

Lévy walkers inside spherical shells with absorbing boundaries: Towards settling the optimal Lévy walk strategy for random searches

L. G. P. Caramês¹, Y. B. Matos², F. Bartumeus^{3,4,5}, C. G. Bezerra¹, T. Macrì^{1,6}, M. G. E. da Luz^{1,2},
E. P. Raposo⁷ and G. M. Viswanathan^{1,8}

¹*Department of Physics, Federal University of Rio Grande do Norte, 59078-900 Natal-RN, Brazil*

²*Departamento de Física, Universidade Federal do Paraná, 81531-980 Curitiba-PR, Brazil*

³*Centre d'Estudis Avançats de Blanes-CEAB-CSIC, Girona 17300, Spain*

⁴*CREAF, Universitat Autònoma de Barcelona, Cerdanyola del Vallès 08193, Spain*

⁵*ICREA, Institució Catalana de Recerca i Estudis Avançats, Barcelona 08010, Spain*

⁶*ITAMP, Harvard-Smithsonian Center for Astrophysics, Cambridge, Massachusetts 02138, USA*

⁷*Laboratório de Física Teórica e Computacional, Departamento de Física, Universidade Federal de Pernambuco, Recife-PE 50670-901, Brazil*

⁸*National Institute of Science and Technology of Complex Systems, Federal University of Rio Grande do Norte, 59078-900 Natal-RN, Brazil*



(Received 17 August 2022; accepted 29 October 2022; published 22 November 2022)

The Lévy flight foraging hypothesis states that organisms must have evolved adaptations to exploit Lévy walk search strategies. Indeed, it is widely accepted that inverse square Lévy walks optimize the search efficiency in foraging with unrestricted revisits (also known as nondestructive foraging). However, a mathematically rigorous demonstration of this for dimensions $D \geq 2$ is still lacking. Here we study the very closely related problem of a Lévy walker inside annuli or spherical shells with absorbing boundaries. In the limit that corresponds to the foraging with unrestricted revisits, we show that inverse square Lévy walks optimize the search. This constitutes the strongest formal result to date supporting the optimality of inverse square Lévy walks search strategies.

DOI: [10.1103/PhysRevE.106.054147](https://doi.org/10.1103/PhysRevE.106.054147)

I. INTRODUCTION

Lévy stable distributions, Lévy walks and flights, have attracted wide attention since the 1990s in areas as diverse as particle kinetics [1,2] and random lasers [3,4]. In particular, almost a quarter of a century ago it was proposed [5] that Lévy walks with an inverse square law distribution for the step lengths can lead to optimal search strategies since they maximize the encounter rate with sparse, randomly distributed, revisitable targets when the search restarts in the vicinity of the previously visited target (thus available for further visits) and with no information about the past behavior—an uninformed process. This key fact about “nondestructive foraging” has motivated the formulation of the Lévy flight foraging hypothesis in ecology which holds that for many species such optimization may have led to adaptations for Lévy walk search strategies [6,7]. When originally published, the result caused surprise, because it questioned and then overturned the assumption that organisms move solely according to Brownian motion. In the last couple of decades, however, these results have been exhaustively verified in many different instances [8], becoming widely accepted (see, e.g., Ref. [9]).

More recently there has been renewed interest in certain fundamental and formal aspects of the problem. For example, eventual theoretical findings against the optimality of inverse square Lévy walk searches for any spatial dimensions D [10] have been shown not to be applicable to the paradigmatic

nondestructive random search context [11]. But this debate is understandable given that although several concrete situations have pointed to the aforementioned optimality, it is mathematically very hard to prove (or disprove) and no general developments have appeared so far in the literature.

Aiming to provide an important advance towards finally settling positively the issue, here we study a very closely related problem that is hopefully much easier to deal with. Specifically, we investigate in detail a Lévy walker inside a two-dimensional annulus with absorbing boundaries. The analysis is also valid for higher dimensions by considering hyperspherical shells. The essential point is that the original foraging problem can thus be analyzed indirectly by proxy, through characterizing the first passage time (FPT) in annuli and shell geometries. Indeed, the first passage time can be thought as the basic building block, representing the finding of successive targets in the random search for many targets (see Fig. 1). The inner boundary represents the previously found target site in the foraging problem. The outer boundary replaces all other targets.

As shown below, our main result is that inverse square Lévy walks strategies become optimal for extremizing the mean FPT when the initial position becomes arbitrarily close to the inner border. In other words, the first passage time for a Lévy walker to reach the boundary of the annulus or shell is minimized under certain initial conditions that correspond

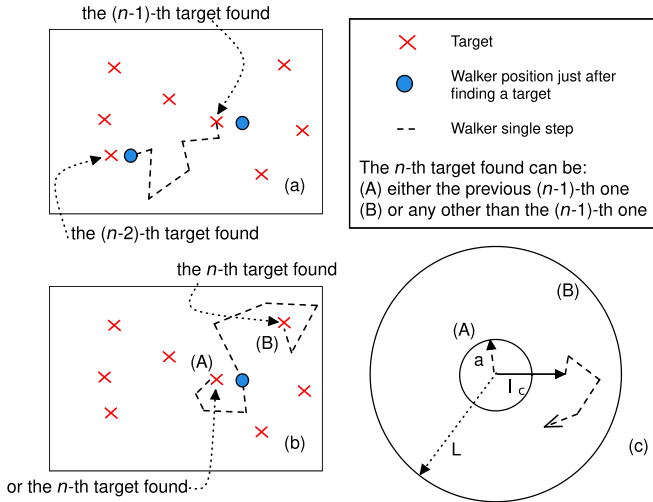


FIG. 1. The random search model is depicted in (a) and (b). (a) Always leaving from a position close to the previously found target, the walker follows the Lévy walk strategy (main text) looking for the next target. (b) Once the $(n-1)$ th target has been found, the n th one can be either a revisit to the previous $(n-1)$ th target [case (A)] or the finding of any other target [case (B)]. (c) A random walker (of similar locomotion rules of the random search model) inside an annulus geometry. The circumferences (A) and (B) represent the inner and outer annuli of radii a and L . l_c marks the restart point each time the walker reaches one of the annuli borders. There is a proxy between (A) and (B) in (b) and (A) and (B) in (c), where being absorbing means finding a target.

to the case of foraging with unrestricted revisits, i.e., for the nondestructive limit of the original foraging problem.

But before continuing, a few words about commonly used jargon would be in order. The traditional definition of “non-destructive foraging” seems not to capture the most general conditions for the relevant search scenario here. Instead, we adopt the term “foraging with (unrestricted) revisits.” Here, by “unrestricted” we mean, e.g., that there is no regenerative (or waiting) time [12,13] to approach any previously visited target, nor are some targets more difficult to reach than others, apart from their Euclidean distance to the walker. Although a subtle difference, we shall make sure that our analysis encompasses a large number of instances, so that the revisits could occur for several reasons besides just retracking regions or areas previously scanned: sensory errors, reemergent or replenishable targets, heterogeneous target distributions in patches that can be exploited in several visits, and so forth.

The foraging model and the model of search inside annuli and spherical shells are presented in Sec. II. Sections III and IV present analytical and numerical results, respectively. We end with concluding remarks in Sec. V.

II. THE MODELS

A. Foraging

The *foraging* model consists of a general strategy rule for the walker (of unit velocity, so that the distance traveled $L = t$, with t the traveled time) to search for targets randomly distributed in a D -dimensional space [Figs. 1(a) and 1(b)]:

(i) If there are targets within a radius of detection (or sight range) r_v , then the searcher moves in a straight line to the nearest target.

(ii) If there is no target at distance r_v from the searcher, then for its j th step, the searcher chooses a random direction and draws a distance from a power-law tailed probability distribution

$$p(\ell_j) \sim 1/\ell_j^\mu. \quad (1)$$

So, the walker starts moving incrementally, continually looking for a new target within sight radius r_v along the way. If no target is detected, the searcher stops after covering the distance ℓ_j and (ii) resumes. Otherwise, it proceeds according to rule (i).

The time-averaged search efficiency η is defined as

$$\eta = \lim_{t \rightarrow \infty} N/t, \quad (2)$$

where N is the number of targets found in time t .

In one dimension, the present foraging model can be solved analytically for the target density going to zero [5,14,15]. Specifically, the “mean-field” treatment in Ref. [5] was later rigorously established in Refs. [14,15] using fractional differential equations with a Riesz kernel. From such framework, one finds that inverse square Lévy walk searches are optimal in the limit of very low target density (ideally vanishing).

In two dimensions, the problem is substantially more difficult. Numerical simulations have strongly suggested that the one-dimensional (1D) result for the optimality of inverse square Lévy walks extends to 2D and 3D (see, e.g., Refs. [8,9,16], and references therein). At present, the rigorous analytical treatment of the general 2D problem is considered to be extremely hard, with the eventual exception of the limit that corresponds to the foraging with revisits (which just happens to be the most important case in real applications) as discussed in [11].

B. Walk inside absorbing annuli and spherical shells

A natural way to tame the original foraging problem, while still retaining most of its important ingredients, is to consider the simpler situation of a Lévy walker inside a 2D annulus, with absorbing boundaries at the inner (border A) and outer (border B) radii [see (c) in Fig. 1]. Indeed, to find either the previously found target (the closest one) or any other farther away bears a close relation to reaching one of these two “frontiers” (Fig. 1). Indeed, the inner circle or sphere represents the previously visited target, whereas the outer circle or sphere represents all the other targets averaged out.

Regarding the borders geometry, given the searcher detection radius and that it starts very close to the previously visited target, then in passing from a random search to a walk with absorbing borders, it is natural to map the closest target to a circle of radius associated to r_v (see below). By the same token, the distribution of the distances to all other nearest targets along any given direction from such previous target is replaced by its mean value, which is independent on the direction because of the distribution rotational symmetry. But in D dimensions the locus of points with a fixed distance from a reference—the previously found target—is the surface of a D -dimensional sphere, whose radius should be related to the

mean distance between targets in the original problem. Hence, the random walk takes place within the annuli or spherical shells region until absorption (akin to finding a target in the foraging process).

The dynamics of this simplified model obeys the following rules:

(1) The walker always starts (or restarts) at $(\ell_c, \theta = 0)$, close to the border of the inner circle ($\ell_c - a$ is positive but small compared to the outer radius L). It then moves between the inner and outer spherical shells, being absorbed upon hitting any of the two.

(2) At the j th step, the walker follows rule (ii) of the random searcher, with the difference that instead of looking for a target, the step terminates if ℓ_j is enough to reach one of the borders, when (1) then resumes.

The FPT efficiency of the above model is characterized as in Eq. (2), but with N now being the number of times the walker is absorbed by the annuli borders. Very importantly, it is simple to realize that there is a direct correspondence between optimizing the foraging η and minimizing the FPT for the absorbing shells problem.

C. The step lengths distribution

For the walk step lengths we consider the α -stable Lévy distribution, whose probability density $P(\ell; \alpha, \beta, d, s)$ reads

$$P(\ell; \alpha, \beta, d, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[\phi(t)] \exp[-i\ell t] dt, \quad (3)$$

with

$$\phi(t) = \begin{cases} itd - |st|^\alpha (1 - i\beta \operatorname{sgn}[t] \tan[\frac{\pi}{2}\alpha]), & \text{if } \alpha \neq 1, \\ itd - |st|(1 + i\beta \frac{2}{\pi} \operatorname{sgn}[t] \ln[|t|]), & \text{if } \alpha = 1, \end{cases}$$

for d, s real and $\beta \in [-1, 1]$. The Lévy index $\alpha \in (0, 2]$ governs the asymptotic behavior of $P(\ell; \alpha, \beta, \mu, s)$ in the form of the power-law tail $\sim 1/\ell^\mu$, with $\mu = \alpha + 1$ for $\alpha < 2$. For $\alpha = 2$ one recovers the Gaussian, since then the second moment is finite and the usual central limit theorem holds. Further, β, d , and s represent, respectively, the distribution asymmetry or skewness, shift or location, and the scaling for the ℓ variable.

In what follows, we assume without loss of generality that $p(\ell)$ is given by $P(\ell; \alpha, 0, 0, s)$. Taking $\beta = d = 0$ is justified because the model should have rotational symmetry [e.g., with $p(\ell) = p(-\ell)$] and then s can be interpreted as a width. With this choice, $\mu = \alpha + 1$ for $\alpha < 2$.

We should note that there is more than one way of generating Lévy walks in higher dimensions. The method we use here bears a resemblance to the uniform model of Lévy walks considered by Zaburdaev *et al.* [17]. Further detailed discussions lie beyond the scope of the present contribution. Hopefully, this rather technical aspect will be addressed in a future study.

But most importantly, due to the generalized central limit theorem for Lévy α -stable distributions [18–20], there are many power-law tailed distributions which should lead to similar results in the relevant particular context we shall address, namely, to solve the absorbing borders model.

III. ANALYTICAL RESULTS

In the foraging problem, there is a well-behaved relationship between the mean free path λ between targets and the target density ρ . With a being the radius of detection, in 2D we have $2a\lambda\rho = 1$. In D dimensions, we similarly have $\rho \sim 1/(a^{D-1}\lambda)$. Noting that for annuli and shells the mean free path goes with L , we can define the effective density according to

$$\rho = \frac{1}{a^{D-1}L}. \quad (4)$$

Equation (4) agrees with the definition for the foraging problem up to a constant factor.

Based on Fig. 1(c), Eqs. (1)–(4), remembering that s is the scaling factor of the step lengths distribution, and observing that in fact $\eta = \eta(\alpha, \delta, \rho, \sigma)$, we can define

$$\delta = \frac{\ell_c}{a} - 1, \quad \sigma = \frac{s}{a}, \quad \eta_0 = \frac{\eta}{\rho a^{D-1}} = \eta L. \quad (5)$$

Note that η_0 is adimensional.

The asymptotic behavior of our random walk inside annuli or D -dimensional spherical shells of inner radius a can be obtained analytically in the triple limit

$$\ell_c \rightarrow a \text{ (so } \delta \rightarrow 0), \quad L \rightarrow \infty, \quad s \rightarrow 0 \text{ with } s < \ell_c - a. \quad (6)$$

As discussed in [11], this corresponds to the key case of foraging with revisits. In fact, $\delta \rightarrow 0$ means, for the foraging model, that the previously visited target becomes revisitable again almost immediately, whereas in the annulus model, it corresponds to the walker starting extremely close to the inner circle (we recall the inner radius corresponds to the radius of detection for the foraging case). Further, in both models $\sigma \rightarrow 0$ implies that the smallest individual random walk steps go to zero. If this limit is not taken, for δ sufficiently small, the very first Lévy walk step might be larger than the distance to the previously found target in the foraging model, or else to the inner radius in the annulus model. In fact, we need $\ell_c - a \gg s$ to avoid difficulties with the first step being the only dominant, and this condition is equivalent to $\delta \gg \sigma$. Finally, $\rho \rightarrow 0$ (i.e., $L \rightarrow \infty$) corresponds to the target site density (outer circle radius) to be extremely low (large) in the foraging (annulus) model. We stress that in the next analysis, we have been careful to take the limits in the right order, i.e., first σ , then δ , and finally ρ .

Note that for $\delta \rightarrow 0$, the curvature of the inner circumference can be neglected: in the very few first steps, the walker sees the surface of the inner circle or sphere as a “flat wall” regardless of the dimension D . This is the reason why the walker behavior is well approximated by a one-dimensional description. In this way, the rigorous theory of the Riesz operator [15] on an interval of length L with absorbing ends becomes applicable [11]. Thence, for $\sigma > \delta$ the efficiency increases when σ decreases because there are fewer large jumps away from the previous target that makes reencountering it difficult [11]. When $\sigma \approx \delta$ the efficiency saturates and should reach its maximum. In fact, for $\sigma \approx \delta \rightarrow 0$ we must have the same scaling behavior as in $D = 1$.

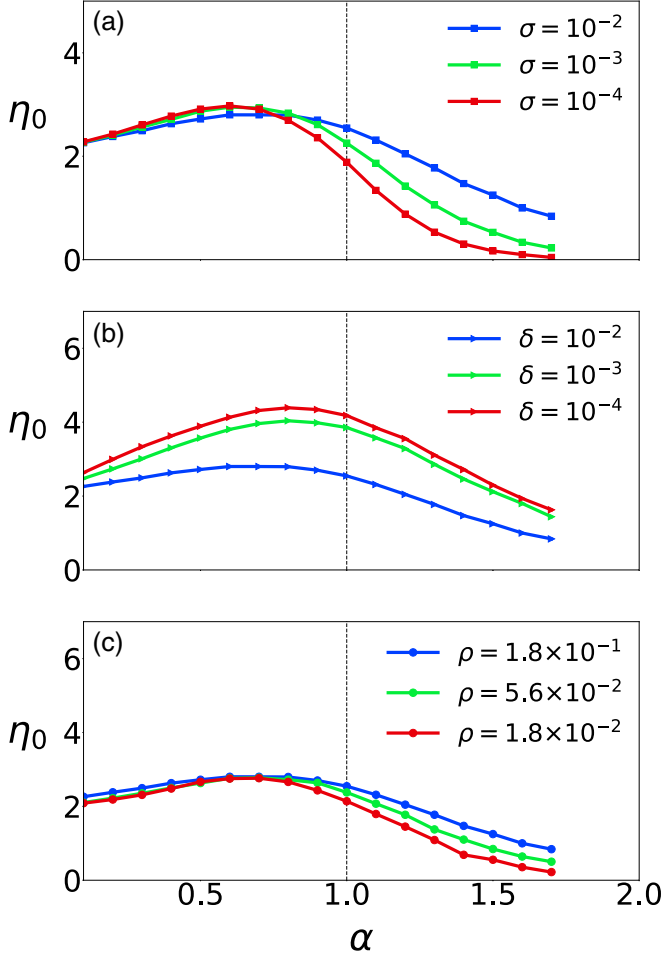


FIG. 2. Efficiency η_0 as a function of the Lévy index α for different values of (a) Lévy scale parameter σ ($\rho = 5.6 \times 10^{-2}$ and $\delta = 10^{-2}$), (b) the relative distance from the inner radius δ ($\sigma = 10^{-2}$ and $\rho = 5.6 \times 10^{-2}$), and (c) the effective density ρ ($\sigma, \delta = 10^{-2}$). The vertical line $\alpha = 1$ is a guide for the eyes.

The 1D behavior has been known exactly for a few decades (for details, see Refs. [14,15]). Extending it to the present case, we get

$$\eta_0 \sim \begin{cases} \delta^{-\alpha/2}, & \alpha < 1, \\ \delta^{-1+\alpha/2}, & \alpha > 1. \end{cases} \quad (7)$$

Here it is worth presenting a simplified summary of this optimization result. On the one hand, for $\alpha > 1$ the mean step size is finite and as α increases, the searcher spends larger and larger amounts of time backtracking, which increases the time to reach both outer and inner boundaries. The minimum time to reach the boundary is thus given by $\alpha \leq 1$. On the other hand, for $\alpha < 1$ the mean step size diverges and as α decreases, the probability of reaching the outer boundary on the first step increases, which reduces the probability of reaching the inner boundary which is extremely close. Hence, the optimal efficiency is given by $\alpha \geq 1$. The two inequalities, $\alpha \geq 1$ and $\alpha \leq 1$, are of course satisfied only if $\alpha = 1$.

In summary, as a fundamental result from Eq. (7), one has that η_0 displays an arbitrarily strong maximum at $\alpha = 1$

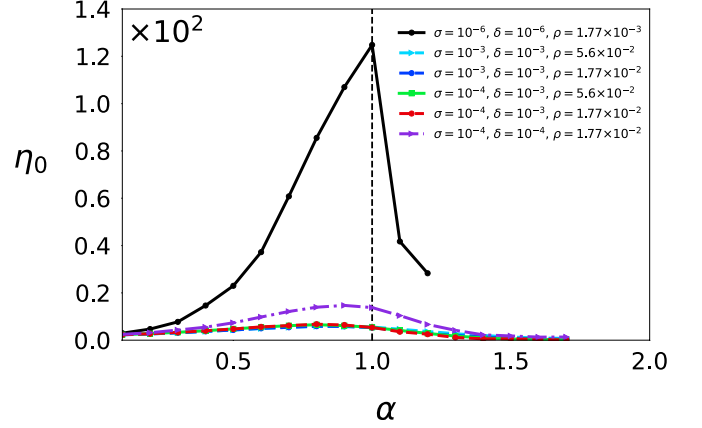


FIG. 3. Efficiency η_0 as a function of the Lévy index for various values of σ , δ , and ρ for search inside a 2D annulus. As these parameters tend to zero, the optimal Lévy index goes to 1, corresponding to inverse square Lévy walks. The efficiency η_0 is so large for the case $\sigma = \delta = 10^{-6}$ and $\rho = 1.77 \times 10^{-3}$ (black curve) that the other curves appear to be zero on a linear scale. For large $\alpha > 1.2$ some points are not shown because the computational runtime can become extremely large. What is important to note, however, is the behavior near the peak.

when $\sigma \approx \delta \rightarrow 0$ in any dimension. This is the key advance reported here for the annulus problem. See Appendices A and B for very similar results obtained via alternative arguments, respectively, for the foraging and annulus models. Moreover, a renormalization group analysis of how the efficiency scales with ρ is presented in Appendix C.

IV. NUMERICAL RESULTS

In what follows we verify the analytical derivations via numerical calculations (whose details about implementation are given in Appendix D). We first perform simulations for the parameter values still not corresponding to the proper limits discussed in the previous section. Then, we consider some representative situations to show that in these limits the simulations do corroborate our theoretical predictions.

We consider η_0 for small values of pertinent parameters, but not yet fully corresponding to the limit in Eq. (6). In Fig. 2 we plot η_0 as a function of α assuming different σ , ρ , and δ . For instance, when δ and ρ are fixed and σ is decreased, the efficiency η_0 for large α decreased, as seen in Fig. 2(a). This behavior should be expected since smaller σ 's implies smaller step sizes, hence making it less probable to reach the target in the first few steps. On the other hand, for σ and δ fixed, as ρ increases the relative prominence of the peak for η_0 increases [Fig. 2(c)]; again an expected result. Actually, an ever-increasing maximum for η_0 is only possible in the limit $\rho \rightarrow 0$. Lastly, by decreasing δ , while keeping ρ and σ , we observe an overall increase of the curve η_0 versus α . This agrees with the fact that for smaller δ the searcher will more frequently find the closer target regardless of α , thus increasing the total efficiency.

If σ , δ , and ρ , Eq. (5), go to zero in a proper manner [see Eq. (6)] we should expect η_0 to become arbitrarily large near $\alpha = 1$. Exactly as predicted by the theory, Fig. 3 shows how

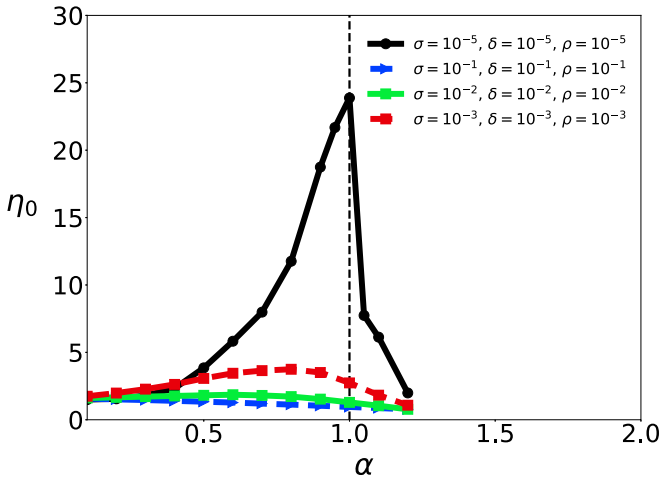


FIG. 4. Efficiency η_0 as a function of the Lévy index for various values of σ , δ , and ρ for search inside a 3D spherical shell. As predicted theoretically, a maximum emerges near $\alpha = 1$ as $\sigma, \delta, \rho \rightarrow 0$. Again, for $\alpha > 1.2$ the points are not shown because of the computational runtime, even more critical in 3D.

the limit $\delta, \sigma, \rho \rightarrow 0$ influences the shape of the efficiency curve. Observe that the peak shifts to $\alpha = 1$ for the case of search with unrestricted revisits. Finally, Fig. 4 shows that likewise the maximum at $\alpha = 1$ also occurs for search in 3D spherical shells, in precise agreement with our findings.

V. CONCLUSION

We have presented solid analytical results, checked through numerical simulations, showing that the inverse square Lévy walks optimize the time to reach the absorbing boundaries of annuli and spherical shells. Given the relationship of this problem to the original foraging model, these results should be expected to extend to the latter by proxy. Most importantly, our analysis unveils the real reason for the optimality of inverse square Lévy walk search strategies for foraging with unrestricted revisits (hence also nondestructive foraging) in any dimension. Regardless of the dimension D , the general process essentially reduces to the well-understood 1D model in the case of scarce distribution of targets, not unlike how the approximately spherical earth appears locally flat for small enough organisms. We hope that such findings can finally settle positively the key fact in the theory of Lévy random search.

ACKNOWLEDGMENTS

We thank Sergey V. Buldyrev for very carefully reading our manuscript and giving us very helpful feedback. We thank CAPES, CNPq, and FACEPE for funding. F.B. acknowledges Ministry of Science (MINECO) in Spain through the research Grant No. CGL2016-78156-C2-1-R.

APPENDIX A: THE FORAGING PROBLEM REVISITED

The foraging problem in 1D is equivalent to a Lévy walker inside an interval with absorbing boundaries. As explained in the main text, the one-dimensional case was rigorously

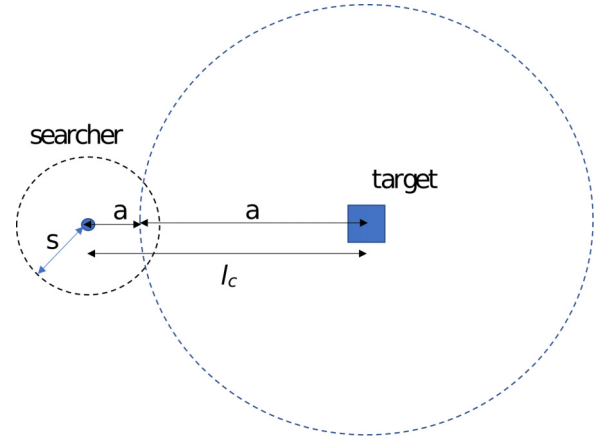


FIG. 5. The spatial disposition and relevant distances of the searcher (dot) and nearest target (blue square). The searcher's detection radius is a , the initial distance from the searcher to the target is l_c , and s is the scaling factor of the distribution of step lengths.

solved in 2001 (see Refs. [14,15]). In higher dimensions, however, the mathematical difficulties seem to be extremely high, probably demanding the development of new methods and tools for a proper solution. As far as we know, to date the only exception is the triple limit discussed in the main article. However, it is possible to treat the problem approximately, thus obtaining some insight into the main physically important features determining the optimal search strategy. So, in the following we give such an approximate treatment.

In Fig. 5 we show the initial condition of a nondestructive search in 2D. We define the dimensionless parameter $\delta = (l_c - a)/a$, with l_c the searcher initial distance to the nearest target and δa its distance to the circumference of radius a around the target. We assume the typical length scale $(\rho a)^{-1}$ calculated for a Lévy walker in 2D (Ref. [10]), so that $\eta = \eta_0 \rho a$. Comparing with Eq. (5) of Ref. [10], we have that $\eta_0 = K_d/a$ (the notation K_d is borrowed from Ref. [10], and represents the efficiency gain). The dimensionless searching efficiency is $\eta_0(\alpha, \delta, a, s) = f/\langle L \rangle$, where $f(\alpha, a, s)$ has dimensions of length (leaving η_0 dimensionless) and $\langle L \rangle(\alpha, \delta, a, s)$ is the average distance traversed until the encounter of a target. We shall calculate η_0 when the search is nondestructive, $\delta \rightarrow 0$, and in the scarce regime, $\rho \rightarrow 0$.

For δ and ρ going to zero, in 2D the encounter of the very close target (hereafter CT) essentially determines η_0 . In fact, in the 2D scarce regime, the probability for the faraway target—at a typical distance $\lambda \gg \delta a$ —to be the first one to be found is much lower than $[\delta a/\lambda]^{\alpha/2}$, the corresponding probability in 1D. Thus, in 2D with $\delta \rightarrow 0$ and $\rho \rightarrow 0$, to first order one should be concerned essentially with the finding of the CT. Now, to reach this nearby target in the first walk step we need $\sigma = s/a \approx \delta \rightarrow 0$, with s the scale of the α -stable Lévy distribution (see main text). However, if the searcher eventually misses the CT in the very first step, the next few successive steps still will lead to the CT provided the searcher does not move away too much from the CT location, say by continuously wandering around within a small region of radius na for n few units (see Fig. 5).

Therefore, it is a good approximation to assume that in the above-mentioned small region, the fractional diffusion equation that governs the Lévy searcher dynamics leads to solutions displaying basically the same qualitative behavior in 1D and 2D. This way, for $\delta \rightarrow 0$, $\rho \rightarrow 0$ in 2D we have $\eta_0 \sim f/(\langle n \rangle_{1D}(\ell))$, where the 1D result for a Lévy walker starting from a distance $\delta a \rightarrow 0$ to the absorbing CT within a distance na can be approximated as in Ref. [5], or

$$\langle n \rangle_{1D} \sim \left(\frac{\delta n a^2}{s^2} \right)^{\alpha/2}$$

and

$$\langle \ell \rangle \sim s \left[\left(\frac{\delta a}{s} \right)^{1-\alpha} + b \right],$$

where $b \sim 1$. We thus get

$$\frac{\langle n \rangle_{1D}(\ell)}{f} \sim \frac{a}{f} \delta^{1-\alpha/2}, \quad \text{with } \delta \rightarrow 0, \quad \rho \rightarrow 0, \quad \alpha > 1$$

and

$$\frac{\langle n \rangle_{1D}(\ell)}{f} \sim \frac{a}{f} \left(\frac{a}{s} \right)^{\alpha-1} \delta^{\alpha/2}, \quad \text{with } \delta \rightarrow 0, \quad \rho \rightarrow 0, \quad \alpha < 1.$$

Then, we obtain in the nondestructive ($\delta \rightarrow 0$) scarce ($\rho \rightarrow 0$) regime that η_0 scales with δ in the form

$$\eta_0 \sim \begin{cases} \delta^{-1+\alpha/2}, & \alpha > 1, \\ \delta^{-\alpha/2}, & \alpha < 1. \end{cases}$$

We once more shall stress that this is an approximate calculation. A far more grounded procedure is developed in Refs. [14,15]. Nonetheless, the above analysis gives the same result as the more rigorous approach in Ref. [11].

APPENDIX B: AN APPROXIMATE ANALYTICAL TREATMENT FOR THE ANNULUS PROBLEM

The 1D equivalent of an annulus or spherical shell is of course just an interval, which has already been discussed in the main text. Moreover, the foraging problem in 1D is identical to the problem of the walker inside an interval (with absorbing boundaries). But although in higher dimensions these two models are not exactly mapped into each other, they are still very closely related.

In the same spirit of the foraging problem in Appendix A, below we present an approximate solution for the absorbing annulus model. It is especially noteworthy that this treatment gives the same answer for the optimality of inverse square Lévy walks in the triple limit $\sigma, \rho, \delta \rightarrow 0$ obtained in the main text through mathematically more well founded arguments. Nonetheless, an interesting aspect of the approach below is that it is a kind of mean-field method for the present problem.

The searcher starts from a distance l_c of the center. For $\delta = (l_c - a)/a$ as before, we are interested in the $l_c \rightarrow a \ll b$ limit, i.e., $\delta \rightarrow 0^+$ for which the probability of the walker reaching the border $r = b$ is very small, $P \ll P_{1D}$. Note that we can approximate $P_{1D} = [(l_c - a)/(b - l_c)]^{\alpha/2}$ as the probability of finding the distant site in 1D, with the step lengths ℓ distributed as a power law of exponent $\mu = \alpha + 1$ (i.e., of

Lévy index α). The efficiency can then be written as

$$\eta \approx \frac{1}{\langle n \rangle \langle |\ell| \rangle}. \quad (\text{B1})$$

Thus, we focus on the case $\delta \rightarrow 0^+$ for the encounter with the $r = a$ ring and take into account only the walks that do not stray too far from the $r = a$ ring. In other words, we restrict the random walk to the region $r < \gamma a$, where γ must be of the order of unit. In such framework, the average number of steps is fairly given by the 1D result, but with $\lambda \rightarrow \gamma a$. Using $\eta \simeq 1/[\langle n \rangle \langle |\ell| \rangle]$, we obtain

$$\eta \sim \begin{cases} \delta^{-\alpha/2}, & \alpha < 1, \\ \delta^{\alpha/2-1}, & \alpha > 1, \end{cases}$$

which coincides with the 2D result in Ref. [11] for the $\delta \rightarrow 0^+$ limit (assuming a Poissonian distribution of targets whose density $\rho \rightarrow 0$). Thus [for x_0 the near ($\lambda - x_0$ the far away) target initial distance to a searcher in 1D],

$$\begin{aligned} \langle \eta \rangle_{1D} &= f(\alpha) \left[\frac{x_0(\lambda - x_0)}{s^2} \right]^{\alpha/2} \rightarrow \\ \langle \eta \rangle &\sim f_{2D}(\alpha) \left[\frac{\delta a(\gamma a - \delta a)}{s^2} \right]^{\alpha/2}, \end{aligned} \quad (\text{B2})$$

where s is the scale parameter of the Lévy distribution $P(\ell)$, defined from its characteristic function $\bar{P}(k) = \exp[-|sk|^\alpha]$ [see also $\phi(t)$ in the main text].

For a power law instead of a Lévy stable distribution, we can take $s = l_0$. The function $f(\alpha)$ in 1D depends on whether $p(\ell)$ is Lévy or power law, e.g., $f(\alpha) = 1/\Gamma(\alpha + 1)$ in the Lévy case (for Γ the gamma function). But for purposes of scaling laws of $\langle \eta \rangle$ when $\delta \rightarrow 0^+$, this prefactor is not important once it does not depend on δ .

Lastly, to compute $\langle |\ell| \rangle$ we also can adapt the calculations for 1D by just supposing walks in the region $r < \gamma a$. We can approximate following Ref. [5],

$$\langle |\ell| \rangle \sim c \left[\left(\frac{\delta a}{c} \right)^{1-\alpha} g(\eta, \alpha) + h(\alpha) \right]. \quad (\text{B3})$$

Here the functions g and h do not depend on δ . Thence

$$\langle \eta \rangle \langle |\ell| \rangle \sim f_{2D} \left[\frac{\delta a^2(\gamma - \delta)}{c^2} \right]^{\alpha/2} c \left[\left(\frac{\delta a}{c} \right)^{1-\alpha} g + h \right]. \quad (\text{B4})$$

Taking the limit $\delta \rightarrow 0^+$,

$$\langle \eta \rangle \langle |\ell| \rangle \sim \begin{cases} \delta^{\alpha/2}, & \alpha < 1, \\ \delta^{-\alpha/2+1}, & \alpha > 1, \end{cases}$$

by retaining only the dependence of $\langle \eta \rangle \langle |\ell| \rangle$ on δ .

Remarkably, such straightforward and easy-to-understand considerations yield a similar result to the procedure followed in the main text. Indeed, recall that there the analysis is as follows: (1) The problem of a 2D walk inside the annulus region is first established. Observe that in the general case, this problem has not been solved. (2) The limit is then taken of $\sigma \rightarrow 0$, $\delta \rightarrow 0$, $\rho \rightarrow 0$. Therefore, the problem becomes effectively one-dimensional. (3) Equation (7) is thus obtained in this limit.

But of course, from a fundamental point of view the findings in the main text represent a much more important and rigorous achievement.

APPENDIX C: RENORMALIZATION GROUP DERIVATION FOR THE EFFICIENCY SCALING WITH ρ

As should be clear from the main text, the association between the foraging and annulus models arises from the fact that the latter represents one target (CT) at the center, the origin, and the mean of all other targets (the FTs) are at the outer radius b , so that $\lambda = b$. Rigorously one should have $\lambda = b - a$, but we can neglect a in the low-density limit since then $b \gg a$.

Consider now an annulus with outer radius $L_0 = 1$ and inner radius a . In 2D we will define the effective density as usual, according to

$$\rho = 1/(2a\lambda), \quad (\text{C1})$$

where λ is the mean free path $\lambda = b$. Further, let T denote the mean first passage time and $v = 1$ the adimensional unity velocity, so that

$$\eta = \frac{1}{T}. \quad (\text{C2})$$

If now we have an absorbing annulus system of outer radius $L \gg L_0$, obviously its ρ will be much smaller than the density corresponding to L_0 . We then can use renormalization to map the two cases as the following. We set $\phi = L/L_0$ and suppose the mapping

$$L \mapsto L/\phi = L_0 = 1, \quad (\text{C3})$$

$$\lambda \mapsto \lambda/\phi = L_0 = 1, \quad (\text{C4})$$

$$a \mapsto a/\phi, \quad (\text{C5})$$

$$\rho \mapsto \rho \phi^D, \quad (\text{C6})$$

$$s \mapsto s/\phi, \quad (\text{C7})$$

$$l_c \mapsto l_c/\phi, \quad (\text{C8})$$

$$\delta \mapsto \delta, \quad (\text{C9})$$

$$T \mapsto T/\phi, \quad (\text{C10})$$

$$\eta \mapsto \eta \phi. \quad (\text{C11})$$

Now, assume that the problem for L_0 has been solved. By recalling that

$$\eta_0 = \eta L, \quad (\text{C12})$$

we have

$$\eta_0 \mapsto \eta_0[\phi/\phi] = \eta_0. \quad (\text{C13})$$

Since

$$\eta = \eta_0 \rho a^{D-1} \quad (\text{C14})$$

and given that η_0 is an invariant of the renormalization group flow map, we can expect to find $\eta \sim \rho$ for fixed a and δ (so fixed l_c).

The above is a nonrigorous scaling argument for the claim first published in Ref. [10] [and stated as proposition (i) in Ref. [11]], namely, that the efficiency η is linear in the density asymptotically. Note that only Eqs. (C3) and (C11) are needed; the other mappings are shown only for generality.

APPENDIX D: ALGORITHM

```
double rng_levy48(double alpha, double rr){
    double ee, phi;
    double mu=alpha-1;
    double mu1=mu-1;
    double xmu=1/mu;
    double xmu1=xmu-1;
    phi=(drand48()-0.5)*PI;
    ee=-log(drand48());
    return rr*sin(mu*phi)/
        pow(cos(phi),xmu)*pow(cos(phi*mu1)/ee,xmu1);
}
```

The random variables are generated from the Lévy α -stable distribution with asymmetry parameter $\beta = 0$ and zero mean; also, the scale is $s = rr$. The simulations were performed with a homemade code written in C (the language C has been chosen due to speed). The Lévy distributed random numbers were generated using the C code displayed in the chart.

For the computational simulations, the 2D algorithm checks, at each walk, whether or not the Lévy searcher has intercepted the inner or the outer annuli. Figure 6 depicts the algorithm flowchart. We use as parametrization $r(t) = r_0 + lt$, where r_0 is the walker starting position.

The algorithm consists of repeating many iterations of the walker always starting a distance l_c from the center, and performing successive Lévy walk steps one at a time. For each step, possible intersection points of the trajectory and the inner or outer annuli (or shells) are calculated, by simultaneously solving the equations for the line of the trajectory and the circles or shells, considered as quadratic equations for D -dimensional conic sections. The full procedure allows accessing intermediate values between the points r_0 and $r_f = r(t = 1)$ from the function $r(t)$, associating this parametrization to the equations of each annulus. We derive the t value necessary for the intersection from

$$t_{\text{inner}} = \frac{1}{2A}(-B \pm \sqrt{B^2 - 4A(C - a^2)}), \quad (\text{D1})$$

$$t_{\text{outer}} = \frac{1}{2A}(-B \pm \sqrt{B^2 - 4A(C - L^2)}), \quad (\text{D2})$$

with $A = 2l^2$, $B = 2(x + y)l$, and $C = x^2 + y^2$. Therefore, we have two solutions for the inner and two for the outer annulus. Those with values in the interval $(0, 1]$ indicate that there were one or more intersections. We choose the lowest t in the interval to compute the distance traveled $d = lt$. At the end of

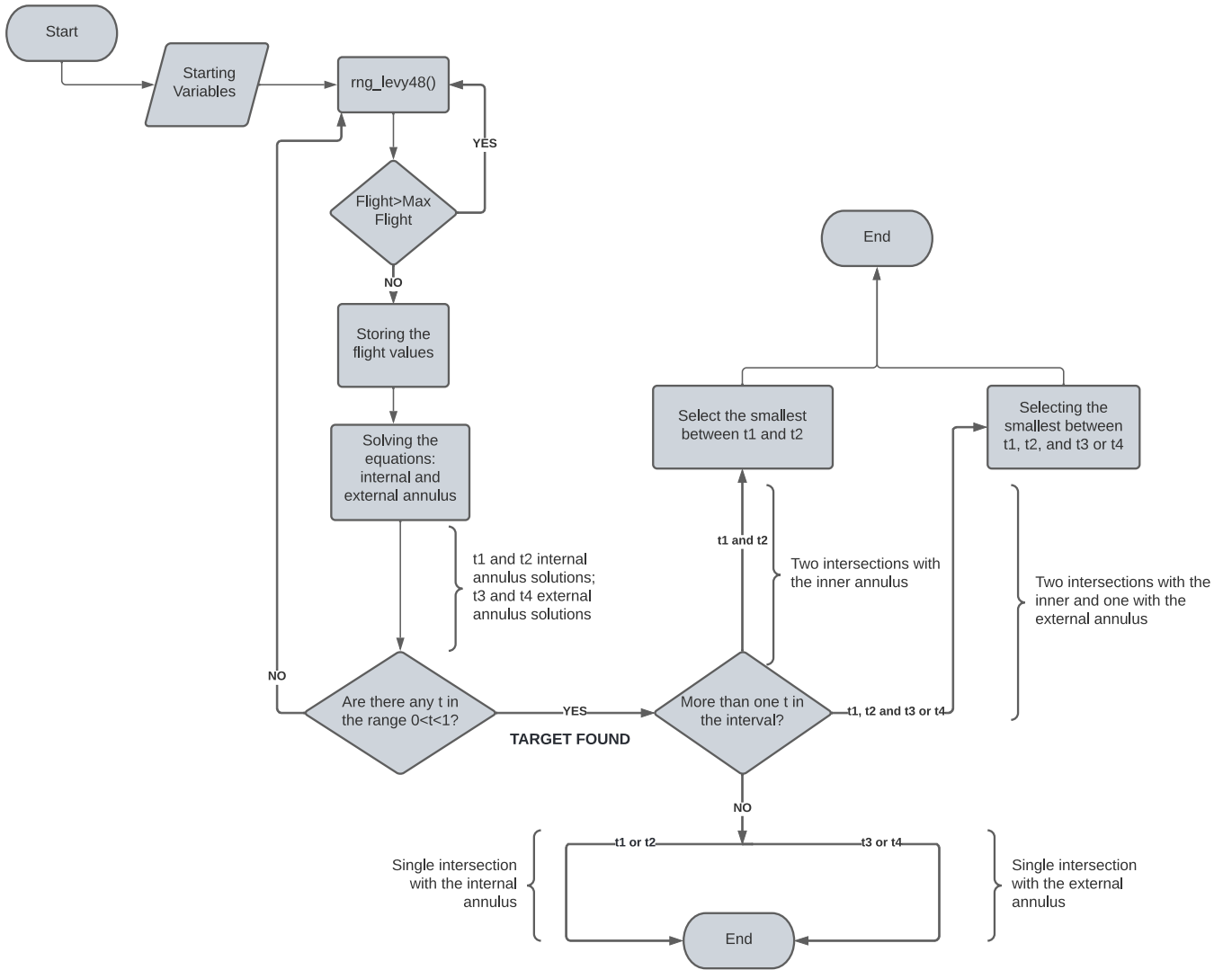


FIG. 6. Flowchart of the search algorithm for concentric annuli with absorbing boundaries.

the flight, the distance traveled is counted, and there are three possible subsequent actions: (i) if there is an intersection, the searcher will return to the starting point of the simulation, $(x, y) = (l_c, 0)$; (ii) if there is no intersection, the next flight

will depart from r_f ; (iii) if the total distance value exceeds a certain threshold, then the simulation ends.

The 3D version of the code was written by adapting the 2D code.

- [1] V. Zaburdaev, S. Denisov, and J. Klafter, *Rev. Mod. Phys.* **87**, 483 (2015).
- [2] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1982).
- [3] A. S. L. Gomes, A. L. Moura, C. B. de Araújo, and E. P. Raposo, *Prog. Quantum Electron.* **78**, 100343 (2021).
- [4] E. P. Raposo and A. S. L. Gomes, *Phys. Rev. A* **91**, 043827 (2015).
- [5] G. M. Viswanathan, S. V. Buldyrev, S. Havlin, M. G. E. da Luz, E. P. Raposo, and H. E. Stanley, *Nature (London)* **401**, 911 (1999).
- [6] M. G. E. da Luz, E. P. Raposo, and G. V. Viswanathan, *Phys. Life Rev.* **14**, 94 (2015).
- [7] M. E. Wosniack, M. C. Santos, E. P. Raposo, G. M. Viswanathan, and M. G. E. da Luz, *PLoS Comput. Biol.* **13**, e1005774 (2017).
- [8] F. Bartumeus and S. A. Levin, *Proc. Natl. Acad. Sci. USA* **105**, 19072 (2008).
- [9] G. M. Viswanathan, M. G. E. da Luz, E. P. Raposo, and H. E. Stanley, *The Physics of Foraging: An Introduction to Random Searches and Biological Encounters*, 1st ed. (Cambridge University Press, Cambridge, 2011).
- [10] N. Levernier, J. Textor, O. Bénichou, and R. Voituriez, *Phys. Rev. Lett.* **124**, 080601 (2020).
- [11] S. V. Buldyrev, E. P. Raposo, F. Bartumeus, S. Havlin, F. R. Rusch, M. G. E. da Luz, and G. M. Viswanathan, *Phys. Rev. Lett.* **126**, 048901 (2021).
- [12] E. P. Raposo, S. V. Buldyrev, M. G. E. da Luz, M. C. Santos, H. E. Stanley, and G. M. Viswanathan, *Phys. Rev. Lett.* **91**, 240601 (2003).
- [13] M. C. Santos, E. P. Raposo, G. M. Viswanathan, and M. G. E. da Luz, *Europhys. Lett.* **67**, 734 (2004).

- [14] S. V. Buldyrev, S. Havlin, A. Ya. Kazakov, M. G. E. da Luz, E. P. Raposo, H. E. Stanley, and G. M. Viswanathan, [Phys. Rev. E](#) **64**, 041108 (2001).
- [15] S. V. Buldyrev, M. Gitterman, S. Havlin, A. Ya. Kazakov, M. G. E. da Luz, E. P. Raposo, H. E. Stanley, and G. M. Viswanathan, [Physica A](#) **302**, 148 (2001).
- [16] F. Bartumeus, P. Fernández, M. G. E. da Luz, J. Catalan, R. V. Solé, and S. A. Levin, [Eur. Phys. J.: Spec. Top.](#) **157**, 157 (2008).
- [17] V. Zaburdaev, I. Fouxon, S. Denisov, and E. Barkai, [Phys. Rev. Lett.](#) **117**, 270601 (2016).
- [18] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables* (Addison-Wesley, Cambridge, 1954).
- [19] P. Lévy, *Théorie de l'Addition des Variables Aléatoires*, (Gauthier-Villars, Paris, 1937).
- [20] A. M. Reynolds and F. Bartumeus, [J. Theor. Biol.](#) **260**, 98 (2009).