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Deformation theory of perfect complexes and traces

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We show that the deformation theory of a perfect complex and that of its determinant are related by the trace map, in a general setting of sheaves on a site. The key technical step, in passing from the setting of modules over a ring where one has global resolutions to the general setting, is achieved using K-theory and higher category theory.

1. Introduction

1.1. Our goal in this article is to prove the following piece of folklore.

Theorem 1.2 (folk theorem, informally stated). *If E is a perfect complex on an algebraic stack X with a first-order thickening $X \hookrightarrow X'$, then the trace of the obstruction class of E, with respect to the thickening, is the obstruction class for the determinant* det E *of E. Moreover, the trace map from the torsor of deformations of E to the torsor of deformations of* det E *coincides with the determinant map.*

This is a geometric generalization of something familiar from a first multivariable analysis course: the derivative of the determinant map on the space of matrices is the trace function.

In our earlier work [Honigs et al. 2021; Lieblich and Olsson 2015; 2017], we needed this result in a level of generality not explicitly available in the literature. Indeed, as we discuss in 1.7, there are many incarnations of Theorem 1.2, in the context of both classical and derived algebraic geometry. However, no source of which we are aware treats the crucial case of a scheme over a mixed characteristic base ring, or a gerbe over such a scheme. Some of the classical arguments written over the complex numbers generalize easily to our needs, while others do not. Moreover, the literature discussing this result in derived algebraic geometry generally starts with a blanket characteristic 0 assumption, and it is not apparent to us which results generalize as written. At the very least, the literature as written is inadequate for the applications that presently exist.

In this article we prove this folk theorem for deformations of perfect complexes in a ringed topos, which suffices for all applications of which we know.

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1.3. Let $\boldsymbol{S}$ be a site and let $\mathcal{O}' \to \mathcal{O}$ be a surjective morphism of rings on $\boldsymbol{S}$ with square-zero kernel K.

Let $E \in D^b(\mathcal{O})$ be a perfect complex of $\mathcal{O}$-modules on $\boldsymbol{S}$ [Stacks, Tag 08G4]. A deformation of E to $\mathcal{O}'$ is a pair (E', σ), where $E' \in D^-(\mathcal{O}')$ is a complex and $\sigma : E' \otimes^{\boldsymbol{L}}_{\mathcal{O}'} \mathcal{O} \xrightarrow{\sim} E$ is an isomorphism in $D(\mathcal{O})$. Such a complex E' is automatically perfect (see, for example, [Lieblich 2006, Lemma 3.2.4]). It is well-known, and documented in various levels of generality in the literature, that the following then hold:

(i) There is an obstruction $\omega(E) \in \mathrm{Ext}^2_{\mathcal{O}}(E, E \otimes^{\boldsymbol{L}}_{\mathcal{O}} K)$ which vanishes if and only if there exists a deformation of E to $\mathcal{O}'$.

(ii) If $\omega(E) = 0$ then the set of deformations of E to $\mathcal{O}'$ form a torsor under $\mathrm{Ext}^1_{\mathcal{O}}(E, E \otimes^{\boldsymbol{L}}_{\mathcal{O}} K)$. We denote this action by

$$(E', \sigma) \mapsto \alpha * (E', \sigma)$$

for a deformation (E', σ) of E and a class $\alpha \in \mathrm{Ext}^1_{\mathcal{O}}(E, E \otimes^{\boldsymbol{L}}_{\mathcal{O}} K)$.

(iii) If furthermore $\mathrm{Ext}^{-1}_{\mathcal{O}}(E, E) = 0$ then the set of automorphisms of any deformation of E to $\mathcal{O}'$ is canonically isomorphic to $\mathrm{Ext}^0_{\mathcal{O}}(E, E \otimes^{\boldsymbol{L}}_{\mathcal{O}} K)$.

The purpose of this article is to elucidate the compatibility of these three facts with traces. In the course of the article we also review the construction of the obstruction $\omega(E)$ in the required degree of generality, as well as points (ii) and (iii). We furthermore explain how to modify (iii) in the case when the vanishing of negative Ext-groups does not hold.

Remark 1.4. One can generalize the definition of the obstruction to bounded above complexes, which are not necessarily perfect. This can be done using the construction of Gabber, discussed in Section 9.14, or in the more general context of spectral algebraic geometry as discussed in [Lurie 2018, §16.2].

1.5. For a perfect complex E as above we can consider its determinant $\det(E)$, which is an invertible $\mathcal{O}$-module. This is again documented in various levels of generality in the literature, for example in [Knudsen 2002]. We explain in this article how to define the determinant in our general setting. On the other hand we can also consider the trace map

$$\mathrm{tr} : \mathrm{Ext}^i_{\mathcal{O}}(E, E \otimes^{\boldsymbol{L}}_{\mathcal{O}} K) \to H^i(\boldsymbol{S}, K),$$

defined in [Illusie 1971, Chapter V, (3.7.3)]. The main result of this article is the compatibility of determinants and traces in the following sense:

Theorem 1.6. (i) $\mathrm{tr}(\omega(E)) = \omega(\det(E))$ *in* $H^2(\boldsymbol{S}, K)$.

(ii) *If* (E', σ) *is a deformation of* E *and* $\alpha \in \mathrm{Ext}^1_{\mathbb{O}}(E, E \otimes^{\boldsymbol{L}}_{\mathbb{O}} K)$ *is a class, then*

$$\det(\alpha * (E', \sigma)) = \mathrm{tr}(\alpha) * (\det(E'), \det(\sigma)).$$

(iii) *If furthermore we have* $\mathrm{Ext}^{-1}(E, E) = 0$, *then for a deformation* (E', σ) *the map on automorphism groups*

$$\mathrm{Ext}^0_{\mathbb{O}}(E, E \otimes^{\boldsymbol{L}}_{\mathbb{O}} K) \to \mathrm{Ext}^0_{\mathbb{O}}(\det(E), \det(E) \otimes^{\boldsymbol{L}}_{\mathbb{O}} K) \simeq H^0(\boldsymbol{S}, K)$$

induced by the determinant agrees with the trace map.

1.7. This compatibility seems to be well-known to experts and appears in the literature in the case of complexes of coherent sheaves admitting a global resolution in [Huybrechts and Thomas 2010; Langholf 2013; Thomas 2000]. The case of perfect complexes on quasiprojective schemes over a field of characteristic 0 (which themselves always admit global resolutions) also appears in [Schürg et al. 2015, Proposition 3.2]. As we explain in Section 11, these cases of complexes admitting global resolutions also follow from the additivity of traces for morphisms in the filtered derived category [Illusie 1971, Chapter V, 3.7.7]. The results in [Gaitsgory and Rozenblyum 2017, Chapter 7, §3.3], which concern the cotangent complex of the stack of perfect complexes for derived schemes over fields of characteristic 0, are also closely related to the work in this article. Our approach in this article is close in spirit to [Gaitsgory and Rozenblyum 2017; Schürg et al. 2015].

1.8. Fundamentally, Theorem 1.6 is a reflection of a more basic statement in the context of "formal moduli problems" in the sense of [Lurie 2018, Chapter IV]. Both perfect complexes and line bundles form such moduli problems and the determinant map defines a morphism between them for which the induced map on tangent complexes is the trace map. While we do not use the language of formal moduli problems, this framework captures the approach taken here.

There are two main issues to be dealt with in proving Theorem 1.6. The first is the definition of the determinant of a perfect complex. In classical treatments, such as [Knudsen 2002], one presents a complex locally using a resolution, takes the alternating tensor product of the determinants of the sheaves in the complex, and then has to argue that this globalizes and enjoys various good properties. This approach to defining the determinant is difficult to work with in the general context of this article. The second issue is that the deformation problem we are concerned with is fundamentally higher-categorical in nature. We should consider not only complexes and isomorphisms between them, but homotopies and higher homotopies. Both these issues are addressed by considering the problem from an ∞-categorical perspective.

1.9. In order to understand the obstruction class $\omega(E)$ we employ the following basic idea, discussed at length in [Lurie 2018, Proposition 0.1.3.5 and surrounding text]. For a map of $\mathscr{O}$-modules $\rho : K \to J$ one can pushout $\mathscr{O}'$ along ρ to get a new surjection

$$\mathscr{O}'_\rho \to \mathscr{O}$$

with kernel J, and equipped with a morphism $\mathscr{O}' \to \mathscr{O}'_\rho$. If we could find an inclusion $\rho : K \hookrightarrow J$ of K into an injective J such that E lifts to a perfect complex E'_ρ over $\mathscr{O}'_\rho$, then the obstruction class can be understood as follows. The pushout of $\mathscr{O}'$ along the composition

$$K \to J \to J/K$$

is isomorphic to $\mathscr{O}[J/K]$ (the ring of dual numbers on J/K), and we get by pushing out E'_ρ along $\mathscr{O}'_\rho \to \mathscr{O}[J/K]$ a class in

$$\mathrm{Ext}^1_{\mathscr{O}}(E, E \otimes^{\boldsymbol{L}} J/K).$$

The image of this class under the boundary map

$$\mathrm{Ext}^1_{\mathscr{O}}(E, E \otimes^{\boldsymbol{L}} J/K) \to \mathrm{Ext}^2_{\mathscr{O}}(E, E \otimes^{\boldsymbol{L}} K)$$

arising from the short exact sequence

$$0 \to K \to J \to J/K \to 0$$

is then the obstruction $\omega(E)$. Unfortunately it is not always possible to find such an inclusion ρ. However, we can always choose an inclusion $K \hookrightarrow J$ into an injective $\mathscr{O}$-module and consider the induced inclusion

$$K \hookrightarrow I := (J \xrightarrow{\mathrm{id}} J),$$

where the complex I on the right is concentrated in degrees -1 and 0.

Applying the Dold–Kan correspondence to I we obtain an inclusion of simplicial $\mathscr{O}$-modules $K \hookrightarrow I_\bullet$, and we can form the pushout of $\mathscr{O}'$ along $K \to I_\bullet$ in the category of simplicial rings. This leads us to consider perfect complexes over simplicial rings and their determinants, which is the context for our discussion of the determinant.

1.10. The article is organized as follows.

In Section 2 we review the basic definitions pertaining to the ∞-category of modules over a sheaf of simplicial rings. We then explain how to understand the fiber, in the sense of ∞-categories, of the reduction functor obtained from a surjection of simplicial algebras with square-zero kernel, such as that which arises in our deformation problem for complexes.

Section 3 contains a brief review of the various approaches to K-theory and comparisons between them that play a role in this article.

Sections 4 and 5 are devoted to a discussion of the determinant functor from perfect complexes to a suitable Picard category of line bundles. While classically one defines the determinant using resolutions and gluing, the ∞-categorical approach to the determinant is easier to work with in our context (in fact, we do not know how to define the determinant in the necessary generality without it). The key point is that the (connective) K-theory of the category of perfect complexes is realized as the universal map to a grouplike $\boldsymbol{E}_\infty$-monoid from the $\boldsymbol{E}_\infty$-monoid of projective modules. We can then use variants of Quillen's plus construction to describe the K-theory of perfect complexes in more explicit ways that allow us to define the determinant directly. We then globalize the discussion by taking global sections in the ∞-categorical sense (homotopy limits).

As pointed out to us by Bhargav Bhatt, the construction of the determinant used in this article also enables us to prove a compatibility with ring structure on K-theory. We explain this in Section 6. The reader may wish to omit this section as it is not used in the rest of the article.

Section 7 gives a description of the trace map of [Illusie 1971, Chapter V, (3.7.3)] from an ∞-categorical perspective, which plays a role in comparing it with the determinant map.

In Section 8 we prove the fundamental compatibility of the determinant and trace maps. The main result is Proposition 8.9. In Sections 9 and 10 we then apply the theory to the deformation theory of perfect complexes and prove the theorems discussed in this introduction. In Section 9 we also verify the equivalence of the definition of the obstruction to deforming a perfect complex to one due to Gabber.

Finally in Section 11 we give an alternate proof of Theorem 1.6(i) in the case when one has global resolutions. The approach in this section does not use ∞-categories but rather the filtered derived category and the compatibility of the trace map with passing to the associated graded proven in [Illusie 1971, Chapter V, Corollary 3.7.7].

We have also included an Appendix wherein we establish the basic relationship between sheaves of dg-modules over the normalization of a simplicial ring and the ∞-category of modules in the sense of Section 2. This result seems well-known to experts but we include it here for lack of a suitable reference.

1.11. ***Conventions.*** We use the language of ∞-categories as developed by Lurie [2009; 2017; 2018], and differential graded (dg) categories as in [Toën 2007].

We often pass from a stable ∞-category $\mathscr{C}$ to its underlying ∞-groupoid, which we denote by $\mathscr{C}^{\simeq}$. This is the ∞-category obtained from $\mathscr{C}$ by considering only morphisms which induce isomorphisms in the homotopy category (see [Lurie 2009, Proposition 1.2.5.3]).

Throughout this article all simplicial rings considered are commutative, and we usually omit the adjective "commutative".

For a ring A and simplicial A-module $I_\bullet$ we often consider the simplicial ring $A[I_\bullet]$ of dual numbers given by $A \oplus I_\bullet$ and multiplication

$$(a, i) \cdot (b, j) = (ab, aj + bi).$$

There is a surjection $\pi : A[I_\bullet] \to A$ sending $I_\bullet$ to 0, and a retraction $A \to A[I_\bullet]$ given by $a \mapsto (a, 0)$.

We write Sp for the ∞-category of spectra and $\mathrm{Sp}^{\geq 0}$ for the ∞-category of connective spectra (see [Lurie 2018, Construction 0.2.3.10 and Definition 0.2.3.12]).

2. The ∞-category of perfect complexes

2.1. ***Animated rings.*** For a commutative ring R we follow [Lurie 2018, Definition 25.1.1.1] and consider an ∞-category CAlg^{Δ}_R. In [Lurie 2018] this is referred to as the ∞-category of simplicial commutative rings, but we prefer to reserve this term for simplicial commutative rings in the classical sense, and use the terminology of [Česnavičius and Scholze 2019, Example 5.1.6(3)] and refer to CAlg^{Δ}_R as the ∞-category of *animated rings*. As noted in [loc. cit.] the ∞-category CAlg^{Δ}_R can be viewed as the ∞-category obtained from simplicial commutative R-algebras by inverting weak equivalences. The category of animated rings can also be viewed as the ∞-category obtained by starting with the category of simplicial commutative R-algebras, endowing this category with the simplicial model category structure described in [Lurie 2009, Proposition 5.5.9.1], and applying the nerve to the subcategory of cofibrant-fibrant objects [Lurie 2018, Remark 25.1.1.3]. Because of these descriptions of CAlg^{Δ}_R, we use simplicial notation in describing objects of CAlg^{Δ}_R (e.g., $A_\bullet \in \mathrm{CAlg}^{\Delta}_R$).

2.2. ***Modules over animated rings.*** For an animated R-algebra $A_\bullet \in \mathrm{CAlg}^{\Delta}_R$ we have the associated stable ∞-category of R-modules [Lurie 2018, Notation 25.2.1.1], which we denote by $\mathfrak{D}(A_\bullet)$ (in [loc. cit.] this category is denoted $\mathrm{Mod}_{A_\bullet}$).

By [Illusie 1971, Chapter I, 3.1.3], for a simplicial R-algebra $A_\bullet$ the normalization $N(A_\bullet)$ is a strictly commutative differential graded algebra, and normalization defines a functor from $A_\bullet$-modules to differential graded $N(A_\bullet)$-modules. As noted in [Kerz et al. 2018, Remark 2.5], by an argument similar to [Lurie 2017, proof of Theorem 7.1.2.13], this defines an equivalence between $\mathfrak{D}(A_\bullet)$ and the ∞-category obtained from the category of dg-modules over $N(A_\bullet)$ by inverting quasi-isomorphisms (this is the approach taken for example in [Toën 2014, §3.1]; see also [Shipley 2007, Theorem 1.1]).

2.3. ***Topology.*** Let Λ be a ring and let $\boldsymbol{S}$ be a site.

2.4. As in [Lurie 2018, §1.3.5] we can consider sheaves of animated Λ-algebras, defined as sheaves on $\boldsymbol{S}$ taking values in the ∞-category $\mathrm{CAlg}^{\Delta}_\Lambda$. For a sheaf of animated Λ-algebras $A_\bullet$ we can consider, as in [Lurie 2018, Definition 2.1.0.1], the associated sheaf of $\boldsymbol{E}_\infty$-algebras and the associated module category, which we denote $\mathrm{Mod}_{(S,A_\bullet)}$. If $A_\bullet$ is a simplicial object in the category of sheaves of Λ-algebras then we also denote by $\mathrm{Mod}_{(S,A_\bullet)}$ the module category of the associated sheaf of $\boldsymbol{E}_\infty$-algebras.

As discussed in the Appendix, for a simplicial sheaf of Λ-algebras $A_\bullet$ we can also consider its normalized complex $N(A_\bullet)$, which is a sheaf of strictly commutative differential graded algebras, and its associated category of sheaves of differential graded modules $\mathrm{Mod}^{\mathrm{dg}}_{(S,N(A_\bullet))}$, viewed as a model category with the flat model category structure. This is a differential graded category and by the general construction of [Lurie 2017, Construction 1.3.1.6] we get an ∞-category

$$\mathscr{D}(S, A_\bullet) := N_{\mathrm{dg}}(\mathrm{Mod}^{\mathrm{dg},\circ}_{(S,N(A_\bullet))}),$$

where $\mathrm{Mod}^{\mathrm{dg},\circ}_{(S,N(A_\bullet))} \subset \mathrm{Mod}^{\mathrm{dg}}_{(S,N(A_\bullet))}$ denotes the subcategory of fibrant-cofibrant objects. We write $D(S, A_\bullet)$ for the associated homotopy category. As noted in Theorem A.9 (in the case of a discrete ring A this is [Lurie 2018, Corollary 2.1.2.3]) the ∞-category $\mathscr{D}(S, A_\bullet)$ is naturally identified with the hypercomplete objects in $\mathrm{Mod}_{(S,A_\bullet)}$. For our purposes studying perfect complexes, this distinction between $\mathscr{D}(S, A_\bullet)$ and $\mathrm{Mod}_{(S,A_\bullet)}$ is not important, and it is a matter of preference as to which ∞-categorical version of the derived category one works with.

Remark 2.5. For two objects $M, N \in \mathrm{Mod}^{\mathrm{dg},\circ}_{(S,N(A_\bullet))}$ defining objects of $\mathscr{D}(S, A_\bullet)$, a description of the mapping space

$$\mathrm{Map}_{\mathscr{D}(S,A_\bullet)}(M, N)$$

is provided by [Lurie 2017, Remark 1.3.1.12], which shows that

$$\mathrm{Map}_{\mathscr{D}(S,A_\bullet)}(M, N) \simeq \mathrm{DK}(\tau_{\leq 0}\,\mathrm{Hom}^\bullet_{N(A_\bullet)}(M, N)),$$

where on the right we consider truncation of the mapping complex followed by the Dold–Kan functor (see for example [Lurie 2017, Construction 1.2.3.5]). Note furthermore that since any object of $\mathrm{Mod}^{\mathrm{dg},\circ}_{(S,N(A_\bullet))}$ is fibrant the complex $\mathrm{Hom}^\bullet_{N(A_\bullet)}(M, N)$ is calculating the internal Hom-complex in the homotopy category $\mathrm{Ho}(\mathrm{Mod}^{\mathrm{dg}}_{(S,N(A_\bullet))})$.

2.6. *Deformations.*

2.7. For a simplicial sheaf of Λ-algebras $A_\bullet$ we can also (by Lemma A.5) describe $\mathscr{D}(S, A_\bullet)$ as the ∞-category

$$N_{\mathrm{dg}}(\mathrm{Mod}^{\mathrm{dg,cof}}_{(S,N(A_\bullet))})[W^{-1}],$$

obtained by localizing the dg-nerve of cofibrant objects in $\mathrm{Mod}^{\mathrm{dg}}_{(S,N(A_\bullet))}$ along weak equivalences.

2.8. If $A_\bullet \to B_\bullet$ is a map of sheaves of simplicial Λ-algebras then there is an induced functor

$$B_\bullet \otimes^{\boldsymbol{L}}_{A_\bullet} (-) : \mathcal{D}(\boldsymbol{S}, A_\bullet) \to \mathcal{D}(\boldsymbol{S}, B_\bullet).$$

This functor is induced by the tensor product

$$N(B_\bullet) \otimes_{N(A_\bullet)} (-) : \mathrm{Mod}^{\mathrm{cof}}_{(\boldsymbol{S},N(A_\bullet))} \to \mathrm{Mod}^{\mathrm{cof}}_{(\boldsymbol{S},N(B_\bullet))}.$$

To prove that this gives a well-defined functor on localizations, we must show that if $a : M \to N$ is an equivalence in $\mathrm{Mod}^{\mathrm{cof}}_{(\boldsymbol{S},N(B_\bullet))}$ then

$$N(B_\bullet) \otimes_{N(A_\bullet)} M \to N(B_\bullet) \otimes_{N(A_\bullet)} N$$

is an equivalence in $\mathrm{Mod}^{\mathrm{cof}}_{(\boldsymbol{S},N(B_\bullet))}$. For this it suffices to show that the adjoint forgetful functor

$$\mathrm{Mod}_{(\boldsymbol{S},N(B_\bullet))} \to \mathrm{Mod}_{(\boldsymbol{S},N(A_\bullet))}$$

preserves fibrations and trivial fibrations — this is immediate from the definitions. Note also that this functor induces the usual derived tensor product on the homotopy categories.

2.9. Let $A'_\bullet \to A_\bullet$ be a surjective map of simplicial Λ-algebras with kernel $I_\bullet$ satisfying $I_\bullet^2 = 0$. Let $E \in \mathcal{D}(\boldsymbol{S}, A_\bullet)$ be an object. We denote by $\mathrm{Def}_\infty(E)$ the homotopy fiber product of the diagram

$$\begin{array}{ccc} & & \mathcal{D}(\boldsymbol{S}, A'_\bullet) \\ & & \downarrow {\scriptstyle A_\bullet \otimes^{\boldsymbol{L}}_{A'_\bullet}(-)} \\ \star & \xrightarrow{E} & \mathcal{D}(\boldsymbol{S}, A_\bullet) \end{array} \tag{2.10}$$

2.11. Let $\mathrm{Def}(E)$ denote the category whose objects are pairs (E', σ), where $E' \in D(\boldsymbol{S}, A'_\bullet)$ is an object and

$$\sigma : E' \otimes^{\boldsymbol{L}}_{N(A'_\bullet)} N(A_\bullet) \to E$$

is an isomorphism in $D(\boldsymbol{S}, A_\bullet)$. A morphism

$$q : (E'_1, \sigma_1) \to (E'_2, \sigma_2) \tag{2.12}$$

in $\mathrm{Def}(E)$ is given by a morphism $\rho : E'_1 \to E'_2$ in $D(\boldsymbol{S}, A'_\bullet)$ such that the diagram in $D(\boldsymbol{S}, A_\bullet)$

$$\begin{array}{ccc} E'_1 \otimes^{\boldsymbol{L}}_{N(A'_\bullet)} N(A_\bullet) & \xrightarrow{\rho} & E'_2 \otimes^{\boldsymbol{L}}_{N(A'_\bullet)} N(A_\bullet) \\ & {\scriptstyle\sigma_1} \searrow \quad \swarrow {\scriptstyle\sigma_2} & \\ & E & \end{array}$$

commutes.

There is a natural map

$$\mathrm{Ho}(\mathrm{Def}_\infty(E)) \to \mathrm{Def}(E). \tag{2.13}$$

Indeed the category $\mathrm{Def}(E)$ is the categorical fiber product of the diagram

$$\begin{array}{ccc} & & \mathrm{Ho}(\mathscr{D}(\boldsymbol{S}, A'_\bullet)) \\ & & \Big\downarrow{\scriptstyle A_\bullet \otimes^{\boldsymbol{L}}_{A'_\bullet}(-)} \\ \star & \xrightarrow{E} & \mathrm{Ho}(\mathscr{D}(\boldsymbol{S}, A_\bullet)) \end{array} \tag{2.14}$$

Now by general adjunction properties of passing to the homotopy category [Lurie 2009, Proposition 1.2.3.1], the diagram (2.10) maps to the diagram obtained by applying the nerve to (2.14). By passing to the homotopy categories of the associated homotopy fibers we get the map (2.13).

We can understand the ∞-category $\mathrm{Def}_\infty(E)$ and its relationship with $\mathrm{Def}(E)$ as follows.

2.15. Note first of all that $\mathrm{Def}_\infty(E)$ is a groupoid in the sense of [Lurie 2009, §1.2.5] (that is, its homotopy category is a groupoid). This follows from observing that if $E' \in \mathscr{D}(\boldsymbol{S}, A'_\bullet)$ is an object with an equivalence $\sigma : E' \otimes^{\boldsymbol{L}}_{N(A'_\bullet)} N(A_\bullet) \to E$ then tensoring the sequence of $N(A'_\bullet)$-modules

$$0 \to N(I_\bullet) \to N(A'_\bullet) \to N(A_\bullet) \to 0$$

with E' we get a distinguished triangle

$$E \otimes^{\boldsymbol{L}}_{N(A_\bullet)} N(I_\bullet) \to E' \to E \to E \otimes^{\boldsymbol{L}}_{N(A_\bullet)} N(I_\bullet)[1]$$

in the triangulated category $D(\boldsymbol{S}, A'_\bullet) := \mathrm{Ho}(\mathscr{D}(\boldsymbol{S}, A'_\bullet))$.

It follows that for a morphism

$$\rho : (E'_1, \sigma_1) \to (E'_2, \sigma_2)$$

of pairs (that is, a morphism in $\mathscr{D}(\boldsymbol{S}, A'_\bullet)$ compatible with the identifications with E in $\mathscr{D}(\boldsymbol{S}, A_\bullet)$) we get an induced morphism of distinguished triangles in $D(\boldsymbol{S}, A'_\bullet)$

$$\begin{array}{ccccccc} E \otimes^{\boldsymbol{L}}_{N(A_\bullet)} N(I_\bullet) & \longrightarrow & E'_1 & \longrightarrow & E & \longrightarrow & E \otimes^{\boldsymbol{L}}_{N(A_\bullet)} N(I_\bullet)[1] \\ \Big\| & & \Big\downarrow{\scriptstyle\rho} & & \Big\| & & \Big\| \\ E \otimes^{\boldsymbol{L}}_{N(A_\bullet)} N(I_\bullet) & \longrightarrow & E'_2 & \longrightarrow & E & \longrightarrow & E \otimes^{\boldsymbol{L}}_{N(A_\bullet)} N(I_\bullet)[1] \end{array}$$

and therefore ρ is an equivalence.

It follows that $\mathrm{Def}_\infty(E)$ can also be described as the fiber product of the underlying ∞-groupoids

$$\begin{array}{ccc} & & \mathfrak{D}(\boldsymbol{S}, A'_\bullet)^{\simeq} \\ & & \big\downarrow {\scriptstyle A_\bullet \otimes^{\boldsymbol{L}}_{A'_\bullet}(-)} \\ \star & \xrightarrow{E} & \mathfrak{D}(\boldsymbol{S}, A_\bullet)^{\simeq} \end{array} \tag{2.16}$$

From this, and looking at the associated long exact sequences of homotopy groups associated to (2.14) and (2.16) one also gets that (2.12) induces a bijection on isomorphism classes of objects, so we can think of objects of $\mathrm{Def}_\infty(E)$ as pairs (E', ρ) as above.

Remark 2.17. The above argument shows, in fact, that a morphism in $\mathfrak{D}(\boldsymbol{S}, A'_\bullet)$ is an equivalence if and only if its image in $\mathfrak{D}(\boldsymbol{S}, A_\bullet)$ is an equivalence.

2.18. This remark implies that for an object $(E', \rho) \in \mathrm{Def}_\infty(E)$ the derived automorphism group

$$\mathrm{Aut}^\infty(E', \rho) = \Omega_{(E',\rho)}(\mathrm{Def}_\infty(E))$$

can be described as the homotopy fiber over 0 of the map

$$\mathrm{DK}(\tau_{\leq 0}\,\mathrm{RHom}_{N(A'_\bullet)}(E', E')) \to \mathrm{DK}(\tau_{\leq 0}\,\mathrm{RHom}_{N(A_\bullet)}(E^\bullet, E^\bullet)).$$

Applying $\mathrm{RHom}_{N(A'_\bullet)}(E', -)$ to the distinguished triangle

$$E \otimes^{\boldsymbol{L}}_{N(A_\bullet)} N(I_\bullet) \to E' \to E \to E \otimes^{\boldsymbol{L}}_{N(A_\bullet)} N(I_\bullet)[1]$$

we see that

$$\mathrm{Aut}^\infty(E', \rho) \simeq \mathrm{DK}(\tau_{\leq 0}\,\mathrm{RHom}_{N(A_\bullet)}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet))).$$

From this we conclude that the ∞-groupoid $\mathrm{Def}_\infty(E)$ is equivalent to

$$\mathrm{DK}\big(\tau_{\leq 0}\,\mathrm{RHom}_{N(A_\bullet)}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet))[1]\big) \times \pi_0(\mathrm{Def}(E)). \tag{2.19}$$

2.20. In the case when $\mathrm{Def}_\infty(E)$ is nonempty we can describe the set of isomorphism classes as follows. Fix one lifting $(E'_0, \rho_0) \in \mathrm{Def}(E)$, and define a map

$$\pi_0(\mathrm{Def}(E)) \to \mathrm{Ext}^1_{N(A_\bullet)}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet)) \tag{2.21}$$

by sending an object (E', ρ) to the image of $\rho^{-1} \circ \rho_0$ under the map

$$\begin{aligned}\mathrm{Hom}_{N(A_\bullet)}(E'_0 \otimes^{\boldsymbol{L}} N(A_\bullet), E' \otimes^{\boldsymbol{L}} N(A_\bullet)) \simeq \mathrm{Hom}_{N(A'_\bullet)}(E'_0, E' \otimes^{\boldsymbol{L}} N(A_\bullet)) \\ \to \mathrm{Ext}^1_{N(A_\bullet)}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet))\end{aligned}$$

obtained from the distinguished triangle

$$E \otimes^{\boldsymbol{L}} N(I_\bullet) \to E' \to E \to E \otimes^{\boldsymbol{L}} N(I_\bullet)[1]$$

by applying

$$\mathrm{Hom}_{N(A'_\bullet)}(E'_0, -).$$

The image of the class of (E', ρ) under (2.21) is by construction zero if and only if the morphism $\rho_0^{-1} \circ \iota$ lifts to $N(A'_\bullet)$, which implies that (2.21) is injective.

The map is also surjective. This can be seen as follows. For a class

$$\alpha \in \mathrm{Ext}^1_{N(A_\bullet)}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet))$$

we can represent α by a map of $N(A_\bullet)$-modules

$$\widetilde{\alpha} : E_0'^{\bullet} \otimes_{N(A'_\bullet)} N(A_\bullet) \to E \otimes N(I_\bullet)[1].$$

Viewing this as a map of $N(A'_\bullet)$-modules and taking cones we get a short exact sequence of $N(A'_\bullet)$-modules

$$0 \to E \otimes N(I_\bullet) \to T_\alpha \to E'_0 \otimes_{N(A'_\bullet)} N(A_\bullet) \to 0.$$

Taking the direct sum with E'_0 we get a short exact sequence

$$0 \to (E \otimes N(I_\bullet))^{\oplus 2} \to T_\alpha \oplus E'_0 \to E'_0 \otimes N(A_\bullet) \oplus E \to 0.$$

Pulling this back along the graph

$$E'_0 \otimes N(A_\bullet) \to E'_0 \otimes N(A_\bullet) \oplus E$$

of ρ and pushing out along the summation map

$$(E \otimes N(I_\bullet))^{\oplus 2} \to E \otimes N(I_\bullet)$$

we get an extension of $N(A'_\bullet)$-modules

$$0 \to E \otimes N(I_\bullet) \to E'_\alpha \to E'_0 \otimes N(A_\bullet) \to 0.$$

We leave it to the reader to check that this defines an object of $\mathrm{Def}(E)$ with class α.

2.22. Combining this with (2.19) we find that in the case when $\mathrm{Def}_\infty(E)$ is nonempty the pullback of (2.10) can be described as

$$\mathrm{DK}\big(\tau_{\leq 0}(\mathrm{Hom}_{N(A_\bullet)}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet))[1])\big).$$

2.23. In what follows we also consider the subcategory

$$\mathscr{D}_{\mathrm{perf}}(\boldsymbol{S}, A_\bullet) \subset \mathscr{D}(\boldsymbol{S}, A_\bullet)$$

of perfect $A_\bullet$-modules [Lurie 2017, Definition 7.2.4.1]. This is again a stable ∞-category.

We consider this as a symmetric monoidal stable ∞-category with the monoidal structure given by direct sums.

In the case when the site $\boldsymbol{S}$ is trivial (e.g., one object and one morphism) we write simply $\mathcal{D}(A_\bullet)$ for $\mathcal{D}(\boldsymbol{S}, A_\bullet)$ and $\mathcal{D}_{\mathrm{perf}}(A_\bullet)$ for $\mathcal{D}_{\mathrm{perf}}(\boldsymbol{S}, A_\bullet)$.

Remark 2.24. In many cases the notion of perfect complex coincides with the notion of dualizable object but we do not know the relationship between the two notions in general.[1]

3. Various descriptions of K-theory

In this section we summarize for the convenience of the reader a few basic approaches to and results about algebraic K-theory that we will need.

3.1. ***K-theory as group completion.*** In this approach to K-theory one starts with the ∞-category of $\boldsymbol{E}_\infty$-monoids and its subcategory of grouplike $\boldsymbol{E}_\infty$-monoids [Lurie 2017, Definition 5.2.6.6]. By [Lurie 2017, Remark 5.2.6.26], this ∞-subcategory is equivalent to the ∞-category of connective spectra $\mathrm{Sp}^{\geq 0}$. We can then consider the group completion functor

$$(\boldsymbol{E}_\infty\text{-monoids}) \to (\text{grouplike } \boldsymbol{E}_\infty\text{-monoids}) \simeq \mathrm{Sp}^{\geq 0}.$$

If $\mathcal{P}$ is a symmetric monoidal category the nerve of the underlying groupoid $\mathcal{P}^{\simeq}$ is an $\boldsymbol{E}_\infty$-monoid and the K-theory of $\mathcal{P}$, denoted $K(\mathcal{P})$, is defined as the associated group completion.

Example 3.2. For a ring R the category $\mathrm{Proj}(R)$ of projective R-modules is symmetric monoidal under $\oplus$ and $K(R)$ is defined to be the group completion of the nerve of the underlying groupoid of $\mathrm{Proj}(R)$.

3.3. ***K-theory as universal additive invariant.*** The main reference for this approach is [Blumberg et al. 2013]. Let $\mathrm{Cat}_\infty^{\mathrm{ex}}$ denote the ∞-category of small, idempotent complete, stable ∞-categories, with morphisms given by exact functors (see [Blumberg et al. 2013, Definition 2.12]). The main result of [Blumberg et al. 2013] is then that there is a universal "additive" invariant

$$U : \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathcal{M}_{\mathrm{add}},$$

where the target is again a presentable stable ∞-category. In fact, $\mathcal{M}_{\mathrm{add}}$ is monoidal with unit object $\mathbf{1}_{\mathcal{M}_{\mathrm{add}}}$ given by applying U to the compact objects in the stable ∞-category of spectra. Given an object $\mathcal{D} \in \mathrm{Cat}_\infty^{\mathrm{ex}}$ we can form the mapping spectrum (see [Blumberg et al. 2013, Definition 2.15])

$$\mathrm{Map}(\mathbf{1}_{\mathcal{M}_{\mathrm{add}}}, U(\mathcal{D})).$$

[1] This question was earlier asked by Daniel Bergh; see https://mathoverflow.net/questions/313318/are-dualizable-objects-in-the-derived-category-of-a-ringed-topos-perfect.

By [Blumberg et al. 2013, Theorem 7.13] this defines connective algebraic K-theory

$$K(\mathcal{D}) := \mathrm{Map}(\mathbf{1}_{\mathcal{M}_{\mathrm{add}}}, U(\mathcal{D})).$$

Because the functor U is monoidal we have an induced map of spectra

$$\mathrm{Map}_{\mathrm{Cat}_\infty^{\mathrm{ex}}}(\mathbf{1}_{\mathrm{Cat}_\infty^{\mathrm{ex}}}, \mathcal{D}) \to \mathrm{Map}(\mathbf{1}_{\mathcal{M}_{\mathrm{add}}}, U(\mathcal{D})).$$

This induces a map

$$\mathcal{D}^{\simeq} \to K(\mathcal{D}),$$

from the underlying ∞-groupoid $\mathcal{D}^{\simeq}$ of $\mathcal{D}$.

One can also describe the algebraic K-theory of a small, idempotent complete, stable ∞-category using an appropriate version of the Waldhausen construction. This is discussed in [Blumberg et al. 2013, §7.1 and §7.2]. See also [Barwick 2016]. This explicit construction makes the functoriality of K-theory clear, and in particular enables us to consider presheaves of small, idempotent complete, stable ∞-categories and their associated K-theory.

Example 3.4. If $A_\bullet$ is a simplicial ring then $K(A_\bullet)$ is defined to be the K-theory of the stable ∞-category $\mathcal{D}_{\mathrm{perf}}(A_\bullet)$ defined in 2.23.

If $A_\bullet = R$ is a ring, then this recovers the K-theory defined in Example 3.2. Namely, the inclusion $\mathrm{Proj}(R)^{\simeq} \hookrightarrow \mathcal{D}_{\mathrm{perf}}(R)^{\simeq}$ induces a monoidal map

$$\mathrm{Proj}(R)^{\simeq} \to K(\mathcal{D}_{\mathrm{perf}}(R))$$

(where the target is defined using the definition in [Blumberg et al. 2013]). By the universal property of group completion this induces a map

$$K(R) \to K(\mathcal{D}_{\mathrm{perf}}(R))$$

(where the left side is defined using group completion). That this map is an equivalence follows, for example, by comparison with Waldhausen K-theory.

3.5. ***K-theory and Picard groupoids.*** The relationship between K-theory and determinants in the setting of functors to Picard categories as developed in [Knudsen 2002] is discussed in [Muro et al. 2015].

The setting here is that of a Waldhausen category $\mathcal{W}$ [Waldhausen 1985, §1.2]. Let $w(\mathcal{W})$ denote the category with the same objects as $\mathcal{W}$ but morphisms the weak equivalences of $\mathcal{W}$, and let $\mathrm{cof}(\mathcal{W})$ denote the category whose objects are cofiber sequences

$$A \hookrightarrow B \twoheadrightarrow C$$

and whose morphisms are commutative diagrams

$$\begin{array}{ccccc} A_1 & \hookrightarrow & B_1 & \twoheadrightarrow & C_1 \\ \downarrow & & \downarrow & & \downarrow \\ A_2 & \hookrightarrow & B_2 & \twoheadrightarrow & C_2 \end{array}$$

where the vertical morphisms are weak equivalences. For a commutative Picard category $\mathcal{P}$ (see for example [Deligne 1987, §4]) a notion of determinant functor from $\mathcal{W}$ to $\mathcal{P}$ is defined in [Muro et al. 2015, §1.2]. This is a functor

$$\delta : w(\mathcal{W}) \to \mathcal{P}$$

together with an isomorphism σ between the two induced functors

$$\mathrm{cof}(\mathcal{W}) \to \mathcal{P}$$

given by

$$(A \hookrightarrow B \twoheadrightarrow C) \mapsto \delta(B)$$

and

$$(A \hookrightarrow B \twoheadrightarrow C) \mapsto \delta(A) +_{\mathcal{P}} \delta(C).$$

The data (δ, σ) are required to satisfy various natural compatibilities detailed in [Muro et al. 2015, Definition 1.2.3].

A Picard category $\mathcal{P}$ defines a grouplike $\boldsymbol{E}_\infty$-monoid. This is explained in [Bhatt and Scholze 2017, Construction 12.5 and Proposition 12.15]. By [Patel 2012, Theorem 5.3] this defines an equivalence between homotopy categories of Picard categories and 1-truncated connective spectra.

On the other hand, we can consider the 1-truncation $\tau_{\leq 1} K(\mathcal{W})$ of the Waldhausen K-theory of $\mathcal{W}$, which comes with a map

$$\delta : w(\mathcal{W}) \to \tau_{\leq 1} K(\mathcal{W}).$$

It is shown in [Muro et al. 2015, Theorem 1.6.3] that this functor has the structure of a universal determinant functor.

The relationship between this approach and the perspective on K-theory as a universal additive invariant is discussed in [Blumberg et al. 2013, §7.2].

We do not need to develop the full theory here. In order to have the appropriate functoriality, however, it is important to note that the comparison map is induced by an explicit map of ∞-categories in our context. Namely, let A be a ring and let $\mathrm{Perf}(A)$ denote the Waldhausen category of perfect complexes of A-modules, which is also a dg-category. Then there is an induced functor of ∞-categories

$$N(\mathrm{Perf}(A)) \to N_{\mathrm{dg}}(\mathrm{Perf}(A))$$

and the comparison between the two approaches is obtained by then applying the ∞-categorical version of the Waldhausen S-construction. In particular, the comparison map is functorial in A.

Remark 3.6. We have an inclusion $\mathrm{Proj}(A) \subset \mathrm{Perf}(A)$ of the category of projective modules, which induces an isomorphism on K-theory. It follows from this that restriction defines an equivalence of categories between the category of determinant functors on $\mathrm{Perf}(A)$ and determinant functors on $\mathrm{Proj}(A)$.

3.7. ***K-theory and left Kan extension.*** The K-theory of animated rings can be described using the K-theory of ordinary rings as a Kan extension from smooth $\boldsymbol{Z}$-algebras. This result is due to Bhatt and Lurie (see [Elmanto et al. 2020, Example A.0.6]).

Let $\mathrm{Alg}_{\boldsymbol{Z}}$ denote the category of commutative rings. Algebraic K-theory defines a functor

$$K : \mathrm{Alg}_{\boldsymbol{Z}} \to \mathrm{Sp}^{\geq 0}.$$

We can then consider the left Kan extension of this functor to get a functor

$$\mathrm{CAlg}^{\Delta}_{\boldsymbol{Z}} \to \mathrm{Sp}^{\geq 0}$$

from animated rings to spaces. The result of Bhatt and Lurie states that this gives K-theory of animated rings, defined as a universal additive invariant.

4. Determinants: punctual case

We are ultimately interested in studying determinants and traces for complexes on a general site, but as a first step in that direction we develop some preliminary material in this section in the case of the punctual site.

4.1. Let $A_\bullet$ be a simplicial ring with associated K-theory $K(A_\bullet)$.

Define $\mathrm{GL}_n(A_\bullet)$ to be the fiber product of simplicial monoids of the diagram

$$\begin{array}{ccc} & & M_n(A_\bullet) \\ & & \downarrow \\ \mathrm{GL}_n(\pi_0(A_\bullet)) & \longrightarrow & M_n(\pi_0(A_\bullet)) \end{array}$$

and let $\widehat{\mathrm{GL}}(A_\bullet)$ denote the colimit of the $\mathrm{GL}_n(A_\bullet)$ along the standard inclusions

$$\mathrm{GL}_n(A_\bullet) \hookrightarrow \mathrm{GL}_{n+1}(A_\bullet).$$

The free modules $A_\bullet^{\oplus n}$ define a map of $\boldsymbol{E}_\infty$-monoids

$$\coprod_{n\geq 0} \mathrm{BGL}_n(A_\bullet) \to \mathscr{D}_{\mathrm{perf}}(A_\bullet)^{\simeq}.$$

Here the monoidal structure on the left is induced by the natural isomorphisms

$$A_\bullet^{\oplus n} \oplus A_\bullet^{\oplus m} \simeq A^{\oplus(n+m)}.$$

Lemma 4.2. *The induced map on K-theory*

$$K\Big(\coprod_{n\geq 0} \mathrm{BGL}_n(A_\bullet)\Big) \to K(A_\bullet)$$

induces an isomorphism after Zariski sheafification on $\mathrm{Spec}(\pi_0(A_\bullet))$.

Remark 4.3. The notion of a sheaf taking values in the ∞-category of spaces is introduced in [Lurie 2009, Definition 6.2.2.6]. In [Lurie 2009, Proposition 6.2.2.7] it is shown that the ∞-category of sheaves can be viewed as a localization of the ∞-category of presheaves, which implies that the inclusion of sheaves into presheaves has a left adjoint — this is what we refer to as *sheafification*.

Proof. As in [Blumberg et al. 2013, Lemma 9.39] the infinite loop space associated to $K\big(\coprod_n \mathrm{BGL}_n(A_\bullet)\big)$ is isomorphic to

$$\boldsymbol{Z} \times \mathrm{B}\widehat{\mathrm{GL}}(A_\bullet)^+,$$

where $\mathrm{B}\widehat{\mathrm{GL}}(A_\bullet)^+$ denotes Quillen's plus construction. In particular, we get a map

$$\boldsymbol{Z} \times \mathrm{B}\widehat{\mathrm{GL}}(A_\bullet)^+ \to \boldsymbol{Z} \times_{K_0(A_\bullet)} K(A_\bullet).$$

By [Blumberg et al. 2013, Proposition 9.40] this map is an equivalence. To prove the lemma it therefore suffices to observe that every object of $K_0(A_\bullet) = K_0(\pi_0(A_\bullet))$ (isomorphism given as in [Kerz et al. 2018, Theorem 2.16]) is locally in the image of $\boldsymbol{Z}$, which is immediate. □

Remark 4.4. Note that the functors $\mathrm{GL}_n(-)$ and $\widehat{\mathrm{GL}}(-)$ reflect weak equivalence and therefore induce functors

$$\mathrm{CAlg}_{\boldsymbol{Z}}^{\Delta} \to (\boldsymbol{E}_\infty\text{-monoids}).$$

By [Elmanto et al. 2020, Proposition A.0.4] these functors are equal to the Kan extensions of their restrictions to smooth algebras.

4.5. Let $\mathscr{P}ic^{\boldsymbol{Z}}(A_\bullet)$ denote the Zariski sheafification of the grouplike $\boldsymbol{E}_\infty$-monoid given by

$$\boldsymbol{Z} \times \mathrm{BGL}_1(A_\bullet)$$

with monoidal structure given by addition on $\boldsymbol{Z}$ and multiplication on $\mathrm{BGL}_1(A_\bullet)$ but with signed commutativity constraint as in [Bhatt and Scholze 2017, Example 12.2(iii)].

There is a projection map

$$\mathscr{P}ic^{\boldsymbol{Z}}(A_\bullet) \to \boldsymbol{Z}$$

with fiber $\mathrm{BGL}_1(A_\bullet)$.

Here $\mathbf{Z}$ should be understood as the global sections of the Zariski sheaf associated to the constant sheaf on $\mathrm{Spec}(\pi_0(A_\bullet))$.

Note that with this definition $\mathcal{P}ic^{\mathbf{Z}}(A_\bullet)$ forms a Zariski sheaf on $\mathrm{Spec}(\pi_0(A_\bullet))$. One can also describe $\mathcal{P}ic^{\mathbf{Z}}(A_\bullet)$ as the Picard spectrum associated to the symmetric monoidal ∞-category of finitely generated projective $A_\bullet$-modules (see Lemma 6.5 below). Note also that $\mathcal{P}ic^{\mathbf{Z}}(-)$ extends to a functor on $\mathrm{CAlg}^{\Delta}_{\mathbf{Z}}$.

4.6. The determinant map on $\mathrm{GL}_n(A_\bullet)$ defines a map of symmetric monoidal ∞-groupoids

$$\det : \coprod_{n\geq 0} \mathrm{BGL}_n(A_\bullet) \to \mathcal{P}ic^{\mathbf{Z}}(A_\bullet).$$

For $A_\bullet$ an ordinary ring this is immediate, and since both sides are left Kan extensions of their restrictions to smooth algebras we get the map also for simplicial rings. Since the target is grouplike this induces a map

$$K\Big(\coprod_{n\geq 0} \mathrm{BGL}_n(A_\bullet)\Big) \to \mathcal{P}ic^{\mathbf{Z}}(A_\bullet).$$

Passing to the associated sheafifications we get a map

$$\det : K(A_\bullet) \to \mathcal{P}ic^{\mathbf{Z}}(A_\bullet).$$

We call this map, as well as the corresponding map

$$\det : \mathcal{D}_{\mathrm{perf}}(A_\bullet)^{\simeq} \to \mathcal{P}ic^{\mathbf{Z}}(A_\bullet), \tag{4.7}$$

the *determinant map*.

5. Perfect complexes on ringed sites

5.1. Let $(\boldsymbol{S}, \mathcal{O})$ be a ringed site. For a simplicial $\mathcal{O}$-algebra $A_\bullet$ we have the corresponding stable ∞-category $\mathcal{D}(\boldsymbol{S}, A_\bullet)$. Let

$$\mathcal{D}^{\mathrm{strict}}_{\mathrm{perf}}(\boldsymbol{S}, A_\bullet) \subset \mathcal{D}(\boldsymbol{S}, A_\bullet)$$

denote the smallest stable ∞-subcategory containing $A_\bullet$ and which is closed under retracts, and define

$$\mathcal{D}_{\mathrm{perf}}(\boldsymbol{S}, A_\bullet) \subset \mathcal{D}(\boldsymbol{S}, A_\bullet)$$

to be the ∞-subcategory of objects M for which there exists a collection of objects $\{U_i\}_{i\in I}$ covering the final object of the topos associated to $\boldsymbol{S}$ such that the restriction of M to each U_i lies in $\mathcal{D}^{\mathrm{strict}}_{\mathrm{perf}}(\boldsymbol{S}|_{U_i}, A_\bullet|_{U_i}) \subset \mathcal{D}(\boldsymbol{S}|_{U_i}, A_\bullet|_{U_i})$. So we have

$$\mathcal{D}^{\mathrm{strict}}_{\mathrm{perf}}(\boldsymbol{S}, A_\bullet) \subset \mathcal{D}_{\mathrm{perf}}(\boldsymbol{S}, A_\bullet) \subset \mathcal{D}(\boldsymbol{S}, A_\bullet).$$

5.2. Let

$$\underline{\mathscr{D}}^{\text{strict}}_{\text{perf}}, \qquad \underline{\mathscr{D}}_{\text{perf}}, \qquad \underline{\mathscr{D}}'_{\text{perf}} \tag{5.3}$$

be the presheaves of symmetric monoidal ∞-categories which to any $U \in \boldsymbol{S}$ associate

$$\mathscr{D}^{\text{strict}}_{\text{perf}}(\boldsymbol{S}|_U, A_\bullet|_U), \qquad \mathscr{D}_{\text{perf}}(\boldsymbol{S}|_U, A_\bullet|_U), \qquad \mathscr{D}_{\text{perf}}(A_\bullet(U)),$$

respectively. There are natural maps

$$\underline{\mathscr{D}}'_{\text{perf}} \to \underline{\mathscr{D}}^{\text{strict}}_{\text{perf}} \to \underline{\mathscr{D}}_{\text{perf}}. \tag{5.4}$$

Remark 5.5. In the case when $A_\bullet = \mathscr{O}$ is a sheaf of (ordinary) rings, the sheaf $\underline{\mathscr{D}}_{\text{perf}}$ is equivalent to the sheaf that associates to any U the ∞-category of perfect complexes of $\mathscr{O}$-modules on U in the sense of [Stacks, Tag 08FL].

Lemma 5.6. *The presheaf $\underline{\mathscr{D}}_{\text{perf}}$ is a sheaf, and both the maps*

$$\underline{\mathscr{D}}'_{\text{perf}} \to \underline{\mathscr{D}}_{\text{perf}}, \qquad \underline{\mathscr{D}}^{\text{strict}}_{\text{perf}} \to \underline{\mathscr{D}}_{\text{perf}}$$

induce equivalences upon sheafification.

Proof. The statements that $\underline{\mathscr{D}}_{\text{perf}}$ is a sheaf and that

$$\underline{\mathscr{D}}^{\text{strict}}_{\text{perf}} \to \underline{\mathscr{D}}_{\text{perf}}$$

induces an equivalence upon sheafification follow immediately from the definition of $\underline{\mathscr{D}}_{\text{perf}}$.

To prove the lemma we therefore show that the map

$$\underline{\mathscr{D}}'_{\text{perf}} \to \underline{\mathscr{D}}_{\text{perf}}$$

induces an equivalence upon sheafification. Let $\underline{\mathscr{D}}'^{a}_{\text{perf}}$ denote the sheaf associated to $\underline{\mathscr{D}}'_{\text{perf}}$.

Since sheafification commutes with finite limits (see [Lurie 2009, Proposition 6.2.2.7]), for two objects $x, y \in \underline{\mathscr{D}}'_{\text{perf}}(U)$ with associated objects $x^a, y^a \in \underline{\mathscr{D}}'^{a}_{\text{perf}}$ and $F_x, F_y \in \underline{\mathscr{D}}_{\text{perf}}(U)$, the sheaf

$$\underline{\text{Map}}_{\underline{\mathscr{D}}'^{a}_{\text{perf}}}(x^a, y^a)$$

is equal to the sheaf associated to the presheaf

$$V \mapsto \text{Map}_{\mathscr{D}_{\text{perf}}(A_\bullet(V))}(x \otimes_{A_\bullet(U)} A_\bullet(V), y \otimes_{A_\bullet(U)} A_\bullet(V)). \tag{5.7}$$

Let $\mathscr{S}$ denote the subcategory of $\underline{\mathscr{D}}'_{\text{perf}}(U)$ of those objects $x \in \underline{\mathscr{D}}'_{\text{perf}}(U)$ for which the map

$$\underline{\text{Map}}_{\underline{\mathscr{D}}'^{a}_{\text{perf}}}(x^a, y^a) \to \underline{\text{Map}}_{\underline{\mathscr{D}}_{\text{perf}}}(F_x, F_y)$$

is an equivalence for all $y \in \underline{\mathscr{D}}'_{\text{perf}}(U)$. Then $\mathscr{S}$ is a stable subcategory closed under retracts, so to show that $\mathscr{S}$ is equal to all of $\underline{\mathscr{D}}'_{\text{perf}}(U)$ it suffices to show that $A_\bullet(U)$ is in $\mathscr{S}$.

For $y \in \underline{\mathscr{D}}'_{\mathrm{perf}}(U)$ with associated object $F_y \in \underline{\mathscr{D}}_{\mathrm{perf}}(U)$ and $x = A_\bullet(U)$ the presheaf (5.7) is given by sending V to

$$\tau_{\leq 0}(y \otimes_{A_\bullet(U)} A_\bullet(V)),$$

where $y \otimes_{A_\bullet(U)} A_\bullet(V)$ is viewed as an object of $\mathscr{D}(\mathrm{Ab})$ (the derived ∞-category of abelian groups). On the other hand, the sheaf

$$\underline{\mathrm{Map}}_{\underline{\mathscr{D}}_{\mathrm{perf}}}(A_\bullet^a, F_y)$$

is given by applying $\tau_{\leq 0}$ to the sheafification of

$$V \mapsto y \otimes_{A_\bullet(U)} A_\bullet(V).$$

Thus the statement that $A_\bullet(U) \in \mathscr{S}$ amounts to the observation that sheafification commutes with the functor $\tau_{\leq 0}$. We conclude that $\mathscr{S} = \underline{\mathscr{D}}'_{\mathrm{perf}}(U)$.

It follows that

$$\underline{\mathscr{D}}'^{a}_{\mathrm{perf}} \to \underline{\mathscr{D}}_{\mathrm{perf}}$$

induces an equivalence on mapping spaces, and since locally every object is evidently in the image we conclude that this map is an equivalence. □

5.8. The ∞-category $\mathscr{D}_{\mathrm{perf}}(\boldsymbol{S}, A_\bullet)$ is given by the global sections

$$\Gamma(\boldsymbol{S}, \underline{\mathscr{D}}_{\mathrm{perf}}).$$

Let $\mathscr{P}ic^{\boldsymbol{Z}}_{(\boldsymbol{S}, A_\bullet)}$ denote the global sections of the sheaf associated to the presheaf $\underline{\mathscr{P}ic}^{\boldsymbol{Z}}_{A_\bullet}$ given by

$$U \mapsto \mathscr{P}ic^{\boldsymbol{Z}}(A_\bullet(U)).$$

Using (4.7) we then obtain a diagram

$$\begin{array}{ccc} \underline{\mathscr{D}}'^{,\simeq}_{\mathrm{perf}} & \longrightarrow & \underline{\mathscr{D}}^{\simeq}_{\mathrm{perf}} \\ \Big\downarrow{\scriptstyle \det} & & \\ \underline{\mathscr{P}ic}^{\boldsymbol{Z}}_{\mathscr{O}} & & \end{array}$$

Passing to the associated sheaves and using Lemma 5.6 we get an induced map

$$\det : \mathscr{D}_{\mathrm{perf}}(\boldsymbol{S}, A_\bullet)^{\simeq} \to \mathscr{P}ic^{\boldsymbol{Z}}_{(\boldsymbol{S}, A_\bullet)}. \tag{5.9}$$

By functoriality of K-theory this map factors through the K-theory of $\mathscr{D}_{\mathrm{perf}}(\boldsymbol{S}, A_\bullet)$, which we denote by $K(\boldsymbol{S}, A_\bullet)$.

5.10. In the case when $A_\bullet = \mathscr{O}$ we can describe $\mathscr{P}ic^{\boldsymbol{Z}}_{(\boldsymbol{S}, \mathscr{O})}$ more explicitly as follows. Let $\boldsymbol{Z}_{(\boldsymbol{S}, \mathscr{O})}$ be the sheaf associated to the presheaf which to any $U \in \boldsymbol{S}$ associated the set of locally constant $\boldsymbol{Z}$-valued functions on $\mathrm{Spec}(\mathbb{O}(U))$. Note that for any section $r \in \boldsymbol{Z}_{(S,\mathbb{O})}(U)$ the expression

$$(-1)^r \in \mathbb{O}^*(U)$$

makes sense. Then the sheaf $\underline{\mathcal{P}ic}^{\boldsymbol{Z}}_{(S,\mathbb{O})}$ associated to the presheaf $\underline{\mathcal{P}ic}^{\boldsymbol{Z}}_{\mathbb{O}}$ can be described as the stack in groupoids which to any U associates the groupoid of pairs $(r, \mathcal{L})$, where $r \in \boldsymbol{Z}_{(S,\mathbb{O})}(U)$ and $\mathcal{L}$ is an invertible module on $(\boldsymbol{S}|_U, \mathbb{O})$, in the sense of [Stacks, Tag 0408]. The monoidal structure is given by

$$(r, \mathcal{L}) * (r', \mathcal{L}') := (r + r', \mathcal{L} \otimes \mathcal{L}')$$

and the commutativity constraint

$$(r, \mathcal{L}) * (r', \mathcal{L}') \simeq (r', \mathcal{L}') * (r, \mathcal{L})$$

is given by the isomorphism

$$\mathcal{L} \otimes \mathcal{L}' \simeq \mathcal{L}' \otimes \mathcal{L}$$

obtained by multiplying the isomorphism switching the factors with $(-1)^{rr'}$.

In particular, for any perfect complex E on $(\boldsymbol{S}, \mathbb{O})$ we can speak about its determinant $\det(E)$, an invertible $\mathbb{O}$-module.

5.11. It is useful to have a variant description of $\underline{\mathcal{D}}'_{\mathrm{perf}}$.

Let $\boldsymbol{S}^{\mathrm{zar}}$ denote the category whose objects are pairs (U, V), where $U \in \boldsymbol{S}$ and $V \subset \mathrm{Spec}(\mathbb{O}(U))$ is an affine open set. A morphism

$$(U', V') \to (U, V)$$

is defined to be a morphism $f : U' \to U$ in $\boldsymbol{S}$ such that the induced morphism

$$\mathrm{Spec}(\mathbb{O}(U')) \to \mathrm{Spec}(\mathbb{O}(U))$$

sends V' to V. The *Zariski topology* on $\boldsymbol{S}^{\mathrm{zar}}$ is defined by declaring a collection of morphisms

$$\{f_i : (U_i, V_i) \to (U, V)\}$$

a covering if each $f_i : U_i \to U$ is an isomorphism, and the collection of maps

$$\{V_i \to V\}$$

is an open covering of V. With this definition the category of sheaves on $\boldsymbol{S}^{\mathrm{zar}}$ is equivalent to the category of collections of sheaves $\{F_U\}_{U \in \boldsymbol{S}}$, where F_U is a sheaf on $\mathrm{Spec}(\mathbb{O}(U))$, together with transition morphisms $\theta_f : f^{-1}F_U \to F_{U'}$ for each morphism $f : U' \to U$ in $\boldsymbol{S}$, satisfying the natural cocycle condition.

A presheaf F (of sets, ∞-categories, etc.) on $\boldsymbol{S}^{\mathrm{zar}}$ induces a presheaf on $\boldsymbol{S}$ by composing F with the functor

$$\boldsymbol{S} \to \boldsymbol{S}^{\mathrm{zar}}, \quad U \mapsto (U, \mathrm{Spec}(\mathbb{O}(U))).$$

We denote this presheaf on $\boldsymbol{S}$ by $\gamma(F)$.

The presheaves of symmetric monoidal ∞-categories on $\boldsymbol{S}$

$$\underline{\mathcal{D}}'_{\mathrm{perf}}, \qquad \underline{\mathcal{P}ic}^{\boldsymbol{Z}}_{\mathbb{O}}$$

then extend to presheaves

$$\underline{\mathcal{D}}'^{\mathrm{zar}}_{\mathrm{perf}}, \qquad \underline{\mathcal{P}ic}^{\boldsymbol{Z},\mathrm{zar}}_{\mathbb{O}}$$

on $\boldsymbol{S}^{\mathrm{zar}}$, where $\underline{\mathcal{D}}'^{\mathrm{zar}}_{\mathrm{perf}}$ sends (U, V) to the category of strictly perfect complexes of $\mathbb{O}_{\mathrm{Spec}(\mathbb{O}(U))}(V)$-modules and $\underline{\mathcal{P}ic}^{\boldsymbol{Z},\mathrm{zar}}_{\mathbb{O}}$ sends (U, V) to the groupoid of $\boldsymbol{Z}$-graded line bundles on the scheme V. Observe that we have

$$\underline{\mathcal{D}}'_{\mathrm{perf}} = \gamma(\underline{\mathcal{D}}'^{\mathrm{zar}}_{\mathrm{perf}}), \qquad \underline{\mathcal{P}ic}^{\boldsymbol{Z}}_{\mathbb{O}} = \gamma(\underline{\mathcal{P}ic}^{\boldsymbol{Z},\mathrm{zar}}_{\mathbb{O}}).$$

Note also that $\mathbb{O}$ extends to a presheaf of rings on $\boldsymbol{S}^{\mathrm{zar}}$, which we denote by $\mathbb{O}^{\mathrm{zar}}$, given by

$$\mathbb{O}^{\mathrm{zar}}(U, V) = \Gamma(V, \mathbb{O}_{\mathrm{Spec}(\mathbb{O}(U))}(V)).$$

Similarly, any complex $I^\bullet$ of presheaves of $\mathbb{O}$-modules extends to a complex of $\mathbb{O}^{\mathrm{zar}}$-modules, which we denote by $I^{\mathrm{zar},\bullet}$, given by

$$I^{\mathrm{zar},\bullet}(U, V) = \widetilde{I(U)}(V),$$

where $\widetilde{I(U)}$ denotes the complex of quasicoherent sheaves on $\mathrm{Spec}(\mathbb{O}(U))$ associated to the complex of $\mathbb{O}(U)$-modules $I(U)$.

Define $\mathrm{BGL}_n(\mathbb{O}^{\mathrm{zar}})$ to be the presheaf on $\boldsymbol{S}^{\mathrm{zar}}$ which to any (U, V) associates $\mathrm{BGL}_n(\mathbb{O}^{\mathrm{zar}}(U, V))$. We then have a natural map

$$\coprod_n \mathrm{BGL}_n(\mathbb{O}^{\mathrm{zar}}) \to \underline{\mathcal{D}}'^{\mathrm{zar},\simeq}_{\mathrm{perf}}$$

of presheaves of symmetric monoidal ∞-categories. The determinant map also extends to a map

$$\det{}^{\mathrm{zar}} : \underline{\mathcal{D}}'^{\mathrm{zar},\simeq}_{\mathrm{perf}} \to \underline{\mathcal{P}ic}^{\boldsymbol{Z},\mathrm{zar}}_{\mathbb{O}}$$

inducing the previously defined determinant map after applying γ. The advantage of working with presheaves on $\boldsymbol{S}^{\mathrm{zar}}$ is that by Lemma 4.2 the map $\det^{\mathrm{zar}}$, and hence also the determinant map (5.9), is determined by the induced map

$$\coprod_n \mathrm{BGL}_n(\mathbb{O}^{\mathrm{zar}}) \to \underline{\mathcal{P}ic}^{\boldsymbol{Z},\mathrm{zar}}_{\mathbb{O}}.$$

More generally, for a complex $I^\bullet$ of presheaves of $\mathcal{O}$-modules we can consider $\mathcal{O}^{\text{zar}}[I^{\text{zar},\bullet}]$ on S^{zar} and the determinant defines a map

$$\det{}^{\text{zar}} : \underline{\mathcal{D}}'^{\text{zar},\simeq}_{\text{perf},\mathcal{O}[I^\bullet]} \to \underline{\mathcal{P}ic}^{\mathbf{Z},\text{zar}}_{\mathcal{O}[I^\bullet]}.$$

6. Ring structure

As pointed out to us by Bhargav Bhatt, the results of the previous section can profitably be upgraded to include statements about the ring structure on algebraic K-theory. The following is a modification of an argument communicated to us by Bhatt, which answers, in particular, a question of Rössler[2] which was also discussed in the Stacks Project[3]. The results of this section are not used in what follows.

6.1. The ring structure on algebraic K-theory can be described in a few ways. Most convenient for us is the description in [Gepner et al. 2015, Theorem 8.6] (see also [Blumberg et al. 2013]). This result is obtained from [Gepner et al. 2015, Theorem 5.1], which gives that the group completion functor

$$(\boldsymbol{E}_\infty\text{-monoids}) \to (\text{grouplike } \boldsymbol{E}_\infty\text{-monoids}) \simeq \text{Sp}^{\geq 0}$$

extends to a symmetric monoidal functor. Here the monoidal structure on $\text{Sp}^{\geq 0}$ is given by the smash product of spectra (see [Gepner et al. 2015, Example 5.3(ii)]).

For a ringed site $(S, \mathcal{O})$ the underlying groupoids of the objects (5.3) have the structure of presheaves of $\boldsymbol{E}_\infty$-semirings (see [Gepner et al. 2015, page 2, (iii)]), and the diagram (5.4) is compatible with this structure.

This implies, in particular, that the K-theory $K(S, \mathcal{O})$ has the structure of an $\boldsymbol{E}_\infty$-ring spectrum [Gepner et al. 2015, Example 8.11(i) and Corollary 8.12].

6.2. The Picard category $\mathcal{P}ic^{\mathbf{Z}}_{(S,\mathcal{O})}$ also has a multiplicative structure given by

$$(r, \mathcal{L}) \otimes (r', \mathcal{L}') := (rr', \mathcal{L}^{\otimes r'} \otimes \mathcal{L}'^{\otimes r}). \tag{6.3}$$

Note here that $\mathcal{L}^{\otimes r'}$ and $\mathcal{L}'^{\otimes r}$ are defined by first defining them on the level of modules over rings and then globalizing.

This multiplicative structure can be upgraded to a structure of an $\boldsymbol{E}_\infty$-ring spectrum as follows.

6.4. Let S^{zar} be as in 5.11, and consider again the functors

$$\coprod_n \text{BGL}_n : S^{\text{zar},\text{op}} \to (\boldsymbol{E}_\infty\text{-monoids}), \quad (U, V) \mapsto \coprod_n \text{BGL}_n(\mathcal{O}^{\text{zar}}(U, V)),$$

[2] See https://mathoverflow.net/questions/354214/determinantal-identities-for-perfect-complexes.
[3] See https://www.math.columbia.edu/ dejong/wordpress/?p=4474.

and

$$\boldsymbol{Z} \times \mathrm{BGL}_1 : \boldsymbol{S}^{\mathrm{zar,op}} \to (\boldsymbol{E}_\infty\text{-monoids}), \quad (U, V) \mapsto \boldsymbol{Z} \times \mathrm{BGL}_1(\mathcal{O}^{\mathrm{zar}}(U, V)).$$

Let

$$\mathcal{K}^{\mathrm{zar}} : \boldsymbol{S}^{\mathrm{zar,op}} \to (\boldsymbol{E}_\infty\text{-monoids})$$

be the sheafification of the group completion of $\coprod_n \mathrm{BGL}_n$ and note that $\underline{\mathcal{P}ic}_{\mathcal{O}}^{\boldsymbol{Z},\mathrm{zar}}$ is the sheafification of $\boldsymbol{Z} \times \mathrm{BGL}_1$. Then the determinant defines a map of sheaves of $\boldsymbol{E}_\infty$-monoids

$$\det : \mathcal{K}^{\mathrm{zar}} \to \underline{\mathcal{P}ic}_{\mathcal{O}}^{\boldsymbol{Z},\mathrm{zar}}.$$

Lemma 6.5. *The determinant map induces an equivalence*

$$\tau_{\leq 1}\mathcal{K}^{\mathrm{zar}} \simeq \underline{\mathcal{P}ic}_{\mathcal{O}}^{\boldsymbol{Z},\mathrm{zar}}.$$

Proof. It suffices to show that for a given $U \in \boldsymbol{S}$ the induced map of sheaves on $\mathrm{Spec}(\mathcal{O}(U))$ is an equivalence. This reduces the proof of the lemma to the case of the Zariski topology of an affine scheme. The verification in this case reduces immediately to the calculation of K_0 and K_1 for a local ring, and for such a ring R we have $K_0(R) = \boldsymbol{Z}$ and $K_1(R) = R^*$. □

6.6. Now observe that $\mathcal{K}^{\mathrm{zar}}$ has the structure of a sheaf of $\boldsymbol{E}_\infty$-rings, and therefore so does $\tau_{\leq 1}\mathcal{K}^{\mathrm{zar}}$. In this way, $\underline{\mathcal{P}ic}_{\mathcal{O}}^{\boldsymbol{Z},\mathrm{zar}}$, and therefore also $\underline{\mathcal{P}ic}_{\mathcal{O}}^{\boldsymbol{Z}}$ and $\mathcal{P}ic_{(\boldsymbol{S},\mathcal{O})}^{\boldsymbol{Z}}$, are given $\boldsymbol{E}_\infty$-ring structures. Note also that the underlying multiplication map is induced by the natural maps on $\coprod_n \mathrm{BGL}_n$ and $\coprod_n \mathrm{BGL}_1$, and therefore the underlying multiplicative structure on $\mathcal{P}ic_{(\boldsymbol{S},\mathcal{O})}^{\boldsymbol{Z}}$ is given by (6.3).

Furthermore, if $\mathcal{K}$ denotes the sheaf on $\boldsymbol{S}$ associated to the presheaf $\gamma(\mathcal{K}^{\mathrm{zar}})$, we then get an equivalence

$$\tau_{\leq 1}\mathcal{K} \simeq \underline{\mathcal{P}ic}_{(\boldsymbol{S},\mathcal{O})}^{\boldsymbol{Z}}.$$

This discussion implies the following:

Theorem 6.7. *The Picard category $\mathcal{P}ic_{(\boldsymbol{S},\mathcal{O})}^{\boldsymbol{Z}}$ has the structure of an $\boldsymbol{E}_\infty$-ring spectrum with multiplicative structure given by (6.3) and such that the map*

$$K(\boldsymbol{S}, \mathcal{O}) \to \mathcal{P}ic_{(\boldsymbol{S},\mathcal{O})}^{\boldsymbol{Z}} \tag{6.8}$$

is a map of $\boldsymbol{E}_\infty$-rings.

Proof. This follows from the preceding discussion, and the observation that the map

$$K(\boldsymbol{S}, \mathcal{O}) \to R\Gamma(\boldsymbol{S}, \mathcal{K})$$

is a map of $\boldsymbol{E}_\infty$-rings, by the universal property of group completion and the fact that the isomorphism

$$\mathcal{D}_{\mathrm{perf}}(\boldsymbol{S}, \mathcal{O})^{\simeq} = R\Gamma(\boldsymbol{S}, \underline{\mathcal{D}}_{\mathrm{perf}}^{\simeq})$$

is an isomorphism of $\boldsymbol{E}_\infty$-semirings. □

7. The trace map

7.1. Let $(S, \mathcal{O})$ be a ringed site and let E be a perfect complex of $\mathcal{O}$-modules. In this section we record some observations about the trace map

$$\mathrm{tr}_{I_\bullet} : \mathcal{R}\mathrm{Hom}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet)) \to N(I_\bullet),$$

defined in [Illusie 1971, Chapter V, (3.7.3)], for a simplicial $\mathcal{O}$-module $I_\bullet$. This map is obtained as the composition of the inverse of the isomorphism

$$E^\vee \otimes^{\boldsymbol{L}} (E \otimes^{\boldsymbol{L}} N(I_\bullet)) \to \mathcal{R}\mathrm{Hom}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet))$$

(using the perfection of E) and the evaluation map

$$E^\vee \otimes^{\boldsymbol{L}} (E \otimes^{\boldsymbol{L}} N(I_\bullet)) \to N(I_\bullet).$$

Observe that under the natural identification

$$\mathcal{R}\mathrm{Hom}(E, E) \otimes^{\boldsymbol{L}} N(I_\bullet) \simeq \mathcal{R}\mathrm{Hom}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet))$$

the map $\mathrm{tr}_{I_\bullet}$ is identified with the map $\mathrm{tr}_{\mathcal{O}} \otimes N(I_\bullet)$. We often drop the subscript and write simply tr for $\mathrm{tr}_{I_\bullet}$ if no confusion seems likely to arise.

7.2. Fix a perfect complex E. Denote by $\mathrm{Mod}_{\mathcal{O}}$ the category of sheaves of $\mathcal{O}$-modules, and by $\mathrm{Mod}_{\mathcal{O}}^{\Delta^{\mathrm{op}}}$ the category of sheaves of simplicial $\mathcal{O}$-modules. We then have two functors

$$F_1, F_2 : \mathrm{Mod}_{\mathcal{O}}^{\Delta^{\mathrm{op}}} \to (\text{Sp-valued sheaves})$$

given by

$$F_1(I_\bullet) := \mathrm{DK}\big(\mathcal{R}\mathrm{Hom}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet))[1]\big)$$

and

$$F_2(I_\bullet) := BI_\bullet.$$

The trace map defines a morphism of ∞-functors

$$\mathrm{tr} : F_1 \to F_2. \tag{7.3}$$

Now observe that F_1 is the left Kan extension of its restriction to $\mathrm{Mod}_{\mathcal{O}}$ (this follows from the observation that the normalization of a simplicial abelian group is quasi-isomorphic to the homotopy colimit), and therefore tr is determined by the restrictions of these functors to $\mathrm{Mod}_{\mathcal{O}}$.

In fact, from the perfection of E we get slightly more. Namely, note that since E is perfect we have locally

$$\mathcal{R}\mathrm{Hom}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet)[n+1]) \simeq \tau_{\leq 0}\mathcal{R}\mathrm{Hom}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet)[n+1])$$

for n sufficiently large. Therefore, if we write $F_1^{\leq 0}$ for the functor

$$I_\bullet \mapsto \mathrm{DK}\big(\tau_{\leq 0}\mathscr{R}\mathrm{Hom}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet))[1]\big)$$

then F_1 is isomorphic to the functor

$$I_\bullet \mapsto \mathrm{colim}_n\, \Omega^n F_1^{\leq 0}(B^n I_\bullet),$$

where $B^n I_\bullet$ is the n-fold delooping of $I_\bullet$, corresponding under the Dold–Kan correspondence to $N(I_\bullet)[n]$. It follows that a morphism of ∞-functors $F_1^{\leq 0} \to F_2$ can be extended to a morphism $F_1 \to F_2$, and therefore is determined by its restriction to $\mathrm{Mod}_{\mathbb{O}}$.

7.4. To understand the map tr on modules, consider first the case of the punctual topos, a ring A, and an A-module I. Let $\mathrm{Perf}^{\mathrm{strict}}_{A[I]}$ denote the category of strictly perfect $A[I]$-modules, viewed as a Waldhausen category as in [Thomason and Trobaugh 1990, Definition 3.1], and let $\mathrm{EXT}(A, I)$ be the Picard category of short exact sequences of A-modules

$$0 \to I \to T \to A \to 0,$$

as in [SGA 4½ 1977, Exposé XVIII, 1.4.22]. Note that by [SGA 4½ 1977, Exposé XVIII, 1.4.23] the object of $\mathrm{Sp}^{\geq 0}$ associated to $\mathrm{EXT}(A, I)$ is BI.

There is a determinant map (in the sense of Section 3.5)

$$\delta : \mathrm{Perf}^{\mathrm{strict}}_{A[I]} \to \mathrm{EXT}(A, I).$$

For an object $E' \in \mathrm{Perf}^{\mathrm{strict}}_{A[I]}$ with reduction $E \in \mathrm{Perf}^{\mathrm{strict}}_{A}$ we get an exact sequence of complexes of A-modules

$$0 \to E \otimes I \to E' \to E \to 0.$$

Tensoring with $E^\vee$, pulling back along $\mathrm{id} : A \to E \otimes E^\vee$ and pushing out along the trace map we get an object of $\mathrm{EXT}(A, I)$; in a diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & T & \longrightarrow & A & \longrightarrow & 0 \\
 & & \big\uparrow{\scriptstyle \mathrm{tr}} & & \big\uparrow & & \big\| & & \\
0 & \longrightarrow & E \otimes E^\vee \otimes I & \longrightarrow & \mathscr{E}' & \longrightarrow & A & \longrightarrow & 0 \\
 & & \big\| & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & E \otimes E^\vee & \longrightarrow & E' \otimes E^\vee & \longrightarrow & E \otimes E^\vee & \longrightarrow & 0
\end{array}$$

To extend this construction to a determinant functor, note that by [Knudsen 2002, Theorem 2.3] it suffices to define a determinant functor on the category of projective $A[I]$-modules $\mathrm{Proj}_{A[I]}$ with appropriate properties.

The preceding construction defines a functor

$$\mathrm{iso}(\mathrm{Proj}_{A[I]}) \to \mathrm{EXT}(A, I).$$

Next consider a short exact sequence of projective $A[I]$-modules

$$0 \to E_1' \to E' \to E_2' \to 0$$

with reduction

$$0 \to E_1 \to E \to E_2 \to 0.$$

Set

$$\Sigma := \mathrm{Ker}(E \otimes E^\vee \to E_2 \otimes E_1^\vee).$$

Then the two maps

$$E \otimes E^\vee \to E_2 \otimes E^\vee, \qquad E \otimes E^\vee \to E \otimes E_1^\vee$$

induce a map

$$\rho : \Sigma \to E_2 \otimes E_2^\vee \oplus E_1 \oplus E_1^\vee.$$

Furthermore, if

$$0 \to E \otimes E^\vee \otimes I \to \mathscr{E} \to A \to 0$$

is the extension obtained from E' then the pushout of this extension along the map

$$E \otimes E^\vee \to E_2 \otimes E_1^\vee$$

is canonically trivialized, which implies that $\mathscr{E}$ is obtained from an extension

$$0 \to \Sigma \otimes I \to \mathscr{E}_\Sigma \to A \to 0.$$

Furthermore, this identifies the pushout of $\mathscr{E}_\Sigma$ along ρ with the sum of the extensions obtained from E_1' and E_2'. In this way we obtain a predeterminant functor in the sense of [Knudsen 2002, Definition 1.2]. We leave it to the reader to verify that this in fact defines a determinant.

7.5. Combining this with the discussion in Section 3.5 we obtain a map

$$\bar{t} : K(A[I]) \to BI$$

from which one can recover the trace map

$$\mathrm{tr}_{I[1]} : \mathrm{DK}(\tau_{\leq 0}\mathscr{R}\mathrm{Hom}(E, E \otimes^{\boldsymbol{L}} I)[1]) \to BI$$

as the composition

$$\mathrm{DK}(\tau_{\leq 0}\mathscr{R}\mathrm{Hom}(E, E \otimes^{\boldsymbol{L}} I)[1]) \xrightarrow{2.22} \mathscr{D}^{\mathrm{strict}}_{\mathrm{perf}}(A[I])^{\simeq} \to K(A[I]) \xrightarrow{\bar{t}} BI.$$

Combining 7.2 and Section 3.7 this also defines, by passing to left Kan extensions, a map for every simplicial A-module $I_\bullet$

$$\bar{t} : K(A[I_\bullet]) \to BI_\bullet$$

inducing the trace map $\mathrm{tr}_{I_\bullet[1]}$.

7.6. In the case of a general ringed topos $(S, \mathcal{O})$ and an $\mathcal{O}$-module I we get by functoriality of the preceding constructions a morphism of presheaves taking values in $\mathrm{Sp}^{\geq 0}$

$$\bar{t} : \mathcal{K}_{\mathcal{O}[I]} \to BI,$$

where $\mathcal{K}_{\mathcal{O}[I]}$ is the presheaf sending $U \in S$ to the K-theory of strictly perfect complexes of $\mathcal{O}(U)[I(U)]$-modules, such that the composition

$$\mathrm{DK}(\tau_{\leq 0}\mathcal{R}\mathrm{Hom}(E, E \otimes^{L} I[1])) \to \mathcal{D}^{\mathrm{strict},\simeq}_{\mathrm{perf},\mathcal{O}[I]} \to \mathcal{K}_{\mathcal{O}[I]} \xrightarrow{\bar{t}} BI$$

is the trace map. Using the method of 7.2 we then also get a map for a simplicial $\mathcal{O}$-module $I_\bullet$

$$\bar{t} : \mathcal{K}_{\mathcal{O}[I_\bullet]} \to BI_\bullet$$

inducing the trace map on $\tau_{\leq 0}\mathcal{R}\mathrm{Hom}(E, E \otimes^{L} N(I_\bullet)[1])$.

8. Determinants and traces

In this section we elucidate the relationship between the determinant map and the trace map constructed in [Illusie 1971, Chapter V, (3.7.3)].

8.1. We begin the discussion in the punctual case. Let A be a ring and let $I_\bullet$ be a simplicial A-module with associated simplicial ring of dual numbers $A[I_\bullet]$.

Note that $\pi_0(A[I_\bullet]) \simeq A[\pi_0(I_\bullet)]$ and we have a short exact sequence of simplicial monoids

$$1 \to 1 + I_\bullet \to \mathrm{GL}_1(A[I_\bullet]) \to A^* \to 1.$$

In particular, $\mathrm{GL}_1(A[I_\bullet])$ is a simplicial group. Furthermore, the retraction r defines a splitting of this sequence giving a homomorphism

$$\mathrm{GL}_1(A[I_\bullet]) \to I_\bullet. \tag{8.2}$$

Concretely this is given by writing an element $\alpha \in (A[I_n])^*$ as $\bar{\alpha}(1 + x)$, where $\bar{\alpha} \in A^*$ is the image of α in A^*, and then sending α to x.

8.3. Note that because $-1 \in A^* \subset \mathrm{GL}_1(A[I_\bullet])$, the projection map (8.2) induces a map

$$\mathcal{P}ic^{\mathbf{Z}}(A[I_\bullet]) \to BI_\bullet,$$

compatible with the symmetric monoidal structure. Furthermore, the induced map

$$\mathcal{P}ic^{\mathbf{Z}}(A[I_\bullet]) \to \mathcal{P}ic^{\mathbf{Z}}(A) \times BI_\bullet \tag{8.4}$$

is an equivalence. The determinant map

$$\det_{A[I_\bullet]} : \mathcal{D}_{\mathrm{perf}}(A[I_\bullet])^{\simeq} \to \mathcal{P}ic^{\mathbf{Z}}(A[I_\bullet])$$

can therefore be written as

$$(\det_A, t) : \mathcal{D}_{\mathrm{perf}}(A[I_\bullet])^{\simeq} \to \mathcal{P}ic^{\boldsymbol{Z}}(A) \times BI_\bullet,$$

where the first component is given by the projection

$$\mathcal{D}_{\mathrm{perf}}(A[I_\bullet])^{\simeq} \to \mathcal{D}_{\mathrm{perf}}(A)^{\simeq}$$

followed by the determinant map for A-modules, and t is a symmetric monoidal map

$$t : \mathcal{D}_{\mathrm{perf}}(A[I_\bullet])^{\simeq} \to BI_\bullet.$$

8.5. Now consider a ringed site $(\boldsymbol{S}, \mathbb{O})$, and a simplicial $\mathbb{O}$-module $I_\bullet$. Write $R\Gamma^{\Delta}(I)$ for the simplicial object of the derived ∞-category obtained by taking derived functors of the global section functor. In terms of the normalization functor from simplicial modules to complexes we have

$$N(R\Gamma^{\Delta}(I_\bullet)) \simeq \tau_{\leq 0} R\Gamma(\boldsymbol{S}, N(I_\bullet)),$$

where the right side denotes the usual derived functor cohomology [Illusie 1971, Chapter I, 3.2.1.11].

Proceeding object by object and taking limits we get an equivalence

$$\mathcal{P}ic^{\boldsymbol{Z}}_{(\boldsymbol{S},\mathbb{O}[I_\bullet])} \simeq \mathcal{P}ic^{\boldsymbol{Z}}_{(\boldsymbol{S},\mathbb{O})} \times R\Gamma^{\Delta} B(I_\bullet),$$

which gives a description of the determinant map

$$\det_{(\boldsymbol{S},\mathbb{O}[I_\bullet])} = (\det_{(\boldsymbol{S},\mathbb{O})}, t) : \mathcal{D}_{\mathrm{perf}}(\boldsymbol{S}, \mathbb{O}[I_\bullet])^{\simeq} \to \mathcal{P}ic^{\boldsymbol{Z}}_{(\boldsymbol{S},\mathbb{O})} \times R\Gamma^{\Delta} B(I_\bullet),$$

where

$$t : \mathcal{D}_{\mathrm{perf}}(\boldsymbol{S}, \mathbb{O}[I_\bullet])^{\simeq} \to R\Gamma^{\Delta} B(I_\bullet)$$

is a map of symmetric monoidal $\boldsymbol{E}_\infty$-categories.

8.6. Let

$$q : \mathcal{D}_{\mathrm{perf}}(\boldsymbol{S}, \mathbb{O}[I_\bullet]) \to \mathcal{D}_{\mathrm{perf}}(\boldsymbol{S}, \mathbb{O})$$

be the projection. By 2.22 the fiber of q over the point given by an object $E \in \mathcal{D}_{\mathrm{perf}}(\boldsymbol{S}, \mathbb{O})$ is given by

$$\mathrm{DK}(\tau_{\leq 0}\, \mathrm{RHom}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet)[1]).$$

Restricting t to the fiber of q and using the Dold–Kan correspondence we get a map

$$\tau_{\leq 0}\, \mathrm{RHom}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet)) \to \tau_{\leq 0} R\Gamma(N(I_\bullet)) \tag{8.7}$$

in the derived category.

Remark 8.8. For $n \geq 0$ write $I_\bullet[n]$ for $\mathrm{DK}(N(I_\bullet)[n])$. Then

$$\tau_{\leq 0}\,\mathrm{RHom}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet[n])) \simeq (\tau_{\leq n}\,\mathrm{RHom}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet)))[n],$$

and therefore by shifting we get a map

$$\tau_{\leq n}\,\mathrm{RHom}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet)) \to R\Gamma(N(I_\bullet))$$

for all n. One can show directly that these maps are compatible and therefore by taking colimits define a map

$$\mathrm{RHom}(E, E \otimes^{\boldsymbol{L}} N(I_\bullet)) \to R\Gamma(N(I_\bullet)).$$

This compatibility follows, however, from Proposition 8.9 below so we do not elaborate further on this point here.

Proposition 8.9. *The map* (8.7) *agrees with the trace map defined in* [Illusie 1971, Chapter V, (3.7.3)].

Proof. The basic idea is to construct a second map

$$t' : K(\boldsymbol{S}, \mathscr{O}[I_\bullet]) \to R\Gamma^\Delta B(I_\bullet)$$

which induces the trace map on the fibers of q, and then show that $t = t'$ using the universal property of group completion.

For this we extend the preceding constructions to presheaves on $\boldsymbol{S}^{\mathrm{zar}}$, defined as in 5.11.

First of all, repeating the construction giving (8.4) we get an equivalence

$$\underline{\mathscr{P}ic}^{\boldsymbol{Z},\mathrm{zar}}_{\mathscr{O}[I_\bullet]} \to \mathscr{P}ic^{\boldsymbol{Z},\mathrm{zar}}_{\mathscr{O}} \times BI^{\mathrm{zar}}_\bullet,$$

and the determinant map on $\underline{\mathscr{D}}'^{\mathrm{zar},\simeq}_{\mathrm{perf},\mathscr{O}[I_\bullet]}$ breaks into two parts

$$(\det{}^{\mathrm{zar}}_{\mathscr{O}}, t^{\mathrm{zar}}) : \underline{\mathscr{D}}'^{\mathrm{zar},\simeq}_{\mathrm{perf},\mathscr{O}[I_\bullet]} \to \mathscr{P}ic^{\boldsymbol{Z},\mathrm{zar}}_{\mathscr{O}} \times BI^{\mathrm{zar}}_\bullet.$$

Similarly, running through the construction of 7.6 we get a map

$$t'^{\mathrm{zar}} : \underline{\mathscr{D}}'^{\mathrm{zar},\simeq}_{\mathrm{perf},\mathscr{O}[I_\bullet]} \to BI^{\mathrm{zar}}_\bullet$$

inducing the trace map after applying γ, sheafifying, and taking global sections.

It therefore suffices to show that t^{zar} and t'^{zar} agree. For this it suffices, in turn, to show that the restrictions along

$$\coprod_n \mathrm{BGL}_n(\mathscr{O}^{\mathrm{zar}}[I^{\mathrm{zar}}_\bullet]) \to BI^{\mathrm{zar}}_\bullet$$

agree. This is immediate from the constructions, and we get Proposition 8.9. □

9. Deformations of complexes

9.1. Let $\boldsymbol{S}$ be a site and let $\mathcal{O}' \to \mathcal{O}$ be a surjection of sheaves of rings on $\boldsymbol{S}$ with kernel K, a square-zero ideal.

Let $E \in D(\boldsymbol{S}, \mathcal{O})$ be a perfect complex of $\mathcal{O}$-modules. As in 2.11 we can then consider the category $\mathrm{Def}(E)$ of deformations of E to $\mathcal{O}'$.

The following is well-known in many cases, e.g., [Illusie 1971, Chapter IV, Proposition 3.1.5; Lieblich 2006, Theorem 3.1.1].

Theorem 9.2. *Let E be a perfect complex of $\mathcal{O}$-modules on $\boldsymbol{S}$.*

(i) *There is a class $\omega(E) \in \mathrm{Ext}^2(E, E \otimes^{\boldsymbol{L}} K)$ which vanishes if and only if E lifts to a perfect complex of $\mathcal{O}'$-modules.*

(ii) *If $\omega(E) = 0$ then the set of isomorphism classes of liftings form a torsor under* $\mathrm{Ext}^1(E, E \otimes^{\boldsymbol{L}} K)$.

(iii) *If* $\mathrm{Ext}^{-1}(E, E) = 0$ *then the set of automorphisms of any lifting is canonically identified with* $\mathrm{Ext}^0(E, E \otimes^{\boldsymbol{L}} K)$.

Remark 9.3. If E' is a deformation of E to $\mathcal{O}'$, then by applying $\mathrm{RHom}_{\mathcal{O}'}(E', -)$ to the distinguished triangle

$$K \otimes^{\boldsymbol{L}}_{\mathcal{O}} E \to E' \to E \to K \otimes^{\boldsymbol{L}}_{\mathcal{O}} E[1]$$

we get a boundary map

$$\partial_{E'} : \mathrm{Ext}^{-1}_{\mathcal{O}}(E, E) \to \mathrm{Ext}^0(E, E \otimes^{\boldsymbol{L}}_{\mathcal{O}} K).$$

If we don't assume that $\mathrm{Ext}^{-1}(E, E) = 0$ then the group of automorphisms of (E', σ) is canonically isomorphic to the cokernel of $\partial_{E'}$. Note that this group may depend on E'.

The proof of Theorem 9.2 occupies the remainder of this section.

9.4. Statements (ii) and (iii) follow from the discussion in 2.11. Indeed, if $\mathrm{Def}_\infty(E)$ denotes the ∞-categorical fiber product of the diagram

$$\begin{array}{ccc} & & \mathcal{D}_{\mathrm{perf}}(\boldsymbol{S}, \mathcal{O}') \\ & & \downarrow \\ \star & \xrightarrow{E} & \mathcal{D}_{\mathrm{perf}}(\boldsymbol{S}, \mathcal{O}) \end{array}$$

then we constructed in 2.11 a functor

$$[\mathrm{Def}_\infty(E)] \to \mathrm{Def}(E),$$

where $[\mathrm{Def}_\infty(E)]$ denotes the underlying 1-category of $\mathrm{Def}_\infty(E)$, which induces a bijection on isomorphism classes of objects, and if $\mathrm{Ext}^{-1}(E, E) = 0$ is an equivalence of categories. On the other hand, if there exists a lifting of E then by 2.22 we have

$$\mathrm{Def}_\infty(E) \simeq \mathrm{DK}(\tau_{\leq 0}\, \mathrm{RHom}(E, E \otimes^{\boldsymbol{L}} K)[1]). \tag{9.5}$$

This identification depends on the choice of a lifting of E to $\mathbb{O}'$, but immediately implies (iii).

9.6. To understand the dependence of (9.5) on the choice of a lifting we can use classical techniques to get an action (suitably defined) of the right side of (9.5) on $\mathrm{Def}_\infty(E)$.

For two surjections $\mathbb{O}'_i \to \mathbb{O}$ $(i = 1, 2)$ with square-zero kernels K_i, the natural functor

$$\mathrm{Def}_{\infty,\mathbb{O}'\times_{\mathbb{O}}\mathbb{O}''}(E) \to \mathrm{Def}_{\infty,\mathbb{O}'}(E) \times \mathrm{Def}_{\infty,\mathbb{O}''}(E) \tag{9.7}$$

is an equivalence if both sides are nonempty, where in the subscripts we indicate which square-zero surjection to $\mathbb{O}$ we are considering. Now observe that

$$\mathbb{O}' \times_{\mathbb{O}} \mathbb{O}' \simeq \mathbb{O}'[K] \simeq \mathbb{O}' \times_{\mathbb{O}} \mathbb{O}[K].$$

We therefore get a map

$$\mathrm{Def}_{\infty,\mathbb{O}'}(E) \times \mathrm{Def}_{\infty,\mathbb{O}[K]}(E) \to \mathrm{Def}_{\infty,\mathbb{O}'}(E). \tag{9.8}$$

The retraction $\mathbb{O} \to \mathbb{O}[K]$ induces a canonical lifting of E and therefore using (9.5) we have a canonical isomorphism

$$\mathrm{Def}_{\infty,\mathbb{O}[K]}(E) \simeq \mathrm{DK}(\tau_{\leq 0}\, \mathrm{RHom}(E, E \otimes^{\boldsymbol{L}} K)[1]),$$

and (9.8) can be written as a map

$$\mathrm{Def}_{\infty,\mathbb{O}'}(E) \times \mathrm{DK}(\tau_{\leq 0}\, \mathrm{RHom}(E, E \otimes^{\boldsymbol{L}} K)[1]) \to \mathrm{Def}_{\infty,\mathbb{O}'}(E).$$

Passing to isomorphism classes we get an action of $\mathrm{Ext}^1(E, E \otimes^{\boldsymbol{L}} K)$ on the set of isomorphism classes in $[\mathrm{Def}_\infty(E)]$, and this action is simply transitive when there exists a lifting in light of the isomorphism (9.5). This gives (ii).

9.9. To define the obstruction we use simplicial techniques. As noted in the introduction, the work in this article is naturally viewed in the context of formal moduli problems in the sense of [Lurie 2018]. The interested reader may wish to consult the introduction to Chapter IV in [Lurie 2018] for more on this perspective.

Choose an inclusion

$$K \hookrightarrow J$$

with J an injective $\mathbb{O}$-module and let $I_\bullet$ denote the simplicial $\mathbb{O}$-module corresponding to the two-term complex

$$J \xrightarrow{\mathrm{id}_J} J \tag{9.10}$$

concentrated in degrees -1 and 0. So we have an inclusion of simplicial $\mathbb{O}$-modules $K \hookrightarrow I_\bullet$. Let $\bar{I}_\bullet$ denote the cokernel. The simplicial module $\bar{I}_\bullet$ is the simplicial module associated to the two-term complex

$$J \to J/K,$$

which is quasi-isomorphic to $K[1]$.

Let $\widetilde{\mathbb{O}}$ denote the simplicial ring obtained by pushout from the diagram

$$\begin{array}{ccc} K & \hookrightarrow & \mathbb{O}' \\ \downarrow & & \\ I_\bullet & & \end{array}$$

So $\widetilde{\mathbb{O}}$ comes equipped with a surjective map to $\mathbb{O}$ with kernel $I_\bullet$. Note also that the further pushout of $\widetilde{\mathbb{O}}$ along $I_\bullet \to \bar{I}_\bullet$ is canonically isomorphic to $\mathbb{O}[\bar{I}_\bullet]$.

Remark 9.11. Note that here we are using the Dold–Kan correspondence applied to the complex (9.10) and then forming the pushout in the category of simplicial rings to obtain $\widetilde{\mathbb{O}}$. In characteristic 0 one could also first consider the pushout in the category of commutative differential graded algebras, but in general it is preferable to work in the category of commutative simplicial rings.

Lemma 9.12. *The natural map* $\mathcal{D}_{\mathrm{perf}}(S, \widetilde{\mathbb{O}}) \to \mathcal{D}_{\mathrm{perf}}(S, \mathbb{O})$ *is an equivalence.*

Proof. This follows from [Schwede and Shipley 2000, Theorem 4.4], which implies that the functor induces an equivalence of homotopy categories, and the description of the fibers given in 2.22. □

9.13. Given a perfect complex E of $\mathbb{O}$-modules, we therefore get a perfect complex $\widetilde{E}$ of $\widetilde{\mathbb{O}}$-modules. Pushing out this complex $\widetilde{E}$ along the natural map

$$\widetilde{\mathbb{O}} \to \mathbb{O}[\bar{I}_\bullet]$$

we get a class in

$$\mathrm{Ext}^1_{\mathbb{O}}(E, E \otimes^{\boldsymbol{L}} \bar{I}_\bullet) \simeq \mathrm{Ext}^1_{\mathbb{O}}(E, E \otimes^{\boldsymbol{L}} K[1]) \simeq \mathrm{Ext}^2_{\mathbb{O}}(E, E \otimes^{\boldsymbol{L}} K).$$

We define

$$\omega(E) \in \mathrm{Ext}^2_{\mathbb{O}}(E, E \otimes^{\boldsymbol{L}} K)$$

to be this class. We see in Section 9.18 below that the class $\omega(E)$ vanishes if and only if E lifts to $\mathbb{O}'$.

9.14. ***Gabber's construction of the obstruction.*** In a 2005 email to Illusie, Gabber gave a construction of an obstruction to deforming a perfect complex which is more direct. The equivalence with the definition of the class $\omega(E)$ can be see as follows.

Let $\mathbb{O}' \to \mathbb{O}$ and E be as in 9.1. Gabber defines a class

$$o(E) \in \mathrm{Ext}^2_{\mathbb{O}}(E, E \otimes^{\boldsymbol{L}}_{\mathbb{O}} K),$$

which vanishes if and only if there exists a deformation of E to $\mathbb{O}'$. In fact, Gabber's construction is more general and can also be considered for nonperfect complexes.

9.15. The class $o(E)$ is constructed as follows. Assume that E is represented by a bounded above complex of flat modules. We work directly with complexes.

Choose a bounded above complex G of flat $\mathbb{O}'$-modules which is acyclic and a surjective map

$$G \to E.$$

Let S be the kernel, so we have

$$0 \to S \to G \to E \to 0.$$

From the snake lemma applied to the diagram

$$\begin{array}{ccccccccc} & & S \otimes K & \longrightarrow & G \otimes K & \longrightarrow & E \otimes K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & S & \longrightarrow & G & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

we get an exact sequence of complexes

$$0 \to E \otimes_{\mathbb{O}} K \to S \otimes_{\mathbb{O}'} \mathbb{O} \to G \otimes_{\mathbb{O}'} \mathbb{O} \to E \to 0. \tag{9.16}$$

Let $o(E) \in \mathrm{Ext}^2_{\mathbb{O}}(E, E \otimes_{\mathbb{O}} K)$ denote the class of this Yoneda extension. A straightforward exercise shows that this class is independent of the choice of $G \to E$.

Proposition 9.17. $$o(E) = \omega(E).$$

Proof. We proceed with notation as in 9.9. Let $\widetilde{E} \to E$ be the complex of $\widetilde{\mathbb{O}}$-modules over E provided by the equivalence in Lemma 9.12. Abusing notation, we consider this as a complex of $N(\widetilde{\mathbb{O}})$-modules. Choose $G \to \widetilde{E}$ a surjection of complexes of $\mathbb{O}'$-modules with G an acyclic bounded above complex of flat $\mathbb{O}'$-modules. We then get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & G & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & E \otimes_{\mathbb{O}} N(I_\bullet) & \longrightarrow & \widetilde{E} & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

Tensoring with $\mathscr{O}$ as above we get a commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & E\otimes_{\mathscr{O}} K & \longrightarrow & S\otimes_{\mathscr{O}'}\mathscr{O} & \longrightarrow & G\otimes_{\mathscr{O}'}\mathscr{O} & \longrightarrow & E & \longrightarrow 0 \\
& & \searrow & \downarrow & & \downarrow & & \| & \\
& & & E\otimes N(I_\bullet) & & & & & \\
& & & \downarrow & & & & & \\
& 0 & \longrightarrow & E\otimes N(\bar{I}_\bullet) & \longrightarrow & \widetilde{E}\otimes_{N(\widetilde{\mathscr{O}})} N(\mathscr{O}[\bar{I}_\bullet]) & \longrightarrow & E & \longrightarrow 0
\end{array}$$

From this it follows that $o(E)$ is the image under the boundary map

$$\mathrm{Ext}^1_{\mathscr{O}}(E, E\otimes^{\boldsymbol{L}} N(\bar{I}_\bullet)) \to \mathrm{Ext}^2_{\mathscr{O}}(E, E\otimes^{\boldsymbol{L}} K)$$

of the class in $\mathrm{Ext}^1_{\mathscr{O}}(E, E\otimes^{\boldsymbol{L}} N(\bar{I}_\bullet))$ given by the extension

$$0\to E\otimes_{\mathscr{O}} N(\bar{I}_\bullet)\to \widetilde{E}\otimes_{N(\widetilde{\mathscr{O}})} N(\mathscr{O}[\bar{I}_\bullet])\to E\to 0,$$

which in turn implies that $o(E)=\omega(E)$. □

9.18. $\boldsymbol{\omega(E)=0}$ *if and only if E lifts.*

Proposition 9.19. *The class $\omega(E)$ is 0 if and only if E lifts to $\mathscr{O}'$.*

Proof. We show this using Gabber's description of $\omega(E)$.

If E lifts to a complex E' over $\mathscr{O}'$ then as above we can choose $G\to E$ that factors through a map $G\to E'$. Examining the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S & \longrightarrow & G & \longrightarrow & E & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & E\otimes_{\mathscr{O}} K & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & 0
\end{array}$$

The map $S\to E\otimes_{\mathscr{O}} K$ induces a retraction of the inclusion $E\otimes_{\mathscr{O}} K\hookrightarrow S\otimes_{\mathscr{O}'}\mathscr{O}$ and therefore the corresponding Yoneda class $\omega(E)$ is zero.

Conversely, suppose $\omega(E)=0$. Fix $G\to E$ as above, and let T denote the image of the map

$$S\otimes_{\mathscr{O}'}\mathscr{O}\to G\otimes_{\mathscr{O}'}\mathscr{O}.$$

We then have short exact sequences

$$0\to E\otimes_{\mathscr{O}} K\to S\otimes_{\mathscr{O}'}\mathscr{O}\to T\to 0$$

and

$$0\to T\to G\otimes_{\mathscr{O}'}\mathscr{O}\to E\to 0.$$

The first of these defines a class

$$\alpha\in \mathrm{Ext}^1_{\mathscr{O}}(T, E\otimes_{\mathscr{O}} K)$$

whose image under the boundary map

$$\mathrm{Ext}^1_{\mathscr{O}}(T, E \otimes_{\mathscr{O}} K) \to \mathrm{Ext}^2_{\mathscr{O}}(E, E \otimes_{\mathscr{O}} K)$$

defined by the second sequence is the class $\omega(E)$. Since G is acyclic this boundary map is injective (since $\mathrm{Ext}^1_{\mathscr{O}}(G \otimes_{\mathscr{O}'} \mathscr{O}, E \otimes_{\mathscr{O}} K)$ surjects onto the kernel), so the class α is 0. Thus there exists a morphism $r : S \otimes_{\mathscr{O}'} \mathscr{O} \to E \otimes_{\mathscr{O}} K$ in the derived category splitting the inclusion $E \otimes_{\mathscr{O}} K \hookrightarrow S \otimes_{\mathscr{O}'} \mathscr{O}$. Forming the pushout of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & G & \longrightarrow & E & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle r} & & & & & & \\ & & E \otimes_{\mathscr{O}} K & & & & & & \end{array}$$

we obtain E' lifting E. □

9.20. In fact a bit more is true. Note that the natural map of simplicial rings

$$\mathscr{O}' \to \mathscr{O} \times_{\mathscr{O}[\bar{I}_\bullet]} \widetilde{\mathscr{O}}$$

is an isomorphism, so there is a commutative square

$$\begin{array}{ccc} \mathscr{D}_{\mathrm{perf}}(\boldsymbol{S}, \mathscr{O}') & \longrightarrow & \mathscr{D}_{\mathrm{perf}}(\boldsymbol{S}, \widetilde{\mathscr{O}}) \\ \big\downarrow & & \big\downarrow \\ \mathscr{D}_{\mathrm{perf}}(\boldsymbol{S}, \mathscr{O}) & \longrightarrow & \mathscr{D}_{\mathrm{perf}}(\boldsymbol{S}, \mathscr{O}[\bar{I}_\bullet]) \end{array} \tag{9.21}$$

Lemma 9.22. *The square (9.21) is homotopy cartesian.*

Proof. This follows from the preceding discussion combined with 2.22, which implies that the map on fibers is an equivalence. □

Remark 9.23. In the case of the punctual topos, the above results are special cases of the results in [Lurie 2018, §16.2].

Remark 9.24. We have formulated Theorem 9.2 in classical terms using the category $\mathrm{Def}(E)$. This forces, in particular, the statement of Theorem 9.2(iii) to incorporate the assumption of vanishing Ext^{-1}. The proof, however, shows that if one considers instead the category $[\mathrm{Def}_\infty(E)]$, then statements (i)–(iii) all hold and no assumption of vanishing Ext^{-1} is needed in (iii). Furthermore, if the obstruction is zero then the choice of a lifting of E identifies $[\mathrm{Def}_\infty(E)]$ with the Picard category associated to the two-term complex

$$\tau_{\geq -1}\tau_{\leq 0}(\mathrm{RHom}(E, E \otimes^{\boldsymbol{L}} K)[1]).$$

10. Proof of Theorem 1.6

We continue with the notation of the preceding section.

10.1. For statement (i) of Theorem 1.6 note that there is a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{D}_{\mathrm{perf}}(S,\mathbb{O})^{\simeq} & \xleftarrow{\simeq} & \mathcal{D}_{\mathrm{perf}}(S,\widetilde{\mathbb{O}})^{\simeq} & \longrightarrow & \mathcal{D}_{\mathrm{perf}}(S,\mathbb{O}[\bar{I}_\bullet])^{\simeq} & \longleftarrow & \mathrm{DK}\,\tau_{\leq 0}\,\mathrm{RHom}(E, E\otimes^{L} K[2]) \\
\downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \mathrm{tr}} \\
\mathcal{P}ic^{Z}_{\mathbb{O}} & \xleftarrow{\simeq} & \mathcal{P}ic^{Z}_{\widetilde{\mathbb{O}}} & \longrightarrow & \mathcal{P}ic^{Z}_{\mathbb{O}[\bar{I}_\bullet]} & \longleftarrow & R\Gamma^{\Delta}(BK[1])
\end{array} \tag{10.2}$$

By definition the obstruction class $\omega(E)$ is obtained as follows: Let $\widetilde{E}\in[\mathcal{D}_{\mathrm{perf}}(S,\widetilde{\mathbb{O}})]$ be the corresponding object. Then the pushout of $\widetilde{E}$ to $\mathbb{O}[\bar{I}_\bullet]$ is a deformation of E to $\mathbb{O}[\bar{I}_\bullet]$ and therefore defines an isomorphism class in the fiber

$$\mathrm{DK}(\tau_{\leq 0}\,\mathrm{RHom}(E, E\otimes^{L} K[2])).$$

The class $\omega(E)\in \mathrm{Ext}^2(E, E\otimes^{L} K)$ is the class of this isomorphism class.

The obstruction class $\omega(\det(E))$ is obtained similarly from the bottom row of (10.2). Statement (i) in Theorem 1.6 therefore follows from the commutativity of (10.2).

Remark 10.3. Note that the construction of the obstruction class $\omega(\mathcal{L})$ for an invertible $\mathbb{O}$-modules is additive in $\mathcal{L}$ in the sense that for two invertible $\mathbb{O}$-modules $\mathcal{L}$ and $\mathcal{M}$ we have

$$\omega(\mathcal{L}\otimes\mathcal{M})=\omega(\mathcal{L})+\omega(\mathcal{M}).$$

10.4. Next we turn to statements (ii) and (iii) in Theorem 1.6. Write $\mathcal{P}ic^{Z}_{\det(E),\mathbb{O}'}$ for the fiber product of the diagram

$$\begin{array}{ccc}
 & & \mathcal{P}ic^{Z}_{\mathbb{O}'} \\
 & & \downarrow \\
\star & \xrightarrow{\det(E)} & \mathcal{P}ic^{Z}_{\mathbb{O}}
\end{array}$$

and similarly for $\mathbb{O}[K]$ and $\mathbb{O}'\times_{\mathbb{O}}\mathbb{O}'$. By Proposition 8.9 the following diagrams commute:

$$\begin{array}{ccccc}
\mathcal{F}_{E,\mathbb{O}'}\times\mathcal{F}_{E,\mathbb{O}'} & \xleftarrow{(9.7)} & \mathcal{F}_{E,\mathbb{O}'\times_{\mathbb{O}}\mathbb{O}'} & \xrightarrow{\simeq} & \mathcal{F}_{E,\mathbb{O}'}\times\mathcal{F}_{E,\mathbb{O}[K]} \\
\downarrow{\scriptstyle \det\times\det} & & \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det\times\det} \\
\mathcal{P}ic^{Z}_{\det(E),\mathbb{O}'}\times\mathcal{P}ic^{Z}_{\det(E),\mathbb{O}'} & \longleftarrow & \mathcal{P}ic^{Z}_{\det(E),\mathbb{O}'\times_{\mathbb{O}}\mathbb{O}'} & \xrightarrow{\simeq} & \mathcal{P}ic^{Z}_{\det(E),\mathbb{O}'}\times\mathcal{P}ic^{Z}_{\det(E),\mathbb{O}[K]}
\end{array}$$

$$\begin{array}{ccc} \mathscr{F}_{E,\mathbb{O}'} \times \mathscr{F}_{E,\mathbb{O}[K]} & \xrightarrow{\simeq} & \mathscr{F}_{E,\mathbb{O}'} \times \mathrm{DK}\, \tau_{\leq 0}\, \mathrm{RHom}(E, E \otimes^{\boldsymbol{L}} K[1]) \\ \downarrow{\scriptstyle \det\times\det} & & \downarrow{\scriptstyle \det\times\mathrm{tr}} \\ \mathscr{P}ic^{\boldsymbol{Z}}_{\det(E),\mathbb{O}'} \times \mathscr{P}ic^{\boldsymbol{Z}}_{\det(E),\mathbb{O}[K]} & \xrightarrow{\simeq} & \mathscr{P}ic^{\boldsymbol{Z}}_{\det(E),\mathbb{O}'} \times R\Gamma^{\Delta}(BK) \end{array}$$

$$\begin{array}{ccc} \mathscr{F}_{E,\mathbb{O}'\times_{\mathbb{O}}\mathbb{O}'} & \longrightarrow & \mathscr{F}_{E,\mathbb{O}'} \\ {\scriptstyle \det}\downarrow & & {\scriptstyle \det}\downarrow \\ \mathscr{P}ic^{\boldsymbol{Z}}_{\det(E),\mathbb{O}'\times_{\mathbb{O}}\mathbb{O}'} & \longrightarrow & \mathscr{P}ic^{\boldsymbol{Z}}_{\det(E),\mathbb{O}'} \end{array}$$

From this and the construction of the action in 9.6 we get statements (ii) and (iii) in Theorem 1.6.

11. An alternate proof of Theorem 1.6(i) in the case of global resolutions

For basic facts about the filtered derived category, see [Illusie 1971, Chapter V].

As in the previous section we work with a site $\boldsymbol{S}$ and consider a surjective map of sheaves of rings $\mathbb{O}' \to \mathbb{O}$ with square-zero kernel K.

11.1. Let $DF(\mathbb{O})$ denote the filtered derived category of perfect complexes E equipped with locally finite decreasing filtration $F_E^\bullet$ such that $F_E^i = 0$ for $i \ll 0$ and $F_E^i = E$ for $i \gg 0$, and such that each of the graded pieces $\mathrm{gr}^i\, E$ are perfect complexes (see for example [Illusie 1971, Chapter V, 3.1]). The category $DF(\mathbb{O})$ is a triangulated category.

There is a forgetful functor

$$\epsilon : DF(\mathbb{O}) \to D(\mathbb{O}), \quad (E, F_E^\bullet) \mapsto E,$$

and for each i a functor

$$\mathrm{gr}^i : DF(\mathbb{O}) \to D(\mathbb{O}), \quad (E, F_E^\bullet) \mapsto \mathrm{gr}^i\, E.$$

For $(E, F_E^\bullet) \in DF(A)$ we get by the same construction as in the unfiltered case a trace map

$$\mathrm{tr} : E \otimes E^\vee \to \mathbb{O},$$

where $\mathbb{O}$ is viewed as filtered with $F_{\mathbb{O}}^i = 0$ for $i < 0$ and $F_{\mathbb{O}}^0 = \mathbb{O}$.

Let $I \in D(\mathbb{O})$ be an object viewed as a filtered object with $F_I^i = 0$ for $i < 0$ and $F_I^0 = I$.

Proposition 11.2. *For any $u \in \mathrm{Hom}_{DF(\mathbb{O})}(E, E \otimes I)$ we have*

$$\mathrm{tr}(\epsilon(u)) = \sum_i \mathrm{tr}(\mathrm{gr}^i(u))$$

in $H^0(\boldsymbol{S}, I)$.

Proof. This is [Illusie 1971, Chapter V, Corollary 3.7.7]. □

11.3. Gabber's construction of the obstruction to deforming a perfect complex carries through in the filtered context as well.

Let $(E, F^\bullet_E) \in DF(\mathbb{O})$ be an object with each $\mathrm{gr}^i\, E$ a perfect complex. Repeating the discussion in 9.15, choose G to be a filtered complex such that for all i the complex F^i_G is a bounded above acyclic complex of $\mathbb{O}'$-modules and

$$F^i_G \to F^i_E$$

is surjective. Then S is also filtered and the sequence (9.16) becomes an exact sequence of filtered complexes. We then get a class

$$\tilde{o}(E, F^\bullet_E) \in \mathrm{Ext}^2_{DF(\mathbb{O})}(E, E \otimes^{\boldsymbol{L}}_{\mathbb{O}} K)$$

mapping to

$$o(E) \in \mathrm{Ext}^2_{D(\mathbb{O})}(E, E \otimes^{\boldsymbol{L}} K).$$

Moreover, by the construction the class

$$\mathrm{gr}^i(\tilde{o}(E, F^\bullet_E)) \in \mathrm{Ext}^2_{\mathbb{O}}(\mathrm{gr}^i\, E, \mathrm{gr}^i\, E \otimes^{\boldsymbol{L}} K)$$

is equal to the obstruction $o(\mathrm{gr}^i\, E)$.

11.4. Now by Proposition 11.2 we have

$$\mathrm{tr}(\epsilon(\tilde{o}(E))) = \sum \mathrm{tr}(\mathrm{gr}^i(\tilde{o}(E))).$$

Furthermore, since each gr^i is a triangulated functor we have

$$\mathrm{gr}^i(\tilde{o}(E)) = o(\mathrm{gr}^i\, E).$$

We conclude that

$$\mathrm{tr}(o(E)) = \sum_i \mathrm{tr}(o(\mathrm{gr}^i(E))).$$

Now we have

$$\det(E) = \bigotimes_i \det(\mathrm{gr}^i\, E)),$$

and by Remark 10.3 we have

$$o(\det(E)) = \sum_i o(\det(\mathrm{gr}^i\, E)).$$

We conclude the following:

Proposition 11.5. *If Theorem 1.6(i) holds for* $\mathrm{gr}^i\, E$ *for all* i *then Theorem 1.6(i) holds for* E.

Proof. Indeed we have

$$\mathrm{tr}(o(E)) = \sum_i \mathrm{tr}(o(\mathrm{gr}^i(E))) = \sum_i o(\det(\mathrm{gr}^i(E))) = o(\det(E)). \qquad \square$$

Remark 11.6. In particular, if E is a strictly perfect complex then we can consider the filtration on E whose successive quotients are E^i and conclude that Theorem 1.6(i) holds by Proposition 11.5 and the case of locally free sheaves (which is straightforward).

Appendix: $\mathcal{D}(S, A_\bullet)$ and dg-modules

Let Λ be a commutative ring, let S be a site, and let $\mathbb{O}$ be a sheaf of Λ-algebras on S.

For a sheaf of simplicial $\mathbb{O}$-algebras $A_\bullet$ on S there are different approaches to defining the associated ∞-categorical derived category of $A_\bullet$-modules. For the convenience of the reader we summarize here how one can compare the different approaches.

Let $\mathrm{Sh}(S, \mathbb{O})$ denote the category of sheaves of $\mathbb{O}$-modules, and let $C(S, \mathbb{O})$ denote the category of complexes of $\mathbb{O}$-modules.

A.1. ***Differential graded modules.*** For a strictly commutative differential graded algebra $\mathbb{O}$-algebra $B^\bullet$ (see for example [Stacks, Tag 061V] and [Stacks, Tag 061W]), we can consider its associated category of differential graded modules [Stacks, Tag 09JI], which we denote by $\mathrm{Mod}^{\mathrm{dg}}_{B^\bullet}$.

There is a forgetful functor

$$\Sigma : \mathrm{Mod}^{\mathrm{dg}}_{B^\bullet} \to C(S, \mathbb{O}).$$

We consider the flat model category structure on $C(S, \mathbb{O})$, defined in this generality in [Liu and Zheng 2012, Proposition 2.1.3]. Since the flat model category structure is monoidal, with respect to the usual tensor product of complexes, we can then use [Schwede and Shipley 2000, Theorem 4.1] to get a model category structure on $\mathrm{Mod}^{\mathrm{dg}}_{B^\bullet}$.

Note that $\mathrm{Mod}^{\mathrm{dg}}_{B^\bullet}$ is again a Λ-linear dg-category. We can therefore apply the construction of [Lurie 2017, Construction 1.3.1.6] to get an ∞-category

$$\mathcal{D}(S, B^\bullet) := N_{\mathrm{dg}}(\mathrm{Mod}^{\mathrm{dg},\circ}_{B^\bullet}),$$

where $\mathrm{Mod}^{\mathrm{dg},\circ}_{B^\bullet} \subset \mathrm{Mod}^{\mathrm{dg}}_{B^\bullet}$ denotes the subcategory of cofibrant-fibrant objects.

Remark A.2. One can show directly that the category $\mathrm{Mod}^{\mathrm{dg},\circ}_{B^\bullet}$ is pretriangulated in the sense of [Bondal and Kapranov 1990]. It follows from this and [Faonte 2017, Theorem 3.18] that $\mathcal{D}_{\mathrm{dg}}(S, B^\bullet)$ is a stable ∞-category.

A.3. The stable ∞-category $\mathcal{D}(S, B^\bullet)$ can also be described using the dg-category of cofibrant objects $\mathrm{Mod}^{\mathrm{dg,cof}}_{B^\bullet}$ as follows.

The inclusion

$$\mathrm{Mod}^{\mathrm{dg},\circ}_{B^\bullet} \hookrightarrow \mathrm{Mod}^{\mathrm{dg,cof}}_{B^\bullet}$$

induces a morphism of ∞-categories

$$N_{\mathrm{dg}}(\mathrm{Mod}_{B^\bullet}^{\mathrm{dg},\circ}) \to N_{\mathrm{dg}}(\mathrm{Mod}_{B^\bullet}^{\mathrm{dg,cof}}). \tag{A.4}$$

Lemma A.5. *The inclusion (A.4) admits a left adjoint in the sense of* [Lurie 2009, Definition 5.2.2.1].

Proof. Let $M \in \mathrm{Mod}_{B^\bullet}^{\mathrm{dg,cof}}$ be an object and let $i : M \to I$ be a trivial cofibration with I fibrant. Then $I \in \mathrm{Mod}_{B^\bullet}^{\mathrm{dg},\circ}$ and for any $E \in \mathrm{Mod}_{B^\bullet}^{\mathrm{dg},\circ}$ the natural map

$$\mathrm{Hom}^\bullet_{B^\bullet}(I, E) \to \mathrm{Hom}^\bullet_{B^\bullet}(M, E)$$

is an equivalence. Indeed it suffices that this map induces an isomorphism on H^i for each i, and by shifting for this it suffices to verify that it holds for H^0, where it holds since both sides are calculating morphisms in the homotopy category. It follows that i exhibits I as a localization of M in the sense of [Lurie 2009, Definition 5.2.7.6], and therefore by [Lurie 2009, Proposition 5.2.7.8] there exists a left adjoint of the inclusion (A.4). □

Lemma A.6. *Let W be the collection of equivalences in* $\mathrm{N}_{\mathrm{dg}}(\mathrm{Mod}_{B^\bullet}^{\mathrm{dg,cof}})$, *and consider the associated localization* $\mathrm{N}_{\mathrm{dg}}(\mathrm{Mod}_{B^\bullet}^{\mathrm{dg,cof}})[W^{-1}]$ (for the existence of this localization, see [Lurie 2017, Remark 1.4.3.2]). *Then the composition*

$$N_{\mathrm{dg}}(\mathrm{Mod}_{B^\bullet}^{\mathrm{dg},\circ}) \to \mathrm{N}_{\mathrm{dg}}(\mathrm{Mod}_{B^\bullet}^{\mathrm{dg,cof}}) \to \mathrm{N}_{\mathrm{dg}}(\mathrm{Mod}_{B^\bullet}^{\mathrm{dg,cof}})[W^{-1}]$$

is an equivalence of ∞-categories

$$N_{\mathrm{dg}}(\mathrm{Mod}_{B^\bullet}^{\mathrm{dg},\circ}) \to \mathrm{N}_{\mathrm{dg}}(\mathrm{Mod}_{B^\bullet}^{\mathrm{dg,cof}})[W^{-1}].$$

Proof. This follows from [Lurie 2009, Proposition 5.2.7.12]. □

A.7. ***Comparison.*** By [Lurie 2017, Proposition 1.3.5.3] there is a model category structure on $C(\boldsymbol{S}, \mathbb{O})$ in which a morphism $f : M \to N$ is a weak equivalence (resp. cofibration) if f is a quasi-isomorphism (resp. termwise injection), and fibrations are defined by the right lifting property with respect to trivial cofibrations. We refer to this as the *injective model structure*.

The derived ∞-category $\mathfrak{D}(\boldsymbol{S}, \mathbb{O})$ is defined in [Lurie 2017, Definition 1.3.5.8] as the dg-nerve

$$\mathfrak{D}(\boldsymbol{S}, \mathbb{O}) := N_{\mathrm{dg}}(C(\boldsymbol{S}, \mathbb{O})^\circ_{\mathrm{inj}})$$

of the fibrant-cofibrant objects $C(\boldsymbol{S}, \mathbb{O})^\circ_{\mathrm{inj}} \subset C(\boldsymbol{S}, \mathbb{O})$ with respect to the injective model structure. By [Lurie 2018, Corollary 2.1.2.3] the ∞-category $\mathfrak{D}(\boldsymbol{S}, \mathbb{O})$ is identified with the hypercomplete objects in $\mathrm{Mod}_{(\boldsymbol{S},\mathbb{O})}$. By [Lurie 2017, Proposition 1.3.5.15], and the observation that every object of $C(\boldsymbol{S}, \mathbb{O})$ is cofibrant, we deduce that

$$N(C(\boldsymbol{S}, \mathbb{O}))[W^{-1}] \to \mathfrak{D}(\boldsymbol{S}, \mathbb{O})$$

is an equivalence, where W denotes the set of quasi-isomorphisms. We therefore obtain an equivalence between $N(C(\boldsymbol{S}, \mathbb{O}))[W^{-1}]$ and the ∞-category of hypercomplete objects in $\mathrm{Mod}_{(\boldsymbol{S},\mathbb{O})}$.

By [Liu and Zheng 2012, Remark 2.1.4(1)] the identity functor is a Quillen equivalence between $C(\boldsymbol{S}, \mathbb{O})$ with the flat model category structure and $C(\boldsymbol{S}, \mathbb{O})$ with the injective model structure, and therefore by [Lurie 2017, Lemma 1.3.4.21] we can also describe $\mathcal{D}(\boldsymbol{S}, \mathbb{O})$ as

$$N(C(\boldsymbol{S}, \mathbb{O})^{\mathrm{cof}}_{\mathrm{fl}})[W^{-1}],$$

where $C(\boldsymbol{C}, \mathbb{O})^{\mathrm{cof}}_{\mathrm{fl}} \subset C(\boldsymbol{S}, \mathbb{O})$ are the cofibrant objects with respect to the flat model category structure.

If $A_\bullet$ is a simplicial $\mathbb{O}$-algebra with each A_n flat over $\mathbb{O}$, then the corresponding differential graded algebra $N(A_\bullet)$ is a cofibrant monoid in the monoidal model category $C(\boldsymbol{S}, \mathbb{O})$. Combining this with [Lurie 2017, Theorem 4.3.3.17] taking $B = \mathbb{O}$, using [Lurie 2017, Corollary 4.3.2.8], we find that

$$N(\mathrm{Mod}^{\mathrm{dg,cof}}_{(\boldsymbol{S},N(A_\bullet))})[W^{-1}] \simeq \mathrm{Mod}_{A_\bullet}(\mathcal{D}(\boldsymbol{S}, \mathbb{O})). \tag{A.8}$$

That is, the ∞-category associated to the model category of dg-modules over $N(A_\bullet)$ is equivalent to the ∞-category of hypercomplete objects in $\mathrm{Mod}_{(\boldsymbol{S},A_\bullet)}$.

In fact, the assumption that the A_n are flat over $\mathbb{O}$ is unnecessary. If $B_\bullet \to A_\bullet$ is an equivalence, then it follows from [Lurie 2017, Corollary 4.3.2.8] (applied to the ∞-category of $(A_\bullet, B_\bullet)$-bimodules in the category of $B_\bullet$-modules) that the restriction functor

$$\mathrm{Mod}_{A_\bullet}(\mathcal{D}(\boldsymbol{S}, \mathbb{O})) \to \mathrm{Mod}_{B_\bullet}(\mathcal{D}(\boldsymbol{S}, \mathbb{O}))$$

is an equivalence, and similarly by [Schwede and Shipley 2000, Theorem 4.4] the restriction

$$\mathrm{Mod}^{\mathrm{dg}}_{(\boldsymbol{S},N(A_\bullet))} \to \mathrm{Mod}^{\mathrm{dg}}_{(\boldsymbol{S},N(B_\bullet))}$$

is a Quillen equivalence. Thus if $B_\bullet \to A_\bullet$ is a cofibrant replacement we see that we also have (A.8) without assuming that $A_\bullet$ is flat. We summarize as follows (using also [Lurie 2017, Proposition 1.3.1.17]).

Theorem A.9. *Let $A_\bullet$ be a simplicial $\mathbb{O}$-algebra with associated differential graded algebra $N(A_\bullet)$. Then the ∞-category*

$$N_{\mathrm{dg}}(\mathrm{Mod}^{\mathrm{dg},\circ}_{(\boldsymbol{S},N(A_\bullet))}) \simeq N_{\mathrm{dg}}(\mathrm{Mod}^{\mathrm{dg,cof}}_{(\boldsymbol{S},N(A_\bullet))})[W^{-1}]$$

is equivalent to the ∞-category of hypercomplete objects in $\mathrm{Mod}_{(\boldsymbol{S},A_\bullet)}$. □

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