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Abstract

We describe a method to show that certain elliptic surfaces do not admit purely inseparable multisections (equivalently, that genus 1 curves over function fields admit no points over the perfect closure of the base field) and use it to show that any non-Jacobian elliptic structure on a very general supersingular K3 surface has no purely inseparable multisections. We also describe specific examples of genus 1 fibrations on supersingular K3 surfaces without purely inseparable multisections.

Contents

1. Introduction

In this paper we study the following question. Fix an algebraically closed field k of characteristic at least 5.

Question 1.1. When does an elliptic surface $f : X \to \mathbf{P}^1$ admit a purely inseparable multisection? Equivalently, when does the corresponding genus 1 curve $C/k(t)$ have points over the perfect closure $k(t)^{\mathrm{perf}}$?

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There has been some previous work on purely inseparable points on curves [Kim97], abelian varieties [Rös15, Rös20], and torsors [Yua21]. Roughly speaking, these authors have found that for non-isotrivial fibrations the set of purely inseparable points tends to be small. Thus, one might suspect that it is quite often the case that this set is empty.

We will give a complete answer to Question 1.1 below for X that are suitably general in the moduli space of supersingular K3 surfaces. One of the main consequences of our techniques is the following theorem.

Theorem 1.2. *Suppose i is 8, 9, or 10. For a very general supersingular K3 surface X of Artin invariant i, every elliptic fibration $f : X \to \mathbf{P}^1$ has the property that it has a purely inseparable multisection if and only if it has a section.*

That is, for genus 1 curves $C/k(t)$ whose minimal model is a suitably general supersingular K3 surface, the presence of a purely inseparable point implies the existence of a rational point.

If X is a supersingular K3 surface of Artin invariant 10, then no elliptic fibration on X has a section (see, for example, [EvdG15, Proposition 13.1]). Thus, Theorem 1.2 implies the following result.

Corollary 1.3. *If X is a very general supersingular K3 surface, then no elliptic fibration $f : X \to \mathbf{P}^1$ has a purely inseparable multisection.*

1.1 The idea

The basic idea of the proof is as follows. We can cover the moduli space of supersingular K3 surfaces by Artin–Tate families (as described in [BL18]), so, by restricting to the first-order deformations of universal formal Brauer classes, we can reduce the existence of purely inseparable multisections on elliptic fibrations on general supersingular K3 surfaces to a problem about the action of powers of Frobenius on coherent cohomology. We briefly go into more detail and give a simultaneous outline of the paper.

Let $f : X \to \mathbf{P}^1$ be a non-Jacobian elliptic supersingular K3 surface. The fibration $f : X \to \mathbf{P}^1$ can be thought of as a torsor under the Jacobian fibration $g : J \to \mathbf{P}^1$. Since X is a supersingular K3 surface, this torsor must have index p. The Artin–Tate isomorphism [Tat95, § 3] tells us that X corresponds to an element α of the Brauer group of J. Moreover, by the special properties of supersingular K3 surfaces, the fibration f can be put in a one-parameter family of J-torsors that contains J itself. That is, α fits into a family of Brauer classes $\tilde{\alpha} \in \mathrm{Br}(J \times \mathbf{A}^1)$ whose restriction $\tilde{\alpha}|_{t=0}$ is trivial. (These are called *Artin–Tate families* and are a particular case of the *twistor spaces* studied in [BL18].) The tangent value of α over $k[t]/(t^2)$ can naturally be thought of as an element of $\mathrm{H}^2(J, \mathcal{O}_J)$.

As explained in § 6, if X is to have a purely inseparable multisection, we must have that α vanishes after passing to a minimal model Y of the pullback of $J \to \mathbf{P}^1$ by some power of the Frobenius morphism $F : \mathbf{P}^1 \to \mathbf{P}^1$. The action of the Frobenius on the restriction $\tilde{\alpha}|_{k[t]/(t^2)}$ is identified with the pullback action $\mathrm{H}^2(J, \mathcal{O}_J) \to \mathrm{H}^2(Y, \mathcal{O}_Y)$. We are thus led to try to understand when this pullback map is injective, for if it is, then a general restriction of $\tilde{\alpha}$ cannot become zero upon such a pullback. (This last statement involves an understanding of the way in which the Artin–Tate families trace out the moduli space of supersingular K3 surfaces.)

As we discuss in § 4 (using technical inputs about divisors explained in § 3), the properties of the Frobenius pullback map on cohomology are determined in large part by the configuration of singular fibers of g. Combining Lemma 4.6 and Proposition 4.9, we describe a condition on the singular fibers of an elliptic fibration which implies that this pullback map is injective. Our condition can be easily checked given a Weierstrass model for the elliptic surface. In § 5,

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we record some explicit examples of elliptic supersingular K3 surfaces verifying this condition and hence deduce the existence of elliptic supersingular K3 surfaces without purely inseparable multisections (see Corollaries 6.7 *ff.* and 6.9).

We next wish to demonstrate the existence of supersingular K3 surfaces that have no (non-Jacobian) elliptic fibrations with purely inseparable multisections. (Any supersingular K3 surface admits many elliptic structures, so this is a stronger claim). Consider a Jacobian elliptic fibration, with canonically associated Weierstrass model given by an equation $y^2 = x^3 + a(t)x + b(t)$. The singular fibers of additive reduction (which determine the behavior of the Frobenius pullback) impose conditions on the polynomials a and b. One can thus compute (or at least bound the dimension of) the locus of Weierstrass equations where the pullback by powers of the Frobenius acts trivially on $\mathrm{H}^2(J, \mathcal{O}_J)$. On the other hand, one also knows the dimension of the locus of Weierstrass equations corresponding to supersingular K3 surfaces of fixed Artin invariant. Putting these dimension bounds together, we can show that the Artin invariant 7 (or higher) locus in the space of Weierstrass equations is too large to be contained in the locus of Weierstrass equations for which some power of Frobenius is non-injective on H^2. We carry this out in §§ 7 and 8. In § 9, we show how the theory of twistor lines developed in [BL18] then gives Theorem 1.2.

Since writing this paper, we have learned that Fakhruddin seems to be the first person to observe that one can create a geometric realization of the universal formal Brauer element of a K3 surface as a deformation of a Jacobian elliptic fibration. The reader is referred to [Fak02, Lemma 4] for details. We thank Fakhruddin for making us aware of his work.

1.2 Consequences for the Artin conjecture

In [Lie15], the existence of purely inseparable multisections for generic members of Artin–Tate families (in the notation of [BL18]) is asserted as the 'technical heart' [Lie15, Proposition 3.5 and Theorem 3.6] of the work on Artin's conjecture that supersingular K3 surfaces are unirational. Theorem 1.2 shows that, in fact, such purely inseparable multisections almost never exist, in the sense that a very general supersingular K3 surface admits no fibrations that admit such multisections. We also include explicit examples in § 5 that directly contradict [Lie15, Proposition 3.5] (see Corollaries 6.7 *ff.* and 6.9). The examples in § 5 and the more general result in Theorem 1.2 invalidate [Lie15, Theorems 5.1 and 5.3] (cf. [Lie15, p. 981]).

The errors in [Lie15] have now been acknowledged in [Lie21]. In particular, the Artin conjecture remains a conjecture.

2. Definitions and notation

We work over a fixed algebraically closed field k of positive characteristic p. We assume throughout that $p \geq 5$. This assumption is made to avoid the more complicated behavior of singular fibers of elliptic surfaces in characteristics 2 and 3.

Let X be a smooth proper surface over k. An *elliptic fibration* on X is a flat proper generically smooth morphism $X \to C$ to a smooth proper curve whose generic fiber has genus 1. An *elliptic surface* is a smooth proper surface X equipped with an elliptic fibration.

An elliptic fibration is *Jacobian* if it admits a section and *non-Jacobian* otherwise.

A *multisection* of an elliptic fibration is an integral subscheme $\Sigma \subset X$ such that the map $\Sigma \to C$ is finite and flat. A multisection Σ is *purely inseparable* if $\Sigma \to C$ induces a purely inseparable map on function fields.

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3. Frobenius split divisors

In this section we will study a certain condition on divisors $D \subset \mathbf{P}^1$. We will use this in the following sections to study relative Frobenius splittings of elliptic K3 surfaces.

We consider $\mathbf{P}^1_{\mathbf{Z}} := \operatorname{Proj} \mathbf{Z}[s,t]$ equipped with its *canonical* system of homogeneous coordinates. Given a positive integer m, let $\rho^{(m)} : \mathbf{P}^1 \to \mathbf{P}^1$ denote the morphism given in homogeneous coordinates by $[s:t] \mapsto [s^m : t^m]$.

DEFINITION 3.1. A divisor $D \in |\mathcal{O}_{\mathbf{P}^1}(n)|$ is *m-split* if the morphism

$$\mathcal{O}_{\mathbf{P}^1} \to \rho^{(m)}_* \mathcal{O}_{\mathbf{P}^1}(n)$$

adjoint to the map

$$(\rho^{(m)})^* \mathcal{O}_{\mathbf{P}^1} = \mathcal{O}_{\mathbf{P}^1} \to \mathcal{O}_{\mathbf{P}^1}(n)$$

associated to D admits a splitting (as $\mathcal{O}_{\mathbf{P}^1}$-modules).

Remark 3.2. Since k is algebraically closed, a divisor is p^e-split if and only if the natural morphism $\mathcal{O}_{\mathbf{P}^1} \to F^e_* \mathcal{O}_{\mathbf{P}^1}(n)$ is split, where F is the *absolute* Frobenius. We will use this observation implicitly in what follows.

Given a divisor $D \in |\mathcal{O}(n)|$, there is a naturally associated homogeneous form $f(s,t)$ of degree n whose vanishing locus is D; f is unique up to scaling.

PROPOSITION 3.3. *Let $D \in |\mathcal{O}(n)|$ be a divisor with associated homogeneous form $f(s,t)$. Fix a positive integer m.*

(i) *If $0 \le n < m$ then any divisor D is m-split.*
(ii) *If $m \le n < 2m$ then the divisor D is m-split if and only if the form f cannot be written as $s^m g + t^m h$ for homogeneous forms g and h of degree $n - m$. (In other words, some term in f is not divisible by s^m or t^m.)*

Proof. Write $n = m + \delta$ and ρ for $\rho^{(m)}$. We have

$$\rho_* \mathcal{O}(\delta) = \mathcal{O}^{\delta+1} \oplus \mathcal{O}(-1)^{m-\delta-1}.$$

This is shown for instance in the introduction to [Ach12] when ρ is a power of the Frobenius, but the proof given applies unchanged in our situation. In terms of graded modules over $k[s,t]$, the corresponding graded module

$$M_\bullet = \bigoplus_n \mathrm{H}^0(\mathbf{P}^1, (\rho_* \mathcal{O}(\delta))(n))$$

admits the following description: M_n is the free k-vector space spanned by $s^i t^j$ with $i + j = \delta + mn$. Given positive integers i and j such that $i + j = \delta$, let $N^{i,j}$ be the graded submodule of $M_\bullet$ with $N^{i,j}_n$ the submodule spanned by $s^a t^b$ such that $a + b = \delta + mn$, $a \equiv i \pmod m$, and $b \equiv j \pmod m$. There is a graded splitting

$$M_\bullet \to N^{i,j}_\bullet$$

defined as follows: given α and β such that $\alpha + \beta = \delta + mn$, send $s^\alpha t^\beta$ to 0 if $\alpha \not\equiv i \pmod m$ or $\beta \not\equiv j \pmod m$, and to $x^\alpha y^\beta$ otherwise. There results a free summand

$$N_\bullet = \bigoplus_{i+j=a} N^{i,j}_\bullet \subset M_\bullet.$$

Since the rank of N is δ, we see that this gives the full free summand of $\rho_* \mathcal{O}(\delta)$.

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To compute the sections of $\rho_*\mathcal{O}(m+\delta) = \rho_*\mathcal{O}(n)$, we shift the decomposition $M_\bullet = N_\bullet \oplus K_\bullet$ by 1. That is, the non-split divisors will correspond to the elements of $N_\bullet$ of degree 1. These are precisely the forms of degree n in s and t each of whose terms are divisible by s^m or t^m. This gives the desired result. □

Remark 3.4. Geometrically, the m-split divisors in $|\mathcal{O}(n)|$ are given by the linear span of the copy of $\mathbf{P}^1 \times |\mathcal{O}(n-m)| \subset |\mathcal{O}(n)|$ corresponding to divisor sums $\Phi + E$ with Φ a fiber of ρ and E a member of $|\mathcal{O}(n-m)|$.

We end this section with a crude estimate on the dimension of the m-split locus in a particular space of polynomials. Suppose given a non-increasing sequence $(\lambda_1, \ldots, \lambda_n)$ of positive integers such that $\sum \lambda_i = m + \delta$ with $0 \leq \delta < m$. Consider the polynomials

$$P_{\lambda_1,\ldots,\lambda_n}(t,s) := \prod (t - z_i s)^{\lambda_i}$$

parametrized by the affine space $\mathbf{A}^n$ with coordinates $z_1, \ldots, z_n$. When the λ_i and z_j are clear, we will write simply P. The polynomial P corresponds to the divisor

$$D_{\lambda_1,\ldots,\lambda_n}(z_1, \ldots, z_n) := \sum \lambda_i z_i$$

in $\mathbf{P}^1$. Write $\tau_j(P) = \sum_{i=1}^{j} \lambda_i$ for the partial sums of the λ_i. When P is understood, we will write simply τ_j. There results a strictly increasing sequence $(\tau_1, \ldots, \tau_n)$. Let

$$B_m(\lambda_1, \ldots, \lambda_n) = \#\{i \mid \delta + 1 \leq \tau_i \leq m - 1\}. \tag{3.4.1}$$

Proposition 3.5. *With the above notation, the set A of $D(z_1, \ldots, z_n)$ that are not m-split has codimension at least $B_m(\lambda_1, \ldots, \lambda_m)$ in $\mathbf{A}^n$.*

Proof. First, we note that the coefficients of $P_{\lambda_1,\ldots,\lambda_n}$ are homogeneous in $z_1, \ldots, z_n$, so it suffices to prove the statement for the image of A in the projective space $\mathbf{P}^{n-1}$ with homogeneous coordinates $z_1, \ldots, z_n$. Let j be minimal such that $\tau_j \in [\delta + 1, m - 1]$. Consider the projective subspace $L \subset \mathbf{P}^{n-1}$ given by the vanishing of $z_1, \ldots, z_j$. This is a codimension j subspace, corresponding to the polynomials

$$P_{\lambda_1+\cdots+\lambda_j,\lambda_{j+1},\ldots,\lambda_n}(s,t) = t^{\tau_j}(t - z_{j+1}s)^{\lambda_{j+1}} \cdots (t - z_n s)^{\lambda_n}$$

in the variables $z_{j+1}, \ldots, z_n$. The lowest t-degree term of this polynomial is

$$z_{j+1}^{\lambda_{j+1}} \cdots z_n^{\lambda_n} t^{\tau_j} s^{p+\delta-\tau_j}.$$

Since $\tau_n < 2m$, we see that this term must vanish in order for the polynomial not to be m-split. This means that one of $z_{j+1}, \ldots, z_n$ must vanish, giving a union of hyperplanes in L. The restriction of P to each of these hyperplanes has the form

$$P_{\tau_j+\lambda_q,\lambda_{j+1},\ldots,\lambda_n}$$

(with λ_q taken out of the sequence). Since the λ_i are non-increasing, the shortest sequence of such polynomials whose lowest t-degree term lies in $[\delta + 1, m - 1]$ has length $B := B_m(\lambda_1, \ldots, \lambda_m)$. It follows that $A \cap L$ has codimension at least B in L. Since the codimension can only go down upon intersection, we conclude that A has codimension at least B, as desired. □

Corollary 3.6. *In the above notation, suppose $\sum \lambda_i \leq 24$. For any positive integer q we have*

$$B_{12q+1}(q\lambda_1, \ldots, q\lambda_n) = B_{13}(\lambda_1, \ldots, \lambda_n).$$

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Proof. Write $\sum \lambda_i = 13 + \delta$ and $\sum q\lambda_i = 12q + 1 + \delta'$. We have $13q + \delta q = 12q + 1 + \delta'$, so that $q(1 + \delta) = \delta' + 1$. Note that the partial sums $\tau_j(q\lambda_1, \ldots, q\lambda_n)$ satisfy

$$\tau_j(q\lambda_1, \ldots, q\lambda_n) = q\tau_j(\lambda_1, \ldots \lambda_n).$$

The bounds on the partial sums $\tau_j(q\lambda_1, \ldots, q\lambda_n)$ to compute $B_{12q+1}(q\lambda_1, \ldots, q\lambda_n)$ are $\delta' + 1$ and $12q$. On the other hand, by the above calculation we have $\delta' + 1 = q(1 + \delta)$. The set of $\tau_j(q\lambda_1, \ldots, q\lambda_n)$ lying between $\delta' + 1$ and $12q$ is thus the same as the set of $\tau_j(\lambda_1, \ldots, \lambda_n)$ lying between $\delta + 1$ and 12. This gives the desired equality. □

COROLLARY 3.7. *Suppose* $\sum_{i=1}^n \lambda_i \leq 24$ *and let* $B = B_{13}(\lambda_1, \ldots, \lambda_n)$. *Then the locus of* $(z_1, \ldots, z_n) \in \mathbf{A}^n$ *such that*

$$\frac{p^{2e} - 1}{12} D(z_1, \ldots, z_n)$$

is not p^{2e}*-split for some positive integer* e *can be written as an ascending union of closed subschemes* Z_{2e} *each of codimension at least* B. *In particular,* $\bigcup_e Z_{2e}$ *cannot contain any closed subspace* $Y \subset D(z_1, \ldots, z_n)$ *of codimension smaller than* B.

Proof. This follows immediately from Corollary 3.6, once we note that $p^{2e} \equiv 1 \pmod{12}$ for all positive integers e and all primes $p > 3$. □

Example 3.8. Consider the polynomial

$$(t - z_1 s)^2 (t - z_2 s)^2 \cdots (t - z_8 s)^2$$

of total degree 16. The partial sums are $(2, 4, 6, 8, 10, 12, 14, 16)$. We have

$$B_{13}(2, 2, 2, 2, 2, 2, 2, 2) = \#\{i \mid 4 \leq \tau_i \leq 12\} = 5.$$

We conclude that the codimension of the non-13-split locus in $\mathbf{A}^8$ is at least 5. As we will see below, this implies that the locus of Weierstrass fibrations associated to elliptic K3 surfaces whose additive fiber configuration is 8 II and which are not ∞-Frobenius split (see § 4) has codimension at least 13 in the space of Weierstrass fibrations (see § 7). Note that the locus admitting configuration 8 II has codimension 8 in the space of Weierstrass fibrations, as explained in § 7. This is the smallest codimension bound on a non-∞-Frobenius split locus to come out of our methods.

4. Frobenius split elliptic fibrations

In this section we introduce a notion of Frobenius splitting of an elliptic fibration. Let $f : X \to C$ be an elliptic surface. For any $e > 0$, there is an induced diagram

$$\begin{array}{ccccc} Y_e & \xrightarrow{g_e} & X^{(e)} & \xrightarrow{W^e} & X \\ & \searrow^{\varphi_e} & \downarrow{\scriptstyle f^{(e)}} & & \downarrow{\scriptstyle f} \\ & & C & \xrightarrow{F^e} & C \end{array} \tag{4.0.1}$$

where F^e is the eth power of the absolute Frobenius of C, the square is cartesian, and $g_e : Y_e \to X^{(e)}$ is the minimal resolution of $X^{(e)}$. For the sake of symmetry, we will write $Y_0 = X$ and $\varphi_0 = f$. There is an induced map

$$\mathbf{R}^1 f_* \mathcal{O}_X \to F^e_* \mathbf{R}^1 (\varphi_e)_* \mathcal{O}_{Y_e} \tag{4.0.2}$$

of locally free sheaves on C. We note that $\mathbf{R}^1 f_* \mathcal{O}_X$ is an invertible sheaf, while the sheaf on the right-hand side is locally free of rank p^e.

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Definition 4.1. An elliptic fibration $f : X \to C$ is *e-Frobenius split* if the map (4.0.2) is split. If f is e-Frobenius split for all $e > 0$ then we say that f is *∞-Frobenius split*.

Note that both sheaves in (4.0.2) are unchanged upon replacing X and Y_e with their relatively minimal models. In particular, if $f' : X' \to C$ is the relatively minimal model of f, then f' is e-Frobenius split if and only if f is e-Frobenius split.

Recall that a variety Z is e-Frobenius split if the map $\mathcal{O}_Z \to F^e_{Z*}\mathcal{O}_Z$ is split.

Proposition 4.2. *Let $f : X \to C$ be an elliptic surface and e a positive integer.*

(i) *If X is e-Frobenius split, then f is e-Frobenius split.*
(ii) *If f is e-Frobenius split, then C is e-Frobenius split.*

Proof. The eth absolute Frobenius $F^e_X \colon X \to X$ factors through $X^{(e)}$. By minimality of Y_e, it also factors through the map g_e, giving a map $h : X \to Y_e$, and hence a map of sheaves $\mathcal{O}_{Y_e} \to h_*\mathcal{O}_X$. Applying $F^e_*\mathbf{R}^1(\varphi_e)_*$, we find a map

$$F^e_*\mathbf{R}^1(\varphi_e)_*\mathcal{O}_{Y_e} \to F^e_*\mathbf{R}^1(\varphi_e)_*h_*\mathcal{O}_X \to \mathbf{R}^1 f_*F^e_{X*}\mathcal{O}_X$$

where the second map is induced by the appropriate Leray spectral sequence. We obtain the following diagram.

$$\begin{array}{ccccc} \mathbf{R}^1 f_*\mathcal{O}_X & & \longrightarrow & & \\ \downarrow & \searrow & & \searrow & \\ F^e_*\mathbf{R}^1 f^{(e)}_*\mathcal{O}_{X^{(e)}} & \longrightarrow & F^e_*\mathbf{R}^1(\varphi_e)_*\mathcal{O}_{Y_e} & \longrightarrow & \mathbf{R}^1 f_*F^e_{X*}\mathcal{O}_X \end{array} \tag{4.2.1}$$

Note that the vertical arrow is identified with the canonical map $\mathcal{O}_C \to F^e_*\mathcal{O}_C$ tensored with the invertible sheaf $\mathbf{R}^1 f_*\mathcal{O}_X$. Now, a splitting of $\mathcal{O}_X \to F^e_*\mathcal{O}_X$ gives a splitting of the long diagonal arrow of (4.2.1), which induces a splitting of the diagonal arrow (4.0.2), and a splitting of (4.0.2) induces a splitting of the vertical arrow. □

Remark 4.3. Neither the converse of (i) nor the converse of (ii) holds (see Examples 5.1 and 5.3). Indeed, we will show that if X is a very general supersingular K3 surface of Artin invariant ≤ 9, then every Jacobian elliptic fibration on X is e-Frobenius split for all e. However, a supersingular K3 surface is never e-Frobenius split for any e.

Example 4.4. Suppose $f : X \to C$ is an elliptic surface that is semistable (that is, f has no additive fibers). This implies, in the notation of the diagram (4.0.1), that the Frobenius pullback $X^{(e)}$ has only rational singularities. By construction, the map (4.0.2) always factors as

$$\mathbf{R}^1 f_*\mathcal{O}_X \to F^e_*\mathbf{R}^1 f^{(e)}_*\mathcal{O}_{X^{(e)}} \xrightarrow{\sim} F^e_*\mathbf{R}^1 f^{(e)}_* g_{e*}\mathcal{O}_{Y_e} \to F^e_*\mathbf{R}^1(\varphi_e)_*\mathcal{O}_{Y_e}. \tag{4.4.1}$$

As $X^{(e)}$ has rational singularities, the map $\mathbf{R}^1 f^{(e)}_* g_{e*}\mathcal{O}_{Y_e} \to \mathbf{R}^1(\varphi_e)_*\mathcal{O}_{Y_e}$ is an isomorphism, and hence the rightmost arrow of (4.4.1) is an isomorphism. But the map

$$\mathbf{R}^1 f_*\mathcal{O}_X \to F^e_*\mathbf{R}^1 f^{(e)}_*\mathcal{O}_{X^{(e)}}$$

is identified with the map

$$\mathbf{R}^1 f_* f^*(\mathcal{O}_C) \to \mathbf{R}^1 f_* f^*(F^e_*\mathcal{O}_C).$$

We conclude that a semistable elliptic surface over an e-Frobenius split curve is itself e-Frobenius split. In particular, a semistable elliptic K3 surface is ∞-Frobenius split. We will strengthen this observation in Proposition 4.10.

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LEMMA 4.5. *If $f\colon X \to C$ is e-Frobenius split for some $e \geq 1$, then f is also e'-Frobenius split for all $1 \leq e' \leq e$.*

Proof. Using the universal property of the minimal resolution, we find a factorization

$$\mathbf{R}^1 f_* \mathcal{O}_X \to \mathbf{R}^1(\varphi_{e'})_* \mathcal{O}_{Y_{e'}} \to \mathbf{R}^1(\varphi_e)_* \mathcal{O}_{Y_e}.$$

Thus, a splitting of (4.0.2) for e gives rise to a splitting for e'. □

We record a cohomological consequence of being e-Frobenius split. Consider the semilinear morphism

$$\gamma_e(f) : \mathrm{H}^2(X, \mathcal{O}_X) \to \mathrm{H}^2(Y_e, \mathcal{O}_{Y_e}) \tag{4.5.1}$$

arising from the composition of g_e^* with the canonical isomorphism

$$\mathrm{H}^2(Y_e, \mathcal{O}_{Y_e}) \xrightarrow{\sim} \mathrm{H}^2(X', \mathcal{O}_{X'}).$$

LEMMA 4.6. *$f : X \to \mathbf{P}^1$ is e-Frobenius split. Then the map $\gamma_e(f)$ (4.5.1) is injective.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(C, \mathbf{R}^1 f_* \mathcal{O}_X) & \longrightarrow & \mathrm{H}^1(C, \mathbf{R}^1(\varphi_e)_* \mathcal{O}_{Y_e}) \\ \downarrow & & \downarrow \\ \mathrm{H}^2(X, \mathcal{O}_X) & \xrightarrow{\gamma_e(f)} & \mathrm{H}^2(Y_e, \mathcal{O}_{Y_e}) \end{array}$$

where the vertical arrows are the isomorphisms induced by the Leray spectral sequence, and the upper horizontal arrow is the map on H^1 induced by the map (4.0.2). A split injection of sheaves induces an injection on cohomology, which gives the result. □

4.1 Weierstrass fibrations and Tate's algorithm

We briefly recall how to determine the fiber type given the (local) Weierstrass equation for an elliptic fibration. Suppose R is a dvr with uniformizer t, fraction field K, and algebraically closed residue field κ of characteristic at least 5. Given an elliptic curve X_K over K, there is a minimal Weierstrass equation

$$y^2 = x^3 + gt^\alpha x + ht^\beta, \tag{4.6.1}$$

where g and h are units of R and $\alpha, \beta \in \mathbf{N} \cup \{\infty\}$, with the convention that $t^\infty = 0$. Minimality is equivalent to the assertion that $\alpha < 4$ or $\beta < 6$ (or both). The discriminant of the curve is given by the formula

$$\Delta = 4g^3 t^{3\alpha} + 27h^2 t^{2\beta}.$$

We let δ denote the t-adic valuation of Δ.

Tate's algorithm [Tat75] implies that the quantities α, β, and δ determine the Kodaira type Φ of the special fiber. We record the relations between the basic data α, β, δ and the Kodaira type Φ in the first three rows of Table 1. In particular, we highlight the following consequences:

(i) the special fiber is smooth if and only if $\delta = 0$;
(ii) the special fiber is semistable if and only if either $\alpha = 0$ or $\beta = 0$ and is additive otherwise;
(iii) if the fiber is additive, then unless $\alpha = 2$ and $\beta = 3$, the fiber type Φ is determined by δ;
(iv) if $\alpha = 2$ and $\beta = 3$, the fiber has type I_n^* where $n = \delta - 6$.

For instance, if the special fiber of the minimal Weierstrass equation above has Kodaira fiber type $\Phi = \mathrm{III}$, then according to Table 1 we must have $\alpha = 1$, $\beta \geq 2$, and $\delta = 3$. We also introduce

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Table 1. Basic data associated to additive fiber types.

Φ	I_0	$\mathrm{I}_{n\ (n\geq 1)}$	II	III	IV	I_0^*	$\mathrm{I}_{n\ (n\geq 1)}^*$	IV^*	III^*	II^*
α	≥ 0	0	≥ 1	1	≥ 2	≥ 2	2	≥ 3	3	≥ 4
β	≥ 0	0	1	≥ 2	2	≥ 3	3	4	≥ 5	5
δ	0	n	2	3	4	6	$n+6$	8	9	10
$\alpha(\Phi)$	0	0	1	1	2	2	2	3	3	4
$\beta(\Phi)$	0	0	1	2	2	3	3	4	5	5
$\delta(\Phi)$	0	0	2	3	4	6	6	8	9	10
$\zeta(\Phi)$	0	0	1	2	3	4	4	6	7	8

the quantities $\alpha(\Phi), \beta(\Phi), \delta(\Phi)$, and $\zeta(\Phi)$, defined according to the fourth through seventh rows of Table 1. These are respectively the minimum possible values of the quantities α, β, and δ among all fibrations with a fiber of type Φ. In particular, these depend only on the fiber type Φ, and not on the specifics of a minimal Weierstrass equation having reduction type Φ. For the additive fiber types we have

$$\zeta(\Phi) = \alpha(\Phi) + \beta(\Phi) - 1.$$

Furthermore, we observe that if Φ is the Kodaira type of the special fiber of the fibration with the above Weierstrass (4.6.1), then we have

$$\min\{3\alpha, 2\beta\} = \delta(\Phi). \tag{4.6.2}$$

In particular, the left-hand side of (4.6.2) depends only on the Kodaira type Φ. (Note that we use α and β, not $\alpha(\Phi)$ and $\beta(\Phi)$, so this could *a priori* depend upon more than just the fiber type Φ.)

Given a list of Kodaira fiber types $\underline{\Phi}$ (possibly with multiplicities), we let $\alpha(\underline{\Phi})$ denote the sum of the $\alpha(\Phi_i)$ where Φ_i ranges over the elements of $\underline{\Phi}$ (with multiplicity). We similarly define $\beta(\underline{\Phi}), \delta(\underline{\Phi})$, and $\zeta(\underline{\Phi})$.

4.2 A criterion for Frobenius splitting

We now consider a Jacobian elliptic surface $f : X \to \mathbf{P}^1$. Using our computations with divisors in $\mathbf{P}^1$ of § 3, we will derive explicit conditions for f to be Frobenius split. These conditions will be easily computable in practice, given a Weierstrass equation for f. In particular, we will see that the Frobenius splitting of f is controlled by its additive singular fibers. For technical reasons, we will focus on the question of when f is $2e$-Frobenius split. A similar analysis is possible for the odd iterates, with a few modifications arising from the fact that fiber types can change under pullback by odd powers of Frobenius. Given this, the analysis is substantially the same.

Given a point $x \in \mathbf{P}^1$, we let Φ_x denote the Kodaira fiber type of the fiber of X over x.

Definition 4.7. We define a divisor $\Delta^+(f)$ on $\mathbf{P}^1$ by

$$\Delta^+(f) := \sum_{x \in \mathbf{P}^1} \delta(\Phi_x) x.$$

The significance of this divisor is in the following result.

Lemma 4.8. *For each $e > 0$, there is a canonical isomorphism*

$$\iota_{2e} : \mathbf{R}^1(\varphi_{2e})_* \mathcal{O}_{Y_{2e}} \xrightarrow{\sim} ((F^{2e})^* \mathbf{R}^1 f_* \mathcal{O}_X)\left(\frac{p^{2e}-1}{12}\Delta^+(f)\right).$$

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Proof. The base change map gives a natural injection

$$\psi : (F^{2e})^*\mathbf{R}^1 f_*\mathcal{O}_X \to \mathbf{R}^1\varphi_{2e,*}\mathcal{O}_{Y_{2e}},$$

from which we conclude that there is an expression

$$\mathbf{R}^1\varphi_{2e,*}\mathcal{O}_{Y_{2e}} \cong (F^{2e})^*\mathbf{R}^1 f_*\mathcal{O}_X(c)$$

for some c. More precisely, the base change map is an isomorphism over the locus of smooth fibers (in fact, we will see that it is an isomorphism also over the locus of semistable fibers, because the corresponding singularities are rational). To find c, we can work locally and describe the length of the cokernel of ψ around each additive fiber. (This also produces an explicit divisor, which is stronger than simply knowing that the latter sheaf is a twist.)

Let $x \in \mathbf{P}^1$ be a point supporting an additive fiber of X. Let R be the local ring of $\mathbf{P}^1$ at x, let $t \in R$ be a uniformizer, and let K be the field of fractions of R. If the elliptic curve X_K is described by the Weierstrass (4.6.1), then the pulled back elliptic curve $(X_K)^{(p^{2e}/K)}$ is given by the Weierstrass equation

$$y^2 = x^3 + gt^{p^{2e}\alpha}x + ht^{p^{2e}\beta}. \tag{4.8.1}$$

Let $\alpha' = p^{2e}\alpha$ and $\beta' = p^{2e}\beta$ be the exponents of the pulled back Weierstrass equation. This equation may no longer be minimal. That is, we may have $\alpha' \geq 4$ and $\beta' \geq 6$. To make a minimal equation, we repeatedly apply the change of variables

$$x \mapsto t^2x, \quad y \mapsto t^3y$$

and divide the resulting equation by t^6, which has the effect of lowering α' by 4 and β' by 6, until at least one of $0 \leq \alpha' < 4$ and $0 \leq \beta' < 6$ holds. The number of times we need to apply this change of variables is the unique positive integer λ such that at least one of the inequalities

$$0 \leq p^{2e}\alpha - 4\lambda < 4 \quad \text{and} \quad 0 \leq p^{2e}\beta - 6\lambda < 6$$

holds, which is

$$\lambda = \frac{p^{2e}-1}{12}\min\{3\alpha, 2\beta\}.$$

It follows from Table 1 that $\delta(\Phi_x) = \min\{3\alpha, 2\beta\}$ (4.6.2). Moreover, as explained by Schröer [Sch21, Theorem 10.1], this change of variables corresponds geometrically to performing a certain blowup and blowdown of the corresponding surface, and each iteration increments the local value of c by 1. Adding up the local contributions, we get the result. □

The following result gives an explicit criterion for Frobenius splitness of a Jacobian elliptic surface.

PROPOSITION 4.9. *Let $f : X \to \mathbf{P}^1$ be a Jacobian elliptic surface.*

(i) *f is $2e$-Frobenius split if and only if the divisor $((p^{2e}-1)/12)\Delta^+(f)$ is $2e$-Frobenius split.*
(ii) *f is ∞-Frobenius split if and only if the divisor $((p^{2e}-1)/12)\Delta^+(f)$ is $2e$-Frobenius split for all $e > 0$.*

Proof. Consider the base change map

$$\mathbf{R}^1 f_*\mathcal{O}_X \to F^{2e}_*\mathbf{R}^1(\varphi_{2e})_*\mathcal{O}_{Y_{2e}}.$$

By the projection formula and Lemma 4.8, this morphism is identified with the twist of the canonical morphism

$$\chi : \mathcal{O}_{\mathbf{P}^1} \to F^{2e}_*\mathcal{O}_{\mathbf{P}^1}\left(\frac{p^{2e}-1}{12}\Delta^+(f)\right)$$

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attached to the divisor $((p^{2e}-1)/12)\Delta^+(f)$ by the invertible sheaf $\mathbf{R}^1 f_*\mathcal{O}_X$. We thus see that f is $2e$-Frobenius split if and only if $((p^{2e}-1)/12)\Delta^+(f)$ is $2e$-Frobenius split, which proves (1). Claim (2) follows from (1) and Lemma 4.5. □

Proposition 4.10. *If $f : X \to \mathbf{P}^1$ is a Jacobian elliptic surface with at most two additive fibers, then f is ∞-Frobenius split.*

Proof. By composing with an automorphism of $\mathbf{P}^1$, we may assume without loss of generality that all additive fibers of f occur over 0 and ∞. The divisor $\Delta^+(f)$ is then represented by a polynomial of the form $P(s,t) = s^{\lambda_1}t^{\lambda_2}$, and we have $\lambda_1, \lambda_2 \leq 10$ (see Table 1). Let e be a positive integer. We have that

$$\lambda_i \frac{p^{2e}-1}{12} < p^{2e}$$

which implies that $P(s,t)^{(p^{2e}-1)/12}$ is not divisible by $s^{p^{2e}}$ or $t^{p^{2e}}$. By Proposition 3.3, $((p^{2e}-1)/12)\Delta^+(f)$ is p^{2e}-split. By Proposition 4.9 (2) we conclude that f is ∞-Frobenius split. □

5. Some explicit examples

In this section, we discuss the Frobenius splitting behavior of some explicit elliptic fibrations. In light of our applications in the following sections we will focus on examples of fibrations on supersingular K3 surfaces. We will consider the equations for such fibrations obtained by Shioda [Shi87], which not only are supersingular but also exhibit a range of Artin invariants. We continue to assume $p \geq 5$.

Example 5.1 (Shioda). Consider the Weierstrass equation

$$y^2 = x^3 + t^7 \cdot x + t^2.$$

The minimal resolution of the corresponding projective surface is a K3 surface X. Let $f : X \to \mathbf{P}^1$ be the resulting elliptic fibration over the projective line $\mathbf{P}^1$ with homogeneous coordinates $[s : t]$. If $p \neq 17$, then this fibration has one singular fiber of Kodaira type IV (over $[1 : 0]$), one fiber of type III (over $[0 : 1]$), and 17 fibers of type I_1 (over the points $[1 : \zeta]$, where ζ is a 17th root of $-27/4$). Thus, the divisor $\Delta^+(f)$ is represented by the polynomial s^3t^4. As in Proposition 4.10, we see that f is ∞-Frobenius split.

This example was studied by Shioda [Shi87, Example 7], who showed that if p is not congruent to 1 or 17 modulo 34 then X is supersingular, with Artin invariant determined by the residue class of p modulo 34 according to the following list:

$$\sigma_0(X) = \begin{cases} 8 & \text{if } p \equiv 3,5,7,11,23,27,29,31 \pmod{34} \\ 4 & \text{if } p \equiv 9,15,19,25 \pmod{34} \\ 2 & \text{if } p \equiv 13,21 \pmod{34} \\ 1 & \text{if } p \equiv 33 \pmod{34}. \end{cases}$$

Example 5.2 (Shioda). Consider the Weierstrass equation

$$y^2 = x^3 + t \cdot x + t^8.$$

We let $X \to \mathbf{P}^1$ be the corresponding elliptic K3 surface, where, as before, the fibration is over the projective line $\mathbf{P}^1$ with homogeneous coordinates $[s : t]$. If $p \neq 13$, then this fibration has one singular fiber of Kodaira type III (over $[1 : 0]$), one fiber of type IV^* (over $[0 : 1]$), and 13

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of type I_1 (over the points $[1:\zeta]$, where ζ is a 13th root of $-4/27$). Thus, the divisor $\Delta^+(f)$ is represented by the polynomial s^8t^3. As in Proposition 4.10, we see that f is ∞-Frobenius split.

Moreover, Shioda [Shi87, Example 9] shows that, if $p \equiv 7, 11, 15, 19 \pmod{26}$, then X is a supersingular K3 surface of Artin invariant $\sigma_0 = 6$.

Example 5.3 (Shioda). Consider the Weierstrass equation

$$y^2 = x^3 + t^{11} + 1$$

and, as before, let $X \to \mathbf{P}^1$ be the corresponding elliptic K3 surface. If $p \neq 11$, then this is an isotrivial elliptic fibration with 12 singular fibers of Kodaira type II, located at $[0:1]$ and the points $[1:\zeta]$ where ζ is an 11th root of -1. The divisor $\Delta^+(f)$ is given by

$$P(s,t) = s^2(s^{11} + t^{11})^2 = s^2t^{22} + 2s^{13}t^{11} + s^{24}.$$

Let e be a positive integer. Every term of the product

$$P(s,t)^{(p^{2e}-1)/12} = s^{(p^{2e}-1)/6}(s^{11} + t^{11})^{(p^{2e}-1)/6}$$

is divisible by $s^{p^{2e}}$ or $t^{p^{2e}}$. By Proposition 4.9, f is not $2e$-Frobenius split. By Lemma 4.5, we conclude that f is not e-Frobenius split for any $e \geq 2$.

As explained in Example 8 of [Shi87], if $p \equiv 17, 29, 35, 41 \pmod{66}$, then X is a supersingular K3 surface of Artin invariant $\sigma_0 = 5$, and if $p \equiv 65 \pmod{66}$ then X is a supersingular K3 surface of Artin invariant $\sigma_0 = 1$.

6. Purely inseparable multisections on elliptic supersingular K3 surfaces

In this section we derive some properties of pullback maps on cohomology for e-Frobenius split elliptic supersingular K3 surfaces.

We begin by recalling the Artin–Tate isomorphism. Let $f\colon X \to C$ be a Jacobian elliptic fibration. The Leray spectral sequence for the sheaf $\mathbf{G}_m$ on the morphism $f\colon X \to C$ induces an isomorphism

$$\mathrm{Br}(X) \xrightarrow{\sim} \mathrm{H}^1(C, \mathrm{Pic}^\circ_{X/C}) \tag{6.0.1}$$

(see [Tat95, § 3]). Given any Brauer class $\alpha \in \mathrm{Br}(X)$, the Artin–Tate isomorphism (6.0.1) produces a $\mathrm{Pic}^\circ_{X/C}$-torsor $X^\circ_\alpha \to C$. Among all compactifications of X°_α to elliptic surfaces over C, there exists a unique minimal elliptic surface $f_\alpha\colon X_\alpha \to C$ such that the inclusion $X^\circ_\alpha \subset X_\alpha$ is equal to the smooth locus of f_α (see [ASD73, § 1], especially the material after Proposition 1.4). We refer to the image of α under the map

$$\alpha \mapsto (f_\alpha\colon X_\alpha \to C) \tag{6.0.2}$$

as its *associated minimal elliptic surface*. Note that the associated minimal surface of the zero class is isomorphic to the relatively minimal model of X.

Lemma 6.1. *Let $f\colon X \to C$ be a Jacobian elliptic surface and $\alpha \in \mathrm{Br}(X)$ a Brauer class. The following assertions are equivalent.*

(i) $\alpha = 0$.
(ii) *The associated elliptic surface $f_\alpha\colon X_\alpha \to C$ is Jacobian.*
(iii) *The generic fiber $X_{\alpha,k(C)} \to \operatorname{Spec} k(C)$ has a $k(C)$-rational point.*

Proof. It is immediate from the Artin–Tate isomorphism (6.0.1) that $\alpha = 0$ if and only if the torsor $X^\circ_\alpha \to C$ admits a section. If this is the case, then $f_\alpha\colon X_\alpha \to C$ is Jacobian.

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Conversely, any section of f_α must be contained in the smooth locus of f_α, and hence gives rise to a section of the torsor, so (i) is equivalent to (ii). Clearly (ii) implies (iii), and conversely taking the closure of a rational point shows (iii) implies (ii). □

Proposition 6.2. *Let $f\colon X \to C$ be a Jacobian elliptic surface and $\alpha \in \mathrm{Br}(X)$ a Brauer class. The associated minimal elliptic surface $f_\alpha\colon X_\alpha \to C$ admits a purely inseparable multisection if and only if there exists a smooth proper curve C' and a purely inseparable finite cover $C' \to C$ such that for any resolution $Y \to X \times_C C'$ of the pullback the class α is in the kernel of the pullback map*

$$\mathrm{Br}(X) \to \mathrm{Br}(Y).$$

Proof. Note that it suffices to prove the result for any resolution, since the Brauer group is a birational invariant for regular schemes of dimension 2. (Birational invariance of the p-primary part is subtle in higher dimension.) Given any such C' and Y, functoriality of the Leray spectral sequence yields the following diagram.

$$\begin{array}{ccccc}
\mathrm{Br}(X) & \hookrightarrow & \mathrm{Br}(X_{k(C)}) & \xrightarrow{\sim} & \mathrm{H}^1(\operatorname{Spec} k(C), \mathrm{Pic}^\circ_{X_{k(C)}/k(C)}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Br}(Y) & \hookrightarrow & \mathrm{Br}(Y_{k(C')}) & \xrightarrow{\sim} & \mathrm{H}^1(\operatorname{Spec} k(C'), \mathrm{Pic}^\circ_{Y_{k(C')}/k(C')})
\end{array}$$

Here, the right vertical arrow is deduced from the isomorphism $Y_{k(C')} \xrightarrow{\sim} X_{k(C')}$ of generic fibers, the injectivity of the left horizontal arrows follows from the regularity of X and Y, and the isomorphy of the right horizontal arrows follows from the Leray spectral sequence. It follows from Lemma 6.1 that α is in the kernel of the pullback map if and only if there exists a map Σ fitting into the following diagram.

$$\begin{array}{ccc}
 & & X_\alpha \\
 & \overset{\Sigma}{\nearrow} & \downarrow f_\alpha \\
C' & \longrightarrow & C
\end{array}$$

Letting $C' \to C$ range over all purely inseparable covers gives the result. □

Lemma 6.3. *If $f\colon X \to C$ is a relatively minimal Jacobian elliptic surface with section σ such that $\sigma(C)$ has negative self-intersection, then the induced map $\mathrm{Pic}^\circ_C \to \mathrm{Pic}^\circ_X$ is an isomorphism and the quotient $\mathrm{Pic}_X / \mathrm{Pic}^\circ_X$ is torsion-free.*

Proof. This is [Mir89, Lemma VII.1.2]. Note that while Miranda seems to make a uniform (often unstated) assumption in [Mir89] that the base field is $\mathbf{C}$, the results we quote in this paper do not depend upon that assumption, as one can see from the proofs. We do freely use the assumption that the base field has characteristic at least 5. □

Lemma 6.4. *If $f : X \to \mathbf{P}^1$ is a Jacobian elliptic fibration with at least one singular fiber, then, in the notation of diagram (4.0.1), for all $e > 0$ we have that*

$$\deg \mathbf{R}^1(\varphi_e)_* \mathcal{O}_{Y_e} < 0.$$

Proof. As explained in [Mir89, Lemma II.5.7], the degree is at most 0, and it is 0 if and only if the family has all smooth fibers. (The proof there, while it may be implicitly stated over a field of characteristic 0, is true over any field of characteristic at least 5.) On the other hand, we can compute the discriminant of $Y_e \to \mathbf{P}^1$ around a point $x \in \mathbf{P}^1$ (with local uniformizer t) as follows. If Δ is the discriminant of a minimal Weierstrass model for $X \to \mathbf{P}^1$, pullback by

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F^e gives a discriminant of Δ^{p^e} for the naïve pullback of the Weierstrass equation. As in the proof of Lemma 4.8, to make the pulled back curve minimal near x, one scales the coordinates x and y by powers of t so that the discriminant gets scaled by a power of t^{12}. Since p is prime to 12, it is impossible to make $\Delta^{p^e} t^{12n}$ invertible at x, whence Y_e still has a singular fiber at x. □

We now consider a Jacobian elliptic supersingular K3 surface $f\colon X \to \mathbf{P}^1$. Suppose

$$\alpha \in \mathrm{Br}(X \otimes k[\![t]\!])$$

is a Brauer class (which will eventually be the class defined by [BL18, Proposition 2.2.4], representing the universal formal Brauer class of X). By the modular description of the Artin–Tate isomorphism [BL18, Proposition 4.3.16], α comes with an associated family of elliptic K3 surfaces

$$\mathfrak{X} \to \operatorname{Spec} k[\![t]\!] \times \mathbf{P}^1$$

over $\operatorname{Spec} k[\![t]\!]$ whose smooth locus is a family of torsors under the smooth locus of the original Jacobian fibration $X \to \mathbf{P}^1$. Restricting to a geometric generic fiber over a chosen algebraic closure $k((t)) \subset \overline{k((t))}$, we find an elliptic K3 surface

$$\mathfrak{X}_\infty = \mathfrak{X} \otimes \overline{k((t))} \to \mathbf{P}^1_{\overline{k((t))}}$$

over the field $\overline{k((t))}$, which is non-Jacobian whenever α is non-trivial (and thus the associated torsor is non-trivial). We wish to understand when this elliptic surface admits a purely inseparable multisection.

Write $\alpha_\eta \in \mathrm{Br}(X \otimes_k k((t)))$ for the generic value of the Brauer class, and let $\alpha_\infty \in \mathrm{Br}(X \otimes \overline{k((t))})$ denote its restriction to the given geometric generic fiber.

PROPOSITION 6.5. *Suppose that the elliptic surface $\mathfrak{X}_\infty \to \mathbf{P}^1_{\overline{k((t))}}$ admits a purely inseparable multisection. There exists a finite purely inseparable morphism $C \to \mathbf{P}^1$ from a smooth proper curve over k and a resolution $Y \to X \times_{\mathbf{P}^1} C$ of the pullback such that the class α_η is in the kernel of the pullback map*

$$\mathrm{Br}(X \otimes_k k((t))) \to \mathrm{Br}(Y \otimes_k k((t))).$$

Proof. Suppose that $\mathfrak{X}_\infty$ admits a purely inseparable multisection. By Proposition 6.2, there exist a smooth proper curve C_∞ and a finite purely inseparable morphism $\pi\colon C_\infty \to \mathbf{P}^1_{\overline{k((t))}}$ such that, for any resolution of the pullback of $X \otimes \overline{k((t))}$ by π, the class α_∞ is in the kernel of the pullback map. The curve C_∞ has genus 0, so by Tsen's theorem is isomorphic to $\mathbf{P}^1_{\overline{k((t))}}$. By [Sta17, Tag 0CCZ], we can find isomorphisms identifying π with the nth relative Frobenius of $\mathbf{P}^1_{\overline{k((t))}}$ over $\overline{k((t))}$.

Let Y be a smooth resolution of the pullback of X by the nth relative Frobenius of $\mathbf{P}^1$ over k. Consider the following diagram.

$$\begin{array}{ccc} \mathrm{Br}(X \otimes_k \overline{k((t))}) & \longrightarrow & \mathrm{Br}(Y \otimes_k \overline{k((t))}) \\ \uparrow & & \uparrow \\ \mathrm{Br}(X \otimes_k k((t))) & \longrightarrow & \mathrm{Br}(Y \otimes_k k((t))) \end{array}$$

By Lemmas 6.4 and 6.3, we have that Pic_X and Pic_Y are both finitely generated free constant group schemes. Thus,

$$\mathrm{H}^1(\operatorname{Spec} k((t)), \mathrm{Pic}_X) = 0 = \mathrm{H}^1(\operatorname{Spec} k((t)), \mathrm{Pic}_Y),$$

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and it follows from the Leray spectral sequence that the vertical maps are injective. Note that $Y \otimes \overline{k(\!(t)\!)}$ is a resolution of the pullback of $X \otimes \overline{k(\!(t)\!)}$ along π (since Y and X are defined over the algebraically closed field k, where regular and of finite type implies smooth), and therefore α_∞ is in the kernel of the upper horizontal map. It follows that α_η is in the kernel of the lower horizontal map, as desired. □

We come to our main result of this section.

Theorem 6.6. *Let X be an elliptic supersingular K3 surface over k. Suppose*

$$f : X \to \mathbf{P}^1$$

is a Jacobian elliptic fibration such that the map $\gamma_e(f)$ in (4.5.1) is injective for all $e \geq 1$. Suppose

$$\alpha \in \mathrm{Br}(X \otimes k[\![t]\!])$$

is a Brauer class such that

$$\alpha_{X \otimes k[t]/(t^2)} \neq 0$$

but

$$\alpha_X = 0.$$

If $\mathfrak{X} \to \mathbf{P}^1_{k[\![t]\!]}$ is the Artin–Tate family of elliptic surfaces associated to α then the geometric generic fiber $\mathfrak{X}_\infty \to \mathbf{P}^1_{\overline{k(\!(t)\!)}}$ has no purely inseparable multisections.

Proof. Suppose $C \to \mathbf{P}^1$ is a purely inseparable cover. By [Sta17, Tag 0CCZ], we may assume that $C = \mathbf{P}^1$ and the map is the eth power of the Frobenius map. Let $Y = Y_e \to \mathbf{P}^1$ be the resolution studied in § 4. Consider the kernel of the pullback map

$$\mathrm{Br}(X \otimes k[\![t]\!]) \to \mathrm{Br}(Y \otimes k[\![t]\!]).$$

Restricting to $k[\varepsilon] := k[t]/(t^2)$, we get a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^2(X, \mathcal{O}) & \longrightarrow & \mathrm{Br}(X \otimes k[\varepsilon]) & \longrightarrow & \mathrm{Br}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^2(Y, \mathcal{O}) & \longrightarrow & \mathrm{Br}(Y \otimes k[\varepsilon]) & \longrightarrow & \mathrm{Br}(Y) & \longrightarrow & 0 \end{array}$$

in which the vertical arrows are the usual pullback maps. It follows from the hypotheses on α that the restriction

$$\alpha_{k[\varepsilon]} \in \mathrm{Br}(X \otimes k[\varepsilon])$$

of α is the image of a generator for $\mathrm{H}^2(X, \mathcal{O})$. Since each $\gamma_e(f)$ is injective, the left vertical map is injective, and we conclude that $\alpha_{Y \otimes k[\![t]\!]}$ is non-zero. Since Y is smooth over k, we have that $Y \otimes k[\![t]\!]$ is regular, and thus

$$\mathrm{Br}(Y \otimes k[\![t]\!]) \to \mathrm{Br}(Y \otimes k(\!(t)\!))$$

is injective.

Putting everything together, we conclude that $\alpha|_{Y_e \otimes k(\!(t)\!)} \neq 0$ for all e. By Proposition 6.5, we see that the original fibration $\mathfrak{X}_\infty \to \mathbf{P}^1_{\overline{k(\!(t)\!)}}$ cannot have any purely inseparable multisections, as claimed. □

Corollary 6.7. *Suppose $f : X \to \mathbf{P}^1$ is a Jacobian elliptic fibration on a supersingular K3 surface such that, for all $e > 0$, the map $\gamma_e(f)$ of (4.5.1) is injective. Suppose $\alpha \in \mathrm{Br}(X \otimes k[\![t]\!])$*

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is an algebraization of the universal formal Brauer class. If $\mathfrak{X} \to \mathbf{P}^1_{k[\![t]\!]}$ is the Artin–Tate family associated to α, then the geometric generic fibration

$$\mathfrak{X}_\infty \to \mathbf{P}^1_{\overline{k((t))}}$$

does not admit any purely inseparable multisections.

Proof. This follows immediately from Theorem 6.6, using the fact that

$$\alpha_{X \otimes k[t]/(t^2)} \in \widehat{\mathrm{Br}}(k[t]/(t^2)) = \mathrm{H}^2(X, \mathcal{O}_X)$$

is a generator. □

Note that the surfaces of Examples 5.1 and 5.2 satisfy the assumptions of Corollary 6.7. Hence, we have shown in particular that there exists a non-Jacobian elliptic supersingular K3 surface over the algebraically closed field $\overline{k((t))}$ with no purely inseparable multisections. We next show that there exist such surfaces over k.

LEMMA 6.8. *Let $X \subset \mathbf{P}^n$ be a projective scheme of finite presentation over a scheme U that admits a map $X \to \mathbf{P}^1$. Fix a morphism $f : \mathbf{P}^1 \to \mathbf{P}^1$. For a fixed integer e, let $V_e^f \subset \mathrm{Hom}(\mathbf{P}^1_U, X)$ be the closed subscheme parametrizing morphisms $\mathbf{P}^1 \to X$ such that the composition $\mathbf{P}^1 \to X \to \mathbf{P}^1$ equals f and the image $\mathbf{P}^1 \to X \to \mathbf{P}^n$ has degree e. The scheme V_e^f is of finite type over U.*

Proof. Since $\mathbf{P}^1$ is separated, the condition that the composition $\mathbf{P}^1 \to X \to \mathbf{P}^1$ equal f is closed. Thus it suffices to show that the scheme $\mathrm{Hom}_e(\mathbf{P}^1_U, X)$ parametrizing maps of degree e is of finite type. Since $\mathrm{Hom}(\mathbf{P}^1_U, X) \subset \mathrm{Hom}(\mathbf{P}^1_U, \mathbf{P}^n_U)$ is a closed immersion, it suffices to show the same thing for

$$\mathrm{Hom}_e(\mathbf{P}^1_U, \mathbf{P}^n_U) = \mathrm{Hom}_e(\mathbf{P}^1, \mathbf{P}^n) \times U,$$

whence it suffices to show it for the absolute scheme $\mathrm{Hom}_e(\mathbf{P}^1, \mathbf{P}^n)$. By the universal property of projective space, this is given by the scheme of quotients $\mathcal{O}^{n+1} \to \mathcal{O}(e)$ on $\mathbf{P}^1$. This Quot-scheme is of finite type by [Sta17, Tag 0DPA]. □

COROLLARY 6.9. *If k is uncountable, then there exist a supersingular K3 surface X over k and a non-Jacobian elliptic fibration $f : X \to \mathbf{P}^1$ with no purely inseparable multisections.*

Proof. Suppose that $f : X \to \mathbf{P}^1$ is an elliptic supersingular K3 surface such that f is ∞-Frobenius split (see, for instance, Examples 5.1 and 5.2). As shown in § 4.3 of [BL18], there exists a class $\alpha' \in \mathrm{H}^2(X \times \mathbf{A}^1, \boldsymbol{\mu}_p)$ which restricts to a class over $X \otimes k[[t]]$ whose associated Brauer class is the universal Brauer class. Moreover, there is a corresponding elliptic surface $\widetilde{\mathfrak{X}} \to \mathbf{P}^1_{\mathbf{A}^1}$ over $\mathbf{A}^1$.

Let $\mathcal{O}(1)$ be a relative polarization of $\widetilde{\mathfrak{X}}$ over $\mathbf{A}^1$. Given a positive integer e, let $H_{e,b} \subset \mathrm{Hom}_{\mathbf{A}^1}(\mathbf{P}^1, \widetilde{\mathfrak{X}})$ be the Hom scheme parametrizing morphisms whose composition $\mathbf{P}^1 \to \widetilde{\mathfrak{X}} \to \mathbf{P}^1$ is the eth relative Frobenius and whose image has $\mathcal{O}(1)$-degree at most b in each fiber. By Lemma 6.8, $H_{e,b} \to \mathbf{A}^1$ is of finite type. By Corollary 6.7, the image of $H_{e,b}$ does not contain the generic point of $\mathbf{A}^1$, and hence the image is finite. Applying this for all e and b, we see that the fiber of $\widetilde{\mathfrak{X}}$ over a very general point k-point of $\mathbf{A}^1$ is a non-Jacobian elliptic supersingular K3 surface that does not admit a purely inseparable multisection. □

We will show in § 9 that in fact a very general supersingular K3 surface has the property that every non-Jacobian elliptic fibration has no purely inseparable multisections.

As a positive result, we note the following proposition.

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Proposition 6.10. *If $f : X \to \mathbf{P}^1$ is an isotrivial Jacobian elliptic fibration on a supersingular K3 surface with supersingular generic fiber, then*

(i) *any form of f admits a purely inseparable multisection of degree p^2, and*
(ii) *f is not e-Frobenius split for any $e \geq 2$.*

Proof. Let $g : X' \to \mathbf{P}^1$ be an elliptic K3 surface that is a form of f. The Brauer group of a supersingular K3 surface is p-torsion [Art74, Theorem 4.3], so g admits a section after pullback along $X[p] \to \mathbf{P}^1$, where $X[p]$ is the p-torsion group scheme of the smooth locus of f. The generic fiber of f is supersingular, so $X[p] \to \mathbf{P}^1$ is purely inseparable of degree p^2. It follows that the map $Y_2 \to X$ induces the zero map on formal Brauer groups, and therefore the pullback $\mathrm{H}^2(X, \mathcal{O}_X) \to \mathrm{H}^2(Y_2, \mathcal{O}_{Y_2})$ vanishes. By Lemma 4.6, the fibration f is not e-Frobenius split for any $e \geq 2$. □

Remark 6.11. In fact, we do not know of a single example of a non-isotrivial fibration on a supersingular K3 surface that is not Jacobian and admits a purely inseparable multisection.

7. Moduli of Weierstrass data

In this section we recall some of the basic theory of Weierstrass fibrations and their associated data. This is inspired by the discussion in [Mir81] (see also [ASD73]) and mainly serves to fix notation. Fix an algebraically closed field k of characteristic not equal to 2 or 3. For technical reasons, we briefly describe how to give a relative construction over k; this is purely linguistic.

Definition 7.1. Fix a k-scheme T. A *Weierstrass fibration parametrized by T* is a pair $(f : X \to Y, \sigma)$ with

(i) $Y \to T$ a flat morphism of schemes with integral geometric fibers;
(ii) $f : X \to Y$ a proper flat morphism of schemes of finite presentation;
(iii) $\sigma : Y \to X$ a section with image in the smooth locus of f, such that
(iv) each (pointed) geometric fiber of f is an elliptic curve, a nodal rational curve, or a cuspidal rational curve.

If Y is fixed, we will called f a *Weierstrass fibration over Y*. If $Y = Z \times T$ for a fixed k-scheme Z, we will call the pair a *Weierstrass fibration over Z parametrized by T*. (In particular, a Weierstrass fibration over Z parametrized by T is the same thing as a Weierstrass fibration over $Z \times T$.)

An *isomorphism* of Weierstrass fibrations is a commutative diagram of T-schemes

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X' \\ {\scriptstyle\sigma}\uparrow\downarrow{\scriptstyle f} & & {\scriptstyle f'}\downarrow\uparrow{\scriptstyle\sigma'} \\ Y & \xrightarrow{\beta} & Y' \end{array} \tag{7.1.1}$$

in which α and β are isomorphisms. If $Y = Y'$ and f and f' are considered as Weierstrass fibrations over Y, then we require that $\beta = \mathrm{id}$ in (7.1.1).

Given a Weierstrass fibration $(f : X \to Y, \sigma)$, we will write $S \subset X$ for the image of σ and call this the *zero section*.

Notation 7.2. We will write $\mathcal{WF}_Y$ for the category of Weierstrass fibrations over Y. Given a fixed k-scheme Z, we will write $\mathcal{WF}(Z)$ for the fppf k-stack whose fiber over a scheme T is the groupoid $\mathcal{WF}_{Z \times_k T}$. There is an open substack $\mathcal{WF}(Y)^\circ$ whose objects over T are Weierstrass

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fibrations over Y parametrized by T such that for each geometric point $t \to T$, the induced fibration $X_t \to Y_t$ has a smooth fiber.

PROPOSITION 7.3. *If Z is a proper k-scheme then the stack $\mathcal{WF}(Z)$ is a separated Artin stack with finite diagonal locally of finite type over k.*

Proof. It follows from the various definitions that there is a canonical isomorphism $\mathcal{WF}(Z) = \mathrm{Hom}(Z, \mathcal{WF}(\mathrm{Spec}\, k))$. Applying [Ols06, Theorem 1.1], we see that it suffices to establish the result for $Z = \mathrm{Spec}\, k$. We will write $\mathcal{M} := \mathcal{WF}(\mathrm{Spec}\, k)$ for the sake of notational simplicity.

Given a family $\pi : X \to T$, $\sigma : T \to X$ in $\mathcal{M}$, cohomology and base change tells us that $\pi_*\mathcal{O}(3\sigma(T))$ is a locally free sheaf of rank 3, and that the natural morphism $X \to \mathrm{Proj}(\mathrm{Sym}^*\, \pi_*\mathcal{O}(3\sigma(T)))$ is a closed immersion. Let $G \subset \mathrm{GL}_3$ be the subgroup whose image in PGL_2 is the stabilizer of the point $(0:1:0)$. We can construct a natural G-torsor $M \to \mathcal{M}$ whose fiber over (π, σ) is the scheme of isomorphisms $\pi_*\mathcal{O}(3S) \to \mathcal{O}^{\oplus 3}$ with the property that the composition

$$\mathcal{O} \to \pi_*\mathcal{O}(3S) \to \mathcal{O}^{\oplus 3}$$

lands in the span of $(0, 1, 0)$, where the first map above is the adjoint of the map $\mathcal{O}_X \to \mathcal{O}_X(3S)$ corresponding to the divisor $3S$. The G-torsor M admits an open immersion into the space of cubic curves in $\mathbf{P}^2$ passing through $(0:1:0)$.

Consider the diagonal of $\mathcal{M}$. By well-known results (for example, [Gro95, paragraph 4.c]), we know that $\Delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is representable by schemes of finite presentation. We claim that Δ is finite. To see this, it suffices to work with two families $\pi : X \to T$, $\sigma : T \to X$ and $\pi' : X' \to T$, $\sigma' : T \to X$ with T the spectrum of a dvr. The divisors $\mathcal{O}(3S)$ and $\mathcal{O}(3S')$ define embeddings $X, X' \hookrightarrow \mathbf{P}^2_T$. An isomorphism of the generic fibers over T gives a pointed isomorphism, which gives a change of coordinates that conjugates the Weierstrass equations of X_K and X'_K. Since the minimal Weierstrass models over T are unique, this shows that the isomorphism extends uniquely to an isomorphism $X \to X'$, giving properness of Δ. On the other hand, we know that the automorphism group of any fiber has size at most 6, when we conclude that Δ is finite.

It follows that we can identify $\mathcal{M}$ with the quotient stack $[M/G]$, which has finite diagonal. This shows the desired result. (One could also prove this result using Artin's representability theorem, and checking various deformation-theoretic conditions, but the proof given here is more concrete and informative.) □

Now we define what we need in order to globalize the classical Weierstrass equations of elliptic curves and their models.

DEFINITION 7.4. Fix a k-scheme T. A *Weierstrass datum parametrized by T* is a triple (L, a, b) with L an invertible sheaf on T, $a \in \Gamma(T, L^{\otimes -4})$, and $b \in \Gamma(T, L^{\otimes -6})$. The section

$$\Delta := 4a^{\otimes 3} + 27b^{\otimes 2} \in \Gamma(T, L^{\otimes -12})$$

is called the *discriminant* of (L, a, b). An *isomorphism* $(L, a, b) \to (L', a', b')$ of Weierstrass data is given by an isomorphism $\varphi : L \to L'$ of invertible sheaves such that $\varphi^{\otimes -4}(a) = a'$ and $\varphi^{\otimes -6}(b) = b'$.

Notation 7.5. We will write $\mathcal{WD}_T$ for the groupoid of Weierstrass data parametrized by T. Given an integral k-scheme Y of finite type, we will write $\mathcal{WD}(Y)$ for the fppf stack whose fiber over a k-scheme T is the subgroupoid $\mathcal{WD}^\circ_{Y \times_k T} \subset \mathcal{WD}_{Y \times_k T}$ consisting of those Weierstrass data such that the discriminant Δ is not identically zero in any geometric fiber over T.

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Proposition 7.6. *If Y is proper over k then $\mathcal{WD}(Y)$ is an Artin stack locally of finite type over k.*

Proof. There is a representable forgetful morphism $\mathcal{WD}(Y) \to \mathrm{Pic}_{Y/\operatorname{Spec} k}$ whose fiber over an invertible sheaf L on $Y \times T$ is the space of pairs (a, b) of sections of $L^{\otimes -4}$ and $L^{\otimes -6}$ such that the discriminant does not identically vanish, which is itself open in the space of pairs. The scheme of sections of an invertible sheaf is representable by a geometric line bundle. Since $\mathrm{Pic}_{Y/\operatorname{Spec} k}$ is an Artin stack locally of finite type over k, we conclude that the same is true for $\mathcal{WD}(Y)$, as desired. □

Given a Weierstrass datum (L, a, b) parametrized by $Y \times T$, one can associate a Weierstrass fibration $X \to Y \times T$ parametrized by T as follows. Let $f : X \to Y \times T$ be the projection morphism associated to the relative divisor

$$y^2 z = x^3 + axz^2 + bz^3$$

in $\mathbf{P}(L^{\otimes -2} \oplus L^{\otimes -3} \oplus \mathcal{O}_{Y \times T})$. The section σ is defined by $(x : y : z) = (0 : 1 : 0)$ (that is, the projection on the second factor).

Proposition 7.7. *Suppose Y is a proper integral k-scheme. The morphism*

$$\mathcal{WD}(Y) \to \mathcal{WF}(Y)$$

described above is an isomorphism of stacks.

Proof. To prove the statement it suffices to prove that the functor

$$\mathcal{WD}(Y)_T \to \mathcal{WF}(Y)_T$$

is an equivalence (respecting the indicated subcategories), where T is a strictly local ring. This then reduces to [Mir81, Theorem 2.1], whose proof invokes [MS72, Theorem 1′], whose proof is, in turn, an unrecorded relativization of [MS72, Theorem 1]). Furthermore, one can see that the proof of [MS72, Theorem 1] is a consequence of cohomology and base change for the fibers of Weierstrass fibrations, so it applies more generally over non-reduced bases (which is not *a priori* allowed by the hypotheses of [MS72, Theorem 1] and [Mir81, Theorem 2.1]). □

Definition 7.8. A Weierstrass datum (L, a, b) parametrized by a smooth curve Y is *minimal* if for all points $y \in Y$ we have that $\operatorname{ord}_y a < 4$ or $\operatorname{ord}_y b < 6$.

Lemma 7.9. *Given an object (L, a, b) of $\mathcal{WD}(Y)_T$, there is an open subscheme $U \subset T$ such that a T-scheme $Z \to T$ factors through U if and only if, for all geometric points $z \to Z$, the restriction $(L, a, b)_z$ is minimal in the sense of Definition 7.8.*

In other words, the minimal locus is open.

Proof. This is a topological condition, so it suffices to show that the locus R of points $t \in T$ such that $(L, a, b)_t$ is minimal is open. First, note that any Weierstrass datum is locally induced by a Weierstrass datum over a Noetherian scheme. Thus, we may assume T is Noetherian. The locus R is closed under generization, since the order of vanishing of a section of an invertible sheaf can only increase under specialization. To show the desired openness it suffices to show that R is constructible. To do this, we may assume T is integral. There is an open subscheme $U \subset T$ over which the vanishing loci of a and b are flat. Shrinking if necessary and making a finite flat extension of U, we may assume that in fact we can write the vanishing locus of a as $\sum_{i \in I} a_i s_i$ and the vanishing locus of b as $\sum_{j \in J} b_j t_j$, where s_i and t_j are sections of the projection $Y_U \to U$ and $a_i, b_j \in \mathbf{N}$. The points in R_U parametrizing minimal Weierstrass data can be described

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as follows. Given $i \in I$ and $j \in J$, let $Z_{ij} \subset U$ be the image of $s_i \cap t_j$, which is a closed subset. If $Z_{ij} \neq \emptyset$, then R contains Z_{ij} if and only if $a_i < 4$ or $b_j < 6$. We can thus write R_U as the complement of finitely many Z_{ij}, depending upon the values of a_i and b_j. This defines an open subset of U. By Noetherian induction, we see that R is constructible, as desired. □

In this paper we will focus on the case $Y = \mathbf{P}^1$, and we will work with fibrations that are generically smooth.

Notation 7.10. We will write $\mathcal{WF}$ for $\mathcal{WF}(\mathbf{P}^1)^\circ$ and $\mathcal{WD}$ for $\mathcal{WD}(\mathbf{P}^1)$ in what follows. We will write $\mathcal{WD}^n$ for the substack consisting of minimal Weierstrass data (L, a, b) where L has degree n (that is, $L \cong \mathcal{O}_{\mathbf{P}^1}(n)$).

7.1 A concrete cover of $\mathcal{WD}^n$

We can describe $\mathcal{WD}^n$ concretely as follows. Let V_i be the affine space whose underlying k-vector space is $\Gamma(\mathbf{P}^1, \mathcal{O}(i))$, and write $V_i^* = V^i \setminus \{0\}$ and $V_i^{\mathrm{aff}} \subset V_i^*$ for the set of sections that do not vanish at $\infty \in \mathbf{P}^1$. There is an open subset $U_n \subset V_{-4n}^* \times V_{-6n}^*$ consisting of pairs $a \in V_{-4n}^*$, $b \in V_{-6n}^*$ such that $4a^3 + 27b^2 \neq 0$.

LEMMA 7.11. *The universal pair (a, b) over U_n gives a smooth cover $\chi : U_n \to \mathcal{WD}^n$. In addition, the fibers of χ have dimension* 1.

Proof. By Proposition 7.7, any deformation of a family is induced by a deformation of the sections a and b, so χ is smooth. On the other hand, the fibers of χ are precisely the $\mathbf{G}_m$-orbits in U, so χ has relative dimension 1. □

COROLLARY 7.12. *The stack $\mathcal{WD}^{-2}$ is a tame Deligne–Mumford (DM) stack of dimension* 21.

Proof. The automorphism group of a datum $(\mathcal{O}(-2), a, b)$ is given by scalar multiplications $s : \mathcal{O}(-2) \to \mathcal{O}(-2)$ such that the action of s on preserves a and b. In particular, if a and b are both non-zero, then $s^4 = 1 = s^6$. Thus, $s^2 = 1$, so the automorphism group is $\boldsymbol{\mu}_2$. Similarly, if $b = 0$ then the automorphism group is $\boldsymbol{\mu}_4$ and if $a = 0$ then the automorphism group is $\boldsymbol{\mu}_6$. Since $p \geq 5$, we see that $\mathcal{WD}^{-2}$ is a tame DM stack.

To compute the dimension, note that $V_8 \times V_{12}$ has dimension 22. By Lemma 7.11, we see that $\mathcal{WD}^{-2}$ has dimension 21, as desired. □

7.2 The cover by universal polynomials

For the sake of computation, there is a further flat covering that will be useful.

DEFINITION 7.13. Let $\mathcal{P} \subset \mathbf{A}^{-10n+2}$ be the space of tuples

$$(a_0, a_1, \ldots, a_{-4n}, b_0, b_1, \ldots, b_{-6n})$$

such that the polynomial

$$\Delta = 4a_0^3\left(\prod_{i=1}^{-4n}(t - a_i)\right)^3 - 27b_0^2\left(\prod_{j=1}^{-6n}(t - b_j)\right)^2$$

is not identically 0 and the associated Weierstrass fibration is relatively minimal. (This is open by Lemma 7.9.)

There is a universal Weierstrass equation defined over $\mathcal{P}$, namely

$$y^2 = x^3 + a_0\left(\prod_{i=1}^{-4n}(t - a_i)\right)x + b_0\prod_{j=1}^{-6n}(t - b_j). \tag{7.13.1}$$

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It has the property that the fiber over $t = \infty$ is not an additive fiber. There is a natural decomposition

$$\mathcal{P} = \mathcal{P}^{ab\neq 0} \sqcup \mathcal{P}^{a=0} \sqcup \mathcal{P}^{b=0}.$$

The open locus $\mathcal{P}^{ab\neq 0}$ corresponds to fibrations where $a \neq 0$ and $b \neq 0$. The closed subsets $\mathcal{P}^{a=0}$ and $\mathcal{P}^{b=0}$ are given by the vanishing of a_0 and b_0 (and hence the corresponding coefficient in the Weierstrass equation), respectively. One sees that $\mathcal{P}^{a=0}$ corresponds to Weierstrass fibrations that are isotrivial with fiber of j-invariant 0, while $\mathcal{P}^{b=0}$ parametrizes Weierstrass fibrations that are isotrivial with fiber of j-invariant 1728. Since these properties are invariant under the action of PGL_2, we see that this stratification covers a similar decomposition

$$\mathcal{WD}^n = \mathcal{WD}^{n,ab\neq 0} \sqcup \mathcal{WD}^{n,a=0} \sqcup \mathcal{WD}^{n,b=0}.$$

We describe a useful covering of the various strata $\mathcal{WD}^{n,\star}$ by copies of the corresponding stratum $\mathcal{P}^\star$.

Lemma 7.14. *Fix a value of $\star \in \{ab \neq 0, a = 0, b = 0\}$. There is a set of flat morphisms of finite presentation*

$$\rho_s^\star : \mathcal{P}^\star \to \mathcal{WD}^{n,\star}, \quad s \in S_\star$$

such that:

(i) *the $\rho_s^\star$ form an fppf covering of $\mathcal{WD}^{n,\star}$;*
(ii) *for each $\rho_s^\star$, the pullback of the universal Weierstrass equation is isomorphic to (7.13.1) (with the appropriate coefficient equal to 0, depending upon the value of $\star$).*

Proof. We claim that the morphism

$$(a_0, a_1, \ldots, a_{-4n}, b_0, b_1, \ldots, b_{-6n}) \mapsto \left(a_0 \prod_{i=1}^{-4n} (x - a_i y), b_0 \prod_{j=1}^{-6n} (x - b_j y) \right)$$

defines flat morphisms of finite presentation

$$\mathcal{P}^\star \to \mathcal{WD}^{n,\star}$$

whose image consists of those Weierstrass data corresponding to fibrations that do not have an additive fiber over ∞. Assuming this, we see that composing with the elements $s \in \mathrm{PGL}_2(k)$ gives a set of morphisms $\rho_s^\star$ covering all of $\mathcal{WD}^{n,\star}$, since any fibration possesses a smooth (in particular, non-additive) k-fiber.

To show flatness, we first note that we can write this as a product of two maps (one involving the a_i and one involving the b_j). Since the cases $a = 0$ or $b = 0$ add a trivial affine factor, it suffices to show that, in general, the morphism

$$(c_0, c_1, \ldots, c_m) \mapsto c_0 \prod_{\ell=1}^{m} (x - c_\ell y)$$

from the open subspace $c_0 \neq 0$ in $\mathbf{A}^{m+1}$ to the space of forms V_m is flat. The map is equivariant for the free scaling action of $\mathbf{G}_m$ on c_0, so it suffices to show that the induced map

$$(c_1, \ldots, c_m) \to \prod (x - c_\ell y)$$

is flat. The coordinates on V_m are given by the coefficients, which are the elementary symmetric functions in the c_i. This is then a morphism $\mathbf{A}^m \to \mathbf{A}^m$ with finite fibers (since $k[c_1, \ldots, c_m]$ is a unique factorization domain), whence it must be flat (since both domain and codomain are regular). □

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This decomposition will allow us to study more easily the locus of fibrations with a given additive fiber configuration.

7.3 Weierstrass data attached to a family of marked surfaces

Let $X \to M$ be a family of K3 surfaces equipped with a pair of divisor classes $a, b \in \mathrm{Pic}(X)$ such that, for all geometric points $m \to M$, we have that

(i) $a|_{X_m}$ is the class associated to $\mathcal{O}_{X_m}(E)$ for a smooth irreducible curve $E \subset X_m$ of genus 1, and
(ii) $a|_{X_m} \cdot b|_{X_m} = 1$.

Let $M' \to M$ be the stack whose fiber over $T \to M$ is the groupoid of Weierstrass fibrations

$$(f : X \times T \to \mathbf{P}^1 \times T, \sigma)$$

such that $[f^*\mathcal{O}(1)] = a$ and $\mathcal{O}_{X\times T}(\mathrm{im}\,\sigma) = b$ as sections of $\mathrm{Pic}_{X/M}$.

Lemma 7.15. *The morphism $M' \to M$ is smooth with fibers of dimension* 3.

Proof. Let $M'' \to M$ be the stack whose objects over $T \to M$ are elliptic fibrations $X_T \to \mathbf{P}^1_T$ with fibers of class a, without a chosen section s.

The morphism $M' \to M$ factors as

$$M' \to M'' \to M$$

by sending $(f : X \to \mathbf{P}^1, \sigma)$ to the fibration $(f : X \to \mathbf{P}^1)$ and then to the invertible sheaf $f^*\mathcal{O}(1)$. We claim that $M' \to M''$ is étale and $M'' \to M$ is a PGL_2-torsor.

To see the first assertion, note that b is assumed to exist everywhere over M, whence the class of the section s is defined everywhere. Since $X \to M$ is a fibration of K3 surfaces, any section in a fiber with class b will locally deform uniquely. It follows that $M' \to M''$ is étale.

It remains to prove that $M'' \to M$ is a PGL_2-torsor. Given a fibration $f : X \to \mathbf{P}^1$ with fibers of class a, there is a natural action of $\alpha \in \mathrm{PGL}_2$ that sends f to αf. This is an action on M'' over M. To establish that $M = M''/\mathrm{PGL}_2$, it thus suffices to work locally and assume that T is a strictly Henselian local scheme. In this case, every fibration $f : X_T \to \mathbf{P}^1_T$ with $[f^*\mathcal{O}(1)] = a$ is given by choosing a basis for $\Gamma(X_T, L)$, up to scaling. The bases are permuted simply transitively by $\mathrm{GL}_2(T)$. Dividing by scalars gives the desired result. □

Suppose S_n is the Ogus moduli space of marked supersingular K3 surfaces of fixed Artin invariant n; a point of S_n is a pair $(X, \tau : \Lambda_n \xrightarrow{\sim} \mathrm{NS}(X))$ consisting of a supersingular K3 surface and an isomorphism of the standard lattice of Artin invariant n with the Néron–Severi lattice of X. Let $N \to S_n$ be the space of triples $((X, \tau), a, b)$ where (X, τ) is in S_n and $a, b \in \Lambda_n$ represent a Jacobian elliptic fibration: a is the class of a smooth genus 1 curve, b is the class of a smooth rational curve, and $a \cdot b = 1$. There is a countable open cover $V_i \subset N$ such that V_i is of finite type over k and $V_i \to S_n$ is étale. There is a PGL_2-torsor $M'_i \to V_i$ parametrizing Weierstrass fibrations whose fiber over $((X, \tau), a, b)$ is the space of fibrations $f : X \to \mathbf{P}^1, \sigma$ with fiber of class a and section of class b. Via the equivalence of Proposition 7.7, there is a resulting morphism $w : M'_i \to \mathcal{WD}^{-2}$. Moreover, every Weierstrass datum corresponding to a Jacobian elliptic fibration on a supersingular K3 surface of Artin invariant n is in the image of one of the M'_i.

Corollary 7.16. *In the notation of the preceding paragraph, the morphism $w : M'_i \to \mathcal{WD}^{-2}$ is quasi-finite.*

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Proof. It suffices to show that the geometric fibers of w are countable, since M_i' and $\mathcal{WD}^2$ are of finite type over k. Suppose

$$w((X,\tau),a,b,f,\sigma) = w((X',\tau'),a',b',f',\sigma').$$

This implies that there is an isomorphism $g : X \to X'$ such that $f' \circ g = f$ and

$$g^*(\tau'(b')) = \tau(b).$$

Since (-2)-curves are rigid, we have that $\sigma \circ g = \sigma'$. In addition, for tautological reasons, we have

$$g^*(\tau'(a')) = \tau(a).$$

(In fact, writing both a and f in the notation is redundant, but we are trying to respect M_i''s origin in a tower of morphisms.)

We thus find that $((X',\tau'),a',b',f',\sigma')$ is isomorphic to $((X,\tau''),a,b,f,\sigma)$ for some other marking $\tau'' : \Lambda_n \xrightarrow{\sim} \mathrm{NS}(X)$ such that $\tau''(a) = \tau(a)$ and $\tau''(b) = \tau(b)$. Conversely, any marking that preserves the classes of a and b gives rise to a point with the same image in $\mathcal{WD}^{-2}$. Thus, we find that the (geometric) fiber of w over $w((X,\tau),a,b,f,\sigma)$ is identified with the subset of the automorphism group $\mathrm{Aut}(\Lambda_n)$ that fixes a and b and such that the resulting point $((X,\tau''),a,b,f,\sigma)$ lies in M_i'. Since $\mathrm{Aut}(\Lambda_n)$ is countable, we conclude that the fibers of w are countable, as desired. □

Corollary 7.17. *The locus of Artin invariant s surfaces in $\mathcal{WD}^{-2}$ is a countable union of constructible subsets, each of which contains a locally closed subspace of dimension $s+2$ (equivalently, codimension $19-s$).*

Proof. By Corollary 7.16, the space of such surfaces is a countable union of images of quasi-finite maps from PGL_2-torsors over an étale morphisms to the moduli space of Artin invariant s surfaces. The latter space has dimension $s-1$, so each torsor has dimension $s+2$. By Chevalley's theorem, the image contains a locally closed subspace, which is necessarily of dimension $s+2$, as claimed. □

7.4 Conditions imposed on $\mathcal{WD}^n$ by Kodaira fiber types

To a minimal Weierstrass datum $(\mathcal{O}_{\mathbf{P}^1}(n),a,b)$ we can associate a minimal resolution of the associated Weierstrass fibration, yielding a smooth relatively minimal elliptic surface

$$X(\mathcal{O}(n),a,b) \to \mathbf{P}^1.$$

The singular fibers of such a surface were described by Kodaira in characteristic 0 and Tate in positive characteristic [Tat75]. Here we consider the conditions imposed on $\mathcal{WD}^n$ by the presence of specific singular fibers. We fix a negative integer n and write simply $\mathcal{WD}$ instead of $\mathcal{WD}^n$.

In the following, we will write Φ for a single Kodaira fiber type and $\underline{\Phi}$ for an unordered list of fiber types (possibly with multiplicities). We focus on fibers of additive type in the Weierstrass fibration.

Notation 7.18. Given a list of additive Kodaira fiber types $\underline{\Phi}$, we will say that a minimal Weierstrass datum $d := (\mathcal{O}(n),a,b)$ *realizes* $\underline{\Phi}$ if for each additive Kodaira fiber type Φ appearing in $\underline{\Phi}$ with non-zero multiplicity k, the associated minimal elliptic surface $X(d) \to \mathbf{P}^1$ has precisely k fibers of type Φ (no restriction is imposed on the semistable fibers of $X(d)$). We will write $\mathcal{WD}[\underline{\Phi}]$ for the locus of Weierstrass data realizing $\underline{\Phi}$.

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We can write the basic data α, β, δ sheaf-theoretically for a Weierstrass datum $(\mathcal{O}(n), a, b)$ in terms of the multiplicity of the divisors associated to the sections $a \in \Gamma(\mathbf{P}^1, \mathcal{O}(-2n))$, $b \in \Gamma(\mathbf{P}^1, \mathcal{O}(-3n))$, and $\Delta \in \Gamma(\mathbf{P}^1, \mathcal{O}(-6n))$.

LEMMA 7.19. *The locus $\mathcal{WD}[\underline{\Phi}] \subset \mathcal{WD}$ is locally closed.*

Proof. It suffices to prove this statement for $\mathcal{P}[\underline{\Phi}] \subset \mathcal{P}$. There is a universal Weierstrass datum $(\mathcal{O}(n), a, b)$ over $\mathcal{P}$. Let $\underline{\Phi} = (\Phi_1, \dots, \Phi_m)$. Let $\Delta \in \Gamma(\mathbf{P}^1 \times \mathcal{P}, \mathcal{O}(-6n))$ be the discriminant, and consider the scheme $Z = Z(a) \cap Z(b) \subset \mathbf{P}^1 \times \mathcal{P}$. The natural morphism $Z \to \mathcal{P}$ is finite but not necessarily flat. There is an open locus $U \subset \mathcal{P}$ over which the geometric fibers of Z have precisely m connected components, corresponding to precisely m additive fibers (because the coefficients a and b of the Weierstrass equation have a common zero at precisely those points). The étale-local structure theory for finite morphisms tells us that there is a factorization $Z_U \to Y \to U$ with $Y \to U$ finite étale of degree m and $Z_U \to Y$ radicial. Replacing U with $\mathrm{Isom}(Y, \sqcup_{i=1}^m U)$ gives an S_m-torsor $I \to U$ over which Y splits. Replacing U with I, we may assume that there are sections $c_1, \dots, c_m \in \mathbf{P}^1(U)$ that support the m additive fibers.

Choose a bijection $\sigma \in S_m$. For each fiber type Φ_i appearing in $\underline{\Phi}$, the condition that $a_{\sigma(i)}$ supports Φ_i is equivalent to a certain requirement on the orders of vanishing of a, b, and Δ at c_i, as determined by Table 1. For each Φ_i the resulting vanishing conditions determine a locally closed locus in I. Taking the union over all $\sigma \in S_m$ gives an S_m-invariant locally closed set of I whose image in $\mathcal{P}$ is precisely $\mathcal{P}[\underline{\Phi}]$. This shows that $\mathcal{P}[\underline{\Phi}]$ is locally closed in U, as desired. □

The theory of § 7.2 gives a way to calculate the codimension of $\mathcal{WD}[\underline{\Phi}]$. Given an additive fiber configuration $\underline{\Phi}$, let $\mathcal{P}[\underline{\Phi}] \subset \mathcal{P}$ denote the closed subspace over which (7.13.1) has additive fiber configuration of type $\underline{\Phi}$, so that $\mathcal{P}[\underline{\Phi}]$ maps to $\mathcal{WD}[\underline{\Phi}]$.

LEMMA 7.20. *For any additive fiber configuration $\underline{\Phi}$, we have*

$$\mathrm{codim}(\mathcal{WD}[\underline{\Phi}] \subset \mathcal{WD}) = \mathrm{codim}(\mathcal{P}[\underline{\Phi}] \subset \mathcal{P}).$$

Proof. This follows immediately from Lemma 7.14. □

PROPOSITION 7.21. *For a list $\underline{\Phi}$ of additive fibers, the codimension of $\mathcal{WD}^{ab\neq 0}[\underline{\Phi}] \subset \mathcal{WD}^{ab\neq 0}$ is at least equal to $\zeta(\underline{\Phi})$ (see Table 1).*

Proof. It suffices to prove the corresponding statement for $\mathcal{P}^{ab\neq 0}[\underline{\Phi}] \subset \mathcal{P}^{ab\neq 0}$. Consider the universal Weierstrass equation of type $\underline{\Phi}$, given as follows:

$$y^2 = x^3 + a_0' \left(\prod_{i=1}^{m} (t - a_i')^{\alpha_i} \prod_{j=m+1}^{-4n+m-\sum \alpha_i} (t - a_j') \right) x + b_0' \prod_{i=1}^{m} (t - a_i')^{\beta_i} \prod_{j=m+1}^{-6n+m-\sum \beta_i} (t - b_j').$$

Here the first m roots of both coefficients are specified by the fiber types and the α and β values of Table 1. The first factor in each coefficient is required to have distinct roots. The second factors in each coefficient are allowed to have roots coinciding with the first factor whenever the corresponding entry in Table 1 has a $\geq \alpha_i$ or $\geq \beta_i$ (depending upon the fiber type).

The locus $F(\underline{\Phi})$ so described is open in the affine space $\mathbf{A}^{m-10n+2-\sum(\alpha_i+\beta_i)}$ with coordinates

$$(a_0', a_1', \dots, a'_{m-4n-\sum \alpha_i}, b'_{m+1}, \dots, b'_{m-6n-\sum \beta_i}).$$

Choose a partition π of the variables $(a_1, \dots, a_{-4n})$ into m subsets A_i, $i = 1, \dots, m$, of sizes α_i, $i = 1, \dots, m$, and $-4n - \sum \alpha_i$ singletons A_j', and a partition of $(b_1, \dots, b_{-6n})$ into m subsets B_i, $i = 1, \dots, m$, of sizes β_i, $i = 1, \dots, m$, and $-6n - \sum \beta_i$ singletons B_j'. Given such a partition,

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we can define a closed immersion

$$\iota_\pi : \mathbf{A}^{m-10n+2-\sum(\alpha_i+\beta_i)} \to \mathbf{A}^{-10n+2}_{a_0,\dots,a_{-4n},b_0,\dots,b_{-6n}}$$

by the ring map sending all elements of A_i and B_i to a'_i, all of the singletons A'_j to the a'_j, and all of the singletons B'_j to b'_j. The union of the images of $F(\underline{\Phi})$ under all of these maps is precisely $\mathcal{P}[\underline{\Phi}]$. The codimension is thus

$$-2+\sum(\alpha_i+\beta_i) = \sum(\alpha_i+\beta_i-1) \geq \sum \zeta(\Phi_i).$$

The last equality follows from the definition of $\zeta(\Phi)$, as one can see in Table 1. □

8. The non-∞-Frobenius split locus in $\mathcal{WD}^{-2}$

In this section, we establish estimates on the codimension of the supersingular and non-∞-Frobenius split loci in $\mathcal{WD}^{-2}$. We split this analysis into various pieces, which will be assembled in Proposition 8.8 to obtain our final estimate.

Notation 8.1. Suppose that $\underline{\Phi} = (\Phi_1, \dots, \Phi_n)$ is a configuration of additive fibers. Let $\lambda_i = \delta(\Phi_i)$, and suppose that the Φ_i are ordered such that the sequence $(\lambda_1, \dots, \lambda_n)$ is non-increasing. We write $B = B_{13}(\lambda_1, \dots, \lambda_n)$ for the integer defined by (3.4.1).

Proposition 8.2. *The locus in $\mathcal{WD}^{-2,ab\neq 0}[\underline{\Phi}]$ parametrizing fibrations realizing $\underline{\Phi}$ that are not ∞-Frobenius split is a countable union of locally closed subspaces of codimension at least $B+\zeta(\underline{\Phi})$ at every point.*

Proof. Let $\mathcal{P}^{ab\neq 0}[\underline{\Phi}]$ be the preimage of $\mathcal{WD}^{-2,ab\neq 0}[\underline{\Phi}]$ under the flat cover of Lemma 7.14. As in the proof of Proposition 7.21, it is covered by copies of $\mathbf{A}^{22-\zeta(\underline{\Phi})}$ with coordinates

$$(a'_0, a'_1, \dots, a'_{8+n-\sum \alpha_i}, b'_0, b'_1, \dots, b'_{12+n-\sum \beta_i}).$$

To such a point we associate the polynomial

$$\prod_{i=1}^{n}(t-a'_i s)^{\lambda_i}.$$

This is a projection onto the affine space $\mathbf{A}^n$ parametrizing polynomials of the form

$$P(s,t) = \prod_{i=1}^{n}(t-z_i s)^{\lambda_i}.$$

We can thus cover $\mathcal{P}^{ab\neq 0}[\underline{\Phi}]$ by components that are flat over $\mathbf{A}^n$. To prove the proposition, it thus suffices to prove a corresponding codimension statement for certain subsets of $\mathbf{A}^n$.

By Corollary 3.7, the locus in $\mathbf{A}^n$ parametrizing polynomials P such that $P^{(p^{2e}-1)/12}$ is not p^{2e}-split for some e is an ascending union of closed subschemes of codimension at least B. The preimage of this locus in $\mathcal{P}^{ab\neq 0}[\underline{\Phi}]$ is exactly equal to the preimage of the locus in $\mathcal{WD}^{-2,ab\neq 0}[\underline{\Phi}]$ parametrizing fibrations that are not $2e$-Frobenius split for some e. By Lemma 4.5, a fibration is not $2e$-Frobenius split for some e if and only if it is not ∞-Frobenius split. □

Given a configuration $\underline{\Phi}$ of additive fibers, the integer $B+\zeta(\underline{\Phi})$ can be easily computed. We will show that, in almost all cases, this codimension is at least 13 (the remaining cases will be dealt with separately below). To reduce the necessary computations, we first observe that certain configurations can never contain non-∞-Frobenius split fibrations. For example, suppose that

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f is a fibration in $\mathcal{WD}^{-2}$ realizing $\underline{\Phi}$ such that $\deg(\Delta^+(f)) \leq 12$. For every $e > 0$ we have

$$\deg\left(\frac{p^{2e}-1}{12}\Delta^+(f)\right) \leq p^{2e}-1.$$

By Proposition 3.3, this divisor is p^{2e}-split. Therefore, by Proposition 4.9(2) f is ∞-Frobenius split. In fact, something slightly stronger is true.

LEMMA 8.3. *Let $\underline{\Phi}$ be a configuration of additive fiber types and let f be a fibration in $\mathcal{WD}^{-2}$ realizing $\underline{\Phi}$. If there exists an i such that $\delta(\underline{\Phi}) - \delta(\Phi_i) \leq 12$, then f is ∞-Frobenius split.*

Proof. For ease of notation, suppose that $i = 1$. Let $p_1 \in \mathbf{P}^1$ be the point supporting the fiber of type Φ_1. Applying an automorphism of $\mathbf{P}^1$, we may assume that $p_1 = [1:0]$, and that the fiber over $[0:1]$ is smooth. We then have that the divisor $\Delta^+(f)$ corresponds to the polynomial

$$P(s,t) = t^{\delta(\Phi_1)}\prod_{i=2}^{n}(t - z_i s)^{\delta(\Phi_i)}$$

where the z_i are non-zero. We see that the monomial with the highest power of s appearing in the expansion of $P(s,t)$ is a scalar times $s^{\delta(\underline{\Phi})-\delta(\Phi_1)}t^{\delta(\Phi_1)}$. It follows that the monomial with the highest power of s appearing in the expansion of $P(s,t)^{(p^{2e}-1)/12}$ is the $((p^{2e}-1)/12)$th power of this term. Hence, $P(s,t)^{(p^{2e}-1)/12}$ contains a non-zero term of the form $s^\gamma t^\epsilon$ where $\gamma, \epsilon \leq p^{2e}-1$. As before, it follows from Propositions 3.3 and 4.9(ii) that f is ∞-Frobenius split. □

We make the following definition.

Conditions 8.4. Let $\underline{\Phi}$ be a configuration of additive fibers. Let Φ be a fiber type which achieves the maximal value of $\delta(\Phi)$ among all fiber types appearing in $\underline{\Phi}$. The fiber configuration $\underline{\Phi}$ is *critical* if it satisfies the following conditions:

(i) $\alpha(\underline{\Phi}) \leq 8$;
(ii) $\beta(\underline{\Phi}) \leq 12$;
(iii) $\delta(\underline{\Phi}) - \delta(\Phi) \geq 13$.

The set of critical $\underline{\Phi}$ can be easily computed using Table 1. We have included the resulting list in Table 2. In particular, we notice that every critical configuration has non-∞-Frobenius split locus of codimension at least 13.

PROPOSITION 8.5. *The locus in $\mathcal{WD}^{-2,ab\neq 0}$ parametrizing fibrations that are not ∞-Frobenius split is a countable union of locally closed subspaces of codimension at least 13 at every point.*

Proof. We will show that for each configuration $\underline{\Phi}$ of additive fiber types the non-∞-Frobenius split locus in $\mathcal{WD}^{-2,ab\neq 0}[\underline{\Phi}]$ is a countable union of locally closed subspaces of codimension at least 13 at every point. This will prove the result.

By Lemma 8.3, it suffices to consider configurations $\underline{\Phi}$ that are critical. Using Table 2, we see that, for each such configuration, the codimension bound $B + \zeta(\underline{\Phi})$ produced by Proposition 8.2 is at least 13. (To simplify this computation slightly, one could observe that, if $\zeta(\underline{\Phi}) \geq 13$, then by Proposition 7.21 the codimension of the entire locus of fibrations in $\mathcal{WD}^{-2,ab\neq 0}$ realizing $\underline{\Phi}$ is already at least 13. This leaves only 27 configurations in Table 2 for which we must compute the integer B.) □

PROPOSITION 8.6. *The supersingular locus in $\mathcal{WD}^{-2,a=0}$ obeys the following dimensional constraints.*

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(i) *If* $p \equiv 1 \pmod 3$, *then the supersingular locus in* $\mathcal{WD}^{-2,a=0}$ *is empty.*
(ii) *If* $p \equiv 2 \pmod 3$, *then the supersingular locus in* $\mathcal{WD}^{-2,a=0}$ *is a closed subspace whose codimension in* $\mathcal{WD}^{-2}$ *is at least* 14.

Proof. We recall that, given any K3 surface X, the image of the map $\mathrm{Aut}(X) \to \mathrm{GL}(\mathrm{H}^0(X, \Omega^2_X))$ is a finite cyclic group, whose order is referred to as the *non-symplectic index* of X. Consider a Weierstrass equation $y^2 = x^3 + b(t)$ parametrized by a point in $\mathcal{WD}^{-2,a=0}$. The corresponding projective surface admits an action of μ_3 where ω acts by $x \mapsto \omega x$ and $y \mapsto y$. The action extends to an action of μ_3 on the associated K3 surface X. If $E_t \subset X$ is a smooth fiber of the elliptic fibration on X, then restriction induces an isomorphism $\mathrm{H}^0(X, \omega_X) \xrightarrow{\sim} \mathrm{H}^0(E, \omega_E)$, and the latter cohomology group is generated by the invariant differential $\mathrm{d}x/2y$ (in Weierstrass coordinates). Thus, μ_3 acts non-trivially on $\mathrm{H}^0(X, \Omega^2_X)$, and therefore the non-symplectic index of X is divisible by 3.

The possible non-symplectic indices of supersingular K3 surfaces have been investigated by Jang in [Jan19]. Table 1 of [Jan19] shows that if $p \equiv 1 \pmod 3$, then no supersingular K3 surface admits a non-symplectic automorphism of order 3, and if $p \equiv 2 \pmod 3$, then the locus in S_{10} consisting of supersingular K3 surfaces with a non-symplectic automorphism of order 3 has dimension at most 4. Here, S_{10} is Ogus's moduli space parametrizing supersingular K3 surfaces X together with a marking of $\mathrm{Pic}(X)$ by the supersingular K3 lattice of Artin invariant 10. The result now follows from Lemma 7.15. □

Remark 8.7. In fact, Jang's results show more: the locus of points in S_{10} with a non-symplectic automorphism of order 3 cannot contain any point of even Artin invariant, and for each $0 \leq n \leq 4$, has intersection with the Artin invariant $2n+1$ locus of dimension at most n.

Proposition 8.8. *The locus* $N \subset \mathcal{WD}^{-2}$ *parametrizing fibrations* $(f : X \to \mathbf{P}^1, \sigma)$ *such that* X *is supersingular and* f *is not* ∞*-Frobenius split is a countable union of locally closed subspaces of codimension at least* 13 *at every point.*

Proof. It suffices to show the result separately for the corresponding loci in the space of Weierstrass data with $ab \neq 0$, $a = 0$, and $b = 0$.

The first case follows from Proposition 8.5, and the second follows from Proposition 8.6. Finally, consider $\mathcal{P}^{b=0}$. This subspace itself has codimension 13, so $N \cap \mathcal{WD}^{-2,b=0}$ has codimension at least 13, as well. □

Remark 8.9. In the case $ab \neq 0$, it seems quite difficult to obtain much information on the supersingular locus, at least in this generality. Fortunately, Proposition 8.5 shows that the Frobenius non-split locus is already too small to contain general supersingular K3 surfaces. In the case $a = 0$, the reverse is true: the condition of being not ∞-Frobenius split imposes only one relation, and hence the non-split locus could *a priori* contain a general supersingular K3 surface. However, Proposition 8.6 shows that in fact the supersingular locus is quite small.

9. Very general elliptic supersingular K3 surfaces

In this section we show that for very general supersingular K3 surfaces there are no non-Jacobian elliptic structures that admit purely inseparable multisections. We proceed by first studying Jacobian fibrations and then using supersingular twistor lines to deduce consequences for non-Jacobian fibrations.

Theorem 9.1. *Every Jacobian elliptic structure on a very general supersingular K3 surface of Artin invariant at least* 7 *is* ∞*-Frobenius split.*

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Proof. With the notation introduced in the paragraph preceding Corollary 7.16, we consider the countable collection of quasi-finite morphisms $w : M_i' \to \mathcal{WD}^{-2}$. Each M_i' is a PGL_2-torsor over V_i, which is an open subset of the space of quadruples $((X, \tau), a, b)$ where X is a supersingular K3 surface, τ is a lattice polarization of X by a fixed supersingular K3 lattice, and a, b are the divisor classes of a fiber or section respectively of a Jacobian elliptic fibration on X.

By Corollary 7.17, the Artin invariant s locus in $\mathcal{WD}^{-2}$ has dimension at least $s+2$ at every point. By Proposition 8.8, a very general point of the Artin invariant s locus in $\mathcal{WD}^{-2}$ is ∞-Frobenius split for any s at least 7. The same follows for each V_i. As there are only countably many V_i, the very general point of Artin invariant at least 7 in the Ogus moduli space is not in the image of the non-∞-Frobenius split locus in any of the V_i. In other words, every Jacobian elliptic structure on a very general supersingular K3 surface of Artin invariant at least 7 is ∞-Frobenius split. □

In order to explain consequences of Theorem 9.1, we briefly discuss twistor lines through generic surfaces. Given a positive integer i between 1 and 10, let Λ_i denote the supersingular K3 lattice of Artin invariant i. We will write M_i for the Ogus moduli space of characteristic subspaces which parametrizes supersingular K3 crystals H together with an isometric embedding $\Lambda_i \to H$. As Ogus describes, M_i is smooth over $\mathbf{F}_p$ and irreducible, with $\Gamma(M_i, \mathcal{O}) = \mathbf{F}_{p^2}$. Let S_i denote the algebraic space parametrizing lattice-polarized supersingular K3 surfaces $(X, \Lambda_i \hookrightarrow \mathrm{NS}(X))$. Furthermore, there is an étale morphism $S_i \to M_i$ and a covering family of Zariski opens $U \subset M_i$ that admit sections $U \to S_i$ over M_i (see the proof of [Ogu83, Proposition 1.16]). Write κ_i for the function field of M_i.

Notation 9.2. Suppose $\eta \to M_i$ is a geometric generic point, and choose a marked supersingular K3 surface $(X, \tau : \Lambda_i \xrightarrow{\sim} \mathrm{NS}(X))$ realizing η.

LEMMA 9.3. *Suppose given any class $a \in \Lambda_i$ that represents an elliptic fibration $X \to \mathbf{P}^1$. For any marking $\tau' : \Lambda_{i-1} \xrightarrow{\sim} \mathrm{NS}(J)$ on the Jacobian surface $J \to \mathbf{P}^1$ the resulting map $\eta \to M_{i-1}$ is a geometric generic point.*

Remark 9.4. Note that a marking τ' as in Lemma 9.3 can only exist in the case where the fibration is not itself Jacobian, since X must have Artin invariant i.

Proof of Lemma 9.3. Suppose the image of $\eta \to M_{i-1}$ has residue field κ. We can descend the pair (J, τ') to a pair defined over a finite extension $\kappa' \supset \kappa$.

The Artin–Tate construction (see § 4.3 of [BL18]) produces a family

$$\mathfrak{X} \to \mathbf{P}^1_{\mathcal{U}_J}$$

where $\mathcal{U}_J$ is a scheme over κ' that is a flat form of $\mathbf{G}_a$. There are two points $x_1, x_2 : \eta \to \mathcal{U}_J$ and diagrams

$$\begin{array}{ccc} \mathfrak{X}_{x_1} & \longrightarrow & J \\ \downarrow & & \downarrow \\ \mathbf{P}^1_\eta & \longrightarrow & \mathbf{P}^1_\eta \end{array}$$

and

$$\begin{array}{ccc} \mathfrak{X}_{x_2} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathbf{P}^1_\eta & \longrightarrow & \mathbf{P}^1_\eta \end{array}$$

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in which all horizontal arrows are isomorphisms. (That is, the chosen fibration structures may only be defined over η, even though the cohomology classes are defined over κ.) There is an open subset $U \subset \mathcal{U}_J$ containing x_2 such that $\mathfrak{X}_U \to U$ is a family of supersingular K3 surfaces of Artin invariant i. There is a global marking $m : \Lambda_i \xrightarrow{\sim} \mathrm{Pic}_{\mathfrak{X}_U/U}$. (We must therefore have that $m|_{x_2}$ differs from τ by an isomorphism of Λ_i.) This gives a map $U \to M_i$. Since x_2 factors through U, we have that U must hit the generic point of M_i. Since $\dim M_i = \dim M_{i-1} + 1 = \dim M_{i-1} + \dim U$, we have that κ must have transcendence degree $i-1$ over $\mathbf{F}_p$, and thus $\eta \to \mathrm{M}_{i-1}$ hits the generic point. □

Corollary 9.5. *Let (X, τ) be as in Notation 9.2 and let $(Y, \tau' : \Lambda_{i-1} \xrightarrow{\sim} \mathrm{NS}(Y))$ be a marked supersingular K3 surface of Artin invariant $i-1$ over η such that $\eta \to M_{i-1}$ is generic. If every Jacobian elliptic fibration $Y \to \mathbf{P}^1$ is ∞-Frobenius split, then every elliptic fibration $X \to \mathbf{P}^1$ admitting a purely inseparable multisection is Jacobian.*

Proof. Fix a non-Jacobian elliptic structure $f : X \to \mathbf{P}^1_\eta$. By [BL18, Lemma 3.2.4] and Lemma 9.3, the Jacobian fibration $J \to \mathbf{P}^1_\eta$ is generic, so there is an isomorphism $J \xrightarrow{\sim} Y$ of (unmarked) supersingular K3 surfaces. By assumption, then, the Jacobian fibration $J \to \mathbf{P}^1$ is ∞-Frobenius split.

Consider the Artin–Tate family $\mathfrak{X} \to \mathbf{P}^1_{\mathbf{A}^1_\eta}$, and let $U \subset \mathbf{A}^1_\eta$ be the locus over which the fibers of $\mathfrak{X}$ have Artin invariant i. There is a point $u \in U$ such that $\mathfrak{X}_u \to \mathbf{P}^1_u$ is isomorphic to the chosen elliptic fibration $X \to \mathbf{P}^1$. The marking τ on X extends to a marking $T : \Lambda_i \xrightarrow{\sim} \mathrm{Pic}_{\mathfrak{X}_U/U}$, inducing the following diagram.

$$\begin{array}{ccc} U & \longrightarrow & S_i \\ \downarrow & \nearrow & \downarrow \\ \eta & \longrightarrow & M_i \end{array}$$

The dotted arrow arises from the fact that $S_i \to M_i$ is étale, $\kappa(\eta)$ is algebraically closed, and U is connected. This says precisely that there is an isomorphism

$$(\mathfrak{X}_U, T) \xrightarrow{\sim} (X, \tau) \times U. \tag{9.5.1}$$

We now argue as in Corollary 6.9. Specifically, given a positive integer e and a relative polarization $\mathcal{O}_{\mathfrak{X}_U}(1)$ (over U), let $H_{e,b} \subset \mathrm{Hom}_U(\mathbf{P}^1, \mathfrak{X}_U)$ be the open subscheme parametrizing morphisms whose composition $\mathbf{P}^1_U \to \mathfrak{X}_U \to \mathbf{P}^1_U$ is the eth relative Frobenius and whose image has $\mathcal{O}(1)$-degree at most b in each fiber. By Lemma 6.8, $H_{e,b} \to U$ is of finite type. On the other hand, by Corollary 6.7, the image of $H_{e,b}$ does not contain the generic point of U. It follows that the image is finite, hence that there is a point $t \to U(\eta)$ such that $\mathfrak{X}_t \to \mathbf{P}^1_t$ does not admit a purely inseparable multisection of degree p^e over $\mathbf{A}^1$ and $\mathcal{O}(1)$-degree bounded by b. But by (9.5.1) we then see that this is true of the chosen elliptic structure $f : X \to \mathbf{P}^1$. Applying this for all e and b, we see that f admits no purely inseparable multisection. □

Corollary 9.6. *Suppose i is a positive integer between 8 and 10. If X is a very general supersingular K3 surface of Artin invariant i, then every elliptic structure on X admitting a purely inseparable multisection is Jacobian.*

Proof. This follows from Theorem 9.1 and Corollary 9.5. □

Corollary 9.7. *No elliptic structure on a very general supersingular K3 surface admits a purely inseparable multisection.*

Proof. This follows immediately from Corollary 9.6. □

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TABLE 2. Critical fiber configurations and associated bounds on codimensions.

Configuration	$\zeta(\underline{\Phi})$	$B+\zeta(\underline{\Phi})$
$8\,\mathrm{II}$	8	13
$5\,\mathrm{II}+2\,\mathrm{III}$	9	13
$7\,\mathrm{II}+\mathrm{III}$	9	14
$2\,\mathrm{II}+4\,\mathrm{III}$	10	13
$5\,\mathrm{II}+\mathrm{III}+\mathrm{IV}$	10	14
$4\,\mathrm{II}+3\,\mathrm{III}$	10	14
$6\,\mathrm{II}+2\,\mathrm{III}$	10	14
$3\,\mathrm{II}+\mathrm{III}+2\,\mathrm{IV}$	11	14
$2\,\mathrm{II}+3\,\mathrm{III}+\mathrm{IV}$	11	14
$\mathrm{II}+5\,\mathrm{III}$	11	14
$5\,\mathrm{II}+\mathrm{III}+\mathrm{I}_n^*$	11	14
$4\,\mathrm{II}+2\,\mathrm{III}+\mathrm{IV}$	11	14
$3\,\mathrm{II}+4\,\mathrm{III}$	11	14
$5\,\mathrm{II}+3\,\mathrm{III}$	11	14
$2\,\mathrm{II}+\mathrm{III}+2\,\mathrm{I}_n^*$	12	13
$\mathrm{II}+\mathrm{III}+3\,\mathrm{IV}$	12	14
$3\,\mathrm{III}+2\,\mathrm{IV}$	12	14
$4\,\mathrm{II}+2\,\mathrm{I}_n^*$	12	13
$3\,\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{I}_n^*$	12	14
$2\,\mathrm{II}+3\,\mathrm{III}+\mathrm{I}_n^*$	12	14
$2\,\mathrm{II}+2\,\mathrm{III}+2\,\mathrm{IV}$	12	14
$\mathrm{II}+4\,\mathrm{III}+\mathrm{IV}$	12	15
$6\,\mathrm{III}$	12	15
$4\,\mathrm{II}+2\,\mathrm{III}+\mathrm{I}_n^*$	12	14
$3\,\mathrm{II}+3\,\mathrm{III}+\mathrm{IV}$	12	15
$2\,\mathrm{II}+5\,\mathrm{III}$	12	14
$4\,\mathrm{II}+4\,\mathrm{III}$	12	14
$\mathrm{II}+3\,\mathrm{I}_n^*$	13	14
$\mathrm{III}+\mathrm{IV}+2\,\mathrm{I}_n^*$	13	14
$2\,\mathrm{II}+\mathrm{IV}+2\,\mathrm{I}_n^*$	13	14
$\mathrm{II}+2\,\mathrm{III}+2\,\mathrm{I}_n^*$	13	14
$\mathrm{II}+\mathrm{III}+2\,\mathrm{IV}+\mathrm{I}_n^*$	13	14
$3\,\mathrm{III}+\mathrm{IV}+\mathrm{I}_n^*$	13	15
$2\,\mathrm{III}+3\,\mathrm{IV}$	13	15
$3\,\mathrm{II}+\mathrm{III}+2\,\mathrm{I}_n^*$	13	14
$2\,\mathrm{II}+2\,\mathrm{III}+\mathrm{IV}+\mathrm{I}_n^*$	13	15
$\mathrm{II}+4\,\mathrm{III}+\mathrm{I}_n^*$	13	15
$\mathrm{II}+3\,\mathrm{III}+2\,\mathrm{IV}$	13	15
$5\,\mathrm{III}+\mathrm{IV}$	13	16
$3\,\mathrm{II}+3\,\mathrm{III}+\mathrm{I}_n^*$	13	15
$2\,\mathrm{II}+4\,\mathrm{III}+\mathrm{IV}$	13	15

Configuration	$\zeta(\underline{\Phi})$	$B+\zeta(\underline{\Phi})$
$\mathrm{III}+3\,\mathrm{I}_n^*$	14	15
$2\,\mathrm{IV}+2\,\mathrm{I}_n^*$	14	15
$2\,\mathrm{II}+\mathrm{III}+\mathrm{I}_n^*+\mathrm{IV}^*$	14	14
$2\,\mathrm{II}+3\,\mathrm{I}_n^*$	14	15
$\mathrm{II}+\mathrm{III}+\mathrm{IV}+2\,\mathrm{I}_n^*$	14	15
$3\,\mathrm{III}+2\,\mathrm{I}_n^*$	14	15
$2\,\mathrm{III}+2\,\mathrm{IV}+\mathrm{I}_n^*$	14	15
$2\,\mathrm{II}+3\,\mathrm{III}+\mathrm{IV}^*$	14	15
$2\,\mathrm{II}+2\,\mathrm{III}+2\,\mathrm{I}_n^*$	14	15
$\mathrm{II}+3\,\mathrm{III}+\mathrm{IV}+\mathrm{I}_n^*$	14	16
$4\,\mathrm{III}+2\,\mathrm{IV}$	14	16
$\mathrm{II}+\mathrm{III}+2\,\mathrm{IV}^*$	15	15
$\mathrm{II}+2\,\mathrm{I}_n^*+\mathrm{IV}^*$	15	15
$\mathrm{III}+\mathrm{IV}+\mathrm{I}_n^*+\mathrm{IV}^*$	15	15
$\mathrm{IV}+3\,\mathrm{I}_n^*$	15	16
$2\,\mathrm{II}+\mathrm{III}+\mathrm{I}_n^*+\mathrm{III}^*$	15	15
$\mathrm{II}+2\,\mathrm{III}+\mathrm{I}_n^*+\mathrm{IV}^*$	15	15
$\mathrm{II}+\mathrm{III}+3\,\mathrm{I}_n^*$	15	16
$3\,\mathrm{III}+\mathrm{IV}+\mathrm{IV}^*$	15	16
$2\,\mathrm{III}+\mathrm{IV}+2\,\mathrm{I}_n^*$	15	16
$\mathrm{I}_n^*+2\,\mathrm{IV}^*$	16	16
$2\,\mathrm{II}+2\,\mathrm{III}^*$	16	16
$\mathrm{II}+\mathrm{III}+\mathrm{IV}^*+\mathrm{III}^*$	16	16
$\mathrm{II}+2\,\mathrm{I}_n^*+\mathrm{III}^*$	16	16
$2\,\mathrm{III}+2\,\mathrm{IV}^*$	16	16
$\mathrm{III}+\mathrm{IV}+\mathrm{I}_n^*+\mathrm{III}^*$	16	16
$\mathrm{III}+2\,\mathrm{I}_n^*+\mathrm{IV}^*$	16	16
$4\,\mathrm{I}_n^*$	16	17
$\mathrm{IV}+2\,\mathrm{III}^*$	17	17
$\mathrm{I}_n^*+\mathrm{IV}^*+\mathrm{III}^*$	17	17

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10. Table of critical configurations

Let $\underline{\Phi} = (\Phi_1, \ldots, \Phi_n)$ be a configuration of additive fiber types. We have defined two associated integers: $\zeta(\underline{\Phi})$ (by definition, $\zeta(\underline{\Phi}) = \sum_i \zeta(\Phi_i)$, where $\zeta(\Phi_i)$ is computed using Table 1) and B (defined in Notation 8.1). By Proposition 7.21, the locus $\mathcal{WD}^{-2,ab\neq 0}[\underline{\Phi}]$ of fibrations realizing $\underline{\Phi}$ has codimension at least $\zeta(\underline{\Phi})$. By Proposition 8.2, the sublocus of $\mathcal{WD}^{-2,ab\neq 0}[\underline{\Phi}]$ parametrizing fibrations that are not ∞-Frobenius split is a countable union of locally closed subspaces of codimension at least $B + \zeta(\underline{\Phi})$ at every point.

Table 2 shows the full list of all additive fiber configurations that are critical (in the sense of Condition 8.4), along with the bounds $\zeta(\underline{\Phi})$ and $B + \zeta(\underline{\Phi})$. The key observation is that for each critical fibration $\underline{\Phi}$ the bound $B + \zeta(\underline{\Phi})$ for the codimension of the non-∞-Frobenius split locus is at least 13, so the full Artin invariant s locus cannot fit for s at least 7, as claimed in Proposition 8.5.

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