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Pseudo-Hermitian Levin–Wen models from non-semisimple TQFTs

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ABSTRACT

We construct large classes of exactly solvable pseudo-Hermitian 2D spin Hamiltonians. The ground states of these systems depend only on the spatial topology of the system. We identify the ground state system on a surface with the value assigned to the surface by a non-semisimple TQFT generalizing the Turaev–Viro model. A non-trivial example arises from a non-semisimple subcategory of representations of quantum $sl(2)$ where the quantum parameter is specialized to a root of unity.

1. Introduction

There is a well developed correspondence connecting topologically ordered phases in 2-dimensional systems with the theory of unitary modular tensor categories (UMTCs). Many of the key properties of these systems, such as topological ground-state degeneracy and non-abelian braiding statistics of low-energy point-like excitations, make them candidates for topological quantum computing. These features can be formalized through UMTCs and their associated $2+1$-dimensional topological quantum field theories (TQFTs) [1–5].

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Levin and Wen constructed an important class of doubled, or time reverse symmetric, 2-dimensional phases utilizing a discretized model defined on a trivalent graph. The graph need not be planar and can be regarded as living on a genus g surface. Their 'string-net' model produces exactly solvable gapped Hamiltonians where the low-energy physics of a 2-dimensional topologically ordered phase is described by the condensation of string-like objects [4]. These models have been further generalized by modifying various assumptions [6–8]. The result is the general belief that the most general class of bosonic topological orders with gapped boundaries are described by the Drinfeld center of a fusion category. Indeed, in [9] the generalized string-net models producing exactly solvable Hermitian Hamiltonians are classified.

In this work, we extend the paradigm of bosonic topological order by introducing a new class of theories not arising as the Drinfeld center of a fusion category. A key difference in our approach is that, while the Hamiltonians in our systems have positive spectrum and normalizable wave functions, these Hamiltonians are not Hermitian. They are however *pseudo-Hermitian*, so that time evolution is given by the exponential of the Hamiltonian that is self adjoint with respect to an indefinite inner product. These inner products are also invariant under the time-evolution generated by the Schrodinger equation.

Quantum mechanics with an indefinite inner product has been studied going back to Dirac [10] and Pauli [11]. Even with the indefinite norms, they observed a formalism consistent with deterministic quantum mechanics. More recently, there has been a resurgence of interest in indefinite quantum mechanics coming from the study of $\mathcal{PT}$-symmetric quantum theory [12]. $\mathcal{PT}$-symmetric quantum theory removes the assumption that the Hamiltonian is Hermitian and replaces it by the condition that the Hamiltonian commutes with the antilinear operator $\mathcal{PT}$ of parity and time reversal, so that $[H, \mathcal{PT}] = 0$. Hermiticity is then replaced by the condition $H = H^{\mathcal{PT}}$ (see [13] for a good introduction). $\mathcal{PT}$-symmetric quantum theory was proposed as a way to measure physical phenomena in the absence of Hermitian Hamiltonians. Indeed, non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians have already been used to describe such phenomena as the ground state of a quantum system of hard spheres [14], Reggeon field theory [15], and the Lee–Yang edge singularity [16]. In each of these examples, the Hamiltonians have spectral positivity and the associated quantum theories are unitary because the Hamiltonians are $\mathcal{PT}$-symmetric.

It was discovered that $\mathcal{PT}$-symmetric quantum theory was not the most general criteria that would ensure that a given non-Hermitian Hamiltonian would have a real spectrum and unitary evolution [17–19]. Rather, the notion of *pseudo-Hermiticity* provides such a criteria. Any diagonalizable Hamiltonian admitting a symmetry generated by an invertible antilinear operator is pseudo-Hermitian, including the $\mathcal{PT}$ operator. A linear operator $H : \mathcal{H} \to \mathcal{H}$ acting on a Hilbert space $\mathcal{H} = (\mathcal{H}, \langle\cdot,\cdot\rangle_+)$ (here we are assuming positive-definite inner product $\langle\cdot,\cdot\rangle_+$) is called *pseudo-Hermitian*, or η-pseudo-Hermitian, if there exists a linear, invertible, Hermitian operator $\eta : \mathcal{H} \to \mathcal{H}$ such that

$$H^\sharp = \eta H \eta^{-1},$$

where $H^\sharp$ above is the usual Hermitian conjugate $\langle H^\sharp\psi, \phi\rangle_+ = \langle\psi, H\phi\rangle_+$ determined by the positive-definite form $\langle\cdot,\cdot\rangle_+$. We use this nonstandard notation since we will be primarily interested in a different inner product and the Hermitian adjoint with respect to that form.

If H is pseudo-Hermitian, the choice of such η is not unique. Each choice of η determines a possibly indefinite inner product, or *pseudo-inner product*, on $\mathcal{H}$ given by

$$\langle\psi, \phi\rangle := \langle\psi, \eta\phi\rangle_+ \,. \tag{1.1}$$

When a Hamiltonian H is η-pseudo-Hermitian, then it becomes an actual Hermitian operator with respect to the indefinite form $\langle,\rangle$. We write $H^\dagger = H$ to mean

$$\langle\psi, H\phi\rangle = \langle H\psi, \phi\rangle.$$

If H is pseudo-Hermitian, exponentiating iH produces an operator U satisfying

$$U^\sharp = \eta U^{-1} \eta^{-1}.$$

Such an operator is called pseudo-unitary (see [20]). The group of such operators is controlled by the group $U(n, m)$ with $n, m \in \mathbb{Z}^+$ determined from the signature of the inner product.

In this article, we show that η-pseudo-Hermitian Hamiltonians extending the Levin–Wen models arise naturally from recent non-semisimple modifications of the theory of $2+1$-dimensional state-sum TQFTs. As argued in [21], the key distinction between indefinite metric quantum mechanics and pseudo-Hermitian quantum mechanics is that the choice of η is additional data that is fixed ahead of time. Here we show that such an η arises naturally from topological considerations.

A key observation from [22–25] establishes a link between the exactly solvable Levin–Wen Hamiltonians defined on trivalent graphs on a surface Σ and the Turaev–Viro topological quantum field theory in three spatial dimensions. In this interpretation, the plaquette operators used to define the Hamiltonian arise from three-dimensional tetrahedra glued onto the triangulated surface of the model. Key properties of these plaquette operators, such as being projectors and mutually commuting, also have natural topological interpretations as change of triangulations in $2+1$-dimensions, (see Section 3.6 for more details). The projection onto the ground state is then given by the image of the operator assigned to $\Sigma \times [0, 1]$ by the Turaev–Viro TQFT.

More recently, there have been developments in the study of topological quantum field theories, so called, non-semisimple TQFTs that live outside the usual unitary modular tensor category framework. 'Non-semisimple' refers to the fact that they are built on tensor categories that do not satisfy the usual semisimple assumptions prevalent in nearly all categorical descriptions of topological phases. These TQFTs are governed by relative $\mathcal{G}$-spherical categories and depend on extra data including a Hamiltonian link in the 3-manifold M and a cohomology class $[\Phi] \in H_1(M, \mathcal{G})$. The key examples of such non-semisimple categories have an infinite number of nonisomorphic simple objects, all having vanishing quantum dimensions. Nevertheless, these non-semisimple TQFTs have remarkable properties, often proving more powerful than their semisimple analogs. For example, non-semisimple TQFTs lead to mapping class group representations with the notable property that the action of a Dehn twist has infinite order, and thus the representation could be faithful, (see [26]). This is in contrast with the usual quantum mapping class group representations where all Dehn twists have finite order and the representations are not faithful. Also, after projectivization, these TQFTs correspond to the Lyubashenko projective mapping class group representations given in [27], (also see [28,29]). Related work of Chang [30] considers non-semisimple generalization of Turaev–Viro TQFTs and their lattice model realizations based on non-semisimple quantum groupoids.

In this article we define pseudo-Hermitian Levin–Wen models from relative $\mathcal{G}$-spherical categories satisfying certain Hermitian properties. Any unitary modular tensor category provides an example of a relative $\mathcal{G}$-spherical category satisfying our assumptions, where $\mathcal{G}$ is the trivial group. We give an example with nontrivial $\mathcal{G}$ in Section 5 coming from the non-semisimple representation theory of quantum $\mathfrak{sl}(2)$ at a root of unity.

1.1. Outline

Section 2 contains the definition of a relative $\mathcal{G}$-spherical category along with extra data needed to construct a pseudo-Hermitian Hamiltonian from it. We give a non-semisimple version of the Levin–Wen construction in Section 3. Various properties of the Hamiltonian are proved here. In Section 4, we construct a pseudo-unitary operator on our Hilbert space, which may be of independent interest. Finally in Section 5 we provide an example of a relative $\mathcal{G}$-spherical category satisfying the extra data required to define a Levin–Wen model. This category comes from quantum $\mathfrak{sl}(2)$, but is not semisimple. This is a point of departure from other papers in the area. The Appendix contains some explicit formulas for morphisms in this category.

2. System Hilbert space

The input for defining the system Hilbert space is a triangulated surface Σ and certain categorical data organized in the notion of a *relative $\mathcal{G}$-spherical category*. The main examples of relative spherical categories are the categories of finite-dimensional weight modules over semi-restricted quantum

groups. In contrast to the usual modular tensor categories used to study topological phases, these categories are not semisimple and have an infinite number of nonisomorphic irreducible modules, all having vanishing quantum dimensions. Nevertheless, such categories have been shown to give rise to nonabelian braiding statistics and lead to new TQFTs.

We make some additional assumptions on the relative $\mathcal{G}$-spherical category. We list these conditions below as 'Hermitian structure'. They are motivated by a Hermitian structure on the non-semisimple category of modules for unrolled quantum $\mathfrak{sl}(2)$ constructed in [31] and recalled in Section 5 and the Appendix. We also assume the multiplicity spaces between generic string types are one dimensional and that the group $\mathcal{G}$ is abelian. It should be relatively straightforward to remove the first condition, but it is less clear how to handle non-abelian groups.

2.1. Basic input

The categorical input described above is the following data.

String data:

- $\mathcal{G}$ is an abelian group with identity 0 containing a small subset $\mathcal{X} \subset \mathcal{G}$, such that $\mathcal{X}$ is symmetric ($-\mathcal{X} = \mathcal{X}$).
- For each $g \in \mathcal{G}$ there is a finite set of string labels I_g.

Set $I = \sqcup I_g$ and $\mathsf{A} = \cup_{g\in\mathcal{G}\setminus\mathcal{X}} I_g$. We say a string type is *generic* if it is in A. The decomposition of string labels by elements of $\mathcal{G}$ and the ability to separate out the set A of generic string types is vital to our new construction of string net Hamiltonians from nonsemisimple categorical data. By restricting to generic string types, we are able to work with semisimple data within nonsemisimple contexts. Most of the additional data described below can only be defined for these generic string types, including the branching rules, modified dimensions d, and modified 6j-symbols.

If $i \in I_g$ we say the degree of i is g and write $\deg(i) = g$. Each element $j \in I_g$ has a *conjugate* string type $j^* \in I_{-g}$, which satisfies $j^{**} = j$. An There is unique *vacuum* string type $j = 0$ in I_0 satisfying $0^* = 0$. In Section 2.2, string types will be associated to edges of labeled graphs, so that an oriented edge labeled j will have label j^* if the orientation is reversed.

Branching Rules: To each triple of strings $i, j, k \in \mathsf{A}$, we associate a branching rule δ_{ijk} that equals 1 if the triple is allowed to meet at a vertex and 0 otherwise. Categorically, these branching rules arise from the dimension of the space of maps from the unit object $\mathbb{1}$ into the tensor product of simple objects $V_i \otimes V_j \otimes V_k$. For simplicity, we have considered the multiplicity one case. However, in future work we will extend this construction to higher multiplicity allowing for 'multiple branching channels'.

Motivated by the cyclic symmetry of trivalent vertices, we consider the triple $\{i, j, k\}$ up to cyclic ordering and require δ_{ijk} is symmetric under cyclic permutations of the three labels: $\delta_{ijk} = \delta_{jki} = \delta_{kij}$. To be compatible with the conjugation structure of labels given by reversing orientation of graph edges, the branching rule satisfies $\delta_{0jj^*} = \delta_{0j^*j} = 1$, $\delta_{ijk} = \delta_{k^*j^*i^*}$ and $\delta_{ijk} = 0$ if $\deg(i) + \deg(j) + \deg(k) \neq 0 \in \mathcal{G}$.

Modified dimensions: There exist functions $\mathsf{d} : \mathsf{A} \to \mathbb{R}^\times$ and $\mathsf{b} : \mathsf{A} \to \mathbb{R}$ satisfying $\mathsf{d}(i^*) = \mathsf{d}(i)$, $\mathsf{b}(i^*) = \mathsf{b}(i)$, and for $g, g_1, g_2 \in \mathcal{G} \setminus \mathcal{X}$ with $-g + g_1 + g_2 = 0$ we have

$$\mathsf{b}(j) = \sum_{j_1\in I_{g_1}, j_2\in I_{g_2}} \mathsf{b}(j_1)\mathsf{b}(j_2)\delta_{j^*j_1j_2} \tag{2.1}$$

for all $j \in I_g$.

Modified 6j **symbols**: There are symmetrized 6j symbols N_{lmn}^{ijk} defined to be zero unless

$$\delta_{ijk^*} = \delta_{i^*mn^*} = \delta_{j^*nl^*} = \delta_{klm^*} = 1. \tag{2.2}$$

These symbols are required to satisfy

$$N_{lmn}^{ijk} = N_{mnl}^{jk^*i^*} = N_{n^*ij^*}^{klm}; \tag{2.3}$$

$$\sum_j \mathrm{d}(j) N^{j_1 j_2 j_5}_{j_3 j_6 j} N^{j_1 j i_6}_{j_4 j_0 j_7} N^{j_2 j_3 j}_{j_4 j_7 j_8} = N^{j_5 j_3 j_6}_{j_4 j_0 j_8} N^{j_1 j_2 j_5}_{j_8 j_0 j_7}; \tag{2.4}$$

$$\sum_n \mathrm{d}(n) N^{ijp}_{lmn} N^{kj^* i}_{nml} = \frac{\delta_{k,p}}{\mathrm{d}(k)} \delta_{ijk^*} \delta_{klm^*}. \tag{2.5}$$

We refer to these identities as *tetrahedral symmetry*, *pentagon identity*, and *orthogonality* respectively.

Hermitian Structure: There exist maps $\beta : \mathsf{A} \to \mathbb{R}^\times$ and $\gamma : \mathsf{A}^3 \to \mathbb{R}^\times$ such that γ is invariant under cyclic permutations of its three arguments. Furthermore,

$$\beta(j^*) = \beta(j); \tag{2.6}$$

$$\gamma(i, j, k)\gamma(k^*, j^*, i^*)\beta(i)\beta(j)\beta(k) = 1 \text{ if } \delta_{ijk} = 1; \tag{2.7}$$

$$\overline{(N^{j_1 j_2 j_3}_{j_4 j_5 j_6})} = N^{j_2^* j_1^* j_3^*}_{j_5 j_4 j_6} \gamma(j_1, j_2, j_3^*)\gamma(j_1^*, j_5, j_6^*) \times \gamma(j_2^*, j_6, j_4^*)\gamma(j_3, j_4, j_5^*) \prod_{i=1}^{6} \beta(j_i), \tag{2.8}$$

where all string types are generic in the above formulas.

2.2. Definition of the state space

Let Σ be a compact, connected, oriented surface. Let $\mathcal{T}$ be a triangulation of Σ and Γ be a finite trivalent graph dual to $\mathcal{T}$. Each vertex of the graph Γ acquires a cyclic ordering compatible with the orientation of Σ. A *$\mathcal{G}$-coloring* of Γ is a map Φ from the set of oriented edges of Γ to $\mathcal{G}$ such that

1. $\Phi(-e) = -\Phi(e)$ for any oriented edge e of Γ, where $-e$ is e with opposite orientation, and
2. if e_1, e_2, e_3 are edges of a vertex v of Γ with a cyclic ordering compatible with the orientation of Σ and each edge is oriented towards the vertex v, then $\Phi(e_1) + \Phi(e_2) + \Phi(e_3) = 0$.

The $\mathcal{G}$-colorings of Γ form a group isomorphic via Poincaré duality to the group of $\mathcal{G}$-valued simplicial 1-cocycles on $\mathcal{T}$. We denote by $[\Phi] \in H^1(\Sigma, \mathcal{G})$ the associated cohomology class. A $\mathcal{G}$-coloring of Γ is admissible if $\Phi(e) \in \mathcal{G} \setminus \mathcal{X}$ for any oriented edge e of Γ.

A state of an admissible $\mathcal{G}$-coloring Φ is a map σ assigning to every oriented edge e of Γ an element $\sigma(e) \in I_{\Phi(e)}$ such that $\sigma(-e) = \sigma(e)^*$. Denote by $\mathrm{St}(\Phi)$ the set of such states. For an admissible $\mathcal{G}$-coloring Φ we define the Hilbert space $\mathcal{H} = \mathcal{H}(\Gamma, \Phi)$ as the span of all elements corresponding to the states of Φ

$$\mathcal{H}(\Gamma, \Phi) = \bigoplus_{\sigma \in \mathrm{St}(\Phi)} \mathbb{C}\,|\Gamma, \sigma\rangle\,.$$

We write σ^* for the state of the $\mathcal{G}$-coloring $-\Phi$ assigning $\sigma(e)^*$ to each oriented edge e of Γ.

2.3. Inner products

We equip $\mathcal{H} = \mathcal{H}(\Gamma, \Phi)$ with a complete Hilbert space structure by defining a positive definite Hermitian inner product

$$\langle \cdot \mid \cdot \rangle_+ : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C} \tag{2.9}$$

in which the states $|\Gamma, \sigma\rangle$ form an orthonormal basis:

$$\langle \Gamma, \sigma \mid \Gamma, \sigma' \rangle_+ = \delta_{\sigma, \sigma'}. \tag{2.10}$$

We will see that from the TQFT perspective, this is not the most natural inner product. The most natural inner product can be obtained from this one utilizing a Hermitian operator $\eta : \mathcal{H} \to \mathcal{H}$.

Define an invertible operator η by

$$\begin{aligned} \eta : \mathcal{H}(\Gamma, \Phi) &\longrightarrow \mathcal{H}(\Gamma, \Phi) \\ |\Gamma, \sigma\rangle &\mapsto \frac{\prod_{e\in\Gamma_1} \mathrm{d}(\sigma(e))}{\prod_{v\in\Gamma_0} \gamma(\sigma(v)) \prod_{e\in\Gamma_1} \beta(\sigma(e))} |\Gamma, \sigma\rangle \,. \end{aligned} \tag{2.11}$$

where $\sigma(v) = (j_1, j_2, j_3) \in I^3, j_k = \sigma(e_k)$, and e_1, e_2, e_3 are the 3 ordered edges adjacent to v oriented toward v. It is clear from the orthogonality of states $|\Gamma, \sigma\rangle$ and the fact that $\gamma(\sigma(v))$, $\mathrm{d}(\sigma(e))$ and $\beta(\sigma(e))$ are all real that η is Hermitian with respect to the form $\langle \cdot \mid \cdot \rangle_+$

$$\langle \psi \mid \eta\phi \rangle_+ = \langle \eta\psi \mid \phi \rangle_+ \tag{2.12}$$

for all $\psi, \phi \in \mathcal{H}$.

Define a new inner product on $\mathcal{H} = \mathcal{H}(\Gamma, \Phi)$ via

$$\langle \cdot \mid \cdot \rangle := \langle \cdot \mid \eta^{-1}(\cdot) \rangle_+ . \tag{2.13}$$

This new pairing is Hermitian since

$$\begin{aligned} \langle \psi \mid \phi \rangle &= \langle \psi \mid \eta^{-1}\phi \rangle_+ = \overline{\langle \eta^{-1}\phi \mid \psi \rangle_+} \\ &\overset{(2.12)}{=} \overline{\langle \phi \mid \eta^{-1}\psi \rangle_+} = \overline{\langle \phi \mid \psi \rangle}. \end{aligned}$$

However, it is indefinite since

$$\langle \Gamma, \sigma \mid \Gamma, \sigma' \rangle = \frac{\prod_{v\in\Gamma_0} \gamma(\sigma(v)) \prod_{e\in\Gamma_1} \beta(\sigma(e))}{\prod_{e\in\Gamma_1} \mathrm{d}(\sigma(e))} \cdot \delta_{\sigma,\sigma'} . \tag{2.14}$$

Indeed, for the example studied in Section 5, the coefficient on the right-hand-side of (2.14) takes both positive and negative values. It follows that the η is Hermitian with respect to the new form $\langle \cdot \mid \cdot \rangle$ since

$$\begin{aligned} \langle \psi \mid \eta\phi \rangle &:= \langle \psi \mid \eta^{-1}\eta\phi \rangle_+ = \langle \psi \mid \eta\eta^{-1}\phi \rangle_+ \\ &\overset{(2.12)}{=} \langle \eta\psi \mid \eta^{-1}\phi \rangle_+ =: \langle \eta\psi \mid \phi \rangle \,. \end{aligned} \tag{2.15}$$

Observe that if a Hamiltonian H is Hermitian with respect to the form $\langle \cdot \mid \cdot \rangle$, so that $\langle \psi \mid H\phi \rangle = \langle H\psi \mid \phi \rangle$, then we have

$$\begin{aligned} \langle \psi \mid H\phi \rangle_+ &:= \langle \psi \mid \eta H\phi \rangle = \langle (\eta H)^\dagger \psi \mid \phi \rangle \\ &\overset{(2.15)}{=} \langle H\eta\psi \mid \phi \rangle := \langle H\eta\psi \mid \eta^{-1}\phi \rangle_+ \\ &\overset{(2.12)}{=} \langle \eta^{-1}H\eta\psi \mid \phi \rangle_+ \,. \end{aligned}$$

Thus the Hermitian conjugate $H^\sharp$ of H with respect to the form $\langle \cdot \mid \cdot \rangle_+$ satisfies

$$H^\sharp = \eta^{-1}H\eta. \tag{2.16}$$

Thus the resulting Hamiltonian then becomes *pseudo-Hermitian* with respect to the inner product $\langle \cdot \mid \cdot \rangle_+$.

2.4. Topological origin of the indefinite inner product

In this section we show how the indefinite inner product from (2.13) arises naturally from topological considerations. This helps to illustrate the distinction between indefinite quantum mechanics and η-pseudo-Hermitian quantum mechanics articulated in [21]. The key distinction is that an explicit map η as in (2.11) is chosen as part of the data. From the TQFT perspective discussed in Section 3.6, this choice arises naturally.

Given a tuple $(\Sigma, \mathcal{T}, \Gamma, \Phi)$ of the objects described in Section 2.2, consider the tuple $(\Sigma^*, \mathcal{T}, \overline{\Gamma}, \overline{\Phi})$ where Σ^* is the surface Σ with opposite orientation, $\overline{\Gamma}$ is the graph Γ dual to the triangulation $\mathcal{T}$ whose cyclic order at each vertex had been reversed to be compatible with Σ^* and where the choice of orientation of each edge has been reversed. The $\mathcal{G}$-coloring $\overline{\Phi}(e) = -\Phi(e)$ for any oriented edge e of Γ. In particular the dual of a state of Φ is a state of $\overline{\Phi}$. For any oriented edge e of Γ, $\overline{\Phi}(e) = -\Phi(e)$ but $\overline{\Phi}$ and Φ represent the same 1-cocycle of on the 1-skeleton of $\mathcal{T}$.

There is a natural pairing $(\cdot,\cdot) : \mathcal{H}(\overline{\Gamma}, \overline{\Phi}) \otimes \mathcal{H}(\Gamma, \Phi) \to \mathbb{C}$ arising from ribbon graph evaluations associated to the relative $\mathcal{G}$-spherical category defined on

$$x = \sum_{\sigma \in \mathrm{St}(\Phi)} x_\sigma \, |\overline{\Gamma}, \sigma^*\rangle \text{ and } y = \sum_{\sigma \in \mathrm{St}(\Phi)} y_\sigma \, |\Gamma, \sigma\rangle$$

by

$$(x, y) = \sum_{\sigma \in \mathrm{St}(\Phi)} x_\sigma y_\sigma \prod_{e \in \Gamma_1} \frac{1}{\mathrm{d}(\sigma(e))} \, .$$

We then define $\dagger : \mathcal{H}(\Gamma, \Phi) \to \mathcal{H}(\overline{\Gamma}, \overline{\Phi})$ by

$$|\Gamma, \sigma\rangle^\dagger = \prod_{v \in \Gamma_0} \gamma(\sigma(v)) \prod_{e \in \Gamma_1} \beta(\sigma(e)) \, |\overline{\Gamma}, \sigma^*\rangle \tag{2.17}$$

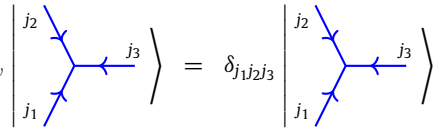
where $\sigma(v) = (j_1, j_2, j_3) \in \mathbb{R}^3$, $j_k = \sigma(e_k)$, and e_1, e_2, e_3 are the 3 ordered edges adjacent to v oriented toward v. Then the inner product from (2.13) is given by $\langle \psi | \phi \rangle := \left(\psi^\dagger, \phi\right)$.

3. Non-semisimple Levin-Wen Hamiltonian

In this section we define operators acting on states associated to the dual graph Γ of a triangulation $\mathcal{T}$ of the surface Σ. Recall that vertices of the graph Γ correspond to triangles of $\mathcal{T}$, while regions (plaquettes), of the dual graph Γ can be identified with vertices of the original triangulation. More precisely, by a plaquette p, we mean a region of $\Sigma \setminus \Gamma$. Its boundary δp is a union of oriented edges of Γ. We write Γ_0 for the set vertices of the graph Γ. Using operators associated with vertices and plaquettes, we introduce a pseudo-Hermitian Hamiltonian in Section 3.5.

3.1. Vertex operators

Associated to each vertex $v \in \Gamma_0$ we have *vertex operators* $Q_v : \mathcal{H}(\Gamma, \Phi) \to \mathcal{H}(\Gamma, \Phi)$ that act locally to impose the branching constraints from Section 2.1.



It is straightforward to see that these operators are mutually commuting projectors, so that

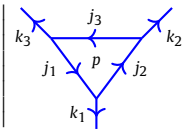
$$Q_v^2 = Q_v, \quad Q_v Q_{v'} = Q_{v'} Q_v \qquad \text{for } v, v' \in \Gamma_0.$$

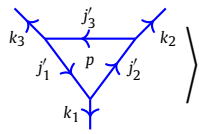
Furthermore, since Q_v is a delta function, it is immediate that these operators are Hermitian with respect to the inner product $\langle \cdot \mid \cdot \rangle$ from (2.14).

3.2. Plaquette operators

If $g \in \mathcal{G}$ and p is a plaquette in $\Sigma \setminus \Gamma$, then we write $g.\delta p$ for the coloring that sends oriented edges of δp to g, their opposites to $-g$, and other edges to 0.

For $g \in \mathcal{G} \setminus \mathcal{X}$ let Φ be an admissible $\mathcal{G}$-coloring such that $\Phi + g.\delta p$ is also admissible. Let $s \in I_g$ and define the operator $B_p^s : \mathcal{H}(\Gamma, \Phi) \to \mathcal{H}(\Gamma, \Phi + g.\delta p)$ which acts on the boundary edges of the plaquette p and is given on a triangle plaquette by

B_p^s 

$= \sum_{j_1', j_2', j_3'} \mathrm{d}(j_1')\mathrm{d}(j_2')\mathrm{d}(j_3') N^{j_3' s j_3}_{j_1^* k_3 j_1'^*} N^{j_1' s j_1}_{j_2^* k_1 j_2'^*} N^{j_2' s j_2}_{j_3^* k_2 j_3'^*}$ 

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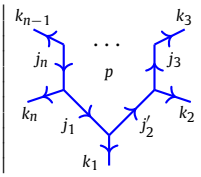
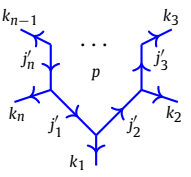
More generally, for a plaquette with n sides, we define the plaquette operator by:

B_p^s 

$:= \sum_{j_1', \dots, j_n'} \prod_{i=1}^{n} \mathrm{d}(j_i') N^{j_i' s j_i}_{j_{i+1}^* k_i j_{i+1}'^*}$ 

.

Note that the conditions on $6j$ symbols (2.2) ensure that $\deg(j_i') = \deg(j_i) - \deg(s)$ otherwise the right hand side is zero in the above formula.

Then we define $\mathcal{G}$-indexed plaquette operators

$$B_p^g = \sum_{s \in I_g} \mathrm{b}(s) B_p^s : \mathcal{H}(\Gamma, \Phi) \to \mathcal{H}(\Gamma, \Phi + g.\delta p). \tag{3.1}$$

Proposition 3.1. *If g_1, g_2 and $g_1 + g_2$ are generic, then*

$$B_p^{g_1} B_p^{g_2} = B_p^{g_1 + g_2}.$$

Moreover, if g_1 and g_2 are generic then $B_p^{g_1} B_p^{-g_1} = B_p^{g_2} B_p^{-g_2}$.

Proof. For a plaquette p and elements g_2 and g_1, we compute the composition of plaquette operators:

$$\begin{aligned} B_p^{g_2} B_p^{g_1}(x) &= \sum_{\substack{s \in I_{g_1} \\ t \in I_{g_2}}} \mathrm{b}(s)\mathrm{b}(t) \sum_{j_1', \dots, j_n'} \sum_{j_1'', \dots, j_n''} \prod_{i=1}^{n} \mathrm{d}(j_i')\mathrm{d}(j_i'') N^{j_i' s j_i}_{j_{i+1}^* k_i j_{i+1}'^*} N^{j_i'' t j_i'}_{j_{i+1}'^* k_i j_{i+1}''^*} . \\ &\overset{(2.4)}{=} \sum_{\substack{s \in I_{g_1} \\ t \in I_{g_2}}} \mathrm{b}(s)\mathrm{b}(t) \sum_{j_1', \dots, j_n'} \sum_{j_1'', \dots, j_n''} \prod_{i=1}^{n} \mathrm{d}(j_i')\mathrm{d}(j_i'') \sum_{a_i} \mathrm{d}(a_i) N^{j_i'' t j_i'}_{s j_i a_i} N^{j_i'' a_i j_i}_{j_{i+1}^* k_i j_{i+1}''^*} N^{t s a_i}_{j_{i+1}^* j_{i+1}''^* j_{i+1}'^*} \\ &\overset{(2.3)}{=} \sum_{\substack{s \in I_{g_1} \\ t \in I_{g_2}}} \mathrm{b}(s)\mathrm{b}(t) \sum_{j_1', \dots, j_n'} \sum_{j_1'', \dots, j_n''} \prod_{i=1}^{n} \mathrm{d}(j_i')\mathrm{d}(j_i'') \sum_{a_i} \mathrm{d}(a_i) N^{j_i'' a_i j_i}_{j_{i+1}^* k_i j_{i+1}''^*} N^{t s a_i}_{j_{i+1}^* j_{i+1}''^* j_{i+1}'^*} N^{a_i s^* t}_{j_i'^* j_i''^* j_i^*} . \end{aligned} \tag{3.2}$$

Reindexing the product for the last factor in (3.2) yields that $B_p^{g_2} B_p^{g_1}(x)$ is equal to

$$\sum_{\substack{s \in I_{g_1} \\ t \in I_{g_2}}} \mathrm{b}(s)\mathrm{b}(t) \sum_{j_1', \dots, j_n'} \sum_{j_1'', \dots, j_n''} \prod_{i=1}^{n} \mathrm{d}(j_i')\mathrm{d}(j_i'') \sum_{a_i} \mathrm{d}(a_i) N^{j_i'' a_i j_i}_{j_{i+1}^* k_i j_{i+1}''^*} N^{t s a_i}_{j_{i+1}^* j_{i+1}''^* j_{i+1}'^*} N^{a_{i+1} s^* t}_{j_{i+1}'^* j_{i+1}''^* j_{i+1}^*} . \tag{3.3}$$

Using the orthogonality property (2.5) for the last two factors in (3.3) yields

$$B_p^{g_2} B_p^{g_1}(x) = \sum_{\substack{s\in I_{g_1}\\ t\in I_{g_2}}} \mathrm{b}(s)\mathrm{b}(t) \sum_{j_1'',\dots,j_n''} \prod_{i=1}^{n} \mathrm{d}(j_i'') \sum_{a_i} \mathrm{d}(a_i) N_{j_{i+1}^* k_i j_{i+1}''^*}^{j_i'' a_i j_i} \frac{\delta_{a_i,a_{i+1}}}{\mathrm{d}(a_{i+1})} \delta_{tsa_{i+1}^*} \delta_{a_{i+1} j_{i+1}^* j_i''} \tag{3.4}$$

$$= \sum_{\substack{s\in I_{g_1}\\ t\in I_{g_2}}} \mathrm{b}(s)\mathrm{b}(t) \sum_{j_1'',\dots,j_n''} \prod_{i=1}^{n} \mathrm{d}(j_i'') \sum_{a} N_{j_{i+1}^* k_i j_{i+1}''^*}^{j_i'' a j_i} \delta_{tsa^*} \delta_{a j_{i+1}^* j_i''}$$

$$\overset{(2.1)}{=} \sum_{u\in I_{g_1+g_2}} \mathrm{b}(u) \sum_{j_1'',\dots,j_n''} \prod_{i} \mathrm{d}(j_i'') N_{j_{i+1}^* k_i j_{i+1}''^*}^{j_i'' a j_i} = B_p^{g_1+g_2}. \tag{3.5}$$

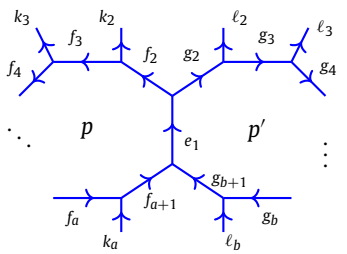
For the second statement, first note that for generic colors $B_p^{g_1} = B_p^{g_3} B_p^{g_1-g_3}$. Then

$$B_p^{g_1} B_p^{-g_1} = B_p^{g_3} B_p^{g_1-g_3} B_p^{-g_1} = B_p^{g_3} B_p^{-g_3}. \quad \square$$

3.3. Commutativity of operators

In this subsection we show that plaquette operators commute following a proof given in [8].

Proposition 3.2. *Let p and p' be two plaquettes which share exactly one common edge e_1 in a state $|\Gamma,\sigma\rangle$ as in (3.6). Then $B_{p'}^t B_p^s = B_p^s B_{p'}^t$.*

$|\Gamma,\sigma\rangle :=$ 

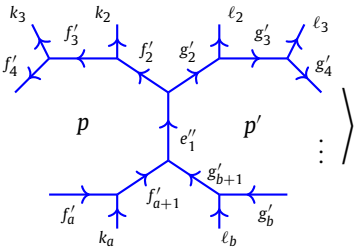
(3.6)

Proof. First we compute $B_{p'}^t B_p^s$:

$$B_{p'}^t B_p^s |\Gamma,\Phi\rangle =$$

$$\sum_{e_1''} \sum_{g_2',\dots,g_{b+1}'} \sum_{e_1'} \sum_{f_2',\dots,f_{a+1}'} \left[\mathrm{d}(e_1') N_{f_2^* g_2 f_2'^*}^{e_1' s e_1} \mathrm{d}(f_{a+1}') N_{e_1^* g_{b+1}^* e_1'^*}^{f_{a+1}' s f_{a+1}} \prod_{i=1}^{a-1} \mathrm{d}(f_{i+1}') N_{f_{i+2}^* k_{i+1} f_{i+2}'^*}^{f_{i+1}' s f_{i+1}} \right]$$

$$\times \left[\mathrm{d}(e_1''^*) N_{g_{b+1} f_{a+1}'^* g_{b+1}'}^{e_1''^* t e_1'^*} \mathrm{d}(g_2'^*) N_{e_1' f_2' e_1''}^{g_2'^* t g_2^*} \prod_{i=2}^{b} \mathrm{d}(g_{i+1}'^*) N_{g_i' \ell_i g_i}^{g_{i+1}'^* t g_{i+1}} \right] \Bigg| \Bigg\rangle.$$

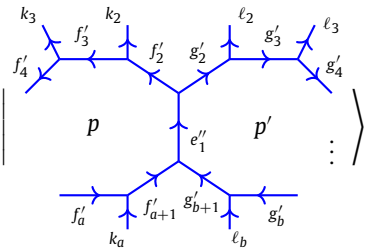


Compare with the computation of $B_p^s B_{p'}^t$:

$$B_p^s B_{p'}^t \,|\Gamma, \Phi\rangle =$$

$$\sum_{e_1''} \sum_{f_2',\dots,f_{a+1}'} \sum_{r_1'} \sum_{g_2',\dots,g_{b+1}'} \left[\mathrm{d}(r_1'^*) N_{g_{b+1} f_{a+1}^* g_{b+1}'}^{r_1'^* t e_1^*} \mathrm{d}(g_2'^*) N_{e_1 f_2 r_1'}^{g_2'^* t g_2^*} \prod_{i=2}^{b} \mathrm{d}(g_{i+1}'^*) N_{g_i' \ell_i g_i}^{g_{i+1}'^* t g_{i+1}^*} \right]$$

$$\times \left[\mathrm{d}(e_1'') N_{f_2^* g_2' f_2'^*}^{e_1'' s r_1'} \mathrm{d}(f_{a+1}') N_{r_1'^* g_{b+1}'^* e_1''^*}^{f_{a+1}' s f_{a+1}} \prod_{i=1}^{a-1} \mathrm{d}(f_{i+1}') N_{f_{i+2}^* k_{i+1} f_{i+2}'^*}^{f_{i+1}' s f_{i+1}} \right] \Bigg|$$



$$\Bigg\rangle.$$

Fixing $e_1'', f_2', \dots, f_{a+1}', g_2', \dots, g_{b+1}'$, and canceling identical factors leaves us with the task of verifying:

$$\sum_{e_1'} (\mathrm{d}(f_{a+1}') N_{e_1^* g_{b+1}^* e_1'^*}^{f_{a+1}' s f_{a+1}}) (\mathrm{d}(e_1') N_{f_2^* g_2 f_2'^*}^{e_1' s e_1}) (\mathrm{d}(e_1''^*) N_{g_{b+1} f_{a+1}^* g_{b+1}'}^{e_1''^* t e_1'^*}) (\mathrm{d}(g_2'^*) N_{e_1' f_2' e_1''}^{g_2'^* t g_2^*}) \tag{3.7}$$

$$= \sum_{r_1'} (\mathrm{d}(r_1'^*) N_{g_{b+1} f_{a+1}^* g_{b+1}'}^{r_1'^* t e_1^*}) (\mathrm{d}(g_2'^*) N_{e_1 f_2 r_1'}^{g_2'^* t g_2^*}) (\mathrm{d}(f_{a+1}') N_{r_1'^* g_{b+1}'^* e_1''^*}^{f_{a+1}' s f_{a+1}}) (\mathrm{d}(e_1'') N_{f_2^* g_2' f_2'^*}^{e_1'' s r_1'}).$$

Using tetrahedral symmetry, we need to verify

$$\sum_{e_1'} \mathrm{d}(f_{a+1}') \mathrm{d}(e_1') \mathrm{d}(e_1''^*) \mathrm{d}(g_2'^*) N_{f_{a+1}^* f_{a+1}'^* g_{b+1}}^{e_1'^* e_1 s} N_{g_{b+1} f_{a+1}'^* g_{b+1}'}^{e_1''^* t e_1'^*} N_{s^* f_2' e_1'}^{g_2^* e_1 f_2} N_{e_1' f_2' e_1''}^{g_2'^* t g_2^*} \tag{3.8}$$

$$= \sum_{r_1'} \mathrm{d}(r_1'^*) \mathrm{d}(g_2'^*) \mathrm{d}(f_{a+1}') \mathrm{d}(e_1'') N_{g_{b+1} f_{a+1}^* g_{b+1}'}^{r_1'^* t e_1^*} N_{g_{b+1}' f_{a+1}^* f_{a+1}'^*}^{s^* e_1''^* r_1'^*} N_{e_1 f_2 r_1'}^{g_2'^* t g_2^*} N_{r_1' f_2 s}^{f_2' e_1''^* g_2'^*}.$$

Now we perform the pentagon identity on the last two factors of the left-hand side of (3.8) and on the first two factors of the right-hand side of (3.8), we obtain the following equality to be verified.

$$\sum_{e_1', z_2} \mathrm{d}(f_{a+1}') \mathrm{d}(e_1') \mathrm{d}(e_1''^*) \mathrm{d}(g_2'^*) \mathrm{d}(z_2) N_{f_{a+1}^* f_{a+1}'^* g_{b+1}}^{e_1'^* e_1 s} N_{g_{b+1} f_{a+1}'^* g_{b+1}'}^{e_1''^* t e_1'^*} N_{e_1 f_2 z_2}^{g_2'^* t g_2^*} N_{s^* f_2' e_1''}^{g_2'^* z_2 f_2} N_{s^* e_1'' e_1'}^{t e_1 z_2} \tag{3.9}$$

$$= \sum_{r_1', z_1} \mathrm{d}(r_1'^*) \mathrm{d}(g_2'^*) \mathrm{d}(f_{a+1}') \mathrm{d}(e_1'') \mathrm{d}(z_1) N_{t e_1^* z_1}^{s^* e_1''^* r_1'^*} N_{g_{b+1} f_{a+1}^* f_{a+1}'^*}^{s^* z_1 e_1^*} N_{g_{b+1} f_{a+1}'^* g_{b+1}'}^{e_1''^* t z_1} N_{e_1 f_2 r_1'}^{g_2'^* t g_2^*} N_{r_1' f_2 s}^{f_2' e_1''^* g_2'^*}.$$

Letting $z_1 = e_1'^*$, and $z_2 = r_1'$, and using symmetry, we see that (3.9) holds by matching up the first, second third, fourth, and fifth factors on the left-hand side with the second, third, fourth, fifth, and first factors on the right-hand side respectively. □

Corollary 3.3. *Let p and p' be two plaquettes. Then $B_{p'}^t B_p^s = B_p^s B_{p'}^t$.*

Proof. If p and p' are identical, or if they share no edges, the result is obvious. The case that they share exactly one edge is proved in Proposition 3.2. □

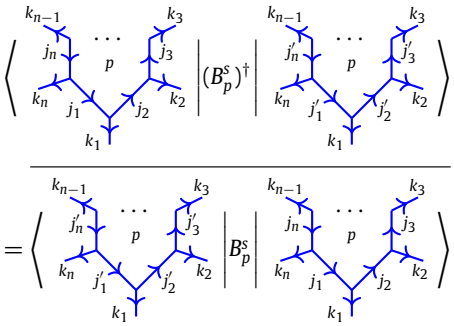
3.4. Hermitian adjoints of plaquette operators

See [9] for a closely related calculation in the semisimple case. In the computations below, we assume that the states are identical except in a neighborhood of the graph shown.

Proposition 3.4. *The Hermitian adjoint of B_p^s is $B_p^{s^*}$. In particular,*

$$\left(B_p^g\right)^\dagger = B_p^{-g}.$$

Proof. We begin by computing matrix elements of $(B_p^s)^\dagger$, where we ignore the contributions to the inner product from edges not in the neighborhood of the plaquette, as these will cancel with the identical contributions in (3.13).



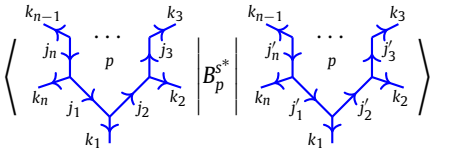
$$= \tag{3.10}$$

$$\overset{(2.14)}{=} \prod_{i=1}^{n} \overline{\mathrm{d}(j'_i)} \overline{N^{j'_i s j_i}_{j^*_{i+1} k_i j'^*_{i+1}}} \frac{\beta(j'_i)\beta(k_i)}{\mathrm{d}(k_i)\mathrm{d}(j'_i)} \gamma(j'^*_{i+1}, k^*_i, j'_i) \tag{3.11}$$

This simplifies to

$$\prod_{i=1}^{n} \overline{N^{j'_i s j_i}_{j^*_{i+1} k_i j'^*_{i+1}}} \frac{\beta(j'_i)\beta(k_i)}{\mathrm{d}(k_i)} \gamma(j'^*_{i+1}, k^*_i, j'_i). \tag{3.12}$$

On the other hand, the matrix elements of $B_p^{s^*}$ are given by



$$= \prod_{i=1}^{n} \frac{\beta(j_i)\beta(k_i)}{\mathrm{d}(k_i)} \gamma(j^*_{i+1}, k^*_i, j_i) N^{j_i s^* j'_i}_{j'^*_{i+1} k_i j^*_{i+1}} \tag{3.13}$$

Showing that (3.12) and (3.13) are equal reduces to checking the following equality:

$$\prod_{i=1}^{n} \overline{N^{j'_i s j_i}_{j^*_{i+1} k_i j'^*_{i+1}}} \beta(j'_i) \gamma(j'^*_{i+1}, k^*_i, j'_i)$$
$$= \prod_{i=1}^{n} \beta(j_i) \gamma(j^*_{i+1}, k^*_i, j_i) N^{j_i s^* j'_i}_{j'^*_{i+1} k_i j^*_{i+1}} \tag{3.14}$$

Using (2.8), and a symmetry from (2.3) on $N^{j_i s j'_i}_{j'^*_{i+1} k_i j^*_{i+1}}$, this amounts to checking:

$$\prod_{i=1}^{n} [\beta(j'_i)\gamma(j'^*_{i+1}, k^*_i, j'_i)]\gamma(j'_i, s, j^*_i)\gamma(j'^*_i, k_i, j'_{i+1})$$
$$\times \gamma(s^*, j'^*_{i+1}, j_{i+1})\gamma(j_i, j^*_{i+1}, k^*_i)$$

$$\times\ \beta(j_i')\beta(s)\beta(j_i)\beta(j_{i+1}^*)\beta(k_i)\beta(j_{i+1}'^*)N_{k_i j_{i+1}^* j_{i+1}'^*}^{s^* j_i'^* j_i^*}$$
$$= \prod_{i=1}^{n} \beta(j_i)\gamma(j_{i+1}^*, k_i^*, j_i)N_{k_i j_{i+1}^* j_{i+1}'^*}^{s^* j_i'^* j_i^*} \tag{3.15}$$

Canceling $\beta(j_i)$ from both sides of (3.15) and reorganizing factors, (3.15) is equivalent to

$$\prod_{i=1}^{n}[\gamma(j_{i+1}'^*, k_i^*, j_i')\gamma(j_i'^*, k_i, j_{i+1}')\beta(j_i')\beta(j_{i+1}^*)\beta(k_i)]$$
$$\times\ [\gamma(j_i', s, j_i^*)\gamma(s^*, j_{i+1}'^*, j_{i+1})\beta(j_i')\beta(s)\beta(j_{i+1}^*)]$$
$$\times\ \gamma(j_i, j_{i+1}^*, k_i^*)N_{k_i j_{i+1}^* j_{i+1}'^*}^{s^* j_i'^* j_i^*}$$
$$= \prod_{i=1}^{n} \gamma(j_{i+1}^*, k_i^*, j_i)N_{k_i j_{i+1}^* j_{i+1}'^*}^{s^* j_i'^* j_i^*} \tag{3.16}$$

Using (2.7) on each of the brackets in the first line of (3.16) verifies that equality. This proves the first part of the proposition. For the second part, recall that $B_p^g = \sum_{s\in I_g} \mathrm{b}(s)B_p^s$. Thus

$$(B_p^g)^\dagger = \sum_{s\in I_g} \mathrm{b}(s)B_p^{s^*} = B_p^{-g}. \quad \square$$

3.5. Properties of the non-semisimple Levin-Wen Hamiltonian

Define the *plaquette operator*

$$B_p := B_p^g B_p^{-g} : \mathcal{H}(\Gamma, \Phi) \to \mathcal{H}(\Gamma, \Phi). \tag{3.17}$$

A priori, this operator depends on g, but the following lemma shows that it does not.

Lemma 3.5. *The plaquette operator defined by $B_p = B_p^g B_p^{-g} : \mathcal{H}(\Gamma, \Phi) \to \mathcal{H}(\Gamma, \Phi)$ does not depend on $g \in \mathcal{G} \setminus \mathcal{X}$.*

Proof. This follows from Proposition 3.1. $\square$

Theorem 3.6. *The plaquette operators have the following properties.*

1. *The plaquette operators are projectors $B_p^2 = B_p$.*
2. *The plaquette operators are mutually commuting:*
$$B_p B_{p'} = B_{p'} B_p.$$
3. *The operators B_p are Hermitian with respect to the indefinite norm (2.14).*

Proof. For the first item, recall that $B_p = B_p^g B_p^{-g}$, so if we assume $g_1, g_2, g_1 + g_2$ are generic, by Proposition 3.1 we have

$$B_p^2 = B_p^{g_1} B_p^{-g_1} B_p^{-g_2} B_p^{g_2} = B_p^{g_1} B_p^{-g_1-g_2} B_p^{g_2}$$
$$= B_p^{-g_2} B_p^{g_2} = B_p.$$

The second item follows from Corollary 3.3. The third item is a consequence of Proposition 3.4. $\square$

Remark 3.7. The second claim above follows immediately from the topological interpretation of the plaquette operators in Section 3.6.

Define a Hamiltonian

$$H = -\sum_p B_p - \sum_v Q_v, \tag{3.18}$$

or we can shift this Hamiltonian so that the ground state has zero energy.

$$H = \sum_p (1 - B_p) + \sum_v (1 - Q_v). \tag{3.19}$$

Theorem 3.8. *The Hamiltonian H from (3.19) has the following properties.*

1. *The Hamiltonian is Hermitian with respect to the indefinite inner product.*
2. *The Hamiltonian is gapped and local.*
3. *The ground state of H corresponding to the simultaneous $+1$-eigenspace of the operators B_p and Q_v over all vertices v and plaquettes p, so that H has spectrum $\mathbb{Z}_{\geq 0}$.*

Proof. The first claim follows from Theorem 3.6 and the Hermicity of the vertex operators Q_v. The second claim follows since the Hamiltonian is constructed from local commuting projectors acting locally only on vertices and plaquettes of Γ. Since the Hamiltonian is comprised of mutually commuting projectors, it is clear that the spectrum of H is $\mathbb{Z}_{\geq 0}$ with ground state corresponding to the simultaneous $+1$ eigenspace of the vertex and plaquette operators. □

Observe, that the simultaneous $+1$-eigenspace of all the vertex operators Q_v over all vertices defines a subspace of the Hilbert space $\mathcal{H}(\Gamma, \Phi)$ consisting of $\mathcal{G}$-colored graphs where each labeled trivalent graph is consistent with the branching rules δ_{ijk}. The simultaneous $+1$-eigenspace B_p operators consists of those basis vectors $|\psi\rangle$ in the image of the projector $\Pi_p B_p$. Indeed, if any $B_p |\psi\rangle = 0$, this operator maps $|\psi\rangle \mapsto 0$ and fixed any vector with $B_p |\psi\rangle = |\psi\rangle$ for all plaquettes p.

We now show that the ground state of the Hamiltonian is described by the modified Turaev–Viro invariant coming from the relative $\mathcal{G}$-spherical category.

3.6. Topological invariance of ground state

In this section we explain the topological origins of the plaquette operators defined in Section 3.2. The modified Turaev–Viro TQFT from [32] is a $2+1$-dimensional TQFT defined on a 3-manifold M with the additional data of an isotopy class of an embedded graph Y in M and a cohomology class $[\Phi] \in H^1(M, \mathcal{G})$. Given a triangulation $\mathcal{T}_M$ of M, the graph Y is assumed to be Hamiltonian, meaning that it intersects every vertex of the triangulation exactly once, and its edges are part of the triangulation.

When the manifold M has boundary, the triangulation of M induces a triangulation $\mathcal{T}$ on the boundary Σ and the cohomology class $[\Phi] \in H^1(M, \mathcal{G})$ restricts to $[\Phi_{|\Sigma}] \in H^1(\Sigma, \mathcal{G})$. In [32], it is assumed the graph Y is embedded transversely to the boundary surface. Thus, the modified Turaev–Viro invariant for the surface Σ includes the additional data of a finite set of marked points $\mathsf{m} \subset \Sigma$ on the surface corresponding to where the graph intersects the surface. Since the graph must go through every vertex of the triangulation, in [32, Section 4], the notion of a "oscillating path" on a surface is used to define enriched cobordisms between triangulated surfaces with oscillating paths. The oscillating path allows vertices of the triangulation of a surface to be outside the set of marked points. In this paper we do not consider oscillating paths but instead assume that the set of vertices is exactly the set of marked points. This seems natural from the perspective of the B_p operators because as we will see, they generate the cylinder of the modified Turaev–Viro invariant, and they are only defined for a vertex at a marked point (also see Remark 3.9). It would be interesting to consider oscillating paths and graphs which are not transverse to the surface, but we leave this for future work.

Next, we give a quick description of the modified Turaev–Viro invariant TV with the assumptions and notation of this paper. Let $(\Sigma, \mathcal{T}, \Gamma, \Phi)$ and $(\Sigma, \mathcal{T}', \Gamma', \Phi')$ be tuples of objects described

in Section 2.2 with $[\Phi] = [\Phi'] \in H^1(\Sigma, \mathcal{G})$. The vertices $\mathcal{T}_0$ of the triangulation $\mathcal{T}$ and $\mathcal{T}_0'$ of the triangulation $\mathcal{T}'$ are the marked points m. Let M be the cylinder $\Sigma \times [0, 1]$. Define a graph $Y = \mathcal{T}_0 \times [0, 1] = \mathsf{m} \times [0, 1]$ in M. The cohomology class $[\Phi] = [\Phi'] \in H^1(\Sigma, \mathcal{G}) \simeq H^1(M, \mathcal{G})$ induces a unique cohomology class on M.

Let $(\mathcal{T}^M, \mathcal{Y}, \Phi^M)$ be an admissible Hamiltonian triangulation of (M, Y) extending the triangulation $\mathcal{T}$ of $\Sigma \times \{0\}$ and the triangulation $\mathcal{T}'$ of $\Sigma \times \{1\}$, see [32, Section 3.2]. The cohomology class $[\Phi] \in H^1(M, \mathcal{G})$ can always be represented by an admissible $\mathcal{G}$-coloring Φ^M of $\mathcal{T}^M$ which restricts to Φ on $\Sigma \times \{0\}$ and to Φ' on $\Sigma \times \{1\}$ [33, Lemma 23]. Likewise, every admissible $\mathcal{G}$-coloring of $\mathcal{T}^M$ gives rise to a representative of a cohomology class $[\Phi^M] \in H^1(M, \mathcal{G})$. In particular, Φ^M is an admissible $\mathcal{G}$-coloring of $\mathcal{T}^M$ representing $[\Phi^M] \in H^1(M, \mathcal{G})$ which is a map on the set of oriented edges of $\mathcal{T}^M$ such that for any oriented edge e of $\mathcal{T}^M$, $\Phi^M(e) \in \mathcal{G} \setminus \mathcal{X}$, $\Phi^M(-e) = -\Phi^M(e)$, and the sum of the values of Φ^M on the oriented edges forming the boundary of any face of $\mathcal{T}^M$ is zero. Also, $\mathcal{Y}$ is a set of edges of $\mathcal{T}^M$ such that the union of these edges is the graph Y in M and all the vertices of $\mathcal{T}$ are contained in Y (i.e. all the vertices are incident to an edge of $\mathcal{Y}$).

A state ϕ of Φ^M is a map assigning to every oriented edge e of $\mathcal{T}^M$ an element $\phi(e) \in I_{\Phi^M(e)}$ such that $\phi(-e) = -\phi(e)$. Given a state ϕ of Φ^M, for each tetrahedron T of $\mathcal{T}^M$ whose vertices v_1, v_2, v_3, v_4 are ordered so that the (ordered) triple of oriented edges $(\overrightarrow{v_1v_2}, \overrightarrow{v_1v_3}, \overrightarrow{v_1v_4})$ is positively oriented with respect to the orientation of M, set $|T|_\phi := N_{lmn}^{ijk}$ where

$$\begin{cases} i = \phi(\overrightarrow{v_2v_1}) & j = \phi(\overrightarrow{v_3v_2}) & k = \phi(\overrightarrow{v_3v_1}) \\ l = \phi(\overrightarrow{v_4v_3}) & m = \phi(\overrightarrow{v_4v_1}) & n = \phi(\overrightarrow{v_4v_2}) \end{cases}.$$

Let $\Phi_{\mathcal{T}}$ be a $\mathcal{G}$-coloring of $\mathcal{T}$ representing $[\Phi] \in H^1(\Sigma, \mathcal{G})$. Let $\mathcal{H}(\mathcal{T}, \Phi) = \bigoplus_{\sigma \in \mathrm{St}(\Phi)} \mathbb{C}\,|\mathcal{T}, \sigma\rangle$ be the span of all colorings corresponding to the states $\mathrm{St}(\Phi)$ of Φ. Similarly, $\mathcal{H}(\mathcal{T}', \Phi') = \bigoplus_{\sigma \in \mathrm{St}(\Phi')} \mathbb{C}\,|\mathcal{T}', \sigma\rangle$

Now fix a state σ of Φ and let $\mathrm{St}(\Phi^M, \sigma)$ be the set of states of Φ^M which restrict to σ on $\mathcal{T} = \mathcal{T}^M \cap (\Sigma \times \{0\})$. The modified Turaev–Viro invariant $TV(M, Y, [\Phi^M]) : \mathcal{H}(\mathcal{T}, \Phi) \to \mathcal{H}(\mathcal{T}', \Phi')$ is defined on vectors as

$$\begin{aligned} &|\mathcal{T}, \sigma\rangle \\ &\mapsto \sum_{\phi \in \mathrm{St}(\Phi^M, \sigma)} \prod_{e \in \mathcal{E}} \mathsf{d}(\phi(e)) \prod_{e \in \mathcal{Y}} \mathsf{b}(\phi(e)) \prod_{T \in \mathcal{T}_3^M} |T|_\phi \cdot |\mathcal{T}', \phi_{|\mathcal{T}'}\rangle \end{aligned} \tag{3.20}$$

where $\mathcal{E}$ is the set of unoriented edges of $\mathcal{T}^M$ that are not in $\mathcal{T}$ nor in $\mathcal{Y}$ and $\mathcal{T}_3^M$ is the set of tetrahedra of $\mathcal{T}^M$.

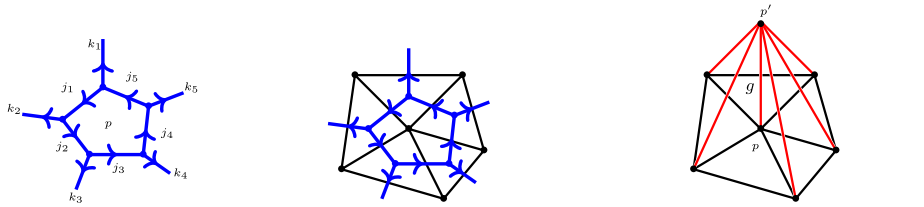
In [32] it is shown that the mapping (3.20) is independent of triangulation, coloring, and basis. However, it does depend on M, $Y = \mathsf{m} \times [0, 1]$, the cohomology class $[\Phi^M] = [\Phi]$ and the triangulation of ∂M. For a fixed triple $(\Sigma, [\Phi], \mathsf{m})$, let us denote the associated mapping $C_{\mathcal{T},\Phi}^{\mathcal{T}',\Phi'}$. Since the composition of two cylinders is topologically equivalent to a single cylinder, [32] implies that $C_{\mathcal{T}',\Phi'}^{\mathcal{T}'',\Phi''} C_{\mathcal{T},\Phi}^{\mathcal{T}',\Phi'} = C_{\mathcal{T},\Phi}^{\mathcal{T}'',\Phi''}$. In particular the maps $C_{\mathcal{T},\Phi}^{\mathcal{T},\Phi}$ are projectors with canonically isomorphic image. Then the definition of the modified Turaev–Viro invariant on surfaces with marked points is

$$TV(\Sigma, \mathsf{m}, [\Phi]) := \mathrm{Im} C_{\mathcal{T},\Phi}^{\mathcal{T},\Phi}, \tag{3.21}$$

for any choice of $\mathcal{T}$, Φ.

A key distinction between the modified invariant (3.20) and the usual Turaev–Viro invariant is the use of a modified dimension d in place of the usual quantum dimension. As noted, the modified dimension is real, but need not be positive. Another difference is the role the Hamiltonian graph Y plays, allowing the modified dimension d to be used on edges not contained in Y, while b is used on those edges of Y. This subtle difference is what enables non-semisimple categories to be used for state-sum topological invariants.

Much like the semisimple case [22,23], the plaquette operators B_p^g from (3.1) used in the definition of the non-semisimple Levin–Wen Hamiltonian have a topological origin via the modified Turaev–Viro invariant. Each plaquette in the dual graph Γ corresponds to a vertex p of the original triangulation see Fig. 1(b). By adding an additional vertex p' over the point p and connecting it by



(a) A plaquette p in the dual graph Γ.

(b) The dual graph Γ superimposed over the original triangulation $\mathcal{T}$.

(c) 3-dimensional tetrahedra obtained by adding a vertex over the vertex in $\mathcal{T}$ corresponding to the plaquette p in Γ.

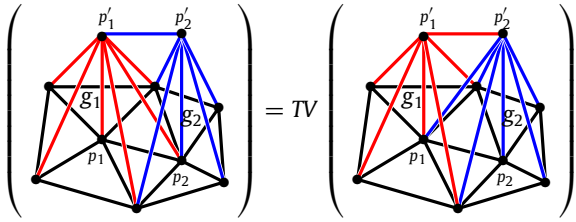
Fig. 1. The plaquette operator B_p^g has a topological interpretation as the operator assigned to the tetrahedra in 1(c) by the modified Turaev–Viro TQFT. The initial edge labels in (1(a)) are mapped to new edge labels by the Turaev–Viro operator from (1(c)). The index $g \in \mathcal{G} \setminus \mathcal{X}$ labels the internal edge common in all the tetrahedra obtained by adding the point p'.

an edge to p and all vertices adjacent to p, we obtain a collection of tetrahedra X_p over our original triangulation (see Fig. 1(c)). Labeling the edge connecting p to p' by g, we can then evaluate the tetrahedra $TV(X_p)$ to produce an operator mapping the Hilbert space $\mathcal{H}(\Gamma, \Phi)$ to the Hilbert space $\mathcal{H}(\Gamma, \Phi + g.\delta p)$. We have defined B_p^g so that

$$B_p^g = TV(X_p). \tag{3.22}$$

Note that several of the key properties of the operators B_p^g become apparent from the topological perspective. For example, the commutativity $[B_{p_1}^{g_1}, B_{p_2}^{g_2}] = 0$ follows from the independence of the modified Turaev–Viro invariant under change of triangulation.

$$TV\left(\quad\right) = TV\left(\quad\right) \tag{3.23}$$



Remark 3.9. Those familiar with the usual Turaev–Viro construction and its connection to Levin–Wen models, might have expected the plaquette operators to be defined using the modified dimension in place of the usual quantum dimension so that $\widetilde{B}_p^g = \sum_{s \in I_g} \mathsf{d}(s) B_p^s$, rather than (3.1). This is a key distinction in the non-semisimple version of the Turaev–Viro construction and is closely related to the role the embedded graph plays in the theory. In Fig. 1(c) the graph was assumed to be transverse to the surface at each vertex of the original triangulation and the resulting triangulation. This means the new edge labeled g now contains the embedded graph, and as such, the Turaev–Viro invariant utilized $\mathsf{b}(s)$, rather than $\mathsf{d}(s)$ for $s \in I_g$. The operators $\widetilde{B}_p^g$ produce nilpotent operators that cannot be used to define projectors.

Theorem 3.10. *The degenerate ground state of the Hamiltonian in (3.18) defined on a surface Σ is isomorphic to the vector space assigned to the surface by the modified TQFT from [32] defined from the corresponding relative $\mathcal{G}$-spherical category:*

$$\operatorname{Ker} H \simeq TV(\Sigma, \mathsf{m}, [\Phi])$$

where $[\Phi] \in H^1(\Sigma, \mathcal{G})$ and the cardinality of m is the number of vertices in the triangulation $\mathcal{T}$ of Σ.

In particular, the ground state of the system is a topological invariant of the surface Σ equipped with the set of points m and the cohomology class $[\Phi] \in H^1(\Sigma, \mathcal{G})$.

Proof. Observe that the image of the operator $\prod_v Q_v$ on $\mathcal{H}(\Gamma, \Phi)$ consists of those states $|\Gamma, \sigma\rangle$ where the $\mathcal{G}$-coloring satisfies the additional constraint dictated by the branching rules δ_{ijk} at each

vertex. Denote this space by $\mathcal{H}'$. The ground state of the Hamiltonian is then the subspace of $\mathcal{H}'$ given by the image of $\prod B_p$. We identify the operator $\prod B_p$ with the operator associated to a cylinder $\Sigma \times [0, 1]$ by the modified Turaev–Viro invariant.

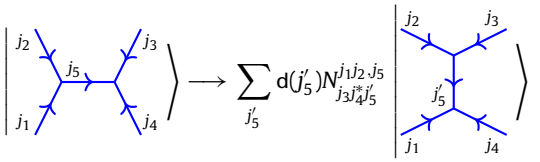
As above, let M be the cylinder $\Sigma \times [0, 1]$ with graph $Y = \mathsf{m} \times [0, 1] \subset M$. The triangulation $\mathcal{T}$ on $\Sigma \times \{0\}$ can be extended to a Hamiltonian triangulation of (M, Y). First, add a new vertex p' over a vertex p of $\mathcal{T}$, making several tetrahedra as in Fig. 1(c) (here the edge between p and p' is in $\mathcal{Y}$). Iterating this process (similar to the picture in (3.23)) for all vertices of $\mathcal{T}$, we obtain the desired triangulation. Now computing the modified Turaev–Viro invariant with this triangulation, we have $TV(M, Y, [\Phi^M]) = \prod B_p$, (see (3.20)). The theorem then follows from (3.21). □

4. Maps between Hilbert spaces for bistellar operator

Here we give an example of a pseudo-unitary transformation relating Hilbert spaces for different triangulations that share the same ground state. These correspond to a change of triangulation according given by the 2-2 Pachner move.

4.1. Local operators

Let $(\Sigma, \{p\}, \mathcal{T}, \Gamma, \Phi)$ be as above with Φ an admissible coloring of Γ. Given an edge e of the triangulation, we can modify $\mathcal{T}$ and Γ to $\mathcal{T}'$, Γ' by doing a bistellar move $I \leftrightarrow H$ on e. The cocycle condition of Φ implies that there is a unique coloring Φ' of Γ' which coincide with Φ outside Γ. In particular, it satisfies $[\Phi] = [\Phi'] \in H^1(\Sigma, \mathcal{G})$. If Φ' is admissible, we define the operator $T : \mathcal{H}(\Gamma, \Phi) \to \mathcal{H}(\Gamma', \Phi')$ by

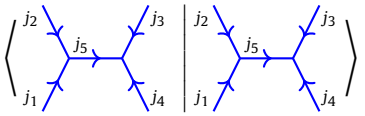
$T:$ 

$$\longrightarrow \sum_{j_5'} \mathsf{d}(j_5') N^{j_1 j_2 j_5}_{j_3 j_4^* j_5'} \tag{4.1}$$

In what follows below, we sometimes write T_{j_5} for the operator above to clarify algebraic computations.

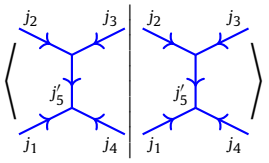
4.2. Bistellar operators are Hermitian

Much like the work in [34], we can define certain local change of triangulation operators that act on our Hilbert space, though the computations in the non-semisimple case require modification. Using the Hermitian inner product from (2.14), we have



$$= \gamma(j_1, j_2, j_5^*)\gamma(j_3, j_4, j_5)\beta(j_5)\frac{1}{\mathsf{d}(j_5)}, \tag{4.2}$$

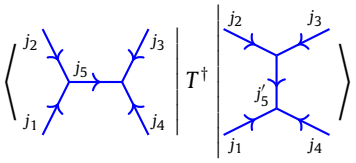
Similarly, we can compute that for a state that is identical to the one above, except in a neighborhood of j_5 that we have



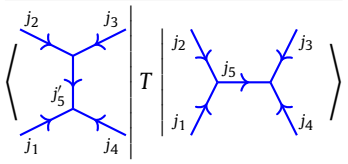
$$= \gamma(j_2, j_3, j_5'^*)\gamma(j_1, j_5'^*, j_4)\beta(j_5')\frac{1}{\mathsf{d}(j_5')} \tag{4.3}$$

Proposition 4.1. *The operator T is Hermitian and unitary.*

Proof. Consider the matrix entries of $T^\dagger$ with respect to the Hermitian form:



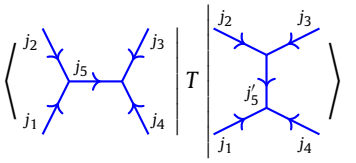
$$:= \overline{\quad}$$



$$\overset{(4.3)}{=} \overline{\mathrm{d}(j_5')}\overline{N^{j_1 j_2, j_5}_{j_3 j_4^* j_5'}} \cdot \gamma(j_2, j_3, j_5'^*)\gamma(j_1, j_5'^*, j_4)\beta(j_5')\frac{1}{\overline{\mathrm{d}(j_5')}}$$

$$= \overline{N^{j_1 j_2, j_5}_{j_3 j_4^* j_5'}}\gamma(j_2, j_3, j_5'^*)\gamma(j_1, j_5'^*, j_4)\beta(j_5')$$

Compare with the matrix coefficients of T which you get by rotating (4.1) 90 degrees



$$= \mathrm{d}(j_5) N^{j_2 j_3, j_5'}_{j_4 j_1^* j_5^*} \cdot \gamma(j_1, j_2, j_5^*)\gamma(j_3, j_4, j_5)\beta(j_5)\frac{1}{\mathrm{d}(j_5)}$$

Therefore, we have shown that the T operator is Hermitian provided the following equality holds.

$$\overline{N^{j_1 j_2, j_5}_{j_3 j_4^* j_5'}}\gamma(j_2, j_3, j_5'^*)\gamma(j_1, j_5'^*, j_4)\beta(j_5') = N^{j_2 j_3, j_5'}_{j_4 j_1^* j_5^*}\gamma(j_1, j_2, j_5^*)\gamma(j_3, j_4, j_5)\beta(j_5). \tag{4.4}$$

Recall from (2.8) that

$$\overline{N^{j_1 j_2, j_5}_{j_3 j_4^* j_5'}} = N^{j_2^* j_1^* j_5^*}_{j_4^* j_3 j_5'}\gamma(j_1, j_2, j_5^*)\gamma(j_1^*, j_4^*, j_5'^*)\gamma(j_2^*, j_5', j_3^*)$$
$$\times \gamma(j_5, j_4, j_4)\beta(j_1)\beta(j_2)\beta(j_1)\beta(j_3)\beta(j_4)\beta(j_5)\beta(j_5') \tag{4.5}$$

and using the symmetries of the tetrahedron we have

$$N^{j_2 j_3 j_5'}_{j_4 j_1^* j_5^*} = N^{j_3 j_5' j_2}_{j_1^* j_5^* j_4} = N^{j_2^* j_1^* j_5^*}_{j_4^* j_4 j_5'}. \tag{4.6}$$

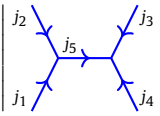
Hence, we have

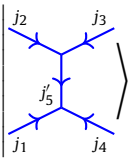
$$\overline{N^{j_1 j_2, j_5}_{j_3 j_4^* j_5'}}\gamma(j_2, j_3, j_5'^*)\gamma(j_1, j_5'^*, j_4)\beta(j_5')$$
$$\overset{(4.5)}{=} N^{j_2^* j_1^* j_5^*}_{j_4^* j_3 j_5'}\gamma(j_1, j_2, j_5^*)\gamma(j_1^*, j_4^*, j_5'^*)\gamma(j_2^*, j_5', j_3^*)$$
$$\times\gamma(j_5, j_3, j_4) \cdot \gamma(j_2, j_3, j_5'^*)\gamma(j_1, j_5'^*, j_4)\beta(j_5')$$
$$\times\beta(j_1)\beta(j_2)\beta(j_1)\beta(j_3)\beta(j_4)\beta(j_5)\beta(j_5')$$

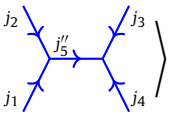
$$
\begin{aligned}
&\overset{(4.6)}{=} N^{j_2 j_3 j_5'}_{j_4 j_1^* j_5^*}\gamma(j_1,j_2,j_5^*)\gamma(j_5,j_3,j_4)\beta(j_5)\cdot\\
&\quad\times\left(\gamma(j_1^*,j_4^*,j_5'^*)\gamma(j_1,j_5'^*,j_4)\beta(j_1)\beta(j_4)\beta(j_5')\right)\\
&\quad\times\left(\gamma(j_2^*,j_5',j_3^*)\gamma(j_2,j_3,j_5'^*)\beta(j_2)\beta(j_3)\beta(j_5')\right)\\
&= N^{j_2 j_3 j_5'}_{j_4 j_1^* j_5^*}\gamma(j_1,j_2,j_5^*)\gamma(j_5,j_3,j_4)\beta(j_5)
\end{aligned}
$$

where we have used (2.7) twice in the last equality. Thus, (4.4) holds and $T = T^\dagger$.

Next we will show that $T^\dagger T_1 = \mathrm{Id}$.

$T^\dagger T$ 

$= \sum_{j_5'} \mathrm{d}(j_5') N^{j_1 j_2 j_5}_{j_3 j_4^* j_5'} T^\dagger$ 

$= \sum_{j_5'} \mathrm{d}(j_5') N^{j_1 j_2 j_5}_{j_3 j_4^* j_5'} \sum_{j_5''} \mathrm{d}(j_5'') N^{j_2 j_3 j_5'}_{j_4 j_1^* j_5''^*}$ 

(The above works since we have established that $T^\dagger = T$.) Observe that using the symmetries from (2.3) we have

$$
N^{j_2 j_3 j_5'}_{j_4 j_1^* j_5''^*} = N^{j_5' j_4 j_1^*}_{j_5'' j_2 j_3^*} = N^{j_1^* j_5'' j_2}_{j_3 j_5' j_4^*} = N^{j_5'' j_2^* j_1}_{j_5' j_4^* j_3}. \tag{4.7}
$$

So unitarity follows from (2.5) since

$$
\begin{aligned}
&\sum_{j_5''}\sum_{j_5'} \mathrm{d}(j_5'')\mathrm{d}(j_5') N^{j_1 j_2 j_5}_{j_3 j_4^* j_5'} \; N^{j_2 j_3 . j_5'}_{j_4 j_1^* j_5''^*}\\
&\quad= \sum_{j_5''}\sum_{j_5'} \mathrm{d}(j_5'')\mathrm{d}(j_5') N^{j_1 j_2 j_5}_{j_3 j_4^* j_5'} \; N^{j_5'' j_2^* j_1}_{j_5' j_4^* j_3}\\
&\quad= \sum_{j_5''} \mathrm{d}(j_5'')\left(\sum_{j_5'} \mathrm{d}(j_5') N^{j_1 j_2 j_5}_{j_3 j_4^* j_5'} \; N^{j_5'' j_2^* j_1}_{j_5' j_4^* j_3}\right)\\
&\quad= \sum_{j_5''} \mathrm{d}(j_5'')\left(\frac{\delta_{j_5 . j_5''}}{\mathrm{d}(j_5)}\delta_{j_1 j_2 j_5''^*}\delta_{j_5 j_3 j_4}\right) = \delta_{j_5 . j_5''}. \quad \square
\end{aligned}
$$

4.3. Local operators commute with the Hamiltonian

Lemma 4.2. *When all involved $\mathcal{G}$-coloring are admissible, plaquette and bistellar operators commute:*

$$
B_p^g T_{j_n} = T_{j_n'} B_p^g.
$$

Proof. By direct computation we have

$$
\begin{aligned}
B_p^s T_{j_n} = \sum_{z_{n-1}} \sum_{j_1',\dots,j_{n-1}'} \prod_{i=1}^{n-2} &\mathrm{d}(z_{n-1}) N^{k_{n-1}^* j_{n-1} j_n}_{j_1^* k_n z_{n-1}}\\
&\times \mathrm{d}(j_i') N^{j_i' s j_i}_{j_{i+1}^* k_i j_{i+1}^*} \mathrm{d}(j_{n-1}') N^{j_{n-1}' s j_{n-1}}_{j_1^* z_{n-1} j_1'^*}
\end{aligned}
$$

$$T_{j'_n} B^s_p = \sum_{z_{n-1}} \sum_{j'_1,\dots,j'_{n-1}} \sum_{j'_n} \prod_{i=1}^{n-2} \mathrm{d}(z_{n-1})\mathrm{d}(j'_i) N^{j'_i s j_i}_{j^*_{i+1} k_i j'^*_{i+1}}$$
$$\times\, \mathrm{d}(j'_{n-1})\mathrm{d}(j'_n) N^{j'_{n-1} s j_{n-1}}_{j^*_n k_{n-1} j'^*_n} N^{j'_n s j_n}_{j^*_1 k_n j'^*_1} N^{k^*_{n-1} j'_{n-1} j'_n}_{j'^*_1 k_n z_{n-1}}$$

In order to verify the equality, we need to check

$$N^{k^*_{n-1} j_{n-1} j_n}_{j^*_1 k_n z_{n-1}} N^{j'_{n-1} s j_{n-1}}_{j^*_1 z_{n-1} j'^*_1}$$
$$= \sum_{j'_n} \mathrm{d}(j'_n) N^{j'_{n-1} s j_{n-1}}_{j^*_n k_{n-1} j'^*_n} N^{j'_n s j_n}_{j^*_1 k_n j'^*_1} N^{k^*_{n-1} j'_{n-1} j'_n}_{j'^*_1 k_n z_{n-1}}$$

which follows from (2.3) and (2.4). □

5. Example: quantum $\mathfrak{sl}(2)$ at roots of unity

In this section we provide a non-trivial example of a Hermitian relative $\mathcal{G}$-spherical category. It is a certain category of representations of quantum $\mathfrak{sl}(2)$ where the quantum parameter q is specialized to a root of unity. There are two versions of the quantum group that we consider. The first is the unrolled quantum group. This larger algebra plays an auxiliary role here. It was proved in [31] that a category of representations for this algebra is Hermitian. This leads to some nice properties about 6j-symbols for the category of representations. However, the category of representations for the unrolled quantum group does not have certain finiteness properties that are required for a relative $\mathcal{G}$-spherical category, which are necessary to construct the modified Turaev–Viro invariant. The semi-restricted quantum group for $\mathfrak{sl}(2)$ does give rise to a relative $\mathcal{G}$-spherical category. There is a forgetful functor between the categories allowing us to transport the desired properties of the 6j-symbols for the unrolled quantum group over to the semi-restricted quantum group.

Throughout the section, fix an odd positive integer $r = 2r' + 1$ and let $q = e^{\frac{\pi\sqrt{-1}}{r}}$ be a $2r^{th}$-root of unity. Set

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad \{n\} := q^n - q^{-n}.$$

5.1. Unrolled quantum group

Let $U^H_q\mathfrak{sl}(2)$ be the $\mathbb{C}$-algebra given by generators E, F, K, K^{-1}, H and relations:

$$KK^{-1} = K^{-1}K = 1, \qquad KEK^{-1} = q^2 E,$$
$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}} \qquad KFK^{-1} = q^{-2} F, \tag{5.1}$$
$$HK = KH, \quad [H, E] = 2E, \qquad [H, F] = -2F. \tag{5.2}$$

The algebra $U^H_q\mathfrak{sl}(2)$ is a Hopf algebra where the coproduct, counit and antipode are defined by

$$\Delta(E) = 1 \otimes E + E \otimes K, \qquad \varepsilon(E) = 0, \qquad S(E) = -EK^{-1}, \tag{5.3}$$
$$\Delta(F) = K^{-1} \otimes F + F \otimes 1, \qquad \varepsilon(F) = 0, \qquad S(F) = -KF, \tag{5.4}$$
$$\Delta(K) = K \otimes K, \qquad \varepsilon(K) = 1, \qquad S(K) = K^{-1}, \tag{5.5}$$
$$\Delta(H) = H \otimes 1 + 1 \otimes H, \qquad \varepsilon(H) = 0, \qquad S(H) = -H. \tag{5.6}$$

Define the *unrolled quantum group* $\overline{U}^H_q\mathfrak{sl}(2)$ to be the Hopf algebra $U^H_q\mathfrak{sl}(2)$ modulo the relations $E^r = F^r = 0$.

We now consider the following class of finite-dimensional highest weight modules for $\overline{U}_q^H\mathfrak{sl}(2)$. For each $\alpha \in \mathbb{R}$, we let V_α be the r-dimensional highest weight $\overline{U}_q^H\mathfrak{sl}(2)$-module of highest weight $\alpha + r - 1$. The module V_α has a basis $\{v_0, \dots, v_{r-1}\}$ whose action is given by

$$H.v_d = (\alpha + r - 1 - 2d)v_d, \quad E.v_d = \frac{\{d\}\{d-\alpha\}}{\{1\}^2} v_{d-1},$$
$$F.v_d = v_{d+1}. \tag{5.7}$$

For all $\alpha \in \mathbb{R}$, the quantum dimension of V_α is zero, but is possible to define a modified dimension that is nonzero.

For α and β sufficiently generic, $V_\alpha \otimes V_\beta$ decomposes into a direct sum of irreducible modules.

Proposition 5.1 ([35, Theorem 5.2]). *If $\alpha, \beta \in (\mathbb{R} \setminus \mathbb{Z}) \cup r\mathbb{Z}$, and $\alpha + \beta \notin \mathbb{Z}$, then*

$$V_\alpha \otimes V_\beta \cong \bigoplus_{\gamma=-r'}^{r'} V_{\alpha+\beta+2\gamma}.$$

Remark 5.2. The modules V_α also exist when $\alpha \in \mathbb{C}$. We restrict to $\mathbb{R}$ here in order to ensure that the required Hermitian structures exist.

The category of $\overline{U}_q^H\mathfrak{sl}(2)$-modules generated by the V_α for $\alpha \in \mathbb{R}$ possesses a set of 6j-symbols N_{klm}^{ijk} that have explicit formulas given in [36, Theorem 29].

Proposition 5.3. *The following identities hold.*

1. $N_{lmn}^{ijk} = N_{mnl}^{jk^*i^*} = N_{n^*ij^*}^{klm}$;
2. $\sum_j \mathsf{d}(j) N_{j_3,j_6,j}^{j_1,j_2,j_5} N_{j_4j_0j_7}^{j_1jj_6} N_{j_4j_7j_8}^{j_2j_3j} = N_{j_4j_0j_8}^{j_5,j_3,j_6} N_{j_8j_0j_7}^{j_1j_2j_5}$;
3. $\sum_n \mathsf{d}(n) N_{lmn}^{ijp} N_{nml}^{kj^*i} = \frac{\delta_{k,p}}{\mathsf{d}(k)} \delta_{ijk^*} \delta_{klm^*}$ *where δ_{ijk} is the branching rule that equals 1 if the triple is allowed and 0 otherwise.*

Proof. The first claim is from [33, Section 4]. The second claim is [33, Theorem 7]. The last claim is [33, Theorem 8]. □

For $i, j, k \in (\mathbb{R} - \mathbb{Z}) \cup r\mathbb{Z}$, with $i + j + k = 2F$, define coefficients

$$\mathsf{d}(i) = \mathsf{d}_0 \frac{q^i - q^{-i}}{q^{ri} - q^{-ri}} \tag{5.8}$$

$$\beta(i) = \frac{\{1\}^{2(r-1)}\mathsf{d}(i)}{r\mathsf{d}_0} \tag{5.9}$$

$$\gamma(i,j,k) = \beta(-k) \prod_{f=1}^{r'+F} \{i-f\}\{j-f\} \times \prod_{g=1}^{r'+F} \frac{\{2r'-g+1\}\{-k+g\}}{\{1\}}, \tag{5.10}$$

where we set $\mathsf{d}_0 = r$.

Using the explicit formulas for the 6j-symbols from [36, Theorem 29] and the definitions above, one can verify the following result.

Proposition 5.4. *The following identities hold.*

1. $\underline{\gamma(i,j,k)}\gamma(k^*,j^*,i^*)\beta(i)\beta(j)\beta(k)=1$;
2. $(N_{j_4j_5j_6}^{j_1j_2j_3})=N_{j_5j_4j_6}^{j_2^*j_1^*j_3^*}\gamma(j_1,j_2,j_3^*)\gamma(j_1^*,j_5,j_6^*)\times\gamma(j_2^*,j_6,j_4^*)\gamma(j_3,j_4,j_5^*)\prod_i\beta(j_i)$.

In Appendix we give a conceptual derivation of Proposition 5.4 by studying elementary intertwiners between $\overline{U}_q^H\mathfrak{sl}(2)$-modules. There is a ribbon category $\mathcal{D}^\dagger$ of $\overline{U}_q^H\mathfrak{sl}(2)$-modules that are generated by a certain set of objects A containing the V_i for $i\in\mathbb{R}$ (see [31, Equation 18]). In [31, Theorem 4.18] it is shown that the category $\mathcal{D}^\dagger$ is a Hermitian ribbon category, so it possesses an indefinite inner product $\langle\cdot,\cdot\rangle$ allowing one to compute Hermitian adjoints $f^\dagger$ for any morphism in $\mathcal{D}^\dagger$.

In this formulation the coefficients $\beta(i)$ and $\gamma(i,j,k)$ have natural interpretations via the Hermitian structure. There is an isomorphism $w_i\colon V_i\to V_{-i}^*$ determined by mapping a highest weight vector $v_0\in V_i$ by $w_i(v_0)=-q^{-1}y_{r-1}^*$, where y_{r-1} is a lowest weight vector of V_{-i}. The coefficient $\beta(i)$ is the unique constant satisfying

$$w^\dagger=\beta(i)w^{-1}. \tag{5.11}$$

Likewise, it is possible to specify a specific basis vector h_{ijk} in the 1-dimensional space $\mathrm{Hom}(1,V_i\otimes V_j\otimes V_k)$. Then using the pivotal structure and the isomorphisms w_α, these distinguished basis vectors determine basis vectors

$$\begin{aligned}h^{ijk}&\in\mathrm{Hom}(V_i\otimes V_j\otimes V_k,1)\cong\mathrm{Hom}(V_i\otimes V_j\otimes V_k,1)\\&\cong\mathrm{Hom}(1,V_k^*\otimes V_j^*\otimes V_i^*)\\&\cong\mathrm{Hom}(1,V_{k^*}\otimes V_{j^*}\otimes V_{i^*}).\end{aligned}$$

The coefficients $\gamma(i,j,k)$ are the unique coefficients satisfying

$$h_{ijk}^\dagger=\gamma(i,j,k)h^{ijk}. \tag{5.12}$$

5.2. Semi-restricted quantum group and relative spherical data

Let $\mathcal{G}$ be the abelian group $\mathbb{R}/2\mathbb{Z}$. Let $\mathcal{X}$ be the subset $\mathbb{Z}/2\mathbb{Z}$ in $\mathcal{G}$. Consider the natural morphisms $\mathbb{R}\to\mathcal{G}$. We denote the image of $\alpha\in\mathbb{R}$ in $\mathcal{G}$ as $\bar\alpha$.

Let $\widetilde{\mathcal{G}}$ be the group $\mathbb{R}/2r\mathbb{Z}$ and $\widetilde{\mathcal{X}}$ be the subset $\mathbb{Z}/2r\mathbb{Z}$ in $\widetilde{\mathcal{G}}$. Consider the natural morphisms $\mathbb{R}\to\widetilde{\mathcal{G}}\to\mathcal{G}$. We denote the image of $\alpha\in\mathbb{R}$ in $\widetilde{\mathcal{G}}$ as $\widetilde{\alpha}$. For $\bar\alpha\in\mathcal{G}$ let

$$\begin{aligned}I_{\bar\alpha}&=\{\widetilde{\alpha}\in\widetilde{\mathcal{G}}\mid\bar\alpha=\mathrm{im}\widetilde{\alpha}\text{ under the morphism }\widetilde{\mathcal{G}}\to\mathcal{G}\}\\&=\{\widetilde{\alpha},\widetilde{\alpha}+\widetilde{2},\dots,\widetilde{\alpha}+\widetilde{2(r-1)}\}.\end{aligned}$$

Note that $\widetilde{\mathcal{G}}$ comes with an involution $\widetilde{\alpha}\mapsto\widetilde{\alpha}^*=-\widetilde{\alpha}$ with $I_{-\bar\alpha}=(I_{\bar\alpha})^*$.

Define

$$\mathsf{d}(\widetilde{\alpha})=\mathsf{d}_0\frac{q^\alpha-q^{-\alpha}}{q^{r\alpha}-q^{-r\alpha}}$$

for a fixed real number d_0. If $\widetilde{i}\in I_{\bar\alpha}$, then we define the constant quantity

$$\mathsf{b}(\widetilde{i}):=\frac{1}{r^2}.$$

Let $U_q\mathfrak{sl}(2)$ be the $\mathbb{C}$-algebra generated by E,F,K,K^{-1} subject to relations (5.1). Let $\overline{U}_q\mathfrak{sl}(2)$ be the algebra $U_q\mathfrak{sl}(2)$ modulo the relations $E^r=F^r=0$. This algebra becomes a Hopf algebra via (5.3)–(5.5).

The category $\overline{U}_q\mathfrak{sl}(2)$−mod of weight $\overline{U}_q\mathfrak{sl}(2)$-modules is a relative $\mathcal{G}$-spherical category with basic data leading to 6j-symbols $N_{\widetilde{l}\widetilde{m}\widetilde{n}}^{\widetilde{i}\widetilde{j}\widetilde{k}}$, see Section 9 of [33]. As we now explain, these 6j-symbols are essentially the same as the values of the 6j-symbols derived from the category $\overline{U}_q^H\mathfrak{sl}(2)$−mod.

Consider the highest weight $\overline{U}_q\mathfrak{sl}(2)$-module $V_{\widetilde{i}}$ determined by a highest weight vector of weight $\widetilde{i+r-1} \in \mathbb{R}/2r\mathbb{Z}$. This module is simple if $\widetilde{i} \in \widetilde{\mathcal{G}} \setminus \widetilde{\mathcal{X}}$.

There is a forgetful functor

$$\mathcal{F}: \overline{U}_q^H\mathfrak{sl}(2)\text{−mod} \to \overline{U}_q\mathfrak{sl}(2)\text{−mod}. \tag{5.13}$$

This functor maps V_{i+2rk} to $V_{\widetilde{i}}$ for all $k \in \mathbb{Z}$. The branching rules $\delta_{\widetilde{i}\,\widetilde{j}\,\widetilde{k}}$ are defined to be

$$\begin{aligned}\delta_{\widetilde{i}\,\widetilde{j}\,\widetilde{k}} &= \dim \mathrm{Hom}_{\overline{U}_q\mathfrak{sl}(2)-\mathrm{mod}}(1, V_{\widetilde{i}} \otimes V_{\widetilde{j}} \otimes V_{\widetilde{k}}) \\ &= \begin{cases} 1, & \widetilde{i}+\widetilde{j}+\widetilde{k} \in \{-2r', -2r'+2, \dots, 2r'\}, \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

Note it is possible that $\delta_{\widetilde{i}\,\widetilde{j}\,\widetilde{k}} = 1$ but $\delta_{ijk} = 0$. However, in such a case there exist lifts (via $\mathcal{F}$) of $V_{\widetilde{i}}$, $V_{\widetilde{j}}$, and $V_{\widetilde{k}}$ such that the corresponding hom-space in $\overline{U}_q^H\mathfrak{sl}(2)$−mod is non-zero and $\delta_{ijk} = 1$. Moreover, there exist lifts of i, j, k, l, m, n of $\widetilde{i}, \widetilde{j}, \widetilde{k}, \widetilde{l}, \widetilde{m}, \widetilde{n}$ such that

$$N_{\widetilde{l}\,\widetilde{m}\,\widetilde{n}}^{\widetilde{i}\,\widetilde{j}\,\widetilde{k}} = N_{lmn}^{ijk}$$

see [33, Remark 21].

Also note that the quantities $\beta(i)$ and $\gamma(i, j, k)$ computed in Appendices A.2 and A.3 are independent of the pre-images of i, j, and k in $\mathbb{R}/2r\mathbb{Z}$. Thus, by choosing any lifts, these formulas define $\beta(\widetilde{i})$ and $\gamma(\widetilde{i}, \widetilde{j}, \widetilde{k})$. Combining this discussion with Proposition 5.4 we have

$$\gamma(\widetilde{i}, \widetilde{j}, \widetilde{k})\gamma(\widetilde{k}^*, \widetilde{j}^*, \widetilde{i}^*)\beta(\widetilde{i})\beta(\widetilde{j})\beta(\widetilde{k}) = 1;$$

$$\begin{aligned}\overline{(N_{\widetilde{j}_4\widetilde{j}_5\widetilde{j}_6}^{\widetilde{j}_1\widetilde{j}_2\widetilde{j}_3})} = N_{\widetilde{j}_5\widetilde{j}_4\widetilde{j}_6}^{\widetilde{j}_2^*\widetilde{j}_1^*\widetilde{j}_3^*}\, &\gamma(\widetilde{j}_1, \widetilde{j}_2, \widetilde{j}_3^*)\gamma(\widetilde{j}_1^*, \widetilde{j}_5, \widetilde{j}_6^*) \\ &\times \gamma(\widetilde{j}_2^*, \widetilde{j}_6, \widetilde{j}_4^*)\gamma(\widetilde{j}_3, \widetilde{j}_4, \widetilde{j}_5^*)\prod_i \beta(\widetilde{j}_i).\end{aligned}$$

The discussion of this subsection can be summarized in the following theorem.

Theorem 5.5. *The category of weight modules $\overline{U}_q\mathfrak{sl}(2)$−mod is a relative $\mathcal{G}$-spherical category giving rise to input data*

$$\mathsf{d}(\widetilde{i}),\ \mathsf{b}(\widetilde{i}),\ N_{\widetilde{l}\,\widetilde{m}\,\widetilde{n}}^{\widetilde{i}\,\widetilde{j}\,\widetilde{k}},\ \beta(\widetilde{i}),\ \gamma(\widetilde{i}, \widetilde{j}, \widetilde{k}),\ \delta_{\widetilde{i}\,\widetilde{j}\,\widetilde{k}}$$

for $\widetilde{i}, \widetilde{j}, \widetilde{k} \in \mathsf{A} = \cup_{\tilde{\alpha} \in \mathcal{G}\setminus\mathcal{X}} I_{\tilde{\alpha}}$ satisfying the axioms of Section 2.1.

6. Conclusion

We have shown that using data associated to a relative $\mathcal{G}$-spherical category, it is possible to use non-semisimple categories to construct new classes of two-dimensional topological phases described by local, gapped, commuting projector Hamiltonians. We showed that the ground state of these models agreed with the modified, or non-semisimple, Turaev–Viro TQFT for the surface that the string net Hamiltonian is defined on. A novel feature of these new theories is that they are pseudo-Hermitian with respect to a specified metric η, providing a new link between non-semisimple TQFTs and pseudo-Hermitian phenomena in topological phases.

We gave a family of examples associated with a non-semisimple representation category of quantum $\mathfrak{sl}_2$ at a root of unity. These models would not be possible in the traditional Levin–Wen theory based on semisimple categories, as all of the simple objects utilized in our theory have vanishing quantum dimensions. By utilizing the modified quantum dimension included in the data of a relative $\mathcal{G}$-spherical category, our theories have continuous parameters worth of simple objects in contrast to the usual theory with a finite set of simple objects. These new non-semisimple Levin–Wen models make use of a $\mathcal{G}$-valued cohomology class, allowing us to select a finite set of allowed string labels for each edge of our string net graph.

We anticipate that the continuous nature of the non-semisimple Levin–Wen models introduced here will provide additional flexibility in realizing these models physically. As explained in the introduction, the non-semisimple TQFTs with which these theories are related, are significantly more powerful than their non-semisimple counterparts. We are optimistic that these more sophisticated topological tools, such as the infinite order mapping class group representations arising in non-semisimple TQFTs, will lead to novel new topological gates and universal quantum computation at smaller roots of unity than can be realized in the standard theory. This again may be physically relevant as semisimple representation categories associated with quantum $\mathfrak{sl}_2$ are believed to describe fractional quantum Hall states, where the order of the root of unity corresponds to the filling fraction of the quantum Hall state [3].

CRediT authorship contribution statement

Nathan Geer: Writing, Formal analysis. **Aaron D. Lauda:** Writing, Formal analysis. **Bertrand Patureau-Mirand:** Writing, Formal analysis. **Joshua Sussan:** Writing, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. Some graphical calculus for the unrolled quantum group

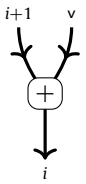
The aim of this section is to give representation-theoretic proofs for Proposition 5.4. Here we recall the evaluation of a network and extend the definition to uni-trivalent graphs. Let Γ be an admissible uni-trivalent network in a disk with its n univalent vertices on the boundary circle. One can associate to Γ a $\mathcal{D}^\dagger$-ribbon graph which can be evaluated with the Turaev functor to some invariant tensor $\in \mathrm{Hom}(1, V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n})$. Note that we read our graphs from bottom to top.

A.1. The module v

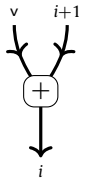
Let v denote the two-dimensional irreducible highest weight module of $\overline{U}_q^H\mathfrak{sl}(2)$ with basis $\{v_0, v_1\}$. The algebra acts as follows:

$$Ev_1 = v_0, \quad Fv_0 = v_1, \quad Hv_0 = v_0, \quad Hv_1 = -v_1.$$

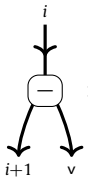
Using definitions from [36, Section 3.3], one could deduce that the following formulas determine maps of $\overline{U}_q^H\mathfrak{sl}(2)$-modules:



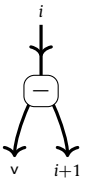
$$: \begin{array}{l} V_i \to V_{i+1} \otimes \mathsf{v}, \\ v_d \mapsto -q^d\{i-d\}(v_d \otimes v_1) - q^{-1}\{1\}v_{d+1} \otimes v_0, \end{array} \tag{A.1}$$



$$: \begin{array}{l} V_i \to \mathsf{v} \otimes V_{i+1}, \\ v_d \mapsto \frac{1}{q}\{i-d\}(v_1 \otimes v_d) - q^{i-d-1}\{1\} v_0 \otimes v_{d+1}, \end{array} \tag{A.2}$$



$$: \begin{array}{l} V_{i+1} \otimes \mathsf{v} \to V_i, \\ v_d \otimes v_0 \mapsto [2r' - d - 1] q^{-2r'+d+1-i} v_{d-1}, \\ v_d \otimes v_1 \mapsto -q v_d, \end{array} \tag{A.3}$$



$$: \begin{array}{l} \mathsf{v} \otimes V_{i+1} \to V_i, \\ v_0 \otimes v_d \mapsto -q[d] v_{d-1}, \\ v_1 \otimes v_d \mapsto q^{-d} v_d. \end{array} \tag{A.4}$$

Proposition A.1 ([36, Equation 13]). *For $i \in \mathbb{R} \setminus \mathbb{Z}$, there is an isomorphism $\mathsf{v} \otimes V_i \cong V_{i+1} \oplus V_{i-1}$.*

Proposition A.2 ([36, Section 3.2]). *There is an isomorphism $w_{\mathsf{v}} : \mathsf{v} \to \mathsf{v}^*$, where*

$$v_0 \mapsto -q v_1^*, \quad v_1 \mapsto v_0^*.$$

A.2. Isomorphisms w_i

There are isomorphisms $w_i : V_i \to V_{-i}^*$ determined by mapping a highest weight vector $v_0 \in V_i$ by $w_i(v_0) = -q^{-1} y_{r-1}^*$, where y_{r-1} is a lowest weight vector of V_{-i}. We depict w_i in the graphical calculus as follows:

$w_i \quad =$  . (A.5)

We want to compute $w_i^\dagger$. In order to do this, consider the Hermitian pairing $(,)_{V_i}$ in [31, Proposition 4.5] and the Hermitian pairing $(,)_{V_{-i}^*}$ in [31, Proposition 4.16]. Using those results, we compute

$$(y_b^*, w v_a)_{V_{-i}^*} = (w^\dagger y_b^*, v_a)_{V_i}. \tag{A.6}$$

So we now need to recall the form on V_{-i}^*. By [31, Proposition 4.16],

$$(y_b^*, y_b^*) = \sum_k y_b^*(e_k) y_b^*(e_k') \tag{A.7}$$

where $\{e_k\}$ and $\{e_k'\}$ are dual bases of V_{-i}. If $e_k = y_k$, we calculate e_k' as follows. Note that

$$\begin{aligned} c_k &:= (y_k, y_k) = (F^k y_0, y_k) = (y_0, E^k y_k) \\ &= \prod_{s=1}^{k} \frac{\{s\}\{s+i\}}{\{1\}^2}. \end{aligned} \tag{A.8}$$

Then set $e'_k := \frac{1}{\bar{c}_k} e_k$. Now a specialization of (A.6) is:

$$(y^*_{r-1}, wv_0)_{V^*_{-i}} = (w^\dagger y^*_{r-1}, v_0)_{V_i}. \tag{A.9}$$

Now we define $\beta(i)$ through the equation

$$w^\dagger = \beta(i) w^{-1}. \tag{A.10}$$

Then $w^\dagger y^*_{r-1} = -q\beta(i)v_0$. Then the righthand side of (A.9) is $-q^{-1}\overline{\beta(i)}$. The lefthand side of (A.9) is

$$\begin{aligned} -q^{-1}(y^*_{r-1}, y^*_{r-1}) &= -q^{-1}\sum_k y^*_{r-1}(e_k) y^*_{r-1}(e'_k) \\ &= \quad -q^{-1} y^*_{r-1}(y_{r-1}) y^*_{r-1}(\frac{1}{\bar{c}_{r-1}} y_{r-1}) = -q^{-1}\frac{1}{\bar{c}_{r-1}}. \end{aligned}$$

This implies

$$\beta(i) = \frac{1}{c_{r-1}}.$$

Lemma A.3. *There is an equality* $\{r-1\}! = \sqrt{-1}^{r-1} r$.

Proof. We compute as follows:

$$\begin{aligned} \{r-1\}! &= \prod_{j=1}^{r-1}(q^j - q^{-j}) = \prod_{j=1}^{r-1}(2\sqrt{-1})\sin(\frac{\pi j}{r}) \\ &= (2\sqrt{-1})^{r-1}\frac{r}{2^{r-1}} = \sqrt{-1}^{r-1} r. \quad \square \end{aligned}$$

Proposition A.4. *One has* $\beta(i) = \frac{\{1\}^{2(r-1)}\mathsf{d}(i)}{r\mathsf{d}_0}$.

Proof. Recall from (A.8) that

$$\begin{aligned} c_{r-1} &= \prod_{s=1}^{r-1}\frac{\{s\}\{s+i\}}{\{1\}^2} = \frac{\{r-1\}!}{\{1\}^{2(r-1)}}\prod_{s=1}^{r-1}\{s+i\} \\ &= \frac{(\{r-1\}!)^2}{\{1\}^{2(r-1)}\{r-1\}!}\prod_{s=1}^{r-1}\{s+i\} \\ &= \frac{(\{r-1\}!)^2}{\{1\}^{2(r-1)}}\begin{bmatrix} r-1+i \\ i \end{bmatrix}. \end{aligned}$$

By [37, Equation 6], this is equal to

$$\frac{(\{r-1\}!)^2}{\{1\}^{2(r-1)}}\left(\frac{\mathsf{d}(i)}{(-1)^{r-1}}\right)^{-1}.$$

By Lemma A.3, this is equal to

$$\frac{(-1)^{r-1}r^2}{\{1\}^{2(r-1)}}\left(\frac{\mathsf{d}(i)}{(-1)^{r-1}}\right)^{-1} = \frac{r^2}{\{1\}^{2(r-1)}\mathsf{d}(i)}.$$

Recall that we take $\mathsf{d}_0 = r$. Thus we get

$$\beta(i) = \frac{(-1)^{r-1}\{1\}^{2(r-1)}\mathsf{d}(i)}{r\mathsf{d}_0} = \frac{\{1\}^{2(r-1)}\mathsf{d}(i)}{r\mathsf{d}_0}. \quad \square$$

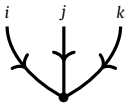
A.3. Coefficients γ

Consider the invariant space

$$H(i,j,k) = \mathrm{Hom}(1, V_i \otimes V_j \otimes V_k).$$

This space is 1-dimensional and is spanned by a specific basis element h_{ijk} that we depict using the graphical notation

h_{ijk} :=



(A.11)

defined explicitly in (A.22). Then using the pivotal structure, a choice of h_{ijk} determines uniquely basis elements of the 1-dimensional spaces

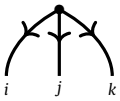
$$\begin{aligned}\mathrm{Hom}(V_i^*, V_j \otimes V_k) &\cong \mathrm{Hom}(V_j^* \otimes V_i^*, V_k)\\ &\cong \mathrm{Hom}(V_k^* \otimes V_j^* \otimes V_i^*, 1).\end{aligned}$$

Combining with the isomorphisms (A.5), a choice of h_{ijk} determines unique basis elements in

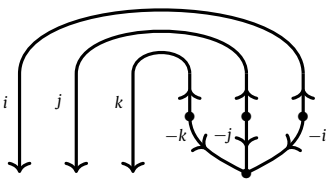
$$\begin{aligned}\mathrm{Hom}(V_{-i}, V_j \otimes V_k) &\cong \mathrm{Hom}(V_{-j} \otimes V_{-i}, V_k)\\ &\cong \mathrm{Hom}(V_{-k} \otimes V_{-j} \otimes V_{-i}, 1).\end{aligned}$$

Then we have h^{ijk} as a distinguished basis element of the 1-dimensional space $\mathrm{Hom}(V_i \otimes V_j \otimes V_k, 1) \cong \mathrm{Hom}(1, V_k^* \otimes V_j^* \otimes V_i^*) \cong \mathrm{Hom}(1, V_{-k} \otimes V_{-j} \otimes V_{-i})$.

h^{ijk} =



:=

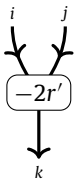


(A.12)

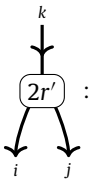
Since $(h_{ijk})^\dagger \in \mathrm{Hom}(V_i \otimes V_j \otimes V_k, 1)$ we define nonzero scalars $\gamma(i,j,k)$ by

$$h_{ijk}^\dagger = \gamma(i,j,k) h^{ijk}. \tag{A.13}$$

Proposition A.5. *If $i + j - k = -2r'$, then there are maps of representations*



$: V_k \to V_i \otimes V_j$

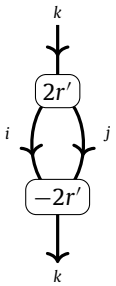


$: V_i \otimes V_j \to V_k$ (A.14)

determined by $v_0 \mapsto v_0 \otimes v_0$ and $v_0 \otimes v_0 \mapsto \frac{1}{\mathrm{d}(k)} v_0$, respectively.

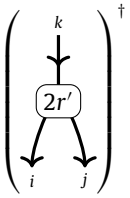
Proof. It is trivial to see that the first map defines a morphism of modules. The second morphism is defined using the first one and the pivotal structure of the category. Both spaces of morphisms are 1-dimensional. The scalar for the second morphism is determined by the identity ([36, Figure

5])

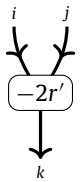


$$= \frac{1}{\mathrm{d}(k)}. \quad \square$$

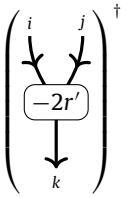
Proposition A.6. *Assume $i + j - k = -2r'$. Then there are equalities of maps*



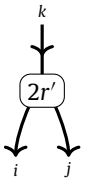
$$= \frac{1}{\mathrm{d}(k)}$$



$$: V_k \to V_i \otimes V_j, \tag{A.15}$$



$$=$$



$$: V_i \otimes V_j \to V_k. \tag{A.16}$$

Proof. Denote the maps in (A.14) by Y_k^{ij} and Y_{ij}^k respectively. In order to compute $(Y_{ij}^k)^\dagger$, we use the pairing from [31, Theorem 4.14]. Let us assume that $(Y_{ij}^k)^\dagger = MY_k^{ij}$, where M is a complex scalar. Then we have the chain of equalities

$$\begin{aligned}
&(v_0, Y_{ij}^k(v_0 \otimes v_0))_{V_k} = ((Y_{ij}^k)^\dagger v_0, v_0 \otimes v_0)_{V_i \otimes V_j} \\
&= (MY_k^{ij} v_0, v_0 \otimes v_0)_{V_i \otimes V_j} \\
&= \overline{M}(v_0 \otimes v_0, v_0 \otimes v_0)_{V_i \otimes V_j} \\
&= \overline{M}(v_0 \otimes v_0, \tau X(v_0 \otimes v_0))_p \\
&= \overline{M}(v_0 \otimes v_0, \tau \sqrt{\theta_{V_j \otimes V_i}}^{-1} c_{V_i, V_j} \sqrt{\theta_{V_i}} \otimes \sqrt{\theta_{V_j}}(v_0 \otimes v_0))_p \\
&= \overline{M}
\end{aligned}$$

where $(,)_{V_i \otimes V_j}$ denotes the form defined in [31, Theorem 4.14] and $(,)_p$ denotes the product of the pairings on each tensor factor (see [31, Equation 31]). The fourth equality follows from [31, Theorem 4.14]. The fifth equality is a consequence of [31, Equation 29]. The sixth equality is a consequence of [31, Equation 28].

On the other hand,

$$(v_0, Y_{ij}^k(v_0 \otimes v_0)) = \frac{1}{\mathrm{d}(k)}.$$

Thus

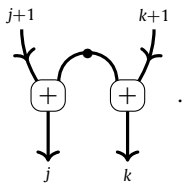
$$M = \frac{1}{\mathrm{d}(k)}$$

establishing (A.15). Eq. (A.16) follows from (A.15), [36, Figure 5], and [31, Lemma 4.19]. □

There are maps (see [36, Lemma 12])

$$H_{jk}: \begin{array}{l} V_j \otimes V_k \to V_{j+1} \otimes V_{k+1}, \\ q^{k+c-d-1}\{j-c\} v_c \otimes v_{d+1} + \frac{1}{q}\{k-d\} v_{c+1} \otimes v_d, \end{array} \tag{A.17}$$

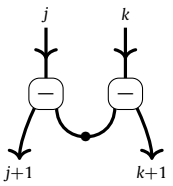
defined as a composite of maps introduced earlier, and graphically denoted by



(A.18)

Remark A.7. A priori, there is an ambiguity in the diagram above since it is not clear which object the morphism w (denoted by a dot) is acting on. However, by the definition of the basic data in a relative $\mathcal{G}$-spherical category, the dot may move anywhere along a cup or cap morphism (see [36, Lemma 7]).

There are maps defined in [36, Section 3.6] (called $\overline{X}$ there)

$H^{jk} =$ 

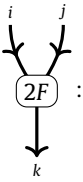
$: V_{j+1} \otimes V_{k+1} \to V_j \otimes V_k.$ (A.19)

Proposition A.8. *We have an equality of maps*

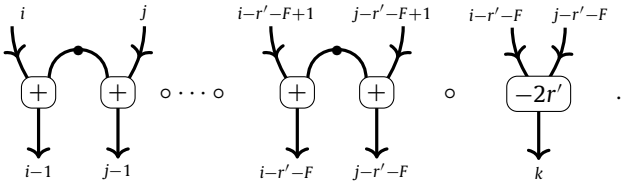
$$H_{jk}^{\dagger} = \{j\}\{k\} H^{jk}.$$

Proof. This follows in a similar fashion to the proof of Proposition A.6. □

Letting $i + j - k = 2F$, we now define the map of representations

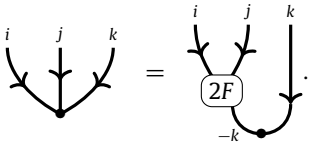


$: V_k \to V_i \otimes V_j$ (A.20)

$:=$ 

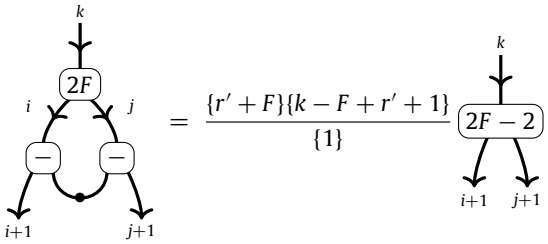
. (A.21)

For $i+j+k=2F$, this allows us to fix an explicit choice of basis element $h_{ijk} \in \mathrm{Hom}(1, V_i \otimes V_j \otimes V_k)$ by



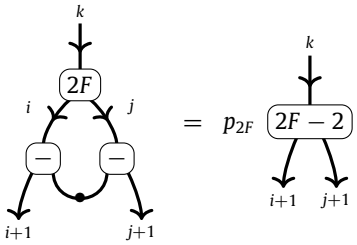
(A.22)

Proposition A.9. *Let $k-i-j=2F$. Then there is an equality of maps $V_{i+1} \otimes V_{j+1} \to V_k$*



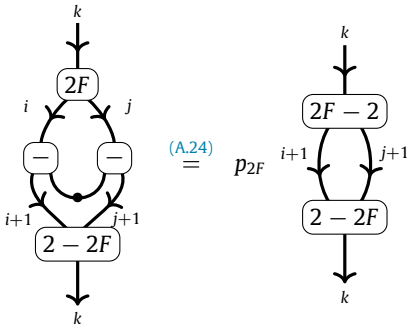
(A.23)

Proof. Since these morphism spaces are 1-dimensional, we must have



(A.24)

for some constant p_{2F}. Precomposing yields the equality



(A.25)

$\prod_{g=1}^{r'+F}\{i-g\}\{j-g\}$

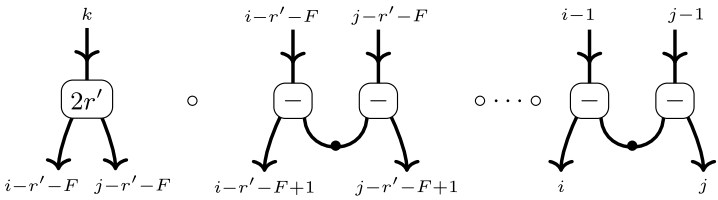
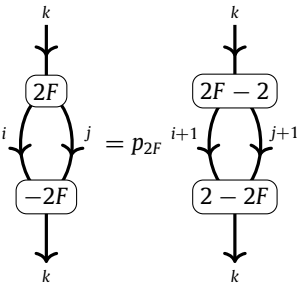


Fig. 2. A decomposition of the left-hand-side of Proposition A.10.

By [36, Proposition 24], (A.25) becomes

$$\frac{\{r'+F\}\{k-F+r'+1\}}{\{1\}}$$

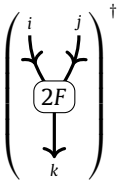


(A.26)

By [36, Figure 5], both diagrams in (A.26) evaluate to $\frac{1}{\mathrm{d}(k)}$. This implies that

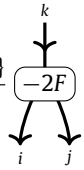
$$p_{2F} = \frac{\{r'+F\}\{k-F+r'+1\}}{\{1\}}. \quad \square$$

Proposition A.10. *There is an equality of maps*



$$= \prod_{g=1}^{r'+F}\{i-g\}\{j-g\}$$

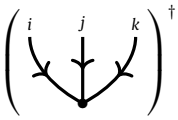
$$\times \prod_{t=1}^{r'+F} \frac{\{2r'-t+1\}\{k+t\}}{\{1\}}$$



Proof. Recall the definition of the map (A.20). Using Proposition A.8 $r'+F$ times and (A.16), we get that the left-hand side of the proposition is shown in Fig. 2 Applying (A.23) $r'+F$ times yields the result. $\square$

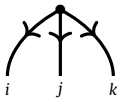
The next proposition yields the value of $\gamma(i,j,k)$. Note that an explicit formula for $\beta(-k)$ is given in Proposition A.4.

Proposition A.11. *Assume $i+j+k=2F$. Then*



$$= \beta(-k)\prod_{g=1}^{r'+F}\{i-g\}\{j-g\} \tag{A.27}$$

$$\times \prod_{t=1}^{r'+F}\frac{\{2r'-t+1\}\{-k+t\}}{\{1\}}$$

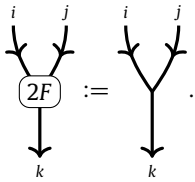


. (A.28)

Proof. Taking the dagger of the right-hand side of (A.22) using Proposition A.10 and the definition of the dagger of the bullet (the map w_{-k}) from Appendix A.2 yield the result. □

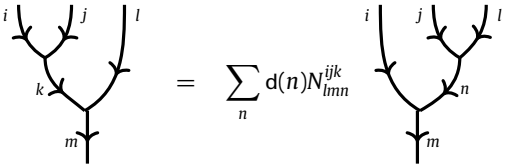
A.4. Modified 6j symbols

In order to reduce the clutter on the diagrams, we will abbreviate the morphism defined earlier as follows:



. (A.29)

The category of representations being monoidal implies that there is an equality of morphisms given below.

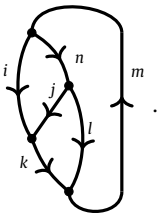


Remark A.12. We note that

$$N^{ijk}_{lmn} := \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \tag{A.30}$$

in the notation from [33]. Since all the vertices in the graphs represent specified maps in a 1-dimensional morphism space, the N^{ijk}_{lmn} are some complex numbers whose formulas could be found in [36].

Note that N^{ijk}_{lmn} could be defined as the value of the following network:



.

In Appendices A.2 and A.3 we defined and computed quantities β and γ:

$$\beta : \mathbb{R} \setminus \mathbb{Z} \to \mathbb{R} \qquad \gamma : (\mathbb{R} \setminus \mathbb{Z})^3 \to \mathbb{R}.$$

This leads to a non involutive semilinear conjugation operator $\Gamma \mapsto \Gamma^{\ddagger}$ on the space of $\mathbb{R}$-colored planar uni-trivalent networks. The image of a graph Γ with state σ, with set of trivalent vertices Γ_0, with set of univalent vertices Γ_0' and with set of edges Γ_1 is given by

$$\Gamma^{\ddagger} = c(\Gamma)\overleftrightarrow{\Gamma} \tag{A.31}$$

where $\overleftrightarrow{\Gamma}$ is the mirror image of Γ with the opposite state and the complex coefficient $c(\Gamma)$ is given by

$$c(\Gamma) = \prod_{v\in\Gamma_0} \gamma(\sigma(v)) \prod_{u\in\Gamma_0'} \beta(\sigma(u))^{-1} \prod_{e\in\Gamma_1} \beta(\sigma(e))$$

where $\sigma(u)$ is σ of the unique edge adjacent to u and $\sigma(v) = (j_1, j_2, j_3) \in \mathbb{R}^3$ where $j_k = \sigma(e_k)$ and e_1, e_2, e_3 are the 3 ordered edges adjacent to v oriented toward v. A network with no univalent vertices is closed.

Remark A.13. Note that for an oriented edge e, $\beta(\sigma(-e)) = \beta(\sigma(e))$. Thus $\beta(\sigma(e))$ make sense even if we forget the orientation on e.

The Poincaré dual of Φ is a $\mathcal{G}$-valued 1-cocycle on the 1-skeleton of $\mathcal{T}$ given on an oriented edge a of $\mathcal{T}$ by $\Phi(a) = \Phi(e)$ where e is the oriented edge of Γ such that the algebraic intersection $a \cap e$ is $+1$. We denote by $[\Phi] \in H^1(\Sigma, \mathcal{G})$ the corresponding cohomology class.

Proposition A.14. *If Γ is a closed planar (or spherical) network, the evaluation of the network $\Gamma^{\ddagger}$ is the conjugate of the value of Γ*

$$\left\langle \Gamma^{\ddagger} \right\rangle = \overline{\langle \Gamma \rangle}. \tag{A.32}$$

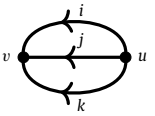
Proof. In [37] it is shown that the graph evaluation arises from the representation theory of the unrolled quantum group for $\mathfrak{sl}(2)$, where the evaluation of a closed graph is determined from a modified trace applied to a cutting of the graph along a projective object. The claim then follows from [31] proving that this category is Hermitian. □

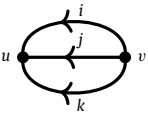
The next result is the goal of the section and is a key ingredient in proving the Hermiticity of the Hamiltonian for the example coming from the semi-restricted quantum group for $\mathfrak{sl}(2)$.

Proposition A.15. *The following identities hold:*

1. $\underline{\gamma(i, j, k)}\gamma(k^*, j^*, i^*)\beta(i)\beta(j)\beta(k) = 1$;
2. $(N_{j_4 j_5 j_6}^{j_1 j_2 j_3}) = N_{j_5 j_4 j_6}^{j_2^* j_1^* j_3^*} \gamma(j_1, j_2, j_3^*)\gamma(j_1^*, j_5, j_6^*) \times \gamma(j_2^*, j_6, j_4^*)\gamma(j_3, j_4, j_5^*) \prod_i \beta(j_i)$.

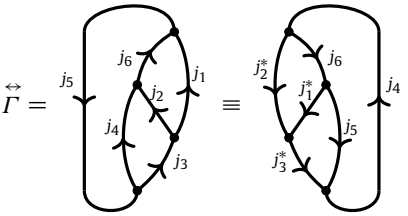
Proof. The first identity comes from $\left\langle \Theta^{\ddagger} \right\rangle = \overline{\langle \Theta \rangle} = 1$ where Θ is the network

$\Theta =$ 

with $\overleftrightarrow{\Theta} =$ 

.

For the second identity we consider the network Γ defining the 6j-symbol $N_{j_4 j_5 j_6}^{j_1 j_2 j_3}$. Then we have



where $\equiv$ is an isotopy of the graph in the sphere. This last graph evaluates to $N_{j_5 j_4 j_6}^{j_2^* j_1^* j_3^*}$. So Eq. (A.32) applied to $\Gamma^{\ddagger}$ implies the proposition. $\square$

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