Dirichlet fractional Gaussian fields on the Sierpinski gasket and their discrete graph approximations

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Abstract

We define and study the Dirichlet fractional Gaussian fields on the Sierpinski gasket and show that they are limits of fractional discrete Gaussian fields defined on the sequence of canonical approximating graphs.

Contents

1 Introduction

Let $K \subset \mathbb{R}^2$ be the Sierpinski gasket fractal. A first goal of the paper is to introduce and study a family of Gaussian fields on K indexed by a parameter $s \ge 0$ and satisfying

$$\mathbb{E}(X_s(f)X_s(g)) = \int_K (-\Delta)^{-s} f (-\Delta)^{-s} g d\mu, \tag{1}$$

where f, g belong to a space of suitable test functions on K, μ is the Hausdorff measure and Δ is the Dirichlet Laplacian on K. Such a field is heuristically defined as the distribution

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$X_s = (-\Delta)^{-s} W$ where W is a white noise on $L^2(K,\mu)$. We will mostly be interested in the regularity properties of those fields and in the convergence of their natural discretizations. Concerning the regularity properties, the value $s = \frac{d_h}{2d_w}$ is a critical value, where $d_h = \frac{\ln 3}{\ln 2}$ is the Hausdorff dimension of K and $d_w = \frac{\ln 5}{\ln 2}$ is called the walk dimension. More precisely, the study of X_s is divided according to two ranges:

- $0 \le s \le \frac{d_h}{2d_w}$: For this range of parameters we show that the Gaussian field X_s can not be defined pointwise but belongs to a Sobolev space of distributions that we identify;
- $s > \frac{d_h}{2d_w}$: For this range, we show that the Gaussian field X_s can be defined pointwise and admits a Hölder regular version.

The critical value $s = \frac{d_h}{2d_w}$ corresponds to a log-correlated field on K that will tentatively be further studied in a later work.

A second goal of the paper is to introduce discrete analogues of the fractional Gaussian fields X_s by using the canonical graph approximation G_m, $m \ge 0$ of the Sierpinski gasket, see Figure 1.

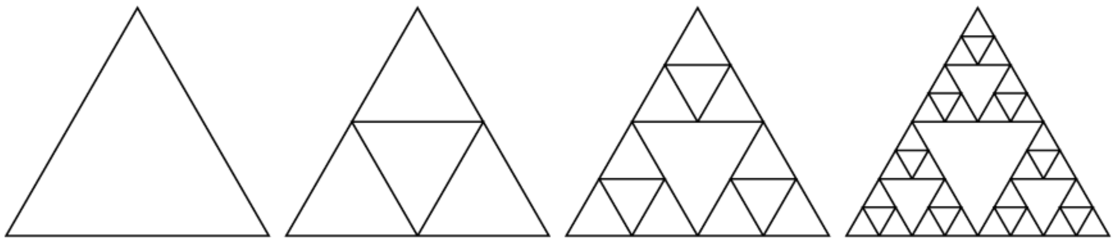

Figure 1: Sierpinski gasket graphs G_0, G_1, G_2 and G_3

Those discrete fields are given by $X_s^m = (-\Delta_m)^{-s} W_m$ where Δ_m is the Dirichlet graph Laplacian on G_m, and W_m is a sequence of i.i.d. standard Gaussian normal on the vertices of G_m. We will show the convergence of X_s^m to X_s, first in law in a space of tempered distributions, and then in law in a suitable Sobolev space.

This paper is natural complement to the recent paper [5] which studied fractional Gaussian fields associated to the Neumann Laplacian on the Sierpinski gasket for the range of parameters $\frac{d_h}{2d_w} < s < 1 - \frac{d_h}{2d_w}$. However, as noted above, in the present paper we are rather interested in the fractional Gaussian fields associated with the Dirichlet Laplacian, study the whole range of parameters $s \ge 0$ and also introduce the family of discrete fields X_s^m for which we prove convergence when $m \to +\infty$. In a subsequent work, we plan to study the maxima of the discrete log-correlated field on the gasket and their possible rescaling limits; we refer for instance to [6] for an introduction and motivation to such questions.

The paper is organized as follows. In Section 2, after some preliminaries, we introduce the discrete and continuous fractional Gaussian fields on the gasket. A highlight result is Theorem 2.12 which states the existence of a Gaussian random variable X_s which takes values in a suitable space of tempered distributions and that satisfies (1). We then prove in Proposition 2.17 that this random tempered distribution X_s defines an L^2 function on K if and only if $s > \frac{d_h}{2d_w}$. Section 3 deals with the regularity theory of the random tempered distribution X_s. For $s \le \frac{d_h}{2d_w}$ we quantify this regularity by introducing a scale of distributional Sobolev spaces and for $s > \frac{d_h}{2d_w}$, using the entropy method as in [1], we study the Hölder regularity property of the L^2 function on K defined by X_s. Finally, in Section 4, we prove the convergence of the discrete fields X_s^m to X_s.

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2 Discrete and continuous FGFs on the Sierpinski gasket

2.1 Discrete and continuous Dirichlet Laplacians

We first define the (Dirichlet) Laplacians on the Sierpinski gasket and Sierpinski gasket graphs. For further references and more general fractals, see for instance [2, 9, 10].

Let $V_0 = \{p_1, p_2, p_3\}$ be a set of vertices of an equilateral triangle of side 1 in $\mathbb{C}$. Define

$$\mathfrak{f}_i(z) = \frac{z - p_i}{2} + p_i$$

for $i = 1, 2, 3$. Then the Sierpinski gasket K is the unique non-empty compact subset in $\mathbb{C}$ such that

$$K = \bigcup_{i=1}^{3} \mathfrak{f}_i(K).$$

The set V_0 is called the boundary of K, we will also denote it by ∂K.

The Hausdorff dimension of K with respect to the Euclidean metric (denoted $d(x, y) = |x-y|$ in this paper) is given by $d_h = \frac{\ln 3}{\ln 2}$. A (normalized) Hausdorff measure on K is given by the Borel measure μ on K such that for any $i_1, \cdots, i_n \in \{1, 2, 3\}$,

$$\mu\left(\mathfrak{f}_{i_1} \circ \cdots \circ \mathfrak{f}_{i_n}(K)\right) = 3^{-n}.$$

This measure μ is d_h-Ahlfors regular, i.e. there exist constants $c, C > 0$ such that for every $x \in K$ and $r \in [0, \mathrm{diam}(K)]$,

$$cr^{d_h} \leq \mu(B(x, r)) \leq Cr^{d_h}. \tag{2}$$

We define a sequence of sets $\{V_m\}_{m\geq 0}$ inductively by

$$V_{m+1} = \bigcup_{i=1}^{3} \mathfrak{f}_i(V_m).$$

Then we have a natural sequence of Sierpinski gasket graphs (or pre-gaskets) $\{G_m\}_{m\geq 0}$ whose edges have length 2^{-m} and whose set of vertices is V_m, see Figure 1. Notice that $\#V_m = \frac{3(3^m+1)}{2}$. We will use the notations $V_* = \cup_{m\geq 0} V_m$ and $V_*^0 = \cup_{m\geq 0} V_m \setminus V_0$.

For any $p \in V_m$, denote by $V_{m,p}$ the collection of neighbors of p in G_m. Then $\#V_{m,p} = 4$ if $p \notin V_0$ and $\#V_{m,p} = 2$ if $p \in V_0$. Let $\ell(V_m)$ be the set of functions $f : V_m \to \mathbb{R}$. Then for any $f \in \ell(V_m)$, we consider the discrete Laplacian on V_m defined by

$$\Delta_m f(p) = 5^m \sum_{q \in V_{m,p}} (f(q) - f(p)), \quad p \in V_m \setminus V_0.$$

The semigroup generated by the discrete Laplacian Δ_m on V_m is denoted by $\{P_t^m\}_{t\geq 0}$.

Let $C(K)$ be the set of continuous functions on K. We define

$$\mathcal{D} = \{f \in C(K), \text{ there exists } g \in C(K) \text{ such that } \lim_{m\to\infty} \max_{p \in V_m \setminus V_0} |\Delta_m f(p) - g(p)| = 0\}.$$

For $f \in \mathcal{D}$, the Kigami Laplacian Δ of f on K is then defined by

$$\Delta f(p) = g(p), \tag{3}$$

where g is in $C(K)$ and satisfies $\lim_{m\to\infty} \max_{p\in V_m\setminus V_0} |\Delta_m f(p) - g(p)| = 0$.

The notations $C_0(K)$ and $\mathcal{D}_0$ denote respectively the sets of functions in $C(K)$ and $\mathcal{D}$ which vanish on ∂K (See for instance [10, Example 3.7.3] and [9, Section 2]). We will also consider the discrete measures on $\{V_m\}_{m\geq 0}$:

$$\mu_m := \frac{2}{3^{m+1}} \sum_{p\in V_m} \delta_p. \tag{4}$$

For later use, we denote by a_m the number $\frac{3^{m+1}}{2}$ and thus $\mu_m = \frac{1}{a_m}\sum_{p\in V_m}\delta_p$.

For any function $f : V_* \to \mathbb{R}$, we consider the quadratic form

$$\mathcal{E}_m(f,f) = \frac{5^m}{a_m} \sum_{x,y\in V_m, x\sim y} (f(x) - f(y))^2,$$

where $x \sim y$ denotes that x, y are neighbors in G_m. Note that $\mathcal{E}_m(f,f)$ is non-decreasing in m. Define

$$\mathcal{E}(f,f) = \lim_{m\to+\infty} \mathcal{E}_m(f,f)$$

and

$$\mathcal{F}_0 = \{f \in C(K) : \lim_{m\to+\infty} \mathcal{E}_m(f,f) < \infty, f = 0 \text{ on } \partial K\}.$$

By Theorem 4.1 and Lemma 4.1 in [9], $(\mathcal{E}, \mathcal{F}_0)$ is a local regular Dirichlet form on $L^2(K,\mu)$. Moreover, for any functions f, g on V_m vanishing on ∂K

$$\mathcal{E}_m(f,g) = -\int_{V_m^0} \Delta_m f(x) g(x) d\mu_m(x),$$

and for $f \in \mathcal{D}_0, g \in \mathcal{F}_0$,

$$\mathcal{E}(f,g) = -\int_K \Delta f(x) g(x) d\mu(x).$$

From [9, Theorem 4.2] the Friedrichs extension of the Kigami Laplacian Δ is the self-adjoint operator on $L^2(K,\mu)$ which is the generator of $(\mathcal{E}, \mathcal{F}_0)$. We still denote this generator by Δ and the operator Δ with domain $\mathcal{D}(\Delta)$ is referred to as the Dirichlet Laplacian on K.

The following lemma shows that any $u \in \mathcal{D}(\Delta)$ is Hölder continuous.

Lemma 2.1. *There exists a constant $C > 0$ such that for every $u \in \mathcal{D}(\Delta)$ and $x, y \in K$,*

$$|u(x) - u(y)| \leq C d(x,y)^{d_w - d_h} \|\Delta u\|_{L^1(K,\mu)},$$

where, as above, the parameter $d_h = \frac{\ln 3}{\ln 2}$ is the Hausdorff dimension and the parameter $d_w = \frac{\ln 5}{\ln 2}$ is the so-called walk dimension.

Proof. Let g^0 be the reproducing kernel of the Dirichlet form $(\mathcal{E}, \mathcal{F}_0)$, see [9, Theorem 4.1, (ii)]. We have for every $y \in K$, and u in $\mathcal{F}_0$,

$$g^0(\cdot, y) \in \mathcal{F}_0, \qquad \mathcal{E}(g^0(\cdot,y), u) = u(y).$$

For $u \in \mathcal{D}_0$, one obtains then

$$\begin{aligned}
|u(x) - u(y)| &= \left|\mathcal{E}(g^0(\cdot,x),u) - \mathcal{E}(g^0(\cdot,y),u)\right| \\
&= \left|\int_K g^0(z,x)\Delta u(z) d\mu(z) - \int_K g^0(z,y)\Delta u(z) d\mu(z)\right| \\
&\leq \int_K \left|g^0(z,x) - g^0(z,y)\right| |\Delta u(z)| d\mu(z).
\end{aligned}$$

Using [11, Theorem 4.1, (GF4)], we have for every $x, y, z \in K$,

$$\left|g^0(z,x) - g^0(z,y)\right| \leq C d(x,y)^{d_w - d_h}.$$

The result follows easily. □

2.2 Discrete Fractional Gaussian Fields

Let $0 < \lambda_1^m \leq \lambda_2^m \leq \cdots \leq \lambda_{N_m}^m$ be the series of increasing eigenvalues (each being repeated according to its multiplicity) of $-\Delta_m$ with zero boundary condition (see Section 3 in [9]). Let $(\Phi_i^m)_{1 \leq i \leq N_m}$ be the corresponding orthonormal eigenfunctions with respect to the measure μ_m defined in (4). The discrete Riesz kernel on V_m with parameter $s \geq 0$ is defined by

$$G_s^m(x,y) = \sum_{i=1}^{N_m} (\lambda_i^m)^{-s} \Phi_i^m(x) \Phi_i^m(y), \quad x, y \in V_m. \tag{5}$$

From this definition it is clear that the matrix $(G_s^m(x,y))_{x,y \in V_m}$ is symmetric and non negative. It is therefore the covariance matrix of a Gaussian vector.

For $f \in \ell(V_m)$, the discrete fractional Laplacian $(-\Delta_m)^{-s}$ is defined by

$$(-\Delta_m)^{-s} f(x) = \frac{1}{a_m} \sum_{y \in V_m} G_s^m(x,y) f(y) = \sum_{i=1}^{N_m} (\lambda_i^m)^{-s} \Phi_i^m(x) \frac{1}{a_m} \sum_{y \in V_m} \Phi_i^m(y) f(y). \tag{6}$$

Note that $\|(-\Delta_m)^{-s} f\|_{L^2(V_m,\mu_m)} \leq (\lambda_1^m)^{-s} \|f\|_{L^2(V_m,\mu_m)}$. Moreover, one has $\inf_m \lambda_1^m > 0$ from [9, Lemma 5.2]. Hence the operators $(-\Delta_m)^{-s} : L^2(V_m, \mu_m) \to L^2(V_m, \mu_m)$ are uniformly bounded.

Definition 2.2 (DFGF)**.** Let $s \geq 0$. A discrete fractional Gaussian field X_s^m with parameter s on V_m is a Gaussian vector indexed by V_m with mean zero and covariance matrix $G_{2s}^m(x,y)$.

Definition 2.3 (Discrete log-correlated fields)**.** We define the discrete log-correlated Gaussian field X^m on V_m as the discrete fractional Gaussian field X_s^m with parameter $s = \frac{d_h}{2d_w}$; see Definition 2.14 and Remark 2.15 for further explanation about this terminology.

Remark 2.4. If $(W_i)_{1 \leq i \leq N_m}$ is a sequence of i.i.d Gaussian random variables with mean zero and variance one, then

$$X_s^m(x) := \sum_{i=1}^{N_m} (\lambda_i^m)^{-s} \Phi_i^m(x) W_i, \quad x \in V_m$$

is easily seen to be a DFGF with parameter s on V_m.

For any $f \in \ell(V_m)$, we will use the notation

$$X_s^m(f) = \frac{1}{a_m} \sum_{p \in V_m} f(p) X_s^m(p).$$

We then note that for $f, g \in \ell(V_m)$

$$\begin{aligned}
\mathbb{E}(X_s^m(f)X_s^m(g)) &= \frac{1}{a_m^2}\sum_{p,q\in V_m} f(p)g(q)\mathbb{E}(X_s^m(p)X_s^m(q)) \\
&= \frac{1}{a_m^2}\sum_{p,q\in V_m} f(p)g(q)G_{2s}^m(p,q) \\
&= \frac{1}{a_m}\sum_{p\in V_m} f(p)\left(\frac{1}{a_m}\sum_{q\in V_m} g(q)G_{2s}^m(p,q)\right) \\
&= \frac{1}{a_m}\sum_{p\in V_m} f(p)(-\Delta_m)^{-2s}g(p) \qquad (7)\\
&= \frac{1}{a_m}\sum_{p\in V_m} (-\Delta_m)^{-s}f(p)(-\Delta_m)^{-s}g(p).
\end{aligned}$$

2.3 Fractional Laplacians and fractional Riesz kernels

The Laplacian Δ with domain $\mathcal{D}(\Delta)$ is the generator of a strongly continuous Markov semigroup $\{P_t\}_{t\geq 0}$ on $L^2(K,\mu)$. This semigroup admits a bicontinuous heat kernel $p_t(x,y)$, $t>0$, $x,y\in K$, with respect to the Hausdorff measure μ. It is called the Dirichlet heat kernel on K.

This heat kernel satisfies for some $c_1, c_2 \in (0,\infty)$,

$$p_t(x,y) \leq c_1 t^{-\frac{d_h}{d_w}} \exp\left(-c_2\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right) \tag{8}$$

for every $(x,y)\in K\times K$ and $t\in\big(0,+\infty)$. The exact values of c_1, c_2 are irrelevant in our analysis. As above, the parameter $d_h = \frac{\ln 3}{\ln 2}$ is the Hausdorff dimension. The parameter $d_w = \frac{\ln 5}{\ln 2}$ is called the walk dimension. The quantity $d_s = \frac{2d_h}{d_w}$ is often referred to as the spectral dimension. Since $d_w > 2$, one speaks of sub-Gaussian heat kernel upper estimates.

The Dirichlet heat kernel $p_t(x,y)$ admits a uniformly convergent spectral expansion:

$$p_t(x,y) = \sum_{j=1}^{+\infty} e^{-\lambda_j t}\Phi_j(x)\Phi_j(y) \tag{9}$$

where $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$ are the eigenvalues of $-\Delta$ and $(\Phi_j)_{j\geq 1} \subset \mathcal{D}(\Delta)$ is an orthonormal basis of $L^2(K,\mu)$ such that

$$\Delta\Phi_j = -\lambda_j\Phi_j.$$

Notice that $\Phi_j \in \mathcal{D}(\Delta)$ and thus is Hölder continuous. It is known from the work of Fukushima and Shima [9] that the counting function of the eigenvalues:

$$N(t) = \mathbf{Card}\{\lambda_j \leq t\}$$

satisfies

$$N(t) \sim \Theta(t)t^{d_h/d_w} \tag{10}$$

when $t\to+\infty$ where Θ is a function bounded away from 0. In particular,

$$\sum_{j=1}^{+\infty}\frac{1}{\lambda_j^{2s}} < +\infty \tag{11}$$

if and only if $s > \frac{d_h}{2d_w}$.

We will consider the following space of test functions

$$\mathcal{S}(K) = \left\{ f \in C_0(K), \forall k \geq 0 \lim_{n \to +\infty} n^k \left| \int_K \Phi_n(y) f(y) d\mu(y) \right| = 0 \right\}.$$

It is clear that if $f \in \mathcal{S}(K)$, then $f \in \bigcap_{k \geq 0} \operatorname{dom}((-\Delta)^k)$ and thus for every $k \geq 0$, $(-\Delta)^k f \in C(K)$ and is Hölder continuous. We also note that from [9, Lemma 4.1(iii)] $\mathcal{S}(K) \subset \mathcal{D}_0$. We consider then on $\mathcal{S}(K)$ the topology defined by the family of norms

$$\|f\|_k = \|(-\Delta)^k f\|_{L^2(K,\mu)}, \quad k \geq 0.$$

From [15, Theorem 2], thanks to (11), $\mathcal{S}(K)$ is a Fréchet nuclear space. The dual space of $\mathcal{S}(K)$ (for the latter topology) will be denoted $\mathcal{S}'(K)$.

Definition 2.5 (Fractional Laplacians)**.** Let $s \geq 0$. For $f \in L^2(K, \mu)$, the fractional Laplacian $(-\Delta)^{-s}$ on f is defined as

$$(-\Delta)^{-s} f = \sum_{j=1}^{+\infty} \frac{1}{\lambda_j^s} \Phi_j \int_K \Phi_j(y) f(y) d\mu(y).$$

For $f \in \mathcal{D}((-\Delta)^s) := \left\{ f \in L^2(K,\mu), \sum_{j=1}^{+\infty} \lambda_j^{2s} \left(\int_K \Phi_j(y) f(y) d\mu(y) \right)^2 < \infty \right\}$, the fractional Laplacian $(-\Delta)^s$ on f is defined as

$$(-\Delta)^s f = \sum_{j=1}^{+\infty} \lambda_j^s \Phi_j \int_K \Phi_j(y) f(y) d\mu(y).$$

From the definition, it is clear that $(-\Delta)^{-s} : L^2(K,\mu) \to L^2(K,\mu)$ is a bounded operator. More precisely, one has $\|(-\Delta)^{-s}\|_{L^2(K,\mu) \to L^2(K,\mu)} \leq \lambda_1^{-s}$.

Definition 2.6. For a parameter $s \geq 0$, we define the fractional Riesz kernel G_s by

$$G_s(x,y) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} p_t(x,y) dt, \quad x, y \in K, \, x \neq y. \tag{12}$$

with Γ the gamma function.

Remark 2.7. We note that from (8) the integral (12) is indeed convergent for all values of $s \geq 0$ provided that $x \neq y$.

We will be interested in the growth size of G_s. The following estimates are therefore important.

Proposition 2.8.

1. *If $s \in [0, d_h/d_w)$, there exists a constant $C > 0$ such that for every $x, y \in K$, $x \neq y$,*

$$G_s(x,y) \leq \frac{C}{d(x,y)^{d_h - s d_w}}.$$

2. *If $s = d_h/d_w$, there exists a constant $C > 0$ such that for every $x, y \in K$, $x \neq y$*

$$G_s(x,y) \leq C |\ln d(x,y)|.$$

3. *If $s > d_h/d_w$, there exists a constant $C > 0$ such that for every $x, y \in K$,*

$$G_s(x,y) \leq C.$$

Proof. The proof is similar to the proof of [5, Proposition 2.6] (which dealt with the Neumann fractional Riesz kernels) and thus is omitted for conciseness. $\square$

Lemma 2.9. *Let $s > 0$. For $f \in \mathcal{S}(K)$, and $x \in K$*

$$(-\Delta)^{-s} f(x) = \int_K G_s(x,y) f(y) d\mu(y). \tag{13}$$

Proof. We first note that the integral $\int_K G_s(x,y)f(y)d\mu(y)$ is indeed convergent. Since $\mathcal{S}(K) \subset C(K)$, it is enough to prove that for every $x \in K$, $\int_K G_s(x,y)d\mu(y) < +\infty$ which easily follows from Proposition 2.8 because (2) implies that for $\gamma < d_h$

$$\int_K \frac{d\mu(y)}{d(x,y)^\gamma} < +\infty.$$

Using then Fubini's theorem and the definition of G_s, one gets

$$\int_K G_s(x,y) f(y) d\mu(y) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} P_t f(x) dt,$$

From (9), it is seen that

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} P_t f(x) dt = (-\Delta)^{-s} f(x)$$

and the conclusion follows. $\square$

Remark 2.10. For $s > \frac{d_h}{2d_w}$, we observe that (13) holds for any $f \in L^2(K,\mu)$ and μ a.e. $x \in K$. Indeed, this can been seen from Proposition 2.8 since it implies that $G_s(x,\cdot) \in L^2(K,\mu)$ for every $x \in K$. We refer to [5, Propositions 2.7 and 2.8] for more details. Moreover, in this case we also have

$$G_s(x,y) = \sum_{j=1}^{+\infty} \frac{1}{\lambda_j^s} \Phi_j(x)\Phi_j(y).$$

Corollary 2.11. *Let $s > 0$. For $f, g \in \mathcal{S}(K)$,*

$$\int_K (-\Delta)^{-s} f (-\Delta)^{-s} g d\mu = \int_K \int_K f(x) g(y) G_{2s}(x,y) d\mu(x) d\mu(y).$$

Proof. From the definition of $(-\Delta)^{-s}$ and the previous lemma

$$\begin{aligned}
\int_K (-\Delta)^{-s} f (-\Delta)^{-s} g d\mu &= \sum_{j=1}^{+\infty} \frac{1}{\lambda_j^{2s}} \int_K \Phi_j(y) f(y) d\mu(y) \int_K \Phi_j(y) g(y) d\mu(y) \\
&= \int_K f (-\Delta)^{-2s} g d\mu \\
&= \int_K \int_K f(x) g(y) G_{2s}(x,y) d\mu(x) d\mu(y).
\end{aligned}$$

$\square$

2.4 Fractional Gaussian Fields

The next theorem states the existence of the fractional Gaussian fields.

Theorem 2.12. *Let $s \geq 0$. There exists a centered Gaussian distribution X_s on $\mathcal{S}'(K)$ such that for $f, g \in \mathcal{S}(K)$,*

$$\mathbb{E}(X_s(f)X_s(g)) = \int_K (-\Delta)^{-s} f (-\Delta)^{-s} g d\mu.$$

Proof. The space $\mathcal{S}(K)$ is a nuclear space. By the Bochner-Minlos theorem in nuclear spaces, it is enough to prove that the functional

$$\varphi : f \to \exp\left(-\frac{1}{2} \int_K |(-\Delta)^{-s} f|^2 d\mu \right)$$

which is defined on $\mathcal{S}(K)$ is continuous at 0 and positive definite. Since the quadratic form $\int_K |(-\Delta)^{-s} f|^2 d\mu$ is positive definite on $\mathcal{S}(K)$, it follows from Proposition 2.4 in [13] that φ is indeed definite positive. From Definition 2.5, it is easy to see that there exists a constant $C > 0$ such that for every $f \in \mathcal{S}(K)$,

$$\int_K |(-\Delta)^{-s} f|^2 d\mu \leq C \int_K f^2 d\mu.$$

Since the convergence in $\mathcal{S}(K)$ implies the convergence in L^2, we conclude that φ is indeed continuous at 0. □

Definition 2.13 (FGF)**.** Let $s \geq 0$. A fractional Gaussian field X_s with parameter s on K is a centered Gaussian field $\{X_s(f), f \in \mathcal{S}(K)\}$ such that for any $f, g \in \mathcal{S}(K)$,

$$\mathbb{E}(X_s(f)X_s(g)) = \int_K (-\Delta)^{-s} f (-\Delta)^{-s} g d\mu.$$

Definition 2.14 (Log-correlated field)**.** We define a log-correlated Gaussian field on K as a fractional Gaussian field X_s in Definition 2.13 with the parameter $s = \frac{d_h}{2d_w}$.

Remark 2.15. We use the terminology log-correlated field because of the estimate proved in Proposition 2.8 on the correlation function G_{2s} for $s = \frac{d_h}{2d_w}$.

In the following of this section, our aim is to establish that the FGF has an L^2 density if and only if $s > \frac{d_h}{2d_w}$. We begin with some reminders on Gaussian measures.

Let $\mathcal{K}$ be the Borel σ-field on K. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider a real-valued centered Gaussian random measure $W : \mathcal{K} \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ with density μ on K. Such measure is often referred to as white noise. In other words, W is such that

- W is a measure on $(K, \mathcal{K})$ almost surely;
- For any $A \in \mathcal{K}$ of finite measure, $W(A)$ is a real-valued Gaussian variable with mean zero and variance $\mathbb{E}(W(A)^2) = \mu(A)$;
- For any sequence of pairwise disjoint measurable sets $(A_n)_{n\in\mathbb{N}} \in K^{\mathbb{N}}$, the random variables $W(A_n)$, $n \in \mathbb{N}$, are independent.

Hence for any $f \in L^2(K, \mathcal{K}, \mu)$, the stochastic integral $W(f) = \int_K f dW$ is a well-defined centered Gaussian random variable. Moreover, the Gaussian measure W gives rise to an isonormal Gaussian family $\{W(f), f \in L^2(K, \mu)\}$ with the covariance function

$$\mathbb{E}(W(f)W(g)) = \int_K fg d\mu.$$

Recall that the Riesz kernel $G_s(x, y)$ is square integrable for $s > \frac{d_h}{2d_w}$, see Remark 2.10. We then introduce the following definition.

Definition 2.16. Let $s > \frac{d_h}{2d_w}$. The fractional Brownian field with parameter s is defined as

$$\tilde{X}_s(x) = \int_K G_s(x,y)W(dy), \quad x \in K.$$

From Remark 2.10, one can equivalently define

$$\tilde{X}_s(x) = \sum_{i=1}^{+\infty} \lambda_i^{-s}\Phi_i(x)W_i,$$

where $(W_i := W(\Phi_i))_{i\geq 1}$ is an i.i.d. sequence of Gaussian random variables with mean zero and variance one.

Proposition 2.17. *Let $s > \frac{d_h}{2d_w}$. Then the Gaussian random field defined by*

$$X_s(f) = \int_K f(x)\tilde{X}_s(x)d\mu(x), \qquad f \in \mathcal{S}(K),$$

has the law of a FGF with parameter s. Moreover, if there exists a Gaussian field $(\tilde{Y}_s(x))_{x\in K}$ on K with

$$\mathbb{E}\left(\int_K \tilde{Y}_s(x)^2 d\mu(x)\right) < +\infty$$

such that the Gaussian random field defined by

$$Y_s(f) = \int_K f(x)\tilde{Y}_s(x)d\mu(x), \qquad f \in \mathcal{S}(K),$$

has the law of a FGF with parameter s, then $s > \frac{d_h}{2d_w}$.

Proof. Let us assume that $s > \frac{d_h}{2d_w}$. From Fubini's theorem, for every $f \in \mathcal{S}(K)$, one has a.s.

$$\int_K f(x)\tilde{X}_s(x)d\mu(x) = \int_K f(x)\int_K G_s(x,y)W(dy)d\mu(x) = \int_K (-\Delta)^{-s}f(y)W(dy).$$

Thus, $\int_K f(x)\tilde{X}_s(x)d\mu(x)$ is a Gaussian random variable with mean zero and variance

$$\int_K |(-\Delta)^{-s}f(x)|^2 d\mu(x).$$

On the other hand, assume that there exists a Gaussian field $(\tilde{Y}_s(x))_{x\in K}$ on K with

$$\mathbb{E}\left(\int_K \tilde{Y}_s(x)^2 d\mu(x)\right) < +\infty$$

and such that the Gaussian random field defined by

$$Y_s(f) = \int_K f(x)\tilde{Y}_s(x)d\mu(x), \qquad f \in \mathcal{S}(K),$$

has the law of a FGF with parameter s. Using spectral decomposition, we have

$$\tilde{Y}_s(x) = \sum_{i=1}^{+\infty} Y_s(\Phi_i)\,\Phi_i(x).$$

Notice that the sequence $(Y_s(\Phi_i))_{i\geq 1}$ is a sequence of independent Gaussian random variables with mean zero and variance $(\lambda_i^{-2s})_{i\geq 1}$. Indeed, we recall that $(\Phi_i)_{i\geq 1}$ is an orthonormal basis in $L^2(K,\mu)$ and $\Phi_i \in \mathcal{S}$. Then by Definition 2.13,

$$\mathbb{E}(Y_s(\Phi_i)Y_s(\Phi_j)) = (\lambda_i\lambda_j)^{-s}\int_K \Phi_i\Phi_j d\mu = (\lambda_i\lambda_j)^{-s}\delta_{ij},$$

where δ_{ij} is the Kronecker delta. Since $\mathbb{E}\left(\int_K \tilde{Y}_s(x)^2 d\mu(x)\right) < +\infty$, we must have

$$\sum_{j=1}^{+\infty} \frac{1}{\lambda_j^{2s}} < +\infty$$

and therefore $s > \frac{d_h}{2d_w}$. □

3 Regularity properties of the FGFs

3.1 Sobolev spaces

For any $\alpha \geq 0$, we define the Sobolev space $H^\alpha(K)$ as the closure of $\mathcal{S}(K)$ with respect to the norm

$$\|f\|^2_{H^\alpha(K)} := \sum_{j=1}^{\infty} \lambda_j^\alpha \left(\int_K f\Phi_j d\mu\right)^2,$$

and the corresponding inner product is

$$(f,g)_{H^\alpha(K)} = \sum_{j=1}^{\infty} \lambda_j^\alpha \int_K f\Phi_j d\mu \int_K g\Phi_j d\mu,$$

where we recall that $(\lambda_j)_{j\geq 1}$ are the non-decreasing eigenvalues of the Laplacian Δ on K and $(\Phi_j)_{j\geq 1}$ are the corresponding orthonormal eigenfunctions. Denote by $H^{-\alpha}(K) \subset \mathcal{S}'(K)$ the dual space of $H^\alpha(K)$ in the distributional sense. Then we have the following lemma.

Lemma 3.1. *The canonical norm on $H^{-\alpha}(K)$ induced by $H^\alpha(K)$ is given by*

$$\|\psi\|^2_{H^{-\alpha}(K)} := \sum_{j=1}^{\infty} \lambda_j^{-\alpha}\psi(\Phi_j)^2, \quad \forall \psi \in H^{-\alpha}(K).$$

Proof. The proof is standard, we write it down for the sake of completeness. For every $\psi \in H^{-\alpha}(K)$ there exists $f_\psi \in H^\alpha(K)$ such that $\psi(g) = (f_\psi, g)_{H^\alpha(K)}$ for all $g \in \mathcal{S}(K)$. In particular, the above inner product gives that for every $\Phi_j, j \geq 1$,

$$\psi(\Phi_j) = (f_\psi, \Phi_j)_{H^\alpha(K)} = \lambda_j^\alpha \int_K f_\psi \Phi_j d\mu.$$

Note also that by isometry one has $\|\psi\|_{H^{-\alpha}(K)} = \|f_\psi\|_{H^\alpha(K)}$. Hence

$$\|\psi\|^2_{H^{-\alpha}(K)} = \|f_\psi\|^2_{H^\alpha(K)} = \sum_{j=1}^{\infty} \lambda_j^\alpha \left(\int_K f_\psi\Phi_j d\mu\right)^2 = \sum_{j=1}^{\infty} \lambda_j^{-\alpha}\psi(\Phi_j)^2.$$

□

3.2 Sobolev regularity property of the continuous FGFs in the range $0 \le s \le \frac{d_h}{2d_w}$

Proposition 3.2. *Let $0 \le s \le \frac{d_h}{2d_w}$. Then the FGF X_s a.s. belongs to $H^{-\alpha}(K)$ for every $\alpha > \frac{d_h}{d_w} - 2s$. More precisely, for every $\alpha > \frac{d_h}{d_w} - 2s$, the series*

$$\sum_{j=1}^{\infty} \lambda_j^{-\alpha} X_s(\Phi_j)^2$$

is a.s. convergent.

Proof. The random variables $X_s(\Phi_j)$ are independent Gaussian random variables with mean zero and variance λ_j^{-2s}. Since the series

$$\sum_{j=1}^{\infty} \lambda_j^{-\alpha-2s}$$

converges for $\alpha + 2s > \frac{d_h}{d_w}$, the result follows. □

3.3 Hölder regularity property of the continuous FGFs in the range $s > \frac{d_h}{2d_w}$

In this section our goal is to study the regularity of the density field $(\tilde{X}_s(x))_{x\in K}$ that appeared in Definition 2.16. The following first result is almost immediate.

Proposition 3.3. *Let $s > \frac{d_h}{2d_w}$. The Gaussian field $(\tilde{X}_s(x))_{x\in K}$ on K defined in Definition 2.16 is such that a.s. $\tilde{X}_s \in H^{\alpha}(K)$ for every $\alpha < 2s - \frac{d_h}{d_w}$.*

Proof. As remarked before, one has

$$\tilde{X}_s(x) = \sum_{i=1}^{+\infty} \lambda_i^{-s} \Phi_i(x) W_i,$$

where W_i is an i.i.d. sequence of Gaussian random variables with mean zero and variance one. Therefore,

$$\|\tilde{X}_s\|^2_{H^\alpha(K)} = \sum_{j=1}^{\infty} \lambda_j^{\alpha-2s} W_j^2,$$

which is a.s. finite if $\alpha < 2s - \frac{d_h}{d_w}$. □

Next, we are interested in the Hölder regularity properties of $(\tilde{X}_s(x))_{x\in K}$. This requires a deeper analysis and our main analytical ingredients are the following Hölder regularization estimates for the operators $(-\Delta)^{-s}$.

Theorem 3.4.

- *Let $\frac{d_h}{2d_w} < s < 1 - \frac{d_h}{2d_w}$. There exists a constant $C > 0$ such that for every $f \in L^2(K,\mu)$ and $x, y \in K$,*

$$|(-\Delta)^{-s} f(x) - (-\Delta)^{-s} f(y)| \le C d(x,y)^{sd_w - \frac{d_h}{2}} \|f\|_{L^2(K,\mu)}.$$

- *Let $s = 1 - \frac{d_h}{2d_w}$. There exists a constant $C > 0$ such that for every $f \in L^2(K,\mu)$ and $x, y \in K$ with $d(x,y) \le 1/2$,*

$$|(-\Delta)^{-s} f(x) - (-\Delta)^{-s} f(y)| \le C d(x,y)^{d_w - d_h} |\ln d(x,y)| \|f\|_{L^2(K,\mu)}.$$

- *Let $s > 1 - \frac{d_h}{2d_w}$. There exists a constant $C > 0$ such that for every $f \in L^2(K,\mu)$ and $x, y \in K$,*

$$|(-\Delta)^{-s} f(x) - (-\Delta)^{-s} f(y)| \le C d(x,y)^{d_w - d_h} \|f\|_{L^2(K,\mu)}.$$

The proof of the Theorem is based on the following lemmas. In the first lemma below, we use a slight modification of an argument due to Barlow in [2, Theorem 3.40], but we include the proof for the sake of completeness. For $\lambda \ge 0$, let $U_\lambda = (\lambda - \Delta)^{-1}$ be the resolvent operator and let $u_\lambda(x,y)$, $x, y \in K$ be its kernel.

Lemma 3.5. *There exists a constant $C > 0$ such that for every $x, y, z \in K$ and $\lambda > 0$*

$$|u_\lambda(x,z) - u_\lambda(y,z)| \le C d(x,y)^{d_w - d_h}$$

and

$$\int_K |u_\lambda(x,z) - u_\lambda(y,z)| d\mu(z) \le C \lambda^{-\frac{d_h}{d_w}} d(x,y)^{d_w - d_h}.$$

Proof. We slightly adapt the proof of Theorem 3.40 in [2]. Let $((X_t)_{t\ge 0}, (\mathbb{P}_x)_{x\in K})$ be the Brownian motion on K, see [2]. Note that the Dirichlet Laplacian Δ is (twice) the generator of the process $(X_t)_{t\ge0}$ killed at the boundary V_0. From (3.39) in [2] we have

$$u_\lambda(x,y) = q_\lambda(x,y) u_\lambda(y,y),$$

with

$$q_\lambda(x,y) = \mathbb{P}_x(T_y \le T_{V_0} \wedge R_\lambda)$$

where T_y is the hitting time of y, T_{V_0} is the hitting time of V_0 and R_λ is an independent exponential random variable with parameter $\lambda > 0$. As in the proof of Theorem 3.40 in [2] we have then

$$|u_\lambda(x,z) - u_\lambda(y,z)| \le C(q_\lambda(z,x) + q_\lambda(z,y)) d(x,y)^{d_w - d_h}.$$

The first estimate

$$|u_\lambda(x,z) - u_\lambda(y,z)| \le C d(x,y)^{d_w - d_h}$$

follows easily since $q_\lambda(z,x) \le 1$. Using the on-diagonal lower bound for the transition densities of the Brownian motion $((X_t)_{t\ge0}, (\mathbb{P}_x)_{x\in K})$, see [2, 3], we easily obtain

$$\int_K q_\lambda(z,x) d\mu(z) \le C \lambda^{-\frac{d_h}{d_w}},$$

which yields the second estimate. □

Lemma 3.6. *There exists a constant $C > 0$ such that for every $f \in L^2(K,\mu)$, and $t > 0$,*

$$|P_t f(x) - P_t f(y)| \le C e^{-\frac{\lambda_1}{2} t} \frac{d(x,y)^{d_w - d_h}}{t^{1 - \frac{d_h}{2d_w}}} \|f\|_{L^2(K,\mu)}.$$

Proof. We have from the previous lemma

$$|u_\lambda(x,z) - u_\lambda(y,z)| \le C d(x,y)^{d_w - d_h}$$

and

$$\int_K |u_\lambda(x,z) - u_\lambda(y,z)| d\mu(z) \le C \lambda^{-\frac{d_h}{d_w}} d(x,y)^{d_w - d_h}.$$

This yields

$$\int_K |u_\lambda(x,z) - u_\lambda(y,z)|^2 d\mu(z) \le C \lambda^{-\frac{d_h}{d_w}} d(x,y)^{2(d_w - d_h)},$$

from which we deduce

$$|U_\lambda f(x) - U_\lambda f(y)| \le C\lambda^{-\frac{d_h}{2d_w}} d(x,y)^{d_w-d_h} \|f\|_{L^2(K,\mu)}.$$

This gives that for every $t > 0$ and $\lambda > 0$

$$|P_t f(x) - P_t f(y)| \le C\lambda^{-\frac{d_h}{2d_w}} d(x,y)^{d_w-d_h} \|(\Delta - \lambda)P_t f\|_{L^2(K,\mu)}. \tag{14}$$

From spectral theory, one has

$$\|(\Delta - \lambda)P_t f\|_{L^2(K,\mu)} \le C\left(\frac{1}{t} + \lambda\right) e^{-\frac{\lambda_1}{2}t} \|f\|_{L^2(K,\mu)}.$$

From (14), we deduce then

$$|P_t f(x) - P_t f(y)| \le Ce^{-\frac{\lambda_1}{2}t} \frac{d(x,y)^{d_w-d_h}}{t^{1-\frac{d_h}{2d_w}}} \|f\|_{L^2(K,\mu)}.$$

by choosing $\lambda = \frac{1}{t}$. □

Proof of Theorem 3.4. We first consider the case $\frac{d_h}{2d_w} < s < 1 - \frac{d_h}{2d_w}$. Note that

$$|(-\Delta)^{-s} f(x) - (-\Delta)^{-s} f(y)| \le \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} |P_t f(x) - P_t f(y)| dt.$$

We then split the integral into two parts:

$$\int_0^{+\infty} t^{s-1} |P_t f(x) - P_t f(y)| dt = \int_0^{\delta} t^{s-1} |P_t f(x) - P_t f(y)| dt + \int_\delta^{+\infty} t^{s-1} |P_t f(x) - P_t f(y)| dt,$$

where $\delta > 0$ is a parameter to be chosen later. We first have

$$\begin{aligned}
\int_0^{\delta} t^{s-1} |P_t f(x) - P_t f(y)| dt &\le \int_0^{\delta} t^{s-1} (|P_t f(x)| + |P_t f(y)|) dt \\
&\le \int_0^{\delta} t^{s-1} \frac{C}{t^{\frac{d_h}{2d_w}}} dt \, \|f\|_{L^2(K,\mu)} \\
&\le C\delta^{s-\frac{d_h}{2d_w}} \|f\|_{L^2(K,\mu)}.
\end{aligned}$$

For the second integral, we have

$$\begin{aligned}
\int_\delta^{+\infty} t^{s-1} |P_t f(x) - P_t f(y)| dt &\le C \int_\delta^{+\infty} t^{s-1} e^{-\frac{\lambda_1}{2}t} \frac{d(x,y)^{d_w-d_h}}{t^{1-\frac{d_h}{2d_w}}} dt \, \|f\|_{L^2(K,\mu)} \\
&\le C\delta^{s-1+\frac{d_h}{2d_w}} d(x,y)^{d_w-d_h} \|f\|_{L^2(K,\mu)}.
\end{aligned}$$

Choosing $\delta = d(x,y)^{d_w}$ finishes the proof that

$$|(-\Delta)^{-s} f(x) - (-\Delta)^{-s} f(y)| \le Cd(x,y)^{sd_w - \frac{d_h}{2}} \|f\|_{L^2(K,\mu)}.$$

Next consider the case $s = 1 - \frac{d_h}{2d_w}$. The above method can still be used, but we now estimate the second integral as follows for $\delta \le \frac{1}{2^{d_w}}$

$$\int_\delta^{+\infty} t^{-1} e^{-\frac{\lambda_1}{2}t} dt \le C|\ln \delta|.$$

This yields

$$|(-\Delta)^{-s}f(x)-(-\Delta)^{-s}f(y)|\le Cd(x,y)^{d_w-d_h}|\ln d(x,y)|\|f\|_{L^2(K,\mu)}.$$

Finally, for the case $s>1-\frac{d_h}{2d_w}$, we just argue as follows

$$\begin{aligned}|(-\Delta)^{-s}f(x)-(-\Delta)^{-s}f(y)| &\le \frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}|P_tf(x)-P_tf(y)|dt\\ &\le C\int_0^{+\infty} t^{s-1}e^{-\frac{\lambda_1}{2}t}\frac{d(x,y)^{d_w-d_h}}{t^{1-\frac{d_h}{2d_w}}}dt\,\|f\|_{L^2(K,\mu)}\\ &\le Cd(x,y)^{d_w-d_h}\|f\|_{L^2(K,\mu)}.\end{aligned}$$

□

As a consequence of Theorem 3.4, we obtain the Hölder regularization properties of the Riesz kernels.

Corollary 3.7.

- *Let $\frac{d_h}{2d_w}<s<1-\frac{d_h}{2d_w}$. There exists a constant $C>0$ such that for every $x,y\in K$,*

$$\int_K (G_s(x,z)-G_s(y,z))^2d\mu(z)\le Cd(x,y)^{2sd_w-d_h}.$$

- *Let $s=1-\frac{d_h}{2d_w}$. There exists a constant $C>0$ such that for every $f\in L^2(K,\mu)$ and $x,y\in K$ with $d(x,y)\le 1/2$,*

$$\int_K (G_s(x,z)-G_s(y,z))^2d\mu(z)\le Cd(x,y)^{2(d_w-d_h)}|\ln d(x,y)|^2.$$

- *Let $s>1-\frac{d_h}{2d_w}$. There exists a constant $C>0$ such that for every $x,y\in K$,*

$$\int_K (G_s(x,z)-G_s(y,z))^2d\mu(z)\le Cd(x,y)^{2(d_w-d_h)}.$$

Proof. Recall that for any $f\in L^2(K,\mu)$,

$$\int_K (G_s(x,z)-G_s(y,z))f(z)d\mu(z)=(-\Delta)^{-s}f(x)-(-\Delta)^{-s}f(y).$$

By L^2 duality, we conclude the results from Theorem 3.4. □

We are now ready for the main results of this section:

Theorem 3.8. *Let $s>\frac{d_h}{2d_w}$ and denote $H_s=\min(sd_w-d_h/2,d_w-d_h)$ and*

$$\omega_s(x,y)=\begin{cases} d(x,y)^{H_s}\sqrt{|\ln d(x,y)|},\ s\ne 1-\frac{d_h}{2d_w}\\ d(x,y)^{d_w-d_h}\,|\ln d(x,y)|^{3/2},\ s=1-\frac{d_h}{2d_w}.\end{cases}$$

There exists a continuous Gaussian field $(\tilde X_s(x))_{x\in K}$ such that a.s.

$$\lim_{\delta\to 0}\sup_{\substack{0<d(x,y)\le\delta\\ x,y\in K}}\frac{\left|\tilde X_s(x)-\tilde X_s(y)\right|}{\omega_s(x,y)}<+\infty$$

and such that the Gaussian random field defined by

$$X_s(f)=\int_K f(x)\tilde X_s(x)d\mu(x),\qquad f\in\mathcal S(K),$$

has the law of a FGF with parameter s.

Proof. Let $\tilde{X}_s(x) = \int_K G_s(x,y)W(dy)$, see Definition 2.16. It follows from Proposition 2.17 that $\{X_s(f), f \in \mathcal{S}(K)\}$ has the law of FGF with parameter s.

Next we will use the entropy method as in [1], see also [5, Theorem 3.8] to construct an appropriate continuous modification of $\tilde{X}_s$ that we will still denote by $\tilde{X}_s$. Assume first $s \neq 1 - \frac{d_h}{2d_w}$. We observe that Corollary 3.7 gives

$$\mathbb{E}\big((\tilde{X}_s(x) - \tilde{X}_s(y))^2\big) \leq Cd(x,y)^{2H_s}, \quad \forall x, y \in K.$$

Consider the pseudo distance $\rho_s(x,y)$ defined by

$$\rho_s(x,y) = \sqrt{\mathbb{E}\big((\tilde{X}_s(x) - \tilde{X}_s(y))^2\big)}.$$

Then $\rho_s(x,y) \leq Cd(x,y)^{H_s}$. Denote by $\mathcal{N}_{\rho_s}(\varepsilon)$ the smallest number of ρ_s-balls with radius $r \leq \varepsilon$ that cover K. We set the log-entropy for K by

$$\mathcal{H}_{\rho_s}(\varepsilon) = \ln(\mathcal{N}_{\rho_s}(\varepsilon)).$$

According to [1, Theorem 1.3.5], there exist a random variable η and a universal constant D such that for all $\tau < \eta$,

$$\sup_{\substack{\rho_s(x,y)\leq\tau \\ x,y\in K}} |\tilde{X}_s(x) - \tilde{X}_s(y)| \leq D\int_0^\tau \sqrt{\mathcal{H}_{\rho_s}(\varepsilon)}d\varepsilon.$$

Notice that $\mathcal{N}_{\rho_s}(\varepsilon) = O(\varepsilon^{-d_h/H_s})$. Then up to the change of η and D, one has for all $\delta < \eta$,

$$\sup_{\substack{d(x,y)\leq\delta \\ x,y\in K}} |\tilde{X}_s(x) - \tilde{X}_s(y)| \leq D\int_0^{C\delta^{H_s}} \sqrt{-\ln\varepsilon}\, d\varepsilon.$$

Finally, up the the change of constant D, for all $\delta < \eta$ small enough, we obtain from integration by parts that

$$\sup_{\substack{d(x,y)\leq\delta \\ x,y\in K}} |\tilde{X}_s(x) - \tilde{X}_s(y)| \leq D\Big(\delta^{H_s}\sqrt{-\ln\delta} + \int_0^{C\delta^{H_s}} \frac{1}{\sqrt{-\ln\varepsilon}}\, d\varepsilon\Big) \leq 2D\delta^{H_s}\sqrt{-\ln\delta}.$$

Thus the proof is concluded.

Consider now the critical case $s = 1 - \frac{d_h}{2d_w}$. By Corollary 3.7, we have

$$\rho_s(x,y) \leq d(x,y)^{d_w - d_h}|\ln d(x,y)| =: F(d(x,y)).$$

Observe that $F(t)$ is increasing on the interval $(0, t_0)$ for some small t_0. We denote by F^{-1} the inverse function on the domain $(0, F(t_0))$. Then for any $0 < \varepsilon < F(t_0)$, one has $\mathcal{N}_{\rho_s}(\varepsilon) = O((F^{-1}(\varepsilon))^{-d_h})$. Using the same argument as above, there exist a random variable η and constants $C, D > 0$ such that for all $\delta < \min\{\eta, t_0\}$,

$$\sup_{\substack{d(x,y)\leq\delta \\ x,y\in K}} |\tilde{X}_s(x) - \tilde{X}_s(y)| \leq D\int_0^{CF(\delta)} \sqrt{-\ln F^{-1}(\varepsilon)}\, d\varepsilon.$$

Hence, up to the change of constant D and for all $\delta < \min\{\eta, t_0\}$ small enough, we have

$$\sup_{\substack{d(x,y)\leq\delta \\ x,y\in K}} |\tilde{X}_s(x) - \tilde{X}_s(y)| \leq D\Big(F(\delta)\sqrt{-\ln\delta} - \int_0^{CF(\delta)} \varepsilon\big(\sqrt{-\ln F^{-1}(\varepsilon)}\big)'\, d\varepsilon\Big) \leq DF(\delta)\sqrt{-\ln\delta}.$$

The second inequality follows from elementary computations below where we take $\gamma = d_w - d_h$ and let ε be small enough:

$$\begin{aligned}
\varepsilon\big(\sqrt{-\ln F^{-1}(\varepsilon)}\big)' &= \frac{1}{2\sqrt{-\ln F^{-1}(\varepsilon)}} \frac{-\varepsilon}{F^{-1}(\varepsilon)F'(F^{-1}(\varepsilon))} \\
&= \frac{1}{2\sqrt{-\ln F^{-1}(\varepsilon)}} \frac{-\varepsilon}{(F^{-1}(\varepsilon))^{\gamma}(-\gamma \ln F^{-1}(\varepsilon)-1)} \\
&= \frac{1}{2\sqrt{-\ln F^{-1}(\varepsilon)}} \frac{-\varepsilon}{(\gamma F(F^{-1}(\varepsilon)) - (F^{-1}(\varepsilon))^{\gamma})} = O\left(\frac{-1}{\sqrt{-\ln F^{-1}(\varepsilon)}}\right).
\end{aligned}$$

Thus we conclude that for $s = 1 - \frac{d_h}{2d_w}$

$$\lim_{\delta\to 0} \sup_{\substack{0<d(x,y)\le\delta \\ x,y\in K}} \frac{\left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|}{d(x,y)^{d_w-d_h}(|\ln d(x,y)|)^{3/2}} < +\infty.$$

□

For $s > 1$ the above result can substantially be improved.

Proposition 3.9. *Let $s > 1$. There exists a continuous Gaussian field $(\tilde{X}_s(x))_{x\in K}$ such that*

$$\mathbb{E}\left(\left(\sup_{x,y\in K, x\neq y} \frac{\left|\tilde{X}_s(x) - \tilde{X}_s(y)\right|}{d(x,y)^{d_w-d_h}}\right)^2\right) < +\infty$$

and such that the Gaussian random random field defined by

$$X_s(f) = \int_K f(x)\tilde{X}_s(x)d\mu(x), \qquad f \in \mathcal{S}(K),$$

has the law of a FGF with parameter s.

Proof. Let $s > 1$. As above, let $(\tilde{X}_s(x))_{x\in K}$ be a continuous Gaussian field on K such that the Gaussian random random field defined by

$$X_s(f) = \int_K f(x)\tilde{X}_s(x)d\mu(x), \qquad f \in \mathcal{S}(K)$$

has the law of a FGF with parameter s. We have then

$$\tilde{X}_s(x) = \sum_{i=1}^{+\infty} \lambda_i^{-s}\Phi_i(x)W_i,$$

where the W_i's form an i.i.d. sequence of Gaussian random variables with mean zero and variance one. Let $\alpha > 1 - \frac{d_h}{2d_w}$ such that $s - \alpha > \frac{d_h}{2d_w}$. Since

$$\mathbb{E}\left(\|(-\Delta)^{\alpha}\tilde{X}_s\|^2_{L^2(K,\mu)}\right) = \sum_{i=1}^{+\infty} \lambda_i^{2(\alpha-s)} < +\infty,$$

we deduce that $\tilde{X}_s$ almost surely belongs to the L^2 domain of $(-\Delta)^{\alpha}$, i.e.

$$\mathbb{P}\left(\|(-\Delta)^{\alpha}\tilde{X}_s\|^2_{L^2(K,\mu)} < +\infty\right) = 1.$$

From Theorem 3.4, one deduces

$$\left|\tilde{X}_s(x) - \tilde{X}_s(y)\right| \le C_s d(x,y)^{d_w-d_h}\|(-\Delta)^{\alpha}\tilde{X}_s\|_{L^2(K,\mu)},$$

and the result follows. □

4 Convergence of the discrete fields to the continuous fields

In this section, our first main goal is to show for $s \geq 0$ the convergence in distribution in $\mathcal{S}'(K)$ of the approximations of discrete fractional Gaussian fields on V_m to the fractional Gaussian field on the Sierpinski gasket. Our second goal will be to prove convergence in the Sobolev spaces $H^\alpha(K)$.

4.1 Preliminary lemmas

This section collects several lemmas that will later be needed.

Lemma 4.1 ([3, Lemma 1.1])**.** *The sequence of measures $\{\mu_m\}_{m\geq 0}$ defined in* (4) *converges to the normalized Hausdorff measure μ on the Sierpinski gasket K in the weak topology. That is,*

$$\lim_{m\to\infty} \int_K f d\mu_m = \int_K f d\mu, \quad \forall f \in C(K).$$

Remark 4.2. Without abuse of notation, for any $g \in \ell(V_m)$, we may write

$$\frac{1}{a_m} \sum_{p\in V_m} g(p) = \int_{V_m} g d\mu_m.$$

Hence let $f_m = f|_{V_m}$ for $f \in C(K)$ in the lemma, one also has $\lim_{m\to\infty} \int_{V_m} f_m d\mu_m = \int_K f d\mu$.

Lemma 4.3 (Convergence of discrete semigroups)**.** *For all $f \in \mathcal{S}(K)$ and $t > 0$,*

$$\lim_{m\to\infty} \frac{1}{a_m} \sum_{p\in V_m} f_m(p) P_t^m f_m(p) = \int_K f(x) P_t f(x) d\mu(x),$$

where $f_m = f|_{V_m}$.

Proof. We follow the strategy in [8, Section 3.2.2]. Recall the definition of Laplacian on K in (3) (see also [9, page 6]), then for any $f \in \mathcal{D}_0$.

$$\lim_{m\to\infty} \sup_{p\in V_m} |\Delta_m f_m(p) - \Delta f(p)| = 0.$$

It follows from [12, Theorem 2.1] that for every $t \geq 0$

$$\lim_{m\to\infty} \sup_{p\in V_m} |P_t^m f_m(p) - P_t f(p)| = 0. \tag{15}$$

Indeed, the Laplacian on K coincides with the extended limit of the sequence of operators $\{\Delta_m\}_{m\geq 0}$ defined in [12, page 355]. Write

$$\frac{1}{a_m} \sum_{p\in V_m} f_m(p) P_t^m f_m(p) = \int_{V_m} f_m P_t^m f_m d\mu_m,$$

and further

$$\int_{V_m} f_m P_t^m f_m d\mu_m = \int_{V_m} f_m \left(P_t^m f_m - (P_t f)_m\right) d\mu_m + \int_{V_m} f_m (P_t f)_m d\mu_m.$$

Taking the limit $m \to \infty$, the first term goes to zero from (15). On the other hand, Lemma 4.1 gives that $\int_{V_m} f_m (P_t f)_m d\mu_m \to \int_K f P_t f d\mu$ and the proof is complete. □

Lemma 4.4. *For all $f \in \mathcal{S}(K)$ and $s \geq 0$, when $m \to \infty$*

$$\frac{1}{a_m} \sum_{p \in V_m} f_m(p)(-\Delta_m)^{-2s} f_m(p) \longrightarrow \int_K f(x)(-\Delta)^{-2s} f(x) d\mu(x), \tag{16}$$

where $f_m = f|_{V_m}$.

Proof. For $s = 0$, the result follows immediately from Lemma 4.1. We now assume $s > 0$. Notice that

$$(-\Delta_m)^{-2s} f_m = \frac{1}{\Gamma(2s)} \int_0^{+\infty} t^{2s-1} P_t^m f_m dt.$$

By Fubini's theorem, we therefore have

$$\frac{1}{a_m} \sum_{p \in V_m} f_m(p)(-\Delta_m)^{-2s} f_m(p) = \frac{1}{\Gamma(2s)} \int_0^{+\infty} t^{2s-1} \frac{1}{a_m} \sum_{p \in V_m} f_m(p) P_t^m f_m(p) dt.$$

Let us now note that by spectral theory

$$\frac{1}{a_m} \sum_{p \in V_m} f_m(p) P_t^m f_m(p) \leq e^{-\lambda_1^m t} \frac{1}{a_m} \sum_{p \in V_m} f_m(p)^2.$$

Notice that $\sup_m \frac{1}{a_m} \sum_{p \in V_m} f_m(p)^2 < +\infty$. Hence we deduce from the dominated convergence theorem and Lemma 4.3 that

$$\begin{aligned} \lim_{m \to \infty} \frac{1}{a_m} \sum_{p \in V_m} f_m(p)(-\Delta_m)^{-2s} f_m(p) &= \frac{1}{\Gamma(2s)} \int_0^{+\infty} t^{2s-1} \lim_{m \to \infty} \frac{1}{a_m} \sum_{p \in V_m} f_m(p) P_t^m f_m(p) dt \\ &= \frac{1}{\Gamma(2s)} \int_0^{+\infty} t^{2s-1} \int_K f(x) P_t f(x) d\mu(x) dt \\ &= \int_K f(x)(-\Delta)^{-2s} f(x) d\mu(x). \end{aligned}$$

□

4.2 Convergence in distribution in $\mathcal{S}'(K)$

We are now in position to prove the following result.

Theorem 4.5. *Let $s \geq 0$. When $m \to \infty$, X_s^m converges to X_s in distribution in $\mathcal{S}'(K)$.*

Proof. We aim to prove $X_s^m \to X_s$ in law in $\mathcal{S}'(K)$. Since $\mathcal{S}(K)$ is a nuclear space, it suffices to prove the convergence of the characteristic functional (see for instance [14, Théorème 2]). That is, for every $f \in \mathcal{S}(K)$, when $m \to \infty$

$$\mathbb{E}\left[\exp\left(i X_s^m(f_m)\right)\right] \longrightarrow \mathbb{E}\left[\exp\left(i X_s(f)\right)\right],$$

where $f_m = f|_{V_m}$. It follows from (7) that

$$\mathbb{E}\left[\exp\left(i\, X_s^m(f_m)\right)\right] = \exp\left(-\frac{1}{2}\mathbb{E}\left((X_s^m(f_m))^2\right)\right) = \exp\left(-\frac{1}{2a_m} \sum_{p \in V_m} f_m(p)(-\Delta_m)^{-2s} f_m(p)\right).$$

Similarly, Definition 2.5 gives that $\mathbb{E}\left[\exp\left(i X_s(f)\right)\right] = \exp\left(-\frac{1}{2}\int_K f(-\Delta)^{-2s} f d\mu\right)$. The conclusion therefore follows from Lemma 4.4. □

4.3 Convergence in distribution in Sobolev spaces

Recall the Sobolev space H^α and the dual space $H^{-\alpha}$ defined in Section 3.1. In this section, we aim to prove the convergence of lifted DFGF in the Sobolev space $H^{-\alpha}$ for appropriate $\alpha > 0$. Following the scheme in [8], we first lift X_s^m on V_m to K using Voronoi cells defined by

$$C_p^m = \{x \in K : d(x,p) \le d(x,q), \forall q \in V_m\}, \quad p \in V_m.$$

Equivalently, one has $C_p^m = \{x \in K : d(x,p) \le 2^{-(m+1)}\}$.

Definition 4.6 (DFGF in $H^{-\alpha}(K)$)**.** Let X_s^m be the DFGF on V_m as in Definition 2.2. We define $\bar{X}_s^m \in H^{-\alpha}(K)$ such that for $f \in H^\alpha(K)$

$$\bar{X}_s^m(f) = \frac{1}{a_m} \sum_{p \in V_m} X_s^m(p) \bar{f}_m(p) = X_s^m(\bar{f}_m),$$

where $\bar{f}_m(p) := \frac{1}{\mu(C_p^m)} \int_{C_p^m} f(x) d\mu(x)$ for any $p \in V_m$.

Our main result in this section is the following theorem.

Theorem 4.7. *Let $s \ge 0$. The Gaussian fields $\bar{X}_s^m$ converge in law to X_s in the strong topology of $H^{-\alpha}(K)$ for $\alpha > 2d_h/d_w$.*

Throughout the section we assume that $s \ge 0$. The proof is divided into two parts. We will first show the tightness of the sequence $(\bar{X}_s^m)_{m \ge 1}$ in $H^{-\alpha}(K)$. Thus every sequence has a convergent subsequence. The second part is to show that the limit is unique.

We first state the following lemma for the sequel use. Let $j \ge 1$. Recall that λ_j is the j-th eigenvalue of Δ on K and Φ_j is the corresponding eigenfunction.

Lemma 4.8. *For any $j \ge 1$, we have*

$$\|\Phi_j\|_{L^\infty(K,\mu)} \le C \lambda_j^{d_h/(2d_w)}.$$

Proof. We use spectral theory (as in the proof of [4, Lemma 3.4]). Notice that $P_t \Phi_j = e^{-\lambda_j t} \Phi_j$. Using the Cauchy-Schwartz inequality and (8), we obtain for μ-a.e. $x \in K$,

$$|\Phi_j(x)| = e^{\lambda_j t_0} |P_{t_0} \Phi_j(x)| \le e^{\lambda_j t_0} \left(\int_K p_{t_0}(x,y)^2 d\mu(y) \right)^{1/2} \le C t_0^{-d_h/(2d_w)} e^{\lambda_j t_0}.$$

In particular, taking $t_0 = \lambda_j^{-1}$ leads to

$$|\Phi_j(x)| \le C \lambda_j^{d_h/(2d_w)}, \quad \mu\text{-a.e. } x \in K.$$

□

Proposition 4.9. *The sequence $(\bar{X}_s^m)_{m \ge 1}$ is tight in $H^{-\alpha}(K)$ for any $\alpha > 2d_h/d_w$.*

Proof. We will first prove that for any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that for all $m \ge 0$

$$\mathbb{P}(\|\bar{X}_s^m\|_{H^{-\alpha}(K)}^2 > R) \le \varepsilon. \tag{17}$$

Note that by Chebyshev's inequality,

$$\mathbb{P}(\|\bar{X}_s^m\|_{H^{-\alpha}(K)}^2 > R) \le \frac{1}{R} \mathbb{E}(\|\bar{X}_s^m\|_{H^{-\alpha}(K)}^2).$$

From Lemma 3.1, we can write $\mathbb{E}(\|\bar{X}_s^m\|^2_{H^{-\alpha}(K)})$ as

$$\mathbb{E}\left(\sum_{j=1}^{\infty}\lambda_j^{-\alpha}\left(\bar{X}_s^m(\Phi_j)\right)^2\right)=\sum_{j=1}^{\infty}\lambda_j^{-\alpha}\mathbb{E}\left(\left(X_s^m((\bar{\Phi}_j)_m)\right)^2\right).$$

Noticing that from [9, Lemma 5.2] one has $\inf_m \lambda_1^m > 0$, then applying (7) gives

$$\mathbb{E}\left(\left(X_s^m((\bar{\Phi}_j)_m)\right)^2\right)=\frac{1}{a_m}\sum_{p\in V_m}\bar{\Phi}_j(p)(-\Delta_m)^{-2s}\bar{\Phi}_j(p)\le(\lambda_1^m)^{-2s}\|\bar{\Phi}_j\|^2_{L^2(V_m,\mu_m)}.$$

Observe that by Lemma 4.8 one has

$$\|\bar{\Phi}_j\|^2_{L^2(V_m,\mu_m)}\le 2\|\Phi_j\|^2_{L^\infty(K,\mu)}\le C\lambda_j^{d_h/d_w}.$$

Besides, Weyl's eigenvalue asymptotics (10) yields that $\lambda_j \sim j^{d_w/d_h}$. Hence

$$\mathbb{E}\left(\|\bar{X}_s^m\|^2_{H^{-\alpha}(K)}\right)\le C\sum_{j=1}^{\infty}\lambda_j^{-\alpha+d_h/d_w}\le C\sum_{j=1}^{\infty}j^{\left(\frac{d_h}{d_w}-\alpha\right)\frac{d_w}{d_h}}.$$

The above series is bounded if $1-\frac{\alpha d_w}{d_h}<-1$, i.e., $\alpha>2d_h/d_w$. Hence (17) holds.

Now fix $\alpha>2d_h/d_w$. Then (17) holds for any $\alpha'\in(2d_h/d_w,\alpha)$ and any $\varepsilon>0$. Equivalently, there exists $R>0$ such that

$$\mathbb{P}\left(\bar{X}_s^m\notin\overline{B_{-\alpha'}(0,R)}\right)\le\varepsilon,$$

where $\overline{B_{-\alpha'}(0,R)}$ denotes the closed ball with radius R and center 0 in $H^{-\alpha'}$. To conclude the proof, it suffices to show that $\overline{B_{-\alpha'}(0,R)}$ is compact in $H^{-\alpha}$. Indeed, this can be seen from Rellich's theorem, i.e., the embedding $H^\alpha\hookrightarrow H^\beta$ is compact for $\beta<\alpha$, see the proof of [7, Theorem 3.15].

□

Proposition 4.10. *For any $f\in\mathcal{S}(K)$, one has $\bar{X}_s^m(f)\longrightarrow X_s(f)$ as $m\to\infty$.*

Proof. Recall that $\bar{X}_s^m(f)=X_s^m(\bar{f}_m)$. Since fractional Gaussian fields are centered, it suffices to show that as $m\to\infty$,

$$\mathbb{E}\left(\left(\bar{X}_s^m(f)\right)^2\right)\longrightarrow\int_K f(-\Delta)^{-2s}f\,d\mu.$$

We will use similar proof as [8, Proposition 4.5] for which Lemma 2.1 is a crucial ingredient.

First observe that by (7), one has

$$\mathbb{E}\left(\left(\bar{X}_s^m(f)\right)^2\right)=\mathbb{E}\left(\left(X_s^m(\bar{f}_m)\right)^2\right)=\frac{1}{a_m}\sum_{p\in V_m}\bar{f}_m(p)(-\Delta_m)^{-2s}\bar{f}_m(p).$$

Hence it remains to prove that

$$\frac{1}{a_m}\sum_{p\in V_m}\bar{f}_m(p)(-\Delta_m)^{-2s}\bar{f}_m(p)\longrightarrow\int_K f(-\Delta)^{-2s}f\,d\mu.$$

Recall the convergence (16) with the notation in Remark 4.2, i.e., for $f_m=f|_{V_m}$,

$$\int_{V_m}f_m(-\Delta_m)^{-2s}f_m\,d\mu_m\longrightarrow\int_K f(-\Delta)^{-2s}f\,d\mu.$$

We thus need to show that

$$\int_{V_m} \bar{f}_m(-\Delta_m)^{-2s}\bar{f}_m d\mu_m - \int_{V_m} f_m(-\Delta_m)^{-2s} f_m d\mu_m \longrightarrow 0.$$

Indeed, the triangular inequality and Cauchy-Schwarz inequality yield

$$\begin{aligned}
&\left|\int_{V_m} \bar{f}_m(-\Delta_m)^{-2s}\bar{f}_m d\mu_m - \int_{V_m} f_m(-\Delta_m)^{-2s} f_m d\mu_m\right| \\
\le &\int_{V_m} \left|\left(\bar{f}_m - f_m\right)(-\Delta_m)^{-2s} f_m\right| d\mu_m + \int_{V_m} \left|\bar{f}_m(-\Delta_m)^{-2s}(f_m - \bar{f}_m)\right| d\mu_m \\
\le &\|\bar{f}_m - f_m\|_{L^2(V_m,\mu_m)} \left\|(-\Delta_m)^{-2s} f_m\right\|_{L^2(V_m,\mu_m)} + \|\bar{f}_m\|_{L^2(V_m,\mu_m)} \left\|(-\Delta_m)^{-2s}(f_m - \bar{f}_m)\right\|_{L^2(V_m,\mu_m)}.
\end{aligned}$$

One has then $\left\|(-\Delta_m)^{-2s} f_m\right\|_{L^2(V_m,\mu_m)} \le (\lambda_1^m)^{-2s}\|f_m\|_{L^2(V_m,\mu_m)}$ and

$$\left\|(-\Delta_m)^{-2s}(f_m - \bar{f}_m)\right\|_{L^2(\mu_m)} \le (\lambda_1^m)^{-2s}\|f_m - \bar{f}_m\|_{L^2(\mu_m)}.$$

Note that from Lemma 4.1, $\|f_m\|^2_{L^2(V_m,\mu_m)} \to \|f\|^2_{L^2(K,\mu)}$. Recall also $\inf_m \lambda_1^m > 0$. It remains to show that $\|f_m - \bar{f}_m\|_{L^2(V_m,\mu_m)} \to 0$. By Lemma 2.1,

$$\begin{aligned}
\|f_m - \bar{f}_m\|^2_{L^2(V_m,\mu_m)} &= \frac{1}{a_m}\sum_{p\in V_m} |f_m(p) - \bar{f}_m(p)|^2 \\
&\le \frac{1}{a_m}\sum_{p\in V_m}\left(\fint_{C_p^m} |f_m(p) - f(x)| d\mu(x)\right)^2 \\
&\le C 2^{-2(m+1)(d_w-d_h)}\|\Delta f\|^2_{L^1(K,\mu)}.
\end{aligned}$$

Now combining the above estimates and letting $m \to \infty$, we conclude the desired result. □

Proof of Theorem 4.7. Since $(\bar{X}_s^m)_{m\ge 1}$ is tight in $H^{-\alpha}(K)$ for any $\alpha > 2d_h/d_w$, it is enough to show that every convergent subsequence $(\bar{X}_s^{m_k})_{k\ge 1}$ converges in law to X_s in $H^{-\alpha}(K)$, that is, $\bar{X}_s^{m_k}(f) \to X_s(f)$ as $k \to \infty$ for all $f \in H^\alpha(K)$.

Indeed, let $f \in H^\alpha(K)$, then there exists a sequence $(f_i)_{i\ge 1} \in \mathcal{S}$ such that $f_i \to f$ in $H^\alpha(K)$ and thus $(\bar{f}_i)_{m_k} \to \bar{f}_{m_k}$ as $i \to \infty$. Therefore $\bar{X}_s^{m_k}(f_i)$ and $X_s(f_i)$ converge to $\bar{X}_s^{m_k}(f)$ and $X_s(f)$ respectively as i goes to infinity. Recall also $\bar{X}_s^{m_k}(f_i) \to \bar{X}_s(f_i)$ as $k \to \infty$. The triangle inequality thus concludes our proof.

□

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