

The classical dynamics of gauge theories in the deep infrared

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ABSTRACT: Gauge and gravitational theories in asymptotically flat settings possess infinitely many conserved charges associated with large gauge transformations or diffeomorphisms that are nontrivial at infinity. To what extent do these charges constrain the scattering in these theories? It has been claimed in the literature that the constraints are trivial, due to a decoupling of hard and soft sectors for which the conserved charges constrain only the dynamics in the soft sector. We show that the argument for this decoupling fails due to the failure in infinite dimensions of a property of symplectic geometry which holds in finite dimensions. Specializing to electromagnetism coupled to a massless charged scalar field in four dimensional Minkowski spacetime, we show explicitly that the two sectors are always coupled using a perturbative classical computation of the scattering map. Specifically, while the two sectors are uncoupled at low orders, they are coupled at quartic order via the electromagnetic memory effect. This coupling cannot be removed by adjusting the definitions of the hard and soft sectors (which includes the classical analog of dressing the hard degrees of freedom). We conclude that the conserved charges yield nontrivial constraints on the scattering of hard degrees of freedom. This conclusion should also apply to gravitational scattering and to black hole formation and evaporation.

In developing the classical scattering theory, we show that generic Lorenz gauge solutions fail to satisfy the matching condition on the vector potential at spatial infinity proposed by Strominger to define the field configuration space, and we suggest a way to remedy this. We also show that when soft degrees of freedom are present, the order at which nonlinearities first arise in the scattering map is second order in Lorenz gauge, but can be third order in other gauges.

KEYWORDS: Black Holes, Gauge Symmetry, Space-Time Symmetries

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1 Introduction and summary

1.1 Background and motivation

Recent years have seen significant progress in our understanding of the dynamics of gauge and gravitational theories in the deep infrared. In particular, a remarkable web of relations has been discovered between three seemingly unrelated areas of infrared physics (see the review [1]): soft theorems, which govern universal properties of scattering amplitudes in the limit where the energy of some external massless particles are taken to zero [2–7]; asymptotic symmetries, which are gauge transformations or diffeomorphisms that act on the boundary of spacetime and that are associated with conservation laws [1, 8–12]; and memory effects, zero frequency components of bursts of radiation that are potentially observable for example with future gravitational wave observations [13–17]. New avenues continue to be explored in this rich subject: aspects of the triangular web of relations have been extended to enlarged symmetry groups, to higher dimensional spacetimes, to subleading orders, to new spacetime boundaries and to new theoretical contexts; see, e.g, refs. [18–22].

One interesting aspect of this field has been the discovery of a new kind of black hole hair in general relativity [23, 24]. Asymptotic symmetry transformations at null infinity and

at future horizons are associated with an infinite number of charges called soft hair which are conserved in scattering processes.¹ It has been argued that during black hole evaporation, effects involving the heretofore neglected soft hair open the door to possible new avenues for resolving the information loss paradox [23, 24, 27, 28]. One possible mechanism is that the soft hair acts as a kind of catalyst in its interactions with the Hawking radiation and engenders correlations between different Hawking quanta. Another possibility is that the Hawking radiation is purified at late times by its entanglement with soft hair degrees of freedom [29].

A number of arguments have been made against the relevance of the soft conservation laws to information loss. First, it has been argued that in some contexts the number of accessible soft hair degrees of freedom is too small to be relevant,² for example in anti-deSitter spacetimes [31]. Second, it has been argued that the interactions between the soft (zero-energy) and hard (non-zero energy) sectors of a gauge or gravitational theory in any scattering process are too trivial to impact the information puzzle [32–35].

In particular, it has been argued that soft charges reside in a sector of the theory that is decoupled from all the other dynamics [32–34]. In other words, the soft conservation laws constrain only a sector of the theory that is decoupled from the sector that contains all the hard degrees of freedom, including the Hawking radiation. As a result, the soft charges do not constrain the scattering problem in an essential way. This has been described as soft hair comprising a “soft wig” that is easily removed [33].

While motivated by deep questions in the quantum theory, the question of whether the soft and hard degrees of freedom are dynamically decoupled has a well defined classical limit, and so can be investigated within the classical theory. Note that the claimed decoupling is considerably stronger than properties of the classical theory implied by the soft factorization theorems in quantum field theory [36, 37] (see section 4.3). The claim is that, first, the classical phase space Γ can be expressed as a product $\Gamma = \Gamma_{\text{soft}} \times \Gamma_{\text{hard}}$, for some definitions of soft and hard sectors Γ_{soft} and Γ_{hard} (subject to the loose requirement that Γ_{soft} contains only zero energy degrees of freedom). Second, the classical scattering map³ $\mathcal{S} : \Gamma \rightarrow \Gamma$ is claimed factorize as

$$(s, h) \rightarrow [\bar{s}(s), \bar{h}(h)] \quad (1.1)$$

for $s \in \Gamma_{\text{soft}}$ and $h \in \Gamma_{\text{hard}}$. We will argue that such factorization cannot occur.

1.2 Summary of results

The purpose of this paper is to reexamine the coupling of hard and soft degrees of freedom in classical gauge theories. We focus for simplicity on perhaps the simplest context where these issues arise, electromagnetism coupled to a massless complex scalar field in four dimensional Minkowski spacetime. In this context we argue that for *any* choice of definitions of hard and soft sectors, the two sectors are always dynamically coupled, and that therefore the soft

¹It has been argued that these charges can be used to compute the entropy of Kerr black holes [25, 26].

²See also ref. [30] which argues that a maximal entanglement between hard and soft sectors in black hole evaporation would require involving soft degrees of freedom beyond the leading and subleading orders.

³By scattering map we mean the mapping from the phase space of initial data on past null infinity to the phase space of final data on future null infinity.

conservation laws yield nontrivial constraints on the dynamics. We expect that our results will generalize to nonabelian gauge theories and to general relativity, and will constrain the quantum as well as the classical theories. This would support the original arguments of Hawking, Perry and Strominger that these conservation laws constrain in a nontrivial way the formation and evaporation of black holes [23, 24].

There are two different versions of our analysis. The two versions arise because the scattering map is not really a map from phase space Γ to itself, as described above. Instead it is a mapping $\mathcal{S} : \Gamma_- \rightarrow \Gamma_+$ from the space Γ_- of initial data at past null infinity $\mathcal{I}^-$ to the space Γ_+ of final data at future null infinity $\mathcal{I}^+$. Our analysis allows for adjusting the definitions of hard and soft sectors, but imposes that the “same” definitions of hard and sectors be used at $\mathcal{I}^-$ and at $\mathcal{I}^+$ (otherwise it is always trivially possible to find definitions which decouple the dynamics). To make this requirement precise requires an identification of Γ_- and Γ_+ . The first version of our analysis (used in most of the paper) adopts the identification

$$\varphi_{0*} : \Gamma_- \rightarrow \Gamma_+ \quad (1.2)$$

which is the pullback action of the mapping $\varphi_0 : \mathcal{I}^+ \rightarrow \mathcal{I}^-$ defined by identifying points that are connected by a null geodesic through Minkowski spacetime.⁴ Using this identification effectively means that we are considering the modified scattering map $\mathcal{S}_1 : \Gamma_+ \rightarrow \Gamma_+$ given by

$$\mathcal{S}_1 = \mathcal{S} \circ \varphi_{0*}^{-1}. \quad (1.3)$$

The second version of our analysis instead identifies Γ_- and Γ_+ using the free field scattering map $\mathcal{S}_0$, detailed in section 3.4 below. This differs from the identification (1.2) due to a nontrivial action of the free field scattering in the soft sector [eq. (3.23e) below]. Using the identification $\mathcal{S}_0$ effectively means that we are considering the modified scattering map $\mathcal{S}_2 : \Gamma_+ \rightarrow \Gamma_+$ given by

$$\mathcal{S}_2 = \mathcal{S} \circ \mathcal{S}_0^{-1}. \quad (1.4)$$

This definition is similar to using the interaction representation in quantum mechanics.

In the first version of our analysis, we show that the scattering map $\mathcal{S}_1$ cannot be of the factorized form (1.1), and we also exclude the more general forms

$$(s, h) \rightarrow [\bar{s}(s, h), \bar{h}(h)] \quad (1.5)$$

and

$$(s, h) \rightarrow [\bar{s}(s), \bar{h}(s, h)]. \quad (1.6)$$

A scattering map of the form (1.5) would support the arguments against the relevance of soft variables to information loss, since it would imply that the hard variables evolve independently of the soft variables. Maps of the form (1.5) and (1.6) are automatically

⁴In curved spacetimes this construction is problematic since caustics can occur. An alternative prescription would be to choose any diffeomorphism $\varphi_0 : \mathcal{I}^- \rightarrow \mathcal{I}^+$ that preserves the appropriate asymptotic structures that determine the BMS group (see, e.g., eq. (E.5) of ref. [38]). Such diffeomorphisms are unique up to compositions with BMS transformations. Our results in the Minkowski context would be unchanged if we used any of these more general identifications.

excluded if one assumes that the hard and soft variables are symplectically orthogonal, that is, have vanishing Poisson brackets with one another, a necessary condition for a factorization of the total Hilbert space into hard and soft factors (see appendix A). Our analysis excluding the forms (1.5) and (1.6) does not impose this requirement.

The second version of our analysis yields different results, since the free field scattering map $\mathcal{S}_0$ can in general mix the two sectors together, depending on the definitions of hard and soft sectors chosen. It is possible to define versions of the soft degrees of freedom which are exactly conserved by the scattering map $\mathcal{S}_2$, as pointed out by Bousso and Poratti (BP) [33]. It is natural to use these variables to define the soft sector, in which case the general form of $\mathcal{S}_2$ is

$$(s, h) \rightarrow [s, \bar{h}(s, h)]. \quad (1.7)$$

(Note that this form of mapping is disallowed for $\mathcal{S}_1$.) However we show that it is not possible to adjust the definition of the hard sector to eliminate the dependence of $\bar{h}$ on s in eq. (1.7). That is, we again exclude completely factorized scattering maps of the form (1.1) when we use the conserved soft variables.

1.3 Overview of paper

Our analysis will proceed in three main stages: (i) The development of the classical scattering theory (sections 2 and 3); (ii) An analysis of the arguments that have been made for trivial soft dynamics (section 4); and (iii) Explicit computations of the scattering map in perturbation theory (sections 5 and 6).

The classical scattering theory of electromagnetism in Minkowski spacetime including soft degrees of freedom has been treated in detail in refs. [1, 39–42], so we focus in the brief overview given here on one or two points where we deviate from these previous treatments. The theory is based on the following well-known sequence of steps of the covariant phase space approach [43–46]: (i) Define a field configuration space $\mathcal{F}$ by specifying appropriate boundary conditions at spatial and null infinity. (ii) Define a presymplectic form on the on-shell subspace $\overline{\mathcal{F}}$ of the field configuration space in terms of an integral over Cauchy slices, and ensure that it is independent of choice of Cauchy slice by adjusting the definition and adjusting the choice of $\mathcal{F}$ if necessary. (iii) The gauge freedom is defined in terms of degeneracy directions of the presymplectic form. By suitably fixing all the asymptotic gauge freedom, define a space of initial data Γ_- on past null infinity $\mathcal{I}^-$ and a space of final data Γ_+ on future null infinity $\mathcal{I}^+$. (iv) Mod out the space $\overline{\mathcal{F}}$ by degeneracy directions of the presymplectic form to obtain the phase space Γ , which can be identified with the space of initial data Γ_- and with the space of final data Γ_+ . These identifications define the classical scattering map $\mathcal{S} : \Gamma_- \rightarrow \Gamma_+$ which is a symplectomorphism.

A key role in the analysis is played by the leading order piece of the angular components of the vector potential in an expansion in $1/r$ about null infinity, which we denote by $\mathcal{A}_{+A}(u, \theta, \varphi) = \mathcal{A}_{+A}(u, \theta)$ on $\mathcal{I}^+$ and by $\mathcal{A}_{-A}(v, \theta)$ on $\mathcal{I}^-$. We write the electric parity pieces of these fields in terms of potentials as $D_A \Psi_+^e(u, \theta)$ and $D_A \Psi_-^e(v, \theta)$. Finally we define the (assumed finite) limiting values of the potentials at spatial infinity i^0 or at the timelike

infinities $i^\pm$ as

$$\Psi_+^e = \lim_{u \rightarrow \infty} \Psi_+^e, \quad \Psi_+^e = \lim_{u \rightarrow -\infty} \Psi_+^e, \quad \Psi_-^e = \lim_{v \rightarrow \infty} \Psi_-^e, \quad \Psi_-^e = \lim_{v \rightarrow -\infty} \Psi_-^e. \quad (1.8)$$

It is also convenient to use the following linear combinations of these quantities:

$$\Delta\Psi_+^e = \Psi_+^e - \Psi_-^e, \quad \bar{\Psi}_+^e = \frac{\Psi_+^e + \Psi_-^e}{2}, \quad \Delta\Psi_-^e = \Psi_-^e - \Psi_+^e, \quad \bar{\Psi}_-^e = \frac{\Psi_-^e + \Psi_+^e}{2}. \quad (1.9)$$

These fields are called soft degrees of freedom since they correspond to distributional components of the Fourier transforms of the fields $\Psi_+^e(u, \theta)$ and $\Psi_-^e(v, \theta)$ at zero frequency or zero energy.

Some of the issues that arise in our analysis of the classical scattering theory that are important for our conclusions are as follows (see sections 2 and 3 for more details):

- *Use independent coordinates on phase space.* We decompose the function $\Psi_-^e(v, \theta)$ as

$$\Psi_-^e(v, \theta) = \tilde{\Psi}_-^e(v, \theta) + g(v)\Delta\Psi_-^e(\theta) + \bar{\Psi}_-^e(\theta), \quad (1.10)$$

where $g(v)$ is a fixed smooth monotonic function with $g(-\infty) = -1/2$ and $g(\infty) = 1/2$. Then the field $\tilde{\Psi}_-^e$ obeys the boundary conditions $\tilde{\Psi}_-^e \rightarrow 0$ as $v \rightarrow \pm\infty$. The fields $\tilde{\Psi}_-^e$, $\Delta\Psi_-^e$ and $\bar{\Psi}_-^e$ can be varied independently and are thus good coordinates on phase space. Refs. [1, 33] use instead the coordinates $\Delta\Psi_-^e$, $\bar{\Psi}_-^e$ and $\Psi_-^e - \bar{\Psi}_-^e$, which are not independent since

$$\lim_{v \rightarrow \infty} [\Psi_-^e - \bar{\Psi}_-^e] = \Delta\Psi_-^e/2. \quad (1.11)$$

This issue underlies why our conclusions on soft-hard coupling differ from those of ref. [33].

- *Ensure uniqueness of presymplectic form.* When large gauge transformations that are nontrivial at null infinity are allowed, the standard expression for the presymplectic form evaluated at past null infinity $\mathcal{I}^-$ does not coincide with the version evaluated at $\mathcal{I}^+$ [see eq. (2.32) and appendix D]. Thus it is problematic to choose one of these to define the presymplectic form as in refs. [39–42]. The usual way of dealing with this difficulty, suggested by Strominger [1, 47], is to restrict the field configuration space by imposing a matching condition that relates the fields Ψ_+^e and Ψ_-^e [eq. (2.38)]. While this condition excludes generic Lorenz gauge solutions (see appendix B), it is possible to generalize the framework to encompass these solutions (see section 2.4).
- *Include edge modes.* It is well known that when gauge theories are formulated in finite regions of spacetime, the theory contains physical degrees of freedom on the boundary, so-called edge modes, that are necessary to preserve gauge invariance [48–52]. Whether or not it is necessary to include edge modes that arise at asymptotic boundaries is less clear, with some treatments in the literature including them [1, 47] and others excluding them [33, 39, 40, 42, 53]. We shall find it convenient to include edge modes, but our results would be unchanged if they were excluded. See appendix F for further discussion.

The way edge modes arise in the present context is as follows [1]. After some preliminary gauge fixing [eq. (2.34)], the potentials transform under large gauge transformations as $\Psi_-^e(v, \theta) \rightarrow \Psi_-^e(v, \theta) + \varepsilon_-(\theta)$ and $\Psi_+^e(u, \theta) \rightarrow \Psi_+^e(u, \theta) + \varepsilon_+(\theta)$ for some free functions ε_- and ε_+ . One linear combination of these two functions is fixed by the matching condition referred to above. The other linear combination is not a degeneracy direction of the presymplectic form and thus is not a true gauge degree of freedom. It is the physical symmetry that underlies the soft conservation laws [47]. The quantity that transforms under this symmetry, a corrected version of $\bar{\Psi}_-$ [eq. (2.51)], is essentially the edge mode (see sections 2.4 and 2.5 and appendix F). Note that excluding this edge mode as in refs. [33, 39, 40, 42, 53] does not exclude memory effects.

We now turn to a consideration of the arguments that have been made for the decoupling of hard and soft degrees of freedom [32–35] (section 4). There is a theorem in symplectic geometry in finite dimensions which we review in section 4.2 and appendix H. It says that if a set of phase space functions are exactly conserved by a symplectomorphism, and their Poisson brackets with each other are constants, then the symplectomorphism factorizes as in eq. (1.1). All of the conditions of this theorem, except for the restriction to finite dimensions, are satisfied in the context of the scattering map $\mathcal{S}_2$ defined in eq. (1.4) above [33]. As noted above, it is possible to define versions of the soft degrees of freedom which are conserved by $\mathcal{S}_2$. This arises because of the soft conservation laws and because of the matching condition [33]. Also the Poisson brackets of these soft degrees of freedom are constants on phase space (section 4.2). Thus one expects the factorization result to apply; this is essentially the factorization argument of Bousso and Poratti (BP) [33].⁵ In section 4.2 we show explicitly that this argument fails because the theorem does not apply in our infinite dimensional context. This indicates that the soft and hard degrees of freedom are coupled in a manner which has no analog in finite dimensions.

Another argument for decoupled soft and hard sectors was given in ref. [32], based on the soft factorization theorems for S-matrix elements in quantum field theory [2]. However, this argument is based on an assumption on how the factorization theorems constrain the quantum theory which we argue is invalid in section 4.3.

The last portion of the paper is an explicit perturbative computation of the scattering map that confirms that the hard and soft sectors are always coupled. The scattering is trivial at second order with the gauge fixing we use in this paper, as we show in section 5 and appendix I (although there is a nonlinear hard-soft coupling in Lorenz gauge). Section 6 considers third and fourth order scattering. We first show in section 6.2 and appendix J that there are two particular pieces of the fourth order scattering which are nonzero in general, one of which is the change in electromagnetic memory in the scattering process. The change in memory constitutes a transformation between an incoming purely hard scalar field and an outgoing electromagnetic field with a nontrivial soft component, so it

⁵More precisely, instead of explicitly invoking the theorem BP attempt to compute the phase space coordinates that give rise to the decoupled dynamics (1.1). The resulting coordinates are not independent which invalidates the decoupling conclusion.

is a soft-hard coupling. We then show in section 6.3 that these pieces of the fourth order scattering cannot be removed by any adjustment of the definitions of soft and hard sectors, using the first version $\mathcal{S}_1$ of the scattering map, ruling out the possibilities (1.1), (1.5) and (1.6). Section 6.4 considers the second version $\mathcal{S}_2$ of the scattering map, and again shows that the hard and soft sectors are always coupled, when the soft variable definitions are restricted to quantities which are conserved by the scattering.

2 Electromagnetism coupled to a charged scalar field: phase space of asymptotic data

In this section we review the phase space and symplectic form of electromagnetism coupled to a charged scalar field in Minkowski spacetime, expressed in terms of data at past null infinity or at future null infinity. This subject has been treated in detail by Strominger and collaborators [1, 47], by Ashtekar [40], and by Prabhu, Satishchandran and Wald [41, 42]. We mostly follow these references, but we also highlight below some points where we deviate from them, as discussed in the Introduction.

To construct the phase space we follow the strategy outlined in the Introduction. In particular, we would like to fix the asymptotic gauge degrees of freedom in order to obtain good coordinates on phase space and a nondegenerate symplectic form that can be inverted to obtain Poisson brackets. Gauge transformations can be divided into two categories, those which correspond to degeneracy directions of the presymplectic form (true gauge transformations) and those which do not. The latter category includes the so-called “large gauge transformations” that have non-trivial behavior at infinity and that correspond to the infinity of conserved charges in gauge theories discovered in the past few years [11, 47, 54]. Our strategy here will be to fix all of the true gauge degrees of freedom, but avoid fixing any of the gauge degrees of freedom that correspond to nondegenerate directions of the presymplectic form.

2.1 Foundations

The action of the theory is

$$S = -\frac{1}{4e^2} \int d^4x \sqrt{-g} F_{ab} F^{ab} - \int d^4x \sqrt{-g} (D^a \Phi)^* D_a \Phi, \quad (2.1)$$

where A_a is the vector potential, Φ is a complex scalar field, e is the electric charge, $D_a = \nabla_a - iA_a$, and $F_{ab} = \nabla_a A_b - \nabla_b A_a$. The equations of motion are

$$\square A_a - \nabla_b \nabla_a A^b = e^2 j_a = -ie^2 (\Phi \nabla_a \Phi^* - \Phi^* \nabla_a \Phi) + 2e^2 A_a \Phi^* \Phi, \quad (2.2a)$$

$$\square \Phi = 2iA^a \nabla_a \Phi + A^a A_a \Phi + i\Phi \nabla_a A^a. \quad (2.2b)$$

The theory is invariant under the local gauge transformations

$$A_a \rightarrow A_a + \nabla_a \varepsilon, \quad \Phi \rightarrow e^{i\varepsilon} \Phi, \quad (2.3)$$

which may or may not be true gauge transformations.

We now consider asymptotic conditions near future null infinity $\mathcal{I}^+$. We use retarded coordinates $(u, r, \theta, \varphi) = (u, r, \theta^A)$ in terms of which the metric is

$$ds^2 = -du^2 - 2dudr + r^2 h_{AB} d\theta^A d\theta^B, \quad (2.4)$$

where the unit metric on the two-sphere is $h_{AB} d\theta^A d\theta^B = d\theta^2 + \sin^2 \theta d\varphi^2$. We assume the following asymptotic behavior of the components of the Maxwell tensor in the limit to $\mathcal{I}^+$, that is, $r \rightarrow \infty$ at fixed u :

$$F_{ur} = \frac{1}{r^2} \mathcal{F}_{+ur} + \frac{1}{r^3} \hat{\mathcal{F}}_{+ur} + O\left(\frac{1}{r^4}\right), \quad (2.5a)$$

$$F_{uA} = \mathcal{F}_{+uA} + \frac{1}{r} \hat{\mathcal{F}}_{+uA} + O\left(\frac{1}{r^2}\right), \quad (2.5b)$$

$$F_{rA} = \frac{1}{r^2} \mathcal{F}_{+rA} + \frac{1}{r^3} \hat{\mathcal{F}}_{+rA} + O\left(\frac{1}{r^4}\right), \quad (2.5c)$$

$$F_{AB} = \mathcal{F}_{+AB} + \frac{1}{r} \hat{\mathcal{F}}_{+AB} + O\left(\frac{1}{r^2}\right). \quad (2.5d)$$

We are using a notational convention where caligraphic quantities are used to represent the pieces of fields that appear at leading order in an expansion in $1/r$, the $+$ subscripts denote quantities on $\mathcal{I}^+$, and hatted caligraphic quantities are subleading. The scalings of the leading terms can be deduced from the physical arguments given by Strominger [1], or from demanding smoothness of the solution on the conformal completion of the spacetime [55].

We assume the following asymptotic behavior of the vector potential and scalar field as $r \rightarrow \infty$ at fixed u , slightly more general than that of [1]:

$$A_A = \mathcal{A}_{+A} + \frac{1}{r} \hat{\mathcal{A}}_{+A} + O\left(\frac{1}{r^2}\right), \quad (2.6a)$$

$$A_u = \mathcal{A}_{+u} + \frac{1}{r} \hat{\mathcal{A}}_{+u} + \frac{1}{r^2} \hat{\mathcal{A}}_{+u} + O\left(\frac{1}{r^3}\right), \quad (2.6b)$$

$$A_r = \frac{1}{r^2} \mathcal{A}_{+r} + \frac{1}{r^3} \hat{\mathcal{A}}_{+r} + O\left(\frac{1}{r^4}\right), \quad (2.6c)$$

$$\Phi = \frac{1}{r} \chi_+ + \frac{1}{r^2} \hat{\chi}_+ + O\left(\frac{1}{r^3}\right). \quad (2.6d)$$

The expansion coefficients in the expansions (2.5) and (2.6) are then related by

$$\mathcal{F}_{+ur} = \partial_u \mathcal{A}_{+r} + \hat{\mathcal{A}}_{+u}, \quad \hat{\mathcal{F}}_{+ur} = \partial_u \hat{\mathcal{A}}_{+r} + 2\hat{\mathcal{A}}_{+u}, \quad (2.7a)$$

$$\mathcal{F}_{+uA} = \partial_u \mathcal{A}_{+A} - D_A \mathcal{A}_{+u}, \quad \hat{\mathcal{F}}_{+uA} = \partial_u \hat{\mathcal{A}}_{+A} - D_A \hat{\mathcal{A}}_{+u}, \quad (2.7b)$$

$$\mathcal{F}_{+rA} = -D_A \mathcal{A}_{+r} - \hat{\mathcal{A}}_{+A}, \quad (2.7c)$$

$$\mathcal{F}_{+AB} = D_A \mathcal{A}_{+B} - D_B \mathcal{A}_{+A}, \quad (2.7d)$$

where D_A is a covariant derivative with respect to the two-sphere metric h_{AB} .

The components of the current (2.2a) are

$$j_u = \mathcal{J}_{+u}/r^2 + O(r^{-3}), \quad j_r = \mathcal{J}_{+r}/r^4 + O(r^{-5}), \quad j_{+A} = \mathcal{J}_A/r^2 + O(r^{-3}), \quad (2.8)$$

where

$$\mathcal{J}_{+u} = -i(\chi_+ \partial_u \chi_+^* - \chi_+^* \partial_u \chi_+) + 2\mathcal{A}_{+u} \chi_+^* \chi_+, \quad (2.9a)$$

$$\mathcal{J}_{+r} = i(\chi_+ \hat{\chi}_+^* - \chi_+^* \hat{\chi}_+) + 2\mathcal{A}_{+r} \chi_+^* \chi_+, \quad (2.9b)$$

$$\mathcal{J}_{+A} = -i(\chi_+ \partial_A \chi_+^* - \chi_+^* \partial_A \chi_+) + 2\mathcal{A}_{+A} \chi_+^* \chi_+. \quad (2.9c)$$

The leading order pieces of Maxwell's equations are [1]

$$\partial_u \mathcal{F}_{+ur} + D^A \mathcal{F}_{+Au} = e^2 \mathcal{J}_{+u}, \quad (2.10a)$$

$$\hat{\mathcal{F}}_{+ur} + D^A \mathcal{F}_{+Ar} = e^2 \mathcal{J}_{+r}, \quad (2.10b)$$

$$-\partial_u \mathcal{F}_{+rA} + \hat{\mathcal{F}}_{+uA} + D^C \mathcal{F}_{+CA} = e^2 \mathcal{J}_{+A}, \quad (2.10c)$$

where $D^A = h^{AB} D_B$.

The expansion coefficients of the various fields transform under gauge transformations as follows. Under the gauge transformation (2.3) with

$$\varepsilon = \varepsilon_+ + \frac{1}{r} \hat{\varepsilon}_+ + \frac{1}{r^2} \hat{\hat{\varepsilon}}_+ + O\left(\frac{1}{r^3}\right), \quad (2.11)$$

we have

$$\mathcal{A}_{+u} \rightarrow \mathcal{A}_{+u} + \partial_u \varepsilon_+, \quad \hat{\mathcal{A}}_{+u} \rightarrow \hat{\mathcal{A}}_{+u} + \partial_u \hat{\varepsilon}_+, \quad \hat{\hat{\mathcal{A}}}_{+u} \rightarrow \hat{\hat{\mathcal{A}}}_{+u} + \partial_u \hat{\hat{\varepsilon}}_+, \quad (2.12a)$$

$$\mathcal{A}_{+A} \rightarrow \mathcal{A}_{+A} + D_A \varepsilon_+, \quad \hat{\mathcal{A}}_{+A} \rightarrow \hat{\mathcal{A}}_{+A} + D_A \hat{\varepsilon}_+, \quad (2.12b)$$

$$\mathcal{A}_{+r} \rightarrow \mathcal{A}_{+r} - \hat{\varepsilon}_+, \quad \hat{\mathcal{A}}_{+r} \rightarrow \hat{\mathcal{A}}_{+r} - 2\hat{\hat{\varepsilon}}_+, \quad (2.12c)$$

$$\chi_+ \rightarrow e^{i\varepsilon_+} \chi_+, \quad \hat{\chi}_+ \rightarrow e^{i\hat{\varepsilon}_+} (\hat{\chi}_+ + i\hat{\hat{\varepsilon}}_+ \chi_+). \quad (2.12d)$$

A similar analysis can be carried out for the limiting behavior of the fields near past null infinity $\mathcal{J}^-$. We use advanced coordinates v, r, θ^A given by $v = u + 2r$. The expansion of the Maxwell tensor at $\mathcal{J}^-$, as $r \rightarrow \infty$ at fixed v , is similar to the expansion (2.5) and is given by

$$F_{vr} = \frac{1}{r^2} \mathcal{F}_{-vr} + \frac{1}{r^3} \hat{\mathcal{F}}_{-vr} + O\left(\frac{1}{r^4}\right), \quad (2.13a)$$

$$F_{vA} = \mathcal{F}_{-vA} + \frac{1}{r} \hat{\mathcal{F}}_{-vA} + O\left(\frac{1}{r^2}\right), \quad (2.13b)$$

$$F_{rA} = \frac{1}{r^2} \mathcal{F}_{-rA} + \frac{1}{r^3} \hat{\mathcal{F}}_{-rA} + O\left(\frac{1}{r^4}\right), \quad (2.13c)$$

$$F_{AB} = \mathcal{F}_{-AB} + \frac{1}{r} \hat{\mathcal{F}}_{-AB} + O\left(\frac{1}{r^2}\right). \quad (2.13d)$$

Here the subscripts $-$ denote expansion coefficients of an expansion near $\mathcal{J}^-$. The corresponding expansions of the vector potential and scalar field are

$$A_A = \mathcal{A}_{-A} + \frac{1}{r} \hat{\mathcal{A}}_{-A} + O\left(\frac{1}{r^2}\right), \quad (2.14a)$$

$$A_v = \mathcal{A}_{-v} + \frac{1}{r} \hat{\mathcal{A}}_{-v} + \frac{1}{r^2} \hat{\hat{\mathcal{A}}}_{-v} + O\left(\frac{1}{r^3}\right), \quad (2.14b)$$

$$A_r = \frac{1}{r^2} \mathcal{A}_{-r} + \frac{1}{r^3} \hat{\mathcal{A}}_{-r} + O\left(\frac{1}{r^4}\right), \quad (2.14c)$$

$$\Phi = \frac{1}{r} \chi_- + \frac{1}{r^2} \hat{\chi}_- + O\left(\frac{1}{r^3}\right), \quad (2.14d)$$

and the gauge transformation parameter can be expanded as

$$\varepsilon = \varepsilon_- + \frac{1}{r} \hat{\varepsilon}_- + \frac{1}{r^2} \hat{\hat{\varepsilon}}_- + O\left(\frac{1}{r^3}\right). \quad (2.15)$$

2.2 Space of solutions of the field equations

We will restrict attention to field configurations on $\mathcal{I}^-$ for which satisfy three conditions:

1. The limits

$$\lim_{v \rightarrow \pm\infty} \mathcal{A}_{-A}, \quad \lim_{v \rightarrow \pm\infty} \mathcal{A}_{-r}, \quad \lim_{v \rightarrow \pm\infty} \mathcal{A}_{-v}, \quad (2.16)$$

exist, as functions on the two-sphere.

2. The field $\mathcal{A}_{-A}$ satisfies the fall off conditions near timelike infinity and spatial infinity

$$\partial_v \mathcal{A}_{-A} \sim \frac{1}{|v|^{1+\epsilon}}, \quad v \rightarrow \pm\infty, \quad (2.17)$$

for some $\epsilon > 0$. This condition is sufficient to ensure the convergence of the symplectic form (2.43) below.

3. The initial data χ_- for the scalar field falls off like⁶

$$\chi_- \sim \frac{1}{|v|^{1+\epsilon}}, \quad v \rightarrow \pm\infty, \quad (2.18)$$

for some $\epsilon > 0$. We will assume that the conditions (1) and (2) are preserved by the scattering process, so that solutions which obey these conditions at $\mathcal{I}^-$ also satisfy analogous conditions at $\mathcal{I}^+$.

We introduce the following notations for the limiting values of fields on $\mathcal{I}^-$ at past timelike infinity i^- and at spatial infinity i^0 . For any function $f_- = f_-(v, \theta)$ defined on $\mathcal{I}^-$, where $\theta = (\theta^1, \theta^2)$, we define

$$f_-(\theta) = \lim_{v \rightarrow -\infty} f_-(v, \theta), \quad f_+(\theta) = \lim_{v \rightarrow \infty} f_-(v, \theta). \quad (2.19)$$

Similarly for functions f_+ defined on $\mathcal{I}^+$ we denote the limiting functions at i^+ and at i^0 by

$$f_+(\theta) = \lim_{u \rightarrow \infty} f_+(u, \theta), \quad f_{\pm}(\theta) = \lim_{u \rightarrow -\infty} f_+(u, \theta). \quad (2.20)$$

We now assume that for solutions to the field equations we have the following behavior near past timelike infinity [1]:

$$\mathcal{F}_{-vr}(\theta) = 0, \quad (2.21a)$$

$$\mathcal{F}_{-rA}(\theta) = 0, \quad (2.21b)$$

$$\mathcal{F}_{-AB}(\theta) = 0, \quad (2.21c)$$

$$\hat{\chi}_-(\theta) = 0. \quad (2.21d)$$

⁶Our assumed fall off for the scalar field is stronger than that for the gauge field. We are excluding for simplicity any nontrivial soft behavior on $\mathcal{I}^-$ in the scalar sector.

These conditions can be derived if the initial data for χ_- is of compact support on $\mathcal{I}^-$. In this case there exists a neighborhood of i^- in which Φ vanishes, and so in that neighborhood A_a satisfies the homogeneous Maxwell equations with no sources. Solutions of these equations satisfy the conditions (2.21a)–(2.21c) (see appendix B). Similar arguments can be given for the condition (2.21d). We will assume that the conditions (2.21) continue to be valid under the weaker assumption (2.18) on the behavior of χ_- ; this should follow from continuity of the dependence of the solutions of the equations of motion on the initial data on $\mathcal{I}^-$. Conditions analogous to (2.21) at future timelike infinity i^+ , namely

$$\mathcal{F}_{+ur}(\theta) = 0, \quad (2.22a)$$

$$\mathcal{F}_{+rA}(\theta) = 0, \quad (2.22b)$$

$$\mathcal{F}_{+AB}(\theta) = 0, \quad (2.22c)$$

$$\hat{\chi}_+(\theta) = 0, \quad (2.22d)$$

should similarly follow from the assumptions on the final data on $\mathcal{I}^+$ that we have discussed.

Another key property of the solutions is the validity of the matching conditions

$$\mathcal{F}_{\pm ur}(\theta) = \mathcal{P}_* \mathcal{F}_{\mp vr}(\theta), \quad (2.23a)$$

$$\mathcal{F}_{\pm AB}(\theta) = -\mathcal{P}_* \mathcal{F}_{\mp AB}(\theta). \quad (2.23b)$$

Here $\mathcal{P} : S^2 \rightarrow S^2$ is the antipodal inversion mapping given by $(\theta, \varphi) \rightarrow (\pi - \theta, \pi + \varphi)$, and $\mathcal{P}_*$ is the pullback $(\mathcal{P}_* f)(\theta, \varphi) = f(\pi - \theta, \pi + \varphi)$ for functions f . These identities are related to the existence of charges related to large gauge transformations and were discovered by Strominger [1, 56]. They were proven rigorously in Minkowski spacetime by Campiglia and Eychelalde [57], and generalized to all asymptotically flat spacetimes by Prabhu [58].

We next claim that solutions are determined up to gauge by specifying on $\mathcal{I}^-$ the initial data

$$\mathcal{A}_{-A}, \mathcal{A}_{-v}, \mathcal{A}_{-r}, \chi_-. \quad (2.24)$$

Subleading fields can be obtained from these fields from the analog on $\mathcal{I}^-$ of the asymptotic expansion (2.10) of Maxwell equations. For example, the subleading field $\hat{\mathcal{A}}_{-v}$ can be obtained from the leading fields (2.24) and from $\mathcal{F}_{-vr}$, from the analog of eq. (2.7a) on $\mathcal{I}^-$. In turn, the field $\mathcal{F}_{-vr}$ can be obtained from its evolution equation, the analog on $\mathcal{I}^-$ of eq. (2.10a), together with the initial condition (2.21a) at $v = -\infty$. Similarly $\hat{\mathcal{A}}_{-A}$ is obtained from $\mathcal{F}_{-rA}$ from the analogs of eqs. (2.7c), the evolution equation (2.10c) and the initial condition (2.21b). Similar arguments apply to the subleading scalar field $\hat{\chi}_-$ using an expansion of the scalar field equation (2.2b).

Therefore we can take the four fields (2.24) on $\mathcal{I}^-$ to parameterize the phase space (up to gauge transformations which we discuss below). Similarly the fields

$$\mathcal{A}_{+A}, \mathcal{A}_{+u}, \mathcal{A}_{+r}, \chi_+ \quad (2.25)$$

on $\mathcal{I}^+$ also uniquely determine the solution, up to gauge transformations.

2.3 Presymplectic form

The presymplectic form obtained from the action (2.1) depends on a pair of linearized perturbations $(\delta_1 A_a, \delta_1 \Phi)$ and $(\delta_2 A_a, \delta_2 \Phi)$ about a background solution A_a, Φ [59]. It is given by the integral over any Cauchy surface Σ ,

$$\Omega_\Sigma(A_a, \Phi; \delta_1 A_a, \delta_1 \Phi, \delta_2 A_a, \delta_2 \Phi) = \int_\Sigma \omega_{abc} \quad (2.26)$$

of the 3-form

$$\omega_{abc} = \frac{1}{e^2} \epsilon_{abcd} \delta_1 F^{df} \delta_2 A_f + \epsilon_{abcd} (D^d \delta_1 \Phi^* \delta_2 \Phi + D^d \delta_1 \Phi \delta_2 \Phi^*) - (1 \leftrightarrow 2). \quad (2.27)$$

Here the orientation of Σ is that determined by $t^a \epsilon_{abcd}$ where t^a is any future-pointing timelike vector field. We define

$$\Omega_{\mathcal{I}^-} = \int_{\mathcal{I}^-} \omega_{abc}, \quad \Omega_{\mathcal{I}^+} = \int_{\mathcal{I}^+} \omega_{abc}, \quad (2.28)$$

the limiting integrals over past and future null infinity. The limits to $\mathcal{I}^+$ and $\mathcal{I}^-$ of ω_{abc} exist by virtue of our assumed expansions (2.5), (2.6), (2.13) and (2.14). We will discuss in section 2.5 below the conditions necessary for the integrals (2.28) to converge and be finite.

Before considering the presymplectic forms (2.28) for general perturbations, it will be useful to first specialize to the case where the second perturbation is a pure gauge transformation,

$$\delta_1 A_a = \delta A_a, \quad \delta_1 \Phi = \delta \Phi, \quad \delta_2 A_a = \delta_\epsilon A_a = \nabla_a \epsilon, \quad \delta_2 \Phi = \delta_\epsilon \Phi = i\epsilon \Phi. \quad (2.29)$$

In this case ω_{abc} is always exact [59], and here we have $\omega_{abc} = 3\nabla_{[a} Q_{bc]}$, where

$$Q_{ab} = \frac{1}{2e^2} \epsilon \epsilon_{abcd} \delta F^{cd}. \quad (2.30)$$

We assume expansions of the form (2.11) for the gauge transformation function ϵ near $\mathcal{I}^+$ and $\mathcal{I}^-$, and we further assume that at each angle θ the leading order coefficients ϵ_+, ϵ_- asymptote to constants as $|u|$ or $|v|$ go to infinity, to be consistent the assumed asymptotic behavior of the fields at $i^\pm$ and i^0 discussed above eq. (2.20). Following the notational conventions (2.19) and (2.20) these limiting values will be denoted $\epsilon_+^\pm, \epsilon_+, \epsilon_-^\pm$ and ϵ_- . Converting the integrals over $\mathcal{I}^+$ and $\mathcal{I}^-$ to integrals over their boundaries using eq. (2.30) we obtain

$$\Omega_{\mathcal{I}^+} = \frac{1}{e^2} \int d^2\Omega \left[-\delta \mathcal{F}_{+ru} \epsilon_+^\pm + \delta \mathcal{F}_{\pm ru} \epsilon_+^\pm \right], \quad (2.31a)$$

$$\Omega_{\mathcal{I}^-} = \frac{1}{e^2} \int d^2\Omega \left[-\delta \mathcal{F}_{-rv} \epsilon_-^\pm + \delta \mathcal{F}_{\mp rv} \epsilon_-^\pm \right]. \quad (2.31b)$$

We now simplify using the asymptotic conditions (2.21a) and (2.22a), and the matching condition (2.23a). This gives

$$\Omega_{\mathcal{I}^+} - \Omega_{\mathcal{I}^-} = \frac{1}{e^2} \int d^2\Omega \delta \mathcal{F}_{\pm ru} \left[\epsilon_\pm - \mathcal{P}_* \epsilon_\pm^\pm \right]. \quad (2.32)$$

Now we would like to specialize our definition of the field configuration space to make

$$\Omega_{\mathcal{J}^+} = \Omega_{\mathcal{J}^-} \quad (2.33)$$

for general on-shell perturbations, that is, to make the presymplectic forms on past null infinity and future null infinity coincide. In other words, the scattering map from data at past null infinity to data at future null infinity should be a symplectomorphism (the classical version of unitarity in the quantum theory). From eq. (2.32) we see that the condition (2.33) cannot be preserved under general transformations of the form $A_a \rightarrow A_a + \nabla_a \varepsilon$. In the next section we will discuss a specialization of the definition of the field configuration space suggested by Strominger [47] which makes the quantity (2.32) vanish, and thus removes the obstruction to achieving (2.33) for general perturbations.

2.4 Gauge specialization and scattering map

We now fix some of the gauge degrees of freedom that correspond to degeneracy directions of the presymplectic form (2.26), and so correspond to true gauge degrees of freedom. First, we use the free function ε_+ to set $\mathcal{A}_{+u}$ to zero, using eq. (2.12a). In doing so we specialize to $\varepsilon_{\pm} = \varepsilon_{\pm}(u = -\infty) = 0$, in order to correspond to a degeneracy direction of (2.31a), from eq. (2.22a). Next, we use the free function $\hat{\varepsilon}_+$ to set $\mathcal{A}_{+r}$ to zero, from eq. (2.12c). We perform similar specializations at $\mathcal{J}^-$. Summarizing, we have fixed the gauge so that

$$\mathcal{A}_{+u} = \mathcal{A}_{+r} = \mathcal{A}_{-v} = \mathcal{A}_{-r} = 0. \quad (2.34)$$

The remaining gauge freedom that acts on the data on $\mathcal{J}^+$ and $\mathcal{J}^-$ consists of functions $\varepsilon_+ = \varepsilon_+(\theta)$ and $\varepsilon_- = \varepsilon_-(\theta)$ that are functions of angle only. In particular we have $\varepsilon_{\pm} = \varepsilon_{\pm} = \varepsilon_{\pm}$ and $\varepsilon_{\pm} = \varepsilon_{\pm} = \varepsilon_{\pm}$. We define the even and odd linear combinations of these gauge transformations via

$$\varepsilon_e = \frac{1}{2}(\varepsilon_+ + \mathcal{P}_* \varepsilon_-), \quad (2.35a)$$

$$\varepsilon_o = \frac{1}{2}(\varepsilon_+ - \mathcal{P}_* \varepsilon_-). \quad (2.35b)$$

We next decompose the fields $\mathcal{A}_{+A}$ and $\mathcal{A}_{-A}$ into electric and magnetic parity pieces on the two sphere, as

$$\mathcal{A}_{+A} = D_A \Psi_+^e + \varepsilon_{AB} h^{BC} D_C \Psi_+^m, \quad (2.36a)$$

$$\mathcal{A}_{-A} = D_A \Psi_-^e + \varepsilon_{AB} h^{BC} D_C \Psi_-^m, \quad (2.36b)$$

which determines the potentials $\Psi_{\pm}^e$ etc. up to their $l = 0$ parts which we take to vanish. Under the gauge transformation given by eqs. (2.11) and (2.15) we have

$$\Psi_-^e \rightarrow \Psi_-^e + \varepsilon_-, \quad \Psi_+^e \rightarrow \Psi_+^e + \varepsilon_+, \quad (2.37a)$$

$$\Psi_-^m \rightarrow \Psi_-^m, \quad \Psi_+^m \rightarrow \Psi_+^m, \quad (2.37b)$$

assuming that ε_- and ε_+ have no $l = 0$ pieces.

Finally, following Strominger [47], we specialize our definition of the field configuration space by imposing the condition⁷

$$\Psi_{\pm}^e = \mathcal{P}_* \Psi_{\pm}^e, \quad (2.39)$$

which can be achieved by using the odd gauge transformation (2.35b), from eqs. (2.37a) and (2.36). The condition (2.39) then eliminates the odd gauge transformation freedom, and consequently the quantity (2.32) vanishes as desired, removing the obstruction to the scattering symplectomorphism property (2.33) discussed in the last section. In appendix D we generalize this result and show that the symplectic forms on $\mathcal{S}^-$ and $\mathcal{S}^+$ coincide in general, and not just for field configurations of the form (2.29), using results of Campiglia and Eychalder [57]. We also show there that the condition (2.39) follows from imposing a Lorenz-like gauge condition on the fields to leading order in an expansion around spatial infinity. We note that the condition (2.39) is generally incompatible with Lorenz gauge (see appendix B); this point is discussed further after eq. (2.42) below.

The condition (2.39) is called a “matching condition” in refs. [33, 47]. However its status is very different from that of the matching conditions (2.23), being a specialization of the definition of the configuration space rather than a property of generic physical solutions. Note also that it is not accurate to call it a gauge specialization, even though it is enforced by making use of the transformations (2.35b), since true gauge transformations are defined in terms of degeneracy directions of the presymplectic form, and the specialization (2.39) is needed before a unique presymplectic form can be defined.

The condition (2.39) together with the gauge fixing (2.34) determines a unique gauge for initial data on $\mathcal{S}^-$ and final data on $\mathcal{S}^+$ which we will call the *preferred asymptotic gauge*. From eq. (2.24), the space Γ_- of initial data in this gauge consists of the pairs $(\mathcal{A}_A, \chi_-)$ on $\mathcal{S}^-$. Similarly the space Γ_+ of final data consists of the pairs $(\mathcal{A}_A, \chi_+)$ on $\mathcal{S}^+$.

We do not fix the even transformation freedom (2.35a), as it corresponds to a non-degenerate direction of the presymplectic form, so it is a physical symmetry transformation rather than a gauge freedom. The corresponding charges are the new conserved charges of [1]. Specifically, the variation in the charge is given by [cf. eq. (8) of ref. [59]] $\delta Q_{\varepsilon} = \Omega(A_a, \Phi; \delta A_a, \delta \Phi, \delta_{\varepsilon} A_a, \delta_{\varepsilon} \Phi)$, and using eqs. (2.31) and (2.35a) and integrating in phase space gives [1]

$$Q_{\varepsilon} = \frac{1}{e^2} \int d^2\Omega \mathcal{F}_{\pm ru} \varepsilon_{\pm} = \frac{1}{e^2} \int d^2\Omega \mathcal{F}_{\pm rv} \varepsilon_{\pm}. \quad (2.40)$$

There are also magnetic conserved charges analogous to the electric charges (2.40) [56], which we review in appendix E. For simplicity will restrict attention to the sector of the theory where all the magnetic charges (E.4) vanish, which by eqs. (2.7d), (2.36), (E.5) and (E.6) is equivalent to the requirement that

$$\Psi_{\pm}^m = \Psi_{\pm}^m = 0. \quad (2.41)$$

⁷Note that the corresponding relation for the magnetic potentials Ψ^m ,

$$\Psi_{\pm}^m = \mathcal{P}_* \Psi_{\pm}^m, \quad (2.38)$$

is a consequence of the matching condition (2.23b) together with eqs. (2.7d) and (2.36).

To summarize, the field configuration space $\mathcal{F}$ of the theory is given by the set of fields that obey the asymptotic conditions (2.6) and (2.14), the matching condition (2.39), the vanishing magnetic charges condition (2.41), and for which the initial data on $\mathcal{I}^-$ obeys the conditions (2.16)–(2.18) and the final data satisfies analogous conditions on $\mathcal{I}^+$. The phase space Γ of the theory is the on-shell subspace $\overline{\mathcal{F}}$ of the field configuration space, modded out by degeneracy directions of the presymplectic form [46], which correspond to gauge transformations that act trivially on the boundaries $\mathcal{I}^-$ and $\mathcal{I}^+$. This phase space is in one-to-one correspondence with the space Γ_- of initial data $(\mathcal{A}_-, \chi_-)$ on $\mathcal{I}^-$ in the preferred asymptotic gauge. It is similarly in one-to-one correspondence with the space Γ_+ of final data $(\mathcal{A}_+, \chi_+)$ on $\mathcal{I}^+$ in the preferred asymptotic gauge. In the remainder of the paper we will be concerned with properties of the scattering map

$$\mathcal{S} : \Gamma_- \rightarrow \Gamma_+ \quad (2.42)$$

which is a symplectomorphism.

While our on-shell field configuration space $\overline{\mathcal{F}}$ is well defined, it is also useful to consider the larger space $\overline{\mathcal{F}}_{\text{ext}}$ of solutions that is obtained by relaxing the matching condition (2.39). This larger space is necessary, for example, to accommodate Lorenz gauge solutions, as we show in appendix B. We extend the definition of the presymplectic form Ω from $\overline{\mathcal{F}}$ to $\overline{\mathcal{F}}_{\text{ext}}$ as follows. Given a pair of solutions in $\overline{\mathcal{F}}_{\text{ext}}$, we transform that pair to preferred asymptotic gauge using the transformations (2.35b), and then use the prescription for Ω discussed above. Modding out by degeneracy directions of Ω then yields the same phase space Γ as before. From the point of view of the extended space $\overline{\mathcal{F}}_{\text{ext}}$, the role of the matching condition (2.39) is to modify the definition of the presymplectic form rather than to restrict the field configuration space. In the remainder of the paper we will work mostly within the extended space $\overline{\mathcal{F}}_{\text{ext}}$, transforming back and forth between Lorenz gauge and preferred asymptotic gauge as needed.

2.5 Soft and hard variables and Poisson brackets

The presymplectic form (2.26) when evaluated on the space Γ_- of initial data $(\mathcal{A}_-, \chi_-)$ is now nondegenerate, so from now on we will call it the symplectic form. It can be written as

$$\begin{aligned} \Omega_{\mathcal{I}^-}(A_a, \Phi; \delta_1 A_a, \delta_1 \Phi, \delta_2 A_a, \delta_2 \Phi) &= \int dv \int d^2\Omega \left[\frac{1}{e^2} h^{AB} \partial_v \delta_1 \mathcal{A}_- \delta_2 \mathcal{A}_- - (1 \leftrightarrow 2) \right] \\ &+ \int dv \int d^2\Omega [\partial_v \delta_1 \chi_-^* \delta_2 \chi_- + \partial_v \delta_1 \chi_- \delta_2 \chi_-^* - (1 \leftrightarrow 2)]. \end{aligned} \quad (2.43)$$

In terms of the potentials Ψ^e and Ψ^m defined in eq. (2.36) the symplectic form is

$$\begin{aligned} \Omega_{\mathcal{I}^-} &= -\frac{1}{e^2} \int dv \int d^2\Omega \left[\partial_v \Psi_{1-}^e D^2 \Psi_{2-}^e + \partial_v \Psi_{1-}^m D^2 \Psi_{2-}^m - (1 \leftrightarrow 2) \right] \\ &+ \int dv \int d^2\Omega [\partial_v \delta_1 \chi_-^* \delta_2 \chi_- + \partial_v \delta_1 \chi_- \delta_2 \chi_-^* - (1 \leftrightarrow 2)]. \end{aligned} \quad (2.44)$$

Here for simplicity we have written Ψ_1^e instead of $\delta_1 \Psi^e$ etc.

We next make a change of coordinates on Γ_- , from the set $[\Psi_-^e(v, \theta), \Psi_-^m(v, \theta), \chi_-(v, \theta)]$ to a new set

$$[\tilde{\Psi}_-(v, \theta), \Psi_-^m(v, \theta), \chi_-(v, \theta), \Delta\Psi_-^e(\theta), \bar{\Psi}_-(\theta)] \quad (2.45)$$

defined as follows. We pick a smooth function $g(v)$ which increases monotonically with boundary values

$$g(-\infty) = -1/2, \quad g(\infty) = 1/2. \quad (2.46)$$

We define

$$\Delta\Psi_-^e = \Psi_+^e - \Psi_-^e, \quad (2.47a)$$

$$\bar{\Psi}_-^e = \frac{1}{2} (\Psi_+^e + \Psi_-^e), \quad (2.47b)$$

$$\tilde{\Psi}_-(v, \theta) = \Psi_-^e(v, \theta) - \bar{\Psi}_-^e(\theta) - g(v)\Delta\Psi_-^e(\theta). \quad (2.47c)$$

Note that it follows from these definitions that

$$\tilde{\Psi}_+^e = \tilde{\Psi}_-^e = 0. \quad (2.48)$$

We will use the terminology “hard variables” for the quantities $\tilde{\Psi}_-^e$, Ψ_-^m , and χ_- which depend on u , and “soft variables” for the quantities $\Delta\Psi_-^e$, $\bar{\Psi}_-^e$ which do not and which correspond to zero energy degrees of freedom, following ref. [1]. We similarly define the phase space variables

$$[\tilde{\Psi}_+(u, \theta), \Psi_+^m(u, \theta), \chi_+(u, \theta), \Delta\Psi_+^e(\theta), \bar{\Psi}_+^e(\theta)] \quad (2.49)$$

at future null infinity $\mathcal{I}^+$, using the same function g .

It may seem strange that we have to introduce an arbitrary function $g(v)$ in order to separate out the soft variables from the hard variables. However, it is necessary to make such a choice in order to get a complete separation. Of course, the choice of $g(v)$ is arbitrary and no observable quantities will depend on this choice. Any two phase space coordinate systems corresponding to two different choices of $g(v)$ are related by a symplectomorphism [see eqs. (2.52) below, which are independent of g].

We note that a transformation of phase space variables similar to our transformation (2.47) was used in refs. [1, 33], except that those authors did not include the third term on the right hand side of eq. (2.47c). As a consequence, their variables are not independent, obeying the constraint (in our notation) of $\Delta\Psi_-^e = \tilde{\Psi}_-^e(u = \infty) - \tilde{\Psi}_-^e(u = -\infty)$. This lack of independence of phase space coordinates is the key reason why our results on the coupling of hard and soft degrees of freedom differ from those of ref. [33], as discussed in more detail in section 4.2 below.

Rewriting the symplectic form (2.44) in terms of the new variables (2.45) gives

$$\begin{aligned} \Omega_{\mathcal{J}^-} = & -\frac{1}{e^2} \int dv \int d^2\Omega \left[\partial_v \tilde{\Psi}_{1-}^e D^2 \tilde{\Psi}_{2-}^e + \partial_v \Psi_{1-}^m D^2 \Psi_{2-}^m - (1 \leftrightarrow 2) \right] \\ & - \frac{1}{e^2} \int d^2\Omega \left\{ D^2 \Delta\Psi_{1-}^e \left[\bar{\Psi}_{2-}^e + 2 \int dv g' \tilde{\Psi}_{2-}^e \right] - (1 \leftrightarrow 2) \right\} \\ & + \int dv \int d^2\Omega \left[\partial_v \delta_1 \chi_-^* \delta_2 \chi_- + \partial_v \delta_1 \chi_- \delta_2 \chi_-^* - (1 \leftrightarrow 2) \right]. \end{aligned} \quad (2.50)$$

We see that the soft and hard phase space variables (2.45) are not symplectically orthogonal, that is, there are nonvanishing Poisson brackets between the hard and soft variables. This can be remedied by defining the new soft variable

$$\hat{\Psi}_-(\boldsymbol{\theta}) = \bar{\Psi}_-(\boldsymbol{\theta}) + 2 \int dv g'(v) \tilde{\Psi}_-(v, \boldsymbol{\theta}). \quad (2.51)$$

Then the soft variables $\Delta\Psi_-^e(\boldsymbol{\theta})$, $\hat{\Psi}_-^e(\boldsymbol{\theta})$ and the hard variables $\tilde{\Psi}_-^e(v, \boldsymbol{\theta})$, $\Psi_-^m(v, \boldsymbol{\theta})$, $\chi_-(v, \boldsymbol{\theta})$ are symplectically orthogonal, from eq. (2.50). The corresponding nonzero Poisson brackets can be obtained from eqs. (2.50) and (2.51) by expanding in spherical harmonics and using the boundary conditions (2.18), (2.48), (E.5) and (2.41), and are^{8,9}

$$\left\{ \tilde{\Psi}_-(v, \boldsymbol{\theta}), D^2 \tilde{\Psi}_-(v', \boldsymbol{\theta}') \right\} = \frac{e^2}{2} \left[\Theta(v - v') - \frac{1}{2} \right] \left[\delta^{(2)}(\boldsymbol{\theta}, \boldsymbol{\theta}') - \frac{1}{4\pi} \right], \quad (2.52a)$$

$$\left\{ \Psi_-^m(v, \boldsymbol{\theta}), D^2 \Psi_-^m(v', \boldsymbol{\theta}') \right\} = \frac{e^2}{2} \left[\Theta(v - v') - \frac{1}{2} \right] \left[\delta^{(2)}(\boldsymbol{\theta}, \boldsymbol{\theta}') - \frac{1}{4\pi} \right], \quad (2.52b)$$

$$\left\{ \Delta\Psi_-^e(\boldsymbol{\theta}), D^2 \hat{\Psi}_-^e(\boldsymbol{\theta}') \right\} = e^2 \left[\delta^{(2)}(\boldsymbol{\theta}, \boldsymbol{\theta}') - \frac{1}{4\pi} \right], \quad (2.52c)$$

$$\left\{ \chi_-(v, \boldsymbol{\theta}), \chi_-^*(v', \boldsymbol{\theta}') \right\} = -\frac{1}{2} \left[\Theta(v - v') - \frac{1}{2} \right] \delta^{(2)}(\boldsymbol{\theta}, \boldsymbol{\theta}'). \quad (2.52d)$$

Here $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x < 0$, and $\delta^{(2)}(\boldsymbol{\theta}, \boldsymbol{\theta}') = \delta(\boldsymbol{\theta} - \boldsymbol{\theta}')\delta(\varphi - \varphi')/\sqrt{h}$ is the covariant delta function on the unit sphere. The formulae (2.52a)–(2.52c) contain factors of D^2 inside the Poisson brackets, but they determine the corresponding formulae without factors of D^2 since all of the functions of $\boldsymbol{\theta}$ (except χ_-) have no $l = 0$ components.

Note that it follows from eqs. (2.47) and (2.52) that the field $\Psi_-^e(v, \boldsymbol{\theta})$ satisfies the same Poisson bracket relation (2.52a) as the field $\tilde{\Psi}_-^e(v, \boldsymbol{\theta})$. However, one cannot replace eqs. (2.52a) and (2.52c) with the single equation (2.52a) with $\tilde{\Psi}_-^e(v, \boldsymbol{\theta})$ replaced by $\Psi_-^e(v, \boldsymbol{\theta})$, because limits $u \rightarrow \pm\infty$ of the Poisson brackets do not coincide with Poisson brackets of limits [1].

We remark that the soft degrees of freedom ($\Delta\Psi_-^e, \hat{\Psi}_-^e$) can alternatively be described in the language of edge modes [48–51], as was conjectured in ref. [1]. This equivalence is outlined in appendix F.

Turn now to the situation at future null infinity $\mathcal{I}^+$. An analogous analysis yields versions of the formulae (2.50)–(2.52) with v replaced by u everywhere, and with the subscripts $+$ replaced by subscripts $-$.

Finally, we note that the main differences between the construction of the phase space given here and previous treatments [1, 39, 40] are as follows:

- Ashtekar [39, 40] constructs the phase space by imposing $\Psi_\pm^e = \Psi_\pm^e = 0$ instead of the matching condition (2.39) suggested by Strominger. This eliminates one of the

⁸Note that many of these Poisson bracket relations are not continuous in the limits $v, v' \rightarrow \pm\infty$. That is, for example, the limit $v \rightarrow \pm\infty$ of a Poisson bracket of a field will not coincide with the Poisson bracket of the limit of the field. However, there is no inconsistency.

⁹These Poisson brackets agree with those of ref. [1] but not those of ref. [39]. We believe that the right hand side of eq. (C.5) of ref. [39] should be proportional to the derivative of a delta function instead of a step function.

physical degrees of freedom (the edge mode) and also the symmetry that underlies the conservation laws (2.40). Including the extra degree of freedom will modify the character of the quantum theory constructed in [39]. Similarly, the analysis of infrared scattering of the recent paper [42] is based on an algebra obtained by excluding this degree of freedom. See appendix F for further discussion.

- In fixing the gauge we restrict to degeneracy directions of the presymplectic form, and demonstrate that the gauge conditions (2.34) used are degeneracy directions.¹⁰
- In separating out hard and soft variables, we use a coordinate system on phase space in which all the variables are independent, unlike the set of variables in refs. [1, 33]. This change does not affect the computation of Poisson brackets, but will be important in the decoupling discussions in sections 4 and 6 below.

3 Classical scattering map: foundations, definition and parameterization

Having completed our analysis of the symmetries, charges and asymptotics of the theory, and the definition of the phase space, we now turn to an exploration of the dynamics of the theory in the deep infrared. Our goal is to determine the extent to which the conserved charges (2.40) constrain in a nontrivial way the dynamics of the theory. In this section we will define the scattering map and derive some of its properties as well as some useful parameterizations. We also set up the framework for computing the scattering map in perturbation theory, and compute the free field scattering. In later sections we will explicitly compute higher order contributions, and we will show that the scattering map cannot be factored into maps that act individually on hard and soft sectors.

3.1 Asymptotic field expansions in a more general class of gauges

It will be convenient to use Lorenz gauge for our explicit computations, since the equations of motion reduce to simple wave equations in this gauge. However, the form (2.6) of the asymptotic expansions that we have assumed are insufficiently general for this purpose, as Lorenz gauges generically requires logarithmic terms when sources are present [41, 60]. Indeed, starting from the expansions (2.6) we obtain

$$\begin{aligned}\nabla_a A^a &= -\frac{2}{r} \mathcal{A}_{+u} + \frac{1}{r^2} \left[-\partial_u \mathcal{A}_{+r} - \hat{\mathcal{A}}_{+u} + D^A \mathcal{A}_{+A} \right] + O\left(\frac{1}{r^3}\right) \\ &= -\frac{2}{r} \mathcal{A}_{+u} + \frac{1}{r^2} \left[e^2 \int_u^\infty du' \mathcal{J}_{+u}(u') + D^A \mathcal{A}_{+A} - \int_u^\infty du' D^2 \mathcal{A}_{+u} \right] + O\left(\frac{1}{r^3}\right),\end{aligned}\quad (3.1)$$

where we have used the asymptotic Maxwell's equations (2.10) and the boundary conditions (2.22) to rewrite the coefficient of the $1/r^2$ term. Thus our assumed expansions are incompatible with Lorenz gauges, since when the first term in eq. (3.1) vanishes

¹⁰An exception is the gauge condition (2.39) that we have adopted which is not a degeneracy direction of the presymplectic forms (2.28). However $\Omega_{\mathcal{J}+}$ and $\Omega_{\mathcal{J}-}$ do not coincide until after this condition is imposed, so one can argue that (2.28) is not the correct presymplectic form until after the condition is imposed.

the second term is generically nonzero and cannot be made to vanish using the gauge transformations (2.12).

We therefore generalize the form of the expansion (2.6) to encompass Lorenz gauges, following refs. [41, 60]. In the limit to $\mathcal{J}^+$, we assume

$$A_A = \mathcal{A}_{+A} - \frac{\ln r}{r} D_A \tilde{\mathcal{A}}_{+r} + \frac{1}{r} \hat{\mathcal{A}}_{+A} - \frac{\ln r}{2r^2} D_A \tilde{\tilde{\mathcal{A}}}_{+r} + \frac{1}{r^2} \hat{\tilde{\mathcal{A}}}_{+A} + O\left(\frac{\ln r}{r^3}\right), \quad (3.2a)$$

$$A_u = \mathcal{A}_{+u} - \frac{\ln r}{r} \partial_u \tilde{\mathcal{A}}_{+r} + \frac{1}{r} \hat{\mathcal{A}}_{+u} - \frac{\ln r}{2r^2} \partial_u \tilde{\tilde{\mathcal{A}}}_{+r} + \frac{1}{r^2} \hat{\tilde{\mathcal{A}}}_{+u} + O\left(\frac{\ln r}{r^3}\right), \quad (3.2b)$$

$$A_r = \frac{Q}{r} + \frac{\ln r}{r^2} \tilde{\mathcal{A}}_{+r} + \frac{1}{r^2} \mathcal{A}_{+r} + \frac{\ln r}{r^3} \tilde{\tilde{\mathcal{A}}}_{+r} + \frac{1}{r^3} \hat{\mathcal{A}}_{+r} + O\left(\frac{\ln r}{r^4}\right), \quad (3.2c)$$

$$\Phi = e^{iQ \ln r} \left[\frac{1}{r} \chi_+ - i \frac{\ln r}{r^2} \tilde{\mathcal{A}}_{+r} \chi_+ + \frac{1}{r^2} \hat{\chi}_+ + O\left(\frac{1}{r^3}\right) \right]. \quad (3.2d)$$

Here the notational conventions are as follows. Caligraphic font quantities are coefficients in the double expansion in $1/r$ and $\ln r/r$, functions of u and θ^A . The quantities $\mathcal{A}_{+A}$, $\mathcal{A}_{+u}$ and $\mathcal{A}_{+r}$ without any tildes or carets are the leading order fields discussed in section 2.1 above. Quantities with one or more carets like $\hat{\mathcal{A}}_{+r}$ are coefficients of subleading terms in the $1/r$ expansion, while quantities with one or more tildes like $\tilde{\mathcal{A}}_{+r}$ are coefficients of log terms, new in this section. Finally eqs. (3.2c) and (3.2d) depend on a quantity Q which is a constant (the total ingoing or outgoing charge multiplied by 4π).

The specific relations between the coefficients of the log terms in eqs. (3.2) are chosen to ensure that the expansions (2.5) of the Maxwell tensor and (2.8) of the current are still valid in this context. The formulae (2.7) for the expansion coefficients are replaced by

$$\mathcal{F}_{+ur} = \partial_u \mathcal{A}_{+r} + \partial_u \tilde{\mathcal{A}}_{+r} + \hat{\mathcal{A}}_{+u}, \quad \hat{\mathcal{F}}_{+ur} = \partial_u \hat{\mathcal{A}}_{+r} + 2\hat{\tilde{\mathcal{A}}}_{+u} + \frac{1}{2} \partial_u \tilde{\tilde{\mathcal{A}}}_{+r}, \quad (3.3a)$$

$$\mathcal{F}_{+uA} = \partial_u \mathcal{A}_{+A} - D_A \mathcal{A}_{+u}, \quad \hat{\mathcal{F}}_{+uA} = \partial_u \hat{\mathcal{A}}_{+A} - D_A \hat{\mathcal{A}}_{+u}, \quad \hat{\tilde{\mathcal{F}}}_{+uA} = \partial_u \hat{\tilde{\mathcal{A}}}_{+A} - D_A \hat{\tilde{\mathcal{A}}}_{+u}, \quad (3.3b)$$

$$\mathcal{F}_{+rA} = -D_A \mathcal{A}_{+r} - \hat{\mathcal{A}}_{+A} - D_A \tilde{\mathcal{A}}_{+r}, \quad \hat{\mathcal{F}}_{+rA} = -D_A \hat{\mathcal{A}}_{+r} - 2\hat{\tilde{\mathcal{A}}}_{+A} - \frac{1}{2} D_A \tilde{\tilde{\mathcal{A}}}_{+r}, \quad (3.3c)$$

$$\mathcal{F}_{+AB} = D_A \mathcal{A}_{+B} - D_B \mathcal{A}_{+A}, \quad \hat{\mathcal{F}}_{+AB} = D_A \hat{\mathcal{A}}_{+B} - D_B \hat{\mathcal{A}}_{+A}. \quad (3.3d)$$

The gauge transformation expansion (2.11) is replaced by the more general version

$$\varepsilon = \delta Q \ln r + \varepsilon_+ + \frac{\ln r}{r} \tilde{\varepsilon}_+ + \frac{1}{r} \hat{\varepsilon}_+ + \frac{\ln r}{r^2} \tilde{\tilde{\varepsilon}}_+ + \frac{1}{r^2} \hat{\tilde{\varepsilon}}_+ + O\left(\frac{1}{r^3}\right), \quad (3.4)$$

under which we have

$$Q \rightarrow Q + \delta Q, \quad (3.5a)$$

$$\mathcal{A}_{+u} \rightarrow \mathcal{A}_{+u} + \partial_u \varepsilon_+, \quad \hat{\mathcal{A}}_{+u} \rightarrow \hat{\mathcal{A}}_{+u} + \partial_u \hat{\varepsilon}_+, \quad \hat{\tilde{\mathcal{A}}}_{+u} \rightarrow \hat{\tilde{\mathcal{A}}}_{+u} + \partial_u \hat{\tilde{\varepsilon}}_+, \quad (3.5b)$$

$$\mathcal{A}_{+A} \rightarrow \mathcal{A}_{+A} + D_A \varepsilon_+, \quad \hat{\mathcal{A}}_{+A} \rightarrow \hat{\mathcal{A}}_{+A} + D_A \hat{\varepsilon}_+, \quad \hat{\tilde{\mathcal{A}}}_{+A} \rightarrow \hat{\tilde{\mathcal{A}}}_{+A} + D_A \hat{\tilde{\varepsilon}}_+, \quad (3.5c)$$

$$\mathcal{A}_{+r} \rightarrow \mathcal{A}_{+r} + \tilde{\varepsilon}_+ - \hat{\varepsilon}_+, \quad \hat{\mathcal{A}}_{+r} \rightarrow \hat{\mathcal{A}}_{+r} + \tilde{\tilde{\varepsilon}}_+ - 2\hat{\tilde{\varepsilon}}_+, \quad (3.5d)$$

$$\tilde{\mathcal{A}}_{+r} \rightarrow \tilde{\mathcal{A}}_{+r} - \tilde{\varepsilon}_+, \quad \tilde{\tilde{\mathcal{A}}}_{+r} \rightarrow \tilde{\tilde{\mathcal{A}}}_{+r} - 2\tilde{\tilde{\varepsilon}}_+, \quad (3.5e)$$

$$\chi_+ \rightarrow e^{i\varepsilon_+} \chi_+, \quad \hat{\chi}_+ \rightarrow e^{i\hat{\varepsilon}_+} (\hat{\chi}_+ + i\hat{\varepsilon}_+ \chi_+). \quad (3.5f)$$

The Lorenz gauge condition can be written using the expansion (3.2) as

$$\begin{aligned}
 0 = \nabla_a A^a &= \frac{1}{r} [-2\mathcal{A}_{+u}] + \frac{1}{r^2} [Q - \mathcal{A}_{+r,u} - \hat{\mathcal{A}}_{+u} + D^A \mathcal{A}_{+A} + \tilde{\mathcal{A}}_{+r,u}] \\
 &+ \frac{\ln r}{r^3} [-\tilde{\mathcal{A}}_{+r,u} - D^2 \tilde{\mathcal{A}}_{+r}] + \frac{1}{r^3} [\tilde{\mathcal{A}}_{+r} - \hat{\mathcal{A}}_{+r,u} - \hat{\mathcal{A}}_{+u} + D^A \hat{\mathcal{A}}_{+A} + \frac{1}{2} \tilde{\mathcal{A}}_{+r,u}] \\
 &+ O\left(\frac{\ln r}{r^4}\right),
 \end{aligned} \tag{3.6}$$

where $D^2 = h^{AB} D_A D_B$. Setting the coefficients of this expansion to zero gives four conditions for Lorenz gauge, and starting from a general gauge of the form (3.2) one can check that it is possible to use the transformation freedom (3.5) to satisfy these conditions.

3.2 Transformation to preferred asymptotic gauge

We now discuss the transformation from Lorenz gauge to the preferred asymptotic gauge discussed in section 2 above, which is defined by the expansion (2.6) and the conditions (2.34) and (2.39). We start from an expansion of the form (3.2), specialized to Lorenz gauge (which implies $\mathcal{A}_{+u} = 0$):

$$\underline{A}_A = \underline{\mathcal{A}}_{+A} - \frac{\ln r}{r} D_A \tilde{\mathcal{A}}_{+r} + \frac{1}{r} \hat{\mathcal{A}}_{+A} - \frac{\ln r}{2r^2} D_A \tilde{\mathcal{A}}_{+r} + \frac{1}{r^2} \hat{\mathcal{A}}_{+A} + O\left(\frac{\ln r}{r^3}\right), \tag{3.7a}$$

$$\underline{A}_u = \frac{\ln r}{r} \partial_u \tilde{\mathcal{A}}_{+r} + \frac{1}{r} \hat{\mathcal{A}}_{+u} - \frac{\ln r}{2r^2} \partial_u \tilde{\mathcal{A}}_{+r} + \frac{1}{r^2} \hat{\mathcal{A}}_{+u} + O\left(\frac{\ln r}{r^3}\right), \tag{3.7b}$$

$$\underline{A}_r = \frac{Q}{r} + \frac{\ln r}{r^2} \tilde{\mathcal{A}}_{+r} + \frac{1}{r^2} \mathcal{A}_{+r} + \frac{\ln r}{r^3} \tilde{\mathcal{A}}_{+r} + \frac{1}{r^3} \hat{\mathcal{A}}_{+r} + O\left(\frac{\ln r}{r^4}\right), \tag{3.7c}$$

$$\underline{\Phi} = e^{iQ \ln r} \left[\frac{1}{r} \underline{\chi}_+ - i \frac{\ln r}{r^2} \tilde{\mathcal{A}}_{+r} \underline{\chi}_+ + \frac{1}{r^2} \hat{\chi}_+ + O\left(\frac{\ln r}{r^3}\right) \right]. \tag{3.7d}$$

Here and throughout underlined quantities refer to quantities in Lorenz gauge. We now make a gauge transformation of the form (3.4) with the expansion coefficients chosen to be

$$\delta Q = -Q, \quad \varepsilon_+ = 0, \quad \tilde{\varepsilon}_+ = \tilde{\mathcal{A}}_{+r}, \quad \hat{\varepsilon}_+ = \underline{\mathcal{A}}_{+r} + \tilde{\mathcal{A}}_{+r}, \quad \tilde{\tilde{\varepsilon}}_+ = \frac{1}{2} \tilde{\mathcal{A}}_{+r}, \tag{3.8}$$

which enforces the required conditions (2.6) and the first two equations of (2.34) by eqs. (3.5). We make a similar gauge transformation near $\mathcal{I}^-$ to enforce (2.14) and the last two equations of (2.34).

We have not yet enforced the condition (2.39) of the preferred asymptotic gauge. To do so we use an odd transformation of the form (2.35b). The gauge transformation function is determined by the condition (2.39) of preferred asymptotic gauge, together with the condition (B.13) of asymptotic Lorenz gauge, which is valid for interacting solutions as well as free solutions as discussed in appendix C. The resulting transformation between Lorenz gauge fields and preferred asymptotic gauge fields is

$$\Psi_+^e = \underline{\Psi}_+^e + \varepsilon_+, \quad \Psi_-^e = \underline{\Psi}_-^e - \mathcal{P}_* \varepsilon_+, \tag{3.9a}$$

$$\Psi_+^m = \underline{\Psi}_+^m, \quad \Psi_-^m = \underline{\Psi}_-^m, \tag{3.9b}$$

$$\chi_+ = e^{i\varepsilon_+} \underline{\chi}_+, \quad \chi_- = e^{-i\mathcal{P}_* \varepsilon_+} \underline{\chi}_-. \tag{3.9c}$$

with

$$\varepsilon_+ = \frac{1}{2} \left(\mathcal{P}_* \underline{\Psi}_+^e - \underline{\Psi}_+^e \right). \quad (3.10)$$

The inverse transformation is given by the same formulae (3.9) but with ε_+ now expressed in terms of the preferred asymptotic gauge fields:

$$\varepsilon_+ = \frac{1}{2} \left(\Psi_+^e - \mathcal{P}_* \Psi_-^e \right). \quad (3.11)$$

3.3 Perturbative framework

The scattering map (2.42) can be written schematically as

$$\chi_+(u, \theta) = \chi_+[u, \theta; \chi_-, \mathcal{A}_{-A}], \quad (3.12a)$$

$$\mathcal{A}_{+A}(u, \theta) = \chi_+[u, \theta; \chi_-, \mathcal{A}_{-A}], \quad (3.12b)$$

where the functional dependence on the initial data is indicated by the square brackets. We will compute this map perturbatively by considering general Lorenz gauge solutions, and by transforming from Lorenz gauge to preferred asymptotic gauge using eqs. (3.9).

We make the following ansatz for the scalar field and vector potential

$$\Phi = \alpha \Phi^{(1)} + \alpha^2 \Phi^{(2)} + \alpha^3 \Phi^{(3)} + O(\alpha^4), \quad (3.13a)$$

$$A^a = \alpha A^{(1)a} + \alpha^2 A^{(2)a} + \alpha^3 A^{(3)a} + O(\alpha^4), \quad (3.13b)$$

where α is the perturbative expansion parameter.¹¹ There is a corresponding expansion for the initial data $\mathcal{A}_{-A}$, χ_- on $\mathcal{I}^-$ in the preferred asymptotic gauge given by eqs. (2.14), (2.34) and (2.39):

$$\chi_- = \alpha \chi_-^{(1)} + \alpha^2 \chi_-^{(2)} + \alpha^3 \chi_-^{(3)} + O(\alpha^4), \quad (3.14a)$$

$$\mathcal{A}_{-A} = \alpha \mathcal{A}_{-A}^{(1)} + \alpha^2 \mathcal{A}_{-A}^{(2)} + \alpha^3 \mathcal{A}_{-A}^{(3)} + O(\alpha^4). \quad (3.14b)$$

We will take the second order initial data $\chi_-^{(2)}$ and $\mathcal{A}_{-A}^{(2)}$ and higher order initial data to vanish. This is done for convenience and incurs no loss in generality, since terms linear, quadratic, cubic etc. in the first order fields $\chi_-^{(1)}$ and $\mathcal{A}_{-A}^{(1)}$ give complete information about the scattering map. The corresponding expansion of the final data at $\mathcal{I}^+$ in preferred asymptotic gauge is

$$\chi_+ = \alpha \chi_+^{(1)} + \alpha^2 \chi_+^{(2)} + \alpha^3 \chi_+^{(3)} + O(\alpha^4), \quad (3.15a)$$

$$\mathcal{A}_{+A} = \alpha \mathcal{A}_{+A}^{(1)} + \alpha^2 \mathcal{A}_{+A}^{(2)} + \alpha^3 \mathcal{A}_{+A}^{(3)} + O(\alpha^4). \quad (3.15b)$$

We will denote the Lorenz-gauge fields with underlines, and write them as $\underline{\Phi}$, $\underline{A}_a$. The general equations of motion (2.2) reduce in this gauge to

$$\square \underline{\Phi} = 2i \underline{A}_a \nabla^a \underline{\Phi} + \underline{A}^a \underline{A}_a \underline{\Phi}, \quad (3.16a)$$

$$\square \underline{A}^a = -ie^2 (\underline{\Phi} \nabla^a \underline{\Phi}^* - \underline{\Phi}^* \nabla^a \underline{\Phi}) + 2e^2 \underline{A}^a \underline{\Phi}^* \underline{\Phi}. \quad (3.16b)$$

¹¹This expansion is equivalent to expanding in powers of the charge e at fixed Φ and fixed A_a/e .

Now using the expansions (3.13) yields the leading order equations of motion

$$\square \underline{\Phi}^{(1)} = 0, \quad (3.17a)$$

$$\square \underline{A}^{(1)a} = 0, \quad (3.17b)$$

the subleading order equations

$$\square \underline{\Phi}^{(2)} = 2i \underline{A}^{(1)a} \nabla_a \underline{\Phi}^{(1)}, \quad (3.18a)$$

$$\square \underline{A}^{(2)a} = -ie^2 \underline{\Phi}^{(1)} \nabla^a \underline{\Phi}^{(1)*} + ie^2 \underline{\Phi}^{(1)*} \nabla^a \underline{\Phi}^{(1)}, \quad (3.18b)$$

and the subsubleading equations

$$\square \underline{\Phi}^{(3)} = 2i \underline{A}^{(1)a} \nabla_a \underline{\Phi}^{(2)} + 2i \underline{A}^{(2)a} \nabla_a \underline{\Phi}^{(1)} + \underline{A}^{(1)a} \underline{A}_a^{(1)} \underline{\Phi}^{(1)}, \quad (3.19a)$$

$$\begin{aligned} \square \underline{A}^{(3)a} = & -ie^2 \underline{\Phi}^{(1)} \nabla^a \underline{\Phi}^{(2)*} + ie^2 \underline{\Phi}^{(1)*} \nabla^a \underline{\Phi}^{(2)} - ie^2 \underline{\Phi}^{(2)} \nabla^a \underline{\Phi}^{(1)*} + ie^2 \underline{\Phi}^{(2)*} \nabla^a \underline{\Phi}^{(1)} \\ & + 2e^2 \underline{A}^{(1)a} \underline{\Phi}^{(1)*} \underline{\Phi}^{(1)}. \end{aligned} \quad (3.19b)$$

3.4 First order solutions and scattering map

Appendix B reviews the general solutions of the leading order Lorenz gauge equations of motion (3.17) which have nontrivial soft charges. These solutions satisfy our assumed expansions (2.6) and (2.14) near $\mathcal{I}^+$ and $\mathcal{I}^-$, and our asymptotic gauge conditions (2.34). They do not satisfy the matching condition (2.39), a reflection of the fact that the Lorenz and preferred asymptotic gauges do not coincide in general.

From these solutions one can evaluate the free field scattering map. One might expect this map to be trivial for free solutions, and to reduce essentially to the identity map (up to antipodal identification). However the presence of the nontrivial soft charges makes the situation slightly more complicated, and in particular the identity map would not be consistent with the matching condition (2.39) at spatial infinity. The scattering map when written in terms of the potentials Ψ^e and Ψ^m , and denoting Lorenz gauge fields with underlines, is [cf. eq. (B.15)]

$$\underline{\Psi}_+^e(u, \theta) = \mathcal{P}_* \left[\underline{\Psi}_-^e(u, \theta) - \underline{\Psi}_+^e(\theta) + \underline{\Psi}_-^e(\theta) \right] + O(\alpha^2), \quad (3.20a)$$

$$\underline{\Psi}_+^m(u, \theta) = -\mathcal{P}_* \underline{\Psi}_-^m(u, \theta) + O(\alpha^2), \quad (3.20b)$$

$$\underline{\chi}_+(u, \theta) = -\mathcal{P}_* \underline{\chi}_-(u, \theta) + O(\alpha^2). \quad (3.20c)$$

We can rewrite this scattering map in terms of the preferred asymptotic gauge fields using the gauge transformation (3.9) applied to both the initial and final fields. The result for the scalar field is

$$\chi_+ = -\exp \left[2i \mathcal{P}_* (\Psi_+^e - \Psi_-^e) \right] \mathcal{P}_* \chi_- + O(\alpha^2). \quad (3.21)$$

However the phase factor here is a nonlinear effect that should be discarded in a perturbative expansion. Discarding this phase factor the full free field scattering map is, from eqs. (3.9) and (3.20),

$$\Psi_+^e = \mathcal{P}_* \left[\Psi_-^e - \Psi_-^e + \Psi_+^e \right] + O(\alpha^2), \quad (3.22a)$$

$$\Psi_+^m = -\mathcal{P}_* \Psi_-^m + O(\alpha^2), \quad (3.22b)$$

$$\chi_+ = -\mathcal{P}_* \chi_- + O(\alpha^2). \quad (3.22c)$$

Note that some of the signs in eq. (3.22a) differ from those in eq. (3.20a). The scattering map (3.22) manifestly satisfies the matching condition (2.39). It also preserves the symplectic form (2.44), as it should, which provides a nontrivial consistency check of some of the coefficients of the u -independent terms.

We can rewrite the free field scattering map (3.22) in terms of the hard variables $(\tilde{\Psi}^e, \Psi^m, \chi)$ and soft variables $(\Delta\Psi^e, \bar{\Psi}^e)$ defined in section 2.5 above. The result is

$$\mathcal{P}_* \tilde{\Psi}_+^e = \tilde{\Psi}_-^e + O(\alpha^2), \quad (3.23a)$$

$$\mathcal{P}_* \Psi_+^m = -\Psi_-^m + O(\alpha^2), \quad (3.23b)$$

$$\mathcal{P}_* \chi_+ = -\chi_- + O(\alpha^2), \quad (3.23c)$$

$$\mathcal{P}_* \Delta\Psi_+^e = \Delta\Psi_-^e + O(\alpha^2), \quad (3.23d)$$

$$\mathcal{P}_* \bar{\Psi}_+^e = \bar{\Psi}_-^e + \Delta\Psi_-^e + O(\alpha^2). \quad (3.23e)$$

Note that the hard and soft sectors are decoupled to this order,¹² with the soft sector evolving via eqs. (3.23d)–(3.23e) and the hard sector via eqs. (3.23c)–(3.23b).

3.5 General parameterization of the scattering map

The scattering map $\mathcal{S} : \Gamma_- \rightarrow \Gamma_+$ must satisfy a number of constraints. In this section we derive the most general form of the map that obeys all the constraints, as a foundation for the non-decoupling analysis of the later sections.

The various constraints are:

- The scattering map must satisfy the conservation law (2.40) for the soft charges Q_ϵ . This conservation law can be written as [1]

$$Q_+(\theta) = \mathcal{P}_* Q_-(\theta), \quad (3.24)$$

where the charges can be decomposed into hard and soft pieces as

$$Q_+(\theta) = Q_+^{\text{hard}}(\theta) + \frac{1}{e^2} D^2 \Delta\Psi_+^e(\theta), \quad Q_-(\theta) = Q_-^{\text{hard}}(\theta) + \frac{1}{e^2} D^2 \Delta\Psi_-^e(\theta), \quad (3.25)$$

from eqs. (2.10a), (2.21a), (2.22a), (2.7b) (2.34), (2.36) and (2.47a). Here

$$Q_-^{\text{hard}} = \int dv \mathcal{J}_{-v}, \quad Q_+^{\text{hard}} = \int du \mathcal{J}_{+u} \quad (3.26)$$

are the total ingoing and outgoing hard charges per unit angle. Since we have specialized to the sector where the magnetic charges (E.4) vanish, there is no corresponding constraint from the magnetic charges.

- It must be compatible with the transformations of initial and final data associated with the residual even transformations ε_e discussed in section 2.4 above, which act on

¹²This remains true if we express the mapping in terms of the symplectically orthogonal variables $(\tilde{\Psi}^e, \Psi^m, \chi, \Delta\Psi^e, \hat{\Psi}^e)$ obtained by replacing $\bar{\Psi}^e$ with $\hat{\Psi}^e$ using eq. (2.51), instead of the variables (3.23).

the physical phase space since they are not degeneracy directions of the presymplectic form. Specifically, under the transformation of initial data

$$\chi_- \rightarrow e^{i\varepsilon} \chi_-, \quad \mathcal{A}_{-A} \rightarrow \mathcal{A}_{-A} + D_A \varepsilon, \quad (3.27)$$

where $\varepsilon = \varepsilon(\boldsymbol{\theta})$, the final data must transform as

$$\chi_+ \rightarrow e^{i\mathcal{P}_* \varepsilon} \chi_+, \quad \mathcal{A}_{+A} \rightarrow \mathcal{A}_{+A} + D_A \mathcal{P}_* \varepsilon. \quad (3.28)$$

- It must obey the gauge specialization condition (2.39) that was imposed in order to correlate the gauge freedom on $\mathcal{J}^-$ and $\mathcal{J}^+$ in such a way as to allow the scattering map to be a symplectomorphism, as discussed in section 2.4.
- It must transform appropriately under Poincaré symmetries. While this is an important constraint we will not make it explicit in this section.
- It must be a symplectomorphism on phase space. We will impose this requirement in section 3.6 below where we parameterize the scattering map in terms of a generating functional.

Taken together, these requirements strongly constrain the scattering map.

For the analysis in this section it will be convenient to use as the basic variables particular combinations of the preferred asymptotic gauge fields, namely Ψ^m , χ , $\bar{\Psi}^e$ and [cf. eqs. (2.47) above]

$$\check{\Psi}^e \equiv \Psi^e - \bar{\Psi}^e = \tilde{\Psi}^e + g\Delta\Psi^e. \quad (3.29)$$

The general scattering map (3.12) can be written in terms of these variables as

$$\mathcal{P}_* \check{\Psi}_+^e = \check{\Psi}_-^e + \mathcal{H}^e \left[u, \check{\Psi}_-^e, \bar{\Psi}_-^e, \Psi_-^m, \chi_- \right], \quad (3.30a)$$

$$\mathcal{P}_* \bar{\Psi}_+^e = \bar{\Psi}_-^e + \Delta\Psi_-^e + \mathcal{I} \left[\check{\Psi}_-^e, \bar{\Psi}_-^e, \Psi_-^m, \chi_- \right], \quad (3.30b)$$

$$\mathcal{P}_* \Psi_+^m = -\Psi_-^m + \mathcal{H}^m \left[u, \check{\Psi}_-^e, \bar{\Psi}_-^e, \Psi_-^m, \chi_- \right], \quad (3.30c)$$

$$\mathcal{P}_* \chi_+ = -\chi_- + \mathcal{K} \left[u, \check{\Psi}_-^e, \bar{\Psi}_-^e, \Psi_-^m, \chi_- \right], \quad (3.30d)$$

in terms of some functionals $\mathcal{H}^e$, $\mathcal{H}^m$, $\mathcal{I}$ and $\mathcal{K}$, where the functional dependence on the initial data is indicated by the square brackets. Here for convenience we have separated out the terms that arise in the free evolution (3.23), so that the functionals parameterize the nonlinear interactions. The functionals do depend on the angles $\boldsymbol{\theta}$ but we have suppressed this dependence for simplicity.

We start by imposing the transformation property (3.27) and (3.28). The functionals $\mathcal{H}^e$, $\mathcal{H}^m$ and $\mathcal{I}$ need to be invariant under the transformation, while $\mathcal{K}$ needs to transform by a phase. Choosing $\varepsilon = -\bar{\Psi}_-^e$, the invariance implies for example that

$$\mathcal{H}^e \left[u, \check{\Psi}_-^e, \bar{\Psi}_-^e, \Psi_-^m, \chi_- \right] = \mathcal{H}^e \left[u, \check{\Psi}_-^e, 0, \Psi_-^m, e^{-i\bar{\Psi}_-^e} \chi_- \right]. \quad (3.31)$$

Hence by redefining the functionals the scattering map can be written in the general form

$$\mathcal{P}_* \check{\Psi}_+^e = \check{\Psi}_-^e + \mathcal{H}^e \left[u, \check{\Psi}_-^e, \Psi_-^m, e^{-i\bar{\Psi}_-^e} \chi_- \right], \quad (3.32a)$$

$$\mathcal{P}_* \bar{\Psi}_+^e = \bar{\Psi}_-^e + \Delta \Psi_-^e + \mathcal{I} \left[\check{\Psi}_-^e, \Psi_-^m, e^{-i\bar{\Psi}_-^e} \chi_- \right], \quad (3.32b)$$

$$\mathcal{P}_* \Psi_+^m = -\Psi_-^m + \mathcal{H}^m \left[u, \check{\Psi}_-^e, \Psi_-^m, e^{-i\bar{\Psi}_-^e} \chi_- \right], \quad (3.32c)$$

$$\mathcal{P}_* \chi_+ = -\chi_- + \exp \left[i\bar{\Psi}_-^e \right] \mathcal{K} \left[u, \check{\Psi}_-^e, \Psi_-^m, e^{-i\bar{\Psi}_-^e} \chi_- \right]. \quad (3.32d)$$

Next we impose the matching condition (2.39). From the definition (3.29) of $\check{\Psi}^e$ we have that $\check{\Psi}_+^e(u = \infty) = -\check{\Psi}_+^e(u = -\infty)$, and $\mathcal{H}^e$ must also have this property by eq. (3.32a). Defining the functional

$$\mathcal{H}_\infty^e = \lim_{u \rightarrow \infty} \mathcal{H}^e(u) = - \lim_{u \rightarrow -\infty} \mathcal{H}^e(u), \quad (3.33)$$

we find from eqs. (2.39) and (3.32) that $\mathcal{I} = \mathcal{H}_\infty^e$. Therefore the scattering map can be written as

$$\mathcal{P}_* \check{\Psi}_+^e = \check{\Psi}_-^e + \mathcal{H}^e \left[u, \check{\Psi}_-^e, \Psi_-^m, e^{-i\bar{\Psi}_-^e} \chi_- \right], \quad (3.34a)$$

$$\mathcal{P}_* \bar{\Psi}_+^e = \bar{\Psi}_-^e + \Delta \Psi_-^e + \mathcal{H}_\infty^e \left[\check{\Psi}_-^e, \Psi_-^m, e^{-i\bar{\Psi}_-^e} \chi_- \right], \quad (3.34b)$$

$$\mathcal{P}_* \Psi_+^m = -\Psi_-^m + \mathcal{H}^m \left[u, \check{\Psi}_-^e, \Psi_-^m, e^{-i\bar{\Psi}_-^e} \chi_- \right], \quad (3.34c)$$

$$\mathcal{P}_* \chi_+ = -\chi_- + \exp \left[i\bar{\Psi}_-^e \right] \mathcal{K} \left[u, \check{\Psi}_-^e, \Psi_-^m, e^{-i\bar{\Psi}_-^e} \chi_- \right]. \quad (3.34d)$$

Finally we impose the conservation laws (3.24). Taking the limits $u \rightarrow \pm\infty$ of eq. (3.34a) and using the definitions (2.47a), (3.29) and (3.33) yields

$$\mathcal{P}_* \Delta \Psi_+^e = \Delta \Psi_-^e + 2\mathcal{H}_\infty^e \left[\check{\Psi}_-^e, \Psi_-^m, e^{-i\bar{\Psi}_-^e} \chi_- \right]. \quad (3.35)$$

It follows that $\mathcal{H}_\infty^e$ is essentially the change in the electromagnetic memory $\Delta \Psi^e$ [15, 16, 61] between $\mathcal{I}^-$ and $\mathcal{I}^+$. Now combining this with the conservation law (3.24) gives

$$\mathcal{H}_\infty^e = -\frac{1}{2} e^2 \mathcal{P}_* D^{-2} \Delta Q^{\text{hard}}(\theta), \quad (3.36)$$

where the total change in the charge per unit angle is $\Delta Q^{\text{hard}} = Q_+^{\text{hard}}(\theta) - \mathcal{P}_* Q_-^{\text{hard}}(\theta)$, which can be computed from the functional $\mathcal{K}$ from eqs. (2.9a), (3.24), (3.26) and (3.34d). This tells us that the functionals are not all independent, as $\mathcal{H}_\infty^e$ can be computed from $\mathcal{K}$.

To summarize, we have derived a general parameterization of the scattering map that is consistent with all of the constraints listed above, given by eq. (3.34), assuming that the various functionals transform appropriately under Poincaré transformations.

We can rewrite the scattering map (3.34) in terms of the symplectically orthogonal variables $(\check{\Psi}^e, \Psi^m, \chi, \Delta \Psi^e, \hat{\Psi}^e)$ introduced in section 2.5 using the definitions (2.47), (2.51) and (3.29). The resulting map is given by eqs. (3.34c), (3.34d), and (3.35) together with

$$\mathcal{P}_* \check{\Psi}_+^e = \check{\Psi}_-^e + \mathcal{H}^e(u) - 2g(u) \mathcal{H}_\infty^e, \quad (3.37a)$$

$$\mathcal{P}_* \hat{\Psi}_+^e = \hat{\Psi}_-^e + \Delta \Psi_-^e + \mathcal{H}_\infty^e + 2 \int du g'(u) \mathcal{H}^e(u), \quad (3.37b)$$

where for convenience we have suppressed the arguments of the functionals $\mathcal{H}^e$ and $\mathcal{H}_\infty^e$.

We can also rewrite the scattering map (3.34) in terms of asymptotic Lorenz gauge fields using the gauge transformation (3.9). The arguments of the functionals in eq. (3.34) are invariant under the transformation, and so we can simply insert underlines on all of these arguments to indicate asymptotic Lorenz gauge fields. The gauge transformation function (3.11) evaluates to $\varepsilon_+ = \mathcal{P}_* \Delta \Psi_-^e + \mathcal{P}_* \mathcal{H}_\infty^e$, and the final result is

$$\mathcal{P}_* \check{\Psi}_+^e = \check{\Psi}_-^e + \mathcal{H}^e \left[u, \check{\Psi}_-^e, \underline{\Psi}^m, e^{-i\bar{\Psi}_-^e} \underline{\chi}_- \right], \quad (3.38a)$$

$$\mathcal{P}_* \bar{\Psi}_+^e = \bar{\Psi}_-^e - \Delta \Psi_-^e - \mathcal{H}_\infty^e \left[\check{\Psi}_-^e, \underline{\Psi}^m, e^{-i\bar{\Psi}_-^e} \underline{\chi}_- \right], \quad (3.38b)$$

$$\mathcal{P}_* \underline{\Psi}_+^m = -\underline{\Psi}_-^m + \mathcal{H}^m \left[u, \check{\Psi}_-^e, \underline{\Psi}^m, e^{-i\bar{\Psi}_-^e} \underline{\chi}_- \right], \quad (3.38c)$$

$$\begin{aligned} \mathcal{P}_* \underline{\chi}_+ = & \exp \left\{ -2i\mathcal{H}_\infty^e \left[\check{\Psi}_-^e, \underline{\Psi}^m, e^{-i\bar{\Psi}_-^e} \underline{\chi}_- \right] - 2i\Delta \Psi_-^e \right\} \\ & \times \left\{ -\underline{\chi}_- + \exp \left[i\bar{\Psi}_-^e \right] \mathcal{K} \left[u, \check{\Psi}_-^e, \underline{\Psi}^m, e^{-i\bar{\Psi}_-^e} \underline{\chi}_- \right] \right\}. \end{aligned} \quad (3.38d)$$

This has the same form as the original scattering map (3.34) except for sign flips in two of the terms in eq. (3.38b) and the overall phase factor in the scalar field (3.38d).

3.6 Scattering map in terms of a generating functional

In this section we discuss a more efficient parameterization of the scattering map, in terms of a generating functional on phase space. This representation will be used extensively in the discussions in sections 4.1, 4.2 and 6.4 below.

We start by decomposing the scattering map $\mathcal{S} : \Gamma_- \rightarrow \Gamma_+$ as described in the introduction:

$$\mathcal{S} = \mathcal{S}_2 \circ \mathcal{S}_0, \quad (3.39)$$

where $\mathcal{S}_0 : \Gamma_- \rightarrow \Gamma_+$ is the linear order scattering map (3.23) and $\mathcal{S}_2 : \Gamma_+ \rightarrow \Gamma_+$ encodes the nontrivial part of the scattering.¹³ Here Γ_- and Γ_+ are the spaces of initial and final data on $\mathcal{I}^-$ and $\mathcal{I}^+$ defined in section 2.4. Since $\mathcal{S}$ and $\mathcal{S}_0$ are both symplectomorphisms, so is $\mathcal{S}_2$. We now define the generating functional $G : \Gamma_+ \rightarrow \mathbf{R}$ by demanding that for any functional $f : \Gamma_+ \rightarrow \mathbf{R}$ we have

$$f \circ \mathcal{S}_2 = \exp \left[\{ \dots, G \} \right] f \equiv f + \{ f, G \} + \frac{1}{2} \{ \{ f, G \}, G \} + \dots \quad (3.40)$$

Here the notation $\{ \dots, G \}$ means the differential operator that acts on functionals f and returns $\{ f, G \}$. The definition (3.40) uses the fact that a general symplectomorphism can be obtained by exponentiating, as for any Lie group. The functional G is similar to a Hamilton-Jacobi functional in that it encodes the dynamics (except that it is a functional of original coordinates and original momenta instead of being a functional of original coordinates and new momenta).

The results of sections 5 and 6 below imply that $G = O(\alpha^3)$, and it follows that through $O(\alpha^5)$ it is sufficient to use a truncated version of the formula (3.40) that retains only the

¹³This is similar to using the interaction representation in quantum mechanics.

first two terms. In particular, if y^a are arbitrary coordinates on phase space, the scattering map $\mathcal{S}_2$ can be represented by

$$y^a \circ \mathcal{S}_2 = y^a + \{y^a, G\} + O(\alpha^6). \quad (3.41)$$

In appendix G we relate the generating functional G to the various functionals $\mathcal{H}^e$, $\mathcal{H}^m$ and $\mathcal{K}$ defined in section 3.5 above. We also show in appendix G that G has the specific form

$$G \left[\tilde{\Psi}_+^e, \Psi_+^m, \chi_+, \Delta\Psi_+^e, \hat{\Psi}_+^e \right] = \mathcal{G} \left[\check{\Psi}_+^e, \Psi_+^m, e^{-i\tilde{\Psi}_+^e} e^{i\Delta\Psi_+^e/2} \chi_+ \right]. \quad (3.42)$$

Here on the left hand side the arguments

$$\left(\tilde{\Psi}_+^e, \Psi_+^m, \chi_+, \Delta\Psi_+^e, \hat{\Psi}_+^e \right) \quad (3.43)$$

are the independent phase space coordinates defined in section 2.5, on the right hand side $\mathcal{G}$ is some functional, and its arguments are given in terms of the fields on the left hand side by

$$\check{\Psi}_+^e(u, \theta) = \tilde{\Psi}_+^e(u, \theta) + g(u) \Delta\Psi_+^e(\theta), \quad (3.44a)$$

$$\bar{\Psi}_+^e(\theta) = \hat{\Psi}_+^e(\theta) - 2 \int du g'(u) \tilde{\Psi}_+^e(u, \theta), \quad (3.44b)$$

[cf. eqs. (2.51) and (3.29)]. The functional $\mathcal{G}$ must obey the constraint (G.10), whose interpretation is that its functional derivative with respect to its first argument $\check{\Psi}_+^e$ involves just a bulk term and no boundary term. It must also obey the constraint (G.7). These constraints together with the form (3.42) of the generating functional guarantee that the corresponding symplectomorphism (3.40) satisfies the required conditions discussed in section 3.5: it preserves the conserved quantities (3.35) and (2.39) and transforms appropriately under the symmetry (3.28).

4 Coupling of soft and hard degrees of freedom: general arguments and their validity

We now turn to an assessment of some of the arguments that have been made in the literature about the nature of the dynamics of the soft degrees of freedom, using as a foundation properties of the scattering map derived in the previous section. In the following three subsections, we will discuss three different lines of reasoning for why the soft dynamics is either trivial or completely decoupled from the dynamics of a hard sector. We will argue that these conclusions are unfounded.

4.1 Argument for triviality of soft dynamics

One argument that might be made about the dynamics of the soft degrees of freedom is that they are a trivial extension of the dynamics of the hard degrees of freedom. Specifically, the full dynamics can be derived from the generating functional [cf. eq. (3.42) above]

$$\mathcal{G} \left[\check{\Psi}_+^e, \Psi_+^m, e^{-i\tilde{\Psi}_+^e} e^{i\Delta\Psi_+^e/2} \chi_+ \right], \quad (4.1)$$

where we recall that the definition (3.29) of the field $\check{\Psi}_+^e$ requires the boundary condition

$$\lim_{u \rightarrow \pm\infty} \check{\Psi}_+^e(u, \theta) = \pm \Delta \Psi_+^e(\theta)/2. \quad (4.2)$$

Suppose now that one considers scattering with purely hard initial data, for which $\bar{\Psi}_+^e = \Delta \Psi_+^e = 0$.¹⁴ Such scattering is defined by a functional $\mathcal{G}[\check{\Psi}_+^e, \Psi_+^m, \chi_+]$, and fully general scattering is defined by the same functional (4.1) except that one of the arguments is modified by a phase factor. Therefore fully general scattering is a trivial extension of purely hard scattering.

A potential loophole in this argument is the fact that the functionals in the two different cases are defined on different function spaces. The purely hard scattering functional is defined on a function space where $\check{\Psi}_+^e$ satisfies the boundary conditions (4.2) with vanishing right hand side, whereas the fully general functional is defined on the larger function space where the right hand side is nonzero. However, it turns out that within the context of perturbative classical scattering, the general functional $\mathcal{G}$ is determined by continuity by its restriction to the smaller function space. This follows from the constraint (G.10) that the generating functional must satisfy. See section 5 below for further discussion of this point within the context of second order perturbation theory. Thus the more general functional is determined uniquely by the restricted functional.

There is, however, a flaw in the argument. Scattering with initial data with $\bar{\Psi}_+^e = \Delta \Psi_+^e = 0$ is not “purely hard” scattering, since these variables will generically be nonzero after the scattering due to the electromagnetic memory effect (see section 6.2 and appendix J below for more details). Thus, even within the restricted functional, the hard and soft degrees of freedom are intertwined.

4.2 Argument for decoupling based on theorem in symplectic geometry

A more fruitful approach, following Bousso and Porrati [33] (henceforth BP), is to focus on the quantities that are preserved under the scattering map (1.4), instead of on $\Delta \Psi_+^e$ and $\bar{\Psi}_+^e$. These quantities are the charge $Q_+(\theta)$ given by eq. (3.24), and the quantity $\Psi_\pm^e(\theta)$ given by eqs. (2.47) and (2.49) which is preserved by eq. (2.39). From eqs. (2.52) these quantities have the Poisson brackets

$$\{Q_+(\theta), Q_+(\theta')\} = 0, \quad (4.3a)$$

$$\{\Psi_\pm^e(\theta), \Psi_\pm^e(\theta')\} = 0, \quad (4.3b)$$

$$\{Q_+(\theta), \Psi_\pm^e(\theta')\} = \delta^{(2)}(\theta, \theta') - \frac{1}{4\pi}. \quad (4.3c)$$

One can now attempt to make an argument for the decoupling of hard and soft degrees of freedom based on the following theorem in symplectic geometry, which is proved in appendix H:

¹⁴This condition is equivalent to $\bar{\Psi}_-^e = \Delta \Psi_-^e = 0$, since we are working within a framework where the $+$ and $-$ variables are related by the linear order scattering map (3.23), cf. eqs. (3.39) and (3.40) above.

Theorem. Let (M, Ω_{ab}) be a symplectic manifold of finite dimension n , with associated Poisson bracket defined by $\{f, g\} = \Omega^{ab} \nabla_a f \nabla_b g$ for any functions f, g on M , where $\Omega^{ab} \Omega_{bc} = \delta^a_c$. Let $\mathcal{S}_2 : M \rightarrow M$ be a symplectomorphism. Suppose that there exists a set of functions $s^A : M \rightarrow \mathbf{R}$ for $1 \leq A \leq t$ which satisfies the two properties:

1. The functions are preserved under the symplectomorphism,

$$s^A \circ \mathcal{S}_2 = s^A. \quad (4.4)$$

2. The Poisson brackets

$$\omega^{AB} = \{s^A, s^B\} \quad (4.5)$$

are constants on M and form an invertible $t \times t$ matrix.

Then locally the symplectomorphism factorizes, that is, locally one can find coordinates $(s^1, \dots, s^t, h^1, \dots, h^{n-t})$ for which the symplectomorphism takes the form

$$s^A \rightarrow s^A, \quad h^\Gamma \rightarrow \bar{h}^\Gamma(h^\Sigma), \quad (4.6)$$

for $1 \leq A \leq t$ and $1 \leq \Gamma, \Sigma \leq n - t$.

Interpreting s^A as soft and h^Γ as hard degrees of freedom, the theorem would seem to give sufficient conditions for the two sets of degrees of freedom to decouple from one another. Indeed, if we take s^A to be the conserved quantities $Q_+(\boldsymbol{\theta})$ and $\Psi_+^e(\boldsymbol{\theta})$ discussed above, and use the definition (1.4) of the scattering map $\mathcal{S}_2$, then both conditions of the theorem are satisfied, from eqs. (4.3). It should follow that there exists a complementary set of hard variables that are completely decoupled from the soft dynamics, as in eq. (4.6). This is essentially the decoupling argument of BP.¹⁵

This argument fails for a subtle reason: the theorem is valid in finite dimensions but not in the infinite dimensional context in which we want to apply it. We can demonstrate this failure as follows. If the theorem were applicable, it would establish the existence of a set of coordinates on phase space which satisfy the following three conditions: (i) The coordinates are independent; (ii) They are symplectically orthogonal (canonical); and (iii) They include the conserved charges $Q_+(\boldsymbol{\theta})$ and $\Psi_+^e(\boldsymbol{\theta})$. Such coordinates do not in fact exist. To see this, we can start from the coordinates (3.43) which satisfy (i) and (ii) but not (iii). Assuming that coordinates exist which satisfy (i), (ii) and (iii), it follows that there exists a symplectomorphism that maps $\Delta\Psi_+^e$ onto Q_+ and $\hat{\Psi}_+^e$ onto Ψ_+^e . We attempt to solve for this symplectomorphism by using a type II generating functional following BP. This yields the

¹⁵BP present their argument in somewhat different terms: instead of invoking the theorem they attempt to directly derive the change of phase space coordinates to the decoupled set satisfying eq. (4.6). They also focus on the gravitational case rather than the electromagnetic case. However the theorem captures the essential idea of their argument.

following coordinates that satisfy (ii) and (iii):

$$\tilde{\Psi}_{\text{new}}^e(u, \theta) = \tilde{\Psi}_+^e(u, \theta) + g(u)\Delta\Psi_+^e(\theta), \quad (4.7a)$$

$$\Psi_{\text{new}}^m(u, \theta) = \Psi_+^m(u, \theta), \quad (4.7b)$$

$$\chi_{\text{new}}(u, \theta) = \exp \left\{ -i \left[\hat{\Psi}_+^e(\theta) - \frac{1}{2} \Delta\Psi_+^e(\theta) - 2 \int d\bar{u} g'(\bar{u}) \tilde{\Psi}_+^e(\bar{u}, \theta) \right] \right\} \chi_+(u, \theta), \quad (4.7c)$$

$$\Delta\Psi_{\text{new}}^e(\theta) = \Delta\Psi_+^e(\theta) + e^2 D^{-2} Q^{\text{hard}}(\theta) [\chi_+], \quad (4.7d)$$

$$\hat{\Psi}_{\text{new}}^e(\theta) = \hat{\Psi}_+^e(\theta) - \frac{1}{2} \Delta\Psi_+^e(\theta) - 2 \int du g'(u) \tilde{\Psi}_+^e(u, \theta), \quad (4.7e)$$

where Q^{hard} is defined by eqs. (3.26) and (2.9a). One can check that these new coordinates have the same Poisson brackets as the original $+$ coordinates, and so are symplectically orthogonal. However these coordinates do not satisfy (i), i.e. they are not independent, since we have

$$\Delta\Psi_{\text{new}}^e(\theta) = 2 \lim_{u \rightarrow \infty} \tilde{\Psi}_{\text{new}}^e(u, \theta) + e^2 D^{-2} Q^{\text{hard}}(\theta) [\chi_{\text{new}}], \quad (4.8)$$

from eqs. (2.46) and (2.48). What has happened is that the original $\tilde{\Psi}$ coordinate satisfied the boundary conditions $\tilde{\Psi}(u) \rightarrow 0$ as $u \rightarrow \pm\infty$, whereas the new $\tilde{\Psi}$ coordinate does not and so effectively encodes more information.¹⁶ Thus the requirements of an autonomous soft sector, independence of phase space coordinates, and symplectic orthogonality of hard and soft sectors are incompatible.

A useful perspective on the dynamics can be obtained by defining an enlarged phase space $\tilde{\Gamma}$ in which the would-be symplectomorphism (4.7) is actually a symplectomorphism. This space is parameterized by the coordinates on the left hand sides of eqs. (4.7), where we demand that

$$\lim_{u \rightarrow \infty} \tilde{\Psi}_{\text{new}}^e(u, \theta) = - \lim_{u \rightarrow -\infty} \tilde{\Psi}_{\text{new}}^e(u, \theta), \quad (4.10)$$

but we do not demand that this limit satisfy the constraint (4.8). The enlarged phase space factors as $\tilde{\Gamma} = \tilde{\Gamma}_{\text{hard}} \times \tilde{\Gamma}_{\text{soft}}$, where $\tilde{\Gamma}_{\text{hard}}$ is parameterized by the coordinates (4.7a)–(4.7c), and $\tilde{\Gamma}_{\text{soft}}$ by the coordinates (4.7d) and (4.7e). We can define a scattering map $\tilde{\mathcal{S}}_2 : \tilde{\Gamma} \rightarrow \tilde{\Gamma}$, which factorizes, acts on $\tilde{\Gamma}_{\text{soft}}$ as the identity, and acts on $\tilde{\Gamma}_{\text{hard}}$ as determined by the generating functional (3.42) [note that the arguments of the generating functional $\mathcal{G}$ coincide with the coordinates (4.7a)–(4.7c)]. The physical phase space Γ is then given by restricting to the subspace of $\tilde{\Gamma}$ given by the constraint (4.8), and the physical scattering map

¹⁶A similar phenomenon arises if one tries to diagonalize the Hamiltonian functional

$$H = \int du \int d^2\Omega (\partial_u \Psi)^2. \quad (4.9)$$

In finite dimensions one can always find canonical coordinates that diagonalize a positive definite quadratic Hamiltonian (Williamson’s theorem [62, 63]). However Williamson’s theorem does not apply in our infinite dimensional context: applying standard methods to find the canonical phase space coordinates that diagonalize the Hamiltonian (4.9) gives rise to coordinates that are not independent, just as in eq. (4.8). The origin of the difficulty in both cases is that the inverse (2.52) of the symplectic form (2.50) does not lie in $\Gamma \otimes \Gamma$ as it would if the phase space Γ were finite dimensional, instead it lies in a larger space. See Ashtekar [40] for a rigorous treatment of Γ as a Fréchet space.

$\mathcal{S}_2$ is given by restricting the action of $\tilde{\mathcal{S}}_2$ to this subspace. Thus, although the scattering map does not factorize as argued by BP, it descends from a mapping on a larger phase space which does.

In the enlarged phase space, even though the scattering map $\tilde{\mathcal{S}}_2$ factorizes, the map on the hard factor $\tilde{\Gamma}_{\text{hard}}$ is constrained by the existence of the soft conservation laws. In particular, the right hand side of eq. (4.8) is preserved, because of the constraint (G.7) on the generating functional $\mathcal{G}$. This is necessary for the mapping $\tilde{\mathcal{S}}_2$ to preserve the physical phase space Γ . Thus, the argument of BP for the triviality of the soft conservation laws — that they act only on a soft sector and do not affect the dynamics in a hard sector — does not apply.

4.3 Argument for decoupling based on factorization theorem for S matrix elements

Another kind of argument for the triviality of scattering of soft degrees of freedom was given by Mirbabayi and Porrati (henceforth MP) [32], based on the soft theorems for S-matrix elements [2]. MP assumed that

1. The system including soft degrees of freedom can be described by a Hilbert space $\mathcal{H}$.
2. There is a factorization of the Hilbert space into hard and soft factors

$$\mathcal{H} = \mathcal{H}_{\text{hard}} \otimes \mathcal{H}_{\text{soft}}, \quad (4.11)$$

a basis $|a\rangle$ of $\mathcal{H}_{\text{hard}}$, and a set of unitary operators $\Omega(a)$ on $\mathcal{H}_{\text{soft}}$ for which the S matrix can be expressed as

$$S = S_{ab} |a\rangle \langle b| \Omega(a) \Omega(b)^\dagger. \quad (4.12)$$

MP defined the factorization (4.11) in terms of an infrared cutoff on mode energy, and took the basis $|a\rangle$ of hard states to be the usual particle number eigenstates used in Feynman diagrams. They then argued for the decomposition (4.12) from soft theorems [2], with the operators $\Omega(a)$ providing the universal soft factors for S matrix elements when some of the ingoing and outgoing particles become soft.

The assumptions (4.11) and (4.12) are equivalent to a decoupling of two sectors. We can write

$$S = U(S_{\text{hard}} \otimes 1)U^\dagger \quad (4.13)$$

with U being the unitary operator $U = \sum_a \Omega(a) |a\rangle \langle a|$ and $S_{\text{hard}} = S_{ab} |a\rangle \langle b|$. It follows that there exists another factorization of the Hilbert space $\mathcal{H} = \tilde{\mathcal{H}}_{\text{hard}} \otimes \tilde{\mathcal{H}}_{\text{soft}}$ for which the scattering matrix S factorizes and acts as the identity on $\tilde{\mathcal{H}}_{\text{soft}}$. Taking the classical limit it then follows that the scattering map $\mathcal{S}_2$ is of the factorized form (4.6) discussed earlier, for some choice of hard and soft variables.

This conclusion is in conflict with explicit computations within the classical theory detailed in sections 5 and 6 below (where we use an exact decomposition into hard and soft degrees of freedom rather than a decomposition based on letting a cutoff go to zero). We

conclude that the assumptions 1 and 2 above are not valid in such a context: the correct description of the soft theorems within the quantum theory must be more involved than eqs. (4.11) and (4.12). It is possible that no Hilbert space description of the dynamics exists when soft degrees of freedom are included, as argued by Prabhu, Satishchandran, and Wald [42] (see section 7 below for further discussion). Another possibility is that the Hilbert space cannot be expressed as a tensor product of hard and soft factors, the appropriate structure might be something like the fusion product defined by Donnelly and Freidel [48].

5 Explicit computation of scattering map: second order

5.1 Preamble

In the previous section we refuted some general arguments for the existence of decoupled hard and soft sectors. However to show that the sectors are actually always coupled requires an explicit computation of the dynamics. Such an explicit computation in the context of perturbation theory is the goal of the rest of this paper. We start in this section by considering the second order scattering map, and proceed to third and fourth orders in the following section.

5.2 Second order dynamics

The scattering at second order is trivial in the sense that all the functionals defined in eq. (3.34), that parameterize the nonlinear interactions in preferred asymptotic gauge, vanish to this order:

$$\mathcal{H}^e = O(\alpha^3), \quad \mathcal{H}_\infty^e = O(\alpha^3), \quad \mathcal{H}^m = O(\alpha^3), \quad \mathcal{K} = O(\alpha^3). \quad (5.1)$$

As a consequence, the explicit form of the scattering map coincides with the linear order scattering map (3.23),

$$\mathcal{P}_* \tilde{\Psi}_+^e = \tilde{\Psi}_-^e + O(\alpha^3), \quad (5.2a)$$

$$\mathcal{P}_* \Psi_+^m = -\Psi_-^m + O(\alpha^3), \quad (5.2b)$$

$$\mathcal{P}_* \chi_+ = -\chi_- + O(\alpha^3), \quad (5.2c)$$

$$\mathcal{P}_* \Delta \Psi_+^e = \Delta \Psi_-^e + O(\alpha^3), \quad (5.2d)$$

$$\mathcal{P}_* \bar{\Psi}_+^e = \bar{\Psi}_-^e + \Delta \Psi_-^e + O(\alpha^3), \quad (5.2e)$$

from eqs. (2.47) and (3.29). In particular, the hard sector $(\tilde{\Psi}^e, \Psi^m, \chi)$ and the soft sector $(\Delta \Psi^e, \bar{\Psi}^e)$ are decoupled from one another and evolve independently. Thus we will need go to higher order in perturbation theory to find the leading order couplings between the soft and hard sectors. We will return to this story in section 6 below. In the remainder of this section we will discuss some properties of the second order scattering and its derivation.

Although the scattering is trivial in preferred asymptotic gauge, it is not trivial in asymptotic Lorenz gauge, where there is a nonlinear soft-hard interaction at quadratic order. This can be seen by substituting the result (5.1) into the general Lorenz gauge scattering

map (3.38), which yields from eqs. (2.47) and (3.29) that

$$\mathcal{P}_* \tilde{\Psi}_+^e = \tilde{\Psi}_-^e + O(\alpha^3), \quad (5.3a)$$

$$\mathcal{P}_* \Psi_+^m = -\Psi_-^m + O(\alpha^3), \quad (5.3b)$$

$$\mathcal{P}_* \chi_+ = -\chi_- + 2i\Delta\Psi_-^e \chi_- + O(\alpha^3), \quad (5.3c)$$

$$\mathcal{P}_* \Delta\Psi_+^e = \Delta\Psi_-^e + O(\alpha^3), \quad (5.3d)$$

$$\mathcal{P}_* \bar{\Psi}_+^e = \bar{\Psi}_-^e - \Delta\Psi_-^e + O(\alpha^3). \quad (5.3e)$$

The nonlinear term in eq. (5.3c) arises from the phase factor in eq. (3.38d), and can be written in terms of the expansion coefficients (3.14) and (3.15) as

$$\mathcal{P}_* \chi_+^{(2)}(u, \theta) = 2i\Delta\Psi_-^{e(1)}(\theta) \chi_-^{(1)}(u, \theta). \quad (5.4)$$

The nonlinear scattering map (5.3), which is equivalent to the result (5.1), is derived in appendix I.

The nonlinearity in the second order Lorenz gauge scattering map (5.3) arises only because of the presence of soft degrees of freedom. It vanishes if we specialize to incoming data that is purely hard ($\Delta\Psi_-^e = \bar{\Psi}_-^e = 0$). This is a well-known result, familiar in the quantum context from the Feynman rules for scalar QED [64]. The argument is that quadratic scattering corresponds to a three particle interaction and to the operator $A^a \Phi^* \nabla_a \Phi + \text{c.c.}$ in the Lagrangian. If we combine the conservation of four-momentum and the fact that the final momentum must be null to be on-shell, this forces all the momenta to be collinear. Finally the Lorenz gauge condition forces the final result to vanish. In appendix I we translate this argument into the language of perturbative classical scattering used in this paper, and explain how it breaks down in the presence of soft degrees of freedom.

One consequence of the nonlinear soft-hard interaction is that the scattering map (5.3) is not continuous, in the following sense. Consider a sequence $^{(n)}\Psi_-^e$ of incoming configurations, each of which has no soft part [$^{(n)}\Delta\Psi_-^e = ^{(n)}\bar{\Psi}_-^e = 0$], which converges pointwise to Ψ_-^e :

$$\lim_{n \rightarrow \infty} ^{(n)}\Psi_-^e(v, \theta) = \Psi_-^e(v, \theta). \quad (5.5)$$

For each element in the sequence, the result (5.4) vanishes, and so the $n \rightarrow \infty$ limit of the scattered fields also has this property. However the result (5.4) does not vanish for the scattering of the $n \rightarrow \infty$ pointwise limit (5.5) of the initial data. Thus, the scattering of soft degrees of freedom cannot be obtained by simply taking a naive limit of hard scattering. Equivalently, the scattering map is not determined by continuity by its restriction to initial data that consists of smooth wavepacket states. This property is specific to asymptotic Lorenz gauge, and does not occur in preferred asymptotic gauge where the scattering map is continuous, as can be seen from eqs. (G.10) and (G.11) of appendix G.

6 Explicit computation of scattering map: third and fourth orders

6.1 Overview

In the previous section we showed that the classical scattering map factorizes into decoupled hard and soft sectors at quadratic order. We now proceed to third and fourth orders in

perturbation theory. We will show in this section that no factorization into decoupled sectors is possible at these orders.

It will be sufficient for our arguments to consider only two specific pieces of the fourth order scattering. The first piece is the change in electromagnetic memory between $\mathcal{I}^-$ and $\mathcal{I}^+$. In section 6.2 and appendix J we compute this quantity explicitly and show that it is nonzero, as would be expected on general grounds [1]. The second piece is the contribution to χ_+ that is linear in $\bar{\Psi}_-^e$ and cubic in χ_- , which we show in section 6.2 is nonzero.

Next, in order to show that the hard and soft sectors are always dynamically coupled, we need to address an ambiguity in the definitions of these sectors. Roughly speaking the soft sector consists of zero energy modes, while the hard sector consists of finite energy modes, but these requirements allow for considerable leeway in the definitions. Our strategy will be to show that for *any* definitions of hard and soft sectors, the two sectors are dynamically coupled. However, we do require that the same definitions of hard and soft be used at $\mathcal{I}^-$ and at $\mathcal{I}^+$ (otherwise it is always trivially possible to find definitions which decouple the dynamics). As discussed in the introduction, we adopt two different conventions for identifying the phase spaces Γ_- at $\mathcal{I}^-$ and Γ_+ at $\mathcal{I}^+$, that allow us to enforce this requirement.

Section 6.3 considers the first convention, based on identifying $\mathcal{I}^-$ and $\mathcal{I}^+$ using null geodesics. First, in section 6.3.1, we consider cubic and quartic terms in the scattering map which couple together the soft and hard sectors. Some of these terms are invariant under linear field redefinitions and perturbative field redefinitions that alter the definitions of the two sectors. We will call such terms *invariant interactions*, as they cannot be removed by field redefinitions, and we derive criteria for interactions to be invariant. Then, in section 6.3.2, we show that the two pieces of the fourth order scattering discussed in section 6.2 are such invariant interactions. Since these quantities are generically nonzero, the hard and soft sectors are always coupled. These invariant interactions also disallow scattering maps of the forms (1.5) and (1.6).

In section 6.4 we turn to the second convention, which is based on the free field scattering map. For this case there exist definitions of the soft variables which are conserved, discussed in section 4.2 above, and it is natural to restrict to definitions of soft variables which preserve this property. With this assumption we show that it is not possible to find definitions of soft and hard variables for which the scattering map has the uncoupled form (1.1), again using the results of section 6.2.

6.2 Two nonzero pieces of fourth order scattering

In this section we focus attention on two particular pieces of the fourth order scattering which we will show are nonzero.

The first piece is the total change $\Delta\Psi_+^e - \mathcal{P}_*\Delta\Psi_-^e$ in electromagnetic memory between $\mathcal{I}^-$ and $\mathcal{I}^+$. From the conservation law (3.24) this quantity is proportional to

$$\Delta Q^{\text{hard}}(\theta) = Q_+^{\text{hard}}(\theta) - \mathcal{P}_*Q_-^{\text{hard}}(\theta), \quad (6.1)$$

the change in the total hard charge per unit angle in the scalar field. It also corresponds to the functional $\mathcal{H}_\infty^e$ in the general parameterization (3.34) of the scattering map, from eq. (3.36).

It is clear that this quantity is nonzero in general [1]. For example, for electromagnetism coupled to massive charged particles, if one has two incoming charged particles, there will in general be a nontrivial scattering and the outgoing particles will have different directions of propagation from the incoming ones. In this paper our source is a massless scalar field and so we deal instead with wave packet ingoing and outgoing states, but it is clear that generically there will be nontrivial scattering. In appendix J we verify this explicitly by computing the leading order contribution to $\Delta Q^{\text{hard}}(\boldsymbol{\theta})$, which is quartic in the initial scalar field $\chi_-^{(1)}$ and is given by the expressions (J.4) and (J.6).

The second piece of the fourth order scattering that we consider is the contribution to $\chi_+(u, \boldsymbol{\theta})$ that is linear in $\bar{\Psi}_-^e$ and cubic in $\chi_-(v, \boldsymbol{\theta})$. We can show that this contribution is nonzero as follows. Combining the leading order formula (3.41) for the scattering map $\mathcal{S}_2$ in terms of the generating functional G together with the Poisson bracket (2.52d) gives that the change in $\chi_+(u, \boldsymbol{\theta})^*$ is

$$-\frac{1}{2} \int du' K(u-u') \frac{\delta G}{\delta \chi_+(u', \boldsymbol{\theta})}, \quad (6.2)$$

where $K(u) = \Theta(u) - 1/2$. Combining this with the specific form (3.42) of the generating functional, the definition (1.4) of the scattering map $\mathcal{S}_2$ and the free field scattering map (3.23) we obtain

$$\mathcal{P}_* \chi_+ = -\chi_- - \frac{1}{2} \int du' K(u-u') e^{i\bar{\Psi}_-^e} e^{i\Delta\Psi_-^e/2} \frac{\delta \mathcal{G}}{\delta \chi_+} \left[\tilde{\Psi}_-^e + g\Delta\Psi_-^e, -\Psi_-^m, -e^{-i\bar{\Psi}_-^e} e^{-i\Delta\Psi_-^e/2} \chi_- \right]^*. \quad (6.3)$$

If we now specialize to a vanishing incoming electromagnetic field we obtain

$$\mathcal{P}_* \chi_+ = -\chi_- - \frac{1}{2} \int du' K(u-u') \frac{\delta \mathcal{G}}{\delta \chi_+} [0, 0, -\chi_-]^*. \quad (6.4)$$

The second term here is computed explicitly in the last paragraph of appendix J and is cubic in χ_- and nonzero. Consider now the contribution to the general result (6.3) that is linear in $\bar{\Psi}_-^e$ evaluated at $\Delta\Psi_-^e = \tilde{\Psi}_-^e = 0$. Because of the form of the expression (6.3), this contribution can be computed from the known expression (6.4) by inserting the complex exponential factors. Therefore we conclude that the contribution which is linear in $\bar{\Psi}_-^e$ and cubic in χ_- is nonzero. We write this contribution as

$$\chi_+ = \hat{F}[\bar{\Psi}_-^e, \chi_-, \chi_-, \chi_-] \quad (6.5)$$

where the function $\hat{F}$ is linear in its first, second and third arguments and antilinear in its fourth (see appendix J).

6.3 Hard and soft sectors are coupled for first version of scattering map

We now specialize to the first version (1.3) of the scattering maps discussed in the introduction. For this version the phase spaces Γ_- and Γ_+ are identified using the mapping φ_0 from $\mathcal{S}^-$ to $\mathcal{S}_+$ defined with null geodesics, allowing us to define the scattering map $\mathcal{S}_1$ from Γ_+ to Γ_+ . We will allow general definitions of hard and soft sectors of Γ_+ , and will not require these sectors to be symplectically orthogonal.

6.3.1 Definition of invariant interactions

The general form (3.34) of the scattering map can be written schematically as

$$\bar{y}^a = L^a_b y^b + M^a_{bcd} y^b y^c y^d + N^a_{bcde} y^b y^c y^d y^e + O(y^5), \quad (6.6)$$

where y^a are abstract phase space coordinates and the map is parameterized in terms of some phase space tensors L^a_b , M^a_{bcd} and N^a_{bcde} . Here we are using a notation where the indices $a, b, \dots$ run over the various fields and also encode the dependence of these fields on the coordinates u, θ^A , so that contractions over these indices encompass integrals over these variables. The barred coordinates $\bar{y}^a$ refer to the final data on $\mathcal{I}^+$,

$$\bar{y}^a = \left(\tilde{\Psi}^e_+, \Psi^m_+, \chi_+, \Delta \Psi^e_+, \bar{\Psi}^e_+ \right), \quad (6.7)$$

while the unbarred coordinates y^a refer to the data on $\mathcal{I}^+$ obtained from the identification map (1.2) acting on the initial data on $\mathcal{I}^-$. This mapping amounts to evaluating the initial data at $v = u$ and applying the pullback $\mathcal{P}_*$, so the functions on $\mathcal{I}^+$ are

$$y^a = \mathcal{P}_* \left(\tilde{\Psi}^e_-, \Psi^m_-, \chi_-, \Delta \Psi^e_-, \bar{\Psi}^e_- \right). \quad (6.8)$$

Finally the schematic scattering map (6.6) encodes the fact that there are no quadratic terms when using preferred asymptotic gauge, cf. eqs. (5.2) above, so the leading nonlinearities arise at cubic order.

As in the previous section the phase space coordinates can be decomposed into hard and soft components,

$$y^a = (h^A, s^\Gamma), \quad (6.9)$$

where the hard variables h^A refer to the first three fields in (6.8), which depend on u and θ^A , while the soft variables s^Γ refer to the last two fields in (6.8), which depend only on θ^A . As discussed in section 3.4, the two sectors are uncoupled in the linear order scattering map (3.23), so the tensor L^a_b is block diagonal with vanishing off-diagonal blocks:

$$L^\Gamma_A = 0, \quad L^A_\Gamma = 0. \quad (6.10)$$

The diagonal block L^A_B in the hard sector is given by eqs. (3.23b)–(3.23c), with the pullback operators $\mathcal{P}_*$ removed because of eq. (6.8), while the diagonal block L^Γ_Σ in the soft sector is similarly given by eqs. (3.23d) and (3.23e) with $\mathcal{P}_*$ removed.

As we showed in section 6.2 above, the hard and soft sectors are coupled via the higher order terms in eq. (6.6) which mix the two sectors together. Our goal here is to determine when such interactions can be removed by field redefinitions that alter the definitions of the hard and soft sectors. We first consider perturbative field redefinitions of the form

$$y^a = z^a + \Upsilon^a_{bcd} z^b z^c z^d + \Xi^a_{bcde} z^b z^c z^d z^e + O(z^5), \quad (6.11)$$

which defines new phase space coordinates z^a , together with an identical transformation for the barred variables. Here we have assumed that the transformation is the identity to linear order, as linear transformations are considered separately below. We also exclude

any quadratic terms in the transformation (6.11), since such terms would not generally maintain the property of the scattering map (6.6) of having no quadratic terms.¹⁷

Using the transformation (6.11) and its inverse, we can write the scattering map (6.6) in terms of the new phase space variables z^a . The result is

$$\bar{z}^a = \hat{L}^a_b z^b + \hat{M}^a_{bcd} z^b z^c z^d + \hat{N}^a_{bcde} z^b z^c z^d z^e + O(z^5), \quad (6.14)$$

where the transformed tensors are

$$\hat{L}^a_b = L^a_b, \quad (6.15a)$$

$$\hat{M}^a_{bcd} = M^a_{bcd} + L^a_e \Upsilon^e_{bcd} - \Upsilon^a_{efg} L^e_b L^f_c L^g_d, \quad (6.15b)$$

$$\hat{N}^a_{bcde} = N^a_{bcde} + L^a_f \Xi^f_{bcde} - \Xi^a_{fghi} L^f_b L^g_c L^h_d L^i_e. \quad (6.15c)$$

We denote by $\mathcal{V}$ the linear space of tensors (M^a_{bcd}, N^a_{bcde}) . Consider now linear maps $\ell : \mathcal{V} \rightarrow \mathbf{R}$, elements of the dual space $\mathcal{V}^*$. For example $\ell(M, N)$ could be a particular component of the tensor N . We define $\mathcal{W}_{\text{mixed}}$ to be the subspace of $\mathcal{V}^*$ that is spanned by components of M and N with both hard and soft indices, excluding the purely soft components $(M^\Sigma_{\Gamma\Delta\Upsilon}, N^\Sigma_{\Gamma\Delta\Upsilon\Lambda})$ and purely hard components (M^A_{BCD}, N^A_{BCDE}) . We will call maps ℓ in $\mathcal{W}_{\text{mixed}}$ *interactions*, since they couple the hard and soft sectors together in the dynamics. An example of such an interaction is the change in electromagnetic memory discussed in section 6.2 above. We define a linear map $\mathfrak{A} : \mathcal{V} \rightarrow \mathcal{V}$ that takes

$$\mathfrak{A} : (\Upsilon^a_{bcd}, \Xi^a_{bcde}) \rightarrow (L^a_e \Upsilon^e_{bcd} - \Upsilon^a_{efg} L^e_b L^f_c L^g_d, L^a_f \Xi^f_{bcde} - \Xi^a_{fghi} L^f_b L^g_c L^h_d L^i_e). \quad (6.16)$$

We define the subspace $\mathcal{W}_{\text{invariant}}$ of $\mathcal{V}^*$ to be the set of maps ℓ for which $\ell \circ \mathfrak{A} = 0$, which is the kernel of the transpose of $\mathfrak{A}$. The space $\mathcal{W}_{\text{invariant}}$ depends on the linear order scattering map L^a_b , for example if $L^a_b = \delta^a_b$ then it is the entire space $\mathcal{V}^*$. The key property of this definition is that nonzero interactions in $\mathcal{W}_{\text{invariant}} \cap \mathcal{W}_{\text{mixed}}$ cannot be set to zero using the field redefinitions (6.11), from eqs. (6.15) and (6.16).

¹⁷Terms of the form $\rho^a_{bc} z^b z^c$ in eq. (6.11) with ρ^a_{bc} chosen to satisfy

$$L^a_e \rho^e_{bc} = \rho^a_{ef} L^e_b L^f_c \quad (6.12)$$

will maintain the property of no quadratic terms in the scattering map. However the contributions of such terms to eqs. (6.15) vanish when $\Upsilon^a_{bcd} = \Xi^a_{bcde} = 0$, except for the contribution

$$3M^a_{f(bc}\rho^f_{de}) - 2\rho^a_{fg}M^f_{(bcd}L^g_{e)} \quad (6.13)$$

to eq. (6.15c). These terms do not contribute to the interaction (6.23). This follows from the fact that the condition (6.12) together with eq. (3.23) allow the following components of ρ to be freely specified: $\rho^1_{11}, \rho^1_{14}, \rho^1_{12}, \rho^1_{33}, \rho^1_{44}, \rho^1_{42}, \rho^1_{22}, \rho^3_{13}, \rho^3_{34}, \rho^3_{32}, \rho^4_{14}, \rho^4_{44}, \rho^4_{42}, \rho^5_{11}, \rho^5_{14}, \rho^5_{12}, \rho^5_{33}, \rho^5_{44}, \rho^5_{42}, \rho^5_{22}, \rho^5_{11}, \rho^5_{14}, \rho^2_{12}, \rho^2_{33}, \rho^2_{44}, \rho^2_{42}, \rho^2_{22}$, with the remaining components constrained to vanish except for $\rho^5_{15} = \rho^4_{14}$, $\rho^5_{45} = \rho^4_{44}/2$, $\rho^5_{52} = \rho^4_{42}$ and those determined by $\rho^a_{bc} = \rho^a_{cb}$. Here we are using the convention (6.8) for ordering the fields. Also the components $M^4_{133} = M^{\Delta\Psi^e}_{\bar{\Psi}^e\chi\chi}$ and $M^4_{533} = M^{\Delta\Psi^e}_{\bar{\Psi}^e\chi\chi}$ in eq. (6.13) vanish from eq. (I.28) and an equation analogous to eq. (J.8) at one lower order. Since general transformations can be obtained by composing those with $\rho^a_{bc} = 0$ and those with $\Upsilon^a_{bcd} = \Xi^a_{bcde} = 0$, we can without loss of generality neglect those with nonzero ρ^a_{bc} .

Turn now to linear field redefinitions of the form

$$y^a = \Omega^a_b z^b. \quad (6.17)$$

The transformed scattering map is again of the form (6.14) with the transformed tensors being

$$\hat{L}^a_b = \left(\Omega^{-1}\right)^a_c L^c_d \Omega^d_b, \quad (6.18a)$$

$$\hat{M}^a_{bcd} = \left(\Omega^{-1}\right)^a_e M^e_{fgh} \Omega^f_b \Omega^g_c \Omega^h_d, \quad (6.18b)$$

$$\hat{N}^a_{bcde} = \left(\Omega^{-1}\right)^a_f N^f_{ghij} \Omega^g_b \Omega^h_c \Omega^i_d \Omega^j_e. \quad (6.18c)$$

We restrict attention to transformations Ω^a_b which preserve the decoupling (6.10) of the two sectors at linear order:

$$\hat{L}^\Gamma_A = 0, \quad \hat{L}^A_\Gamma = 0. \quad (6.19)$$

Without loss of generality for analyzing decoupling we can assume that

$$\Omega^\Gamma_\Sigma = \delta^\Gamma_\Sigma, \quad \Omega^A_B = \delta^A_B. \quad (6.20)$$

The conditions (6.19) are then equivalent to, from eqs. (6.10) and (6.18a),

$$\Omega^A_\Sigma L^\Sigma_\Gamma = L^A_B \Omega^B_\Gamma, \quad \Omega^\Gamma_B L^B_A = L^\Gamma_\Sigma \Omega^\Sigma_A. \quad (6.21)$$

For each such map Ω^a_b , we define the map $\mathfrak{A}_\Omega : \mathcal{V} \rightarrow \mathcal{V}$ by

$$\mathfrak{A}_\Omega : (M^a_{bcd}, N^a_{bcde}) \rightarrow \left(M^a_{bcd} - \Omega^{-1a}_e M^e_{fgh} \Omega^f_b \Omega^g_c \Omega^h_d, N^a_{bcde} - \Omega^{-1a}_f N^f_{ghij} \Omega^g_b \Omega^h_c \Omega^i_d \Omega^j_e \right). \quad (6.22)$$

We define the subspace $\mathcal{W}'_{\text{invariant}}$ of $\mathcal{V}^*$ to be the set of maps ℓ for which $\ell \circ \mathfrak{A}_\Omega = 0$ for all Ω^a_b satisfying the conditions (6.20) and (6.21). Equivalently, $\mathcal{W}'_{\text{invariant}}$ is the intersection of the kernels of the transposes of the maps $\mathfrak{A}_\Omega$. From eqs. (6.18) and (6.22) we see that nonzero interactions ℓ in $\mathcal{W}'_{\text{invariant}} \cap \mathcal{W}_{\text{mixed}}$ cannot be set to zero using the field redefinitions (6.17).

We will refer to the interactions in $\mathcal{W}_{\text{invariant}} \cap \mathcal{W}'_{\text{invariant}} \cap \mathcal{W}_{\text{mixed}}$ as *invariant interactions*. These interactions are not altered by either type of transformation, linear or nonlinear. Which specific cubic and quartic interactions are invariant depends on the details of the free field scattering map L^a_b .

6.3.2 Two invariant nonzero interactions at quartic order

We next show that the two quartic interactions discussed in section 6.2 above are invariant interactions.

Consider first the change (6.1) in electromagnetic memory, specialized to the case when there is an incoming scalar field but no electromagnetic field. We show in appendix J that this quantity is of order $O(\alpha^4)$, so it contributes to the tensor N^a_{bcde} . Since the incoming scalar field χ_- is assumed to have no soft part [cf. Eq (2.18) above] the corresponding component of N is

$$\ell(M, N) = N^\Lambda_{ABCD} \quad (6.23)$$

where the Λ index corresponds to the field $\Delta\Psi^e$, and the indices A, B, C and D correspond to the field χ , from eqs. (3.34b), (3.35) and (6.9).

We now consider the action of the map (6.16) on the interaction (6.23). From eqs. (6.10), (6.16) and (6.23) we have that $\ell \circ \mathfrak{A} = 0$ if

$$L_{\Sigma}^{\Lambda} \Xi_{ABCD}^{\Sigma} = \Xi_{EFGH}^{\Lambda} L_A^E L_B^F L_C^G L_D^H, \quad (6.24)$$

where the indices Λ, A, B, C and D have the values discussed after eq. (6.23), and Ξ_{bcde}^a is the transformation tensor defined in eq. (6.11). By using the linear order scattering map (3.23) together with eq. (6.8), we see that the effect of the mappings L_A^E on the right hand side is to replace χ_- with $-\chi_-$. Similarly the mapping L_{Σ}^{Λ} on the left hand side acts as the identity map. Therefore the condition (6.24) can be written as

$$F[\chi_-] = F[-\chi_-], \quad (6.25)$$

where we have defined the quartic functional $F[\chi_-]$ to be $\Xi_{ABCD}^{\Lambda} y^A y^B y^C y^D$ with the same values of Λ and A, B, C, D . On the right hand side, the minus sign in the argument will cancel out, since the term is a quartic function of this argument. Therefore the condition (6.25) is satisfied, and so the interaction (6.23) is invariant under the perturbative redefinitions (6.11) and is an element of $\mathcal{W}_{\text{invariant}} \cap \mathcal{W}_{\text{mixed}}$.

We now turn to invariance under the linear phase space transformations (6.17). These transformations are strongly constrained by the conditions (6.20) and (6.21) and by the explicit form (3.23) of the free field scattering map. The most general linear map consistent with these conditions is¹⁸

$$\tilde{\Psi}^e \rightarrow \tilde{\Psi}^e + \beta[\Delta\Psi^e], \quad (6.26a)$$

$$\Psi^m \rightarrow \Psi^m, \quad (6.26b)$$

$$\chi \rightarrow \chi, \quad (6.26c)$$

$$\Delta\Psi^e \rightarrow \Delta\Psi^e, \quad (6.26d)$$

$$\bar{\Psi}^e \rightarrow \bar{\Psi}^e + \sigma[\tilde{\Psi}^e], \quad (6.26e)$$

where β and σ are linear functionals of their arguments. The electromagnetic memory produced by an incoming scalar field with no incoming electromagnetic field is invariant under these transformations. This follows from the fact that it is the change in the quantity $\Delta\Psi^e$, which is invariant by eq. (6.26d), and it is a functional only of χ , which is invariant by (6.26c).

Since the change in memory $\Delta Q^{\text{hard}}(\theta)$ is both generically nonzero from section 6.2 and an invariant interaction, and since it constitutes a transformation of incoming hard degrees of freedom to outgoing soft degrees of freedom, we conclude that there is an unavoidable coupling between the hard and soft sectors for the scattering map $\mathcal{S}_1$. Furthermore this interaction rules out the form (1.6) of the scattering map.

¹⁸These transformations encompass transformations caused by changes in the choice of function $g(v)$ in the definition (2.47c), and also the change of variables (2.51) used to diagonalize the Poisson brackets.

A similar argument applies to the interaction (6.5) and rules out the form (1.5) of the scattering map. By paralleling the derivation of eq. (6.25), we find that this interaction will be invariant under the perturbative redefinition (6.11) if

$$\tilde{F}[\bar{\Psi}_-, \chi_-] = -\tilde{F}[\bar{\Psi}_-, -\chi_-]. \quad (6.27)$$

Here we have defined the quartic functional $\tilde{F}[\bar{\Psi}_-, \chi_-]$ to be $\Xi_{\Lambda BCD}^A y^\Lambda y^B y^C y^D$, where the index A corresponds to the field χ_+ , Λ to the field $\bar{\Psi}_-$ and B, C and D to the field χ_- . The condition (6.27) is satisfied since $\tilde{F}$ is cubic in χ_- . Similarly one can check that the interaction (6.5) is invariant under the linear field redefinition (6.18c) using the explicit formula (6.26). Finally one can check that the conditions of footnote 17 are satisfied using the conditions on ρ_{bc}^a listed there and the properties of third order scattering

$$M_{\bar{\Psi}^e \bar{\Psi}^e \chi}^\chi = M_{\bar{\Psi}^e \bar{\Psi}^e \chi}^\chi = M_{\bar{\Psi}^e \chi \chi}^{\Delta \Psi^e} = M_{\bar{\Psi}^e \chi \chi}^{\tilde{\Psi}^e} = 0, \quad (6.28)$$

which can be derived using the methods of appendices I and J.

6.4 Hard and soft sectors are coupled for second version of scattering map

We next turn to the second version (1.4) of the scattering maps discussed in the introduction. For this case there exist definitions of the soft variables which are conserved, and it is natural to restrict to definitions of soft variables which preserve this property. We will show that in this context it is not possible to find definitions of hard and soft variables for which the scattering map has the uncoupled form (1.1).

We take our initial definition of hard and soft sectors to be given by the phase space coordinates

$$y^a = (h^A, s^\Gamma) = (\tilde{\Psi}_+^e, \Psi_+^m, \check{\chi}_+, Q_+, \Psi_+^e). \quad (6.29)$$

Here the hard variables h^A consist of the first three fields which depend on u and θ , while the soft variables s^Γ consist of the last two fields, which depend only on θ and which are conserved in the scattering as discussed in section 4.2. Note that these phase space coordinates are not symplectically orthogonal but they are independent, cf. the discussion in section 4.2 above. Also we have defined

$$\check{\chi}_+ = e^{-i\Psi_+^e} \chi_+, \quad (6.30)$$

motivated by the form (3.42) of the generating functional; this definition removes some (but not all) of the soft-hard coupling.

We denote by Σ_s the subspace of the phase space Γ_+ given by fixing values of the soft variables s . Taking the pullback of the symplectic form (2.50) to Σ_s yields a symplectic form Ω_{AB} whose inverse yields a Dirac bracket on Σ_s which we denote by $\{, \}_D$. This Dirac bracket is easiest to compute starting from the Poisson brackets of the variables (4.7). The result is that the Dirac brackets of the hard variables $(\tilde{\Psi}_+^e, \Psi_+^m, \check{\chi}_+)$ coincide with the Poisson brackets (2.52) (with χ_+ replaced by $\check{\chi}_+$) except for the bracket

$$\left\{ D^2 \tilde{\Psi}_+^e(u, \theta), \check{\chi}_+(u', \theta') \right\}_D = ie^2 g(u) \delta^{(2)}(\theta, \theta') \check{\chi}_+(u', \theta'). \quad (6.31)$$

Since the soft variables s are conserved, the restriction of the scattering map $\mathcal{S}_2$ to Σ_s gives a map $\mathcal{S}_s : \Sigma_s \rightarrow \Sigma_s$, which we can write as

$$h^A \rightarrow \bar{h}^A(h, s). \quad (6.32)$$

When the right hand side is independent of s then the two sectors are uncoupled. If we make an s -dependent change of the coordinates h^A on Σ_s (effectively changing the definition of the hard sector), the effect is to make the replacement

$$\mathcal{S}_s \rightarrow \bar{\mathcal{S}}_s = \varphi_s \circ \mathcal{S}_s \circ \varphi_s^{-1}, \quad (6.33)$$

for some diffeomorphism $\varphi_s : \Sigma_s \rightarrow \Sigma_s$. We do not require that φ_s be a symplectomorphism. We would like to show that there is no choice of diffeomorphism that removes the s dependence from the right hand side of eq. (6.32). We will show that attempting to derive such a diffeomorphism runs into the same kind of obstacle as in section 4.2 above: one obtains a unique result but it is not a diffeomorphism, since the resulting coordinates are not independent.

To show that no such diffeomorphism exists it is sufficient to work to linear order in s and to linear order in the deviation of $\mathcal{S}_s$ from the identity map (appropriate to the small h limit), since if it is impossible to remove the coupling in this regime then it is impossible in general. Within this limit we can write the mapping (6.32) as

$$h^A \rightarrow h^A + \xi^A(h) + v^A(h, s), \quad (6.34)$$

where ξ^A is independent of s and v^A is linear in s . The effect of the transformation (6.33) to linear order is to add to the right hand side of eq. (6.34) a term $(\mathcal{L}_{\vec{\eta}} \vec{\xi})^A$ for some vector field η^A on Σ_s . Thus the condition for the dependence on s to be removable is that there exists a η^A which depends linearly on s for which

$$\vec{v} = -\mathcal{L}_{\vec{\eta}} \vec{\xi}. \quad (6.35)$$

Next, the generating function (3.42) for the scattering map can be written in terms of the coordinates (6.29) used in this section as

$$\mathcal{G} \left[\tilde{\Psi}_+^e + ge^2 D^{-2} Q_+ - ge^2 D^{-2} Q^{\text{hard}}[\check{\chi}_+], \Psi_+^m, \check{\chi}_+ \right], \quad (6.36)$$

from eqs. (2.47), (3.24), (3.29) and (6.30). Note that this expression depends on only one of the two soft variables, Q_+ . Evaluating this generating functional at $Q_+ = 0$ and inserting into the linearized transformation¹⁹ (3.41) yields the vector field ξ^A . Evaluating the piece of $\mathcal{G}$ linearized in Q_+ , which is

$$e^2 \int du \int d^2 \Omega g(u) \frac{\delta \mathcal{G}}{\delta \tilde{\Psi}_+^e(u, \theta)} D^{-2} Q_+(\theta), \quad (6.37)$$

¹⁹Using the Dirac brackets discussed around eq. (6.31).

gives instead the vector field v^A . Note that the quantity (6.37) is generally nonzero, by eqs. (3.33), (3.36), (G.6c) and the result of section 6.2. With these inputs we find that the solution to eq. (6.35) for η^A is

$$\vec{\eta} = -e^2 \int du \int d^2\Omega g(u) [D^{-2}Q_+(\boldsymbol{\theta})] \frac{\delta}{\delta \tilde{\Psi}_+^e(u, \boldsymbol{\theta})}. \quad (6.38)$$

However the corresponding coordinate transformation on Σ_s is similar to eq. (4.7a), and leads to a set of coordinates which are not independent. Thus, just as in section 4.2, it is not possible to decouple the two sectors.

7 Discussion and conclusions

In this paper, we computed the perturbative classical scattering map from $\mathcal{I}^-$ to $\mathcal{I}^+$ of electromagnetism coupled to a massless, charged scalar field in four-dimensional Minkowski spacetime. We consider various definitions hard and soft sectors of the theory, and adopt two different conventions for how to ensure that the same definitions of hard and soft are used at $\mathcal{I}^-$ and at $\mathcal{I}^+$. For both conventions, we showed that the soft and hard sectors of the theory evolve independently to quadratic order, but that at higher orders the two sectors are coupled. In the first convention, this coupling between the hard and soft sectors cannot be eliminated using linear or perturbative redefinitions of the two sectors, which includes any possible “dressing” of the hard degrees of freedom. In the second convention, the coupling cannot be eliminated by any field redefinitions which preserve the fact that the soft variables are conserved by the scattering process.

Our conclusions disagree with those of ref. [33], which argued that there exist definitions of the two sectors for which there is an exact decoupling. We showed that the explanation for the disagreement is a property of symplectic geometry which holds in finite dimensions but fails in infinite dimensions.

We expect this nontrivial coupling of soft and hard sectors to generalize to nonabelian gauge theories and to general relativity, and to the quantum as well as classical regimes.²⁰ As a result, the infinitely many conserved soft hair charges should yield non-trivial constraints on gravitational scattering and on black hole formation and evaporation, as argued in refs. [23, 24].

To probe this issue further, it would be interesting to study the quantization of the gauge theory studied in this paper, while carefully accounting for the soft degrees of freedom. A recent detailed study of infrared issues in scattering by Prabhu, Satishchandran and Wald (PSW) [42] shows that for massless fields it is not possible to find in and out Hilbert spaces that accommodate nontrivial changes in memory. However, as mentioned in section 2.5 above, PSW start from an algebra of observables which is different from the algebra obtained from our phase space Γ , due to our inclusion of the edge mode (2.51) that is canonically conjugate to memory which PSW exclude as being pure gauge. It would be interesting to study the consequences for the analysis of PSW of adopting the different algebra used here.

²⁰As discussed in section 4.3, the statement that the quantum theory does not admit a decomposition into two decoupled sectors is not in conflict with the factorization theorems that constrain the soft behavior of S matrix elements [36, 37].

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A Symplectically orthogonal hard and soft sectors exclude certain forms of scattering map

In this appendix we show that a scattering map $\mathcal{S} : \Gamma \rightarrow \Gamma$ on a phase space Γ cannot have the forms (1.5) or (1.6) if the soft and hard variables are symplectically orthogonal. We assume a finite dimensional phase space although we expect the result to generalize.

By assumption the phase space can be expressed as a product of hard and soft factors $\Gamma = \Gamma_{\text{soft}} \times \Gamma_{\text{hard}}$, and we denote by $y^a = (s^A, h^\Gamma)$ corresponding coordinates. Defining $\Omega^{ab} = \{y^a, y^b\}$, we have by assumption that

$$0 = \{s^A, h^\Gamma\} = \Omega^{ab} \partial_a s^A \partial_b h^\Gamma. \quad (\text{A.1})$$

Applying the pullback operator $\mathcal{S}_*$ to this equation gives

$$(\mathcal{S}_* \Omega)^{ab} \partial_a (\mathcal{S}_* s^A) \partial_b (\mathcal{S}_* h^\Gamma) = 0. \quad (\text{A.2})$$

We now use the fact that the scattering map is a symplectomorphism so that $\mathcal{S}_* \Omega^{ab} = \Omega^{ab}$, and use the definitions $\bar{s}^A = \mathcal{S}_* s^A = s^A \circ \mathcal{S}$, $\bar{h}^\Gamma = \mathcal{S}_* h^\Gamma = h^\Gamma \circ \mathcal{S}$. This gives $\Omega^{ab} \partial_a \bar{s}^A \partial_b \bar{h}^\Gamma = 0$, and expanding this out and using eq. (A.1) again gives

$$\Omega^{BC} \frac{\partial \bar{s}^A}{\partial s^B} \frac{\partial \bar{h}^\Gamma}{\partial s^C} = -\Omega^{\Sigma\Lambda} \frac{\partial \bar{s}^A}{\partial h^\Sigma} \frac{\partial \bar{h}^\Gamma}{\partial h^\Lambda}. \quad (\text{A.3})$$

We now assume a scattering map of the form (1.5), which implies that

$$\frac{\partial \bar{h}^\Gamma}{\partial s^C} = 0, \quad (\text{A.4})$$

so the left hand side of eq. (A.3) vanishes. On the right hand side the first factor is invertible by eq. (A.1) since Ω^{ab} is invertible, and similarly the third factor is invertible by eq. (A.4). This implies that

$$\frac{\partial \bar{s}^A}{\partial h^\Sigma} = 0, \quad (\text{A.5})$$

which contradicts the assumed form (1.5). A similar argument excludes the form (1.6).

B Free Lorenz gauge solutions with nontrivial soft charges

In this appendix we review the solutions of the free, noninteracting theory which satisfy everywhere the Lorenz gauge condition. We show that these solutions satisfy the gauge-invariant asymptotic properties at i^+ and i^- given in eqs. (2.21) and (2.22), but fail to satisfy the matching condition (2.39) [cf. eq. (B.11) below]. We then generalize the solutions to a larger class which satisfy Lorenz gauge only asymptotically, and deduce the general form of the Lorenz gauge scattering map for free solutions.

B.1 Global Lorenz gauge

In inertial coordinates (t, x^i) the homogeneous global Lorenz gauge solutions are

$$\underline{A}_0 = 0, \quad (\text{B.1a})$$

$$\begin{aligned} \underline{A}_i = \sum_{l \geq 0} \left\{ \partial_{ijL} \left[\frac{D_{jL}(t-r)}{r} - \frac{D_{jL}(t+r)}{r} \right] - \partial_L \left[\frac{D''_{iL}(t-r)}{r} - \frac{D''_{iL}(t+r)}{r} \right] \right\} \\ + \sum_{l \geq 0} \epsilon_{ipq} \partial_{pL} \left[\frac{C_{qL}(t-r)}{r} - \frac{C_{qL}(t+r)}{r} \right]. \end{aligned} \quad (\text{B.1b})$$

The notation here follows ref. [65] and is as follows. The symbol L is the multi-index $L = (i_1, i_2, \dots, i_l)$, and D_{iL} , C_{iL} are Cartesian tensors which are symmetric and trace-free on all of their indices. The symbol ∂_L means $\partial_{i_1} \dots \partial_{i_l}$. Here and throughout underlines mean that the corresponding quantities are in Lorenz gauge. Note that, for a given solution, the tensors D_{iL} and C_{iL} are not unique, since the solution (B.1) is invariant under transformations of the form

$$D_{iL}(x) \rightarrow D_{iL}(x) + \delta D_{iL}(x), \quad C_{iL}(x) \rightarrow C_{iL}(x) + \delta C_{iL}(x), \quad (\text{B.2})$$

for $l \geq 0$, where δD_{iL} and δC_{iL} are a polynomials in x of degree $2l + 2$.

We will restrict attention to solutions for which the asymptotic behavior of the symmetric tracefree tensors D_{iL} and C_{iL} as $x \rightarrow \infty$ is given by

$$D_{iL}(x) = \tilde{D}_{+iL}(x) + P_{+iL}(x), \quad C_{iL}(x) = \tilde{C}_{+iL}(x) + Q_{+iL}(x), \quad (\text{B.3})$$

where $\tilde{D}_{+iL}, \tilde{C}_{+iL}$ go to 0 as $x \rightarrow \infty$, and P_{+iL} and Q_{+iL} are polynomials in x of degree $l + 2$ and $l + 1$ respectively. Similarly as $x \rightarrow -\infty$ we require that

$$D_{iL}(x) = \tilde{D}_{-iL}(x) + P_{-iL}(x), \quad C_{iL}(x) = \tilde{C}_{-iL}(x) + Q_{-iL}(x), \quad (\text{B.4})$$

where $\tilde{D}_{-iL}, \tilde{C}_{-iL} \rightarrow 0$ as $x \rightarrow -\infty$, and P_{-iL} and Q_{-iL} are again polynomials in x of degree $l + 2$ and $l + 1$ respectively. Because of the invariance property (B.2), the solution (B.1) depends only on the differences $\Delta P_{iL} = P_{+iL} - P_{-iL}$ and $\Delta Q_{iL} = Q_{+iL} - Q_{-iL}$ between these polynomials and not on $P_{\pm iL}$ or $Q_{\pm iL}$ individually. The assumptions (B.3) and (B.4) are compatible with the large r field expansions (2.6) and (2.14) that we have assumed,²¹ and also yield solutions with finite energy.

²¹This would no longer be true if the polynomials $P_{\pm iL}$ and $Q_{\pm iL}$ were of higher degree than $l + 2$ and $l + 1$.

The coefficients of the large r expansions of the fields near $\mathcal{I}^+$ are given by $\underline{\mathcal{A}}_{+u} = \underline{\mathcal{A}}_{+r} = 0$, and

$$\underline{\mathcal{A}}_{+A}(u, \theta) = \sum_l (-1)^{l+1} n^L e_A^i \left[\tilde{D}_{+iL}^{(l+2)}(u) + \epsilon_{ipq} n^p \tilde{C}_{+qL}^{(l+1)}(u) \right], \quad (\text{B.5})$$

where $n^i = x^i/r$, $e_A^i = D_A n^i$, $n^L = n^{i_1} \dots n^{i_l}$, and the superscripts $(l+2)$ and $(l+1)$ indicate the number of derivatives taken. This expression satisfies the condition

$$\underline{\mathcal{A}}_{+A} \rightarrow 0 \quad (\text{B.6})$$

as $u \rightarrow \infty$ at i^+ , which yields the condition (2.22c), from eq. (2.7d). In the other limit $u \rightarrow -\infty$ at i^0 we have, from eqs. (B.3) and (B.4),

$$\underline{\mathcal{A}}_{+A}(u, \theta) \rightarrow \underline{\mathcal{A}}_{+A}(\theta) = \sum_l (-1)^l n^L e_A^i \left[\Delta P_{iL}^{(l+2)} + \epsilon_{ipq} n^p \Delta Q_{qL}^{(l+1)} \right]. \quad (\text{B.7})$$

Here the right hand side is independent of u since ΔP_{iL} is a polynomial of order $l+2$ and ΔQ_{qL} is a polynomial of order $l+1$.

Similar results apply for the limiting behavior of the solutions near $\mathcal{I}^-$. The expansion coefficients are $\underline{\mathcal{A}}_{-v} = \underline{\mathcal{A}}_{-r} = 0$, and

$$\underline{\mathcal{A}}_{-A}(v, \theta) = \sum_l n^L e_A^i \left[\tilde{D}_{-iL}^{(l+2)}(v) - \epsilon_{ipq} n^p \tilde{C}_{-qL}^{(l+1)}(v) \right]. \quad (\text{B.8})$$

This satisfies the condition as $v \rightarrow -\infty$ at i^-

$$\underline{\mathcal{A}}_{-A} \rightarrow 0, \quad (\text{B.9})$$

yielding the condition (2.21c), while as $v \rightarrow \infty$ at i^0 we have, from eqs. (B.3) and (B.4)

$$\underline{\mathcal{A}}_{-A}(v, \theta) \rightarrow \underline{\mathcal{A}}_{-A}(\theta) = \sum_l n^L e_A^i \left[\Delta P_{iL}^{(l+2)} - \epsilon_{ipq} n^p \Delta Q_{qL}^{(l+1)} \right]. \quad (\text{B.10})$$

One can check that these Lorenz gauge solutions (B.1) satisfy the matching conditions (2.23a) and (2.23b) [or equivalently (2.38)] discussed in the body of the paper. Note, however, that they do not satisfy the matching condition (2.39), instead this relation is satisfied with a sign flip, since from eqs. (B.7) and (B.10) we have

$$\underline{\mathcal{A}}_{+A} = -\mathcal{P}_* \underline{\mathcal{A}}_{-A}. \quad (\text{B.11})$$

This is discussed further in section 3.2 above.

By expanding the solution (B.1) to subleading order in $1/r$ near $\mathcal{I}^+$ and $\mathcal{I}^-$ and reading off the subleading coefficients $\hat{\mathcal{A}}_{+u}$, $\hat{\mathcal{A}}_{+A}$, $\hat{\mathcal{A}}_{-v}$ and $\hat{\mathcal{A}}_{-A}$, one can verify the limiting behavior at i^- and i^+ given by eqs. (2.21a), (2.21b), (2.22a) and (2.22b). Similar analyses for free massless scalar field solutions establishes (2.21d) and (2.22d).

B.2 Asymptotic Lorenz gauge

The solutions (B.1) are the most general free solutions that obey the Lorenz gauge condition everywhere in spacetime. However one can obtain a more general class of solutions in *asymptotic Lorenz gauge*, in which one imposes the Lorenz gauge condition only at $r \geq R$ for some R . Specifically for $r \geq R$ one can transform the solutions according to

$$\underline{A}_a \rightarrow \underline{A}_a + \nabla_a \varepsilon, \quad \underline{\Phi} \rightarrow e^{i\varepsilon} \underline{\Phi}, \quad \varepsilon = \sum_{l \geq 1} D_L \partial_L \left(\frac{u^l + v^l}{r} \right), \quad (\text{B.12})$$

where D_L for $l \geq 1$ are some constant traceless symmetric tensors. For $r < R$ one can use any smooth extension of ε . This transformation preserves the asymptotic gauge conditions (2.34), and the initial and final data transform as $\underline{A}_{+A} \rightarrow \underline{A}_{+A} + D_A \varepsilon_+$, $\underline{A}_{-A} \rightarrow \underline{A}_{-A} + D_A \varepsilon_-$, where $\varepsilon_+(\theta)$ is a freely specifiable function with only $l \geq 1$ components²² and $\varepsilon_- = \mathcal{P}_* \varepsilon_+$. The transformation is of the even form (2.35), and thus is not gauge but instead is a mapping from solutions to physically distinct solutions, as discussed in section 2.4.

The general asymptotic Lorenz gauge solutions will no longer satisfy the conditions (B.6) and (B.9) at past and future timelike infinity. However there is a combination of these conditions which is unaffected by the transformation (B.12), and which is still valid for the general solutions, namely

$$\underline{\Psi}_+^e = \mathcal{P}_* \underline{\Psi}_-^e. \quad (\text{B.13})$$

B.3 Scattering map for free solutions

The scattering map that relates the initial data on $\mathcal{I}^-$ to the final data on $\mathcal{I}^+$ for the free global Lorenz gauge solutions can be read off from eqs. (B.5) and (B.8) and their analogs for scalar fields, and is

$$\underline{\chi}_+(u, \theta) = -\mathcal{P}_* \underline{\chi}_-(u, \theta), \quad (\text{B.14a})$$

$$\underline{A}_{+A}(u, \theta) = \mathcal{P}_* \left[\underline{A}_{-A}(u, \theta) - \underline{A}_{+A}(\theta) \right]. \quad (\text{B.14b})$$

This scattering map can be generalized to the class of asymptotic Lorenz gauge solutions generated by the transformation (B.12) by expressing the gauge transformation function ε_+ in terms of the new initial data. Writing the result in terms of the potentials (2.36), making use of the vanishing magnetic charges condition (2.41) and using $\mathcal{P}_* \varepsilon_{AB} = -\varepsilon_{AB}$ gives

$$\underline{\Psi}_+^e(u, \theta) = \mathcal{P}_* \left[\underline{\Psi}_-^e(u, \theta) - \underline{\Psi}_+^e(\theta) + \underline{\Psi}_-^e(\theta) \right], \quad (\text{B.15a})$$

$$\underline{\Psi}_+^m(u, \theta) = -\mathcal{P}_* \underline{\Psi}_-^m(u, \theta), \quad (\text{B.15b})$$

$$\underline{\chi}_+(u, \theta) = -\mathcal{P}_* \underline{\chi}_-(u, \theta). \quad (\text{B.15c})$$

C Properties of interacting Lorenz gauge solutions

In this appendix we deduce some properties of nonlinear Lorenz gauge solutions. Consider first solutions in global Lorenz gauge. We claim that the conditions (B.6) and (B.9) at

²²This function is given explicitly in terms of the coefficients D_L in eq. (B.12) by $\varepsilon_+ = \sum_{l \geq 1} c_l D_L n^L$ with $c_l = 2 \sum_{i=0, i \text{ even}}^l 2(i-1)(i-3) \dots (i-2l+1)$.

future and past timelike infinity are still satisfied by these solutions. To see this, consider the version of the theory on the Einstein static universe obtained by making a conformal transformation. In this version spacetime is compact and all the fields are bounded. Consider now a small neighborhood $\mathcal{V}$ of future timelike infinity i^+ . At each order in perturbation theory, one can obtain the fields inside $\mathcal{V}$ by specifying the initial conditions on the Cauchy surface obtained by taking the intersection of $\partial\mathcal{V}$ with the image of Minkowski spacetime on the Einstein static universe cylinder. The fields inside $\mathcal{V}$ (and in particular the limit to i^+ of the fields on $\mathcal{I}^+$) are given as a sum of a homogeneous solution determined by the initial data, and an inhomogeneous solution determined by the sources inside $\mathcal{V}$ with zero initial data. However, we know from appendix B that the homogeneous solutions must satisfy the vanishing condition (B.6) at i^+ , since they are free Lorenz gauge solutions. The inhomogeneous solution can give a nonvanishing contribution to the limit, however this can be made arbitrarily small by taking the size of the neighborhood to zero, since the sources are bounded. We conclude that the conditions (B.6) and (B.9) are satisfied.

Just as for the free solutions, more general nonlinear solutions in asymptotic Lorenz gauge obtained from the transformation (B.12) will no longer satisfy the conditions (B.6) and (B.9), but will satisfy the condition (B.13).

D Presymplectic form is independent of choice of Cauchy surface

In the body of the paper, we showed that the presymplectic form (2.26) is independent of the choice of Cauchy surface Σ , for the cases $\Sigma = \mathcal{I}^-$ and $\Sigma = \mathcal{I}^+$, and for field perturbations of the special form (2.29) given the condition (2.39). In this appendix we review and modify slightly the results of Campiglia and Eyheralde (CE) [57], that give sufficient conditions for extending the independence of Cauchy surface to arbitrary Cauchy surfaces and to arbitrary field perturbations. A closely related treatment has been given by Henneaux and Troessaert [66].

Consider first spacelike Cauchy surfaces Σ that limit to spatial infinity i^0 . We denote by $\mathcal{H}$ the unit hyperboloid of the space of directions of approach to the point of spatial infinity. This hyperboloid acts like a spacetime boundary, and the presymplectic form Ω_Σ can potentially depend on the Cauchy surface Σ through the location of the two-surface $\mathcal{S}$ where Σ intersects $\mathcal{H}$. Under the appropriate assumptions of the asymptotic fall offs of the fields near $\mathcal{H}$ [eq. (4.2) of CE]), one can compute the pullback θ of the presymplectic potential of the theory to $\mathcal{H}$. This pullback can be decomposed as

$$\theta = -\delta\ell + d\beta + \mathcal{E}, \quad (\text{D.1})$$

the sum of a total variation boundary term $-\delta\ell$, an exact corner term $d\beta$, and a residual term $\mathcal{E}$ called the flux term. The general theory for obtaining a presymplectic form that is independent of Cauchy surface [46, 57] consists of (i) Modifying the definition (2.26) of the presymplectic form to exclude the corner term,

$$\Omega_\Sigma = \delta \int_\Sigma \theta - \delta \int_{\mathcal{S}} \beta; \quad (\text{D.2})$$

and (ii) Specializing the decomposition (D.1) and the definition of the field configuration space to make the flux term vanish.

CE find a nonzero flux term in general, the second term in their eq. (4.5). This term can be eliminated by specializing the definition of the field configuration space. CE impose an asymptotic Lorenz gauge condition, their eq. (4.8). However that Lorenz gauge condition is incompatible with our condition (2.39), and instead yields eq. (2.39) with a sign flip [cf. eq. (B.11) above]. Therefore we instead impose a modified version of their eq. (4.8) with the first term omitted. That modified condition can always be enforced by making an appropriate “gauge transformation” of the form of their eq. (2.38), although as explained after eq. (2.39) above the transformation should not be thought of as a gauge specialization, and instead should be thought of as a specialization of the definition of the field configuration space. Our modified version of their condition (4.8) eliminates the flux term and also implies eq. (2.39).²³

CE also find a nonzero corner term proportional to the pullback to $\mathcal{H}$ of $\epsilon_{abcd}A^{[c}\delta A^{d]}$, their eq. (4.6).²⁴ With the corner term subtraction included, the presymplectic form (D.2) is independent of the choice of spacelike Cauchy surface Σ that limits to i^0 . The corner term on $\mathcal{H}$ vanishes when one takes the limit to the boundaries of $\mathcal{H}$, using their eqs. (4.8) [the modified version], (4.6), (3.9), (B.2), footnote 3, and assuming the specialization given in their footnote 9. It also vanishes on null infinity, with the gauge specializations (2.34). It follows that the corner term subtraction does not affect the presymplectic forms $\Omega_{\mathcal{J}^-}$ and $\Omega_{\mathcal{J}^+}$, and that $\Omega_{\mathcal{J}^-}$ and $\Omega_{\mathcal{J}^+}$ coincide with the common value of Ω_Σ for all the spacelike Cauchy surfaces Σ .

E Field configuration space in the magnetic sector

In this appendix we discuss a difference between the space of solutions for the electric potential Ψ_-^e and the magnetic potential Ψ_-^m . For the class of Lorenz gauge free solutions discussed in appendix B these potentials vanish at i^- , from eq. (B.9):

$$\underline{\Psi}_-^e = 0, \quad \underline{\Psi}_-^m = 0, \quad (\text{E.1})$$

and this generalizes to the solutions of the interacting theory. However, by using the transformation freedom (2.35a) we can make Ψ_-^e be nonzero, from eq. (2.37a). Thus in the full space of solutions with our preferred asymptotic gauge conditions (2.34) and (2.39) [which do not include Lorenz gauge] we have that both Ψ_-^e and Ψ_-^e are nonzero in general.

For the magnetic variables the story is somewhat different. Starting from the class of solutions which satisfy (E.1), one can attempt to obtain a larger class of solutions by analogy with the procedure for the electric variables, by making a transformation of the initial data on $\mathcal{J}^-$ of the form

$$\Psi_-^e \rightarrow \Psi_-^e, \quad \Psi_-^m \rightarrow \Psi_-^m + \tilde{\epsilon}_-, \quad (\text{E.2})$$

²³This follows from the fact that, in their notation, the even parity piece of A_α^0 is of the form $D_\alpha \lambda$ for some λ with $D_\alpha D^\alpha \lambda = 0$, and their eq. (3.2).

²⁴Note that they use different terminology and call it a boundary term, and they also work with $\omega = \delta\theta$ rather than θ .

where $\tilde{\varepsilon}_-$ is a function on $\mathcal{I}^-$ with no $l = 0$ component that is independent of v . Although the transformation (E.2) is not a gauge transformation, it is a kind of magnetic analog of the gauge transformation given by eqs. (2.37) [56]. By solving the equations of motion one can determine the effects of this transformation of the initial data on the entire solution, and in particular the data on $\mathcal{I}^+$ transforms as [cf. eq. (2.35a) above]

$$\Psi_+^e \rightarrow \Psi_+^e, \quad \Psi_+^m \rightarrow \Psi_+^m + \tilde{\varepsilon}_+, \quad (\text{E.3})$$

where $\tilde{\varepsilon}_+ = \mathcal{P}_* \tilde{\varepsilon}_-$. The magnetic charges

$$\tilde{Q}_{\tilde{\varepsilon}} = \frac{1}{e^2} \int d^2\Omega \tilde{\varepsilon}_+ \mathcal{F}_{+AB} = \frac{1}{e^2} \int d^2\Omega \tilde{\varepsilon}_- \mathcal{F}_{-AB} \quad (\text{E.4})$$

associated with these transformations [cf. eq. (2.23b) above] can be derived in exact parallel with the derivation for the electric case discussed in section 2.4 [56]. However, the solutions generated by the transformations (E.2) are generically singular in the interior of the spacetime. One example of such a solution is a static magnetic dipole at the origin $r = 0$. In this paper we restrict attention to initial data on $\mathcal{I}_-$ which evolves into smooth solutions in the interior of the spacetime, which requires that

$$\Psi_-^m = 0, \quad (\text{E.5})$$

and disallows the transformations (E.2). A similar analysis at $\mathcal{I}^+$ yields the condition

$$\Psi_+^m = 0. \quad (\text{E.6})$$

The other limits Ψ_+^m and Ψ_-^m of the magnetic potentials at spatial infinity are generally nonzero (see appendix B). However in this paper we restrict attention to the sector of the theory in which they vanish, as discussed around eq. (2.41).

F Asymptotic edge modes

F.1 Equivalence of soft degrees of freedom and edge modes

In the literature on soft charges it has been conjectured [1] that soft modes can be alternatively described in the language of edge modes used in refs. [48–51]. In this appendix we demonstrate the equivalence explicitly by giving an alternative construction of the phase space and symplectic product of section 2.5 using edge modes. A general treatment of edge modes in electromagnetism at finite boundaries has been given by Freidel and Pranzetti [52].

We slightly generalize the treatment in the body of the paper by considering a region $\mathcal{R}$ of $\mathcal{I}^-$ of points with $v_1 \leq v \leq v_2$. At the end we will specialize to $v_1 = -\infty$, $v_2 = \infty$, the case treated in the body of the paper. The presymplectic product evaluated on this region is not gauge invariant: the derivation that led to eq. (2.31b) gives in this context

$$\Omega_{\mathcal{R}}(\delta A_a, \delta \Phi; \delta_{\varepsilon} A_a, \delta_{\varepsilon} \Phi) = \frac{1}{e^2} \int d^2\Omega [\delta \mathcal{F}_{-rv} \varepsilon_-]_{v_1}^{v_2}. \quad (\text{F.1})$$

Now the field component at the boundary $v = v_1$ is given by, from eq. (2.21a) and an equation analogous to eq. (2.10a),

$$\mathcal{F}_{-rv}(v_1) = \int_{-\infty}^{v_1} dv \left[e^2 \mathcal{J}_{-v} - D^A \mathcal{F}_{-Av} \right]. \quad (\text{F.2})$$

If we now vary just the fields within the region $\mathcal{R}$, as is relevant for the presymplectic form (F.1), the result vanishes, since the field $\mathcal{F}_{-rv}$ at v_1 depends only on fields outside the region:

$$\delta \mathcal{F}_{-rv}(v_1) = 0. \quad (\text{F.3})$$

Hence the one of the two terms in (F.1) vanishes. However the second term at $v = v_2$ is generally nonzero and so the presymplectic form is not gauge invariant.

The key idea of edge modes is to use the Stueckelberg trick²⁵ to restore gauge invariance by introducing new degrees of freedom [48–50]. We introduce a U(1) field $e^{ie^2\lambda}$ on the boundary at $v = v_2$, which transforms under gauge transformations as $\lambda \rightarrow \lambda + \varepsilon_-/e^2$. We add to the presymplectic form $\Omega_{\mathcal{R}}$ a term

$$\int d^2\Omega \left[\delta_1 \lambda \delta_2 \mathcal{F}_{-vr}(v_2) - \delta_2 \lambda \delta_1 \mathcal{F}_{-vr}(v_2) \right], \quad (\text{F.4})$$

so that the total presymplectic form is now gauge invariant.

We next specialize the gauge by using the gauge function ε_- to set $\mathcal{A}_{-v}$ to zero, using the analog of eq. (2.12a). There is a residual gauge freedom given by

$$\varepsilon_- = \varepsilon_-(\theta). \quad (\text{F.5})$$

We now follow the derivation given in the body of the paper that leads to the form (2.50) of the presymplectic form, but keeping the extra term (F.4). The final result is

$$\begin{aligned} \Omega_{\mathcal{R}} = & -\frac{1}{e^2} \int dv \int d^2\Omega \left[\partial_v \tilde{\Psi}_{1-}^e D^2 \tilde{\Psi}_{2-}^e + \partial_v \Psi_{1-}^m D^2 \Psi_{2-}^m - (1 \leftrightarrow 2) \right] \\ & - \frac{1}{e^2} \int d^2\Omega \left\{ D^2 \Delta \Psi_{1-}^e \left[\bar{\Psi}_{2-}^e - \lambda_2 + 2 \int dv g' \tilde{\Psi}_{2-}^e \right] - (1 \leftrightarrow 2) \right\} \\ & + \int dv \int d^2\Omega \left[\partial_v \delta_1 \chi_-^* \delta_2 \chi_- + \partial_v \delta_1 \chi_- \delta_2 \chi_-^* - (1 \leftrightarrow 2) \right]. \end{aligned} \quad (\text{F.6})$$

We may now use the residual gauge freedom (F.5) (which was not a gauge freedom in the body of the paper) to set to zero the variable $\bar{\Psi}_{2-}^e$. Then the phase space and presymplectic form (F.6) coincide exactly with the version (2.50) from the body of the paper (when we specialize to $v_1 = -\infty$, $v_2 = \infty$), except that the variable $\bar{\Psi}_{-}^e$ has been replaced by $-\lambda$. Thus we see that of the two canonically conjugate variables $\Delta \Psi_{-}^e$ and $\bar{\Psi}_{-}^e$ that comprise the soft modes, one can be interpreted as an edge mode at the future boundary at $v = v_2 = \infty$ of $\mathcal{I}^-$.

²⁵A nice review of this technique and its applications in gauge theory as well as gravitational theories can be found in ref. [67].

F.2 Necessity of including edge modes

As discussed in the introduction and in section 2.5, there is some disagreement in the literature on the necessity of including asymptotic edge modes. Ashtekar [39, 40] eliminates these modes by imposing $\Psi_{\pm}^e = \Psi_{\pm}^e = 0$ in the definition of the field configuration space. Bousso and collaborators argue in the gravitational case that the edge modes are unobservable [33, 53]. The issue arises in a different guise in the algebraic approach to the scattering problem, where one focuses on constructing the algebra of observables. The recent analysis of infrared scattering by Prabhu, Satishchandran and Wald [42] is based on an algebra obtained by excluding the edge modes. However that algebra could be modified to incorporate them.²⁶ Finally, the edge modes are included in the treatments by Strominger and collaborators [1, 47].

One argument for including edge modes is that they are required in order for there to be an action on the phase space of the large gauge transformations (2.35a) associated with the new conserved charges (2.40) [1, 47].

Another argument is that it is not possible to eliminate edge modes using a gauge transformation (since the corresponding transformations are not degeneracy directions of the presymplectic form and are thus not true gauge transformations). However, they can be instead eliminated by specializing the definition of the field configuration space.

The strongest argument for including edge modes is as follows. We will give the argument for the gravitational case. The edge modes (or soft hair) in that case are observable only relative to a choice of convention, a BMS frame, equivalent to the choice of zero readings on a set of clocks on an asymptotic two sphere. This is the reason that the edge modes are sometimes argued to be unobservable [33, 53]. However, the same non-observability argument would also apply to the direction in space of the angular momentum of a stationary gravitational source, which is observable only relative to a choice of convention. One could make a diffeomorphism to fix the direction of angular momentum to be in the z direction. However, quantum fluctuations in the direction of angular momentum have physical consequences, and cannot be correctly described in a phase space where angular momentum always points in the z direction. By analogy, one expects quantum fluctuations in gravitational edge modes to have physical consequences, and they would not be correctly described in a phase space in which the edge modes are fixed.

Finally we note that the results of this paper are unchanged if the edge modes are excluded. This can be seen as follows. The general form of the scattering map (3.34) transforms appropriately under the transformation freedom (3.27) and (3.28). Eliminating the edge modes $\bar{\Psi}_{-}^e$ and $\bar{\Psi}_{+}^e$ fixes this transformation freedom and also eliminates eq. (3.34b). However the soft-hard coupling discussed in section 6 is still present in the second term on the right hand side of eq. (3.34a), specifically in the difference between its $u \rightarrow \pm\infty$ limits. As in section 6 this soft-hard coupling cannot be eliminated.

²⁶Specifically the algebra of observables constructed in their section 4.2 could be extended by including the additional quantity (2.51) (suitably smeared over angles), which satisfies the requirement of their eq. (2.5).

G Derivation of generating functional representation of scattering map

In this appendix we derive some properties of the generating functional G for the scattering map which were discussed in section 3.6.

First, in order for the scattering map $\mathcal{S}$ to be of the specific form (3.34) derived in section 3.5, the generating functional G must be of the form

$$G[\tilde{\Psi}_+^e, \Psi_+^m, \chi_+, \Delta\Psi_+^e, \hat{\Psi}_+^e] = \mathcal{G}[\check{\Psi}_+^e, \Psi_+^m, e^{-i\bar{\Psi}_+^e} e^{i\Delta\Psi_+^e} \chi_+] \quad (\text{G.1})$$

for some functional $\mathcal{G}$, where $\check{\Psi}_+^e$ and $\bar{\Psi}_+^e$ are defined in terms of the fields on the left hand side by eqs. (2.51) and (3.29). In deriving the representation (G.1) we used the decomposition (3.39) of the scattering map, the generating functional parameterization (3.41), the Poisson brackets (2.52), and the linear order scattering map (3.23).

We next define variational derivatives of the functional $\mathcal{G}[\check{\Psi}^e, \Psi^m, \chi]$. These are defined by

$$\begin{aligned} \delta\mathcal{G} = & \int du \int d^2\Omega \frac{\delta\mathcal{G}}{\delta\Psi^m(u, \boldsymbol{\theta})} \delta\Psi^m(u, \boldsymbol{\theta}) + \int du \int d^2\Omega \left[\frac{\delta\mathcal{G}}{\delta\chi(u, \boldsymbol{\theta})} \delta\chi(u, \boldsymbol{\theta}) + \text{c.c.} \right] \\ & + \int du \int d^2\Omega \frac{\delta\mathcal{G}}{\delta\check{\Psi}^e(u, \boldsymbol{\theta})} \delta\check{\Psi}^e(u, \boldsymbol{\theta}) + \int d^2\Omega \frac{\delta\mathcal{G}}{\delta\Delta\Psi^e(\boldsymbol{\theta})} \delta\Delta\Psi^e(\boldsymbol{\theta}). \end{aligned} \quad (\text{G.2})$$

The only nontrivial point here is that the field variation

$$\check{\Psi}^e \rightarrow \check{\Psi}^e + \delta\check{\Psi}^e = \check{\Psi}^e + \delta\bar{\Psi}^e + g(u)\delta\Delta\Psi^e, \quad (\text{G.3})$$

generates two terms in the variation $\delta\mathcal{G}$ of the functional, a bulk term [first term on the second line of eq. (G.2)], and a boundary term (second term on the second line), since the variation $\delta\check{\Psi}^e$ is generally nonzero at the boundaries $u = \pm\infty$, with

$$\delta\check{\Psi}^e(\pm\infty) = \pm\delta\Delta\Psi^e/2. \quad (\text{G.4})$$

Also, the integral of $\delta\mathcal{G}/\delta\check{\Psi}^e(u, \boldsymbol{\theta})$ against any test function $f(u, \boldsymbol{\theta})$ is only defined when $f(\infty, \boldsymbol{\theta}) = -f(-\infty, \boldsymbol{\theta})$.

One might worry that there is some inconsistency in the definition (G.2) since the variables $\check{\Psi}^e$ and $\Delta\Psi^e$ are not independent, by eq. (G.4). However there is no inconsistency, as can be seen by formulating the definitions in terms of the independent variables $\bar{\Psi}^e$ and $\Delta\Psi^e$ and then translating using eq. (G.3), which yields

$$\frac{\delta\mathcal{G}}{\delta\check{\Psi}^e(u, \boldsymbol{\theta})} = \left(\frac{\delta\mathcal{G}}{\delta\bar{\Psi}^e(u, \boldsymbol{\theta})} \right)_{\Delta\Psi^e}, \quad (\text{G.5a})$$

$$\frac{\delta\mathcal{G}}{\delta\Delta\Psi^e(\boldsymbol{\theta})} = \left(\frac{\delta\mathcal{G}}{\delta\Delta\Psi^e(\boldsymbol{\theta})} \right)_{\bar{\Psi}^e} - \int du g(u) \left(\frac{\delta\mathcal{G}}{\delta\bar{\Psi}^e(u, \boldsymbol{\theta})} \right)_{\Delta\Psi^e}. \quad (\text{G.5b})$$

We use the variational derivatives on the left hand side rather than those on the right hand side, since they are independent of the choice of function $g(u)$.

We can now compute the various functionals $\mathcal{H}^m$, $\mathcal{K}$, $\mathcal{H}^e$ and $\mathcal{H}_\infty^e$ that appear in the scattering map (3.34) in terms of the generating functional $\mathcal{G}$. By using the generating

functional parameterization (3.41) with y^a taken to be Ψ^m , χ , $\tilde{\Psi}^e$ and $\Delta\Psi^e$, respectively, the Poisson brackets (2.52), the variational derivative definitions (G.2), the decomposition (3.39) of the scattering map, the linear order scattering map (3.23), and comparing with the definitions (3.34c), (3.34d), (3.37a) and (3.35), we can read off the following expressions:

$$\mathcal{H}^m[u, \boldsymbol{\theta}; \tilde{\Psi}^e, \Psi^m, \chi] = \frac{1}{2}e^2 D^{-2} \int du' K(u-u') \frac{\delta\mathcal{G}}{\delta\Psi^m(u', \boldsymbol{\theta})} [\tilde{\Psi}^e, -\Psi^m, -\chi], \quad (\text{G.6a})$$

$$\mathcal{K}[u, \boldsymbol{\theta}; \tilde{\Psi}^e, \Psi^m, \chi] = -\frac{1}{2} \int du' K(u-u') \frac{\delta\mathcal{G}}{\delta\chi(u', \boldsymbol{\theta})} [\tilde{\Psi}^e, -\Psi^m, -\chi]^*, \quad (\text{G.6b})$$

$$\mathcal{H}^e[u, \boldsymbol{\theta}; \tilde{\Psi}^e, \Psi^m, \chi] = \frac{1}{2}e^2 D^{-2} \int du' K(u-u') \frac{\delta\mathcal{G}}{\delta\tilde{\Psi}^e(u', \boldsymbol{\theta})} [\tilde{\Psi}^e, -\Psi^m, -\chi], \quad (\text{G.6c})$$

$$\mathcal{H}_\infty^e[\boldsymbol{\theta}; \tilde{\Psi}^e, \Psi^m, \chi] = \frac{i}{2}e^2 D^{-2} \int du \chi(u, \boldsymbol{\theta}) \frac{\delta\mathcal{G}}{\delta\chi(u, \boldsymbol{\theta})} [\tilde{\Psi}^e, -\Psi^m, -\chi] + \text{c.c.}, \quad (\text{G.6d})$$

where $K(u) = \Theta(u) - 1/2$. These expressions are accurate up to corrections of order $O(\mathcal{G}^2)$ or equivalently $O(\alpha^6)$.

The generating functional $\mathcal{G}$ must obey two additional constraints. First, by imposing consistency of the formulae (G.6c) and (G.6d) with the definition (3.33), and making the substitutions $\Psi^m \rightarrow -\Psi^m$ and $\chi \rightarrow -\chi$, we obtain

$$\int du \frac{\delta\mathcal{G}}{\delta\tilde{\Psi}^e(u, \boldsymbol{\theta})} [\tilde{\Psi}^e, \Psi^m, \chi] = -2i \int du \left\{ \chi(u, \boldsymbol{\theta}) \frac{\delta\mathcal{G}}{\delta\chi(u, \boldsymbol{\theta})} [\tilde{\Psi}^e, \Psi^m, \chi] \right\} + \text{c.c.} + O(\mathcal{G}^2). \quad (\text{G.7})$$

Second, using the generating functional parameterization (3.41) of the scattering map with $y^a = \hat{\Psi}^e$, making use of the Poisson bracket (2.52c) and the variational derivative definitions (G.2), and comparing with the scattering map (3.37b) and making use of the formulae (G.6c) and (G.6d) for the functionals $\mathcal{H}^e$ and $\mathcal{H}_\infty^e$, we obtain the constraint

$$\frac{\delta\mathcal{G}}{\delta\Delta\Psi^e(\boldsymbol{\theta})} [\tilde{\Psi}^e, \Psi^m, \chi] = -\frac{i}{2} \int du \left\{ \chi(u, \boldsymbol{\theta}) \frac{\delta\mathcal{G}}{\delta\chi(u, \boldsymbol{\theta})} [\tilde{\Psi}^e, \Psi^m, \chi] \right\} + \text{c.c.} + O(\mathcal{G}^2). \quad (\text{G.8})$$

By direct substitution one can show that the general solution of this equation is

$$\mathcal{G} [\tilde{\Psi}^e, \Psi^m, \chi] = \mathcal{G} [\tilde{\Psi}^e, \Psi^m, e^{-\frac{i}{2}\Delta\Psi^e} \chi] + O(\alpha^6), \quad (\text{G.9})$$

for some functional $\mathcal{G}$, with the restriction that the boundary variational derivative defined in eq. (G.2) associated with the first argument of $\mathcal{G}$ vanishes:

$$\left(\frac{\delta\mathcal{G}}{\delta\Delta\Psi^e} \right)_{\text{first argument}} = 0. \quad (\text{G.10})$$

Substituting the expression (G.9) into the definition (G.1) of $\mathcal{G}$ gives that the original generating functional G can be written as

$$G [\tilde{\Psi}_+^e, \Psi_+^m, \chi_+, \Delta\Psi_+^e, \hat{\Psi}_+^e] = \mathcal{G} [\tilde{\Psi}_+^e, \Psi_+^m, e^{-i\tilde{\Psi}_+^e} e^{i\Delta\Psi_+^e/2} \chi_+] + O(\alpha^6). \quad (\text{G.11})$$

The form of the generating functional given by eqs. (G.10) and (G.11) is actually valid to all orders in α [cf. eq. (3.42) above], even though we have derived it here only to $O(\alpha^5)$. This is because (i) it guarantees that the quantity $\Psi_+^e = \hat{\Psi}_+^e - 2 \int du g' \tilde{\Psi}_+^e - \Delta\Psi_+^e/2$ is preserved under the transformation (3.40), from the Poisson brackets (2.52) and eqs. (3.39) and (3.23), thus ensuring the matching condition (2.39); and (ii) it similarly guarantees that the conservation law (3.36) is satisfied.

H Sufficient conditions for decoupling in finite dimensions

In this appendix we prove the theorem in symplectic geometry given in section 4.2, which gives sufficient conditions for two sets of degrees of freedom to decouple in a finite dimensional system.

The derivation is essentially an application of Frobenius' theorem for vector fields [68]. We define the symmetry generator (Hamiltonian) vector fields $\vec{v}^{(A)}$ associated with the conserved quantities s^A by

$$v^{(A)a} = -\Omega^{ab}\nabla_b s^A \quad (\text{H.1})$$

for $1 \leq A \leq t$. In order for Frobenius' theorem to be applicable, the Lie brackets of these vector fields with each other must be expressible as linear combinations of the vector fields. However, the Lie bracket $[\vec{v}^{(A)}, \vec{v}^{(B)}]$ is just the Hamiltonian vector field associated with the Poisson bracket (4.5), which vanishes since ω^{AB} are assumed to be constants, $\nabla_a \omega^{AB} = 0$. Thus we obtain $[\vec{v}^{(A)}, \vec{v}^{(B)}] = 0$.

From Frobenius' theorem it now follows that the subbundle of the tangent bundle over M defined by the span of the vector fields $\vec{v}^{(A)}$ is integrable. That is, locally there exist coordinates (x^A, h^Γ) for $1 \leq A \leq t$ and $1 \leq \Gamma \leq n - t$ for which the vector fields $\vec{v}^{(A)}$ are tangent to the surfaces of constant h^Γ :

$$\vec{v}^{(A)} = \chi^{AB}(x, h) \frac{\partial}{\partial x^B} \quad (\text{H.2})$$

for some invertible χ^{AB} . However this implies that $\omega^{AB} = \vec{v}^{(A)}(s^B) = \chi^{AC} \partial s^B / \partial x^C$, and since ω^{AB} and χ^{AC} are both invertible it follows from the inverse function theorem that at fixed h we can express x^A as a function of s^A . Hence without loss of generality we can take $x^A = s^A$.

It now follows that in these coordinates (s^A, h^Γ) we have $\Omega^{AB} = \omega^{AB}$ and $\Omega^{A\Gamma} = \{s^A, h^\Gamma\} = -v^{(A)\Gamma} = 0$ from eq. (H.2). Thus the symplectic form can be expressed as

$$\Omega = \omega_{AB} ds^A \wedge ds^B + \Omega_{\Gamma\Sigma}(s, h) dh^\Gamma \wedge dh^\Sigma, \quad (\text{H.3})$$

where $\omega^{AB} \omega_{BC} = \delta^A_C$. From $d\Omega = 0$ it follows that $\Omega_{\Gamma\Sigma}$ is independent of s , and demanding that the symplectic form (H.3) be preserved under the pull back $\mathcal{S}_{2*}$ now shows that the symplectomorphism $\mathcal{S}_2$ must have the form (4.6).

I Computation of Lorenz gauge scattering map to second order

In this appendix we derive the second order Lorenz gauge scattering map (5.3).

Our starting point is the set of free Lorenz gauge solutions with soft charges described in appendix B. Rather than using the symmetric trace-free tensor expansion used there, it will be convenient to represent those solutions as conventional plane wave expansions, with the soft degrees of freedom encoded in distributional components of the mode coefficients at zero frequency. We write the first order scalar field as

$$\underline{\Phi}^{(1)}(t, \mathbf{x}) = \sum_{\eta=\pm} \int d^3k e^{-i\eta kt} e^{i\mathbf{k}\cdot\mathbf{x}} f_\eta(\mathbf{k}), \quad (\text{I.1})$$

where $k = |\mathbf{k}|$ and the functions $f_+(\mathbf{k})$ and $f_-(\mathbf{k})$ are independent. These functions are related to the initial data $\underline{\chi}^{(1)}(v, \mathbf{n}) = \lim_{r \rightarrow \infty} r \underline{\Phi}^{(1)}(v - r, r\mathbf{n})$ on $\mathcal{I}^-$, with $\mathbf{n} = \mathbf{x}/r$, by [cf. eq. (9.0.29) of ref. [1]]

$$f_\eta(\mathbf{k}) = -\frac{i\eta}{2\pi k} \tilde{\chi}^{(1)}(\eta k, -\eta \hat{\mathbf{k}}). \quad (\text{I.2})$$

Here $\eta = \pm 1$, $\hat{\mathbf{k}} = \mathbf{k}/k$ and $\tilde{\chi}^{(1)}$ is the Fourier transform

$$\tilde{\chi}^{(1)}(\omega, \mathbf{n}) = \frac{1}{2\pi} \int dv e^{i\omega v} \underline{\chi}^{(1)}(v, \mathbf{n}). \quad (\text{I.3})$$

Similarly the first order vector potential is given by $\underline{A}^{(1)t} = 0$ and

$$\underline{\mathbf{A}}^{(1)}(t, \mathbf{x}) = \sum_{\gamma=\pm} \int d^3p e^{-i\gamma p t} e^{i\mathbf{p} \cdot \mathbf{x}} \mathbf{a}_\gamma(\mathbf{p}), \quad (\text{I.4})$$

with $p = |\mathbf{p}|$,

$$\mathbf{p} \cdot \mathbf{a}_\gamma(\mathbf{p}) = 0 \quad (\text{I.5})$$

and $\mathbf{a}_\gamma(-\mathbf{p})^* = \mathbf{a}_{-\gamma}(\mathbf{p})$. The mode coefficients $\mathbf{a}_\gamma(\mathbf{p})$ are related to the initial data $\underline{\mathcal{A}}_A^{(1)}(v, \mathbf{n})$ defined in section 2.1 by

$$a_\gamma^i(\mathbf{p}) = -\frac{i\gamma}{2\pi p} \tilde{\mathcal{A}}_-^{(1)i}(\gamma p, -\gamma \hat{\mathbf{p}}). \quad (\text{I.6})$$

Here $\gamma = \pm 1$, $\hat{\mathbf{p}} = \mathbf{p}/p$, $\tilde{\mathcal{A}}_-^{(1)i}(\omega, \mathbf{n}) = \tilde{\mathcal{A}}_A^{(1)}(\omega, \mathbf{n}) h^{AB} e_B^i$, $\theta^A = (\theta, \varphi)$ are coordinates on the two-sphere, h_{AB} is the metric given by $h_{AB} d\theta^A d\theta^B = d\theta^2 + \sin^2 \theta d\varphi^2$, $h^{AB} h_{BC} = \delta_C^A$, $e_A^i = \partial n^i / \partial \theta^A$ and

$$\tilde{\mathcal{A}}_-^{(1)}(\omega, \mathbf{n}) = \frac{1}{2\pi} \int dv e^{i\omega v} \underline{\mathcal{A}}_-^{(1)}(v, \mathbf{n}). \quad (\text{I.7})$$

We next discuss how the soft charges are encoded in the mode coefficients. For the incoming scalar field, we have excluded by assumption any soft degrees of freedom, and the fall off assumption (2.18) implies that the function $\tilde{\chi}^{(1)}(\omega, \mathbf{n})$ is bounded. For the incoming vector potential, on the other hand, using the fall-off assumption (2.17), the definitions (2.47) and (2.36b), and the vanishing magnetic charges condition (2.41) we obtain for the Fourier transform

$$\tilde{\mathcal{A}}_-^{(1)i}(\omega, \mathbf{n}) = \frac{i}{2\pi} \Delta \underline{\mathcal{A}}^{(1)i}(\mathbf{n}) \text{P.V.} \frac{1}{\omega} + \overline{\mathcal{A}}_-^{(1)i}(\mathbf{n}) \delta(\omega) + \tilde{\mathcal{A}}_-^{(1)i \text{ rest}}(\omega, \mathbf{n}), \quad (\text{I.8})$$

where P.V. means “principal value”,

$$\Delta \underline{\mathcal{A}}^{(1)i}(\mathbf{n}) = e_A^i h^{AB} D_B \Delta \underline{\Psi}^{e(1)}(\mathbf{n}), \quad \overline{\mathcal{A}}_-^{(1)i}(\mathbf{n}) = e_A^i h^{AB} D_B \bar{\Psi}^{e(1)}(\mathbf{n}), \quad (\text{I.9})$$

and

$$\tilde{\mathcal{A}}_-^{(1)i \text{ rest}}(\omega, \mathbf{n}) = O(\omega^{-1+\epsilon}) \quad (\text{I.10})$$

as $\omega \rightarrow 0$ with $\epsilon > 0$. We will see that the second and third terms in eq. (I.8) give a vanishing contribution to the second order scattering while the first term gives a nonvanishing contribution.

We next write the equation of motion (3.18a) for the second order scalar field as $\square \Phi^{(2)} = s$, where the source s is $s = 2i\mathcal{A}^{(1)a}\nabla_a\Phi^{(1)}$. Defining the final data on $\mathcal{I}^+$ as $\chi_+^{(2)}(u, \mathbf{n}) = \lim_{r \rightarrow \infty} r\Phi^{(2)}(u+r, r\mathbf{n})$, we make use of the general identity derived in section 2.2 of ref. [69]

$$\tilde{\chi}_+^{(2)}(\omega, \mathbf{n}) = -2\pi^2 \tilde{s}(\omega, \omega\mathbf{n}), \quad (\text{I.11})$$

where $\tilde{s}$ is the spacetime Fourier transform

$$\tilde{s}(\omega, \mathbf{k}) = \frac{1}{(2\pi)^4} \int dt \int d^3x e^{i\omega t} e^{-i\mathbf{k}\cdot\mathbf{x}} s(t, \mathbf{x}) \quad (\text{I.12})$$

and the Fourier transform $\tilde{\chi}_+^{(2)}$ is defined as in eq. (I.3) but with v replaced by u . We now evaluate the source $s(t, \mathbf{x})$ using the mode expansions (I.1) and (I.4), and multiply by a regulator factor $\exp[-t^2/(2T^2)]$, where we will eventually take the limit $T \rightarrow \infty$. We then evaluate the spacetime Fourier transform, make use of the identity (I.11), and eliminate the mode functions in favor of the initial data on $\mathcal{I}^-$ using eqs. (I.2) and (I.6). This gives

$$\begin{aligned} \tilde{\chi}_+^{(2)}(\omega, \mathbf{n}) = & - \lim_{T \rightarrow \infty} \sum_{\eta, \gamma = \pm} \int d^3p \frac{\eta\gamma}{lp} \delta_T(\nu k - \eta l - \gamma p) \\ & \times \tilde{\chi}_-^{(1)} \left[\eta l, -\eta \frac{(\mathbf{k} - \mathbf{p})}{l} \right] \mathbf{k} \cdot \tilde{\mathcal{A}}_-^{(1)}(\gamma p, -\gamma \hat{\mathbf{p}}), \end{aligned} \quad (\text{I.13})$$

where we have defined

$$\mathbf{k} = \omega\mathbf{n}, \quad k = |\omega|, \quad \nu = \text{sgn}(\omega), \quad l = |\mathbf{k} - \mathbf{p}|, \quad (\text{I.14})$$

and δ_T is the approximate δ -function

$$\delta_T(\omega) = \frac{T}{\sqrt{2\pi}} \exp \left[-\frac{1}{2} \omega^2 T^2 \right]. \quad (\text{I.15})$$

To evaluate the momentum space integral (I.13) it is convenient to switch to prolate spheroidal coordinates. We pick Cartesian coordinates for which $\mathbf{k} = (0, 0, k)$, and define coordinates (σ, τ, φ) with $1 \leq \sigma < \infty$, $-1 \leq \tau \leq 1$, $0 \leq \varphi \leq 2\pi$ by

$$\mathbf{p} = \frac{1}{2}k \left(\sqrt{\sigma^2 - 1} \sqrt{1 - \tau^2} \cos \varphi, \sqrt{\sigma^2 - 1} \sqrt{1 - \tau^2} \sin \varphi, 1 + \sigma\tau \right). \quad (\text{I.16})$$

It follows that $d^3p = (k^3/8)(\sigma^2 - \tau^2)d\sigma d\tau d\varphi$ and that

$$l = |\mathbf{k} - \mathbf{p}| = \frac{1}{2}k(\sigma - \tau), \quad p = \frac{1}{2}k(\sigma + \tau). \quad (\text{I.17})$$

This gives

$$\begin{aligned} \tilde{\chi}_+^{(2)}(\omega, \mathbf{n}) = & -\frac{1}{2} \lim_{T \rightarrow \infty} \sum_{\eta, \gamma = \pm} \int_1^\infty d\sigma \int_{-1}^1 d\tau \int_0^{2\pi} d\varphi \eta\gamma \delta_{kT} \left[1 - \frac{\eta(\sigma - \tau)}{2} - \frac{\gamma(\sigma + \tau)}{2} \right] \\ & \times \tilde{\chi}_-^{(1)} \left[\nu\eta l, -\nu\eta \frac{\mathbf{k} - \mathbf{p}}{l} \right] \mathbf{k} \cdot \tilde{\mathcal{A}}_-^{(1)}(\nu\gamma p, -\nu\gamma \hat{\mathbf{p}}), \end{aligned} \quad (\text{I.18})$$

where we have redefined $\eta \rightarrow \nu\eta$ and $\gamma \rightarrow \nu\gamma$.

We now discuss in turn the four different terms in the expression (I.18). Two of these give vanishing contributions, and the contributions from the remaining two are equal. For the first term with $\eta = \gamma = -1$, the approximate delta function reduces in the limit $T \rightarrow \infty$ to $\delta(\sigma + 1)$, which gives a vanishing contribution since the range of σ is $\sigma \geq 1$. For the second term with $\eta = -1$, $\gamma = 1$, the approximate delta function reduces instead to $\delta(\tau - 1)$, and the expression reduces to an integral over the half line $\mathbf{p} = (0, 0, k + (\sigma - 1)k/2)$ for $1 \leq \sigma < \infty$. It follows that the frequency argument $\nu\gamma p$ of $\tilde{\mathcal{A}}_-^{(1)}$ is bounded away from zero, and therefore the second line of the integrand is bounded, from eq. (I.8); the soft degrees of freedom or distributional components do not contribute. The Lorenz gauge condition (I.5) now makes the result vanish; this is the essentially the Feynman diagram argument given in the last paragraph of section 5.

The third term with $\eta = \gamma = 1$ is a little more involved. Inserting the decomposition (I.8) of the initial data for the vector potential, this term can be written as

$$-\frac{1}{2} \lim_{T \rightarrow \infty} \int_1^\infty d\sigma \int_{-1}^1 d\tau \int_0^{2\pi} d\varphi \delta_{kT}(1 - \sigma) \tilde{\chi}^{(1)} \left[\nu l, -\nu \frac{\mathbf{k} - \mathbf{p}}{l} \right] \\ \times \mathbf{k} \cdot \left[\frac{i\nu}{2\pi} \Delta \mathcal{A}^{(1)}(-\nu \hat{\mathbf{p}}) \text{P.V.} \frac{1}{p} + \overline{\mathcal{A}}^{(1)}(-\nu \hat{\mathbf{p}}) \delta(p) + \tilde{\mathcal{A}}_-^{(1)\text{rest}}(\nu p, -\nu \hat{\mathbf{p}}) \right]. \quad (\text{I.19})$$

Now the limiting $T \rightarrow \infty$ delta function $\delta(\sigma - 1)$ would enforce an integral over the line segment

$$\mathbf{p} = (0, 0, k(1 + \tau)/2) \quad (\text{I.20})$$

for $-1 \leq \tau \leq 1$ that extends from $\mathbf{p} = \mathbf{0}$ to $\mathbf{p} = \mathbf{k}$ and that overlaps with the location $\mathbf{p} = \mathbf{0}$ of the distributional contributions to the integrand. Thus we will need to take the $T \rightarrow \infty$ limit carefully. For the second and third terms in the large square brackets on the second line we can just use the limiting delta function $\delta(\sigma - 1)$. The third term scales as $\sim p^{-(1-\epsilon)} \sim (1 + \tau)^{-(1-\epsilon)}$ with $\epsilon > 0$, from eq. (I.10). This yields a diverging integrand but the integral over τ converges, and thus the result vanishes by the Lorenz gauge condition (I.5), since it is evaluated at $\hat{\mathbf{p}} = \hat{\mathbf{k}}$ by eq. (I.20). Similarly, for the second term, the $\delta(\sigma - 1)$ factor enforces evaluation at $\hat{\mathbf{p}} = \hat{\mathbf{k}}$, and then the $\delta(p)$ factor evaluates at $\tau = -1$ at which the integrand is finite, giving again a vanishing result by eq. (I.5).

Thus we are left with the first term in the large square brackets in the expression (I.19). We simplify this in several stages:

- First, we argue that we can evaluate the factor $\tilde{\chi}^{(1)}$ at $\mathbf{p} = \mathbf{0}$ and pull it outside the integral. To see this we define the variable

$$\mu = \hat{\mathbf{k}} \cdot \hat{\mathbf{p}} = \frac{1 + \sigma\tau}{\sigma + \tau} \quad (\text{I.21})$$

from eq. (I.16), and we note that by the Lorenz gauge condition (I.5) that $\mathbf{k} \cdot \Delta \mathcal{A}^{(1)}(-\nu \hat{\mathbf{p}})$ must vanish at $\mu = 1$ and at $\mu = -1$. Hence it must be expressible as $1 - \mu^2$ times a bounded function, and including the factor of $1/p$ and using eqs. (I.17) and (I.21) we find that the second line in the expression (I.19) can be written as a bounded function times the factor

$$\frac{(\sigma^2 - 1)(1 - \tau^2)}{(\sigma + \tau)^3}. \quad (\text{I.22})$$

Now the approximate delta function $\delta_{kT}(1 - \sigma)$ will restrict the value of σ to $\sigma - 1 = O(\varepsilon_0)$ where $\varepsilon_0 = 1/(kT)$ and we are taking the limit $\varepsilon_0 \rightarrow 0$, from eq. (I.15). Thus when $\tau + 1 = O(1)$ the factor (I.22) scales like $\sim \varepsilon_0$ and the integrand is suppressed, whereas when $\tau + 1 = O(\varepsilon_0)$ the factor (I.22) scales like $\sim \varepsilon_0^{-1}$. Hence for small ε_0 we can evaluate the factor $\tilde{\chi}_{-}^{(1)}$ in the expression (I.19) at $\tau = -1$ and pull it outside the integral, where it can be written as $\tilde{\chi}_{-}^{(1)}(\omega, -\mathbf{n})$ by eq. (I.14).

- Second, we can drop the integral over σ and the approximate delta function, evaluate the remaining integral at $\sigma = 1 + \varepsilon_0$, take the limit $\varepsilon_0 \rightarrow 0$, and then multiply by a correction factor of $1/2$. This is because we can pull the approximate delta function outside the integral over τ and φ , and that integral becomes independent of σ in the limit $\sigma \rightarrow 1$ from above (see below). Then for small ε_0 the integral over σ of the approximate delta function gives an overall factor of $1/2$, since the range of σ is restricted to $\sigma \geq 1$.
- Third, the remaining integral at a fixed value of σ is an integral over a prolate spheroid that limits as $\sigma \rightarrow 1$ to the line segment (I.20). We parameterize this integral as an integral over solid angles centered at $\mathbf{p} = \mathbf{0}$ by changing one of the variables of integration from τ to μ , using

$$\frac{1}{\tau + \sigma} d\tau = \frac{1}{\sigma - \mu} d\mu$$

from eq. (I.21).

The result of all these simplifications is the expression

$$- \frac{i}{4\pi} \tilde{\chi}_{-}^{(1)}(\omega, -\mathbf{n}) \lim_{\sigma \rightarrow 1} \int_{-1}^1 d\mu \int_0^{2\pi} d\varphi \frac{1}{\sigma - \mu} \nu \hat{\mathbf{k}} \cdot \Delta \underline{\mathcal{A}}^{(1)}(-\nu \hat{\mathbf{p}}) \quad (\text{I.23})$$

for the $\eta = \gamma = 1$ piece of eq. (I.18). Defining the unit vectors $\hat{\mathbf{m}} = -\nu \hat{\mathbf{p}}$ and $\hat{\mathbf{m}}_0 = -\nu \hat{\mathbf{k}} = -\mathbf{n}$ from the definitions (I.14), and taking the limit $\sigma \rightarrow 1$, we can rewrite this as

$$\frac{i}{4\pi} \tilde{\chi}_{-}^{(1)}(\omega, -\mathbf{n}) \int d^2 \Omega_{\hat{\mathbf{m}}} \frac{\hat{\mathbf{m}}_0 \cdot \Delta \underline{\mathcal{A}}^{(1)}(\hat{\mathbf{m}})}{1 - \mu}. \quad (\text{I.24})$$

Next we define the Green's function

$$G(\hat{\mathbf{m}}, \hat{\mathbf{m}}_0) = \frac{1}{4\pi} \log(1 - \hat{\mathbf{m}} \cdot \hat{\mathbf{m}}_0) \quad (\text{I.25})$$

which satisfies $D^A D_A G = \delta^2(\hat{\mathbf{m}}, \hat{\mathbf{m}}_0) - 1/(4\pi)$. Using eq. (I.9) we can rewrite the expression (I.24) as

$$- i \tilde{\chi}_{-}^{(1)}(\omega, -\mathbf{n}) \int d^2 \Omega_{\hat{\mathbf{m}}} D^A G(\hat{\mathbf{m}}, \hat{\mathbf{m}}_0) D_A \Delta \underline{\Psi}^{e(1)}(\hat{\mathbf{m}}), \quad (\text{I.26})$$

where we have used $D_A m^i = e_A^i$. Finally integrating by parts and using the fact that $\Delta \underline{\Psi}^{e(1)}$ has no $l = 0$ component [cf. eq. (2.36b) above] gives that the contribution to $\tilde{\chi}_{+}^{(2)}(\omega, \mathbf{n})$ is

$$i \tilde{\chi}_{-}^{(1)}(\omega, -\mathbf{n}) \Delta \underline{\Psi}^{e(1)}(-\mathbf{n}). \quad (\text{I.27})$$

We now turn to the fourth and final term $\eta = 1$, $\gamma = -1$ in eq. (I.18). This can be evaluated using the same methods as for the $\gamma = \eta = 1$ term just described. One ends up integrating over a hyperboloid of revolution of the form $\tau = -1 + O(\varepsilon_0)$ which limits to the half line $\mathbf{p} = (0, 0, (1 - \sigma)k/2)$ for $1 \leq \sigma < \infty$. The final result is the same as the result (I.27), and adding the two contributions gives

$$\tilde{\chi}_+^{(2)}(\omega, \mathbf{n}) = 2i\tilde{\chi}_-^{(1)}(\omega, -\mathbf{n})\Delta\Psi_-^{e(1)}(-\mathbf{n}), \quad (\text{I.28})$$

which is equivalent to eq. (5.4).

The remaining relations in eqs. (5.3) can be evaluated using similar methods; all the second order contributions vanish since they are quadratic in the incoming first order scalar field by eq. (3.18b), and that scalar field has no soft piece.

J Leading order change in electromagnetic memory

In this appendix we compute explicitly the leading order electromagnetic memory produced when there is an incoming scalar field but no incoming vector potential. From the conservation law (3.36) this quantity is directly related to the change $\Delta Q^{\text{hard}}(\mathbf{n})$ between the total hard outgoing charge per solid angle at $\mathcal{I}^+$ and that at $\mathcal{I}^-$. This quantity is the lowest order coupling of soft and hard degrees of freedom that cannot be removed by modifying the definitions of the hard and soft sectors, as demonstrated in section 6. We shall show that it is non-zero in general.

The quantity $\Delta Q^{\text{hard}}(\mathbf{n})$ is quartic in the incoming Lorenz gauge field $\underline{\chi}^{(1)}(v, \mathbf{n})$. For any function $\varepsilon = \varepsilon(\mathbf{n})$ we define $\Delta Q_\varepsilon^{\text{hard}} = \int d^2\Omega \varepsilon(\mathbf{n}) \Delta Q^{\text{hard}}(\mathbf{n})$, and the leading order expression for this quantity in terms of the initial data is (see below for the derivation)

$$\begin{aligned} \Delta Q_\varepsilon^{\text{hard}} &= 2ie^2 \sum_{\gamma, \eta, \nu, \sigma = \pm} \int \frac{d^3p}{\gamma p} \int \frac{d^3q}{\eta q} \int \frac{d^3k}{\nu k} \int \frac{d^3l}{\sigma l} \delta^{(4)}(\vec{p} - \vec{q} + \vec{k} - \vec{l}) \varepsilon(\nu \hat{\mathbf{k}}) \\ &\quad \times \tilde{\chi}_-^{(1)}(\gamma p, -\gamma \hat{\mathbf{p}})^* \tilde{\chi}_-^{(1)}(\eta q, -\eta \hat{\mathbf{q}}) \tilde{\chi}_-^{(1)}(\nu k, -\nu \hat{\mathbf{k}})^* \tilde{\chi}_-^{(1)}(\sigma l, -\sigma \hat{\mathbf{l}}) \\ &\quad \times \frac{(\vec{p} + \vec{q}) \cdot \vec{l}}{(\vec{p} - \vec{q})^2} + \text{c.c.} \end{aligned} \quad (\text{J.1})$$

Here on the right hand side we integrate over four spatial momenta $\mathbf{p}$, $\mathbf{q}$, $\mathbf{k}$, and $\mathbf{l}$ and sum over four corresponding signs γ , η , ν and σ , and we define $p = |\mathbf{p}|$, $\hat{\mathbf{p}} = \mathbf{p}/p$ and $\vec{p} = (\gamma p, \mathbf{p})$, with similar formulae for the other three momenta.

We next discuss some properties of the formula (J.1). First, the divergent factor $(\vec{p} - \vec{q})^{-2}$ on the third line is made integrable by the fact that when $\vec{p} = \vec{q}$, the second line is explicitly real, and thus vanishes by virtue of the factor of i and the +c.c.. Second, the $l = 0$ component of the expression (J.1) vanishes, which is necessary for consistency with the conservation law (3.24) since the memory variables $\Delta\Psi_+^e$ and $\Delta\Psi_-^e$ have no $l = 0$ pieces. This follows from the fact that when $\varepsilon(\mathbf{n}) = 1$, taking the complex conjugate of the integrand is equivalent to making the change of variables $\vec{p} \leftrightarrow \vec{q}$ and $\vec{k} \leftrightarrow \vec{l}$. Third, of the sixteen combinations of signs on the right hand side, ten are identically vanishing. Squaring the 4-momentum conservation equation $\vec{p} + \vec{k} = \vec{q} + \vec{l}$ yields that $\gamma\nu = \eta\sigma$, which eliminates

eight combinations. Also when both sides of this equation are timelike, requiring that they both be either future directed or past directed eliminates the cases $(\gamma, \nu, \eta, \sigma) = (+, +, -, -)$ and $(-, -, +, +)$.

When the momentum transfer $\vec{p} - \vec{q}$ is spacelike, the factor on the third line in eq. (J.1) is proportional to the t -channel²⁷ scattering amplitude for 2 particle to 2 particle scattering in scalar QED [64]. This occurs for the four sign combinations $(\gamma, \nu, \eta, \sigma) = (+, +, +, +), (-, -, -, -), (+, -, +, -)$ and $(-, +, -, +)$. Summing over these sign combinations gives the t -channel and u -channel contributions to the change in memory

$$\begin{aligned} \Delta Q_\varepsilon^{\text{hard, tu-channels}} &= 2ie^2 \int \frac{d^3 p}{p} \int \frac{d^3 q}{q} \int \frac{d^3 k}{k} \int \frac{d^3 l}{l} \delta^{(4)}(\vec{p} - \vec{q} + \vec{k} - \vec{l}) \Xi(\vec{p}, \vec{q}, \vec{k}, \vec{l}) \\ &\quad \times \frac{(\vec{p} + \vec{q}) \cdot \vec{l}}{(\vec{p} - \vec{q})^2} + \text{c.c.} \end{aligned} \quad (\text{J.2})$$

Here on the right hand side we have $\vec{p} = (p, \mathbf{p})$ etc., and

$$\begin{aligned} \Xi(\vec{p}, \vec{q}, \vec{k}, \vec{l}) &= \tilde{\chi}^{(1)}(p, -\hat{\mathbf{p}})^* \tilde{\chi}^{(1)}(q, -\hat{\mathbf{q}}) \tilde{\chi}^{(1)}(k, -\hat{\mathbf{k}})^* \tilde{\chi}^{(1)}(l, -\hat{\mathbf{l}}) \varepsilon(\hat{\mathbf{k}}) \\ &\quad + \tilde{\chi}^{(1)}(-p, \hat{\mathbf{p}})^* \tilde{\chi}^{(1)}(-q, \hat{\mathbf{q}}) \tilde{\chi}^{(1)}(-k, \hat{\mathbf{k}})^* \tilde{\chi}^{(1)}(-l, \hat{\mathbf{l}}) \varepsilon(-\hat{\mathbf{k}}) \\ &\quad - \tilde{\chi}^{(1)}(p, -\hat{\mathbf{p}})^* \tilde{\chi}^{(1)}(q, -\hat{\mathbf{q}}) \tilde{\chi}^{(1)}(-l, -\hat{\mathbf{l}})^* \tilde{\chi}^{(1)}(-k, -\hat{\mathbf{k}}) \varepsilon(\hat{\mathbf{l}}) \\ &\quad - \tilde{\chi}^{(1)}(-p, \hat{\mathbf{p}})^* \tilde{\chi}^{(1)}(-q, \hat{\mathbf{q}}) \tilde{\chi}^{(1)}(l, \hat{\mathbf{l}})^* \tilde{\chi}^{(1)}(k, \hat{\mathbf{k}}) \varepsilon(-\hat{\mathbf{l}}), \end{aligned} \quad (\text{J.3})$$

where to obtain the third and fourth lines we have made the change of variables in the integral $\vec{k} \leftrightarrow -\vec{l}$.

To eliminate the δ -function in eq. (J.2) it is convenient to parameterize the integral in terms of a spatial momentum $\mathbf{P}$ and spatial momentum transfer $\mathbf{\Delta}$ for which the momenta in the center of momentum frame are $\vec{p}' = (P, \mathbf{P} + \mathbf{\Delta})$, $\vec{q}' = (P, \mathbf{P})$, $\vec{k}' = (P, -\mathbf{P} - \mathbf{\Delta})$ and $\vec{l}' = (P, -\mathbf{P})$, and in terms of the boost with rapidity parameter ψ in the direction of the unit vector $\boldsymbol{\beta}$ that takes the center of momentum frame to the frame of eq. (J.2). We also parameterize $\mathbf{\Delta}$ in terms of the unit vector $\mathbf{w}$ via $\mathbf{\Delta} = P\mathbf{w} - \mathbf{P}$. The result is

$$\begin{aligned} \Delta Q_\varepsilon^{\text{hard, tu-channels}} &= \frac{ie^2}{2} \int_{-\infty}^{\infty} d\psi \int d^3 P \int d^2 \Omega_\beta \int d^2 \Omega_w \Xi(\vec{p}, \vec{q}, \vec{k}, \vec{l}) P \frac{g(\Theta, \psi)}{\cosh \psi + \cos \Theta \sinh \psi} \\ &\quad \times \left[\frac{\Delta^2 - 8P^2}{\Delta^2} \right] + \text{c.c.}, \end{aligned} \quad (\text{J.4})$$

where we have defined

$$g(\Theta, \psi) = \frac{\sin \Theta \sinh \psi \tanh \psi \{6 + 2 \cos(2\Theta) + 2[1 - \cos(2\Theta)] \cosh(2\psi)\}}{\cosh^2 \psi - \cos^2 \Theta \sinh^2 \psi} \quad (\text{J.5})$$

and $\cos \Theta = \boldsymbol{\beta} \cdot \mathbf{w}$ and $\cos \bar{\Theta} = \boldsymbol{\beta} \cdot \hat{\mathbf{P}}$. This form of the integral eliminates all the divergences in the integrand of eq. (J.2) except for the divergence at $\mathbf{\Delta} = 0$ which was discussed after eq. (J.1) above.

²⁷This could also be called u -channel, depending on the identifications chosen of the ingoing and outgoing momenta.

The remaining two sign combinations in eq. (J.1) yield a timelike momentum transfer $\vec{p} - \vec{q}$ and the s -channel contribution to the change in memory. These sign combinations are $(\gamma, \nu, \eta, \sigma) = (+, -, -, +)$ and $(-, +, +, -)$. In this case we parameterize the integral in terms of a spatial momentum $\mathbf{P}$ and a momentum transfer Δ in the frame where $\vec{p} - \vec{q}$ has no spatial component, for which $\vec{p}' = (P, \mathbf{P})$, $\vec{q}' = (-P, \mathbf{P})$, $\vec{k}' = (-P, \mathbf{P} + \Delta)$, $\vec{l}' = (P, \mathbf{P} + \Delta)$. The result for the s -channel contribution to the change in memory is

$$\Delta Q_{\varepsilon}^{\text{hard}, s\text{-channel}} = \frac{ie^2}{2} \int_{-\infty}^{\infty} d\psi \int d^3P \int d^2\Omega_{\beta} \int d^2\Omega_w \tilde{\Xi}(\vec{p}, \vec{q}, \vec{k}, \vec{l}) P \frac{g(\Theta, \psi)}{\cosh \psi + \cos \bar{\Theta} \sinh \psi} \times \left[\frac{\Delta^2 - 2P^2}{2P^2} \right] + \text{c.c.}, \quad (\text{J.6})$$

where

$$\begin{aligned} \tilde{\Xi}(\vec{p}, \vec{q}, \vec{k}, \vec{l}) &= \tilde{\chi}_{-}^{(1)}(p, -\hat{\mathbf{p}})^* \tilde{\chi}_{-}^{(1)}(-q, \hat{\mathbf{q}}) \tilde{\chi}_{-}^{(1)}(-k, \hat{\mathbf{k}})^* \tilde{\chi}_{-}^{(1)}(l, -\hat{\mathbf{l}}) \varepsilon(-\hat{\mathbf{k}}) \\ &\quad + \tilde{\chi}_{-}^{(1)}(-p, \hat{\mathbf{p}})^* \tilde{\chi}_{-}^{(1)}(q, -\hat{\mathbf{q}}) \tilde{\chi}_{-}^{(1)}(k, -\hat{\mathbf{k}})^* \tilde{\chi}_{-}^{(1)}(-l, \hat{\mathbf{l}}) \varepsilon(\hat{\mathbf{k}}). \end{aligned} \quad (\text{J.7})$$

The contributions (J.4) and (J.6) to the change in memory are generically nonzero, as can be shown for example by numerical integrations with specific choices of initial data $\tilde{\chi}^{(1)}(\omega, \mathbf{n})$.

Finally we turn to the derivation of the formula (J.1). We are assuming that $\underline{A}_A = 0$, and it follows from the equations of motion (3.16) that $\underline{\chi}_{+}$ will be non-zero at orders $O(\alpha)$, $O(\alpha^3)$ and at higher orders, while $\underline{A}_{+A}$ will be non-zero at $O(\alpha^4)$ and at higher orders (its $O(\alpha^2)$ piece vanishes as argued in section 5). Combining the definitions (6.1) and (3.26) of $\Delta Q^{\text{hard}}(\mathbf{n})$, the expansion (2.9a) of the asymptotic current $\mathcal{J}_{+u}$ and a similar expansion for $\mathcal{J}_{-v}$, the expansion (3.15a) of the scalar field $\underline{\chi}_{+}$, the definition (I.3) of the Fourier transform and the free field scattering map (3.23c) yields

$$\Delta Q^{\text{hard}}(\mathbf{n}) = -4\pi \int_{-\infty}^{\infty} d\omega \omega \left[\tilde{\chi}_{+}^{(3)}(\omega, \mathbf{n}) \tilde{\chi}_{-}^{(1)}(\omega, -\mathbf{n})^* + \text{c.c.} \right] + O(\alpha^5). \quad (\text{J.8})$$

To evaluate the third order scattered field $\tilde{\chi}_{+}^{(3)}$ in this expression we use the same techniques as in appendix I. We solve the equation of motion (3.18b) for $\underline{A}^{(2)}$ using the mode expansion (I.1) to expand the source and using the momentum space retarded Green's function. We then insert the result into the equation of motion (3.19a) for $\underline{\Phi}^{(3)}$, extract the asymptotic form of the solution $\underline{\chi}_{+}^{(3)}$ near $\mathcal{I}^{+}$ using the general identity (I.11), and eliminate the mode functions in favor of the initial data using eq. (I.2). Finally inserting the resulting expression for $\tilde{\chi}_{+}^{(3)}$ into the expression (J.8) yields the result (J.1).

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