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Lower Bounds for Eigenfunction Restrictions in Lacunary Regions

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Abstract: Let (M, g) be a compact, smooth Riemannian manifold and $\{u_h\}$ be a sequence of L^2-normalized Laplace eigenfunctions that has a localized defect measure μ in the sense that $M\backslash \mathrm{supp}(\pi_*\mu) \neq \emptyset$ where $\pi : T^*M \to M$ is the canonical projection. Using Carleman estimates we prove that for any real smooth closed hypersurface $H \subset (M\backslash \mathrm{supp}(\pi_*\mu))$ sufficiently close to $\mathrm{supp}\, \pi_*\mu$ and for all $\delta > 0$,

$$\int_H |u_h|^2 d\sigma_H \geq C_\delta e^{-[\varphi(\tau_H)+\delta]/h},$$

as $h \to 0^+$. Here, $\varphi(\tau) = \tau + O(\tau^2)$ and $\tau_H := d(H, \mathrm{supp}(\pi_*\mu))$. We also show that an analogous result holds for eigenfunctions of Schrödinger operators and give applications to eigenfunctions on warped products and joint eigenfunctions of quantum completely integrable (QCI) systems.

1. Introduction

Let (M, g) be a compact, n-dimensional C^∞ Riemannian manifold, with or without boundary. Let Δ_g be the Laplace operator, and consider L^2-normalized Laplace eigenfunctions $u_h \in C^\infty(M)$,

$$(-h^2\Delta_g - 1)u_h = 0, \qquad \|u_h\|_{L^2} = 1. \tag{1}$$

In the case where $\partial M \neq \emptyset$, we ask that the boundary be smooth and we impose that the u_h satisfy either Dirichlet or Neumann boundary conditions on $\partial\Omega$. That is, we ask that either $u_h = 0$ on ∂M or $\partial_\nu u_h = 0$ on ∂M, where ∂_ν is the normal derivative along the boundary.

Let $H \subset M\backslash\partial M$ be a smooth interior hypersurface. Quantitative unique continuation for eigenfunction restrictions $u_h|_H$ is an important property with applications to the study of eigenfunction nodal sets and has received a lot of attention in the literature over the past

decade, e.g., [DF88,TZ09,BR15,ET15,CT18,GRS13,Jun14,JZ16,TZ21]. Specifically, let $\{u_{h_j}\}_j$ be a sequence of eigenfunctions. Then, the question of whether there exist constants $C_H > 0$ and $h_0 > 0$ such that

$$\int_H |u_{h_j}|^2 d\sigma_H \geq e^{-C_H/h_j} \tag{2}$$

for all $h_j \in (0, h_0]$ is, in general, an open question. Here, $d\sigma_H$ denotes the measure on H induced by the Riemannian metric.

In principle, the validity of (2) depends on *both* the particular eigenfunction sequence and the geometry of the hypersurface H. In the terminology of [TZ09], hypersurfaces H for which (2) is satisfied are said to be *good* for the eigenfunction sequence $\{u_{h_j}\}_{j=1}^{\infty}$. It is a much more subtle problem than the analogue for submanifolds $U \subset M$ with $\dim U = \dim M$; indeed, in the latter case, the estimate $\|u_h\|_{L^2(U)} \geq e^{-C_U/h}$ follows by a well-known argument using Carleman estimates [Zw, Chapter 7]. For the hypersurface analogue in (2), there are comparatively few cases where (2) has been proved.

For a planar domain Ω with real analytic boundary, Toth and Zelditch proved that that $H = \partial\Omega$ is always a good curve [TZ09]. In [GRS13] it is proved that closed horocycles on an arithmetic surface are good curves, with a polynomial lower bound. For compact hyperbolic surfaces, Jung proved that geodesic circles are good curves [Jun14]. On the flat 2-torus, Bourgain and Rudnick proved that if H is a real analytic curve with nowhere vanishing curvature, then H is good [BR15]. For $\Omega \subset \mathbb{R}^2$ bounded, piecewise-smooth convex domain with ergodic billiard flow, El-Hajj and Toth [ET15] proved that if H is a closed real analytic interior curve with strictly positive geodesic curvature, and $(u_{h_j})_{j=1}^{\infty}$ is a quantum ergodic sequence of Neumann or Dirichlet eigenfunctions in Ω, then H is good. Finally, Toth and Zelditch also proved that if H is a microlocally asymmetric hypersurface inside a real-analytic manifold, then there is a large-density sequence of eigenfunctions for which H is good, with a constant lower bound [TZ21].

Our first result in Theorem 1 deals with the case where the u_h's have a defect measure, μ, that is well-localized in the sense that $\operatorname{supp}(\pi_*\mu) \neq M$, where $\pi : T^*M \to M$ is the natural projection. See [Zw, Chapter 5] for background on defect measures. Roughly speaking, under this condition, our main result, Theorem 1, shows that (2) is satisfied for a wide class of smooth separating hypersurfaces $H \subset M \backslash \operatorname{supp}(\pi_*\mu)$ that lie near $\operatorname{supp}(\pi_*\mu)$.

Before stating our first result, we introduce the notion of a *lacunary region*. Throughout the paper, we write $\mathcal{U}_H(\tau) := \{x \in M : d(x, H) < \tau\}$ for a Fermi collar neighbourhood of a hypersurface $H \subset M$ of width $\tau > 0$. In the following, we work with h-pseudodifferential operators in $\Psi_h^0(\mathcal{U}_H(\tau))$ that are properly-supported in the Fermi tube $\mathcal{U}_H(\tau)$ (see Sect. 3.1 for the definition). In addition, given cutoffs $\chi_1, \chi_2 \in C_0^\infty(\mathcal{U}_H(\tau))$, the notation $\chi_2 \Subset \chi_1$ means that $\chi_1 \equiv 1$ on $\operatorname{supp}\chi_2$.

Definition 1.1 (*H-lacunary region*). Let M be a C^∞ manifold, $H \subset M$ a C^∞-interior hypersurface (possibly with boundary), and $\tau > 0$ such that $\mathcal{U}_H(\tau)$ is a Fermi collar neighborhood of H. An h-pseudodifferential operator $Q(h) \in \Psi_h^0(\mathcal{U}_H(\tau))$ that is properly-supported in the Fermi tube $\mathcal{U}_H(\tau)$ is said to be a *lacunary operator* for the sequence $\{u_h\}$ if:
(i) $Q(h)$ is h-elliptic on $T^*\mathcal{U}_H(\tau)$ (see Definition 3.1),
(ii) for every $\chi_1, \chi_2 \in C_0^\infty(\mathcal{U}_H(\tau), [0, 1])$ with $\chi_2 \Subset \chi_1$, there exists a constant $C > 0$, depending only on the sequence $\{u_h\}$, such that

$$\|\chi_2 Q(h) \chi_1 u_h\|_{L^2(\mathcal{U}_H(\tau))} = O(e^{-C/h}), \qquad h \to 0^+. \tag{3}$$

We say that there is an *H-lacunary region* for the sequence $\{u_h\}$ provided there exist $\tau > 0$ and a lacunary operator $Q(h) \in \Psi_h^0(\mathcal{U}_H(\tau))$ for $\{u_h\}$.

Remark 1.2. Since one can always replace $Q(h)$ with $Q(h)^* Q(h)$ in (3) above, without loss of generality, we also assume that the principal symbol q_0 is *real-valued*. Also, we note that in the case where $Q(h)$ is h-differential, since it is then local, one can replace (3) with the condition that $\|\chi_2 Q(h)\, u_h\|_{L^2} \leq C_0\, e^{-C/h} \|u_h\|_{L^2}$.

Remark 1.3. We note that if the eigenfunction sequence $\{u_h\}$ has a defect measure μ associated to it, then $Q(h) \in \Psi_h^0(\mathcal{U}_H(\tau))$ can only be a lacunary operator for $\{u_h\}$ provided $\operatorname{supp}(\pi_*\mu) \cap \mathcal{U}_H(\tau) = \emptyset$.

Remark 1.4. The requirement that the order of the lacunary operator $Q(h)$ be zero is not necessary and is only made for convenience. We note that there is no loss of generality in making this assumption. For example, if $Q(h)$ is an h-elliptic differential operator of order $k \geq 1$ and $L(h)$ is a left-parametrix for $Q(h)$, then one can simply replace $Q(h)$ with $L(h)Q(h) \in \Psi_h^0(\mathcal{U}_H(\tau))$ as the new lacunary operator since by L^2-boundedness,

$$\|L(h)Q(h)u_h\|_{L^2} = O(e^{-C/h}).$$

In Sect. 5 we discuss the existence of lacunary operators in several examples.

The main result of the paper is the following theorem.

Theorem 1. *Let (M, g) be a compact C^∞ Riemannian manifold. Let $\{u_h\}$ be a sequence of eigenfunctions satisfying (1) with an associated defect measure μ such that*

$$K := \operatorname{supp}(\pi_*\mu) \subsetneq M.$$

Let $H \subset K^c$ be an interior C^∞-hypersurface (possibly with boundary) and suppose that for some $\tau > 0$, $\mathcal{U}_H(\tau) \subset K^c$ is an H-lacunary region for $\{u_h\}$. Let $\tau_H := d(H, K)$ and suppose there exists $q_H \in \operatorname{int}(H)$ such that $\tau_H = d(q_H, K)$.

There exists $\tau_0 > 0$ with the property that if $0 < \tau_H < \tau_0$, then for any $\varepsilon > 0$, there are constants $C_0(\varepsilon) > 0$, $h_0(\varepsilon) > 0$, and $\alpha = \alpha(H, \tau_0, \varepsilon) > 0$ such that

$$\|u_h\|_{L^2(H)} \geq C_0(\varepsilon) e^{-[\varphi(\tau_H)+\varepsilon]/h},$$

for $h \in (0, h_0(\varepsilon)]$, where $\varphi(\tau) = \tau + \alpha\tau^2$.

We refer the reader to the proof of Lemma 2.1 for estimates of the constant $\alpha > 0$.

The proof of Theorem 1 involves two main ideas: First, in Theorem 2 we prove a Carleman type estimate adapted to H to obtain an exponential lower bound for $\|u_h\|_{L^2(\mathcal{U}_H(\varepsilon))}$. This estimate shows that if $\operatorname{supp}(\pi_*\mu)$ is close enough to H, then the positive mass detected by μ yields the lower bound on the L^2-tubular mass $\|u_h\|_{L^2(\mathcal{U}_H(\varepsilon))}$. Second, we show that the ellipticity of $Q(h)$ allows us to factorize $Q(h)$ in the form $A(h)(hD_{x_n} - iB_0)$ where $A(h)$ is an h-elliptic operator, B_0 is a positive constant, and x_n denotes the normal direction to H (see Sect. 3.2). Then, a further refinement of the factorization argument (see Proposition 3.4 for a precise statement) roughly speaking allows us to show that, since $Q(h)$ is lacunary for $\{u_h\}$, we can work as if $(hD_{x_n} - iB_0)u_h = O(e^{-C/h})$ in the Fermi tube $\mathcal{U}_H(\varepsilon)$. We then use this observation together with a simple integration argument over the tube to obtain a lower bound on $\|u_h\|_{L^2(H)}$ from the lower bound on $\|u_h\|_{L^2(\mathcal{U}_H(\varepsilon))}$.

The following Carleman estimate adapted to H is crucial to the proof of Theorem 1 and is of independent interest.

Theorem 2. *Let (M, g) be a compact C^∞ Riemannian manifold. Let $\{u_h\}$ be a sequence of eigenfunctions satisfying (1) and let μ be a defect measure associated to it. Let $H \subset K^c$ be an interior C^∞-hypersurface (possibly with boundary), where $K := \text{supp}(\pi_*\mu) \subsetneq M$. Let $\tau_H := d(H, K)$ and suppose there exists $q_H \in int(H)$ such that $\tau_H = d(q_H, K)$.*

There exists $\tau_0 > 0$ such that if $0 < \tau_H < \tau_0$, then for any $\varepsilon > 0$, there are constants $C_0(\varepsilon) > 0$, $h_0(\varepsilon) > 0$, and $\alpha = \alpha(H, \tau_0, \varepsilon) > 0$ such that

$$\|u_h\|_{L^2(\mathcal{U}_H(\varepsilon))} \geq C_0(\varepsilon) e^{-[\varphi(\tau_H)+\varepsilon]/h}$$

for all $h \in (0, h_0(\varepsilon)]$, where $\varphi(\tau) = \tau + \alpha\tau^2$.

Remark 1.5. We note that since we allow H to have possibly non-empty boundary, both Theorems 1 and 2 are *local* results. For example, when $\dim M = 2$, Theorem 1 gives exponential lower bounds for eigenfunction restrictions along lacunary H, where H can be a curve segment of arbitrarily small length fixed independent of h.

Our results also hold for eigenfunctions of a Schrödinger operator $P(h) = -h^2\Delta_g + V - E$ where $V \in C^\infty(M; \mathbb{R})$ is a real smooth potential and E is a regular value of V such that

$$(P(h) - E(h))u_h = 0, \qquad \|u_h\|_{L^2} = 1. \tag{4}$$

with eigenvalues $E(h) \to 0$ as $h \to 0^+$. We indicate the fairly minor changes to the proofs in Sect. 4.

Theorem 3. *Let (M, g) be a compact C^∞ Riemannian manifold. Let $\{u_h\}$ be a sequence of eigenfunctions satisfying (4) with an associated defect measure μ such that $K := \text{supp}(\pi_*\mu) \subsetneq M$. Let $H \subset K^c$ be an interior lacunary C^∞-hypersurface. Let $\tau_H := d(H, K)$ and suppose there exists $q_H \in int(H)$ such that $\tau_H = d(q_H, K)$. Then, there exists $\tau_0 > 0$ such that if $0 < \tau_H < \tau_0$, then for any $\varepsilon > 0$, there are constants $C_0(\varepsilon) > 0$, $h_0(\varepsilon) > 0$ and $\alpha = \alpha(V, H, \tau_0, \varepsilon) > 0$ such that*

$$\|u_h\|_{L^2(H)} \geq C_0(\varepsilon) e^{-[\varphi(\tau_H)+\varepsilon]/h},$$

for all $h \in (0, h_0(\varepsilon)]$, where $\varphi(\tau) = \tau + \alpha\tau^2$.

We refer the reader to the proof of Theorem 4 (see (76)) for estimates on α.

1.1. Outline of the paper. The Carleman estimates required for the proof of Theorem 2 are proved in Sect. 28. In Sect. 3 we combine the result in Theorem 2 with the operator factorization argument in Proposition 3.4 to prove Theorem 1. In Sect. 4, we indicate the relatively minor changes needed to handle the case of Schrödinger operators. Finally, in Sect. 5 we present several examples to which our results apply.

2. Carleman Estimates: Proof of Theorem 2

This section is devoted to the proof of Theorem 2. In Sect. 2.1 we construct a Carleman weight and in Sect. 2.2 we introduce the relevant regions we will use in the proof of Theorem 2 to infer the lower bound on the eigenfunction L^2-mass near H using the assumption on the support of the defect measure μ. The actual proof of Theorem 2 is given in Sect. 2.3.

2.1. Carleman weight. Given an interior hypersurface $H \subset K^c$, using the facts that K is closed and that there is $q_H \in \text{int}(H)$ such that $d(q_H, K)$, we choose $q_0 \in K$ with the property that

$$\tau_H = d(q_0, q_H) = d(K, H) > 0.$$

Let (y', y_n) be shifted geodesic normal coordinates adapted to H (also known as Fermi coordinates), defined for $|y'| < c_*$ and $|y_n| \leq 2\tau_*$, where c_* and τ_* are positive constants that depend on H only. In these coordinates,

$$q_0 = (0, 0), \qquad q_H = (0, \tau_H),$$

and

$$\{(y', \tau_H) : |y'| < c_*\} \subset \text{int}(H). \tag{5}$$

Next, define the hypersurface translate of H that intersects K at the point q_0:

$$Y := \{(y', 0) : |y'| < c_*\}. \tag{6}$$

For $0 < \tau \leq \tau_*$ and $\varepsilon_0 > 0$ we will carry out a Carleman argument in the rectangular domain (see Fig. 1)

$$\mathcal{W}(\tau, \varepsilon_0) := \{(y', y_n) : |y'| < c_*, \quad -2\varepsilon_0 < y_n < \tau + 2\varepsilon_0\}, \tag{7}$$

We also introduce a tangential cutoff $\rho_\varepsilon = \rho_\varepsilon(y') \in C_0^\infty(\{|y'| < c_*\})$ satisfying

(1) $\text{supp}\rho_\varepsilon \subset \{|y'| > 3\varepsilon\}$,
(2) $\rho_\varepsilon \leq 0$,
(3) $\rho_\varepsilon(y) \equiv -1$ on $\{\frac{1}{3}c_* < |y'| < c_*\}$,
(4) There exists $c > 0$, independent of ε, such that

$$|\partial_{y'}\rho_\varepsilon(y')| \leq c, \qquad |y'| < c_*, \qquad 0 < \varepsilon < \varepsilon_0. \tag{8}$$

For $0 < \tau < \tau_*$ and $0 < \varepsilon \leq \varepsilon_0$, we consider a putative weight function $\psi = \psi_{\varepsilon,\tau} \in \psi \in C^\infty(\mathcal{W}(\tau, \varepsilon_0))$ of the form

$$\psi(y', y_n) := \varphi(y_n) + 2\varphi(\tau)\rho_\varepsilon(y'), \tag{9}$$

where the function φ will be determined below in Lemma 2.1.

One then forms the conjugated operator

$$P_\psi(h) := e^{\psi/h} P(h) e^{-\psi/h} : C_0^\infty(\mathcal{W}(\tau, \varepsilon_0)) \to C_0^\infty(\mathcal{W}(\tau, \varepsilon_0)) \tag{10}$$

with principal symbol

$$p_\psi(y, \xi) := p(y, \xi + i\partial_y\psi), \tag{11}$$

where $p(y, \xi) = |\xi|^2_{g(y)} - 1$. We note for future reference that from (9), the weight function ψ implicitly depends on the parameters ε and τ.

Lemma 2.1. *There exists $\tau_0 \in (0, \tau_*]$ such that for $\tau_H < \tau_0$ and*

$$0 < \varepsilon_0 < \tfrac{1}{10}\min(\tau_0, c_*), \tag{12}$$

and $0 < \varepsilon < \varepsilon_0$, there is $\alpha = \alpha(H, \tau_0, \varepsilon) > 0$ such that the function $\psi_{\varepsilon,\tau_H} \in C^\infty(\mathcal{W}(\tau_H, \varepsilon_0))$ in (9) with

$$\varphi(y_n) = y_n + \alpha\, y_n^2,$$

is a Carleman weight with

$$\{\text{Re}\, p_\psi, \text{Im}\, p_\psi\} > 0 \quad \text{on} \quad \{p_\psi = 0\} \cap T^*\mathcal{W}(\tau_H, \varepsilon_0). \tag{13}$$

Proof. To prove the claim in (13) note that (see e.g. [SZ95, Lemma 2.1]) the principal symbol of $P(h) = -h^2\Delta_g - I$ in the (y', y_n) coordinates, with dual variables (ξ', ξ_n), takes the form

$$p(y, \xi) = \xi_n^2 + \omega(y, \xi') - 1 \tag{14}$$

where $\omega(y, \xi')$ is a positive-definite form in the ξ'-variables with

$$\omega(y, \xi') = a(y', \xi') - 2y_n b(y', \xi') + R(y, \xi'), \tag{15}$$

and $R(y, \xi') = O(y_n^2|\xi'|^2)$. The expansion in (15) holds near Y for y_n small. The geometric significance of a and b is that the quadratic form dual to the induced metric on H is of the form $a + O(\tau_H)$ and the quadratic form dual to the second fundamental form for H is $b + O(\tau_H)$.

Choosing the weight function ψ as in (9) with $\varphi(y_n) = y_n + \alpha\, y_n^2$, it follows from (14) that

$$\begin{aligned} \operatorname{Im} p_\psi &= 2\xi_n(1 + 2\alpha y_n) + O(\tau_0|\xi'|), \\ \operatorname{Re} p_\psi &= p(y, \xi) - (1 + 2\alpha y_n)^2 + O(\tau_0^2|\xi'|^2). \end{aligned}$$

As a result, since $-2\varepsilon_0 < y_n < \tau_0 + 2\varepsilon_0$ for $y \in \mathcal{W}(\tau, \varepsilon_0)$, on $\{p_\psi = 0\}$ we have

$$\operatorname{Re} p_\psi = a(y', \xi') - 1 - (1 + 2\alpha y_n)^2 + O(\tau_0), \qquad |\xi_n| = O(\tau_0). \tag{16}$$

Consequently, since $|y_n| < \tau_0 + 2\varepsilon_0 < \frac{12}{10}\tau_0$, it follows from (16) that,

$$\{p_\psi = 0\} \subset \big\{(y, \xi) \in T^*(\mathcal{W}(\tau, \varepsilon_0)) : \ a(y', \xi') = 2 + O(\tau_0), \quad \xi_n = O(\tau_0)\big\}, \tag{17}$$

and so, in particular, $|\xi'| = O(1)$ when $(y, \xi) \in \{p_\psi = 0\}$. A direct computation then shows that for $(y, \xi) \in \{p_\psi = 0\}$,

$$\begin{aligned} \{\operatorname{Re} p_\psi, \operatorname{Im} p_\psi\}(y, \xi) &= 2\,\{\,\xi_n^2 + a(y', \xi') - 2b(y', \xi')y_n - 4\alpha y_n,\ \xi_n(1 + 2\alpha y_n)\,\} \\ &\quad + O(\tau_0) = 4\,\big(b(y', \xi') + 2\alpha\big) + O(\tau_0). \end{aligned} \tag{18}$$

In view of (17), there exists $C_0 = C_0(H, \varepsilon_0) > 0$, such that if

$$\alpha > \max_{a(y', \xi') = 2} \tfrac{1}{2}|b(y', \xi')| + C_0\tau_0, \tag{19}$$

then it follows from (18) that

$$\{\operatorname{Re} p_\psi, \operatorname{Im} p_\psi\}(y, \xi) > C_0' > 0, \qquad (y, \xi) \in \{p_\psi = 0\}. \tag{20}$$

Since $\tau_H = d(Y, H) < \tau_0$ and a, b depend only on the geometry of H for τ_H small, there exists $\alpha = \alpha(H, \varepsilon, \tau_0)$ as in (19) so that (20) holds for τ_0 small. □

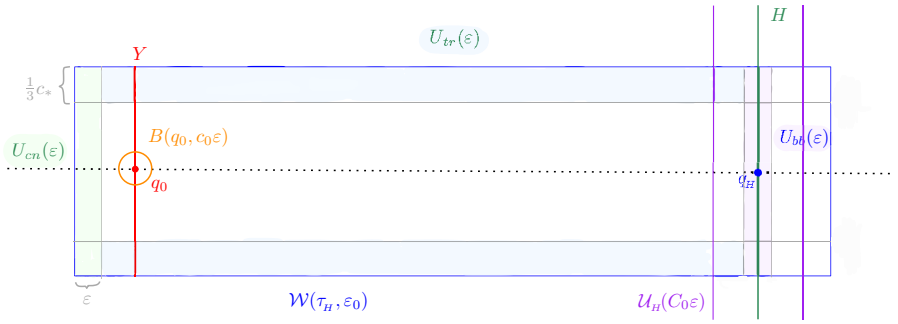

Fig. 1. Depiction of the control, transition, and black-box regions

2.2. Control, transition, and black-box regions. Given $H \subset K^c$ a smooth hypersurface, we choose points $q_0 \in K = \operatorname{supp}(\pi_*\mu)$ and $q_H \in H$ as in Subsection 2.1. Furthermore, in the following we work with $\tau_0 > 0$ as in Lemma 2.1. We assume from now on that

$$\tau_H + 2\varepsilon_0 < \tau_0, \qquad \varepsilon_0 < \tfrac{1}{10}\min(\tau_H, c_*). \tag{21}$$

As in (5), we continue to work in normalized Fermi coordinates (y', y_n) adapted to H.

Note that, since $\tau_H \leq \varphi(\tau_H)$, the bounds in (21) yield

$$0 < \varepsilon < \varepsilon_0 \leq \tfrac{1}{10}\varphi(\tau_H). \tag{22}$$

We carry out the Carleman argument in the rectangular domain $\mathcal{W}(\tau_H, \varepsilon_0)$ defined in (7), where $\tau_H = d(q_0, H)$. Within this set, we identify three key regions: the control region $U_{cn}(\varepsilon)$, the transition region $U_{tr}(\varepsilon)$, and the black-box region $U_{bb}(\varepsilon)$. Here, $U_{cn}(\varepsilon)$ refers to an ε-tube near Y, $U_{bb}(\varepsilon)$ is the region where we wish to prove lower bounds, and $U_{tr}(\varepsilon)$ are the transitional regions connecting the two former regions (see Fig. 1). To define these we need the following cut-off functions.

Let $\varepsilon > 0$ be a small constant satisfying the bound in (22). We define $\chi_{\varepsilon,Y} \in C^\infty(\mathbb{R}; [0, 1])$ with

$$\begin{cases} \chi_{\varepsilon,Y}(y_n) = 1 & \varphi(y_n) > -\frac{1}{2}\varepsilon, \\ \chi_{\varepsilon,Y}(y_n) = 0 & \varphi(y_n) < -2\varepsilon, \end{cases} \qquad \operatorname{supp}\partial\chi_{\varepsilon,Y} \subset \{-2\varepsilon < \varphi(y_n) < -\varepsilon\}.$$

Let $\chi_{\varepsilon,H} \in C^\infty(\mathbb{R}; [0, 1])$ be a cutoff localized around $\{y_n = \tau_H\}$ with

$$\begin{cases} \chi_{\varepsilon,H}(y_n) = 0 & \varphi(y_n) > \varphi(\tau_H) + 2\varepsilon, \\ \chi_{\varepsilon,H}(y_n) = 1 & \varphi(y_n) < \varphi(\tau_H) - 2\varepsilon, \end{cases} \qquad \operatorname{supp}\partial\chi_{\varepsilon,H} \subset \{|\varphi(y_n) - \varphi(\tau_H)| < \varepsilon\}.$$

Let $\chi_{\varepsilon,tr} \in C_0^\infty(\mathbb{R}^{n-1}; [0, 1])$ be a transitional cutoff with

$$\begin{cases} \chi_{\varepsilon,tr}(y') = 0 & |y'| > c_*, \\ \chi_{\varepsilon,tr}(y') = 1 & |y'| < 4\varepsilon, \end{cases} \qquad \operatorname{supp}\partial\chi_{\varepsilon,tr} \subset \{\tfrac{1}{3}c_* < |y'| < c_*\}.$$

Finally, we define the cutoff function $\chi_\varepsilon \in C_0^\infty(\mathcal{W}(\tau_H))$ with

$$\chi_\varepsilon(y', y_n) := \chi_{\varepsilon,Y}(y_n) \cdot \chi_{\varepsilon,H}(y_n) \cdot \chi_{\varepsilon,tr}(y'). \tag{23}$$

By the Leibniz rule it follows that

$$\operatorname{supp} \partial \chi_\varepsilon \subset U_{cn}(\varepsilon) \cup U_{bb}(\varepsilon) \cup U_{tr}(\varepsilon), \tag{24}$$

where, as shown in Fig. 1,

$$\begin{aligned}
U_{cn}(\varepsilon) &:= \operatorname{supp} \partial \chi_{\varepsilon,y} \times \operatorname{supp} \chi_{\varepsilon,tr} \subset \{-2\varepsilon < \varphi(y_n) < -\varepsilon,\ |y'| < c_*\} \\
U_{bb}(\varepsilon) &:= \operatorname{supp} \partial \chi_{\varepsilon,H} \times \operatorname{supp} \chi_{\varepsilon,tr} \subset \{|\varphi(y_n) - \varphi(\tau_H)| < \varepsilon,\ |y'| < c_*\} \\
U_{tr}(\varepsilon) &:= \operatorname{supp}(\chi_{\varepsilon,Y}\, \chi_{\varepsilon,H}) \times \operatorname{supp} \partial \chi_{\varepsilon,tr} \subset \{-2\varepsilon < \varphi(y_n) < \varphi(\tau_H) + \varepsilon,\ \tfrac{1}{3} c_* < |y'| < c_*\}.
\end{aligned}$$

We note that one can refine the containment in (24) slightly by setting

$$\tilde{U}_{tr}(\varepsilon) := U_{tr}(\varepsilon) \setminus \big(U_{bb}(\varepsilon) \cup U_{cn}(\varepsilon)\big),$$

and noting that Leibniz rule actually gives

$$\operatorname{supp} \partial \chi_\varepsilon \subset \tilde{U}_{tr}(\varepsilon) \cup U_{bb}(\varepsilon) \cup U_{cn}(\varepsilon). \tag{25}$$

2.3. Proof of Theorem 2. Let $q_0 \in \operatorname{supp}(\pi_* \mu)$, $\tau_0 > 0$ be as in Lemma 2.1, and $q_H \in H$ be chosen so that $\tau_H = d(q_0, q_H) = d(K, H)$. As in (5), we continue to let (y', y_n) be the (shifted) geodesic normal coordinates adapted to H. For ε_0 and τ_H satisfying (21) let

$$\mathcal{W} := \mathcal{W}(\tau_H, \varepsilon_0).$$

Choose $c_0 > 0$ so that for $0 < \varepsilon < \varepsilon_0$ (see Fig. 1), the *control ball*

$$B(q_0, c_0\varepsilon) \subset \{(y', y_n) :\ |(y', y_n)| < \tfrac{1}{5}\varepsilon\}. \tag{26}$$

We now carry out the Carleman argument. With ε as in (22) and $\chi_\varepsilon \in C_0^\infty(\mathcal{W})$ as defined in (23), set

$$v_h := \chi_\varepsilon e^{\psi/h} u_h, \qquad \psi(y', y_n) = \varphi(y_n) + 2\varphi(\tau_H)\rho_\varepsilon(y'), \tag{27}$$

with ψ as in (9) with τ_H in place of τ and φ as in Lemma 2.1. By Lemma 2.1, since $0 < \tau_H < \tau_0$, $\psi \in C^\infty(\mathcal{W})$ is a Carleman weight. Thus, by the subelliptic Carleman estimates [Zw, Theorem 7.5], there exists $C > 0$ so that, with $P_\psi(h)$ as in (10),

$$\|P_\psi(h) v_h\|^2_{L^2(\mathcal{W})} \geq Ch\, \|v_h\|^2_{L^2(\mathcal{W})}. \tag{28}$$

Note that $\chi_\varepsilon = 1$ on $B(q_0, c_0\varepsilon)$ by (26), $B(q_0, c_0\varepsilon) \subset \mathcal{W}$ and $\rho_\varepsilon = 0$ on $B(q_0, c_0\varepsilon)$. Thus, it follows that

$$\psi(y) = \varphi(y_n) = y_n + \alpha\, y_n^2 \geq y_n \geq -\tfrac{1}{2}\varepsilon, \qquad (y', y_n) \in B(q_0, c_0\varepsilon). \tag{29}$$

Also, since $q_0 \in \operatorname{supp}(\pi_* \mu)$, it follows that for all $r > 0$ there is $C(r) > 0$ such that

$$\pi_* \mu(B(q_0, r)) \geq C(r) > 0.$$

In particular, there exist constants $C(\varepsilon) > 0$ and $h_0(\varepsilon) > 0$ such that for $h \in (0, h_0(\varepsilon)]$,

$$\int_{B(q_0, c_0\varepsilon)} |u_h|^2\, dv_g \geq C(\varepsilon). \tag{30}$$

Thus, from (27), (29) and (30) it follows that there exist $C(\varepsilon) > 0$ and $h_0(\varepsilon) > 0$ such that

$$\|v_h\|^2_{L^2(\mathcal{W})} \geq e^{-\varepsilon/h} \int_{B(q_0, c_0\varepsilon)} |u_h|^2 \, dv_g \geq C(\varepsilon) e^{-\varepsilon/h}, \tag{31}$$

for $h \in (0, h_0(\varepsilon)]$. Here, (31) gives the required lower bound for the RHS in (28).

Next, since $P(h)u_h = 0$, we will use that

$$P_\psi(h)v_h = e^{\psi/h}[P(h), \chi_\varepsilon]u_h. \tag{32}$$

Also, since $[P(h), \chi_\varepsilon]$ is an h-differential operator of order one supported in $\operatorname{supp}\partial\chi_\varepsilon \subset \tilde{U}_{tr}(\varepsilon) \cup U_{bb}(\varepsilon) \cup U_{cn}(\varepsilon)$, where the inclusion was derived in (25). Thus, from (28) and (31) it follows that, after possibly shrinking $C(\varepsilon)$,

$$\begin{aligned} &\|P_\psi(h)v_h\|^2_{L^2(U_{bb}(\varepsilon))} + \|P_\psi(h)v_h\|^2_{L^2(U_{cn}(\varepsilon))} + \|P_\psi(h)v_h\|^2_{L^2(\tilde{U}_{tr}(\varepsilon))} \\ &\quad \geq C(\varepsilon) h e^{-\varepsilon/h}. \end{aligned} \tag{33}$$

We proceed to find upper bounds for each term in the LHS of (33). On the control set $U_{cn}(\varepsilon)$ we have that $-2\varepsilon < \varphi(y_n) < -\varepsilon$ and, since $\rho_\varepsilon \leq 0$,

$$\psi(y) \leq \varphi(y_n), \qquad y \in U_{cn}(\varepsilon). \tag{34}$$

From (32) and (34), it follows by L^2-boundedness that there are constants $\tilde{C} > 0$ and $\tilde{h}_0 > 0$ such that

$$\|P_\psi(h)v_h\|^2_{L^2(U_{cn}(\varepsilon))} \leq \|e^{\varphi(y_n)/h}[P(h), \chi_\varepsilon]u_h\|^2_{L^2(U_{cn}(\varepsilon))} \leq \tilde{C}h^2 e^{-2\varepsilon/h}, \tag{35}$$

for all $0 < h < \tilde{h}_0$.

On the transition set $\tilde{U}_{tr}(\varepsilon)$ we have $\rho_\varepsilon(y') = -1$ and $\varphi(y_n) < \varphi(\tau_H) + \varepsilon$. Thus, from (22) it follows that

$$\psi(y) = \varphi(y_n) + 2\varphi(\tau_H)\rho_\varepsilon(y') = \varphi(y_n) - 2\varphi(\tau_H) \leq -\tfrac{9}{10}\varphi(\tau_H) < -9\varepsilon,$$

when $y \in U_{tr}(\varepsilon)$. Therefore, after possibly adjusting $\tilde{C}$ and $\tilde{h}_0$, and recalling (12) and (32),

$$\|P_\psi(h)v_h\|^2_{L^2(U_{tr}(\varepsilon))} \leq \tilde{C}h^2 e^{-18\varepsilon/h}, \tag{36}$$

for all $0 < h < \tilde{h}_0(\varepsilon)$.

In view of (35) and (36), both the transition and control terms on the LHS of (33) can be absorbed into the RHS for $h > 0$ small. The result is that there are constants $C(\varepsilon) > 0$ and $h_0(\varepsilon) > 0$ such that for all $0 < h < h_0(\varepsilon)$

$$\|e^{\psi/h}[P(h), \chi_\varepsilon]u_h\|^2_{L^2(U_{bb}(\varepsilon))} = \|P_\psi(h)v_h\|^2_{L^2(U_{bb}(\varepsilon))} \geq C(\varepsilon) h e^{-\varepsilon/h}. \tag{37}$$

Next, on the black-box set $U_{bb}(\varepsilon)$ we have $\varphi(y_n) < \varphi(\tau_H) + \varepsilon$ and so,

$$\psi(y) \leq \varphi(\tau_H) + \varepsilon, \quad y \in U_{bb}(\varepsilon),$$

since $\rho_\varepsilon \leq 0$. So, (37) implies that

$$\tilde{C}h^2 \|u_h\|^2_{L^2(U_{bb}(\varepsilon))} \geq \|[P(h), \chi_\varepsilon]u_h\|^2_{L^2(U_{bb}(\varepsilon))} \geq Che^{-(2\varphi(\tau_H)+3\varepsilon)/h}, \tag{38}$$

for some $\tilde{C} > 0$.

By Taylor expansion, there is a constant $C_0 > 1$ such that $|y_n - \tau_H| \leq C_0|\varphi(y_n) - \varphi(\tau_H)|$ when $-2\varepsilon_0 < y_n < \tau_H + 2\varepsilon_0$, and so, for all $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned} U_{bb}(\varepsilon) &\subset \{(y', y_n) : \ |\varphi(y_n) - \varphi(\tau_H)| < \varepsilon, \ |y'| < c_*\} \\ &\subset \{(y', y_n) : \ |y_n - \tau_H| < C_0\varepsilon, \ |y'| < c_*\} \subset \mathcal{U}_H(C_0\varepsilon). \end{aligned} \tag{39}$$

Then, relabelling $C_0\varepsilon$ by ε, it follows from (38) that for some $C(\varepsilon) > 0$ and $h_0(\varepsilon) > 0$ such that

$$h^2\|u_h\|^2_{L^2(\mathcal{U}_H(\varepsilon))} \geq C(\varepsilon)h^{-1}e^{-\left(2\varphi(\tau_H)+3\varepsilon C_0^{-1}\right)/h}, \qquad 0 < h < h_0. \tag{40}$$

□

3. Goodness Estimates in Lacunary Regions: Proof of Theorem 1

In this section, we prove Theorem 1. Before carrying out the proof, we briefly recall some background material.

3.1. Semiclassical pseudodifferential operators (h-pseudos). Let $U \subset M$ be open. We say that $a \in S^m_h(U)$ provided $a \sim h^{-m}(a_0 + ha_1 + \dots)$ in the sense that for all $\ell \geq 0$

$$a - h^{-m} \sum_{0 \leq j \leq \ell} h^j a_j \quad \in h^{-m+\ell+1} S^0(U), \tag{41}$$

where (see [Zw, Section 14.2.2])

$$S^0(U) = \left\{ a \in C^\infty(T^*U) : \ \partial_x^\alpha \partial_\xi^\beta a(x, \xi) = O_{\alpha,\beta}(1) \text{ for all } \alpha, \beta \in \mathbb{N}^n, \ (x, \xi) \in T^*U \right\}.$$

Consider now the special case where $H \subset M$ is an interior closed hypersurface and $\mathcal{U}_H(\tau)$ is an open Fermi tube about H of width $\tau > 0$. In the following, we let $x = (x', x_n) : \mathcal{U}_H(\tau) \to \mathbb{R}^n$ be Fermi coordinates centered on the hypersurface $H = \{x_n = 0\}$. We say that that $P(h)$ is an h-pseudodifferential operator (h-pseudo) on the tube $\mathcal{U}_H(\tau)$ if its kernel can be written in the form

$$P(x, y; h) = K_a(x, y; h) + R(x, y; h)$$

where

$$K_a(x, y; h) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} \tilde{\chi}_1(x_n)\, a(x, \xi; h)\, \tilde{\chi}_2(y_n) d\xi, \tag{42}$$

and for all $\alpha, \beta \in \mathbb{N}^n$,

$$|\partial_x^\alpha \partial_y^\beta R(x, y)| = \mathcal{O}_{\alpha,\beta}(h^\infty).$$

Here, $\tilde{\chi}_1, \tilde{\chi}_2 \in C_0^\infty(\mathcal{U}_H(\tau))$ are tubular cutoffs with $\tilde{\chi}_1 \Subset \tilde{\chi}_2$ and $a \in S^m_h(\mathcal{U}_H(\tau))$. As for the corresponding operator, we write $P(h) \in \Psi^m_h(\mathcal{U}_H(\tau))$.

In the following it will also be useful to introduce two other cutoffs $\chi_1, \chi_2 \in C_0^\infty(\mathcal{U}_H(\tau))$ with $\chi_2 \Subset \tilde{\chi}_2 \Subset \chi_1 \Subset \tilde{\chi}_1$. For concreteness, choosing $\varepsilon \in (0, \varepsilon_0)$ we assume that

- $\tilde{\chi}_1 \in C_0^\infty([-2\varepsilon, 2\varepsilon])$ with $\chi_1(x_n) = 1$ when $|x_n| \leq \varepsilon$,
- $\chi_1 \in C_0^\infty([-\varepsilon, \varepsilon])$ with $\chi_1(x_n) = 1$ when $|x_n| \leq \varepsilon/2$,
- $\tilde{\chi}_2 \in C_0^\infty([-\varepsilon/2, \varepsilon/2])$ with $\tilde{\chi}_1(x_n) = 1$ when $|x_n| \leq \varepsilon/4$,
- $\chi_2 \in C_0^\infty((-\varepsilon/4, \varepsilon/4))$ with $\chi_2(x_n) = 1$ when $|x_n| \leq \varepsilon/8$.

For convenience, in the following we will use Fermi coordinates $x = (x', x_n)$, with $H = \{x_n = 0\}$, to represent the h-pseudos without further comment. Given a symbol $a \in S_h^m(\mathcal{U}_H(\tau))$ we write $a(x, hD_x)$ for the operator whose kernel is given by (42). In what follows, we will say that $a \in S_h^m(\mathcal{U}_H(\tau))$ is a *tangential symbol* if $a = a(x, \xi')$ does not depend on the geodesic conormal variable ξ_n. In this case we also say that $a(x, hD_{x'})$ is a tangential operator.

For future reference, we also recall the following basic definition.

Definition 3.1. . We say that $A \in \Psi_h^m(\mathcal{U}_H(\tau))$ is *h-elliptic* if there exists $C_0 > 0$ such that the principal symbol $\sigma(A) := a_0$ satisfies

$$|a_0(x, \xi)| \geq C_0 h^{-m}, \qquad (x, \xi) \in T^*(\mathcal{U}_H(\tau)).$$

For more detail on the calculus of h-pseudos, we refer the reader to [Zw, Mar02].

3.2. Operator factorization. In this section we carry out a factorization of an h-elliptic pseudo $Q(h) \in \Psi_h^0(\mathcal{U}_H(\tau))$ over a Fermi tube $\mathcal{U}_H(\tau)$ in terms of the diffusion operator $hD_{x_n} - iB(x, hD_{x'})$, where $B(x, \xi') \gtrapprox 1$ is a tangential symbol.

Proposition 3.2. *Let $\tau \in (0, \tau_H]$, $Q(h) \in \Psi_h^0(\mathcal{U}_H(\tau))$ be an h-elliptic operator and $\chi_1, \chi_2 \in C_0^\infty(\mathcal{U}_H(\tau), [0, 1])$ be the tubular cutoffs defined above. Let $B \in S^0(\mathcal{U}_H(\tau))$ be a real valued tangential symbol and $c_* > 0$ such that $B(x, \xi') \geq c_*$ when $(x, \xi) \in T^*\mathcal{U}_H(\tau)$.*

Then, there exists an h-elliptic operator $A(h) \in \Psi_h^0(\mathcal{U}_H(\tau))$ such that

$$\left\| \chi_2 \left(Q(h) - A(h) \left(hD_{x_n} - iB(x, hD_{x'}) \right) \right) \chi_1 \right\|_{L^2 \to L^2} = \mathcal{O}(h^\infty). \tag{43}$$

Proof. Let $q \sim \sum_{j=0}^\infty q_{-j} h^j$ be the symbol of $Q(h)$. First, note that

$$|\xi_n - iB(x, \xi')|^2 = \xi_n^2 + |B(x, \xi')|^2 \geq c_*^2, \quad (x, \xi) \in T^*\mathcal{U}_H(\tau),$$

and since by assumption $Q(h)$ is h-elliptic, we have that

$$a_0(x, \xi) := \frac{q_0(x, \xi)}{\xi_n - iB(x, \xi')} \in S^0(T^*\mathcal{U}_H(\tau)), \qquad a_0(x, \xi) \geq C > 0.$$

Our objective is to obtain an operator factorization of the form

$$\begin{aligned} &\chi_2 Q(h) \chi_1 = \chi_2 A(x, hD_x) \left(hD_{x_n} - iB(x, hD_x') \right) \chi_1 + R(h), \\ &\|R(h)\|_{L^2 \to L^2} = \mathcal{O}(h^\infty). \end{aligned} \tag{44}$$

To achieve (44), our ansatz is to use the factorization at the level of principal symbols

$$q_0(x, \xi) = a_0(x, \xi)(\xi_n - iB(x, \xi')), \tag{45}$$

to iteratively construct an h-smooth symbol $a \in S^0_h(T^*\mathcal{U}_H(\tau))$ satisfying

$$q(x,\xi,h) \sim a(x,\xi,h)\#\Big(\xi_n - iB(x,\xi')\Big), \tag{46}$$

where $q \sim \sum_{j=0}^{\infty} q_{-j}h^j$ is the total symbol of $Q(h)$. Since the elliptic symbol $\xi_n - iB(x,\xi')$ is already in the desired form, we perturb the a_0-term only by adding lower order corrections in h to match the total symbol of q_0. The first term a_0 already satisfies the desired equation in (46):

$$q_0 = a_0(\xi_n - iB), \qquad a_0 \in S^0(\mathcal{U}_H(\tau)).$$

For the second term a_{-1} one must solve an equation of the form

$$q_{-1} = a_{-1}(\xi_n - iB) + r_{-1}, \tag{47}$$

where $r_{-1} = -\partial_\xi a_0 \cdot \partial_x B := -\sum_{j=1}^n \partial_{\xi_j} a_0\, \partial_{x_j} B \in S^0(\mathcal{U}_H(\tau))$. Since $\xi_n - iB \neq 0$, we just solve for a_{-1} in (47) and get

$$a_{-1} = (\xi_n - iB)^{-1}(q_{-1} - r_{-1}) = (\xi_n - iB)^{-1}(q_{-1} + \partial_\xi a_0 \partial_x B), \tag{48}$$

where we note that $a_{-1} \in S^0(T^*\mathcal{U}_H(\tau))$. For the subsequent terms a_{-m} with $m \geq 2$, we have

$$a_{-m} := (\xi_n - iB)^{-1}\Big(q_{-m} + i\sum_{1\leq \ell \leq m}(-i)^\ell \sum_{|\alpha|=\ell}\frac{1}{\alpha!}(\partial_\xi^\alpha a_{\ell-m}\,\partial_x^\alpha B)\Big). \tag{49}$$

It follows that the decomposition (46) holds for

$$a \sim \sum_{j=0}^{\infty} a_{-j}h^j \in S^0_h(\mathcal{U}_H(\tau)).$$

Setting $A(h) := a(x, hD_x)$ and noting $hD_{x_n} = Op_h(\xi_n)$, it follows that the remainder

$$R(h) := \chi_2\left(Q(h) - A(h)(hD_{x_n} - iB(x,hD_{x'})\right)\chi_1$$

satisfies $\|R(h)\|_{L^2\to L^2} = O(h^\infty)$. □

Remark 3.3. Since $Q(h)$ is h-elliptic in the Fermi tube about H, the factorization in Proposition 3.2 is by no means unique; indeed, one can factorize Q over the Fermi tube in terms of *any* reference h-elliptic operator. As we will see in Proposition 3.4, the factorization corresponding to the specific choice of the reference diffusion operator $hD_n - iB_0$, where $B_0 > 0$ is constant, is particularly convenient in converting the lower bound for L^2 eigenfunction mass $\|u_h\|^2_{\mathcal{U}_H(\varepsilon)}$ into an actual lower bound for the L^2 restrictions, $\|\gamma_H u_h\|_{L^2(H)}$

3.3. Exploiting the lacunary condition. In this section, we explain how to combine Proposition 3.2 with the lacunary condition on $Q(h)$ for the eigenfunction sequence $\{u_h\}$ to essentially allow us to work as if $(hD_{x_n} - iB_0)u_h = O(e^{-C/h})$ on $\mathcal{U}_H(\varepsilon)$, where B_0 is a positive constant.

In the following, it will be useful to define truncated cutoff functions. Given any cutoff $\chi \in C_0^\infty(\varepsilon)$, we set

$$\chi^+ := \chi \cdot \mathbf{1}_{x_n \geq 0}, \qquad \mathcal{U}_H^+(\varepsilon) := \{x \in \mathcal{U}_H(\varepsilon),\ x_n \geq 0\}.$$

In addition, given an open submanifold $\tilde{H} \subset H$ and $\varepsilon > 0$ sufficiently small, we let $\psi(x') \in C_0^\infty(\tilde{H}; [0, 1])$ with the property that there exists a proper open submanifold $\tilde{H}_\varepsilon \subset \tilde{H}$ with $\max_{x \in \tilde{H}} d(\tilde{H}_\varepsilon, x) < \varepsilon$ such that

$$\psi|_{\tilde{H}_\varepsilon} = 1. \tag{50}$$

Finally, in the following, $\gamma_H : M \to H$ denotes the restriction operator to H.

The proof of Theorem 1 hinges on the following factorization result.

Proposition 3.4. *Let $\{u_h\}$ be a sequence of eigenfunctions satisfying (1), $H \subset M$ be an interior closed C^∞-hypersurface, and suppose there exists an H-lacunary region for $\{u_h\}$ containing the Fermi tube $\mathcal{U}_H(2\varepsilon)$. Then, if $B_0 > 0$ is any positive* constant, *$\tilde{H} \subset H$ is any open submanifold of H and $\psi \in C_0^\infty(\tilde{H})$ is a tangential cutoff satisfying (50), there exist operators $E(h) : C_0^\infty(\mathcal{U}_H(\varepsilon)) \to C^\infty(\mathcal{U}_H^+(\varepsilon))$ such that for some $C > 0$,*

$$\left\| \chi_2^+ \left(hD_{x_n} - iB_0\right) \psi \left(I + E(h)\right) \chi_1 u_h \right\|_{L^2} = O(e^{-C/h}), \tag{51}$$

where $\|\chi_2^+ E(h) \chi_1\|_{L^2 \to L^2} = O(h^\infty)$ and $\gamma_H E(h) = 0$.

Proof. We apply Proposition 3.2 with $B(x, \xi') := B_0$, so that $B(x', hD_{x'}) = B_0 I$ is simply the multiplication operator by $B_0 > 0$.

Since the operator $A(h) \in \Psi_h^0(\mathcal{U}_H(2\varepsilon))$ from Proposition 3.2 is h-elliptic over $\mathcal{U}_H(2\varepsilon)$, there exists a local parametrix $L(h) \in \Psi_h^0(\mathcal{U}_H(2\varepsilon))$ such that

$$\|\chi_1 \big(L(h)A(h) - I \big) \chi_1\|_{L^2 \to L^2} = O(h^\infty). \tag{52}$$

Since $Q(h)$ is lacunary for u_h, there exists $C > 0$, depending only on $\{u_h\}$, such that

$$\|\chi_2 Q(h) \chi_1 u_h\|_{L^2} = O(e^{-C/h}).$$

Therefore, since $\|\chi_1 [L(h), \chi_2]\|_{L^2 \to L^2} = O(h^\infty)$, it follows from Proposition 3.2 (52) that

$$\chi_2 \big(hD_{x_n} - iB(x, hD_{x'})\big) \chi_1 u_h = \chi_2 R'(h) \chi_1 u_h + O(e^{-C/h}), \tag{53}$$

where

$$\|\chi_2 R'(h) \chi_1\|_{L^2 \to L^2} = O(h^\infty). \tag{54}$$

It follows from (53) that

$$\chi_2^+ \psi \big(hD_{x_n} - iB_0\big) \chi_1 u_h = \chi_2^+ \psi R'(h) \chi_1 u_h + O(e^{-C/h}). \tag{55}$$

Moreover, by variation of constants, with $E(h) : C_0^\infty(\mathcal{U}_H(\varepsilon)) \to C^\infty(\mathcal{U}_H^+(\varepsilon))$ given by

$$E(h)f(x', x_n) = -\frac{i}{h}\int_0^{x_n} e^{-(x_n-\tau)B_0/h} R'(h)f(x', \tau)\, d\tau, \quad x_n \in [0, \varepsilon], \tag{56}$$

we obtain that $\gamma_H E(h) = 0$ and

$$\big(hD_{x_n} - iB_0\big)E(h)\chi_1 = -R'(h)\chi_1, \quad x_n \in [0, \varepsilon]. \tag{57}$$

Thus, from (55) it follows that

$$\chi_2^+\psi\big(hD_{x_n} - iB_0\big)\chi_1 u_h = -\chi_2^+\psi\big(hD_{x_n} - iB_0\big)E(h)\chi_1 u_h + O(e^{-C/h}). \tag{58}$$

Since

$$[hD_{x_n} - iB_0, \psi] = 0, \tag{59}$$

the bound in (51) follows from (58). Also, by (54) and the fact that $B_0 > 0$,

$$\|E(h)\|_{L^2(\mathcal{U}_H(\varepsilon))\to L^2(\mathcal{U}_H^+(\varepsilon))} = O(h^\infty).$$

□

Remark 3.5. We note that the final crucial step in the proof of Proposition 3.4 involves showing that the error term $R'(h)$ in (54) can also be factorized as in (55). Proposition 3.4 implies that $\big(hD_{x_n} - iB_0\big)v_h = O(e^{-C/h})$ with $v_h = \psi(I + E(h))\chi_1 u_h$, where we note that $v_h = \psi u_h$ on H since $\gamma_H E(h) = 0$. We will also use that the L^2-mass of v_h is comparable to that of ψu_h since $\|\chi_2^+ E(h)\chi_1\|_{L^\infty\to L^\infty} = O(h^\infty)$.

Remark 3.6. A key step in the proof of Proposition 3.4 that allows us to localize the eigenfunction restriction bounds to an open submanifold $\tilde{H} \subset H$ involves the commutator condition $[hD_{x_n} - iB(x, hD_{x'}), \psi] = 0$ in (59) where $\psi = \psi(x') \in C_0^\infty(\tilde{H})$ is a tangential cutoff satisfying (50). Since trivially $[hD_{x_n}, \psi] = 0$, (59) is equivalent to $[B(x, hD'), \psi] = 0$ and the latter requirement forces us to choose the tangential h-psdo to be a *constant* multiplication operator; that is, $B(x, hD') = B_0$ with $B_0 > 0$.

3.4. Proof of Theorem 1. Let $\tilde{H} \subset H$ be an open submanifold and choose $q_0 \in K = \operatorname{supp}(\pi_*\mu)$ and $q_H \in \tilde{H}$ so that

$$0 < d(q_0, q_H) = d(K, \tilde{H}) < \tau_0,$$

where τ_0 is as in Theorem 2.

In the following we let (x', x_n) be Fermi coordinates adapted to H,

$$H = \{x_n = 0\}, \qquad q_H = (0, 0),$$

and we assume they are well defined for $(x', x_n) \in \mathcal{U}_H(2\varepsilon)$. See Fig. 2.

We continue to let $\chi_j \in C_0^\infty(\mathcal{U}_H(2\varepsilon))$, for $j = 1, 2$, be the nested cutoff functions in Sect. 3.3. In general, for each $0 \le \tau < 2\varepsilon$ we define the level hypersurface

$$H_\tau := \{(x', x_n) : x_n = \tau\}, \qquad H_0 = H.$$

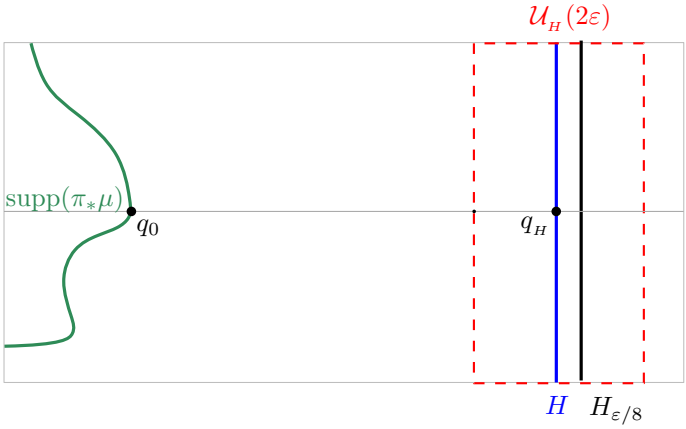

Fig. 2. Depiction of $\mathrm{supp}(\pi * \mu)$ and H

We note that for $0 < x_n < 2\varepsilon$ there is a natural diffeomorphism $\kappa_\tau : H \to H_{x_n}$ that in Fermi coordinates takes the form $\kappa_\tau(x') = (x', \tau)$. Consequently, using κ_τ to parametrize H_{x_n} by H together with the fact that $(\kappa_\tau)_*(d\sigma_H) = d\sigma_{H_\tau}$, for every $v \in L^2(\mathcal{U}_H(2\varepsilon))$

$$\|\gamma_{H_\tau} v\|^2_{L^2(H_\tau)} = \int_H |v(x', \tau)|^2 \, d\sigma_H(x') = \|\kappa_\tau^* \gamma_{H_\tau} v\|^2_{L^2(H)}. \tag{60}$$

Let $\psi \in C_0^\infty(\tilde{H})$ be a tangential cutoff satisfying (50) and $E(h) : C_0^\infty(\mathcal{U}_H(\varepsilon)) \to C^\infty(\mathcal{U}_H^+(\varepsilon))$ as in Proposition 3.4 (Fig. 2). Set

$$v_h := \psi(I + E(h))\chi_1 u_h, \qquad \text{on } \mathcal{U}_H^+(2\varepsilon) \cap \mathrm{supp}\, \psi,$$

where we note that since $\gamma_H \psi E(h)\chi_1 u = \psi \gamma_H E(h)\chi_1 u = 0$ and $\chi_1|_H = 1$, it follows that

$$\gamma_H v_h = \gamma_H \psi \chi_1 u_h = \psi \gamma_H \chi_1 u_h = \psi \gamma_H u_h.$$

Since $\chi_2^+(x_n) = 1$ for $x_n \in [0, \frac{\varepsilon}{8}]$, from Proposition 3.4 it follows that there is $C > 0$, depending only on $\{u_h\}$, such that for $(x', x_n) \in H \times [0, \frac{\varepsilon}{8}]$,

$$hD_{x_n} v_h = iB_0 v_h + r_h, \qquad \|r_h\|_{L^2} = O(e^{-C/h}).$$

Therefore, for $x_n \in [0, \frac{\varepsilon}{8}]$,

$$\int_H h\partial_{x_n} v_h(x' x_n)\overline{v_h(x', x_n)} d\sigma_H(x') = -\int_H B_0 v_h(x', x_n)\overline{v_h(x', x_n)} d\sigma_H(x') + O(e^{-\tilde{C}/h}). \tag{61}$$

Taking real parts of both sides of (61), it follows that

$$\tfrac{1}{2} h\partial_{x_n} \int_H |v_h(x', x_n)|^2 d\sigma_H(x') = -\int_H B_0 v_h(x', x_n)\overline{v_h(x', x_n)}\, d\sigma_H(x') + O(e^{-\tilde{C}/h}). \tag{62}$$

In view of (60) one can rewrite (62) in the form

$$\tfrac{1}{2} h\partial_{x_n} \|\gamma_{H_{x_n}} v_h\|^2_{L^2(H_{x_n})} = -B_0 \|\gamma_{H_{x_n}} v_h\|^2_{L^2(H_{x_n})} + O(e^{-\tilde{C}/h}), \qquad x_n \in [0, \tfrac{\varepsilon}{8}]. \tag{63}$$

Integration of (63) over $0 \leq x_n \leq \frac{\varepsilon}{8}$ and multiplication by -1 gives

$$h\|\gamma_H v_h\|^2_{L^2(H)} - h\|\gamma_{H_{\varepsilon/8}} v_h\|^2_{L^2(H_{\varepsilon/8})} = 2B_0 \left\| v_h \right\|^2_{L^2(\mathcal{U}_H(\varepsilon/8))} + O(e^{-\tilde{C}/h}), \tag{64}$$

and consequently,

$$h\|\gamma_H v_h\|^2_{L^2(H)} \geq 2B_0 \left\| v_h \right\|^2_{L^2(\mathcal{U}_H(\varepsilon/8))} + O(e^{-\tilde{C}/h}). \tag{65}$$

We also note that, by Proposition 3.4, we have $\|\chi_2^+ E(h)\chi_1\|_{L^2\to L^2} = O(h^\infty)$ and so,

$$\left\| v_h \right\|_{L^2(\mathcal{U}_H(\varepsilon/8))} = (1 + O(h^\infty)) \left\| \psi u_h \right\|_{L^2(\mathcal{U}_H(\varepsilon/8))} \geq \tfrac{1}{2} \left\| \psi u_h \right\|_{L^2(\mathcal{U}_H(\varepsilon/8))}.$$

Since $\psi|_{\tilde{H}_\varepsilon} = 1$ on the open submanifold $\tilde{H}_\varepsilon \subset \tilde{H}$ with $\max_{x\in\tilde{H}} d(\tilde{H}_\varepsilon, x) < \varepsilon$ (see (50)), it follows that

$$\left\| v_h \right\|^2_{L^2(\mathcal{U}_H(\varepsilon/8))} \geq \tfrac{1}{4} \left\| u_h \right\|^2_{L^2(\mathcal{U}_{\tilde{H}_\varepsilon}(\varepsilon/8))}. \tag{66}$$

We next find a lower bound for the RHS of (66) by applying Theorem 2. Indeed, Theorem 2 yields that for $\varepsilon > 0$ arbitrarily small, we have

$$\left\| v_h \right\|^2_{L^2(\mathcal{U}_H(\varepsilon/8))} \geq \tfrac{1}{2} \left\| u_h \right\|^2_{L^2(\mathcal{U}_{\tilde{H}_\varepsilon}(\varepsilon/8))} \geq C_\varepsilon e^{-2(\varphi(d(\tilde{H}_\varepsilon,K))+\varepsilon)/h} \geq C_\varepsilon e^{-2\left(\varphi(d(\tilde{H},K))+2\varepsilon\right)/h}. \tag{67}$$

In the last estimate in (67), we use (50) and the fact that $\max_{x\in\tilde{H}} d(\tilde{H}_\varepsilon, x) < \varepsilon$ where $\varepsilon > 0$ is arbitrarily small but fixed independent of h.

Combining (65) and (67), and recalling that $\gamma_H v_h = \psi \gamma_H u_h$, implies that for any $\varepsilon > 0$ and $h \in (0, h_0(\varepsilon)]$ there are constants $C_\varepsilon > 0$ and $C'_\varepsilon > 0$ such that

$$h\|\psi\gamma_H u_h\|^2_{L^2(H)} \geq C_\varepsilon e^{-2\left(\varphi(d(\tilde{H},K))+2\varepsilon\right)/h} - C'_\varepsilon e^{-\tilde{C}/h}. \tag{68}$$

To complete the proof of Theorem 1, we note that, since the second term on the RHS of (68) depends only on the eigenfunction sequence (and not on $\tilde{H}$), it is clear that it can be absorbed in the first term provided one chooses $\tilde{H}$ sufficiently close to K, with $2\,d(\tilde{H}, K) < \tilde{C}$. Thus, for such $\tilde{H}$ it follows from (68), and the fact that $\psi \in C_0^\infty(\tilde{H})$, that

$$h \int_{\tilde{H}} |u_h|^2 \, d\sigma_{\tilde{H}} \geq C'_\varepsilon e^{-2\left(\varphi(d(\tilde{H},K))+2\varepsilon\right)/h}.$$

Since $\varepsilon > 0$ is arbitrarily small, this concludes the proof of Theorem 1. □

4. The Case of Schrödinger Operators

Let (M, g) be a compact C^∞ Riemannian manifold, $V \in C^\infty(M, \mathbb{R})$. Consider the classical Schrödinger operator

$$P(h) = -h^2\Delta_g + V - E,$$

where E is a regular value for V. In the classically forbidden region $\{V > E\}$, the eigenfunctions u_h satisfy the Agmon-Lithner estimates [Zw]: for all $\delta > 0$ there is $C(\delta) > 0$ such that

$$|u_h(x)| \leq C(\delta)e^{-[d_E(x)-\delta]/h}, \qquad x \in \{V > E\}, \tag{69}$$

where $d_E(x)$ is the distance from x to $\{V = E\}$ in the Agmon metric $g_E = (V - E)_+|dx|^2$. As a immediate consequence of (69), it follows that if μ is a defect measure associated to a sequence $\{u_h\}$ of L^2-normalized Schrödinger eigenfunctions, $P(h)u_h = 0$, then its support is localized in the allowable region; that is,

$$\operatorname{supp}(\pi_*\mu) \subset \{x \in M : V(x) \leq E\}. \tag{70}$$

We show that if H lies inside the forbidden region $\{V > E\}$ but it is such that a Fermi neighborhood of it reaches the support $\operatorname{supp}(\pi_*\mu)$, then H is a *good* curve for $\{u_h\}$ in the sense of (2).

The proof of Theorem 3 follows the same outline as in the homogeneous case in Theorem 1. Here, we explain the relatively minor changes required to prove the analogue of the Carleman estimates in Theorem 2 and refer to the previous sections for further details.

Theorem 4. *Let (M, g) be a compact C^∞ Riemannian manifold. Let $\{u_h\}$ be a sequence of eigenfunctions satisfying (4) and let μ be a defect measure associated to it. Let $H \subset M$ be an interior C^∞-hypersurface (possibly with boundary) and suppose there exist $q_0 \in \operatorname{supp}(\pi_*\mu)$ and $q_H \in int(H)$ such that $d(q_0, q_H) = d(q_0, H) = \tau_H$. There exists $\tau_0 > 0$ such that it $0 < \tau_H < \tau_0$, then for any $\varepsilon > 0$, there are constants $C_0(\varepsilon) > 0$, $h_0(\varepsilon) > 0$, and $\alpha = \alpha(H, V, \tau_0, \varepsilon) > 0$, such that*

$$\|u_h\|_{L^2(\mathcal{U}_H(\varepsilon))} \geq C_0(\varepsilon)e^{-(\varphi(\tau_H)+\varepsilon)/h}$$

for all $h \in (0, h_0(\varepsilon)]$, where $\varphi(\tau) = \tau + \alpha\tau^2$.

Proof. The proof of Theorem 4 follows the same outline as the homogeneous analogue in Theorem 2, but we need to adapt the argument slightly by constructing a modified weight function.

Given $H \subset M$ a C^∞-hypersurface we continue to let $q_H \in \operatorname{int}(H)$ such that

$$\tau_H := d(q_0, q_H) = d(q_0, H), \qquad 0 < \tau_H < \tau_0. \tag{71}$$

As in (5), we work with (y', y_n) being (shifted) geodesic normal coordinates adapted to H. In particular, with $Y = \{y_n = 0\}$, ε_0 as in (12), the coordinates are well defined on $\mathcal{W}(\tau_H)$ as in (7).

In analogy with (9) and (27), given $\varepsilon \in (0, \varepsilon_0]$, we define the putative weight function by setting

$$\psi(y', y_n) := \varphi(y_n) + 2\varphi(\tau_H)\rho_\varepsilon(y'), \tag{72}$$

where

$$\varphi(\tau) = \tau + \alpha\tau^2,$$

where $\alpha > 0$ is a constant that is to be determined. As in (15),

$$p(y,\xi) = \xi_n^2 + a(y',\xi') - 2y_n b(y',\xi') + R(y,\xi') + V(y) - E, \tag{73}$$

with $R(y,\xi') = O(y_n^2|\xi'|^2)$. Next, note that provided $|y_n| \le \tau_0$,

$$\begin{aligned}\operatorname{Re} p_\psi &= p(y,\xi) - (\partial_{y_n}\varphi)^2 + O(\tau_0^2|\xi'|^2)\\ &= p(y,\xi) - (1+2\alpha y_n)^2 + O(\tau_0^2)|\xi'|^2)\\ &= \xi_n^2 + a(y',\xi') - 2y_n b(y',\xi') + V(y) - E - (1+2\alpha y_n)^2 + O(\tau_0^2|\xi'|^2),\end{aligned} \tag{74}$$

$$\operatorname{Im} p_\psi = 2\xi_n(\partial_{y_n}\varphi) + O(\tau_0|\xi'|) = 2\,\xi_n(1+2\alpha y_n) + O(\tau_0|\xi'|).$$

Therefore,

$$\{p_\psi = 0\} = \big\{(y,\xi) \in T^*(\mathcal{W}(\tau_H,\varepsilon_0)) : a(y',\xi') = E - V(y) + 1 + O(\tau_0),\ \xi_n = O(\tau_0)\big\},$$

and so

$$\{\operatorname{Re} p_\psi, \operatorname{Im} p_\psi\}(y,\xi) = 4\big(b(y',\xi') + 2\alpha - \tfrac{1}{2}\partial_{y_n}V(y)\big) + O(\tau_0), \quad (y,\xi) \in \{p_\psi = 0\}. \tag{75}$$

Then, with $\tau_0 > 0$ sufficiently small, we choose the constant $\alpha > 0$ so that

$$2\alpha > \max_{a(y',\xi')=E-V(y)+1} b(y',\xi') + \tfrac{1}{2}\max_{|y_n|<\tau_0}|\partial_{y_n}V(y)| + C_0\tau_0. \tag{76}$$

with appropriate constant $C_0 > 0$ to absorb the $O(\tau_0)$-error in (75). Then, (75) implies that $\{\operatorname{Re} p_\psi, \operatorname{Im} p_\psi\}(y,\xi) > 0$ on $\{p_\psi = 0\}$, and so, ψ is a Carleman weight on $\mathcal{W}(\tau_0,\varepsilon_0)$.

One then proceeds exactly as in the proof of Theorem 2 to show that when $0 < \tau_H < \tau_0$,

$$\|u_h\|_{L^2(\mathcal{U}_H(\varepsilon))} \ge C(\varepsilon)e^{-(\varphi(\tau_H)+\varepsilon)/h}. \tag{77}$$

□

The proof of Theorem 3 then follows exactly as in the homogeneous case using the Carleman result in Theorem 4 combined with the factorization argument under the lacunary assumption on H.

Remark 4.1. Let $\{u_h\}$ be a sequence of L^2-normalized eigenfunctions of a Schrödinger operator $P(h) = -h^2\Delta_g + V - E$. Suppose $H \subset \{V > E\}$ and choose $0 < \tau_0 < \tau_0$ such that

$$\mathcal{U}_H(\tau_0) \subset \{x \in M :\ V(x) > E\}.$$

We recall that in this case, by (70), $K = \operatorname{supp}(\pi_*\mu) \subset \{V \le E\}$. Since $P(h)$ is elliptic on $\mathcal{U}_H(\tau_0)$, it has a left parametrix $L(h)$. Thus,

$$Q(h) := L(h)P(h) \in \Psi_h^0(\mathcal{U}_H(\tau_0))$$

is h-elliptic over the set $\{V > E\}$ and $Q(h)u_h = 0$ since the u_h are eigenfunctions. We conclude from Remark 1.2 that $Q(h)$ is automatically a lacunary operator for $\{u_h\}$.

Consequently, both Theorems 3 and 4 are then satisfied for all hypersurfaces $H \subset \{V > E\}$.

5. Examples

In this section we present several examples to which our results apply.

5.1. Warped products. Let (M, g_M) and (N, g_N) be two compact C^∞ Riemannian manifolds. We work on the warp product manifold $M \times_f N$ endowed with the metric $g = g_M \oplus f^2 g_N$, for some function $f \in C^\infty(M, \mathbb{R}\backslash\{0\})$.

Let $\{\varphi_h\}_h \in C^\infty(N)$ be a sequence of normalized eigenfunctions

$$-h^2 \Delta_{g_N} \varphi_h = \varphi_h, \qquad \|\varphi_h\|_{L^2(N)} = 1, \tag{78}$$

and for each φ_h consider the subspace

$$\mathcal{F}_h = \{v \otimes \varphi_h : \ v \in L^2(M)\} \subset L^2(M \times_f N).$$

Since $g = g_M \oplus f^2 g_N$, with $V := f^{-2} > 0$ we have

$$-h^2 \Delta_g = -h^2 \Delta_{g_M} - V h^2 \Delta_{g_N} + hL(h),$$

where

$$W(h) = -nf^{-1} h \nabla_{g_M} f, \qquad n = \dim N.$$

Note that $W(h)$ is a first order differential operator on $L^2(M \times N)$ which acts by differentiating in the M variables only.

In particular, $\mathcal{F}_h \subset L^2(M \times_f N)$ is invariant under $-h^2 \Delta_g$ and

$$P(h) := -h^2 \Delta_g \big|_{\mathcal{F}_h} = -h^2 \Delta_{g_M} + V + hW(h).$$

Using that $-h^2 \Delta_g$ is self-adjoint on $L^2(M \times_f N)$, it is immediate to see that $P(h)$ is self-adjoint when viewed as an operator acting on $(M, \langle \cdot, \cdot \rangle_{\tilde{g}_M})$ where $\langle v_1, v_2 \rangle_{\tilde{g}_M} = \int_M v_1 \overline{v_2} \, f^n dv_{g_M}$.

Lemma 5.1. *Let E be a regular value for V and let $\{v_h\}$ be a sequence of eigenfunctions, $(P(h) - E)v_h = 0$, with defect measure μ. Let $u_h = v_h \otimes \varphi_h$ be the sequence of eigenfunctions $(-h^2 \Delta_g - E)u_h = 0$ with φ_h as in (78).*

Let $H \subset \{x \in M : V(x) > E\}$ be a closed C^∞ interior hypersurface. Then, there exists $\tau_0 > 0$ such that the following holds. If there exists $q_0 \in \operatorname{supp}(\pi_* \mu)$ with $0 < d(q_0, H) < \tau_0$, then for all $\varepsilon > 0$ there are $C_0(\varepsilon) > 0$ and $h_0(\varepsilon) > 0$ such that

$$\|u_h\|_{L^2(H \times N)} \geq C_0 e^{-(d(q_0, H) + \varepsilon)/h},$$

for all $h \in (0, h_0(\varepsilon)]$.

Proof. The operator $Q(h) = P(h)^{-1}(P(h) - E) \in \Psi_h^0(M)$ acts on M and

$$Q(h) v_h = 0. \tag{79}$$

Let (x', x_n) be Fermi coordinates on M adapted to $H = \{x_n = 0\}$. Then, $Q(h) = Op_h(q)$ with

$$q_0(x, \xi) = (\xi_n^2 + r(x, \xi') + V(x))^{-1}(\xi_n^2 + r(x, \xi') + V(x) - E).$$

It follows that $Q(h)$ is h-elliptic, and hence Remark 1.2 and (79) yield that $Q(h)$ is a lacunary operator for $\{v_h\}$ in a Fermi neighborhood of $H \subset \{x \in M : V(x) > E\}$. The result then follows from Theorem 3 and the fact that since M is compact there exists $C > 0$ such that $f \geq C$ and so $\|u_h\|_{L^2(H \times N)} \geq C \|v_h\|_{L^2(H)}$. □

5.2. Eigenfunctions of quantum completely integrable (QCI) systems. Let (M, g) be a compact C^∞ Riemannian manifold of dimension n and let $\{P_j(h)\}_{j=1}^n$ be a QCI system of n real-smooth, self-adjoint h-partial differential operators with

$$[P_i(h), P_j(h)] = 0, \qquad i \neq j,$$

and such that $\sum_{j=1}^n P_j(h)^* P_j(h)$ is h-elliptic with left parametrix $L(h)$. We apply our results to studying restrictions of appropriate subsequences $\{u_h\}$ of joint eigenfunctions of the $P_j(h)$ for $j = 1, \dots, n$. Examples include joint eigenfunctions on spheres and tori of revolution, eigenfunctions on hyperellipsoids with distinct axes, eigenfunctions of Neumann oscillators, Lagrange and Kowalevsky tops and spherical pendulum (see [HW95] for further examples).

Without loss of generality, we assume that $P_j \in \Psi_h^2(M)$ for $j = 1, .., n.$ and also assume that

$$P_1(h) = -h^2 \Delta_g, \quad \text{or} \quad P_1(h) = -h^2 \Delta_g + V.$$

All QCI systems on compact manifolds that we are aware of satisfy these properties.

Let

$$\mathcal{P} = (p_1, ..., p_n) : T^*M \to \mathbb{R}^n$$

be the associated moment map where $p_j = \sigma(P_j(h))$, and suppose $E = (E_1, ..., E_n) \in \mathcal{P}(T^*M)$ is a regular value of the moment map. By Liouville-Arnold, the level set

$$\Lambda_E := \{(x, \xi) \in T^*M : \ p_j(x, \xi) = E_j, \ j = 1, ..., n\}$$

is a finite union of $\mathbb{R}$-Lagrangian tori. To simplify the writing somewhat we assume here that Λ_E is connected. Let $\pi : T^*M \to M$ be the canonical projection and π_{Λ_E} be its restriction to Λ_E.

Let $u_{E,h} \in C^\infty(M)$ be joint eigenfunctions of the $P_j(h)$'s with joint eigenvalues $E_j(h) = E_j + o(1)$. Then, since

$$Q(h)u_{h,E} = 0, \qquad Q(h) := L(h)W_E(h) \in \Psi_h^0(M),$$
$$W_E(h) := \sum_{j=1}^n (P_j(h) - E_j(h))^*(P_j(h) - E_j(h)),$$

it follows that if μ is a defect measure for $\{u_{h,E}\}$, then it concentrates on the torus Λ_E. Indeed, it follows from the quantum Birkhoff normal form expansion for $Q(h)$ near Λ_E [TZ03] that

$$\mu = (2\pi)^{-n} |d\theta_1 \cdots d\theta_n|$$

where θ's are the angle variables on the tori Λ_E and so,

$$K = \text{supp}(\pi_* \mu) = \pi(\Lambda_E). \tag{80}$$

Let $\tilde{K} \ni K$ with a closed hypersurface $H \subset (M \backslash \tilde{K})$ that is is sufficiently close to K. Then, if $\mathcal{U}_H(\tau_0) \subset M \setminus \tilde{K}$, it is not difficult to show that ([GT20] Lemma 3.5)

$$\sigma(W_E(h))(x, \xi) = \sum_{j=1}^n (p_j(x, \xi) - E_j)^2 \geq C\langle \xi \rangle^4, \qquad (x, \xi) \in T^*(M \setminus \tilde{K}),$$

and so, since $C''\langle\xi\rangle^4 \leq \sum_j |p_j(x,\xi)|^2 \leq C'\langle\xi\rangle^4$, it follows that $Q(h) \in \Psi_h^0(\mathcal{U}_H(\tau_0))$ is h-elliptic. Also, since $Q(h)u_{E,h} = 0$, the operator $Q(h)$ is lacunary for the subsequence $\{u_{E,h}\}$ in the Fermi tube $\mathcal{U}_H(\tau_0)$.

An application of Theorem 1 (resp. Theorem 3) in the case where $P_1(h)$ is a Laplacian (resp. Schrödinger operator) yields the following result.

Theorem 5. *Let $\{u_{E,h}\}$ be a sequence of joint eigenfunctions of the $P_j(h)$'s with joint eigenvalues $E_j(h) = E_j + o(1)$, and let μ be the associated defect measure. There exists $\tau_0 > 0$ such that if $H \subset M \setminus \pi(\Lambda_E)$ is a closed C^∞ hypersurface with*

$$d(H, \pi(\Lambda_E)) < \tau_0,$$

then for any $\varepsilon > 0$ there are constants $C_0(\varepsilon) > 0$ and $h_0(\varepsilon) > 0$ such that

$$\|u_h\|_{L^2(H)} \geq C_0(\varepsilon) e^{-[d(H,\pi(\Lambda_E))+\varepsilon]/h},$$

for all $h \in (0, h_0(\varepsilon)]$.

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