ON THE SUPERCONVERGENCE OF A HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD FOR THE CAHN-HILLIARD EQUATION[*]

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Abstract. We propose a hybridizable discontinuous Galerkin (HDG) method effected with the convex-concave splitting temporal discretization for solving the Cahn-Hilliard equation. We establish optimal convergence rates for the scalar variables and the flux variables in the L^2 norm for polynomials of degree $k \geq 0$. The error constants depend on inverse of the interface thickness in polynomial orders, which is obtained by utilizing a spectral-type estimate of the discrete Cahn-Hilliard operator in the HDG framework. In terms of degrees of freedom of the globally coupled unknowns, the scalar variables are superconvergent. Numerical results are reported to corroborate the theoretical convergence rates and the effectiveness of the method.

Key words. Cahn-Hilliard; HDG method; superconvergence; finite element.

1. Introduction.

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a convex polygonal domain with Lipshitz boundary $\partial\Omega$ and T be a positive constant. We consider the following Cahn-Hilliard equation:

$$u_t - \Delta\phi = 0, \quad -\epsilon\Delta u + \epsilon^{-1} f(u) = \phi \ \text{ in } \Omega \times (0, T], \tag{1.1a}$$

$$\nabla u \cdot \boldsymbol{n} = \nabla\phi \cdot \boldsymbol{n} = 0 \ \text{ on } \partial\Omega \times (0, T], \quad u(\cdot, 0) = u^0(\cdot) \ \text{ in } \Omega, \tag{1.1b}$$

where $f(u) = u^3 - u$. Owing to its importance in material science and multiphase flow, many works have been devoted to the design and analysis of numerical schemes for solving the Cahn-Hilliard equation, see, e.g., finite difference methods [36], mixed and nonconforming finite element methods [30, 29, 31, 5, 35, 27] and Fourier-spectral methods [54, 42, 53].

In recent years, the discontinuous Galerkin (DG) method has become popular for solving the Cahn-Hilliard equation. Applications of DG methods to fourth order elliptic problems have been considered by Babuška and Zlámal in [3], by Baker in [4], and more recently by Mozolevski et al. in a series of works [45, 44, 46, 56]. In [32], Feng and Karakashian design and analyze a DG method of interior penalty type based on the fourth order formulation of the Cahn-Hilliard equation. Optimal error estimates in the L^2 and broken H^1 norms are established for polynomials of degrees $k \geq 3$; see [32, 34] for details. Kay et al. propose and analyze a different DG method [40] that treats the Cahn-Hilliard equation as a system of second order equations allowing a relatively smaller penalty term with optimal convergence in the H^1 norm for polynomials of degree $k \geq 1$. A fully adaptive version of the interior penalty

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DG method was recently constructed in [2] for the Cahn-Hilliard equation with a source and optimal L^2 error bounds were derived. The local discontinuous Galerkin (LDG) method has also been proposed for the discretization of the Cahn-Hilliard equation by writing it as a system of four first-order equations. Dong and Shu in [28] analyzed a LDG scheme for fourth-order equations including the linearized Cahn-Hilliard equation and obtained optimal error estimates in the L^2 norm for polynomials of all degrees.

The classical DG method however entails larger amount of degrees of freedom compared to the continuous Galerkin (CG) methods. In the seminal work [18] Cockburn et al. propose a hybridizable discontinuous Galerkin (HDG) method for second order elliptic problems. The HDG method can be viewed as a hybridizable version of the LDG method. In a nutshell, the HDG method locally connects the flux and solution variables with the numerical traces of the solution via a local solver, which are in turn coupled by the continuity of fluxes across inter-element boundaries (a transmission condition). Hence the globally coupled degrees of freedom are those numerical traces, resulting in a significant reduction of the number of unknowns than traditional DG methods. Moreover, the HDG methods possess the same favorable properties as classical mixed methods. In particular, HDG methods provide optimal convergence rates for both the gradient and the primal variables of the mixed formulation. This property enables the construction of superconvergent solutions via postprocessing, contrary to other DG methods. These advantages of the HDG methods have made HDG an attractive alternative for solving problems governed by PDEs and PDE control problems, cf. [23, 19, 24, 22, 7, 51, 50, 21, 52, 55, 11, 38, 37].

Most studies currently focus on establishing optimal and superconvergent rates of HDG methods for second order problems, such as elliptic PDEs [20], convection diffusion equations [14, 15, 48], Stokes equations [24, 19], Oseen equations [7] and Navier-Stokes equations [49, 8]. In [17], the authors utilized an HDG method with polynomial of degree k for all variables for solving the biharmonic equation and obtained an optimal convergence rates for solution variables and suboptimal convergence rates for other variables.

In this work, we propose a HDG method for the Cahn-Hilliard equation with Lehrenfeld-Schöberl type stabilization using polynomials of order $k+1$ for the scalar unknowns, and polynomials of order k for the other unknowns. The time-marching is based on first-order backward Euler method with convex-splitting discretization for the nonlinear term. We establish optimal convergence rates in the L^2 norm for all variables and for polynomials of order $k \geq 0$. Since the globally coupled degrees of freedom (numerical traces) are approximated by polynomials of order k, superconvergence $(k+2)$ is achieved for approximation of the scalar variables. A particular difficulty in error analysis of numerical schemes for solving phase field models is to avoid exponential dependence of the error constants on $1/\epsilon$. On the other hand, it is well-known that the principal eigenvalue of the linearized Cahn-Hilliard operator has a lower bound, cf. [1, 13]. Based on this spectral result an optimal error estimate with the error constants polynomially depending on $1/\epsilon$ is established in [35] for a conforming finite element method. A spectral-type estimate in the DG space is obtained in [34], which is non-trivial since the DG space is not a subspace of H^1. Adapting the perturbative argument in [34] we establish a similar spectral type estimate of the discrete Cahn-Hilliard operator within the framework of the HDG method. This enables us to obtain error constants depending on $1/\epsilon$ in polynomial orders, provided that the spatio-temporal resolution is sufficiently small.

Closely related to our scheme is the hybrid high-order method (HHO) proposed in

[9] which also uses mixed degree approximation and the Lehrenfeld-Schoberl stabilization. It is further pointed out in [16] that HDG methods for elliptic equation mostly resemble standard HHO methods. More recently it is shown that our HDG scheme for the Cahn-Hilliard equation is indeed a HHO method following the full gradient approach, cf. Sec. 4.2 of the book [26]. However, only error estimates in the energy norm was obtained for the HHO method in [9], although L^2 error estimate is alluded in a remark therein. Furthermore, the error constants in [9] exponentially depend on $\frac{1}{\epsilon}$. The HDG framework with reduced stabilization and polynomials of mixed orders was first introduced by Lehrenfeld in [41] where it was alluded that the scheme could be a superconvergent method, i.e., $O(h^{k+2})$ for the solution variables even though polynomials of order k are used for the globally coupled unknowns (numerical traces of the solution). Optimal convergence and hence superconvergence was then rigorously established for convection diffusion problems [48], for the Navier-Stokes equations [49], and more recently for linear elasticity problems [47].

The rest of the article is organized as follows. We provide the HDG formulation for the Cahn-Hilliard equation in Section 2. We then give some preliminary tools essential for the numerical analysis in Section 3. Afterwards we perform stability estimate of the nonlinear HDG methods in Section 4. In Section 5, we establish the optimal convergence rates of the HDG methods. The theoretical convergence rates are further validated by numerical experiments in Section 6.

2. The HDG formulation. To introduce the fully discrete HDG formulation for the Cahn-Hilliard equation, we first fix some notation. Let $\mathcal{T}_h$ be a shape-regular, quasi-uniform triangulation of Ω. Other regular polygonal meshes are applicable too. Let $\mathcal{E}_h$ denote the set of all faces E of all simplexes K of the triangulation $\mathcal{T}_h$. Also let $\mathcal{E}_h^o$ and $\mathcal{E}_h^\partial$ denote the set of interior faces and boundary faces, respectively. Furthermore, we introduce the discrete inner products

$$(w,v)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}(w,v)_K = \sum_{K\in\mathcal{T}_h}\int_K wv, \quad \langle\zeta,\rho\rangle_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}\langle\zeta,\rho\rangle_{\partial K} = \sum_{K\in\mathcal{T}_h}\int_{\partial K}\zeta\rho.$$

For any integer $k\geq 0$, let $\mathcal{P}^k(K)$ denote the set of polynomials of degree at most k on the element K. We introduce the following discontinuous finite element spaces:

$$\begin{aligned}
\boldsymbol{V}_h &:= \{\boldsymbol{v}_h \in [L^2(\Omega)]^d : \boldsymbol{v}_h|_K \in [\mathcal{P}^k(K)]^d, \forall K\in\mathcal{T}_h\},\\
W_h &:= \{w_h \in L^2(\Omega) : w_h|_K \in \mathcal{P}^{k+1}(K), \forall K\in\mathcal{T}_h\},\\
\mathring{W}_h &:= \{w_h \in L_0^2(\Omega) : w_h|_K \in \mathcal{P}^{k+1}(K), \forall K\in\mathcal{T}_h\},\\
M_h &:= \{\mu_h \in L^2(\mathcal{E}_h) : \mu_h|_E \in \mathcal{P}^k(E), \forall E\in\mathcal{E}_h\},
\end{aligned}$$

where $L_0^2(\Omega)$ is the subspace of $L^2(\Omega)$ of mean zero functions.

Since the HDG methods are based on a mixed formulation, we rewrite the PDE as a first order system by setting $\boldsymbol{p}+\nabla\phi = 0$ and $\boldsymbol{q}+\nabla u = 0$ in (1.1). The mixed formulation of (1.1) is

$$\boldsymbol{p}+\nabla\phi = \boldsymbol{0}, \ \ u_t + \nabla\cdot\boldsymbol{p} = 0, \ \ \boldsymbol{q}+\nabla u = 0, \ \ \epsilon\nabla\cdot\boldsymbol{q} + \epsilon^{-1}f(u) = \phi. \tag{2.1}$$

Now we introduce the fully discrete HDG formulation based on the first-order convex-splitting approach. A similar scheme can be constructed utilizing the backward Euler method. For a fixed integer N, let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of $[0,T]$ with $\Delta t = T/N$. The HDG method seeks $(\boldsymbol{p}_h^n, \phi_h^n, \widehat{\phi}_h^n)$, $(\boldsymbol{q}_h^n, u_h^n, \widehat{u}_h^n) \in$

$\boldsymbol{V}_h \times W_h \times M_h$ such that

$$(\partial_t^+ u_h^n, w_1)_{\mathcal{T}_h} + \mathcal{A}(\boldsymbol{p}_h^n, \phi_h^n, \widehat{\phi}_h^n; \boldsymbol{r}_1, w_1, \mu_1) = 0, \tag{2.2a}$$

$$(\epsilon^{-1} F(u_h^n, u_h^{n-1}), w_2)_{\mathcal{T}_h} + \epsilon \mathcal{A}(\boldsymbol{q}_h^n, u_h^n, \widehat{u}_h^n; \boldsymbol{r}_2, w_2, \mu_2) - (\phi_h^n, w_2)_{\mathcal{T}_h} = 0, \tag{2.2b}$$

$$(u_h^0, w_3)_{\mathcal{T}_h} - (u^0, w_3)_{\mathcal{T}_h} = 0 \tag{2.2c}$$

for all $(\boldsymbol{r}_1, w_1, \mu_1), (\boldsymbol{r}_2, w_2, \mu_2) \in \boldsymbol{V}_h \times W_h \times M_h$ and $w_3 \in W_h$, $\mathcal{A} : [\boldsymbol{V}_h \times W_h \times M_h]^2 \to \mathbb{R}$ is defined by

$$\begin{aligned} \mathcal{A}(\boldsymbol{q}_h, u_h, \widehat{u}_h; \boldsymbol{r}_h, w_h, \mu_h) &= (\boldsymbol{q}_h, \boldsymbol{r}_h)_{\mathcal{T}_h} - (u_h, \nabla \cdot \boldsymbol{r}_h)_{\mathcal{T}_h} + \langle \widehat{u}_h, \boldsymbol{r}_h \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} \\ &\quad + (\nabla \cdot \boldsymbol{q}_h, w_h)_{\mathcal{T}_h} - \langle \boldsymbol{q}_h \cdot \boldsymbol{n}, \mu_h \rangle_{\partial \mathcal{T}_h} + \langle h_K^{-1} (\Pi_k^\partial u_h - \widehat{u}_h), \Pi_k^\partial w_h - \mu_h \rangle_{\partial \mathcal{T}_h}. \end{aligned} \tag{2.2d}$$

Here, $\partial_t^+ u_h^n = (u_h^n - u_h^{n-1})/\Delta t$, $F(u_h^n, u_h^{n-1}) = (u_h^n)^3 - u_h^{n-1}$, and Π_k^∂ is the element-wise L^2 projection onto $\mathcal{P}^k(E)$ such that

$$\langle \Pi_k^\partial u_h, \mu_h \rangle_E = \langle u_h, \mu_h \rangle_E, \quad \forall \mu_h \in \mathcal{P}^k(E) \text{ and } E \in \partial K.$$

3. Preliminaries. In Subsection 3.1 we recall and prove some useful inequalities necessary for the error analysis. Then in Subsection 3.2 we introduce the HDG elliptic projection and study its approximation properties. Finally in Subsection 3.4 we introduce the the HDG spectral estimate, which is useful to our error analysis in the next section. Throughout, the generic constant C is independent of $h, \Delta t, \epsilon$ and may change from line to line. For convience of analysis we assume $\epsilon < 1$.

3.1. Useful inequalities . We recall the standard L^2 projections $\boldsymbol{\Pi}_k^o : [L^2(\Omega)]^d \to \boldsymbol{V}_h$ and $\Pi_{k+1}^o : L^2(\Omega) \to W_h$. The following approximation results are classical, cf. [12, Lemma 3.3], [6, Lemma 4.5.3].

$$\|\boldsymbol{q} - \boldsymbol{\Pi}_k^o \boldsymbol{q}\|_{L^2(K)} \le C h_K^{k+1} |\boldsymbol{q}|_{H^{k+1}(K)}, \|u - \Pi_{k+1}^o u\|_{L^2(K)} \le C h_K^{k+2} |u|_{H^{k+2}(K)}, \tag{3.1a}$$

$$\|u - \Pi_{k+1}^o u\|_{L^2(\partial K)} \le C h_K^{k+3/2} |u|_{H^{k+2}(K)}, \tag{3.1b}$$

$$\|w_h\|_{L^2(\partial K)} \le C h_K^{-1/2} \|w_h\|_{L^2(K)}, \forall w_h \in W_h, \tag{3.1c}$$

$$\|u - \Pi_{k+1}^o u\|_{L^q(K)} \le C h_K^{k+2+d(1/q-1/2)} |u|_{H^{k+2}(K)}, \quad q \in [1, +\infty], \tag{3.1d}$$

$$\|w_h\|_{L^q(K)} \le C h_K^{d(1/q-1/2)} \|w_h\|_{L^2(K)}, \quad \forall w_h \in W_h, q \in [1, +\infty]. \tag{3.1e}$$

The following HDG Sobolev inequalities can be readily derived from [26, Theorem 6.5]. A direct proof based on Oswald interpolation operator [39, Page 644, Theorem 2.1] is available in [10].

LEMMA 3.1 (HDG Sobolev inequality). *Suppose $q \in [1, \infty)$ for $d = 2$, and $q \in [1, 6]$ for $d = 3$. For any $\mu_h \in M_h$ there hold*

$$\|w_h\|_{L^q(\Omega)} \le C \left(\|w_h\|_{\mathcal{T}_h} + \|\nabla w_h\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2} (\Pi_k^\partial w_h - \mu_h)\|_{\partial \mathcal{T}_h} \right) \forall w_h \in W_h, \tag{3.2a}$$

$$\|w_h\|_{L^q(\Omega)} \le C \left(\|\nabla w_h\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2} (\Pi_k^\partial w_h - \mu_h)\|_{\partial \mathcal{T}_h} \right), \forall w_h \in \mathring{W}_h. \tag{3.2b}$$

Here and throughout, $h_{\mathcal{T}}$ is a piecewise constant function equal to h_K on each element K.

Next, we present some basic properties of the operator $\mathcal{A}$.

Proposition 3.2. *For all* $(\boldsymbol{q}_h, u_h, \widehat{u}_h), (\boldsymbol{p}_h, \phi_h, \widehat{\phi}_h) \in \boldsymbol{V}_h \times W_h \times M_h$, *one has*

$$\mathcal{A}(\boldsymbol{q}_h, u_h, \widehat{u}_h; \boldsymbol{p}_h, -\phi_h, -\widehat{\phi}_h) = \mathcal{A}(\boldsymbol{p}_h, \phi_h, \widehat{\phi}_h; \boldsymbol{q}_h, -u_h, -\widehat{u}_h), \tag{3.3}$$

$$\mathcal{A}(\boldsymbol{q}_h, u_h, \widehat{u}_h; \boldsymbol{q}_h, u_h, \widehat{u}_h) = \|\boldsymbol{q}_h\|^2_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial u_h - \widehat{u}_h)\|^2_{\partial\mathcal{T}_h}, \tag{3.4}$$

$$\begin{aligned}\left|\mathcal{A}(\boldsymbol{q}_h, u_h, \widehat{u}_h; \boldsymbol{p}_h, \phi_h, \widehat{\phi}_h)\right| &\le C\Big(\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial u_h - \widehat{u}_h)\|_{\partial\mathcal{T}_h} + \|\boldsymbol{q}_h\|_{\mathcal{T}_h} \\ &+ \|\nabla u_h\|_{\mathcal{T}_h}\Big) \times \left(\|\boldsymbol{p}_h\|_{\mathcal{T}_h} + \|\nabla \phi_h\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial \phi_h - \widehat{\phi}_h)\|_{\partial\mathcal{T}_h}\right).\end{aligned} \tag{3.5}$$

In addition, if $\mathcal{A}(\boldsymbol{q}_h, u_h, \widehat{u}_h; \boldsymbol{r}_h, 0, 0) = 0$ *for all* $\boldsymbol{r}_h \in \boldsymbol{V}_h$, *then the following inequality holds*

$$\|\nabla u_h\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(u_h - \widehat{u}_h)\|_{\partial\mathcal{T}_h} \le C\left(\|\boldsymbol{q}_h\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial u_h - \widehat{u}_h)\|_{\partial\mathcal{T}_h}\right). \tag{3.6}$$

The proof of Proposition 3.2 is straightforward, see [38, Lemma 1, Lemma 2] and [48, Lemma 3.2] for proofs of similar results.

Next, we show that $\mathcal{A}$ satisfies the following discrete LBB condition.

Lemma 3.3. *For all* $(\boldsymbol{q}_h, u_h, \widehat{u}_h) \in \boldsymbol{V}_h \times \mathring{W}_h \times M_h$, *we have*

$$\begin{aligned}\sup_{\boldsymbol{0}\neq(\boldsymbol{p}_h,\phi_h,\widehat{\phi}_h)\in \boldsymbol{V}_h\times\mathring{W}_h\times M_h} &\frac{\mathcal{A}(\boldsymbol{q}_h, u_h, \widehat{u}_h; \boldsymbol{p}_h, \phi_h, \widehat{\phi}_h)}{\|\boldsymbol{p}_h\|_{\mathcal{T}_h} + \|\nabla \phi_h\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial \phi_h - \widehat{\phi}_h)\|_{\partial\mathcal{T}_h}} \\ &\ge C\left(\|\boldsymbol{q}_h\|_{\mathcal{T}_h} + \|\nabla u_h\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial u_h - \widehat{u}_h)\|_{\partial\mathcal{T}_h}\right).\end{aligned} \tag{3.7}$$

Proof. First we note that if

$$\|\boldsymbol{p}_h\|_{\mathcal{T}_h} + \|\nabla \phi_h\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial \phi_h - \widehat{\phi}_h)\|_{\partial\mathcal{T}_h} = 0,$$

then $\boldsymbol{p}_h = 0$, and $\phi_h = 0$ by (3.2b), and hence $\widehat{\phi}_h = 0$ as well.

Next, let α be a positive constant that will be specified below. For any fixed $(\boldsymbol{q}_h, u_h, \widehat{u}_h) \in \boldsymbol{V}_h \times \mathring{W}_h \times M_h$, we take $(\boldsymbol{p}_h, \phi_h, \widehat{\phi}_h) = (\boldsymbol{q}_h + \alpha\nabla u_h, u_h, \widehat{u}_h) \in \boldsymbol{V}_h \times \mathring{W}_h \times M_h$ to get

$$\begin{aligned}&\mathcal{A}(\boldsymbol{q}_h, u_h, \widehat{u}_h; \boldsymbol{p}_h, \phi_h, \widehat{\phi}_h) \\ &\quad= (\boldsymbol{q}_h, \boldsymbol{q}_h + \alpha\nabla u_h)_{\mathcal{T}_h} - (u_h, \nabla\cdot(\boldsymbol{q}_h + \alpha\nabla u_h))_{\mathcal{T}_h} \\ &\qquad+ \langle \widehat{u}_h, (\boldsymbol{q}_h + \alpha\nabla u_h)\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + (\nabla\cdot\boldsymbol{q}_h, u_h)_{\mathcal{T}_h} - \langle \boldsymbol{q}_h\cdot\boldsymbol{n}, \widehat{u}_h\rangle_{\partial\mathcal{T}_h} \\ &\qquad+ \langle h_K^{-1}(\Pi_k^\partial u_h - \widehat{u}_h), \Pi_k^\partial u_h - \widehat{u}_h\rangle_{\partial\mathcal{T}_h} \\ &\quad= \|\boldsymbol{q}_h\|^2_{\mathcal{T}_h} + \alpha\|\nabla u_h\|^2_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial u_h - \widehat{u}_h)\|^2_{\partial\mathcal{T}_h} \\ &\qquad+ \alpha(\boldsymbol{q}_h, \nabla u_h)_{\mathcal{T}_h} + \alpha\langle \widehat{u}_h - \Pi_k^\partial u_h, \boldsymbol{n}\cdot\nabla u_h\rangle_{\partial\mathcal{T}_h} \\ &\quad\ge (1 - C\alpha)\left(\|\boldsymbol{q}_h\|^2_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial u_h - \widehat{u}_h)\|^2_{\partial\mathcal{T}_h}\right) + \frac{\alpha}{2}\|\nabla u_h\|^2_{\mathcal{T}_h}.\end{aligned}$$

By choosing $\alpha > 0$ such that $0 < 1 - C\alpha$, we get

$$\mathcal{A}(\boldsymbol{q}_h, u_h, \widehat{u}_h; \boldsymbol{p}_h, \phi_h, \widehat{\phi}_h) \ge C_1\left(\|\boldsymbol{q}_h\|^2_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial u_h - \widehat{u}_h)\|^2_{\partial\mathcal{T}_h} + \|\nabla u_h\|^2_{\mathcal{T}_h}\right).$$

Finally, for the choice of $(\boldsymbol{p}_h, \phi_h, \widehat{\phi}_h) = (\boldsymbol{q}_h + \alpha\nabla u_h, u_h, \widehat{u}_h)$ the triangle inequality implies

$$\begin{aligned}&\|\boldsymbol{p}_h\|^2_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial \phi_h - \widehat{\phi}_h)\|^2_{\partial\mathcal{T}_h} + \|\nabla \phi_h\|^2_{\mathcal{T}_h} \\ &\quad\le C_2(\|\boldsymbol{q}_h\|^2_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial u_h - \widehat{u}_h)\|^2_{\partial\mathcal{T}_h} + \|\nabla u_h\|^2_{\mathcal{T}_h}).\end{aligned}$$

Then (3.7) follows immediately. ∎

The proof of Lemma 3.3 also shows that the operator $\mathcal{A}$ defines a norm on the space $\boldsymbol{V}_h \times \mathring{W}_h \times M_h$. Thanks to the discrete LBB condition in Lemma 3.3, the following inversion of the Laplacian operator equipped with Neumann boundary conditions is well defined.

DEFINITION 3.4. *For all $u_h \in W_h$, define $(\boldsymbol{\Pi_V} u_h, \Pi_W u_h, \Pi_M u_h) \in \boldsymbol{V}_h \times \mathring{W}_h \times M_h$ to be the unique solution of the problem*

$$\mathcal{A}(\boldsymbol{\Pi_V} u_h, \Pi_W u_h, \Pi_M u_h; \boldsymbol{r}_h, w_h, \mu_h) = (u_h, w_h)_{\mathcal{T}_h} \tag{3.8}$$

for all $(\boldsymbol{r}_h, w_h, \mu_h) \in \boldsymbol{V}_h \times \mathring{W}_h \times M_h$.

In particular, by the definition $\mathcal{A}$ in (2.2d) and integration by parts, one can show that for any $u_h \in \mathring{W}_h$ and $\forall (\boldsymbol{r}_h, w_h, \mu_h) \in \boldsymbol{V}_h \times W_h \times M_h$

$$\mathcal{A}(\boldsymbol{\Pi_V} u_h, \Pi_W u_h, \Pi_M u_h; \boldsymbol{r}_h, w_h, \mu_h) = (u_h, w_h)_{\mathcal{T}_h}. \tag{3.9}$$

For all $u_h \in \mathring{W}_h$, we define the semi-norm

$$\|u_h\|_{-1,h}^2 := \mathcal{A}(\boldsymbol{\Pi_V} u_h, \Pi_W u_h, \Pi_M u_h; \boldsymbol{\Pi_V} u_h, \Pi_W u_h, \Pi_M u_h).$$

It follows from (3.4) that

$$\|u_h\|_{-1,h}^2 = \|\boldsymbol{\Pi_V} u_h\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial \Pi_W u_h - \Pi_M u_h)\|_{\partial\mathcal{T}_h}^2 = (u_h, \Pi_W u_h)_{\mathcal{T}_h}. \tag{3.10}$$

Next, we show that $\|\cdot\|_{-1,h}$ is a norm on the space $\mathring{W}_h$.

LEMMA 3.5. *$\|\cdot\|_{-1,h}$ defines a norm on the space $\mathring{W}_h$.*

Proof. Thanks to (3.10), one only needs to show that $\|u_h\|_{-1,h} = 0$ implies $u_h = 0$ for $u_h \in \mathring{W}_h$. It follows readily from (3.10) that

$$\boldsymbol{\Pi_V} u_h = \boldsymbol{0}, \qquad \Pi_k^\partial \Pi_W u_h - \Pi_M u_h = 0.$$

Next, Definition 3.4 and (2.2d) imply that for all $(\boldsymbol{r}_h, w_h) \in \boldsymbol{V}_h \times \mathring{W}_h$ we have

$$(u_h, w_h)_{\mathcal{T}_h} = (\Pi_W u_h, \nabla \cdot \boldsymbol{r}_h)_{\mathcal{T}_h} - \langle \Pi_M u_h, \boldsymbol{r}_h \cdot \boldsymbol{n} \rangle_{\partial\mathcal{T}_h}.$$

Taking $\boldsymbol{r}_h = 0$ and $w_h = u_h$ one obtains $u_h = 0$. This completes the proof. ∎

LEMMA 3.6. *If $u_h \in \mathring{W}_h$ and $(w_h, \mu_h) \in W_h \times M_h$, then we have*

$$(u_h, w_h)_{\mathcal{T}_h} \le C\|u_h\|_{-1,h} \left(\|\nabla w_h\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial w_h - \mu_h)\|_{\partial\mathcal{T}_h} \right). \tag{3.11}$$

Proof. Let $(w_h, \mu_h) \in W_h \times M_h$, $u_h \in \mathring{W}_h$ and $(\boldsymbol{\Pi_V} u_h, \Pi_W u_h, \Pi_M u_h)$ be the solution of (3.8). By Equation (3.9) and (2.2d) we have

$$\begin{aligned}
(u_h, w_h)_{\mathcal{T}_h} &= \mathcal{A}(\boldsymbol{\Pi_V} u_h, \Pi_W u_h, \Pi_M u_h; \boldsymbol{0}, w_h, \mu_h) \\
&= (\nabla \cdot \boldsymbol{\Pi_V} u_h, w_h)_{\mathcal{T}_h} - \langle \boldsymbol{n} \cdot \boldsymbol{\Pi_V} u_h, \mu_h \rangle_{\partial\mathcal{T}_h} \\
&\quad + \langle h_K^{-1}(\Pi_k^\partial \Pi_W u_h - \Pi_M u_h), \Pi_k^\partial w_h - \mu_h \rangle_{\partial\mathcal{T}_h}.
\end{aligned}$$

By integration by parts, (3.10) and the L^2 stability of Π_k^∂ we have

$$\begin{aligned}
(u_h, w_h)_{\mathcal{T}_h} &\le \|\boldsymbol{\Pi_V} u_h\|_{\mathcal{T}_h} \|\nabla w_h\|_{\mathcal{T}_h} + C\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial w_h - \mu_h)\|_{\partial\mathcal{T}_h} \\
&\qquad \times \left(\|\boldsymbol{\Pi_V} u_h\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial \Pi_W u_h - \Pi_M u_h)\|_{\partial\mathcal{T}_h} \right) \\
&\le C\|u_h\|_{-1,h} \left(\|\nabla w_h\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial w_h - \mu_h)\|_{\partial\mathcal{T}_h} \right).
\end{aligned}$$

This completes the proof. ∎

Finally, by the Definition 3.4, the identity (3.10) and (3.6) one can easily establish the following inequality

$$\|\nabla \Pi_W u_h\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}(\Pi_k^{\partial}\Pi_W u_h - \Pi_M u_h)\|_{\partial\mathcal{T}_h} \le C\|u_h\|_{-1,h}, \quad u_h \in \mathring{W}_h, \tag{3.12}$$

$$\|u_h\|_{-1,h} \le C\|u_h\|_{\mathcal{T}_h}, \quad u_h \in \mathring{W}_h. \tag{3.13}$$

3.2. The HDG elliptic projection. Given $\Theta \in L^2(\Omega)$, let $(\boldsymbol{\Psi}, \Phi)$ denote the solution of the following system

$$\boldsymbol{\Psi} + \nabla\Phi = 0, \quad \nabla\cdot\boldsymbol{\Psi} = \Theta \quad \text{in } \Omega, \quad \boldsymbol{\Psi}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \quad \int_\Omega \Phi = 0. \tag{3.14}$$

If Ω is convex, then we have the following regularity result:

$$\|\boldsymbol{\Psi}\|_{H^1(\Omega)} + \|\Phi\|_{H^2(\Omega)} \le C_{\rm reg}\|\Theta\|_{L^2(\Omega)}. \tag{3.15}$$

Recall that $(\boldsymbol{p}, \phi, \boldsymbol{q}, u)$ is the solution of the Cahn-Hilliard equation in mixed form (2.1). For all $t \in [0, T]$, we define the HDG elliptic projections:
finding $(\boldsymbol{p}_{Ih}, \phi_{Ih}, \widehat{\phi}_{Ih}), (\boldsymbol{q}_{Ih}, u_{Ih}, \widehat{u}_{Ih}) \in \boldsymbol{V}_h \times W_h \times M_h$ such that

$$\mathcal{A}(\boldsymbol{p}_{Ih}, \phi_{Ih}, \widehat{\phi}_{Ih}; \boldsymbol{r}_1, w_1, \mu_1) = -(\Delta\phi, w_1)_{\mathcal{T}_h} \qquad \text{and} \ \ (\phi_{I_h} - \phi, 1)_{\mathcal{T}_h} = 0, \tag{3.16a}$$

$$\mathcal{A}(\boldsymbol{q}_{Ih}, u_{Ih}, \widehat{u}_{Ih}; \boldsymbol{r}_2, w_2, \mu_2) = -(\Delta u, w_2)_{\mathcal{T}_h} \qquad \text{and} \ \ (u_{I_h} - u, 1)_{\mathcal{T}_h} = 0, \tag{3.16b}$$

for all $(\boldsymbol{r}_1, w_1, \mu_1), (\boldsymbol{r}_2, w_2, \mu_2) \in \boldsymbol{V}_h \times \mathring{W}_h \times M_h$.

Denote the norm on the Hilbert space $H^s, s > 0$ by $|\cdot|_s$. We have the following approximation property for the HDG elliptic projection (3.16).

Lemma 3.7. *Assume the regularity condition* (3.15) *holds. Let* $(\boldsymbol{p}, \phi, \boldsymbol{q}, u)$ *be smooth enough and* $(\boldsymbol{p}_{Ih}, \phi_{Ih}, \boldsymbol{q}_{Ih}, u_{Ih})$ *be the solutions of* (3.16). *We have*

$$\|u - u_{Ih}\|_{\mathcal{T}_h} \le Ch^{k+2}|u|_{k+2}, \tag{3.17a}$$

$$\|\boldsymbol{q} - \boldsymbol{q}_{Ih}\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^{\partial}u_{Ih} - \widehat{u}_{Ih})\|_{\partial\mathcal{T}_h} \le Ch^{k+1}|u|_{k+2}, \tag{3.17b}$$

$$\|\partial_t u - \partial_t u_{Ih}\|_{\mathcal{T}_h} \le Ch^{k+2}|\partial_t u|_{k+2}, \tag{3.17c}$$

$$\|\phi - \phi_{Ih}\|_{\mathcal{T}_h} \le Ch^{k+2}|\phi|_{k+2}, \tag{3.17d}$$

$$\|\boldsymbol{p} - \boldsymbol{p}_{Ih}\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^{\partial}\phi_{Ih} - \widehat{\phi}_{Ih})\|_{\partial\mathcal{T}_h} \le Ch^{k+1}|\phi|_{k+2}. \tag{3.17e}$$

The proof of Lemma 3.7 follows applications of the 3rd Strang Lemma and the Aubin-Nitsche technique in [25]. For completeness, we present details of the proof as follows. Introducing

$$\varepsilon_h^{\boldsymbol{q}} := \boldsymbol{\Pi}_k^o\boldsymbol{q} - \boldsymbol{q}_{Ih}, \quad \varepsilon_h^u := \Pi_{k+1}^o u - u_{Ih}, \quad \varepsilon_h^{\widehat{u}} := \Pi_k^{\partial}u - \widehat{u}_{Ih}, \tag{3.18}$$

one derives the error equation

$$\begin{aligned} \mathcal{A}(\varepsilon_h^{\boldsymbol{q}}, \varepsilon_h^u, \varepsilon_h^{\widehat{u}}; \boldsymbol{r}_2, w_2, \mu_2) &= -(\Delta u, w_2)_{\mathcal{T}_h} \\ &+ \langle \boldsymbol{q}\cdot\boldsymbol{n} - \boldsymbol{\Pi}_k^o\boldsymbol{q}\cdot\boldsymbol{n}, \mu_2 - w_2\rangle_{\partial\mathcal{T}_h} + \langle h_K^{-1}(\Pi_{k+1}^o u - u), \Pi_k^{\partial}w_2 - \mu_2\rangle_{\partial\mathcal{T}_h}. \end{aligned} \tag{3.19}$$

Proof. We only give a proof of (3.17a) and (3.17b), since the proofs of the remaining inequalities are similar. We split the proof into two steps.

Step 1: an energy argument

Since $(\varepsilon_h^u, 1)_{L^2(\Omega)} = (\Pi_{k+1}^o u - u_{Ih}, 1)_{L^2(\Omega)} = (u - u_{Ih}, 1)_{L^2(\Omega)} = 0$, then $\varepsilon_h^u \in \mathring{W}_h$. We take $(\boldsymbol{r}_2, w_2, \mu_2) = (\varepsilon_h^{\boldsymbol{q}}, \varepsilon_h^u, \varepsilon_h^{\widehat{u}})$ in (3.19) to get

$$
\begin{aligned}
&\|\varepsilon_h^{\boldsymbol{q}}\|_{\mathcal{T}_h}^2 + \|h_K^{-1/2}(\Pi_k^\partial \varepsilon_h^u - \varepsilon_h^{\widehat{u}})\|_{\partial\mathcal{T}_h}^2 \\
&\quad = \langle \boldsymbol{q}\cdot\boldsymbol{n} - \boldsymbol{\Pi}_k^o \boldsymbol{q}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{u}} - \varepsilon_h^u\rangle_{\partial\mathcal{T}_h} + \langle h_K^{-1}(\Pi_{k+1}^o u - u), \Pi_k^\partial \varepsilon_h^u - \varepsilon_h^{\widehat{u}}\rangle_{\partial\mathcal{T}_h} \\
&\quad \le Ch^{k+1}|u|_{k+2}\left(\|\varepsilon_h^{\boldsymbol{q}}\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial \varepsilon_h^u - \varepsilon_h^{\widehat{u}})\|_{\partial\mathcal{T}_h}^2\right)^{1/2},
\end{aligned}
$$

where we have used the fact $\mathcal{A}(\varepsilon_h^{\boldsymbol{q}}, \varepsilon_h^u, \varepsilon_h^{\widehat{u}}; \boldsymbol{r}_2, 0, 0) = 0$ and (3.6). Hence

$$
\left(\|\varepsilon_h^{\boldsymbol{q}}\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial \varepsilon_h^u - \varepsilon_h^{\widehat{u}})\|_{\partial\mathcal{T}_h}^2\right)^{1/2} \le Ch^{k+1}|u|_{k+2}. \tag{3.20}
$$

Now in light of the definitions of the error functions in (3.18), one obtains by the triangle inequality, the L^2 stability of the projection Π_k^∂, the inequalities in (3.1) and the fact that $\boldsymbol{q} = -\nabla u$ that

$$
\|\boldsymbol{q} - \boldsymbol{q}_{Ih}\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial u_{Ih} - \widehat{u}_{Ih})\|_{\partial\mathcal{T}_h} \le Ch^{k+1}|u|_{k+2}.
$$

Thus (3.17b) is established.

Step 2: L^2 error estimate by a duality argument Let $\Theta \in L^2(\Omega)$ and let (Ψ, Φ) be the solution to (3.14). One has for all $(\boldsymbol{r}_2, w_2, \mu_2) \in \boldsymbol{Q}_h \times W_h \times M_h$

$$
\begin{aligned}
&\mathcal{A}(\boldsymbol{\Pi}_k^o \boldsymbol{\Psi}, \Pi_{k+1}^o \Phi, \Pi_k^\partial \Phi; \boldsymbol{r}_2, w_2, \mu_2) = (\Theta, w_2) + \langle \boldsymbol{\Psi}\cdot\boldsymbol{n} - \boldsymbol{\Pi}_k^o\boldsymbol{\Psi}\cdot\boldsymbol{n}, \mu_2 - w_2\rangle_{\partial\mathcal{T}_h} \\
&\quad + \langle h_K^{-1}(\Pi_{k+1}^o \Phi - \Phi), \Pi_k^\partial w_2 - \mu_2\rangle_{\partial\mathcal{T}_h}.
\end{aligned} \tag{3.21}
$$

We take $(\boldsymbol{r}_2, w_2, \mu_2) = (\varepsilon_h^{\boldsymbol{q}}, -\varepsilon_h^u, -\varepsilon_h^{\widehat{u}})$ and $\Theta = -\varepsilon_h^u$ in (3.21) to get

$$
\begin{aligned}
\|\varepsilon_h^u\|_{\mathcal{T}_h}^2 &= \mathcal{A}(\varepsilon_h^{\boldsymbol{q}}, \varepsilon_h^u, \varepsilon_h^{\widehat{u}}; \boldsymbol{\Pi}_k^o\boldsymbol{\Psi}, -\Pi_{k+1}^o\Phi, -\Pi_k^\partial\Phi;) \\
&\qquad + \langle \boldsymbol{\Pi}_k^o\boldsymbol{\Psi}\cdot\boldsymbol{n} - \boldsymbol{\Psi}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{u}} - \varepsilon_h^u\rangle_{\partial\mathcal{T}_h} - \langle h_K^{-1}(\Pi_{k+1}^o\Phi - \Phi), \Pi_k^\partial\varepsilon_h^u - \varepsilon_h^{\widehat{u}}\rangle_{\partial\mathcal{T}_h} \\
&= -\langle \boldsymbol{\Pi}_k^o\boldsymbol{q}\cdot\boldsymbol{n} - \boldsymbol{q}\cdot\boldsymbol{n}, \Pi_k^\partial\Phi - \Pi_{k+1}^o\Phi\rangle_{\partial\mathcal{T}_h} \\
&\qquad - \langle h_K^{-1}(\Pi_{k+1}^o u - u), \Pi_k^\partial\Pi_{k+1}^o\Phi - \Pi_k^\partial\Phi\rangle_{\partial\mathcal{T}_h} \\
&\qquad + \langle \boldsymbol{\Pi}_k^o\boldsymbol{\Psi}\cdot\boldsymbol{n} - \boldsymbol{\Psi}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{u}} - \varepsilon_h^u\rangle_{\partial\mathcal{T}_h} - \langle h_K^{-1}(\Pi_{k+1}^o\Phi - \Phi), \Pi_k^\partial\varepsilon_h^u - \varepsilon_h^{\widehat{u}}\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
$$

Since $\boldsymbol{q} \in H(\mathrm{div}, \Omega)$, $\boldsymbol{q}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ and $\Pi_k^\partial\Phi$ is single-valued on $\partial\mathcal{T}_h$, then $\langle \boldsymbol{q}\cdot\boldsymbol{n}, \Pi_k^\partial\Phi\rangle_{\partial\mathcal{T}_h} = 0 = \langle \boldsymbol{q}\cdot\boldsymbol{n}, \Phi\rangle_{\partial\mathcal{T}_h}$. We have

$$
\begin{aligned}
&-\langle \boldsymbol{\Pi}_k^o\boldsymbol{q}\cdot\boldsymbol{n} - \boldsymbol{q}\cdot\boldsymbol{n}, \Pi_k^\partial\Phi - \Pi_{k+1}^o\Phi\rangle_{\partial\mathcal{T}_h} \\
&\quad = \langle \boldsymbol{q}\cdot\boldsymbol{n}, \Pi_k^\partial\Phi - \Pi_{k+1}^o\Phi\rangle_{\partial\mathcal{T}_h} - \langle \boldsymbol{\Pi}_k^o\boldsymbol{q}\cdot\boldsymbol{n}, \Pi_k^\partial\Phi - \Pi_{k+1}^o\Phi\rangle_{\partial\mathcal{T}_h} \\
&\quad = \langle \boldsymbol{q}\cdot\boldsymbol{n}, \Phi - \Pi_{k+1}^o\Phi\rangle_{\partial\mathcal{T}_h} - \langle \boldsymbol{\Pi}_k^o\boldsymbol{q}\cdot\boldsymbol{n}, \Phi - \Pi_{k+1}^o\Phi\rangle_{\partial\mathcal{T}_h} \\
&\quad = -\langle \boldsymbol{\Pi}_k^o\boldsymbol{q}\cdot\boldsymbol{n} - \boldsymbol{q}\cdot\boldsymbol{n}, \Phi - \Pi_{k+1}^o\Phi\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
$$

By the error estimates in (3.20), the inequality (3.6) and the regularity result (3.15) with $\Theta = -\varepsilon_h^u$, one derives that

$$
\|\varepsilon_h^u\|_{\mathcal{T}_h} \le Ch^{k+2}|u|_{k+2}. \tag{3.22}
$$

The error estimate (3.17a) now follows from the triangle inequality. This completes the proof. ∎

We shall need the uniform estimate of the HDG elliptic projection u_{Ih}. By the triangle inequality and the inverse inequality (3.1e), one obtains

$$\begin{aligned}||u_{Ih}-u||_{L^\infty} &\le ||u_{Ih}-\Pi_{k+1}^o u||_{L^\infty} + ||u-\Pi_{k+1}^o u||_{L^\infty}\\ &\le Ch^{-d/2}\|u_{Ih}-\Pi_{k+1}^o u\|_{\mathcal{T}_h} + Ch^{2-d/2}|u|_2\\ &\le Ch^{2-d/2}|u|_2.\end{aligned}$$

Hence with $||\Delta u||_{L^\infty(L^2)} \le C\epsilon^{-\frac{7}{2}}$ (cf. the stability estimate in the upcoming section) and $||u||_{L^\infty(L^\infty)} \le C$ [34], provided $h \le \epsilon^{\frac{7}{4-d}}$, one has

$$\|u_{Ih}\|_{L^\infty(L^\infty)} \le C. \tag{3.23}$$

3.3. The discrete Laplacian. For any $u_h \in W_h$, we define $\Delta_h u_h \in W_h$, such that for all $(\boldsymbol{r}_h, w_h, \mu_h) \in \boldsymbol{V}_h \times W_h \times M_h$ we have

$$(\Delta_h u_h, w_h)_{\mathcal{T}_h} = -\mathcal{A}(\boldsymbol{q}_h^u, u_h, \widehat{u}_h^u; \boldsymbol{r}_h, w_h, \mu_h), \tag{3.24}$$

where $(\boldsymbol{q}_h^u, \widehat{u}_h^u) \in \boldsymbol{V}_h \times M_h$ satisfy

$$\mathcal{A}(\boldsymbol{q}_h^u, u_h, \widehat{u}_h^u; \boldsymbol{r}_h, 0, \mu_h) = 0 \tag{3.25}$$

for all $(\boldsymbol{r}_h, \mu_h) \in \boldsymbol{V}_h \times M_h$. It is clear there exists a unique solution $(\Delta_h u_h, \boldsymbol{q}_h^u, \widehat{u}_h^u) \in \boldsymbol{W}_h \times V_h \times M_h$, since Eqs. (3.24) and (3.25) define a square linear system of finite dimension.

LEMMA 3.8. *For all $w_h \in W_h$, we have the inequality*

$$\|w_h\|_{L^\infty(\Omega)} \le C\|\Delta_h w_h\|_{\mathcal{T}_h}, \tag{3.26}$$

where C depends on Ω.

Proof. Consider the following continuous problem: find $w \in H^1(\Omega)$, such that

$$-\Delta w = -\Delta_h w_h \text{ in } \Omega, \quad \nabla w \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega, \quad (w,1)_{\mathcal{T}_h} = (w_h, 1)_{\mathcal{T}_h}. \tag{3.27}$$

Since Ω is convex, we have the regularity estimate

$$|w|_{H^2(\Omega)} \le C_{\text{reg}}\|\Delta_h w_h\|_{\mathcal{T}_h}. \tag{3.28}$$

We use the definition (3.24) and (3.27) to get

$$\mathcal{A}(\boldsymbol{q}_h^w, w_h, \widehat{u}_h^w; \boldsymbol{s}_h, v_h, \mu_h) = -(\Delta w, v_h)_{\mathcal{T}_h}, \qquad (w_h - w, 1)_{\mathcal{T}_h} = 0$$

for all $(\boldsymbol{s}_h, v_h, \mu_h) \in \boldsymbol{V}_h \times W_h \times M_h$. Therefore, we may use the HDG elliptic projection result in Lemma 3.7 to get

$$\|w - w_h\|_{\mathcal{T}_h} \le Ch^2|w|_{H^2(\Omega)}. \tag{3.29}$$

By the triangle inequality, we have

$$\begin{aligned}\|w_h\|_{L^\infty(\Omega)} &\le \|w_h - \Pi_{k+1}^o w\|_{L^\infty(\Omega)} + \|\Pi_{k+1}^o w - w\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)}\\ &:= R_1 + R_2 + R_3.\end{aligned}$$

Now we estimate $\{R_i\}_{i=1}^3$ term by term as follows

$$
\begin{aligned}
R_1 &\le Ch^{-d/2}\|w_h-\Pi_{k+1}^o w\|_{\mathcal{T}_h} \le Ch^{-d/2}\left(\|w_h-w\|_{\mathcal{T}_h}+\|w-\Pi_{k+1}^o w\|_{\mathcal{T}_h}\right)\\
&\le Ch^{2-d/2}|w|_{H^2(\Omega)} \le Ch^{2-d/2}\|\Delta_h w_h\|_{\mathcal{T}_h},\\
R_2 &\le Ch^{2-d/2}|w|_{H^2(\Omega)} \le Ch^{2-d/2}\|\Delta_h w_h\|_{\mathcal{T}_h},\\
R_3 &\le CD_\Omega^{2-\frac{2}{d}}|w|_{H^2(\Omega)} \le CD_\Omega^{2-\frac{2}{d}}\|\Delta_h w_h\|_{\mathcal{T}_h},
\end{aligned}
$$

where D_Ω is the diameter of the domain Ω. The inequality (3.26) follows from the above estimates and the fact $h \le D_\Omega$. This completes the proof. □

We remark that it is possible to obtain a Gagliardo–Nirenberg type inequality, cf. [10]:

$$
\|w_h\|_{L^\infty(\Omega)} \le C\left(h^{2-d/2}\|\Delta_h w_h\|_{\mathcal{T}_h}+\|\Delta_h w_h\|_{\mathcal{T}_h}^{\frac{d}{2(6-d)}}\|w_h\|_{L^6(\Omega)}^{\frac{3(4-d)}{2(6-d)}}+\|w_h\|_{L^6(\Omega)}\right).
$$

3.4. The HDG spectral estimate.

Recall that for $f \in L_0^2(\Omega)$, $u=(-\Delta)^{-1}f \in L_0^2(\Omega)$ is such that $-\Delta u = f$, $\frac{\partial u}{\partial n}|_{\partial\Omega}=0$. Introduce an operator $K_h : W_h \times M_h \to \boldsymbol{V}_h$ defined by

$$
(K_h(\psi_h,\widehat{\psi}_h),\boldsymbol{w}_h)_K = -(\psi_h,\nabla\cdot\boldsymbol{w}_h)_K+\langle\widehat{\psi}_h,\boldsymbol{n}\cdot\boldsymbol{w}_h\rangle_{\partial K} \tag{3.30}
$$

for all $\boldsymbol{w}_h \in \boldsymbol{V}_h$ and $K \in \mathcal{T}_h$.

Lemma 3.9. *Assume that the spectral estimate of the Cahn-Hilliard operator holds (Proposition 1 in [35]). Then for any fixed $\beta \in (0,1)$, one has*

$$
\begin{aligned}
&\epsilon\|K_h(\psi_h,\widehat{\psi}_h)\|_{L^2(\Omega)}^2+\epsilon\|h_{\mathcal{T}}^{-\frac{1}{2}}(\Pi_k^\partial\psi_h-\widehat{\psi}_h)\|_{\partial\mathcal{T}_h}^2+\frac{1-\beta}{\epsilon}(f'(u_{Ih})\psi_h,\psi_h)_{\mathcal{T}_h} \qquad (3.31)\\
&\quad\ge -2C_0\|\nabla\Delta^{-1}(\psi_h)\|_{L^2(\Omega)}^2-\|\psi_h\|_{-1,h}^2, \quad \forall(\psi_h,\widehat{\psi}_h)\in \mathring{W}_h\times M_h,\\
&\quad\textit{provided } h \le C\min\{\beta^{1/2}\epsilon^{11/4},\beta^{3/4}\epsilon^{7/4},\epsilon^{\frac{7}{4-d}}\},
\end{aligned}
$$

where C_0 is the constant in the continuous version of the original spectral estimate of the Cahn-Hilliard operator.

Proof. Let $\psi \in H^1(\Omega)\cap L_0^2(\Omega)$ be the solution of

$$
(\nabla\psi,\nabla v)=(K_h(\psi_h,\widehat{\psi}_h),\nabla v), \forall v\in H^1(\Omega).
$$

It follows

$$
\|\nabla\psi\|_{L^2(\Omega)} \le \|K_h(\psi_h,\widehat{\psi}_h)\|_{\mathcal{T}_h}. \tag{3.32}
$$

Consider the dual problem: finding $w \in H^1(\Omega)$ such that

$$
-\Delta w = \psi-\psi_h, \quad \boldsymbol{n}\cdot\nabla w|_{\partial\Omega}=0
$$

Then

$$
\begin{aligned}
\|\psi-\psi_h\|_{L^2(\Omega)}^2 &= (-\Delta w, \psi-\psi_h)\\
&= (\nabla w, \nabla(\psi-\psi_h)) + \langle \boldsymbol{n}\cdot\nabla w, \psi_h\rangle_{\partial\mathcal{T}_h}\\
&= (\nabla w, K_h(\psi_h,\widehat{\psi}_h)-\nabla\psi_h) + \langle \boldsymbol{n}\cdot\nabla w, \psi_h\rangle_{\partial\mathcal{T}_h}\\
&= (\boldsymbol{\Pi}_k^o\nabla w, K_h(\psi_h,\widehat{\psi}_h)-\nabla\psi_h) + \langle \boldsymbol{n}\cdot\nabla w, \psi_h\rangle_{\partial\mathcal{T}_h}\\
&= \langle \boldsymbol{n}\cdot\boldsymbol{\Pi}_k^o\nabla w - \boldsymbol{n}\cdot\nabla w, \widehat{\psi}_h-\psi_h\rangle_{\partial\mathcal{T}_h}\\
&\le Ch\|w\|_{H^2(\Omega)}\|h_K^{-\frac12}(\psi_h-\widehat{\psi}_h)\|_{\partial\mathcal{T}_h}\\
&\le Ch\|\psi-\psi_h\|_{L^2(\Omega)}\left(\|K_h(\psi_h,\widehat{\psi}_h)\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-\frac12}(\Pi_k^\partial\psi_h-\widehat{\psi}_h)\|_{\partial\mathcal{T}_h}\right),
\end{aligned}
$$

hence

$$
\|\psi-\psi_h\|_{L^2(\Omega)} \le Ch\left(\|K_h(\psi_h,\widehat{\psi}_h)\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-\frac12}(\Pi_k^\partial\psi_h-\widehat{\psi}_h)\|_{\partial\mathcal{T}_h}\right). \tag{3.33}
$$

Define

$$
\begin{aligned}
&\mathcal{L}_h := \epsilon\|K_h(\psi_h,\widehat{\psi}_h)\|_{L^2(\Omega)}^2 + \epsilon\|h_{\mathcal{T}}^{-\frac12}(\Pi_k^\partial\psi_h-\widehat{\psi}_h)\|_{\partial\mathcal{T}_h}^2 + \frac{1-\beta}{\epsilon}(f'(u_{Ih})\psi_h,\psi_h)\\
&+ c_0\|\nabla\Delta^{-1}\psi_h\|_{L^2(\Omega)}^2 + \|\psi_h\|_{-1,h}^2,\\
&\mathcal{L} := \epsilon\|\nabla\psi\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon}(f'(u)\psi,\psi) + C_0\|\nabla\Delta^{-1}\psi\|_{L^2(\Omega)}^2,
\end{aligned}
$$

with a constant c_0 to be determined, and C_0 the constant in the continuous version of the original spectral estimate of the Cahn-Hilliard operator, cf. [1, 13, 35]. Since $||u||_{L^\infty(L^\infty)} \le C$ [35] and under the constraint $h \le \epsilon^{7/(4-d)}$, it follows from (3.32) and the uniform bound (3.23) that

$$
\begin{aligned}
&\mathcal{L}_h - (1-\beta)\mathcal{L} \ge \beta\epsilon\left(\|K_h(\psi_h,\widehat{\psi}_h)\|_{L^2(\Omega)}^2 + \|h_{\mathcal{T}}^{-\frac12}(\Pi_k^\partial\psi_h-\widehat{\psi}_h)\|_{\partial\mathcal{T}_h}^2\right)\\
&- C\frac{1-\beta}{\epsilon}\|\psi_h^2-\psi^2\|_{L^1(\Omega)} - C\epsilon^{-1}\|u_{Ih}-u\|_{L^2(\Omega)}||\psi_h||_{L^4(\Omega)}^2\\
&+ (c_0-C_0)\|\nabla\Delta^{-1}\psi_h\|_{L^2(\Omega)}^2 + \|\psi_h\|_{-1,h}^2 - C_0\left|\|\nabla\Delta^{-1}\psi_h\|_{L^2(\Omega)}^2 - \|\nabla\Delta^{-1}\psi\|_{L^2(\Omega)}^2\right|
\end{aligned}
$$

The negative terms on the right-hand side of the above inequality are estimated as follows. First, by the error estimate (3.33), the interpolation inequality (3.11) and the estimate (3.32) there holds

$$
\begin{aligned}
&C\frac{1-\beta}{\epsilon}\|\psi_h^2-\psi^2\|_{L^1(\Omega)} \le \frac{C}{\epsilon}\|\psi_h-\psi\|_{L^2(\Omega)}^2 + \frac{C}{\epsilon}\|\psi_h-\psi\|_{L^2(\Omega)}\|\psi_h\|_{L^2(\Omega)}\\
&\le \frac{C}{\epsilon}h^2\left(\|K_h(\psi_h,\widehat{\psi}_h)\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-\frac12}(\Pi_k^\partial\psi_h-\widehat{\psi}_h)\|_{\partial\mathcal{T}_h}^2\right)\\
&+ \frac{Ch}{\epsilon}\left(\|K_h(\psi_h,\widehat{\psi}_h)\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-\frac12}(\Pi_k^\partial\psi_h-\widehat{\psi}_h)\|_{\partial\mathcal{T}_h}\right)^{3/2}\|\psi_h\|_{-1,h}^{1/2}\\
&\le C(\epsilon^{-1}h^2+\epsilon^{-4/3}h^{4/3})\left(\|K_h(\psi_h,\widehat{\psi}_h)\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-\frac12}(\Pi_k^\partial\psi_h-\widehat{\psi}_h)\|_{\partial\mathcal{T}_h}^2\right) + \|\psi_h\|_{-1,h}^2.
\end{aligned}
$$

Next, with $||u||_{L^\infty(H^2)} \le C\epsilon^{-7/2}$, the error estimate of HDG elliptic projection (3.17a), the estimate (3.32) and the HDG Sobolev imbedding (3.2b) imply

$$
\begin{aligned}
&C\epsilon^{-1}\|u_{Ih}-u\|_{L^2(\Omega)}\|\psi_h\|^2_{L^4(\Omega)}\\
&\le C\epsilon^{-1}h^2\|u\|_{H^2}\left(\|K_h(\psi_h,\widehat{\psi}_h)\|^2_{\mathcal{T}_h}+\|h_K^{-\frac{1}{2}}(\Pi_k^\partial\psi_h-\widehat{\psi}_h)\|^2_{\partial\mathcal{T}_h}\right)\\
&\le C\epsilon^{-9/2}h^2\left(\|K_h(\psi_h,\widehat{\psi}_h)\|^2_{\mathcal{T}_h}+\|h_{\mathcal{T}}^{-\frac{1}{2}}(\Pi_k^\partial\psi_h-\widehat{\psi}_h)\|^2_{\partial\mathcal{T}_h}\right).
\end{aligned}
$$

Finally, utilizing (3.33) gives

$$
\begin{aligned}
&\left|\|\nabla\Delta^{-1}\psi_h\|^2_{L^2(\Omega)}-\|\nabla\Delta^{-1}\psi\|^2_{L^2(\Omega)}\right|\le 2\|\nabla\Delta^{-1}(\psi_h-\psi)\|^2_{L^2(\Omega)}+\|\nabla\Delta^{-1}\psi_h\|^2_{L^2(\Omega)}\\
&\le 2\|\psi_h-\psi\|^2_{L^2(\Omega)}+\|\nabla\Delta^{-1}\psi_h\|^2_{L^2(\Omega)}\\
&\le Ch^2\left(\|K_h(\psi_h,\widehat{\psi}_h)\|^2_{\mathcal{T}_h}+\|h_{\mathcal{T}}^{-\frac{1}{2}}(\Pi_k^\partial\psi_h-\widehat{\psi}_h)\|^2_{\partial\mathcal{T}_h}\right)+\|\nabla\Delta^{-1}\psi_h\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathcal{L}_h-(1-\beta)\mathcal{L}\\
&\ge g(\epsilon,h)\left(\|K_h(\psi,\widehat{\psi}_h)\|^2_{L^2(\Omega)}+\|h_{\mathcal{T}}^{-\frac{1}{2}}(\Pi_k^\partial\psi_h-\widehat{\psi}_h)\|^2_{\partial\mathcal{T}_h}\right)\\
&+(c_0-2C_0)\|\nabla\Delta^{-1}\psi_h\|^2_{L^2(\Omega)}+\|\psi_h\|^2_{-1,h},
\end{aligned}
$$

with $g(\epsilon,h):=\beta\epsilon-C\epsilon^{-1}h^2-C\epsilon^{-4/3}h^{4/3}-C\epsilon^{-9/2}h^2-Ch^2$.

Thus if $c_0=2C_0$ and $h\le Cmin\{\beta^{1/2}\epsilon^{11/4},\beta^{3/4}\epsilon^{7/4}\}$, then

$$
\mathcal{L}_h-(1-\beta)\mathcal{L}\ge 0,
$$

so $\mathcal{L}_h\ge(1-\beta)\mathcal{L}\ge 0$. This establishes the spectral estimate (3.31). ∎

4. Stability of the HDG formulation. In this section we obtain stability estimates of the HDG method (2.2). Throughout, C denotes a generic constant that may depend on the initial condition and final time T but independent of ϵ,h and Δt. We assume that the initial energy is uniformly bounded in terms of ϵ, i.e.

$$
\frac{1}{4\epsilon}\|(u_h^0)^2-1\|^2_{\mathcal{T}_h}+\frac{\epsilon}{2}\|\boldsymbol{q}_h^0\|^2_{\mathcal{T}_h}+\frac{\epsilon}{2}\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial u_h^0-\widehat{u}_h^0)\|^2_{\partial\mathcal{T}_h}\le C, \tag{4.1}
$$

and that $\|\phi_h^0\|_{\mathcal{T}_h}\le\frac{C}{\epsilon}$. This assumption is reasonable from the standpoint of sharp interface limit of the Cahn-Hilliard free energy functional, cf. [43]. Relaxed assumption where the constant C is replaced by $\epsilon^{-\sigma_1}$ is utilized in [35, 34].

By an elementary fixed point argument and energy method, the unconditional unique solvability of the HDG scheme (2.2) is established in [10]. The basic energy stability bounds are provided in the following lemma.

LEMMA 4.1. *For any $h,\Delta t>0$ and $m=1,2,\ldots,N$, the following stability bounds hold for the solution to the HDG scheme*

$$
\begin{aligned}
&\frac{1}{4\epsilon}\|(u_h^m)^2-1\|^2_{\mathcal{T}_h}+\frac{\epsilon}{2}\left(\|\boldsymbol{q}_h^m\|^2_{\mathcal{T}_h}+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial u_h^m-\widehat{u}_h^m)\|^2_{\partial\mathcal{T}_h}\right)\\
&+\Delta t\sum_{n=1}^m\left(\|\boldsymbol{p}_h^n\|^2_{\mathcal{T}_h}+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial\phi_h^n-\widehat{\phi}_h^n)\|^2_{\partial\mathcal{T}_h}\right)\le C.
\end{aligned}
\tag{4.2}
$$

Proof. One takes $(\boldsymbol{r}_2, w_2, \mu_2) = (\mathbf{0}, \partial_t^+ u_h^n, \partial_t^+ \widehat{u}_h^n)$ in (2.2b) to get

$$\begin{aligned}(\epsilon^{-1}F(u_h^n, u_h^{n-1}), \partial_t^+ u_h^n)_{\mathcal{T}_h} + \epsilon(\nabla\cdot \boldsymbol{q}_h^n, \partial_t^+ u_h^n)_{\mathcal{T}_h} - \epsilon\langle \boldsymbol{q}_h^n\cdot\boldsymbol{n}, \partial_t^+ \widehat{u}_h^n\rangle_{\partial\mathcal{T}_h}\\ + \epsilon\langle h_K^{-1}(\Pi_k^\partial u_h^n - \widehat{u}_h^n), \partial_t^+(\Pi_k^\partial u_h^n - \widehat{u}_h^n)\rangle_{\partial\mathcal{T}_h} - (\phi_h^n, \partial_t^+ u_h^n)_{\mathcal{T}_h} = 0.\end{aligned} \tag{4.3}$$

Then one applies ∂_t^+ to (2.2b) and take $(\boldsymbol{r}_2, w_2, \mu_2) = (\boldsymbol{q}_h^n, 0, 0)$ to get

$$\epsilon(\partial_t^+ \boldsymbol{q}_h^n, \boldsymbol{q}_h^n)_{\mathcal{T}_h} - \epsilon(\partial_t^+ u_h^n, \nabla\cdot\boldsymbol{q}_h^n)_{\mathcal{T}_h} + \epsilon\langle \partial_t^+ \widehat{u}_h^n, \boldsymbol{q}_h^n\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0. \tag{4.4}$$

Likewise, taking $(\boldsymbol{r}_1, w_1, \mu_1) = (\boldsymbol{p}_h^n, \phi_h^n, \widehat{\phi}_h^n)$ in (2.2a) one obtains

$$(\partial_t^+ u_h^n, \phi_h^n)_{\mathcal{T}_h} + \|\boldsymbol{p}_h^n\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial \phi_h^n - \widehat{\phi}_h^n)\|_{\partial\mathcal{T}_h}^2 = 0. \tag{4.5}$$

Taking the summation of (4.3), (4.4) and (4.5) gives

$$\begin{aligned}&\epsilon^{-1}(F(u_h^n, u_h^{n-1}), \partial_t^+ u_h^n)_{\mathcal{T}_h} + \epsilon(\partial_t^+ \boldsymbol{q}_h^n, \boldsymbol{q}_h^n)_{\mathcal{T}_h} + \epsilon\langle h_K^{-1}(\Pi_k^\partial u_h^n - \widehat{u}_h^n), \partial_t^+(\Pi_k^\partial u_h^n - \widehat{u}_h^n)\rangle_{\partial\mathcal{T}_h}\\ &+ \|\boldsymbol{p}_h^n\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial \phi_h^n - \widehat{\phi}_h^n)\|_{\partial\mathcal{T}_h}^2 = 0.\end{aligned}$$

Recall that $F(u_h^n, u_h^{n-1}) = (u_h^n)^3 - u_h^{n-1}$, and the identity

$$(a^3 - b)(a - b) = \frac{1}{4}[(a^2-1)^2 - (b^2-1)^2 + (a^2-b^2)^2 + 2a^2(a-b)^2 + 2(a-b)^2].$$

One obtains

$$\begin{aligned}&\frac{1}{4\epsilon}\|(u_h^n)^2 - 1\|_{\mathcal{T}_h}^2 + \frac{\Delta t\epsilon}{2}(\|\boldsymbol{q}_h^n\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial u_h^n - \widehat{u}_h^n)\|_{\partial\mathcal{T}_h}^2)\\ &+ \Delta t(\|\boldsymbol{p}_h^n\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial \phi_h^n - \widehat{\phi}_h^n)\|_{\partial\mathcal{T}_h}^2)\\ &\le \frac{1}{4\epsilon}\|(u_h^{n-1})^2 - 1\|_{\mathcal{T}_h}^2 + \frac{\Delta t\epsilon}{2}(\|\boldsymbol{q}_h^{n-1}\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial u_h^{n-1} - \widehat{u}_h^{n-1})\|_{\partial\mathcal{T}_h}^2).\end{aligned}$$

The inequality (4.2) follows from taking summation of the above equation from $n = 1$ to $n = m$. This completes the proof. ∎

From Lemma 4.1 one derives the following estimates.

LEMMA 4.2. *For $h, \Delta t > 0$ there holds*

$$\|u_h^m\|_{L^4}^4 + \Delta t\sum_{n=1}^m \|\nabla\phi_h^n\|_{\mathcal{T}_h}^2 \le C, \quad \|\nabla u_h^m\|_{\mathcal{T}_h}^2 \le \frac{C}{\epsilon}, \tag{4.6}$$

$$\|\phi_h^m\|_{\mathcal{T}_h}^2 + \sum_{n=1}^m \|\phi_h^n - \phi_h^{n-1}\|_{\mathcal{T}_h}^2 + \epsilon\Delta t\sum_{n=1}^m \|\partial_t^+ u_h^n\|_{\mathcal{T}_h}^2 \le \frac{C}{\epsilon^7}. \tag{4.7}$$

Proof. Inequalities (4.6) are consequences of the estimate (4.2) and the inequality (3.6). Taking $(\boldsymbol{r}_2, w_2, \mu_2) = (\mathbf{0}, 1, 1)$ in (2.2b) yields

$$|(\phi_h^n, 1)_{\mathcal{T}_h}| = \frac{1}{\epsilon}|\big(F(u_h^n, u_h^{n-1}), 1\big)_{\mathcal{T}_h}| \le \frac{1}{\epsilon}(\|u_h^n\|_{L^3(\Omega)}^3 + \|u_h^n\|_{L^1(\Omega)}) \le \frac{C}{\epsilon}.$$

In light of the stability bounds (4.2), it follows from the triangle inequality, the HDG Sobolev inequality (3.2b), and the inequality (3.6) that

$$\Delta t\sum_{n=1}^m \|\phi_h^n\|_{\mathcal{T}_h}^2 \le \frac{C}{\epsilon^2}. \tag{4.8}$$

Applying ∂_t^+ to (2.2b) gives

$$\begin{aligned}\epsilon\mathcal{A}(\partial_t^+\boldsymbol{q}_h^n,\partial_t^+u_h^n,\partial_t^+\widehat{u}_h^n;\boldsymbol{r}_2,w_2,\mu_2)+(\epsilon^{-1}\partial_t^+F(u_h^n,u_h^{n-1}),w_2)_{\mathcal{T}_h}\\-(\partial_t^+\phi_h^n,w_2)_{\mathcal{T}_h}=0.\end{aligned}\tag{4.9a}$$

Taking $(\boldsymbol{r}_1,w_1,\mu_1)=\epsilon(-\partial_t^+\boldsymbol{q}_h^n,\partial_t^+u_h^n,\partial_t^+\widehat{u}_h^n)$ in (2.2a), $(\boldsymbol{r}_2,w_2,\mu_2)=(\boldsymbol{p}_h^n,-\phi_h^n,-\widehat{\phi}_h^n)$ in (4.9a), one obtains

$$\begin{aligned}&\epsilon(\partial_t^+u_h^n,\partial_t^+u_h^n)_{\mathcal{T}_h}-\epsilon\mathcal{A}(\boldsymbol{p}_h^n,\phi_h^n,\widehat{\phi}_h^n;\partial_t^+\boldsymbol{q}_h^n,-\partial_t^+u_h^n,-\partial_t^+\widehat{u}_h^n)=0,\\&\epsilon\mathcal{A}(\partial_t^+\boldsymbol{q}_h^n,\partial_t^+u_h^n,\partial_t^+\widehat{u}_h^n;\boldsymbol{p}_h^n,-\phi_h^n,-\widehat{\phi}_h^n)\\&+\Big(\epsilon^{-1}\partial_t^+F(u_h^n,u_h^{n-1}),-\phi_h^n\Big)_{\mathcal{T}_h}-(\partial_t^+\phi_h^n,-\phi_h^n)_{\mathcal{T}_h}=0.\end{aligned}$$

Taking summation of the two equations, one obtains

$$(\partial_t^+\phi_h^n,\phi_h^n)_{\mathcal{T}_h}+\epsilon\|\partial_t^+u_h^n\|_{\mathcal{T}_h}^2=\epsilon^{-1}\big(\partial_t^+F(u_h^n,u_h^{n-1}),\phi_h^n\big)_{\mathcal{T}_h},\tag{4.10}$$

and hence

$$\begin{aligned}&\|\phi_h^m\|_{\mathcal{T}_h}^2+\sum_{n=1}^m\|\phi_h^n-\phi_h^{n-1}\|_{\mathcal{T}_h}^2+2\Delta t\epsilon\sum_{n=1}^m\|\partial_t^+u_h^n\|_{\mathcal{T}_h}^2\\&\quad=2\epsilon^{-1}\Delta t\sum_{n=1}^m\big(\partial_t^+F(u_h^n,u_h^{n-1}),\phi_h^n\big)_{\mathcal{T}_h}+\|\phi_h^0\|_{\mathcal{T}_h}^2.\end{aligned}\tag{4.11}$$

By Hölder's inequality and the HDG Sobolev inequality in Lemma 3.1 one derives

$$\begin{aligned}&\frac{\Delta t}{\epsilon}\sum_{n=1}^m\Big(\partial_t^+F(u_h^n,u_h^{n-1}),\phi_h^n\Big)_{\mathcal{T}_h}\\&\le\frac{C\Delta t}{\epsilon}\sum_{n=1}^m\|\partial_t^+u_h^n\|_{\mathcal{T}_h}\|\phi_h^n\|_{L^6(\Omega)}\Big(\|u_h^n\|_{L^6(\Omega)}^2+\|u_h^{n-1}\|_{L^6(\Omega)}^2+1\Big)\\&\le\frac{C\Delta t}{\epsilon}\sum_{n=1}^m\|\partial_t^+u_h^n\|_{\mathcal{T}_h}\|\phi_h^n\|_{L^6(\Omega)}\big(\|u_h^n\|_{\mathcal{T}_h}^2+\|u_h^{n-1}\|_{\mathcal{T}_h}^2+\|\boldsymbol{q}_h^n\|_{\mathcal{T}_h}^2+\|\boldsymbol{q}_h^{n-1}\|_{\mathcal{T}_h}^2\\&\qquad+\|h_K^{-1/2}(\Pi_k^\partial u_h^n-\widehat{u}_h^n)\|_{\partial\mathcal{T}_h}^2+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial u_h^{n-1}-\widehat{u}_h^{n-1})\|_{\partial\mathcal{T}_h}+1\Big)\\&\le\frac{C\Delta t}{\epsilon^2}\sum_{n=1}^m\|\partial_t^+u_h^n\|_{\mathcal{T}_h}\|\phi_h^n\|_{L^6(\Omega)}\\&\le\frac{\epsilon\Delta t}{2}\sum_{n=1}^m\|\partial_t^+u_h^n\|_{\mathcal{T}_h}^2+\frac{C\Delta t}{\epsilon^5}\sum_{n=1}^m\|\phi_h^n\|_{L^6(\Omega)}^2.\end{aligned}$$

Hence (4.11) implies that

$$\|\phi_h^m\|_{\mathcal{T}_h}^2+\sum_{n=1}^m\|\phi_h^n-\phi_h^{n-1}\|_{\mathcal{T}_h}^2+\epsilon\Delta t\sum_{n=1}^m\|\partial_t^+u_h^n\|_{\mathcal{T}_h}^2\le\frac{C\Delta t}{\epsilon^5}\sum_{n=1}^m\|\phi_h^n\|_{L^6(\Omega)}^2+\|\phi_h^0\|_{\mathcal{T}_h}^2.$$

An application of the HDG Sobolev embedding inequality Lemma 3.1 then gives

$$
\begin{aligned}
&\|\phi_h^m\|_{\mathcal{T}_h}^2 + \sum_{n=1}^m \|\phi_h^n - \phi_h^{n-1}\|_{\mathcal{T}_h}^2 + \epsilon\Delta t \sum_{n=1}^m \|\partial_t^+ u_h^n\|_{\mathcal{T}_h}^2 \\
&\quad \le \frac{C\Delta t}{\epsilon^5} \sum_{n=1}^m (\|\boldsymbol{p}_h^n\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial \phi_h^n - \widehat{\phi}_h^n)\|_{\partial\mathcal{T}_h}^2 + \|\phi_h^n\|_{\mathcal{T}_h}^2) + \|\phi_h^0\|_{\mathcal{T}_h}^2 \\
&\quad \le \frac{C}{\epsilon^7},
\end{aligned}
$$

where one uses the estimate (4.8) and the bounds in Lemma 4.1 in the derivation of the last step. This finishes the proof. ∎

LEMMA 4.3. *Let u_h^n be the solution of (2.2). For all $n = 1, 2 \ldots, N$, we have*

$$
\|\Delta_h u_h^n\|_{\mathcal{T}_h} \le C\epsilon^{-\frac{7}{2}}, \tag{4.12}
$$

where C depends on ϵ, T and the initial condition.

Proof. We take $(\boldsymbol{r}_2, w_2, \mu_2) = (\mathbf{0}, \Delta_h u_h^n, 0)$ in (2.2b) to get

$$
\left(\epsilon^{-1} F(u_h^n, u_h^{n-1}), \Delta_h u_h^n\right)_{\mathcal{T}_h} - \epsilon\|\Delta_h u_h^n\|_{\mathcal{T}_h}^2 - (\phi_h^n, \Delta_h u_h^n)_{\mathcal{T}_h} = 0.
$$

By the Cauchy-Schwarz inequality, one gets

$$
\begin{aligned}
\epsilon\|\Delta_h u_h^n\|_{\mathcal{T}_h}^2 &= \left(\epsilon^{-1} F(u_h^n, u_h^{n-1}), \Delta_h u_h^n\right)_{\mathcal{T}_h} - (\phi_h^n, \Delta_h u_h^n)_{\mathcal{T}_h} \\
&\le \epsilon^{-1}(\|u_h^n\|_{L^6(\Omega)}^3 + \|u_h^n\|_{\mathcal{T}_h})\|\Delta_h u_h^n\|_{\mathcal{T}_h} + \|\phi_h^n\|_{\mathcal{T}_h}\|\Delta_h u_h^n\|_{\mathcal{T}_h}.
\end{aligned}
$$

Next using the HDG Sobolev inequality (3.2a) and Lemma 4.1 one obtains

$$
\begin{aligned}
\|u_h^n\|_{L^6(\Omega)} &\le C\left(\|u_h^n\|_{\mathcal{T}_h} + \|\nabla u_h^n\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial u_h^n - \widehat{u}_h^n)\|_{\partial\mathcal{T}_h}\right) \\
&\le C\left(\|u_h^n\|_{L^4(\Omega)} + \|\nabla u_h^n\|_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial u_h^n - \widehat{u}_h^n)\|_{\partial\mathcal{T}_h}\right) \\
&\le C\epsilon^{-\frac{1}{2}}.
\end{aligned}
$$

Then the desired result follows from Lemma 4.2 and Young's inequality. ∎

Using Lemma 3.8 and Lemma 4.3 one deduces the uniform estimate of u_h^n.

LEMMA 4.4. *Let u_h^n be the solution of (2.2), then for all $n = 1, 2, \ldots, N$, we have*

$$
\|u_h^n\|_{L^\infty(\Omega)} \le C\epsilon^{-\frac{7}{2}}.
$$

5. Error analysis.

In this section, we establish the optimal convergence result for the HDG scheme for solving the Cahn-Hilliard equation. We prove the main convergence result by performing error estimates first in the negative norm, then in the L^2 norm for the scalar variables and flux variables, respectively. Throughout, we assume the data and the solution of (1.1) are smooth enough. For convenience, we shall also adopt the notation $||a, b||_{L^p} := ||a||_{L^p} + ||b||_{L^p}$.

5.1. The main result.

THEOREM 5.1. *Assume the same conditions as in Proposition 1 in [35] so that the spectral estimate of the Cahn-Hilliard operator holds. Let* $(\boldsymbol{p}, \phi, \boldsymbol{q}, u)$ *and* $(\boldsymbol{p}_h^n, \phi_h^n, \boldsymbol{q}_h^n, u_h^n)$ *be the solutions of* (2.1) *and* (2.2)*, respectively. Furthermore, assume the solution* $(\boldsymbol{p}, \phi, \boldsymbol{q}, u)$ *attains the maximum regularity for the best approximation results in* (3.1)*. Then provided*

$$
\begin{aligned}
&h \le C \min\left\{\epsilon^{\frac{17}{4}}, \quad \epsilon^{\frac{7}{4-d}}, \quad \left(\frac{\epsilon}{C(T,u,\phi)}\right)^{\frac{37}{2(k+2)}}\right\},\\
&\Delta t \le C\epsilon^{60},
\end{aligned}
$$

the following optimal error estimates hold for polynomials of degree $k \ge 0$

$$
\begin{aligned}
&\max_{1\le n\le N} \|u^n - u_h^n\|^2_{L^2(\Omega)} + \Delta t \sum_{n=1}^N \|\phi^n - \phi_h^n\|^2_{L^2(\Omega)} \le \frac{1}{\epsilon^{26}} C(T,\epsilon,u,\phi)(h^{k+2} + \Delta t)^2,\\
&\max_{1\le n\le N} \|\boldsymbol{q}^n - \boldsymbol{q}_h^n\|^2_{L^2(\Omega)} + \Delta t \sum_{n=1}^N \|\boldsymbol{p}^n - \boldsymbol{p}_h^n\|^2_{L^2(\Omega)} \le \frac{1}{\epsilon^{24}} C(T,\epsilon,u,\phi)(h^{k+1} + \Delta t)^2,
\end{aligned}
$$

where $C(T,u,\phi) := Ce^{CT}(||u, \partial_t u||^2_{L^2(H^{k+2})} + ||\phi||^2_{L^2(H^{k+2})} + ||\partial_{tt} u||^2_{L^2(L^2)} + 1)$.

5.2. Proof of Theorem 5.1. To simplify notation, we define

$$
\begin{aligned}
&(5.1a) \qquad e_h^{\boldsymbol{p}^n} := \boldsymbol{p}_{Ih}^n - \boldsymbol{p}_h^n, \quad e_h^{\phi^n} := \phi_{Ih}^n - \phi_h^n, \quad e_h^{\widehat{\phi}^n} := \widehat{\phi}_{Ih}^n - \widehat{\phi}_h^n,\\
&(5.1b) \qquad e_h^{\boldsymbol{q}^n} := \boldsymbol{q}_{Ih}^n - \boldsymbol{q}_h^n, \quad e_h^{u^n} := u_{Ih}^n - u_h^n, \quad e_h^{\widehat{u}^n} := \widehat{u}_{Ih}^n - \widehat{u}_h^n.
\end{aligned}
$$

By the definition of $\mathcal{A}$ in (2.2d) and the HDG elliptic projection (3.16), for all $(\boldsymbol{r}_1, w_1, \mu_1), (\boldsymbol{r}_2, w_2, \mu_2)$ in $\boldsymbol{V}_h \times W_h \times M_h$ one obtains the error equations

$$
\begin{aligned}
&(5.2a) \qquad (\partial_t^+ e_h^{u^n}, w_1)_{\mathcal{T}_h} + \mathcal{A}(e_h^{\boldsymbol{p}^n}, e_h^{\phi^n}, e_h^{\widehat{\phi}^n}; \boldsymbol{r}_1, w_1, \mu_1) = (\partial_t^+ u_{Ih}^n - \partial_t u^n, w_1)_{\mathcal{T}_h},\\
&(5.2b) \qquad \epsilon\mathcal{A}(e_h^{\boldsymbol{q}^n}, e_h^{u^n}, e_h^{\widehat{u}^n}; \boldsymbol{r}_2, w_2, \mu_2)_{\mathcal{T}_h} - (e_h^{\phi^n}, w_2)_{\mathcal{T}_h} = (\phi^n - \phi_{Ih}^n, w_2)_{\mathcal{T}_h}\\
&\qquad\qquad + \epsilon^{-1}\Big(F(u_h^n, u_h^{n-1}) - f(u^n), w_2\Big)_{\mathcal{T}_h}.
\end{aligned}
$$

We divide the error analysis into three lemmas. We first obtain a non-optimal error estimate in the negative norm.

LEMMA 5.2 (Error estimates in the negative norm). *Under the same conditions as the discrete spectral estimate in Lemma 3.9, provided*

$$
\begin{aligned}
&h \le C \min\left\{\epsilon^{\frac{17}{4}}, \quad \epsilon^{\frac{7}{4-d}}, \quad \left(\frac{\epsilon}{C(T,u,\phi)}\right)^{\frac{37}{2(k+2)}}\right\},\\
&\Delta t \le C\epsilon^{60},
\end{aligned}
$$

then

$$
\begin{aligned}
(5.3) \qquad &\max_{1\le n\le N} \|e_h^{u^n}\|^2_{-1,h} + \frac{4\epsilon^4 \Delta t}{1-\epsilon^3} \sum_{n=1}^N \Big(\|e_h^{\boldsymbol{q}^n}\|^2_{\mathcal{T}_h} + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n} - e_h^{\widehat{u}^n})\|^2_{\partial\mathcal{T}_h}\Big)\\
&\le C(T,u,\phi)\epsilon^{-7}\big((\Delta t)^2 + h^{2(k+2)}\big),
\end{aligned}
$$

where $C(T,u,\phi) := Ce^{CT}(||u, \partial_t u||^2_{L^2(H^{k+2})} + ||\phi||^2_{L^2(H^{k+2})} + ||\partial_{tt} u||^2_{L^2(L^2)} + 1)$.

Proof. Taking $(\boldsymbol{r}_1, w_1, \mu_1) = (-\boldsymbol{\Pi_V} e_h^{u^n}, \Pi_W e_h^{u^n}, \Pi_M e_h^{u^n})$ in (5.2a), $(\boldsymbol{r}_2, w_2, \mu_2) = (e_h^{\boldsymbol{q}^n}, e_h^{u^n}, e_h^{\widehat{u}^n})$ in (5.2b), adding the resulting equations, one obtains

$$
\begin{aligned}
&(\partial_t^+ e_h^{u^n}, \Pi_W e_h^{u^n})_{\mathcal{T}_h} + \epsilon \|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \epsilon \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n} - e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2 \\
&+ \epsilon^{-1}\Big(f(u^n) - F(u_h^n, u_h^{n-1}), e_h^{u^n}\Big)_{\mathcal{T}_h} \\
&= (\phi^n - \phi_{Ih}^n, e_h^{u^n})_{\mathcal{T}_h} + (\partial_t^+ u_{Ih}^n - \partial_t u^n, \Pi_W e_h^{u^n})_{\mathcal{T}_h}.
\end{aligned}
$$

Utilizing Definition 3.4 and (2.2d), one has

$$
\begin{aligned}
(\partial_t^+ e_h^{u^n}, \Pi_W e_h^{u^n})_{\mathcal{T}_h} &= \mathcal{A}(\boldsymbol{\Pi_V} \partial_t^+ e_h^{u^n}, \Pi_W \partial_t^+ e_h^{u^n}, \Pi_M \partial_t^+ e_h^{u^n}; \boldsymbol{0}, \Pi_W e_h^{u^n}, \Pi_M e_h^{u^n}) \\
&= (\nabla \cdot \boldsymbol{\Pi_V} \partial_t^+ e_h^{u^n}, \Pi_W e_h^{u^n})_{\mathcal{T}_h} - \langle \boldsymbol{n} \cdot \boldsymbol{\Pi_V} \partial_t^+ e_h^{u^n}, \Pi_M e_h^{u^n} \rangle_{\partial\mathcal{T}_h} \\
&\quad + \langle h_K^{-1}(\Pi_k^\partial \Pi_W e_h^{u^n} - \Pi_M e_h^{u^n}), \partial_t^+(\Pi_k^\partial \Pi_W e_h^{u^n} - \Pi_M e_h^{u^n}) \rangle_{\partial\mathcal{T}_h}.
\end{aligned}
$$

On the other hand, $\mathcal{A}(\boldsymbol{\Pi_V} e_h^{u^n}, \Pi_W e_h^{u^n}, \Pi_M e_h^{u^n}; \boldsymbol{r}_h, w_h, \mu_h) = (e_h^{u^n}, w_h)_{\mathcal{T}_h}$. With $\boldsymbol{r}_h = \boldsymbol{\Pi_V} \partial_t^+ e_h^{u^n}$, $w_h = \mu_h = 0$ it follows

$$
(\Pi_W e_h^{u^n}, \nabla \cdot \boldsymbol{\Pi_V} \partial_t^+ e_h^{u^n})_{\mathcal{T}_h} - \langle \Pi_M e_h^{u^n}, \boldsymbol{n} \cdot \boldsymbol{\Pi_V} \partial_t^+ e_h^{u^n} \rangle_{\partial\mathcal{T}_h} = (\boldsymbol{\Pi_V} e_h^{u^n}, \partial_t^+ \boldsymbol{\Pi_V} e_h^{u^n})_{\mathcal{T}_h}.
$$

Hence $(\partial_t^+ e_h^{u^n}, \Pi_W e_h^{u^n})_{\mathcal{T}_h} = \frac{1}{2}\partial_t^+ \|e_h^{u^n}\|_{-1,h}^2 + \frac{\Delta t}{2}\|\partial_t^+ e_h^{u^n}\|_{-1,h}^2$. Therefore one obtains the following error equation

$$
\begin{aligned}
&\frac{1}{2}\partial_t^+ \|e_h^{u^n}\|_{-1,h}^2 + \frac{\Delta t}{2}\|\partial_t^+ e_h^{u^n}\|_{-1,h}^2 + \epsilon\Big(\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n} - e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\Big) \\
&+\epsilon^{-1}\Big(f(u^n) - F(u_h^n, u_h^{n-1}), e_h^{u^n}\Big)_{\mathcal{T}_h} = (\phi^n - \phi_{Ih}^n, e_h^{u^n})_{\mathcal{T}_h} + (\partial_t^+ u_{Ih}^n - \partial_t u^n, \Pi_W e_h^{u^n})_{\mathcal{T}_h}.
\end{aligned} \tag{5.4}
$$

The right-hand side of (5.4) is estimated as follows

$$
\begin{aligned}
&|(\phi^n - \phi_{Ih}^n, e_h^{u^n})_{\mathcal{T}_h} + (\partial_t^+ u_{Ih}^n - \partial_t u^n, \Pi_W e_h^{u^n})_{\mathcal{T}_h}| \\
&\le C\rho_1^n + \frac{\epsilon^4}{1-\epsilon^3}(\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n} - e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2) + \|e_h^{u^n}\|_{-1,h}^2,
\end{aligned} \tag{5.5}
$$

with $\rho_1^n := \frac{1}{\epsilon^4}\|\phi^n - \phi_{Ih}^n\|_{\mathcal{T}_h}^2 + \|\partial_t^+ u_{Ih}^n - \partial_t u^n\|_{\mathcal{T}_h}^2$. Since $F(u_h^n, u_h^{n-1}) = (u_h^n)^3 - u_h^{n-1}$, the nonlinear term can be written as

$$
\begin{aligned}
&f(u^n) - F(u_h^n, u_h^{n-1}) \\
&= f(u^n) - f(u_{Ih}^n) + f'(u_{Ih}^n) e_h^{u^n} + 3(u_{Ih}^n)^2 (e_h^{u^n})^2 + (e_h^{u^n})^3 - \Delta t \partial_t^+ u_h^n.
\end{aligned}
$$

By the Cauchy-Schwartz inequality, the HDG Sobolev imbedding, the inequality (3.6) and the uniform estimate (3.23) , one obtains

$$
\begin{aligned}
\epsilon^{-1}\Big(f(u^n) - F(u_h^n, u_h^{n-1}), e_h^{u^n}\Big)_{\mathcal{T}_h} &\ge \epsilon^{-1}\big(f'(u_{Ih}^n) e_h^{u^n}, e_h^{u^n}\big)_{\mathcal{T}_h} + \frac{1}{\epsilon}\|e_h^{u^n}\|_{L^4}^4 \\
&- C\rho_2^n - \frac{\epsilon^4}{1-\epsilon^3}(\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n} - e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2) - \frac{C}{\epsilon}\|e_h^{u^n}\|_{L^3}^3,
\end{aligned} \tag{5.6}
$$

where $\rho_2^n := \frac{1}{\epsilon^6}\left[\|f(u^n) - F(u_{Ih}^n, u_{Ih}^{n-1})\|_{\mathcal{T}_h}^2 + \Delta t^2 \|\partial_t^+ u_h^n\|_{\mathcal{T}_h}^2\right]$, and the uniform bound in (3.23) has been utilized.

In light of (5.5) and (5.6), the error equation (5.4) can be written as

$$
\begin{aligned}
(5.7)\quad & \frac{1}{2}\partial_t^+\|e_h^{u^n}\|_{-1,h}^2 + \frac{\Delta t}{2}\|\partial_t^+ e_h^{u^n}\|_{-1,h}^2 + \frac{1}{\epsilon}\|e_h^{u^n}\|_{L^4}^4 \\
& + \frac{1-7\epsilon^3}{1-\epsilon^3}\Big(\epsilon\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \epsilon\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n} - e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2 + \frac{1-\epsilon^3}{\epsilon}\big(f'(u_{Ih}^n)e_h^{u^n}, e_h^{u^n}\big)_{\mathcal{T}_h}\Big) \\
& + \frac{4\epsilon^4}{1-\epsilon^3}\Big(\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n} - e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\Big) \\
& \le \|e_h^{u^n}\|_{-1,h}^2 + \frac{C}{\epsilon^8}\|e_h^{u^n}\|_{L^3}^3 + C(\rho_1^n + \rho_2^n) + C\epsilon^2\big(f'(u_{Ih}^n)e_h^{u^n}, e_h^{u^n}\big)_{\mathcal{T}_h}
\end{aligned}
$$

By the uniform estimate (3.23) and the inequality (3.11), there holds

$$
\begin{aligned}
(5.8)\quad & C\epsilon^2|\big(f'(u_{Ih}^n)e_h^{u^n}, e_h^{u^n}\big)_{\mathcal{T}_h}| \le C\epsilon^2\|e_h^{u^n}\|_{\mathcal{T}_h}^2 \\
& \le C\|e_h^{u^n}\|_{-1,h}^2 + \frac{\epsilon^4}{1-\epsilon^3}\Big(\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n} - e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\Big).
\end{aligned}
$$

Furthermore, the interpolation inequality and the HDG Sobolev imbedding implies

$$
\begin{aligned}
(5.9)\quad & \frac{C}{\epsilon}\|e_h^{u^n}\|_{L^3}^3 \le \frac{C}{\epsilon}\|e_h^{u^n}\|_{L^2}^{\frac{3}{2}}\|e_h^{u^n}\|_{L^6}^{\frac{3}{2}} \\
& + \frac{C}{\epsilon^{12}}\|e_h^{u^n}\|_{L^2}^6 + \frac{\epsilon^4}{1-\epsilon^3}\Big(\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n} - e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\Big).
\end{aligned}
$$

By the definition (3.30) and the error equation (5.2b), one has $K_h(e_h^{u^n}, e_h^{\widehat{u}^n}) = e_h^{\boldsymbol{q}^n}$. The spectral estimate (3.31) with $\beta = \epsilon^3$ then implies

$$
\begin{aligned}
(5.10)\quad & \epsilon\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \epsilon\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n} - e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2 + \frac{1-\epsilon^3}{\epsilon}\big(f'(u_{Ih}^n)e_h^{u^n}, e_h^{u^n}\big)_{\mathcal{T}_h} \\
& \ge -2C_0||\nabla(-\Delta)^{-1}e_h^{u^n}||_{L^2}^2 - \|e_h^{u^n}\|_{-1,h}^2.
\end{aligned}
$$

Defining $(-\Delta)^{-1}e_h^{u^n} = f^n$, it follows from the definition (3.8)

$$
\mathcal{A}(\boldsymbol{\Pi_V} e_h^{u^n}, \Pi_W e_h^{u^n}, \Pi_M e_h^{u^n}; \boldsymbol{r}_h, w_h, \mu_h) = (e_h^{u^n}, w_h)_{\mathcal{T}_h} = -(\Delta f^n, w_h)_{\mathcal{T}_h}.
$$

Hence the error estimate of the HDG elliptic projection in Lemma 3.7 and the interpolation inequality (3.11) implies that

$$
\begin{aligned}
& 2C_0||\nabla(-\Delta)^{-1}e_h^{u^n}||_{L^2}^2 \le C(||\nabla[(-\Delta)^{-1}e_h^{u^n} - \Pi_W e_h^{u^n}]||_{\mathcal{T}_h}^2 + \|\nabla\Pi_W e_h^{u^n}\|_{\mathcal{T}_h}^2) \\
& \le Ch^2||f^n||_{H^2}^2 + C\|e_h^{u^n}\|_{-1,h}^2 \\
& \le Ch^2||e_h^{u^n}||_{L^2}^2 + C\|e_h^{u^n}\|_{-1,h}^2 \\
& \le \frac{\epsilon^4}{1-\epsilon^3}\Big(\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n} - e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\Big) + \frac{Ch^4}{\epsilon^4}\|e_h^{u^n}\|_{-1,h}^2 + C\|e_h^{u^n}\|_{-1,h}^2 \\
& \le \frac{\epsilon^4}{1-\epsilon^3}\Big(\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n} - e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\Big) + C\|e_h^{u^n}\|_{-1,h}^2,
\end{aligned}
$$

provided $h \le \epsilon$. The estimate (5.10) now becomes

$$
\begin{aligned}
(5.11)\quad & \epsilon\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \epsilon\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n} - e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2 + \frac{1-\epsilon^3}{\epsilon}\big(f'(u_{Ih}^n)e_h^{u^n}, e_h^{u^n}\big)_{\mathcal{T}_h} \\
& \ge -\frac{\epsilon^4}{1-\epsilon^3}\Big(\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n} - e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\Big) - C\|e_h^{u^n}\|_{-1,h}^2
\end{aligned}
$$

Taking into account of (5.8), (5.9), and (5.11), the error inequality (5.7) is now

$$\begin{aligned}(5.12)\qquad &\frac{1}{2}\partial_t^+\|e_h^{u^n}\|_{-1,h}^2+\frac{\Delta t}{2}\|\partial_t^+e_h^{u^n}\|_{-1,h}^2+\frac{1}{\epsilon}\|e_h^{u^n}\|_{L^4}^4\\ &+\frac{2\epsilon^4}{1-\epsilon^3}\Big(\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n}-e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\Big)\\ &\le C\|e_h^{u^n}\|_{-1,h}^2+\frac{C}{\epsilon^{12}}\|e_h^{u^n}\|_{L^2}^6+C(\rho_1^n+\rho_2^n).\end{aligned}$$

By the L^2 stability estimate of the numerical solution (4.6) and the L^2 error estimate of the HDG elliptic projection in Lemma 3.7, provided $h\le\epsilon^{\frac{7}{4}}$, one obtains $\|e_h^{u^n}\|_{L^2}^6\le C\|e_h^{u^n}\|_{L^2}^3$. Then the inequality (3.11) implies

$$\begin{aligned}&\frac{C}{\epsilon^{12}}\|e_h^{u^n}\|_{L^2}^3\le C\Big(\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n}-e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\Big)^{\frac{3}{4}}\|e_h^{u^n}\|_{-1,h}^{\frac{3}{2}}\\ &\le\frac{\epsilon^4}{1-\epsilon^3}\Big(\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n}-e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\Big)+C\epsilon^{-60}\|e_h^{u^n}\|_{-1,h}^6,\end{aligned}$$

whence (5.12) becomes

$$\begin{aligned}(5.13)\qquad &\frac{1}{2}\partial_t^+\|e_h^{u^n}\|_{-1,h}^2+\frac{\Delta t}{2}\|\partial_t^+e_h^{u^n}\|_{-1,h}^2+\frac{1}{\epsilon}\|e_h^{u^n}\|_{L^4}^4\\ &+\frac{\epsilon^4}{1-\epsilon^3}\Big(\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n}-e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\Big)\\ &\le C(\|e_h^{u^n}\|_{-1,h}^2+\epsilon^{-60}\|e_h^{u^n}\|_{-1,h}^6)+C(\rho_1^n+\rho_2^n).\end{aligned}$$

Multiplying (5.13) by $2\Delta t$ and taking summation over n from 1 to m, one derives

$$\begin{aligned}(5.14)\qquad &\|e_h^{u^m}\|_{-1,h}^2+\frac{2\epsilon^4\Delta t}{1-\epsilon^3}\sum_{n=1}^m\Big(\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n}-e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\Big)\\ &\le\|e_h^{u^0}\|_{-1,h}^2+C\Delta t\sum_{n=1}^m(\|e_h^{u^n}\|_{-1,h}^2+\epsilon^{-60}\|e_h^{u^n}\|_{-1,h}^6)+C\Delta t\sum_{n=1}^m(\rho_1^n+\rho_2^n).\end{aligned}$$

In view of the imbedding (3.13) and the fact $\|e_h^{u^n}\|_{L^2}\le C$, one chooses $\Delta t\le C\epsilon^{60}$ such that

$$C\Delta t(\|e_h^{u^n}\|_{-1,h}^2+\epsilon^{-60}\|e_h^{u^n}\|_{-1,h}^6)\le\frac{1}{2}\|e_h^{u^n}\|_{-1,h}^2.$$

It follows

$$\begin{aligned}(5.15)\qquad &\|e_h^{u^m}\|_{-1,h}^2+\frac{4\epsilon^4\Delta t}{1-\epsilon^3}\sum_{n=1}^m\Big(\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n}-e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\Big)\\ &\le 2\|e_h^{u^0}\|_{-1,h}^2+C\Delta t\sum_{n=1}^{m-1}(\|e_h^{u^n}\|_{-1,h}^2+\epsilon^{-60}\|e_h^{u^n}\|_{-1,h}^6)+C\Delta t\sum_{n=1}^m(\rho_1^n+\rho_2^n).\\ &\le\frac{C}{\epsilon^6}h^{2(k+2)}(||u,\partial_tu||_{L^2(H^{k+2})}^2+||\phi||_{L^2(H^{k+2})}^2)+\frac{C(\Delta t)^2}{\epsilon^7}(||\partial_{tt}u||_{L^2(L^2)}^2+1)\\ &+C\Delta t\sum_{n=1}^{m-1}(\|e_h^{u^n}\|_{-1,h}^2+\epsilon^{-60}\|e_h^{u^n}\|_{-1,h}^6).\end{aligned}$$

One has applied the stability estimate (4.7), the uniform estimate (3.23), the imbedding (3.13), and the error estimates of the HDG elliptic projection in Lemma 3.7.

Define $d_m \geq 0$ and S_m such that

$$
\begin{aligned}
S_m &:= \|e_h^{u^m}\|_{-1,h}^2 + \frac{2\epsilon^4 \Delta t}{1-\epsilon^3} \sum_{n=1}^{m} \left(\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2} (\Pi_k^{\partial} e_h^{u^n} - e_h^{\widehat{u}^n})\|_{\partial \mathcal{T}_h}^2 \right) + d_m \\
&= \text{right-hand side of (5.15)}.
\end{aligned}
$$

Then

$$
S_{m+1} - S_m \leq C \Delta t S_m + C \Delta t \epsilon^{-60} S_m^3, \quad m \geq 1.
$$

An application of the nonlinear Gronwall's inequality in Lemma 2.3 of [33] gives

$$
\begin{aligned}
S_m &\leq (1 + C\Delta t)^{m-1} S_1 \{1 - C S_1^2 [(1 + C\Delta t)^{2m} - (1 + C\Delta t)^2] \epsilon^{-60} (2 + C\Delta t)^{-1}\}^{-\frac{1}{2}} \\
&\leq C e^{CT} S_1 \leq C(T, \epsilon, u) \epsilon^{-7} [h^{2(k+2)} + (\Delta t)^2]
\end{aligned}
$$

with $C(T, u, \phi) := C e^{CT} (||u, \partial_t u||_{L^2(H^{k+2})}^2 + ||\phi||_{L^2(H^{k+2})}^2 + ||\partial_{tt} u||_{L^2(L^2)}^2 + 1)$, provided that

$$
1 - C S_1^2 [(1 + C\Delta t)^{2m} - (1 + C\Delta t)^2] \epsilon^{-60} (2 + C\Delta t)^{-1} \geq \frac{1}{2},
$$

which is fulfilled if

$$
h^{2(k+2)} + (\Delta t)^2 \leq \frac{1}{C(T, u, \phi)} \epsilon^{37}.
$$

Recalling the constraint on h from (3.31) with $\beta = \epsilon^3$, one concludes that provided

$$
\begin{aligned}
h &\leq C \min \left\{ \epsilon^{\frac{17}{4}}, \quad \epsilon^{\frac{7}{4-d}}, \quad \left(\frac{\epsilon}{C(T, u, \phi)} \right)^{\frac{37}{2(k+2)}} \right\}, \\
\Delta t &\leq C \epsilon^{60},
\end{aligned}
$$

the estimate (5.3) holds. This completes the proof. ∎

Next, we perform the error estimates of the scalar variables in the L^2 norm.

LEMMA 5.3. *Under the same conditions as Lemma 5.2, the following error estimate holds*

$$
\Delta t \sum_{n=1}^{N} \|\phi^n - \phi_h^n\|_{L^2(\Omega)}^2 + \max_{1 \leq n \leq N} \|u^n - u_h^n\|_{L^2(\Omega)}^2 \leq \frac{1}{\epsilon^{26}} C(T, u, \phi) (h^{k+2} + \Delta t)^2.
$$

Proof. Taking $(\boldsymbol{r}_1, w_1, \mu_1) = (-e_h^{\boldsymbol{q}^n}, e_h^{u^n}, e_h^{\widehat{u}^n})$ in (5.2a) and $(\boldsymbol{r}_2, w_2, \mu_2) = (e_h^{\boldsymbol{p}^n}, -e_h^{\phi^n}, -e_h^{\widehat{\phi}^n})$ in (5.2b) respectively one obtains

$$
(\partial_t^+ e_h^{u^n}, e_h^{u^n})_{\mathcal{T}_h} + \mathcal{A}(e_h^{\boldsymbol{p}^n}, e_h^{\phi^n}, e_h^{\widehat{\phi}^n}; -e_h^{\boldsymbol{q}^n}, e_h^{u^n}, e_h^{\widehat{u}^n}) = (\partial_t^+ u_{Ih}^n - \partial_t u^n, e_h^{u^n})_{\mathcal{T}_h}, \tag{5.16}
$$

$$
\begin{aligned}
\|e_h^{\phi^n}\|_{\mathcal{T}_h}^2 + \epsilon \mathcal{A}(e_h^{\boldsymbol{q}^n}, e_h^{u^n}, e_h^{\widehat{u}^n}; e_h^{\boldsymbol{p}^n}, -e_h^{\phi^n}, -e_h^{\widehat{\phi}^n}) &= -(\phi^n - \phi_{Ih}^n, e_h^{\phi^n})_{\mathcal{T}_h} \\
&\quad - \epsilon^{-1} \Big(F(u_h^n, u_h^{n-1}) - f(u^n), e_h^{\phi^n} \Big)_{\mathcal{T}_h}.
\end{aligned} \tag{5.17}
$$

Next, we multiply (5.16) by ϵ and add the result to (5.17) to get

$$\epsilon(\partial_t^+ e_h^{u^n}, e_h^{u^n}) + \|e_h^{\phi^n}\|_{\mathcal{T}_h}^2 = -(\phi^n - \phi_{Ih}^n, e_h^{\phi^n})_{\mathcal{T}_h} \tag{5.18}$$
$$- \epsilon^{-1}\Big(F(u_h^n, u_h^{n-1}) - f(u^n), e_h^{\phi^n}\Big)_{\mathcal{T}_h} + \epsilon(\partial_t^+ u_{Ih}^n - \partial_t u^n, e_h^{u^n})_{\mathcal{T}_h}.$$

By Lemma 4.4, there holds

$$|F(u_h^n, u_h^{n-1}) - f(u^n)| \le C\big(\frac{1}{\epsilon^7} + 1\big)|u_h^n - u^n| + \Delta t|\partial_t^+ u_h^n|. \tag{5.19}$$

Applying Cauchy-Schwarz inequality and taking summation of (5.18) from $n = 1$ to $n = m$ one arrives at

$$\begin{aligned}
&\epsilon\|e_h^{u^m}\|_{\mathcal{T}_h}^2 + 2\epsilon\Delta t\sum_{n=1}^m \|e_h^{\phi^n}\|_{\mathcal{T}_h}^2 + \epsilon(\Delta t)^2\sum_{n=1}^N \|\partial_t^+ e_h^{u^n}\|_{\mathcal{T}_h}^2 \\
&\le C\big(\frac{1}{\epsilon^{14}} + 1\big)\Delta t\sum_{n=1}^m(\|\phi^n - \phi_{Ih}^n\|_{\mathcal{T}_h}^2 + \|e_h^{u^n}\|_{\mathcal{T}_h}^2 + \|u^n - u_{Ih}^n\|_{\mathcal{T}_h}^2) \\
&+ C\Delta t\epsilon\sum_{n=1}^m \|\partial_t^+ u_{Ih}^n - \partial_t u^n\|_{\mathcal{T}_h}^2 + \frac{C}{\epsilon^{10}}(\Delta t)^2.
\end{aligned}$$

Now by Lemma 3.7 we have

$$\begin{aligned}
&\Delta t\sum_{n=1}^m \|\partial_t^+ u_{Ih}^n - \partial_t u^n\|_{\mathcal{T}_h}^2 \\
&\le \Delta t\sum_{n=1}^m \left(\|\partial_t^+ u_{Ih}^n - \partial_t^+ u^n\|_{\mathcal{T}_h}^2 + \|\partial_t^+ u^n - \partial_t u^n\|_{\mathcal{T}_h}^2\right) \\
&\le \frac{1}{\Delta t}\sum_{n=1}^m\int_\Omega\left\{\left[\int_{t_{n-1}}^{t_n}\partial_t(u_{Ih} - u)\,dt\right]^2 + \left[\int_{t_{n-1}}^{t_n}(t - t_{n-1})\partial_{tt}u\,dt\right]^2\right\} \\
&\le \int_0^T \|\partial_t(u_{Ih} - u)\|_{L^2(\Omega)}^2\,dt + (\Delta t)^2\int_0^T \|\partial_{tt}u\|_{L^2(\Omega)}^2\,dt.
\end{aligned} \tag{5.20}$$

Next by the HDG Sobolev imbedding (3.2b), the inequality (3.6), and the error estimate in the negative norm (5.3) one obtains

$$\begin{aligned}
\Delta t\sum_{n=1}^N \|e_h^{u^n}\|_{\mathcal{T}_h}^2 &\le C\Delta t\sum_{n=1}^N\Big(\|e_h^{\boldsymbol{q}^n}\|_{\mathcal{T}_h}^2 + \|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^n} - e_h^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2\Big) \\
&\le \epsilon^{-11}C(T, u, \phi)\big((\Delta t)^2 + h^{2(k+2)}\big).
\end{aligned}$$

Therefore the approximation properties of the elliptic projection in Lemma 3.7 gives

$$\epsilon\|e_h^{u^m}\|_{\mathcal{T}_h}^2 + 2\Delta t\sum_{n=1}^m \|e_h^{\phi^n}\|_{\mathcal{T}_h}^2 \le \frac{1}{\epsilon^{25}}C(T, u, \phi)\big(h^{2(k+2)} + (\Delta t)^2\big).$$

This establishes the optimal error estimates of u and ϕ in the L^2 norm. ∎

Finally one performs the error analysis of the flux variables. One has

LEMMA 5.4. *Under the same conditions as Lemma 5.2 the flux variables satisfy the following error bounds*

$$\max_{1\le n\le N}\|e_h^{\boldsymbol{q}^n}\|^2_{\mathcal{T}_h}+\Delta t\sum_{n=1}^{N}\|e_h^{\boldsymbol{p}^n}\|^2_{\mathcal{T}_h}+\le C(T,u,\phi)\epsilon^{-24}((\Delta t)^2+h^{2(k+1)}). \tag{5.21}$$

Proof. Taking $(\boldsymbol{r}_1,w_1,\mu_1)=(e_h^{\boldsymbol{p}^n},e_h^{\phi^n},e_h^{\widehat{\phi}^n})$ in (5.2a) gives

$$\begin{aligned}(\partial_t^+e_h^{u^n},e_h^{\phi^n})_{\mathcal{T}_h}&+\|e_h^{\boldsymbol{p}^n}\|^2_{\mathcal{T}_h}+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^{\partial}e_h^{\phi^n}-e_h^{\widehat{\phi}^n})\|^2_{\partial\mathcal{T}_h}\\&=(\partial_t^+u_{Ih}^n-\partial_tu^n,e_h^{\phi^n})_{\mathcal{T}_h}.\end{aligned} \tag{5.22}$$

Applying ∂_t^+ to (5.2b) and then setting $(\boldsymbol{r}_2,w_2,\mu_2)=(e_h^{\boldsymbol{q}^n},0,0)$ in the resulting equation, one obtains

$$\epsilon(\partial_t^+e_h^{\boldsymbol{q}^n},e_h^{\boldsymbol{q}^n})_{\mathcal{T}_h}-\epsilon(\partial_t^+e_h^{u^n},\nabla\cdot e_h^{\boldsymbol{q}^n})_{\mathcal{T}_h}+\epsilon\langle\partial_t^+e_h^{\widehat{u}^n},e_h^{\boldsymbol{q}^n}\cdot n\rangle_{\partial\mathcal{T}_h}=0. \tag{5.23}$$

Next with $(\boldsymbol{r}_2,w_2,\mu_2)=(0,\partial_t^+e_h^{u^n},\partial_t^+e_h^{\widehat{u}^n})$ in (5.2b) taking summation of the result with (5.22)–(5.23) yield

$$\begin{aligned}&\epsilon(\partial_t^+e_h^{\boldsymbol{q}^n},e_h^{\boldsymbol{q}^n})_{\mathcal{T}_h}+\epsilon\langle h_{\mathcal{T}}^{-1}(\Pi_k^{\partial}e_h^{u^n}-e_h^{\widehat{u}^n}),\partial_t^+(\Pi_k^{\partial}e_h^{u^n}-e_h^{\widehat{u}^n})\rangle_{\partial\mathcal{T}_h}\\&+\|e_h^{\boldsymbol{p}^n}\|^2_{\mathcal{T}_h}+\|h_K^{-1/2}(\Pi_k^{\partial}e_h^{\phi^n}-e_h^{\widehat{\phi}^n})\|^2_{\partial\mathcal{T}_h}=\big(\partial_t^+u_{Ih}^n-\partial_tu^n,e_h^{\phi^n}\big)_{\mathcal{T}_h}\\&+\big(\phi^n-\phi_{Ih}^n,\partial_t^+e_h^{u^n}\big)_{\mathcal{T}_h}+\epsilon^{-1}\Big(F(u_h^n,u_h^{n-1})-f(u^n),\partial_t^+e_h^{u^n}\Big)_{\mathcal{T}_h}\\&:=I_1+I_2+I_3.\end{aligned} \tag{5.24}$$

The three terms $I_j, j=1\ldots3$ are estimated as follows. For I_1, by the Cauchy-Schwartz inequality, the HDG Sobolev inequality Lemma 3.1, and (3.6), one obtains

$$\begin{aligned}|I_1|&\le\|\partial_t^+u_{Ih}^n-\partial_tu^n\|_{\mathcal{T}_h}\|e_h^{\phi^n}\|_{\mathcal{T}_h}\\&\le C\|\partial_t^+u_{Ih}^n-\partial_tu^n\|_{\mathcal{T}_h}(\|\nabla e_h^{\phi^n}\|_{\mathcal{T}_h}+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^{\partial}e_h^{\phi^n}-e_h^{\widehat{\phi}^n})\|_{\partial\mathcal{T}_h})\\&\le C\|\partial_t^+u_{Ih}^n-\partial_tu^n\|_{\mathcal{T}_h}(\|e_h^{\boldsymbol{p}^n}\|_{\mathcal{T}_h}+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^{\partial}e_h^{\phi^n}-e_h^{\widehat{\phi}^n})\|_{\partial\mathcal{T}_h})\\&\le C\|\partial_t^+u_{Ih}^n-\partial_tu^n\|^2_{\mathcal{T}_h}+\frac14\big(\|e_h^{\boldsymbol{p}^n}\|^2_{\mathcal{T}_h}+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^{\partial}e_h^{\phi^n}-e_h^{\widehat{\phi}^n})\|_{\partial\mathcal{T}_h}\big).\end{aligned}$$

By (5.2a) with $(\boldsymbol{r}_1,\mu_1)=(0,0)$ one has

$$\begin{aligned}&|I_2|=|(\Pi_{k+1}^o\phi^n-\phi_{Ih}^n,\partial_t^+e_h^{u^n})_{\mathcal{T}_h}|\\&\le|\mathcal{A}\big(e_h^{\boldsymbol{p}^n},e_h^{\phi^n},e_h^{\widehat{\phi}^n};0,(\Pi_{k+1}^o\phi^n-\phi_{Ih}^n),0\big)|+|\big(\partial_t^+u_{Ih}^n-\partial_tu^n,(\Pi_{k+1}^o\phi^n-\phi_{Ih}^n)\big)_{\mathcal{T}_h}|\\&\le\|\partial_t^+u_{Ih}^n-\partial_tu^n\|_{\mathcal{T}_h}\|\Pi_{k+1}^o\phi^n-\phi_{Ih}^n\|_{\mathcal{T}_h}\\&\quad+C(\|e_h^{\boldsymbol{p}^n}\|_{\mathcal{T}_h}+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^{\partial}e_h^{\phi^n}-e_h^{\widehat{\phi}^n})\|_{\partial\mathcal{T}_h})\\&\quad\times(\|\nabla(\Pi_{k+1}^o\phi^n-\phi_{Ih}^n)\|_{\mathcal{T}_h}+\|h_{\mathcal{T}}^{-1/2}\Pi_k^{\partial}(\Pi_{k+1}^o\phi^n-\phi_{Ih}^n)\|_{\partial\mathcal{T}_h})\\&\le C(\|\partial_t^+u_{Ih}^n-\partial_tu^n\|^2_{\mathcal{T}_h}+\|\Pi_{k+1}^o\phi^n-\phi_{Ih}^n\|^2_{H^1})+\frac14\big(\|e_h^{\boldsymbol{p}^n}\|^2_{\mathcal{T}_h}+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^{\partial}e_h^{\phi^n}-e_h^{\widehat{\phi}^n})\|_{\partial\mathcal{T}_h}\big),\end{aligned}$$

where one has utilizes the continuity of the operator $\mathcal{A}$ in (3.5), the inverse inequality, the L^2 stability of the projections Π_k^∂ and Π_{k+1}^o. Likewise,

$$
\begin{aligned}
&|I_3| \\
&\le \frac{C}{\epsilon^2}\|\Pi_{k+1}^o\big(F(u_h^n,u_h^{n-1})-f(u^n)\big)\|_{\mathcal{T}_h}^2+\frac{1}{4}\big(\|e_h^{\boldsymbol{p}^n}\|_{\mathcal{T}_h}^2+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{\phi^n}-e_h^{\widehat{\phi}^n})\|_{\partial\mathcal{T}_h}^2\big) \\
&\quad+\frac{C}{\epsilon^2}\big(\|\nabla\Pi_{k+1}^o\big(F(u_h^n,u_h^{n-1})-f(u^n)\big)\|_{\mathcal{T}_h}^2+C\|\partial_t^+u_{Ih}^n-\partial_t u^n\|_{\mathcal{T}_h}^2 \\
&\quad+\|h_{\mathcal{T}}^{-1/2}\Pi_k^\partial\Pi_{k+1}^o\big(F(u_h^n,u_h^{n-1})-f(u^n)-f(u^n)\big)\|_{\partial\mathcal{T}_h}^2\big) \\
&\le C\|\partial_t^+u_{Ih}^n-\partial_t u^n\|_{\mathcal{T}_h}^2+\frac{1}{4}\big(\|e_h^{\boldsymbol{p}^n}\|_{\mathcal{T}_h}^2+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{\phi^n}-e_h^{\widehat{\phi}^n})\|_{\partial\mathcal{T}_h}^2\big) \\
&\quad+\frac{C}{\epsilon^2}\|\Pi_{k+1}^o\big(F(u_h^n,u_h^{n-1})-f(u^n)\big)\|_{H^1}^2.
\end{aligned}
$$

Taking into account of the above estimates of I_1, I_2, I_3, multiplying (5.24) by Δt, then taking summation from $n=1$ to $n=m$, one derives

$$
\begin{aligned}
&\epsilon\|e_h^{\boldsymbol{q}^m}\|_{\mathcal{T}_h}^2+\epsilon\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^m}-e_h^{\widehat{u}^m})\|+\frac{1}{2}\Delta t\sum_{n=1}^{m}\big(\|e_h^{\boldsymbol{p}^n}\|_{\mathcal{T}_h}^2+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{\phi^n}-e_h^{\widehat{\phi}^n})\|_{\partial\mathcal{T}_h}^2\big) \\
&\le \|e_h^{\boldsymbol{q}^0}\|_{\mathcal{T}_h}^2+\|h_{\mathcal{T}}^{-1/2}(\Pi_k^\partial e_h^{u^0}-e_h^{\widehat{u}^0})\|_{\partial\mathcal{T}_h}^2+C\Delta t\sum_{n=1}^{m}\|\partial_t^+u_{Ih}^n-\partial_t u^n\|_{\mathcal{T}_h}^2 \\
&+C\Delta t\sum_{n=1}^{m}(\|\Pi_{k+1}^o\phi^n-\phi_{Ih}^n\|_{H^1}^2+\frac{1}{\epsilon^2}\|\Pi_{k+1}^o\big(F(u_h^n,u_h^{n-1})-f(u^n)\big)\|_{H^1}^2) \\
&\le C(T,u,\phi)\epsilon^{-23}\big(h^{2(k+1)}+(\Delta t)^2\big).
\end{aligned}
$$

We have applied the inverse inequality, the approximation results of the HDG elliptic projection Lemma 3.7, the uniform estimate (5.19), and the error estimate (5.3). This completes the proof. ∎

Theorem 5.1 follows from Lemmas 5.3 to 5.4, the triangle inequality and the error estimates of the HDG elliptic projection (3.7). This finishes the proof of the main convergence result.

6. Numerical Experiments.

We consider two examples on square domains in $\mathbb{R}^2$. The meshes are regular triangulation by right triangles. We remark that more robust tests need to be randomized meshes or polygonal meshes. cf. [9]. These will be considered in future development of the computer codes. In the first example we use manufactured solution of the system (1.1) with explicit forcing terms; in the second example we perform simulation of spinodal decomposition and coarsening of a random field. Since the scheme is nonlinear, we solve the system iteratively by linearizing the nonlinear term using Newton's method.

Example 6.1. We take the domain to be the unit square, and the problem data u^0 is chosen so that the exact solution of the system (1.1) is given by

$$\varepsilon = 1, \quad u = \phi = e^{-t}x^2y^2(1-x)^2(1-y)^2.$$

We report the errors at the final time $T=1$ for polynomial degrees $k=0$ and $k=1$ in Tables 1 and 2 for the energy-splitting scheme. The observed convergence rates match the theory, where $\Delta t = h^{k+1}$.

$h/\sqrt{2}$	1/4	1/8	1/16	1/32	1/64
$\Vert \boldsymbol{q}-\boldsymbol{q}_h\Vert_{\mathcal{T}_h}$	8.6761E-04	4.8768E-04	2.5059E-04	1.2614E-04	6.3177E-05
order	-	0.83111	0.96063	0.99025	0.99757
$\Vert \boldsymbol{p}-\boldsymbol{p}_h\Vert_{\mathcal{T}_h}$	8.9460E-04	4.9143E-04	2.5107E-04	1.2620E-04	6.3185E-05
order	-	0.86427	0.96891	0.99232	0.99809
$\Vert u-u_h\Vert_{\mathcal{T}_h}$	2.5759E-04	6.7122E-05	1.6952E-05	4.2490E-06	1.0629E-06
order	-	1.9402	1.9853	1.9963	1.9991
$\Vert \phi-\phi_h\Vert_{\mathcal{T}_h}$	2.6295E-04	6.7806E-05	1.7076E-05	4.2768E-06	1.0697E-06
order	-	1.9553	1.9894	1.9974	1.9993

TABLE 1

Example 6.1, $k=0$ with energy-splitting scheme: Errors, observed convergence orders for u, ϕ and their fluxes $\boldsymbol{q}$ and $\boldsymbol{p}$.

$h/\sqrt{2}$	1/4	1/8	1/16	1/32	1/64
$\Vert \boldsymbol{q}-\boldsymbol{q}_h\Vert_{\mathcal{T}_h}$	1.5809E-04	4.3945E-05	1.1415E-05	2.8955E-06	7.2935E-07
order	-	1.8470	1.9448	1.9790	1.9891
$\Vert \boldsymbol{p}-\boldsymbol{p}_h\Vert_{\mathcal{T}_h}$	1.5896E-04	4.3991E-05	1.1418E-05	2.8957E-06	7.2940E-07
order	-	1.8534	1.9459	1.9793	1.9891
$\Vert u-u_h\Vert_{\mathcal{T}_h}$	4.9741E-05	6.3026E-06	7.9008E-07	9.8850E-08	1.2358E-08
order	-	2.9804	2.9959	2.9987	2.9998
$\Vert \phi-\phi_h\Vert_{\mathcal{T}_h}$	4.9111E-05	6.1809E-06	7.7336E-07	9.6709E-08	1.2090E-08
order	-	2.9902	2.9986	2.9994	2.9998

TABLE 2

Example 6.1, $k=1$ with energy-splitting scheme: Errors, observed convergence orders for u, ϕ and their fluxes $\boldsymbol{q}$ and $\boldsymbol{p}$.

Example 6.2. In this test, the exact solution is unknown, we take the domain $\Omega=(0,1)\times(0,1)$, $\varepsilon=0.03$, the initial condition is a random field of values that are uniformly distributed between -0.05 and 0.05. We report the solution at the time $T=0.001$, $T=0.1$ and $T=0.5$ for polynomial degrees $k=1$ in Figure 1 with $\Delta t=2.5\times10^{-4}$ and $h=1/100$. The plots demonstrate the dynamics of rapid initial spinodal decomposition and later stage of slow coarsening.

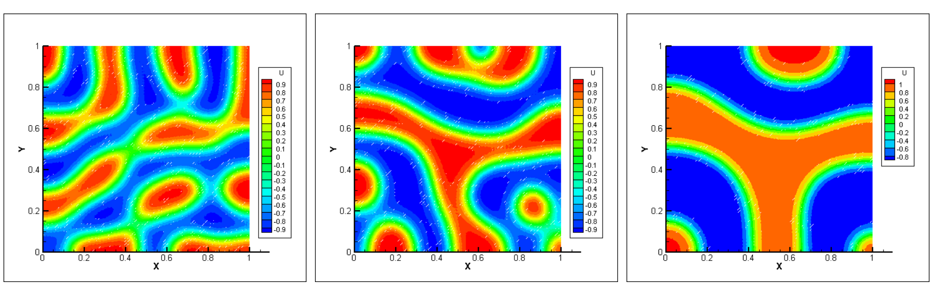

FIG. 1. *Filled contour plots of the time evolution of u. Left is $T=0.001$, middle is $T=0.1$ and right is $T=0.5$.*

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