}, \kappa] \,. \tag{E.2}$$

We call the quantity $\mathfrak{u}_{21}$ an intrinsic geometric structure on $\mathscr{I}^+$. These structures are universal in the sense that given any two such structures on $\mathscr{I}^+$, there exists a diffeomorphism $\varphi : \mathscr{I}^+ \to \mathscr{I}^+$ which maps one onto the other via pullback.[oo]

We will be defining a number of similar equivalence classes throughout this appendix, and our notational conventions for these objects are as follows. In the symbol $\mathfrak{u}_{AB}$, A can be 2 (if the induced metric q_{ij} is present in the set of fields), 1 (if the volume form η_{ijk} is instead present), or 0 (if neither q_{ij} nor η_{ijk} is present). The second index B can be 1 (if the inaffinity κ is present in the set of fields) or 0 (if κ is absent). Thus there will be six types of equivalence class, $\mathfrak{u}_{21}$, $\mathfrak{u}_{11}$, $\mathfrak{u}_{01}$, $\mathfrak{u}_{20}$, $\mathfrak{u}_{10}$ and $\mathfrak{u}_{00}$. Additionally, we will consider structures in which the normal covector n_a is also present in the set of fields. When this is the case, we will use the notation $\mathfrak{p}_{AB}$, while the notation $\mathfrak{u}_{AB}$ is reserved for structures in which n_a is absent. Finally tensor fields in the equivalence classes are barred (e.g. $\bar{n}^i$, $\bar{q}_{ij}, \ldots$) when κ is absent, and are not barred (e.g. n^i, $q_{ij}, \ldots$) when κ is present.

A given asymptotically flat space–time $(\mathcal{M}, \tilde{g}_{ab})$ determines a unique intrinsic structure $\mathfrak{u}_{21} = \mathfrak{u}_{21}[\tilde{g}_{ab}]$, as follows. Choose an unphysical metric g_{ab} and conformal factor Φ for which $\tilde{g}_{ab} = \Phi^{-2} g_{ab}$. Compute the quantities q_{ij}, n^i and κ from the unphysical metric and conformal factor, and take the equivalence class (E.2). The result is independent of which conformal factor and unphysical metric within the equivalence class is chosen, by the equivalence relation (E.1) and the scaling laws (D.8).

We can define a different type of universal intrinsic structure,[97] without the inaffinity κ, as follows. Consider pairs $(\bar{n}^i, \bar{q}_{ij})$ that satisfy Eqs. (D.7a) and (D.7b) with $\kappa = 0$:

$$\bar{n}^i \bar{q}_{ij} = 0 \,, \quad \pounds_{\bar{n}} \bar{q}_{ij} = 0 \,. \tag{E.3}$$

We define two such pairs to be equivalent if they are related by a transformation of the form (D.8) that preserves $\kappa = 0$, that is,

$$(\bar{n}^i, \bar{q}_{ij}) \sim (e^{\sigma} \bar{n}^i, e^{-2\sigma} \bar{q}_{ij}) \tag{E.4}$$

with $\pounds_{\bar{n}} \sigma = 0$. We denote the equivalence class associated with a given pair as

$$\mathfrak{u}_{20} = \left[\bar{n}^i, \bar{q}_{ij}\right] . \tag{E.5}$$

There is a one-to-one correspondence between intrinsic structures of the type $\mathfrak{u}_{21}$ and those of the type $\mathfrak{u}_{20}$. Given an intrinsic structure $\left[n^i, q_{ij}, \kappa\right]$, if we consider the set of representative triples $(\bar{n}^i, \bar{q}_{ij}, 0)$ with vanishing inaffinity, the result is the equivalence class $\left[\bar{n}^i, \bar{q}_{ij}\right]$. Conversely, given the equivalence class $\left[\bar{n}^i, \bar{q}_{ij}\right]$, we can take any element $(\bar{n}^i, \bar{q}_{ij})$, consider the corresponding triple $(\bar{n}^i, \bar{q}_{ij}, 0)$, and

[oo]This can be shown by an argument similar to that given in Subsec. 4.1 of Ref. 32.

then take the equivalence class under the equivalence relation (E.1) to generate the intrinsic structure of type $\mathfrak{u}_{21}$. We will denote this one-to-one correspondence as $\mathfrak{u}_{21} = \mathfrak{u}_{21}(\mathfrak{u}_{20})$.

E.1.2. *Symmetry group of intrinsic structure*

Consider now diffeomorphisms $\varphi : \mathscr{I}^+ \to \mathscr{I}^+$. We define the action of the pullback φ^* on an intrinsic structure $\mathfrak{u}_{21} = \left[n^i, q_{ij}, \kappa\right]$ by acting with the pullback on a representative of the equivalence class:

$$\varphi^* \left[n^i, q_{ij}, \kappa\right] = \left[\varphi^* n^i, \varphi^* q_{ij}, \varphi^* \kappa\right] . \tag{E.6}$$

This action is well defined, since if (n^i, q_{ij}, κ) and $(\hat{n}^i, \hat{q}_{ij}, \hat{\kappa})$ are two triples related by a rescaling function σ, then the pullbacks of these triples are related by the rescaling function $\varphi^*\sigma$. Now given an intrinsic structure $\mathfrak{u}_{21}$, we define the corresponding symmetry group to be the group of diffeomorphisms which preserves the intrinsic structure:

$$\mathcal{D}_{\mathfrak{u}_{21}} = \left\{\varphi : \mathscr{I}^+ \to \mathscr{I}^+ \middle| \, \varphi^* \mathfrak{u}_{21} = \mathfrak{u}_{21}\right\} . \tag{E.7}$$

From the definition (E.6) and the equivalence relation (E.1), given a diffeomorphism φ in this group and a representative (n^i, q_{ij}, κ) of the intrinsic structure, the action of the diffeomorphism is that of a rescaling by some smooth function $\alpha = \alpha(\varphi, n^i)$:

$$\varphi^* n^i = e^{-\alpha} n^i \,, \tag{E.8a}$$

$$\varphi^* q_{ij} = e^{2\alpha} q_{ij} \,, \tag{E.8b}$$

$$\varphi^* \kappa = e^{-\alpha}(\kappa + \pounds_n \alpha) . \tag{E.8c}$$

The dependence of the function α on the choice of representative (or equivalently on the normalization of the normal) is given by

$$\alpha(\varphi, e^\sigma n^i) = \alpha(\varphi, n^i) + \sigma - \varphi^* \sigma \,, \tag{E.9}$$

from Eqs. (E.1) and (E.8).

We similarly define the symmetry group $\mathcal{D}_{\mathfrak{u}_{20}}$ to be the group of diffeomorphisms that preserves a given intrinsic structure $\mathfrak{u}_{20} = \left[\bar{n}^i, \bar{q}_{ij}\right]$:

$$\mathcal{D}_{\mathfrak{u}_{20}} = \left\{\varphi : \mathscr{I}^+ \to \mathscr{I}^+ \middle| \, \varphi^* \mathfrak{u}_{20} = \mathfrak{u}_{20}\right\} . \tag{E.10}$$

Because of the one-to-one correspondence discussed above, this group coincides with the group (E.7), in the sense that

$$\mathcal{D}_{\mathfrak{u}_{21}(\mathfrak{u}_{20})} = \mathcal{D}_{\mathfrak{u}_{20}} \,, \tag{E.11}$$

where the notation is defined after Eq. (E.5). To see this in more detail, if $\varphi \in \mathcal{D}_{\mathfrak{u}_{20}}$ then $\varphi^* \mathfrak{u}_{20} = \mathfrak{u}_{20}$, and so $\varphi^* \mathfrak{u}_{21}(\mathfrak{u}_{20}) = \mathfrak{u}_{21}(\varphi^* \mathfrak{u}_{20}) = \mathfrak{u}_{21}(\mathfrak{u}_{20})$, where we have used covariance, and so $\varphi \in \mathcal{D}_{\mathfrak{u}_{21}(\mathfrak{u}_{20})}$. The converse uses the fact that the mapping $\mathfrak{u}_{20} \to \mathfrak{u}_{21}(\mathfrak{u}_{20})$ is a bijection.

Because of the equality (E.11), we can give an alternative characterization of the symmetries in the group. From the definition (E.6) and the equivalence relation (E.4), given a representative $(\bar{n}^i, \bar{q}_{ij})$ of the intrinsic structure $\mathfrak{u}_{20}$ and a diffeomorphisms φ in $\mathcal{D}_{\mathfrak{u}_{20}}$, the action of the diffeomorphism is that of a rescaling by some smooth function $\alpha = \alpha(\varphi, \bar{n}^i)$:

$$\varphi^* \bar{n}^i = e^{-\alpha} \bar{n}^i \,, \tag{E.12a}$$

$$\varphi^* \bar{q}_{ij} = e^{2\alpha} \bar{q}_{ij} \,, \tag{E.12b}$$

where

$$\pounds_{\bar{n}} \alpha = 0 \,. \tag{E.13}$$

The dependence of the function α on the choice of representative is given by

$$\alpha(\varphi, e^{\sigma} \bar{n}^i) = \alpha(\varphi, \bar{n}^i) + \sigma - \varphi^* \sigma \,, \tag{E.14}$$

from Eqs. (E.1) and (E.12), which coincides with the dependence (E.9) except that here we must have $\pounds_{\bar{n}} \sigma = 0$ from Eq. (E.4). Equations (E.3), (E.12) and (E.13) are the usual definition[pp] of the BMS group. The linearized versions of Eqs. (E.12) and (E.14) are

$$\pounds_{\xi} \bar{n}^i = -\alpha \bar{n}^i \,, \tag{E.15a}$$

$$\pounds_{\xi} \bar{q}_{ij} = 2\alpha \bar{q}_{ij} \,, \tag{E.15b}$$

and

$$\alpha(\xi^i, e^{\sigma} \bar{n}^i) = \alpha(\xi^i, \bar{n}^i) - \pounds_{\xi} \sigma \,, \tag{E.16}$$

where the infinitesimal diffeomorphism is represented by the vector field ξ^i on $\mathscr{I}^+$.

E.1.3. *Definition of field configuration space*

We now turn to the definition of a field configuration space whose symmetry group matches that of the intrinsic structures discussed above. We start by defining a geometric structure on $\mathscr{I}^+$ which we call a *boundary structure*, which is an extension of our previous definition of intrinsic structure. We consider sets of tensor fields on $\mathscr{I}^+$ of the form

$$(n^i, q_{ij}, \kappa, n_a) \,, \tag{E.17}$$

where n_a is a choice of normal covector, the remaining fields satisfy the relations (D.7a) and (D.7b), and the vector field n^i is complete. We define any two such sets

[pp]The induced metric $\bar{q}_{ij}$ induces a unique two-dimensional Riemannian metric on the space of generators of $\mathscr{I}^+$, from Eqs. (E.3). One can specialize the choice of representative in the equivalence class $[\bar{n}^i, \bar{q}_{ij}]$, using the freedom (E.4), to make this metric have constant scalar curvature (i.e. be a round two metric). While this specialization is often used to simplify the presentation of the BMS group, it is not necessary to do so.

to be equivalent if they are related by a rescaling of the form (D.8) for some smooth function σ:

$$(n^i, q_{ij}, \kappa, n_a) \sim (e^\sigma n^i, e^{-2\sigma} q_{ij}, e^\sigma \kappa - e^\sigma \pounds_n \sigma, e^{-\sigma} n_a)\,. \tag{E.18}$$

We denote the equivalence class associated with a given set as

$$\mathfrak{p}_{21} = [n^i, q_{ij}, \kappa, n_a]\,. \tag{E.19}$$

A choice of equivalence class is the desired boundary structure on $\mathscr{I}^+$.

A choice of boundary structure $\mathfrak{p}_{21} = [n^i, q_{ij}, \kappa, n_a]$ determines a unique intrinsic structure $\mathfrak{u}_{21}$: choose a representative $(n^i, q_{ij}, \kappa, n_a)$, discard n_a, and form the equivalence class $\mathfrak{u}_{21} = [n^i, q_{ij}, \kappa]$ under the equivalence relation (E.1). The result is independent of the representative initially chosen, from Eqs. (E.1) and (E.18). We will denote this induced intrinsic structure by $\mathfrak{u}_{21}(\mathfrak{p}_{21})$. The boundary structure contains more information than the intrinsic structure, which is necessary for the definition of the field configuration space.

Just as for intrinsic structures, a given asymptotically flat space–time $(\mathcal{M}, \tilde{g}_{ab})$ determines a unique boundary structure $\mathfrak{p}_{21} = \mathfrak{p}_{21}[\tilde{g}_{ab}]$, as follows. Choose an unphysical metric g_{ab} and conformal factor Φ for which $\tilde{g}_{ab} = \Phi^{-2} g_{ab}$. Compute the quantities q_{ij}, n^i, κ and n_a from the unphysical metric and conformal factor, and take the equivalence class (E.19). The result is independent of which conformal factor and unphysical metric are chosen, by the equivalence relation (E.18) and the scaling laws (D.8).

Just as for intrinsic structures, we can define a different type of boundary structure, without the inaffinity κ, as follows. Consider triplets $(\bar{n}^i, \bar{q}_{ij}, \bar{n}_a)$ that satisfy Eqs. (E.3) for which $\bar{n}_a$ is a complete normal covector. We define two such triplets to be equivalent if they are related by a transformation of the form (D.8) that preserves $\kappa = 0$, that is,

$$(\bar{n}^i, \bar{q}_{ij}, \bar{n}_a) \sim (e^\sigma \bar{n}^i, e^{-2\sigma} \bar{q}_{ij}, e^{-\sigma} \bar{n}_a) \tag{E.20}$$

with $\pounds_{\bar{n}} \sigma = 0$. We denote the equivalence class associated with a given triplet as

$$\mathfrak{p}_{20} = \left[\bar{n}^i, \bar{q}_{ij}, \bar{n}_a\right]. \tag{E.21}$$

Just as above, a boundary structure $\mathfrak{p}_{20}$ determines a unique intrinsic structure $\mathfrak{u}_{20} = \mathfrak{u}_{20}(\mathfrak{p}_{20})$ by dropping the normal covector $\bar{n}_a$. Also, just as for intrinsic structures, there is a one-to-one correspondence between boundary structures of the type $\mathfrak{p}_{21}$ and those of the type $\mathfrak{p}_{20}$, which we will denote as $\mathfrak{p}_{21} = \mathfrak{p}_{21}(\mathfrak{p}_{20})$ and $\mathfrak{p}_{20} = \mathfrak{p}_{20}(\mathfrak{p}_{21})$. Given an asymptotically flat space–time $(\mathcal{M}, \tilde{g}_{ab})$, we define the corresponding boundary structure of the new type to be

$$\mathfrak{p}_{20}(\tilde{g}_{ab}) = \mathfrak{p}_{20}(\mathfrak{p}_{21}(\tilde{g}_{ab}))\,. \tag{E.22}$$

Next, given a boundary structure $\mathfrak{p}_{21}$, we define the corresponding field configuration space to be the set of all unphysical metrics and conformal factors that are compatible with that boundary structure:

$$\Gamma_{\mathfrak{p}_{21}} = \{(\mathcal{M}, g_{ab}, \Phi) \in \Gamma_0 |\ \mathfrak{p}_{21}(g_{ab}, \Phi) = \mathfrak{p}_{21}\}\,. \tag{E.23}$$

Similarly, given a boundary structure $\mathfrak{p}_{20}$, we define the field configuration space

$$\Gamma_{\mathfrak{p}_{20}} = \{(\mathcal{M}, g_{ab}, \Phi) \in \Gamma_0 |\ \mathfrak{p}_{20}(g_{ab}, \Phi) = \mathfrak{p}_{20}\}\,. \tag{E.24}$$

These two spaces coincide, from Eq. (E.22), in the sense that

$$\Gamma_{\mathfrak{p}_{21}(\mathfrak{p}_{20})} = \Gamma_{\mathfrak{p}_{20}}\,. \tag{E.25}$$

An argument analogous to that given in App. B of Ref. 32 can be used to show that the orbit of $\Gamma_{\mathfrak{p}_{21}}$ under diffeomorphisms of $\mathcal{M}$ is the entire space Γ_0 defined in Eq. (4.1).

E.1.4. *Symmetry group of field configuration space*

We now turn to a discussion of the symmetry group of diffeomorphisms that preserve the configuration phase space,

$$\mathcal{G}_{\mathfrak{p}_{21}} = \{\psi : \mathcal{M} \to \mathcal{M} \mid \psi(\mathscr{I}^+) = \mathscr{I}^+,\ \psi^*\Gamma_{\mathfrak{p}_{21}} = \Gamma_{\mathfrak{p}_{21}}\}\,. \tag{E.26}$$

These diffeomorphisms induce diffeomorphisms of $\mathscr{I}^+$: for any ψ in $\mathcal{G}_{\mathfrak{p}_{21}}$ we define

$$\varphi = \psi|_{\mathscr{I}^+}\,, \tag{E.27}$$

and since ψ preserves the boundary, φ is a diffeomorphism from $\mathscr{I}^+$ to $\mathscr{I}^+$. Next, since ψ preserves $\mathscr{I}^+$, the pullback of any normal covector n_a evaluated on $\mathscr{I}^+$ must be a rescaling of that normal, so we have

$$\psi^* n_a \mathrel{\widehat{=}} e^{\gamma} n_a\,, \tag{E.28}$$

where $\gamma = \gamma(\psi, n_a)$ is a smooth function on $\mathscr{I}^+$ which depends on the diffeomorphism and on the normalization of the normal. The dependence on the normalization of the normal is given by

$$\gamma(\psi, e^{-\sigma} n_a) = \gamma(\psi, n_a) + \sigma - \varphi^*\sigma\,, \tag{E.29}$$

from Eqs. (E.27) and (E.28).

The physical asymptotic symmetry group is given by modding out by trivial diffeomorphisms whose asymptotic charges vanish:

$$\mathcal{D}_{\mathfrak{p}_{21}} = \mathcal{G}_{\mathfrak{p}_{21}} / \sim\,. \tag{E.30}$$

Here the equivalence relation $\sim$ is defined so that two difeomorphisms are equivalent if they are related by a trivial diffeomorphism. For space–time boundaries that are null surfaces at a finite location, the trivial diffeomorphisms are those with[32]

$$\varphi = \text{identity}\,, \quad \gamma = 0\,. \tag{E.31}$$

This is also true in the BMS context, and we will assume it remains true for the more general symmetry groups discussed below, pending the explicit computation of the corresponding charges. It follows that the group $\mathcal{D}_{\mathfrak{p}_{21}}$ is in one-to-one correspondence with the set of pairs (φ, γ):

$$\mathcal{D}_{\mathfrak{p}_{21}} \simeq \{(\varphi, \gamma) \mid \psi \in \mathcal{G}_{\mathfrak{p}_{21}}\}\,. \tag{E.32}$$

We now argue that the group $\mathcal{D}_{\mathfrak{p}_{21}}$ coincides with the symmetry group $\mathcal{D}_{\mathfrak{u}_{21}}$ of the intrinsic structure discussed in Subsec. E.1.2. From the condition $\psi^*\Gamma_{\mathfrak{p}_{21}} = \Gamma_{\mathfrak{p}_{21}}$ in the definition (E.26), we obtain that for any $(\mathcal{M}, g_{ab}, \Phi)$ in $\Gamma_{\mathfrak{p}_{21}}$ we have $\mathfrak{p}_{21} = \psi^*\mathfrak{p}_{21}(g_{ab}, \Phi) = \mathfrak{p}_{21}(\psi^* g_{ab}, \psi^*\Phi)$. Using Eqs. (E.27) and (E.28) we can rewrite this as

$$\left[\varphi^* n^i, \varphi^* q_{ij}, \varphi^* \kappa, e^{\gamma} n_a\right] = \left[n^i, q_{ij}, \kappa, n_a\right]. \tag{E.33}$$

Using the equivalence relation (E.18) it follows that there exists a scaling function α on $\mathscr{I}^+$ for which

$$\varphi^* n^i = e^{-\alpha} n^i \,, \tag{E.34a}$$

$$\varphi^* q_{ij} = e^{2\alpha} q_{ij} \,, \tag{E.34b}$$

$$\varphi^* \kappa = e^{-\alpha}(\kappa + \pounds_n \alpha) \,, \tag{E.34c}$$

$$e^{\gamma} n_a = e^{\alpha} n_a \,. \tag{E.34d}$$

The first three equations here coincide with Eqs. (E.12), which imply that φ lies in $\mathcal{D}_{\mathfrak{u}_{21}}$. The last equation implies that $\alpha = \gamma$, which is compatible with the scaling laws (E.9) and (E.29). In particular this implies that γ is determined by φ, $\gamma = \gamma(\varphi)$, which implies from Eq. (E.32) that $\mathcal{D}_{\mathfrak{p}_{21}}$ and $\mathcal{D}_{\mathfrak{u}_{21}}$ are isomorphic.

E.1.5. *Alternative definition of field configuration space with conformal freedom fixed*

The literature has often used an alternative definition of the field configuration space, which differs from the definition (E.23) given above only in that the conformal freedom is fixed.[34,46] This configuration space $\Gamma_{\rm BMS}$ is defined in Eq. (4.2), and depends on a choice of conformal factor Φ_0 on a neighborhood $\mathcal{D}$ of $\mathscr{I}^+$ and a choice of unphysical metric $g_{0\,ab}$ on $\mathscr{I}^+$.

We now show that the orbit of $\Gamma_{\rm BMS}$ under conformal transformations is a particular space $\Gamma_{\mathfrak{p}_{21}}$, where $\mathfrak{p}_{21} = \left[\bar{n}_0^i, \bar{q}_{0\,ij}, 0, \bar{n}_{0\,a}\right]$ and $\bar{n}_0^i$, $\bar{q}_{0\,ij}$ and $\bar{n}_{0\,a}$ are computed from the given data Φ_0 on $\mathcal{D}$ and $g_{0\,ab}$ on $\mathscr{I}^+$. First, it follows from the definitions (E.23) and (4.2) that $\Gamma_{\rm BMS} \subset \Gamma_{\mathfrak{p}_{21}}$. Next, suppose that $(\mathcal{M}, g_{ab}, \Phi)$ lies in $\Gamma_{\mathfrak{p}_{21}}$. It follows that

$$\left[n^i, q_{ij}, \kappa, n_a\right] = \left[\bar{n}_0^i, \bar{q}_{0\,ij}, 0, \bar{n}_{0a}\right], \tag{E.35}$$

where the fields on the left-hand side are computed from g_{ab}, Φ. From the equivalence relation (E.18) there exists a scaling function α on $\mathscr{I}^+$ so that

$$\left(n^i, q_{ij}, \kappa, n_a\right) = \left(e^{-\alpha}\bar{n}_0^i, e^{2\alpha}\bar{q}_{0\,ij}, e^{-\alpha}\pounds_{\bar{n}_0}\alpha, e^{\alpha}\bar{n}_{0a}\right). \tag{E.36}$$

By suitably extending the definition of α from $\mathscr{I}^+$ into the interior of the space–time we can make $e^{-\alpha}\Phi$ coincide with Φ_0 on $\mathcal{D}$, since the gradients of these functions agree on $\mathscr{I}^+$. It then follows that

$$\left(\mathcal{M}, e^{-2\alpha} g_{ab}, e^{-\alpha}\Phi\right) \tag{E.37}$$

lies in $\Gamma_{\rm BMS}$.

From this relation between $\Gamma_{\rm BMS}$ and $\Gamma_{\mathfrak{p}_{21}}$, it follows that the asymptotic symmetry group of $\Gamma_{\rm BMS}$ coincides with $\mathcal{D}_{\mathfrak{p}_{21}}$. Note however that a bulk diffeomorphism ψ acts differently on the two spaces. On $\Gamma_{\rm BMS}$ it acts in tandem with a conformal transformation to preserve the conformal factor,

$$(g_{ab}, \Phi) \to \left[\left(\frac{\Phi}{\psi^*\Phi}\right)^2 \psi^* g_{ab}, \Phi\right], \tag{E.38}$$

while on $\Gamma_{\mathfrak{p}_{21}}$ it acts simply as

$$(g_{ab}, \Phi) \to (\psi^* g_{ab},\, \psi^*\Phi)\,. \tag{E.39}$$

E.2. *Generalized BMS field configuration space and symmetry group*

We now turn to the generalized BMS field configuration space and generalized BMS group of Refs. 29–31. The discussion in this case mirrors exactly the discussion of the BMS case given in the previous section, with the following modifications:

- The induced metric q_{ij} is replaced everywhere by the volume form η_{ijk}. Thus we use Eq. (D.7c) instead of Eqs. (D.7a) and (D.7b), and use the scaling relation (D.8e) everywhere instead of the relation (D.8c).
- The equivalence relation (E.1) is replaced with

$$(n^i, \eta_{ijk}, \kappa) \sim (e^\sigma n^i, e^{-3\sigma}\eta_{ijk}, e^\sigma\kappa - e^\sigma \pounds_n \sigma)\,, \tag{E.40}$$

  and the definition (E.2) of intrinsic structure is replaced by

$$\mathfrak{u}_{11} = [n^i, \eta_{ijk}, \kappa]\,. \tag{E.41}$$

- Similarly the equivalence relation (E.4) is replaced by

$$\left(\bar{n}^i, \bar{\eta}_{ijk}\right) \sim \left(e^\sigma \bar{n}^i, e^{-3\sigma}\bar{\eta}_{ijk}\right) \tag{E.42}$$

  with $\pounds_{\bar{n}}\sigma = 0$ and $\pounds_{\bar{n}}\bar{\eta}_{ijk} = 0$. The definition (E.5) of intrinsic structure is replaced

$$\mathfrak{u}_{10} = \left[\bar{n}^i, \bar{\eta}_{ijk}\right]. \tag{E.43}$$

- The corresponding symmetry groups $\mathcal{D}_{\mathfrak{u}_{11}}$ and $\mathcal{D}_{\mathfrak{u}_{10}}$ are defined as in App. E.1.2, and again coincide in the appropriate sense. The relations (E.12) that define the symmetries are replaced by

$$\varphi^*\bar{n}^i = e^{-\alpha}\bar{n}^i\,, \tag{E.44a}$$

$$\varphi^*\bar{\eta}_{ijk} = e^{3\alpha}\bar{\eta}_{ijk}\,, \tag{E.44b}$$

where $\pounds_{\bar{n}}\alpha = 0$, whose linearized versions are[46]

$$\pounds_\xi \bar{n}^i = -\alpha \bar{n}^i \,, \tag{E.45a}$$

$$\pounds_\xi \bar{\eta}_{ijk} = 3\alpha \bar{\eta}_{ijk} \,. \tag{E.45b}$$

- The definitions (E.19) and (E.21) of boundary structures are replaced by the analogous definitions

$$\mathfrak{p}_{11} = \left[n^i, \eta_{ijk}, \kappa, n_a\right] \tag{E.46}$$

and

$$\mathfrak{p}_{10} = \left[\bar{n}^i, \bar{\eta}_{ijk}, \bar{n}_a\right]. \tag{E.47}$$

- The corresponding field configuration spaces $\Gamma_{\mathfrak{p}_{11}}$ and $\Gamma_{\mathfrak{p}_{10}}$ are defined as before, and the argument that the corresponding symmetry groups $\mathcal{D}_{\mathfrak{p}_{11}}$ and $\mathcal{D}_{\mathfrak{p}_{10}}$ coincide with those of the intrinsic structures is unchanged.
- The definition (4.2) of the conformal-freedom-fixed field configuration space is replaced with the definition (4.3) of the space Γ_{GBMS}. As before, one can show that taking the orbit of Γ_{GBMS} under conformal transformations yields a particular space $\Gamma_{\mathfrak{p}_{11}}$, with $\mathfrak{p}_{11} = \left[\bar{n}^i_0, \bar{\eta}_{0\,ijk}, 0, \bar{n}_{0\,a}\right]$, and that the asymptotic symmetry group of Γ_{GBMS} coincides with $\mathcal{D}_{\mathfrak{p}_{11}}$.

E.3. *Weyl BMS field configuration space and symmetry group*

The field configuration space can be further expanded by omitting both the induced metric q_{ij} and volume form η_{ijk} from the definitions. We call the resulting space the *Weyl BMS* field configuration space, following Ref. 45, since the extra symmetries correspond to conformal transformations of the form (D.8) that are independent of other pieces of the symmetry generator. The resulting symmetry group then coincides with the symmetry group of general null surfaces at finite locations derived in Ref. 32. This coincidence of symmetry groups should facilitate understanding how the asymptotic symmetry group is obtained from a limit of symmetry groups on finite null boundaries. It will also be important in future derivations of global conservation laws in black hole space–times, where analyses analogous to Refs. 162 and 163 at future timelike infinity will be needed in order to determine the appropriate matching of symmetry generators on the future horizon with those on future null infinity; see for example the discussion in Sec. 7 of Ref. 32.

For the Weyl BMS field configuration space, the required modifications to the discussion of the BMS case of App. E.1 are:

- The induced metric q_{ij} is omitted everywhere. Thus the equivalence relation (E.1) is replaced with

$$(n^i, \kappa) \sim (e^\sigma n^i, e^\sigma \kappa - e^\sigma \pounds_n \sigma)\,, \tag{E.48}$$

and the definition (E.2) of intrinsic structure is replaced by

$$\mathfrak{u}_{01} = [n^i, \kappa]\,. \tag{E.49}$$

Similarly the equivalence relation (E.4) is replaced by

$$(\bar{n}^i) \sim (e^\sigma \bar{n}^i) \tag{E.50}$$

with $\pounds_{\bar{n}}\sigma = 0$. The definition (E.5) of intrinsic structure is replaced

$$\mathfrak{u}_{00} = \left[\bar{n}^i\right]. \tag{E.51}$$

- The corresponding symmetry groups $\mathcal{D}_{\mathfrak{u}_{01}}$ and $\mathcal{D}_{\mathfrak{u}_{00}}$ are defined as in App. E.1.2, and again coincide in the appropriate sense. The relations (E.12) that define the symmetries are replaced by

$$\varphi^* \bar{n}^i = e^{-\alpha} \bar{n}^i , \tag{E.52}$$

where $\pounds_{\bar{n}}\alpha = 0$, whose linearized version is

$$\pounds_\xi \bar{n}^i = -\alpha \bar{n}^i . \tag{E.53}$$

The symmetry group (E.52) coincides with that of general finite null surfaces, given by Eqs. (4.4) of Ref. 32 specialized to $\kappa = 0$.

- The definitions (E.19) and (E.21) of boundary structures are replaced by the analogous definitions

$$\mathfrak{p}_{01} = [n^i, \kappa, n_a] \tag{E.54}$$

and

$$\mathfrak{p}_{00} = \left[\bar{n}^i, \bar{n}_a\right]. \tag{E.55}$$

- The corresponding field configuration spaces $\Gamma_{\mathfrak{p}_{01}}$ and $\Gamma_{\mathfrak{p}_{00}}$ are defined as before, and the argument that the corresponding symmetry groups $\mathcal{D}_{\mathfrak{p}_{01}}$ and $\mathcal{D}_{\mathfrak{p}_{00}}$ coincide with those of the intrinsic structures is unchanged.
- The definition (4.2) of the conformal-freedom-fixed field configuration space is replaced with the definition (4.4) of the space $\Gamma_{\rm WBMS}$. As before, one can show that taking the orbit of $\Gamma_{\rm WBMS}$ under conformal transformations yields a particular space $\Gamma_{\mathfrak{p}_{01}}$, with $\mathfrak{p}_{01} = \left[\bar{n}_0^i, 0, \bar{n}_{0\,a}\right]$, and that the asymptotic symmetry group of $\Gamma_{\rm WBMS}$ coincides with $\mathcal{D}_{\mathfrak{p}_{01}}$.

E.4. *Properties of the asymptotic symmetry groups*

We now turn to a characterization of the structure of the symmetry groups discussed in the previous sections and the corresponding algebras.

For convenience, we will specialize to the definitions $\mathcal{D}_{\mathfrak{u}_{20}}$, $\mathcal{D}_{\mathfrak{u}_{10}}$ and $\mathcal{D}_{\mathfrak{u}_{00}}$ of these groups in which the inaffinity κ has been set to zero, given by Eq. (E.10) and its avatars. For each universal structure $\mathfrak{u}_{20}$, $\mathfrak{u}_{10}$ or $\mathfrak{u}_{00}$, we pick a corresponding representative $(\bar{n}^i, \bar{q}_{ij})$, $(\bar{n}^i, \bar{\eta}_{ijk})$, or $(\bar{n}^i)$. The null generator $\bar{n}^i$ is common to all of these representatives, and we construct a coordinate system (u, θ^A) on $\mathscr{I}^+$ using this normal as described in Subsec. 4.2 in the body of the paper. The symmetry transformations are then given by Eqs. (4.7).

To derive these transformations, we start with Eq. (E.12a), which is common to all three groups. Combining this with Eq. (4.6) yields

$$\varphi^* \partial_u = e^{-\alpha} \partial_u = \frac{\partial u}{\partial \hat{u}} \partial_u + \frac{\partial \theta^A}{\partial \hat{u}} \partial_A \,, \tag{E.56}$$

which yields $\theta^A = \theta^A(\hat{\theta}^B)$, and inverting yields Eq. (4.7b). It also yields

$$\partial_u \hat{u}(u, \theta^A) = e^{\alpha(u, \theta^A)} \,. \tag{E.57}$$

Also from Eq. (E.13) which applies to all three groups we obtain that $\partial_u e^{\alpha} = 0$, so that $\alpha = \alpha(\theta^A)$, and now integrating Eq. (E.57) yields Eq. (4.7a).

This completes the derivation for the Weyl BMS case, where the functions χ, α and γ are unconstrained. For the generalized BMS case, it follows from the conditions (E.44) and the definition (C.2) of μ_{ij} that the function α is given by Eq. (4.13). Similarly, for the BMS case, it follows from the condition (E.34b) that the function α is given by Eq. (4.11).

Finally, it can be useful to understand the action of the groups on representatives of the universal structures for which $\kappa \neq 0$. We specialize for simplicity to linearized supertranslations of the form

$$\xi^i = f n^i \,. \tag{E.58}$$

Note that the symmetry generator ξ^i is invariant under the conformal rescalings (D.8) by definition, but that the coefficient f has a nonzero conformal weight, transforming as $f \to e^{-\sigma} f$ from Eq. (D.8a). For the BMS group the coefficient f satisfies the conformally invariant equation

$$(\pounds_n - \kappa) f = 0 \,, \tag{E.59}$$

from Eqs. (D.7a), (D.7b), (E.8a) and (E.8b). This equation is also valid for the generalized BMS group, from Eqs. (D.7c) together with the unbarred version of Eqs. (E.45). Finally, for the Weyl BMS group, Eq. (E.59) is replaced with the conformally invariant equation[qq]

$$\pounds_n (\pounds_n - \kappa) f = 0 \,, \tag{E.60}$$

from Eqs. (E.8a) and (E.8c) which apply to the group $\mathcal{D}_{\mathrm{u}_{01}}$. This equation now admits the two different kinds of supertranslations as solutions.

Appendix F. Details of Holographic Renormalization with a Rigging Vector Field

In this appendix we derive some of the results on holographic renormalization which were discussed in Sec. 5.

[qq]This equation differs from the corresponding Eq. (4.17) of Ref. 32, despite the fact that the underlying algebras of infinitesimal diffeomorphisms $\boldsymbol{\xi}$ on the null surfaces coincide. The difference arises from the fact that the scaling properties of κ and n^i differ in the two cases (see App. C).

We start by inserting the coordinate expansions (5.12) of the Lagrangian and symplectic form into the identity (2.5), and specializing to on-shell field configurations. This yields

$$\theta'^{\,0}_{,0} + \theta'^{\,i}_{,i} = \delta\mathcal{L}\,. \tag{F.1}$$

Similarly the boundary canonical transformation (5.6) can be written in terms of coordinate components as

$$\mathcal{L}_{\rm ren} = \mathcal{L} + B^i_{,i} + B^0_{,0}\,, \tag{F.2a}$$

$$\theta'^{\,0}_{\rm ren} = \theta'^{\,0} + \delta B^0 - \Lambda^i_{,i}\,, \tag{F.2b}$$

$$\theta'^{\,i}_{\rm ren} = \theta'^{\,i} + \delta B^i + \Lambda^i_{,0} + 2\Lambda^{ij}_{,j}\,. \tag{F.2c}$$

It follows that the choices (5.13) of B and Λ yield $\mathcal{L}_{\rm ren} = \theta'^{\,i}_{\rm ren} = 0$, cf. Eq. (5.14a). Also differentiating Eq. (F.2b) with respect to x^0 and combining with Eqs. (5.13) and (F.1) gives $\theta'^{\,0}_{{\rm ren}\,,0} = 0$. Hence to evaluate $\theta'^{\,0}_{\rm ren}$ at $x^0 = 0$ we can evaluate it at $x^0 = \upsilon_0$, at which value it reduces to $\theta'^{\,0}(\upsilon_0)$, from Eqs. (F.2b) and (5.13). This yields the result (5.14c).

We next turn to the computation of the anomalies (5.20). Given the prescription (5.9) for $B[L', \mathbf{v}]$, the anomaly is given by, from the definition (2.4),

$$\Delta_{\hat\xi} B = B[\psi^* L', \mathbf{v}] - B[\psi^* L', \psi^* \mathbf{v}]\,. \tag{F.3}$$

Here it is understood that the right-hand side is to be linearized in the diffeomorphism ψ, whose linear part is parametrized by the vector field $\boldsymbol{\xi}$. (It will be convenient to initially work with the full nonlinear diffeomorphism rather than its linearized version). Acting on both sides with $\psi^{-1\,*}$, we see that the right-hand side is proportional to ξ, and so we can drop the $\psi^{-1\,*}$ on the left-hand side when working to linear order. This gives

$$\Delta_{\hat\xi} B = B[L', \psi^{-1\,*}\mathbf{v}] - B[L', \mathbf{v}]\,. \tag{F.4}$$

Defining $B_1 = B[L', \psi^{-1\,*}\mathbf{v}]$, we have from Eqs. (5.9) that B_1 is given by

$$i_{\tilde v} B_1 = 0\,, \tag{F.5a}$$

$$\tilde\pi^*_\upsilon B_1 = \int_\upsilon^{\upsilon_0} d\bar\upsilon\, \tilde\pi^*_{\bar\upsilon} i_{\tilde v} L'\,, \tag{F.5b}$$

where $\tilde{\mathbf{v}} = \psi^{-1\,*}\mathbf{v}$ and

$$\tilde\pi_\upsilon = \psi \circ \pi_\upsilon \circ \varphi^{-1}\,, \tag{F.6}$$

with $\varphi : \mathcal{N} \to \mathcal{N}$ being the restriction of ψ to the boundary $\mathcal{N}$. Acting on both sides of Eq. (F.5b) with φ^* now gives

$$\pi^*_\upsilon \psi^* B_1 = \int_\upsilon^{\upsilon_0} d\bar\upsilon\, \pi^*_{\bar\upsilon} \psi^* i_{\tilde v} L'\,. \tag{F.7}$$

Using $\psi^* = 1 + \pounds_\xi + \cdots$ together with Eq. (F.4) this can be rewritten as

$$\pi_v^* \Delta_{\hat{\xi}} B = -\pi_v^* \pounds_\xi B + \int_v^{v_0} d\bar{v}\, \pi_{\bar{v}}^* i_v \pounds_\xi L' \,. \tag{F.8}$$

We now switch to using the coordinate notation of Subsec. 5.3.1. First, for any vector field $\mathbf{w} = w^0 \partial_0 + w^i \partial_i$ and any d-form $\chi = \chi^0 \varpi + \chi^i dx^0 \wedge \varpi_i$, the Lie derivative is given by

$$\begin{aligned} \pounds_w \chi = {} & \left[w^0 \chi^0_{,0} - w^0_{,i} \chi^i + (w^i \chi^0)_{,i} \right] \varpi \\ & + \left[-w^i_{,0} \chi^0 - w^i_{,j} \chi^j + (w^0 \chi^i)_{,0} + (w^j \chi^i)_{,j} \right] dx^0 \wedge \varpi_i \,. \end{aligned} \tag{F.9}$$

Using this formula together with Eqs. (5.13), we find Eq. (F.8) reduces to

$$(\Delta_{\hat{\xi}} B)^0 = (\xi^0 \mathcal{L})(v_0) - \left(\xi^i \int_v^{v_0} d\bar{v}\, \mathcal{L} - \int_v^{v_0} d\bar{v}\, \xi^i \mathcal{L} \right)_{,i} \,. \tag{F.10}$$

The other component of $\Delta_{\hat{\xi}} B$ is given by combining Eq. (F.5a) with $\tilde{\mathbf{v}} = \mathbf{v} - \pounds_\xi \mathbf{v}$ and Eq. (F.4), which gives

$$i_v \Delta_{\hat{\xi}} B = i_{\pounds_\xi v} B \,. \tag{F.11}$$

Using $\mathbf{v} = \partial_0$ this yields

$$(\Delta_{\hat{\xi}} B)^i = \xi^i_{,0} \int_v^{v_0} d\bar{v} \mathcal{L} \,. \tag{F.12}$$

Combining the results (F.10) and (F.12) with Eqs. (5.24a), (5.24b) and (5.26a) now shows consistency with the identity (5.20a).

The derivation of $\Delta_{\hat{\xi}} \Lambda$ is exactly analogous. Equations (F.8) and (F.11) are replaced by

$$\pi_v^* \Delta_{\hat{\xi}} \Lambda = -\pi_v^* \pounds_\xi \Lambda - \int_v^{v_0} d\bar{v}\, \pi_{\bar{v}}^* i_v \pounds_\xi \theta' \,, \tag{F.13a}$$

$$i_v \Delta_{\hat{\xi}} \Lambda = i_{\pounds_\xi v} \Lambda \,, \tag{F.13b}$$

which together yield

$$\begin{aligned} \Delta_{\hat{\xi}} \Lambda = {} & - \left[(\xi^0 \theta'^{\,i})(v_0) - \int_v^{v_0} d\bar{v}\, \xi^i_{,0} \theta'^{\,0} + \xi^j \int_v^{v_0} d\bar{v}\, \theta'^{\,i}_{,j} - \int_v^{v_0} d\bar{v}\, \xi^j \theta'^{\,i}_{,j} \right] \varpi_i \\ & + \xi^i_{,0} \int_v^{v_0} d\bar{v}\, \theta'^{\,j}\, dx^0 \wedge \varpi_{ij} \,. \end{aligned} \tag{F.14}$$

Combining this with Eqs. (5.24c), (5.24d), (5.26) and (F.1) now shows consistency with the identity (5.20b).

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