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A Hermitian TQFT from a non-semisimple category of quantum $\mathfrak{sl}(2)$-modules

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Abstract
We endow a non-semisimple category of modules of unrolled quantum $\mathfrak{sl}(2)$ with a Hermitian structure. We also prove that the CGP TQFT constructed in arXiv:1202.3553 using this category is Hermitian. This gives rise to projective representations of the mapping class group in the group of indefinite unitary matrices.

Keywords Quantum groups · Quantum topology · Hermitian structures

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1 Introduction

Unitary topological quantum field theories are closely related to various physical systems. In particular, they are connected to topological phases of matter [11, 13, 15,

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16]. There is hope that these mathematical theories could be realized physically and perhaps be used for fault tolerant quantum computing [7, 14, 20]. Many fundamental examples of these unitary TQFTs come from the representation theory of quantum groups at roots of unity. The standard procedure is to perform some semisimplification on a category of representations and then use the resulting category to construct 3-manifold invariants and their extensions to $(2+1)$-TQFTs.

If one instead works with the full representation category of a quantum group at a root of unity, without passing to the semisimplification, it was not obvious how to construct TQFTs, as the standard quantum trace on projective modules for quantum groups at roots of unity vanish. The first and third authors, in collaboration with Kujawa, introduced in [8] a modified trace. This modified trace has the remarkable property that it does not vanish on projective objects and retains most of the important properties of the standard quantum trace. This construction led to new link invariants, 3-manifold invariants [4], and $(2+1)$-TQFTs [2]. We refer to these invariants collectively as *non-semisimple* invariants.

Unlike the usual semisimple theory where quantum dimensions of simple objects are strictly positive, the modified dimensions of many objects in the non-semisimple theory are real, but not positive. This means that there is no hope that the TQFT constructed in [2] is unitary. However, in this note, we show that the TQFT arising from a non-semisimple category $\mathscr{D}^{\mathrm{Herm}}$ of representations of the unrolled quantum group for $\mathfrak{sl}_2$ is Hermitian. This means that the TQFT will produce nondegenerate bilinear forms with an indefinite signature.

The notion of a Hermitian ribbon category was introduced by Turaev [24]. One of our main results is the following.

Theorem *$\mathscr{D}^{\mathrm{Herm}}$ is a* Hermitian *ribbon category in the sense of Definition 3.1.*

We stress that analogous fundamental results for the semisimplified categories coming from quantum groups were achieved by Kirillov [12] and Wenzl [25]. In order to accommodate projective objects in $\mathscr{D}^{\mathrm{Herm}}$, we needed to modify some arguments of [12, 25].

Finally, in the last section, we apply the Hermicity of $\mathscr{D}^{\mathrm{Herm}}$ to show that the TQFT constructed with it as in [2] is Hermitian in the sense of Turaev [24].

Theorem *The TQFT $(\mathbb{V}, Z)$ introduced in Sect. 5.3 is Hermitian.*

A more precise relation of this TQFT with the TQFT $\widetilde{\mathbb{V}}$ of [2] is as follows. There is a full forgetful functor from $\mathscr{D}^{\mathrm{Herm}}$ to the category of $\overline{U}_q^H\mathfrak{sl}(2)$ weight module which forgets the Hermitian structure. It is surjective on projective modules with real weights. This induces a functor from the cobordism category which is the source of $\mathbb{V}$ onto a subcategory (of cobordisms equipped with real valued cohomology classes) of the source category of $\widetilde{\mathbb{V}}$. As a $\mathcal{V}ect$-valued functor, $\mathbb{V}$ factors through $\widetilde{\mathbb{V}}$ but vector spaces in the image of $\mathbb{V}$ are now equipped with Hermitian forms.

In Proposition 5.8, we show that this implies that the mapping class group action induced by the non-semisimple TQFT produces a projective representation in the group of indefinite unitary matrices.

Even with an indefinite normed inner product, the Hermitian TQFTs defined here may have physical relevance. Indeed, quantum mechanics with indefinite norms have

been studied going back to Dirac [6] and Pauli [22]. Even with the indefinite norms, they observed a formalism consistent with deterministic quantum mechanics, including positive energy eigenvalues, normalizable wave functions, and time evolution by an exponential of the Hamiltonian that is self-adjoint in the indefinite norm. More recently, the study of pseudo-Hermitian quantum mechanics has been intensively studied [17, 18], motivated by connections to $\mathcal{PT}$-symmetric quantum theory [1]. In all of these studies, indefinite normed Hilbert spaces admit Hamiltonians that are Hermitian with respect to the indefinite inner product, yet still have real spectrum, and unitary evolution. In [10], the authors show that the TQFTs studied here give rise large classes of exactly solvable pseudo-Hermitian Hamiltonians generalizing Levin-Wen models into this framework.

2 A quantization of $\mathfrak{sl}(2)$ and its associated ribbon category

In this section, we recall the algebra $\overline{U}_q^H\mathfrak{sl}(2)$ and a category of modules over this algebra. Fix a positive integer r. Let $r' = r$ if r is odd and $r' = \frac{r}{2}$ otherwise. Let $\mathbb{C}$ be the complex numbers and $\ddot{\mathbb{C}} = (\mathbb{C} \setminus \mathbb{Z}) \cup r\mathbb{Z}$. Let $q = e^{\frac{\pi\sqrt{-1}}{r}}$ be a $2r^{th}$-root of unity. We use the notation $q^x = e^{\frac{\pi\sqrt{-1}x}{r}}$. For $n \in \mathbb{N}$, we also set

$$\{x\} = q^x - q^{-x}, \quad [x] = \frac{\{x\}}{\{1\}}, \quad \{n\}! = \{n\}\{n-1\}\cdots\{1\} \quad \text{and} \quad [n]! = [n][n-1]\cdots[1].$$

2.1 The Drinfel'd–Jimbo quantum group

Let $U_q\mathfrak{sl}(2)$ be the $\mathbb{C}$-algebra given by generators E, F, K, K^{-1} and relations:

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,$$
$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{1}$$

The algebra $U_q\mathfrak{sl}(2)$ is a Hopf algebra where the coproduct, counit and antipode are defined by

$$\Delta(E) = 1 \otimes E + E \otimes K, \qquad \varepsilon(E) = 0, \qquad S(E) = -EK^{-1}, \tag{2}$$
$$\Delta(F) = K^{-1} \otimes F + F \otimes 1, \qquad \varepsilon(F) = 0, \qquad S(F) = -KF, \tag{3}$$
$$\Delta(K) = K \otimes K \qquad \varepsilon(K) = 1, \qquad S(K) = K^{-1}. \tag{4}$$

Let $\overline{U}_q\mathfrak{sl}(2)$ be the algebra $U_q\mathfrak{sl}(2)$ modulo the relations $E^r = F^r = 0$.

2.2 A modified version of $U_q\mathfrak{sl}(2)$

Let $U_q^H\mathfrak{sl}(2)$ be the $\mathbb{C}$-algebra given by generators E, F, K, K^{-1}, H and relations in (1) along with the relations:

$$HK = KH, \qquad [H, E] = 2E, \qquad [H, F] = -2F.$$

The algebra $U_q^H\mathfrak{sl}(2)$ is a Hopf algebra where the coproduct, counit and antipode are defined in (2)–(4) and by

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \qquad \varepsilon(H) = 0, \qquad S(H) = -H.$$

Define $\overline{U}_q^H\mathfrak{sl}(2)$ to be the Hopf algebra $U_q^H\mathfrak{sl}(2)$ modulo the relations $E^r = F^r = 0$.

Let V be a finite dimensional $\overline{U}_q^H\mathfrak{sl}(2)$-module. An eigenvalue $\lambda \in \mathbb{C}$ of the operator $H : V \to V$ is called a *weight* of V and the associated eigenspace is called a *weight space*. A vector v in the λ-eigenspace of H is a *weight vector* of *weight* λ, i.e. $Hv = \lambda v$. We call V a *weight module* if V splits as a direct sum of weight spaces and $q^H = K$ as operators on V, i.e. $Kv = q^\lambda v$ for any vector v of weight λ. Let $\mathscr{C}$ be the category of finite dimensional weight $\overline{U}_q^H\mathfrak{sl}(2)$-modules.

Since $\overline{U}_q^H\mathfrak{sl}(2)$ is a Hopf algebra, $\mathscr{C}$ is a tensor category where the unit $\mathbb{I}$ is the 1-dimensional trivial module $\mathbb{C}$. Moreover, $\mathscr{C}$ is $\mathbb{C}$-linear: hom-sets are $\mathbb{C}$-modules, the composition and tensor product of morphisms are $\mathbb{C}$-bilinear, and $\mathrm{End}_{\mathscr{C}}(\mathbb{I}) = \mathbb{C}\,\mathrm{Id}_{\mathbb{I}}$. When it is clear, we denote the unit $\mathbb{I}$ by $\mathbb{C}$. We say a module V is *simple* if it has no proper submodules. For a module V and a morphism $f \in \mathrm{End}_{\mathscr{C}}(V)$, we write $\langle f\rangle_V = \lambda \in \mathbb{C}$ if $f - \lambda\,\mathrm{Id}_V$ is nilpotent. If V is simple, then Schur's lemma implies that $\mathrm{End}_{\mathscr{C}}(V) = \mathbb{C}\,\mathrm{Id}_V$. Thus for $f \in \mathrm{End}_{\mathscr{C}}(V)$, we have $f = \langle f\rangle_V\,\mathrm{Id}_V$.

We will now recall the fact that the category $\mathscr{C}$ is a ribbon category. Let V and W be objects of $\mathscr{C}$. Let $\{v_i\}$ be a basis of V and $\{v_i^*\}$ be a dual basis of $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Then

$$\overrightarrow{\mathrm{coev}}_V : \mathbb{C} \to V \otimes V^*, \text{ given by } 1 \mapsto \sum v_i \otimes v_i^*,$$
$$\overrightarrow{\mathrm{ev}}_V : V^* \otimes V \to \mathbb{C}, \text{ given by } f \otimes w \mapsto f(w)$$

are duality morphisms of $\mathscr{C}$. In [21], Ohtsuki truncates the usual formula of the h-adic quantum $\mathfrak{sl}(2)$ R-matrix to define an operator on $V \otimes W$ by

$$R = q^{H\otimes H/2} \sum_{n=0}^{r-1} \frac{\{1\}^{2n}}{\{n\}!} q^{n(n-1)/2} E^n \otimes F^n. \tag{5}$$

where $q^{H\otimes H/2}$ is the operator given by

$$q^{H\otimes H/2}(v \otimes v') = q^{\lambda\lambda'/2} v \otimes v'$$

for weight vectors v and v' of weights of λ and λ'. The R-matrix is not an element in $\overline{U}_q^H\mathfrak{sl}(2) \otimes \overline{U}_q^H\mathfrak{sl}(2)$. However the action of R on the tensor product of two objects of $\mathscr{C}$ is a well defined linear map. Moreover, R gives rise to a braiding $c_{V,W} : V \otimes W \to W \otimes V$ on $\mathscr{C}$ defined by $v \otimes w \mapsto \tau(R(v \otimes w))$ where τ is the permutation $x \otimes y \mapsto y \otimes x$. This braiding follows from the invertibility of the R-matrix. An

explicit inverse (see [3, Section 2.1.2] and [21]) is given by

$$R^{-1} = \left(\sum_{n=0}^{r-1} (-1)^n \frac{\{1\}^{2n}}{\{n\}!} q^{-n(n-1)/2} E^n \otimes F^n \right) q^{-H\otimes H/2}. \tag{6}$$

Let θ be the operator given by

$$\theta = K^{r-1} \sum_{n=0}^{r-1} \frac{\{1\}^{2n}}{\{n\}!} q^{n(n-1)/2} S(F^n) q^{-H^2/2} E^n \tag{7}$$

where $q^{-H^2/2}$ is an operator defined on a weight vector v_λ by $q^{-H^2/2}.v_\lambda = q^{-\lambda^2/2} v_\lambda$. Ohtsuki shows that the family of maps $\theta_V : V \to V$ in $\mathscr{C}$ defined by $v \mapsto \theta^{-1} v$ is a twist (see [19, 21]).

Now the ribbon structure on $\mathscr{C}$ yields right duality morphisms

$$\overleftarrow{\mathrm{ev}}_V = \overrightarrow{\mathrm{ev}}_V \, c_{V,V^*} (\theta_V \otimes \mathrm{Id}_{V^*}) \text{ and } \overleftarrow{\mathrm{coev}}_V = (\mathrm{Id}_{V^*} \otimes \theta_V) c_{V,V^*} \overrightarrow{\mathrm{coev}}_V \tag{8}$$

which are compatible with the left duality morphisms $\{\overrightarrow{\mathrm{coev}}_V\}_V$ and $\{\overrightarrow{\mathrm{ev}}_V\}_V$. These duality morphisms are given explicitly by

$$\overleftarrow{\mathrm{coev}}_V : \mathbb{C} \to V^* \otimes V, \text{ where } 1 \mapsto \sum v_i^* \otimes K^{r-1} v_i,$$
$$\overleftarrow{\mathrm{ev}}_V : V \otimes V^* \to \mathbb{C}, \text{ where } v \otimes f \mapsto f(K^{1-r} v).$$

The *quantum dimension* $\mathrm{qdim}(V)$ of an object V in $\mathscr{C}$ is defined by

$$\mathrm{qdim}(V) = \left\langle \overleftarrow{\mathrm{ev}}_V \circ \overrightarrow{\mathrm{coev}}_V \right\rangle_{\mathbb{I}} = \sum v_i^* (K^{1-r} v_i) \, .$$

For $g \in \mathbb{C}/2\mathbb{Z}$, define $\mathscr{C}_g$ as the full subcategory of weight modules whose weights are all in the class $g \pmod{2\mathbb{Z}}$. Then, $\mathscr{C} = \{\mathscr{C}_g\}_{g\in\mathbb{C}/2\mathbb{Z}}$ is a $\mathbb{C}/2\mathbb{Z}$-graded category (where $\mathbb{C}/2\mathbb{Z}$ is an additive group). Let $V \in \mathscr{C}_g$ and $V' \in \mathscr{C}_{g'}$. Then, the weights of $V \otimes V'$ are congruent to $g + g' \mod 2\mathbb{Z}$, and so the tensor product is in $\mathscr{C}_{g+g'}$. Also, if $g \neq g'$, then $\mathrm{Hom}_{\mathscr{C}}(V, V') = 0$ since morphisms in $\mathscr{C}$ preserve weights. Finally, if $f \in V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, then by definition the action of H on f is given by $(Hf)(v) = f(S(H)v) = -f(Hv)$ and so $V^* \in \mathscr{C}_{-g}$.

We now consider the following class of finite dimensional highest weight modules. For each $\alpha \in \mathbb{C}$, we let V_α be the r-dimensional highest weight $\overline{U}_q^H \mathfrak{sl}(2)$-module of highest weight $\alpha + r - 1$. The module V_α has a basis $\{v_0, \ldots, v_{r-1}\}$ whose action is given by

$$H.v_i = (\alpha + r - 1 - 2i) v_i, \quad E.v_i = \frac{\{i\}\{i-\alpha\}}{\{1\}^2} v_{i-1}, \quad F.v_i = v_{i+1}. \tag{9}$$

For all $\alpha \in \mathbb{C}$, the quantum dimension of V_α is zero:

$$\mathrm{qdim}(V_\alpha) = \sum_{i=0}^{r-1} v_i^*(K^{1-r}v_i) = \sum_{i=0}^{r-1} q^{(r-1)(\alpha+r-1-2i)} = q^{(r-1)(\alpha+r-1)}\frac{1-q^{2r}}{1-q^2} = 0.$$

For $a \in \mathbb{Z}$, let $\mathbb{C}_{ar}^H$ be the one dimensional module in $\mathscr{C}_{\overline{0}}$ where both E and F act by zero and H acts by ar. For each $n \in \{0, \ldots, r-2\}$, let S_n be the usual $(n+1)$-dimensional simple highest weight $\overline{U}_q^H\mathfrak{sl}(2)$-module with highest weight n. The module S_n has highest weight vector s_0 such that $Es_0 = 0$ and $Hs_0 = ns_0$. Then, $\{s_0, s_1, \ldots, s_n\}$ is a basis of S_n where $Fs_i = s_{i+1}$, $H.s_i = (n-2i)s_i$, $E.s_0 = 0 = F^{n+1}.s_0$ and $E.s_i = \frac{\{i\}\{n+1-i\}}{\{1\}^2}s_{i-1}$. Every simple module of $\mathscr{C}$ is isomorphic to exactly one of the modules in the list:

- $S_n \otimes \mathbb{C}_{ar}^H$, for $n = 0, \cdots, r-2$ and $a \in \mathbb{Z}$,
- V_α for $\alpha \in (\mathbb{C} \setminus \mathbb{Z}) \cup r\mathbb{Z}$.

For $i \in \{0, ..., r-2\}$, let P_i be the projective and indecomposable module with highest weight $2r-2-i$, defined in Proposition 6.2 of [5]. Moreover, any indecomposable projective weight module has a highest weight, and such a module $P \in \mathscr{C}_{\overline{0}} \cup \mathscr{C}_{\overline{1}}$ with highest weight $(k+2)r - i - 2$ is isomorphic to $P_i \otimes \mathbb{C}_{kr}^H$.

3 Categorical preliminaries

3.1 Hermitian ribbon category

Here, we follow [24, Section 5.1]. Let $\mathscr{C}$ be a strict monoidal category. A dagger, or conjugation, on $\mathscr{C}$, assigns to each morphism $f: V \to W$ a morphism $f^\dagger: W \to V$ such that

$$(f^\dagger)^\dagger = f, \quad (f \otimes g)^\dagger = f^\dagger \otimes g^\dagger, \quad (f \circ g)^\dagger = g^\dagger \circ f^\dagger. \tag{10}$$

These relations imply $\mathrm{Id}_V^\dagger = \mathrm{Id}_V$. In other words, $\dagger$ is an object preserving contravariant involution on $\mathscr{C}$.

Definition 3.1 A *Hermitian ribbon category* is a ribbon monoidal category $\mathscr{C}$ equipped with a conjugation satisfying the following conditions:

(i) for any objects V, W of $\mathscr{C}$, we have
For any objects

$$c_{V\otimes W}^\dagger = (c_{V,W})^{-1}, \tag{11}$$

(ii) for any object V of $\mathscr{C}$, we have [1]

$$\theta_V^\dagger = (\theta_V)^{-1}, \overrightarrow{\mathrm{coev}}_V^\dagger = \overrightarrow{\mathrm{ev}}_V\, c_{V,V^*}(\theta_V \otimes \mathrm{Id}_{V^*}),\ \overrightarrow{\mathrm{ev}}_V^\dagger = (\mathrm{Id}_{V^*} \otimes \theta_V)c_{V,V^*}\overrightarrow{\mathrm{coev}}_V\,. \tag{12}$$

[1] We use an equivalent definition to the one Turaev uses for $\overrightarrow{\mathrm{ev}}_V^\dagger$.

3.2 Modified traces on the projective modules

Let Proj be the full subcategory of $\mathscr{C}$ consisting of projective $\overline{U}_q^H \mathfrak{sl}(2)$-modules. The subcategory Proj is an ideal (see also [8]). That is, it is closed under retracts (i.e. if $W \in \mathsf{Proj}$ and $\alpha : X \to W$ and $\beta : W \to X$ satisfy $\beta \circ \alpha = \mathrm{Id}_X$, then $X \in \mathsf{Proj}$) and if X is in $\mathscr{C}$, and Y is in Proj, then $X \otimes Y$ is in Proj.

For any objects V, W of $\mathscr{C}$ and any endomorphism f of $V \otimes W$, set

$$\mathrm{ptr}_L(f) = (\overrightarrow{\mathrm{ev}}_V \otimes \mathrm{Id}_W) \circ (\mathrm{Id}_{V^*} \otimes f) \circ (\overleftarrow{\mathrm{coev}}_V \otimes \mathrm{Id}_W) \in \mathrm{End}_{\mathscr{C}}(W), \tag{13}$$

and

$$\mathrm{ptr}_R(f) = (\mathrm{Id}_V \otimes \overleftarrow{\mathrm{ev}}_W) \circ (f \otimes \mathrm{Id}_{W^*}) \circ (\mathrm{Id}_V \otimes \overrightarrow{\mathrm{coev}}_W) \in \mathrm{End}_{\mathscr{C}}(V). \tag{14}$$

Definition 3.2 A *trace on* Proj is a family of linear functions

$$\{\mathsf{t}_V : \mathrm{End}_{\mathscr{C}}(V) \to K\}$$

where V runs over all objects of Proj, such that the following two conditions hold.

(1) If $U \in \mathsf{Proj}$, and $W \in \mathrm{Ob}(\mathscr{C})$, then for any $f \in \mathrm{End}_{\mathscr{C}}(U \otimes W)$, we have

$$\mathsf{t}_{U\otimes W}(f) = \mathsf{t}_U\left(\mathrm{ptr}_R(f)\right). \tag{15}$$

(2) If $U, V \in \mathsf{Proj}$, then for any morphisms $f : V \to U$, and $g : U \to V$ in $\mathscr{C}$, we have

$$\mathsf{t}_V(g \circ f) = \mathsf{t}_U(f \circ g). \tag{16}$$

There exists up to a scalar a unique trace on Proj. It is non-degenerate (cf Theorem 5.5 of [9]), in the following way. Let $V, W \in \mathscr{C}$ with V projective. Then, the pairing $\langle \cdot, \cdot \rangle_{V,W} : \mathrm{Hom}_{\mathscr{C}}(W, V) \otimes \mathrm{Hom}_{\mathscr{C}}(V, W) \to \mathbb{C}$ given by

$$\langle f, g \rangle_{V,W} = \mathsf{t}_V(fg)$$

is non-degenerate. It is symmetric in the following sense. If W is also projective, then

$$\langle g, f \rangle_{W,V} = \langle f, g \rangle_{V,W}\,. \tag{17}$$

If W is not projective, then we take Equation (17) as a definition.

The modified dimension $\mathsf{d}(M)$, of an object M, is the modified trace of the identity morphism of M. The modified trace on $\mathscr{C}$ in this paper is normalized by

$$\mathsf{d}(V_\alpha) = \frac{\mathsf{d}_0\{\alpha\}}{\{r\alpha\}}$$

for a fixed real number d_0.

4 Hermitian ribbon structure on quantum $\mathfrak{sl}_2$-modules

4.1 Hermitian $\overline{U}_q^H\mathfrak{sl}(2)$-modules

A map $f : V \to W$ between two complex vector spaces is called *antilinear* if $f(av + bv') = \bar{a}f(v) + \bar{b}f(v')$ where $\bar{a}$ and $\bar{b}$ are the complex conjugates of the complex numbers a and b, respectively. Consider the operation $\varrho : x \mapsto \varrho(x) := x^\varrho$ defined on generators of $\overline{U}_q^H\mathfrak{sl}(2)$ by

$$E^\varrho = F, \quad F^\varrho = E, \quad K^\varrho = K^{-1}, \quad H^\varrho = H.$$

Lemma 4.1 *The operator $\varrho : \overline{U}_q^H\mathfrak{sl}(2) \to \overline{U}_q^H\mathfrak{sl}(2)$ induces an antilinear, antialgebra involution which is also a coalgebra antimorphism. That is, for any $a \in \mathbb{Q}(q)$ and $x, y \in \overline{U}_q^H\mathfrak{sl}(2)$,*

$$(ax)^\varrho = \bar{a}x^\varrho \quad (xy)^\varrho = y^\varrho x^\varrho \quad (x^\varrho)^\varrho = x \quad \Delta(x^\varrho) = (\varrho \otimes \varrho)(\tau(\Delta x)).$$

Furthermore, $S(\dagger x) = (S(x))^\varrho$, $(\varrho \otimes \varrho)(R) = \tau(R^{-1})$, and $\dagger\theta = \theta^{-1}$.

Proof This is similar to Lemma 1.3 of [25]. □

A *Hermitian form* on a $\mathbb{C}$-vector space V is a function $f : V \times V \to \mathbb{C}$ such that

(1) $f(v, av' + bv'') = af(v, v') + bf(v, v'')$,
(2) $f(v, v') = \overline{f(v', v)}$,

for all $v, v', v'' \in V$ and $a, b \in \mathbb{C}$. It follows that f is antilinear in the first coordinate. The kernel of f is $\{v \in V : V^* \ni f(v, \cdot) = 0\}$, and we say f is non-degenerate if its kernel is $\{0\}$.

Let $f : V \otimes V \to \mathbb{C}$ be a non-degenerate Hermitian form on a finite dimensional weight $\overline{U}_q^H\mathfrak{sl}(2)$-module V. Then, we say f is *compatible* with the antilinear antialgebra automorphism ϱ if $f(\dagger x v, v') = f(v, xv')$ for all $x \in \overline{U}_q^H\mathfrak{sl}(2)$ and $v, v' \in V$. Equivalently, this means $\rho_V(x)^\varrho = \rho_V(\dagger x)$ where $\rho_V(x)^\varrho$ denotes the Hermitian adjoint on Hermitian vector spaces. In this case, we say V is a *Hermitian* $\overline{U}_q^H\mathfrak{sl}(2)$-module with *Hermitian structure* f.

Lemma 4.2 *Let f_V and f_W be Hermitian structures on $\overline{U}_q^H\mathfrak{sl}(2)$-modules V and W, respectively. Then, there is a well-defined adjoint map $\dagger$, which is an antilinear homomorphism*

$$\begin{aligned} \dagger : \mathrm{Hom}_{\mathscr{C}}(V, W) &\to \mathrm{Hom}_{\mathscr{C}}(W, V) \\ g &\mapsto g^\dagger, \end{aligned}$$

uniquely defined by $f_W(\cdot, g(\cdot)) = f_V(g^\dagger(\cdot), \cdot)$.

Moreover, if U is a third Hermitian module, and $h \in \mathrm{Hom}_{\mathscr{C}}(W, U)$, then $(hg)^\dagger = g^\dagger h^\dagger$.

Proof First, since f_V and f_W are non-degenerate Hermitian forms, $g^\dagger$ is a well-defined linear map. To see that this map is a $\overline{U}_q^H\mathfrak{sl}(2)$-module morphism, let $x \in \overline{U}_q^H\mathfrak{sl}(2)$, $v \in V$ and $w \in W$. Then

$$f_V(g^\dagger(xw), v) = f_W(xw, g(v)) = f_W(w, g(x^\varrho v)) = f_V(g^\dagger(w), x^\varrho v) = f_V(xg^\dagger(w), v).$$

Since v is any element of V and f_V is non-degenerate, it then follows that $g^\dagger(xw) = xg^\dagger(w)$. Similar calculations show that the map $\dagger$ is antilinear and satisfies the last property of the lemma. □

If V is a $\overline{U}_q^H\mathfrak{sl}(2)$ weight module, let $\bar{V}^\varrho = \{\bar{v} : v \in V\}$ be the same real vector space with antilinear scalar multiplication, and the action of $\overline{U}_q^H\mathfrak{sl}(2)$ on $\bar{v} \in \bar{V}^\varrho$ be given by $x.\bar{v} = \overline{S(x)^\varrho.v}$.

A sesquilinear form f on a $\overline{U}_q^H\mathfrak{sl}(2)$-module V is said to be compatible with $\overline{U}_q^H\mathfrak{sl}(2)$ if it is non-degenerate and for any $x \in \overline{U}_q^H\mathfrak{sl}(2)$ and $v_1, v_2 \in V$, we have $f(v_1, \rho_V(x)(v_2)) = f(\rho_V(x^\varrho)(v_1), v_2)$.

Lemma 4.3 *Let V be a $\overline{U}_q^H\mathfrak{sl}(2)$ weight module. Then, there exists a compatible sesquilinear form f on V if and only if $\bar{V}^\varrho \simeq V^*$. Furthermore, if V is simple, then f can be chosen to be Hermitian and is unique up to a constant in $\mathbb{R}^*$.*

Proof For the first part, we follow [12, 25]. The sesquilinear form associated to an isomorphism $\varphi : \bar{V}^\varrho \xrightarrow{\sim} V^*$ is given by

$$f : V \times V \xrightarrow{\bar{\mathrm{Id}}\otimes\mathrm{Id}} \bar{V}^\varrho \otimes V \xrightarrow{\varphi\otimes\mathrm{Id}} V^* \otimes V \xrightarrow{\overrightarrow{\mathrm{ev}}} \mathbb{C}.$$

The compatibility follows from the following string of equalities:

$$\begin{aligned} f(v_1, x.v_2) &= \overrightarrow{\mathrm{ev}}\ (\varphi(\bar{v}_1) \otimes x.v_2) = \overrightarrow{\mathrm{ev}}\ (S^{-1}(x).\varphi(\bar{v}_1), v_2) = \overrightarrow{\mathrm{ev}}\ (\varphi(S^{-1}(x).\bar{v}_1), v_2) \\ &= \overrightarrow{\mathrm{ev}}\ (\varphi(\overline{\dagger x.v_1}), v_2) = f(\dagger x.v_1, v_2). \end{aligned}$$

Next, for a fixed basis (e) of V, let A denote the matrix of f: $A_{i,j} = f(e_i, e_j)$. Let $\mathrm{Mat}_{(e)}(x) = M$ and denote by M^* the conjugate transpose of M. Then, $\mathrm{Mat}_{(e)}(\dagger x) = (A^{-1})^* M^* A^*$ and $M = \mathrm{Mat}_{(e)}((\dagger x)^\varrho) = (A^{-1})^* A M A^{-1} A^*$. Hence, $A^{-1}A^*$ commutes with the image of $\overline{U}_q^H\mathfrak{sl}(2)$ in $\mathrm{End}_{\mathbb{C}}(V)$. Suppose now that V is also simple.

Then, $A^{-1}A^*$ is a scalar because it is the matrix of a $\overline{U}_q^H\mathfrak{sl}(2)$-module morphism and $\mathrm{End}_{\mathscr{C}}(V) = \mathbb{C}\,\mathrm{Id}_V$. Let us write $A^{-1}A^* = \lambda^2$ then $(A^*)^* = A$ implies $\overline{\lambda}\lambda = 1$.

Then, the matrix of λf is Hermitian as $(\lambda A)^* = \overline{\lambda}A^* = \overline{\lambda}\lambda^2 A = |\lambda|^2(\lambda A) = \lambda A$.

Finally, any two compatible sesquilinear forms f, f' differ by an automorphism $g \in \mathrm{Aut}_{\mathscr{C}}(V)$ such that $f' = f(\cdot, g(\cdot))$. So the last statement follows with $\mathrm{Aut}_{\mathscr{C}}(V) = \mathbb{C}^*\,\mathrm{Id}$ and for f Hermitian, λf is Hermitian if and only if $\lambda \in \mathbb{R}$. □

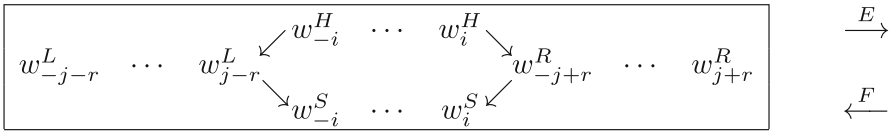

Fig. 1 The weight spaces structure of the module P_i (here $j = r - 2 - i$)

If A is a set of objects in $\mathscr{C}$, then we define the *category generated* by A as the full subcategory of $\mathscr{C}$ which has as objects, all direct sums of retracts of all tensor products of the form:

$$X_1 \otimes X_2 \otimes \cdots \otimes X_p \quad \text{where } X_i \in A \cup A^*.$$

Each simple module S_i, for $i = 0, \ldots, r-2$ has an indecomposable projective cover P_i. The dimension of each P_i is $2r$. A detailed description of this module could be found in [5, Proposition 6.1]. A summary could be found in Figure 1. A vector w_k^Y, for $Y \in \{R, H, S, L\}$ has weight k (under the action of H).

Consider the subcategory $\mathscr{D}$ of $\mathscr{C}$ generated by the following set:

$$\mathsf{A} = \left\{ V_\alpha, S_n, P_i, \mathbb{C}^H_{ar} \;\middle|\; \alpha \in (\mathbb{R} \setminus \mathbb{Z}) \cup r\mathbb{Z}, n, i \in \{0, \cdots, r-2\}, a \in \mathbb{Z} \right\}. \quad (18)$$

Proposition 4.4 *All projectives objects are in the Karoubi envelope (or idempotent completion) of the additive monoidal category generated by the simple objects.*

Proof By [5, Proposition 8.4], one could generate all $P_k \otimes \mathbb{C}^H_{mr}$ where k is even by decomposing $(S_{r-1} \otimes \mathbb{C}^H_{mr}) \otimes S_{r-1}$. Similarly, one could generate all $P_k \otimes \mathbb{C}^H_{mr}$ where k is odd by decomposing $(S_{r-2} \otimes \mathbb{C}^H_{mr}) \otimes S_{r-1}$.

All other indecomposable projective objects are already simple. □

We will now show that each object of $\mathscr{D}$ has a Hermitian structure and prove that $\mathscr{D}$ is a Hermitian ribbon category.

Proposition 4.5 *Any simple module V in A has a Hermitian structure. Moreover, the form $(\cdot, \cdot)$ on V is uniquely determined by $(v_0, v_0) = 1$ where v_0 is a highest weight vector of V.*

Proof Let V be a simple module in A. Since $\dagger H = H$, we have $\bar{V}^\varrho$ is a simple module with character that is the conjugate of that of V^*. Since the weights of modules in A are real, $\bar{V}^\varrho$ and V^* have the same character so they are isomorphic. Then, Lemma 4.3 applies. □

Proposition 4.6 *If $i \in \{0, ..., r-2\}$ then the projective indecomposable P_i has a Hermitian form $(\cdot, \cdot)_{\alpha,\beta}$ making it a Hermitian $\overline{U}_q^H \mathfrak{sl}(2)$-module where α and β are two real nonzero parameters.*

Proof Recall from [5, Proposition 6.1] that the action of $\overline{U}_q^H \mathfrak{sl}(2)$ on the basis $\{w_k^Y\}$ of P_i involves the quantities $\gamma_{n,k} = [k][n-k+1]$.

Now define

$$(w^L_{-j-r}, w^L_{-j-r})_{\alpha,\beta} = \alpha \qquad (w^H_{-i}, w^H_{-i})_{\alpha,\beta} = \beta\ . \tag{19}$$

If the form is Hermitian and $(xv, w)_{\alpha,\beta} = (v, x^{\varrho} w)_{\alpha,\beta}$, then

$$\begin{aligned}(w^L_{j-2k-r}, w^L_{j-2k-r})_{\alpha,\beta} &= \left(\frac{E^{j-k}w^L_{-j-r}}{\prod_{m=k+1}^{j} -\gamma_{j,m}}, \frac{E^{j-k}w^L_{-j-r}}{\prod_{m=k+1}^{j} -\gamma_{j,m}}\right)_{\alpha,\beta}\\ &= \frac{1}{\prod_{m=k+1}^{j} \overline{\gamma_{j,m}}\gamma_{j,m}}(E^{j-k}w^L_{-j-r}, E^{j-k}w^L_{-j-r})_{\alpha,\beta}\\ &= \frac{1}{\prod_{m=k+1}^{j} \overline{\gamma_{j,m}}\gamma_{j,m}}(w^L_{-j-r}, F^{j-k}E^{j-k}w^L_{-j-r})_{\alpha,\beta}\\ &= \frac{1}{\prod_{m=k+1}^{j} -\overline{\gamma_{j,m}}}(w^L_{-j-r}, w^L_{-j-r})_{\alpha,\beta} \;=\; \frac{\alpha}{\prod_{m=k+1}^{j} -\overline{\gamma_{j,m}}}.\end{aligned}$$

Next note that

$$(w^S_{-i}, w^S_{-i})_{\alpha,\beta} = (Ew^L_{j-r}, Ew^L_{j-r})_{\alpha,\beta} = (w^L_{j-r}, FEw^L_{j-r})_{\alpha,\beta} = (w^L_{j-r}, Fw^S_{-i})_{\alpha,\beta} = 0\ .$$

Similarly,

$$\begin{aligned}(w^S_{i-2k}, w^S_{i-2k})_{\alpha,\beta} &= \left(\frac{E^{i-k}w^S_{-i}}{\prod_{m=k+1}^{i} \gamma_{j,m}}, \frac{E^{i-k}w^S_{-i}}{\prod_{m=k+1}^{i} \gamma_{j,m}}\right)_{\alpha,\beta}\\ &= \frac{1}{\prod_{m=k+1}^{i} \overline{\gamma_{i,m}}\gamma_{i,m}}(E^{i-k}w^S_{-i}, E^{i-k}w^S_{-i})_{\alpha,\beta}\\ &= \frac{1}{\prod_{m=k+1}^{i} \overline{\gamma_{i,m}}\gamma_{i,m}}(w^S_{-i}, F^{i-k}E^{i-k}w^S_{-i})_{\alpha,\beta}\\ &= \frac{1}{\prod_{m=k+1}^{i} \overline{\gamma_{i,m}}}(w^S_{-i}, w^S_{-i})_{\alpha,\beta} \;=\; 0.\end{aligned}$$

Next note that

$$(Fw^H_{-i}, Fw^H_{-i})_{\alpha,\beta} = (w^L_{j-r}, w^L_{j-r})_{\alpha,\beta} \tag{20}$$

and

$$(Fw^H_{-i}, Fw^H_{-i})_{\alpha,\beta} = (w^H_{-i}, EFw^H_{-i})_{\alpha,\beta} = (w^H_{-i}, Ew^L_{j-r})_{\alpha,\beta} = (w^H_{-i}, w^S_{-i})_{\alpha,\beta}\ . \tag{21}$$

Thus

$$(w^H_{-i}, w^S_{-i})_{\alpha,\beta} = (w^L_{j-r}, w^L_{j-r})_{\alpha,\beta} = \frac{\alpha}{\prod_{m=1}^{k} -\overline{\gamma_{j,m}}}. \tag{22}$$

Next note that

$$\begin{aligned}(w^R_{-j+r}, w^R_{-j+r})_{\alpha,\beta} &= \left(\frac{E^{i+1}w^H_{-i}}{\prod_{m=1}^{i}\gamma_{i,m}}, \frac{E^{i+1}w^H_{-i}}{\prod_{m=1}^{i}\gamma_{i,m}}\right) = \frac{1}{\prod_{m=1}^{i}\overline{\gamma_{i,m}}}(w^H_{-i}, F^{i+1}w^R_{-j+r})_{\alpha,\beta}\\ &= \frac{1}{\prod_{m=1}^{i}\overline{\gamma_{i,m}}}(w^H_{-i}, w^S_{-i})_{\alpha,\beta} = \frac{\alpha}{\prod_{m=1}^{i}\overline{\gamma_{i,m}}\prod_{m=1}^{j}-\overline{\gamma_{j,m}}}.\end{aligned}$$

From this we compute

$$\begin{aligned}(w^R_{r-j+2k}, w^R_{r-j+2k})_{\alpha,\beta} &= (E^k w^R_{r-j}, E^k w^R_{r-j})_{\alpha,\beta} = (w^R_{r-j}, F^k w^R_{r-j+2k})_{\alpha,\beta}\\ &= (w^R_{r-j}, \prod_{m=j}^{k} -\gamma_{j,m} w^R_{r-j})_{\alpha,\beta} = \frac{\prod_{m=j}^{k}-\gamma_{j,m}}{\prod_{m=1}^{i}\overline{\gamma_{i,m}}\prod_{m=1}^{j}-\overline{\gamma_{j,m}}}\alpha.\end{aligned}$$

Next note that

$$\begin{aligned}(w^H_{i-2k}, w^S_{i-2k})_{\alpha,\beta} &= \left(\frac{E^{i-k}w^H_{-i}}{\prod_{m=1}^{k}\gamma_{i,m}} + \Gamma, \frac{E^{i-k}w^S_{-i}}{\prod_{m=1}^{i}\gamma_{i,m}}\right)_{\alpha,\beta} = \frac{1}{\prod_{m=1}^{k}\overline{\gamma_{i,m}}}(w^H_{-i}, F^{i-k}w^S_{i-2k})_{\alpha,\beta}\\ &= \frac{1}{\prod_{m=1}^{k}\overline{\gamma_{i,m}}}(w^H_{-i}, w^S_{-i})_{\alpha,\beta} = \frac{\alpha}{\prod_{m=1}^{k}\overline{\gamma_{i,m}}\prod_{m=1}^{j}-\overline{\gamma_{j,m}}}\end{aligned}$$

where in the first equality, Γ contains w^S terms which were already show to kill terms on the right.

Next we compute

$$(w^H_{i-2k}, E^{i-k}w^H_{-i})_{\alpha,\beta} = (F^{i-k}w^H_{i-2k}, w^H_{-i})_{\alpha,\beta} = (w^H_{-i}, w^H_{-i})_{\alpha,\beta} = \beta\,. \tag{23}$$

But

$$(w^H_{i-2k}, w^H_{i-2k})_{\alpha,\beta} = (w^H_{i-2k}, e_{i-k}(\gamma_{i,i}, \ldots, \gamma_{i,k+1})w^H_{i-2k} + e_{i-k-1}(\gamma_{i,i}, \ldots, \gamma_{i,k+1})w^S_{i-2k})_{\alpha,\beta} \tag{24}$$

where the e_{i-k} and e_{i-k-1} are elementary symmetric functions. This could be verified with a straightforward induction argument. Thus

$$(w^H_{i-2k}, E^{i-k}w^H_{-i})_{\alpha,\beta} = e_{i-k}(w^H_{i-2k}, w^H_{i-2k})_{\alpha,\beta} + e_{i-k-1}(w^H_{i-2k}, w^S_{i-2k})_{\alpha,\beta}\,. \tag{25}$$

Thus

$$(w^H_{i-2k}, w^H_{i-2k})_{\alpha,\beta} = \frac{\left(\beta - \frac{e_{i-k-1}(\gamma_{i,i},\dots\gamma_{i,k+1})\alpha}{\prod_{m=1}^{k}\overline{\gamma_{i,m}}\prod_{m=1}^{j}-\overline{\gamma_{j,m}}}\right)}{e_{i-k}(\gamma_{i,i},\dots,\gamma_{i,k+1})}. \tag{26}$$

□

Remark 4.7 Consider the submodule Δ_{j+r} of P_i defined as follows:

$$\Delta_{j+r} = \{w^R_{j+r}, \dots, w^R_{j-r}, w^S_i, \dots, w^S_{-i}\}\,. \tag{27}$$

It inherits the form $(\cdot,\cdot)_{\alpha,\beta}$ from P_i. Note that this form is degenerate on Δ_{j+r}. The submodule of Δ_{j+r} spanned by the vectors $\{w^S_i, \dots, w^S_{-i}\}$ is isomorphic to the simple module S_i but is actually a totally isotropic subspace of P_i. So we should not expect an abelian structure on a Hermitian category.

Proposition 4.8 *The simple module S_i is a quotient of P_i by the radical of a specialization of the Hermitian form $(\cdot,\cdot)_{\alpha,\beta}$:*

$$S_i \cong P_i/rad(\cdot,\cdot)_{0,\beta}\,.$$

Proof The preceding analysis shows that all vectors w^L, w^S, w^R are in the radical, while the vectors w^H are not. The quotient of P_i by this radical yields a module of dimension $i+1$ with highest weight i which is isomorphic to S_i. □

4.2 The Hermitian ribbon category $\mathscr{D}^{\mathbf{Herm}}$

Let $\mathscr{C}^{\mathrm{Herm}}$ be the category whose objects are Hermitian weight $\overline{U}^H_q\mathfrak{sl}(2)$-modules and morphisms from $(V,(\cdot,\cdot)_V)$ to $(W,(\cdot,\cdot)_W)$ are given by

$$\mathrm{Hom}_{\mathscr{C}^{\mathrm{Herm}}}((V,(\cdot,\cdot)_V),(W,(\cdot,\cdot)_W)) = \mathrm{Hom}_{\mathscr{C}}(V,W).$$

Let $\mathscr{D}^{\mathrm{Herm}}$ be the full subcategory of $\mathscr{C}^{\mathrm{Herm}}$ whose objects have their underlying module in $\mathscr{D}$. In particular any module in $\mathscr{D}^{\mathrm{Herm}}$ is equipped with a Hermitian form. We will need the following lemmas to define a tensor product in $\mathscr{D}^{\mathrm{Herm}}$.

Lemma 4.9 *Any module in $\mathscr{D}$ has a Hermitian structure.*

Proof Using orthogonal direct sums, it is sufficient to show that any indecomposable module in $\mathscr{D}$ has a Hermitian structure. The category $\mathscr{D}$ has indecomposable modules isomorphic to $V' = V \otimes \mathbb{C}^H_{kr}$ for some $(V,k) \in \mathsf{A}\times\mathbb{Z}$ (see [5] where the tensor decomposition rules are described). Proposition 4.5 and Proposition 4.6 ensure that any $V \in \mathsf{A}$ has a Hermitian structure. Choose such a Hermitian form. As a complex vector space, $V \cong V \otimes \mathbb{C}^H_{kr} = V'$, but they are different as representations. For example, $\rho_{V'}(E) = q^{kr}\rho_V(E)$. We can consider the Hermitian form on V' given by $(x,y)_{V'} = a\big(x, q^{krH/2}y\big)_V$ where $a = \overline{a}^{-1}$ is a square root of $\left\langle K^{-kr}\right\rangle_V$. Note that

the expression $\langle K^{-kr}\rangle_V$ makes sense since the central element K^{-kr} acts as a scalar operator on V. The form is Hermitian since

$$(y,x)_{V'} = a\Big(y, q^{krH/2}x\Big)_V = \overline{a}\Big(a^{-2}q^{-krH/2}y, x\Big)_V = \overline{a}\Big(q^{krH/2}y, x\Big)_V = \overline{(x,y)_{V'}}.$$

The second equality comes from $\big(q^{krH/2}\big)^{\varrho} = q^{-krH/2}$ and $a = \overline{a}\,\overline{a}^{-2}$. The third equality follows from:

$$\rho_V(a^{-2}q^{-krH/2}) = \rho_V(K^{kr}q^{-krH/2}) = \rho_V(q^{krH/2}).$$

Next we check the compatibility of this form with $\overline{U}_q^H\mathfrak{sl}(2)$.

$$\begin{aligned}(y, \rho_{V'}(E)x)_{V'} &= a\Big(y, q^{kr}\rho_V(E)q^{krH/2}x\Big)_V = a\Big(y, q^{krH/2}\rho_V(E)x\Big)_V\\ &= a\Big(\rho_V(E^{\varrho})y, q^{krH/2}x\Big)_V\\ &= a\Big(\rho_V(F)y, q^{krH/2}x\Big)_V = (\rho_{V'}(F)y, x)_{V'} = \big(\rho_{V'}(E^{\varrho})y, x\big)_{V'}.\end{aligned}$$

Thus $\rho_{V'}(E)^{\varrho} = \rho_{V'}(E^{\varrho})$. Similar computations for F, K and H show that the form is compatible with $\overline{U}_q^H\mathfrak{sl}(2)$. □

Lemma 4.10 *Any Hermitian module V in $\mathscr{D}^{\mathrm{Herm}}$ splits as an orthogonal direct sum of Hermitian indecomposable modules.*

Proof We prove that V has an indecomposable Hermitian submodule P, then the proof follows by induction since the non degeneracy of $(\cdot,\cdot)$ on P implies that $P^{\perp}$ is a complementary submodule of P in V.

Consider a direct sum decomposition $V = \bigoplus_i W_i$ with W_i indecomposable and let $W_i' = (\bigoplus_{j\neq i} W_j)^{\perp}$. Then, the proof of Lemma 4.3 implies that $(\bar{W}_i')^{\varrho} \simeq W_i^*$. But as for any indecomposable module of $\mathscr{D}^{\mathrm{Herm}}$ we have $(\bar{W}_i)^{\varrho} \simeq W_i^*$ so $W_i' \simeq W_i$. If we choose such an isomorphism $f : W_i \to W_i'$, then $(f(\cdot),\cdot)$ is a compatible sesquilinear form on W_i. Lemma 4.9 ensures that W_i has a compatible Hermitian form $(\cdot,\cdot)_{W_i}$. Thus, there exists $g \in \mathrm{Aut}_{\mathscr{C}}(W_i)$ such that $(f(\cdot),\cdot) = (g(\cdot),\cdot)_{W_i}$: Then, $h = fg^{-1} : W_i \to W_i'$ is an isomorphism such that $(h(\cdot),\cdot) = (\cdot,\cdot)_{W_i}$ is Hermitian. For $\alpha \in \mathbb{R}$, let $W_i'' = (h + \mathrm{e}^{i\alpha}\,\mathrm{Id})(W_i)$. Fix a basis (e) of W_i, $(e') = h((e))$ and $(e'') = (h + \mathrm{e}^{i\alpha}\,\mathrm{Id})((e))$. Let $B = \mathrm{Mat}_{(e)}((\cdot,\cdot)_{W_i})$, $A = \mathrm{Mat}_{(e)}((\cdot,\cdot))$, $A' = \mathrm{Mat}_{(e')}((\cdot,\cdot))$ and $A'' = \mathrm{Mat}_{(e'')}((\cdot,\cdot))$. Then, $A''_{ij} = (e''_i, e''_j) = (e'_i + \mathrm{e}^{i\alpha}e_i, e'_j + \mathrm{e}^{i\alpha}e_j) = A'_{ij} + A_{ij} + \mathrm{e}^{-i\alpha}\overline{B_{j,i}} + \mathrm{e}^{i\alpha}B_{i,j} = A'_{ij} + A_{ij} + 2\cos\alpha B_{i,j}$ so that $A'' = A + A' + 2\cos\alpha B = B(B^{-1}(A+A') + 2\cos\alpha)$ which is non-degenerate when $-2\cos\alpha$ is not an eigenvalue of $B^{-1}(A + A')$. Then, W_i'' is an indecomposable Hermitian submodule of V. □

Note that the above statement is false in $\mathscr{C}^{\mathrm{Herm}}$. For example, one can check that $V_{1+i} \oplus V_{1-i}$ has a Hermitian structure but $V_{1\pm i}$ does not.

We will now introduce the notion of a half twist and give a construction of it in our category. For related ideas, see [23].

Definition 4.11 A half twist $\sqrt{\theta}$ for $\mathscr{C}$ is a natural isomorphism of the identity functor whose square is the twist. That is for any $f : V \to W$, $\sqrt{\theta}_W f = f\sqrt{\theta}_V$, $\sqrt{\theta}_V^2 = \theta_V \in \mathrm{End}_{\mathscr{C}}(V)$ and $\sqrt{\theta}_{V^*} = \left(\sqrt{\theta}_V\right)^*$. We also assume that $\sqrt{\theta}_{\mathbb{I}} = \mathrm{Id}_{\mathbb{I}}$.

Proposition 4.12 *Any $\mathbb{K}$-linear ribbon category over an algebraically closed field of characteristic 0 has a half twist.*

Proof Consider a maximal block decomposition $\mathscr{C} = \bigoplus_i \mathscr{C}_i$ where for any $(V, W) \in \mathscr{C}_i \times \mathscr{C}_j$, $\mathrm{Hom}_{\mathscr{C}}(V, W) \neq 0 \implies i = j$. Then, the twist has a unique generalized eigenvalue $\theta_i \in \mathbb{K}$ on $\mathscr{C}_i$. That is for any $V \in \mathscr{C}_i$, $\langle \theta_V \rangle_V = \theta_i$. One can choose a square root $\sqrt{\theta}_i$ of θ_i such that $\sqrt{\theta}_{i^*} = \sqrt{\theta}_i$ where $\mathscr{C}_{i^*}$ is the block with the dual objects of $\mathscr{C}_i$. Then, a half twist is given by the formula:

$$\forall V \in \mathscr{C}_i, \sqrt{\theta}_V = \sqrt{\theta}_i \, \mathrm{sqr}(\theta_V \theta_i^{-1})$$

where sqr is the square root function defined on an unipotent x by

$$\mathrm{sqr}(x) = 1 - \sum_{k \geq 0} \frac{(2k)!(1-x)^{k+1}}{2^{2k+1} k!(k+1)!}.$$

Note that $\theta_V \theta_i^{-1}$ is unipotent since $\theta_V \theta_i^{-1} - \mathrm{Id}_V = \theta_i^{-1}(\theta_V - \langle \theta_V \rangle \mathrm{Id}_V)$ is nilpotent. The naturality follows from the uniform choice of $\sqrt{\theta}_i$ for all modules of $\mathscr{C}_i$ and the self duality comes from $\theta_{V^*} = (\theta_V)^*$. □

We now fix a half twist by its values on simples in (28). Note that for any $k \in \mathbb{Z}$, $i = 0, \ldots, r-2$ and $j = r-2-i$, $S_i \otimes \mathbb{C}^H_{kr}$ and $S_j \otimes \mathbb{C}^H_{(k+1)r}$ belong to the same block since they are both simple composition factors in a Jordan-Holder series of $P_i \otimes \mathbb{C}^H_{kr}$.

V	V_α	S_i	$S_i \otimes \mathbb{C}^H_{kr}$, $k \in 2\mathbb{Z}$	$S_i \otimes \mathbb{C}^H_{kr}$, $k \in 2\mathbb{Z}+1$	
$\langle \theta_V \rangle$	$q^{\frac{\alpha^2-(r-1)^2}{2}}$	$q^{\frac{i(i+2-2r)}{2}}$	$q^{\frac{i(i+2-2r)}{2}}$	$q^{\frac{j(j+2-2r)}{2}}$	$\alpha \in \ddot{\mathbb{C}}$, $i = 0, \ldots, r-2$ $j = r-2-i$
$\left\langle \sqrt{\theta}_V \right\rangle$	$q^{\frac{\alpha^2-(r-1)^2}{4}}$	$q^{\frac{i(i+2-2r)}{4}}$	$q^{\frac{i(i+2-2r)}{4}}$	$q^{\frac{j(j+2-2r)}{4}}$	

(28)

Using this half twist, we can define an involutive isomorphism (see also [25]) $\mathsf{X} : (\mathscr{C}, \otimes) \to (\mathscr{C}, \otimes^{op})$ in $\mathscr{C}$ by

$$\mathsf{X}_{V,W} = \left(\sqrt{\theta}_{W \otimes V}\right)^{-1} c_{V,W}(\sqrt{\theta}_V \otimes \sqrt{\theta}_W). \tag{29}$$

Note that X is not a braiding, but we have the following lemma.

Lemma 4.13 *For any $V, V', V'' \in \mathscr{C}$ and any morphisms f, g, we have the following equalities*

$$X_{V',V} X_{V,V'} = \mathrm{Id}, \qquad X(f \otimes g) = (g \otimes f)X,$$

$$X_{V'\otimes V,V''}(X_{V,V'}\otimes \mathrm{Id}_{V''}) = X_{V,V''\otimes V'}(\mathrm{Id}_V \otimes X_{V',V''}) : V\otimes V'\otimes V'' \to V''\otimes V'\otimes V,$$
$$X_{V,V^*}\,\overrightarrow{\mathrm{coev}}_V = \overleftarrow{\mathrm{coev}}_V \quad \textit{and} \quad \overrightarrow{\mathrm{ev}}_V\, X_{V,V^*} = \overleftarrow{\mathrm{ev}}_V \ .$$

Proof In order to prove the first identity, we first recall the following fact

$$\theta_{V\otimes V'} = (\theta_V\otimes\theta_{V'})c_{V',V}c_{V,V'} \ . \tag{30}$$

We have

$$\begin{aligned}
\mathsf{X}_{V',V}\mathsf{X}_{V,V'} &= \left(\sqrt{\theta}_{V\otimes V'}\right)^{-1} c_{V',V}(\sqrt{\theta}_{V'}\otimes\sqrt{\theta}_V)\left(\sqrt{\theta}_{V'\otimes V}\right)^{-1} c_{V,V'}(\sqrt{\theta}_V\otimes\sqrt{\theta}_{V'}) \\
&= \left(\sqrt{\theta}_{V\otimes V'}\right)^{-1}\left(\sqrt{\theta}_{V\otimes V'}\right)^{-1} c_{V',V}(\sqrt{\theta}_{V'}\otimes\sqrt{\theta}_V)c_{V,V'}(\sqrt{\theta}_V\otimes\sqrt{\theta}_{V'}) \\
&= \left(\theta_{V\otimes V'}\right)^{-1}(\sqrt{\theta}_V\otimes\sqrt{\theta}_{V'})^2 c_{V',V}c_{V,V'} \\
&= \left(\theta_{V\otimes V'}\right)^{-1}(\theta_V\otimes\theta_{V'})c_{V',V}c_{V,V'} \\
&= \left(\theta_{V\otimes V'}\right)^{-1}\left(\theta_{V\otimes V'}\right) \\
&= \mathrm{Id}_{V\otimes V'} \ .
\end{aligned}$$

The second equality follows from the naturality of $\sqrt{\theta}$. The third equality follows from the naturality of the braiding. The fifth equality follows from (30).

The second identity of the lemma is also an easy consequence of these naturalities.

For the third identity, we have that both sides are equal to

$$\left(\sqrt{\theta}_{V''\otimes V'\otimes V}\right)^{-1}(c_{V',V''}\otimes\mathrm{Id})(\mathrm{Id}\otimes c_{V,V''})(c_{V,V'}\otimes\mathrm{Id})(\sqrt{\theta}_V\otimes\sqrt{\theta}_{V'}\otimes\sqrt{\theta}_{V''}) \ .$$

For the fourth identity, we have the following equalities.

$$\begin{aligned}
\mathsf{X}_{V,V^*}\,\overrightarrow{\mathrm{coev}}_V &= \left(\sqrt{\theta}_{V^*\otimes V}\right)^{-1} c_{V,V^*}(\sqrt{\theta}_V\otimes\sqrt{\theta}_V^*)\,\overrightarrow{\mathrm{coev}}_V \\
&= c_{V,V^*}(\sqrt{\theta}_V\otimes\sqrt{\theta}_V^*)\,\overrightarrow{\mathrm{coev}}_V\left(\sqrt{\theta}_{\mathbb{I}}\right)^{-1} \\
&= c_{V,V^*}(\sqrt{\theta}_V^2\otimes\mathrm{Id})\,\overrightarrow{\mathrm{coev}}_V\left(\sqrt{\theta}_{\mathbb{I}}\right)^{-1} \\
&= c_{V,V^*}(\theta_V\otimes\mathrm{Id})\,\overrightarrow{\mathrm{coev}}_V \\
&= \overleftarrow{\mathrm{coev}}_V \ .
\end{aligned}$$

The second equality follows from the naturality of $\left(\sqrt{\theta}_{V^*\otimes V}\right)^{-1}$. The third equality comes from moving $\sqrt{\theta}_V^*$ in the second tensor factor to $\sqrt{\theta}_V$ in the first factor (which is allowed in a ribbon category). The final equality is also a standard identity in a ribbon category. The last identity is proved in a similar way. □

Let $W_1, W_2 \in \mathscr{C}^{\mathrm{Herm}}$ and let $(\cdot,\cdot)_{W_i}$ be the Hermitian structures on W_i. Define $(\cdot,\cdot)_p$ on $W_1 \otimes W_2$ by

$$(x \otimes y, x' \otimes y')_p = (x, x')_{W_1}(y, y')_{W_2} \tag{31}$$

for $x, x' \in W_1$ and $y, y' \in W_2$. The sesquilinear form $(\cdot,\cdot)_p$ is Hermitian, non-degenerate but not compatible with ϱ. We modify the pairing to obtain a Hermitian form compatible with ϱ in the next result.

Theorem 4.14 *Let $W_1, W_2 \in \mathscr{D}^{\mathrm{Herm}}$. Then, $W_1 \otimes W_2$ is a Hermitian module with Hermitian structure given by*

$$(v, v') = (v, \tau(\mathsf{X}_{W_1\otimes W_2} v'))_p.$$

This gives $\mathscr{D}^{\mathrm{Herm}}$ the structure of a strict $\mathbb{C}$-linear tensor category.

Proof Let $u \in \overline{U}_q^H \mathfrak{sl}(2)$, write $\Delta(u) = u_1 \otimes u_2$ and consider $v = v_1 \otimes v_2 \in W_1 \otimes W_2$ and $v'' = \mathsf{X}_{W_1\otimes W_2}(v') = v_2'' \otimes v_1'' \in W_2 \otimes W_1$. Note that we have omitted all summation symbols and will continue to do so throughout the course of the proof. Then, we have

$$\begin{aligned}(v, u.v') &= (v, \tau(\mathsf{X}_{W_1\otimes W_2} u.v'))_p = (v, \tau(u.v''))_p = (v_1, u_2 v_1'')_{W_1}(v_2, u_1 v_2'')_{W_2} \\ &= (\dagger u_2 v_1, v_1'')_{W_1}(\dagger u_1 v_2, v_2'')_{W_2} = (\dagger u.v, \tau(v''))_p = (\dagger u.v, v'),\end{aligned}$$

where we used that $\Delta(\dagger u) = \dagger u_2 \otimes \dagger u_1$. Hence, we have defined a compatible sesquilinear form on $W_1 \otimes W_2$.

To prove that this form is Hermitian, first note that by Lemma 4.10, we can assume that W_1 and W_2 are indecomposable. Then, consider a direct sum decomposition $W_1 \otimes W_2 = \bigoplus_i V_i$ with V_i indecomposable and the dual decomposition $V_i' = \left\{ v' \in W_1 \otimes W_2 | (v', \bigoplus_{j\neq i} V_j) = \{0\} \right\}$.

Fix a factor V_i. Then, $(\cdot,\cdot)$ is non-degenerate on $V_i' \times V_i$. Hence, $V_i'^* \simeq \bar{V}_i^{\varrho} \simeq V_i^*$ so in particular, the half twist has the same value $\left\langle \sqrt{\theta}_{V_i} \right\rangle = \left\langle \sqrt{\theta}_{V_i'} \right\rangle$, which we denote by $\sqrt{\theta}_0 \in \mathbb{C}$. Let us call $\sqrt{\theta}_1 = \left\langle \sqrt{\theta}_{W_1} \right\rangle$ and $\sqrt{\theta}_2 = \left\langle \sqrt{\theta}_{W_2} \right\rangle$ and $\theta_i = \sqrt{\theta}_i^2$. Then, let $(v, v') \in V_i \times V_i'$ and write $R = \sum a \otimes b$. Then,

$$\begin{aligned}(v', v) &= (v', \tau\mathsf{X}(v))_p \\ &= (v', \tau\sqrt{\theta}_{W_2\otimes W_1}^{-1} \tau R(\sqrt{\theta}_{W_1} \otimes \sqrt{\theta}_{W_2})(v)) \\ &= \sqrt{\theta}_0^{-1}\sqrt{\theta}_1\sqrt{\theta}_2 \Big(v', \operatorname{sqr}(\theta_0 \Delta^{op}(\theta)) R(\operatorname{sqr}(\theta_1^{-1}\theta^{-1}) \otimes \operatorname{sqr}(\theta_2^{-1}\theta^{-1}))\Big)_p \\ &= \sqrt{\theta}_0^{-1}\sqrt{\theta}_1\sqrt{\theta}_2 \Big(\big(\operatorname{sqr}(\overline{\theta}_1^{-1}\left(\theta^{-1}\right)^{\varrho}) \otimes \operatorname{sqr}(\overline{\theta}_2^{-1}\left(\theta^{-1}\right)^{\varrho})\big) R_{21}^{-1} \operatorname{sqr}(\overline{\theta}_0 \Delta(\dagger\theta)))v', v\Big)_p \\ &= \Big(\sqrt{\theta}_0\sqrt{\theta}_1^{-1}\sqrt{\theta}_2^{-1} (\operatorname{sqr}(\theta_1\theta) \otimes \operatorname{sqr}(\theta_2\theta)) R_{21}^{-1} \operatorname{sqr}(\theta_0^{-1}\Delta(\theta^{-1})))v', v\Big)_p \\ &= \Big(\tau(\sqrt{\theta}_{W_2}^{-1} \otimes \sqrt{\theta}_{W_1}^{-1}) R^{-1} \tau\sqrt{\theta}_{W_1\otimes W_2} v', v\Big)_p = \Big(\tau\mathsf{X}^{-1}(v'), v\Big)_p\end{aligned}$$

$$= \big(\tau \mathsf{X}(v'), v\big)_p = \overline{\big(v, \tau \mathsf{X}(v')\big)_p} = \overline{\big(v, v'\big)}.$$

In the above equalities, the elements θ, θ^{-1}, etc., are all acting on modules. Note that the fifth equality above follows from the fact that $\theta_0, \theta_1, \theta_2$ all have norm one, and Lemma 4.2. □

Proposition 4.15 *The Hermitian structure from Theorem 4.14 defines an associative tensor product turning $\mathscr{D}^{\mathrm{Herm}}$ into a tensor category. Furthermore, for any two morphisms $f, g \in \mathscr{D}^{\mathrm{Herm}}$, one has*

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger.$$

Proof To show the associativity of the tensor product of Hermitian modules, consider the product Hermitian form on $V \otimes V' \otimes V''$ given by

$$\big(v_1 \otimes v_1' \otimes v_1'', v_2 \otimes v_2' \otimes v_2''\big)_t = (v_1, v_2)\big(v_1', v_2'\big)\big(v_1'', v_2''\big)$$

Then, the compatible Hermitian forms obtained by iterating Theorem 4.14 are related to this pairing by

$$(w_1, w_2)_{(V\otimes V')\otimes V''} = \big(w_1, \tau_{13}\mathsf{X}_{V,V',V''}(w_2)\big)_t = (w_1, w_2)_{V\otimes(V'\otimes V'')},$$

where $\mathsf{X}_{V,V',V''} = \mathsf{X}_{V'\otimes V,V''}(\mathsf{X}_{V,V'} \otimes \mathrm{Id}_{V''}) = \mathsf{X}_{V,V''\otimes V'}(\mathrm{Id}_V \otimes \mathsf{X}_{V',V''})$ and τ_{13} is the permutation $x \otimes y \otimes z \mapsto z \otimes y \otimes x$. This gives the associativity of the tensor product of $\mathscr{D}^{\mathrm{Herm}}$. The category also has a strict unit $\mathbb{I} = \mathbb{C}$ with $(1, 1)_{\mathbb{I}} = 1$ because $\mathsf{X}_{\mathbb{I},V} = \mathsf{X}_{V,\mathbb{I}} = \mathrm{Id}_V$.

The last statement is a consequence of the second property of Lemma 4.13. □

Proposition 4.16 *If $V \in \mathscr{C}^{\mathrm{Herm}}$, V^* has a unique Hermitian structure determined by $(\varphi, \psi) = \sum_i \overline{\varphi(e_i)}\psi(e_i')$ where (e_i) and (e_i') are any dual bases of V for the Hermitian form of V. Furthermore, if $V \in \mathscr{D}^{\mathrm{Herm}}$ then one has*

$$\overleftarrow{\mathrm{coev}}_V^{\,\dagger} = \overrightarrow{\mathrm{ev}}_V \quad \text{and} \quad \overleftarrow{\mathrm{ev}}_V^{\,\dagger} = \overrightarrow{\mathrm{coev}}_V\,.$$

Proof Fix a basis $e = (e_i)_i$ of V and let $e^* = (e_i^*)_i$ be the dual basis of V^*. Let $B_{ij} = \big(e_i, e_j\big)$ and let $B' = B^{-1}$. Then, $e_i' = \sum_j \overline{B'_{ij}} e_j$ since $\Big(\sum_j \overline{B'_{ij}} e_j, e_k\Big) = \sum_j B'_{ij}\big(e_j, e_k\big) = \delta_i^k$. Then, the matrix $C_{ij} = \Big(e_i^*, e_j^*\Big)$ is given by

$$C_{ij} = \Big(e_i^*, e_j^*\Big) = \sum_k \overline{e_i^*(e_k)} e_j^*(e_k') = e_j^*(e_i') = \overline{B'_{ij}}.$$

So $C = \overline{B}^{-1}$ is Hermitian. In particular, a change of basis would change B to $P^* B P$ and C to $\big({}^t P^{-1}\big)^* C\, {}^t P^{-1} = \overline{P}^{-1}\overline{B}^{-1}\,\overline{P^*}^{-1} = \overline{P^* B P}^{-1}$ so it would define the same Hermitian form on V^*.

Now for any vector x, one has $x = \sum_i (e_i', x)e_i = ((\cdot, x) \otimes \mathrm{Id})(\sum_i e_i' \otimes e_i)$, so

$$((\cdot, x) \otimes \mathrm{Id}) \left(\sum_i e_i' \otimes u.e_i \right) = u.x = \sum_i (e_i', u.x)e_i = \sum_i (u^{\varrho}.e_i', x)e_i$$
$$= ((\cdot, x) \otimes \mathrm{Id}) \left(\sum_i u^{\varrho}.e_i' \otimes e_i \right).$$

Then,

$$\sum_i e_i' \otimes u.e_i = \sum_i u^{\varrho}.e_i' \otimes e_i \tag{32}$$

because $(\cdot, \cdot)$ is non-degenerate. Finally,

$$(u.\varphi, \psi) = (\varphi(S(u)\cdot), \psi) = \sum_i \overline{\varphi}(S(u).e_i)\psi(e_i') = \sum_i \overline{\varphi}(e_i)\psi(S(u)^{\varrho}.e_i')$$
$$= \big(\varphi, \psi(S(u^{\varrho})\cdot)\big) = \big(\varphi, u^{\varrho}.\psi\big),$$

where in the third equality we used (32). We now compute the Hermitian adjoint of $\overleftarrow{\mathrm{coev}}_V$. Let $(v, \varphi) \in V \times V^*$. Then,

$$\left(\varphi \otimes v, \overleftarrow{\mathrm{coev}}_V\right)_{V^* \otimes V}$$
$$= \left(\varphi \otimes v, \tau \mathsf{X} \overleftarrow{\mathrm{coev}}_V\right)_p = \left(\varphi \otimes v, \tau\, \overrightarrow{\mathrm{coev}}_V\right)_p = \sum_i \overline{(e_i^*, \varphi)}(v, e_i)$$
$$= \sum_i \overline{\varphi(e_i')}(v, e_i) = \overline{\varphi(v)} = \left(\overrightarrow{\mathrm{ev}}_V\ (\varphi \otimes v), 1\right)_{\mathbb{I}}.$$

Hence, $\overleftarrow{\mathrm{coev}}_V^{\dagger} = \overrightarrow{\mathrm{ev}}_V$. Note that the second equality above follows from Lemma 4.13.

Taking the Hermitian adjoint of the zigzag $(\overleftarrow{\mathrm{ev}} \otimes \mathrm{Id})(\mathrm{Id} \otimes \overleftarrow{\mathrm{coev}}) = \mathrm{Id}_V$, we get that

$$\mathrm{Id}_V = (\mathrm{Id}_V \otimes \overleftarrow{\mathrm{coev}}_V^{\dagger})(\overleftarrow{\mathrm{ev}}_V^{\dagger} \otimes \mathrm{Id}_V) = (\mathrm{Id}_V \otimes \overrightarrow{\mathrm{ev}}_V)(\overleftarrow{\mathrm{ev}}_V^{\dagger} \otimes \mathrm{Id}_V).$$

Thus $\overleftarrow{\mathrm{ev}}_V^{\dagger} = \overrightarrow{\mathrm{coev}}_V$. □

Proposition 4.17 *Let $V, W \in \mathscr{C}^{\mathrm{Herm}}$, then*

$$c_{V,W}{}^{\dagger} = (c_{V,W})^{-1} \quad \textit{and} \quad \theta_V{}^{\dagger} = (\theta_V)^{-1}.$$

Proof Taking the Hermitian adjoint of the R-matrix in (5), and applying a permutation to the tensor factors, one recovers R^{-1} from (6). The first statement now follows immediately. The second statement is contained in Lemma 4.2. □

We define the contravariant monoidal antilinear functor $\dagger : \mathscr{D}^{\mathrm{Herm}} \to \mathscr{D}^{\mathrm{Herm}}$ as the identity on each object and as the Hermitian adjoint on each morphism. This functor is contravariant, monoidal, and antilinear. We now have the following result.

Theorem 4.18 *$\mathscr{D}^{\mathrm{Herm}}$ is a* Hermitian *ribbon category in the sense of Definition 3.1.*

We conclude this section with some observations about how the Hermitian structure on $\mathscr{D}^{\mathrm{Herm}}$ interacts with the modified trace.

Lemma 4.19 *Let $(W, (\cdot,\cdot)) \in \mathscr{D}^{\mathrm{Herm}}$ with $W \in$ Proj. Then for any $f \in \mathrm{End}_{\mathscr{C}}(W)$,*

$$t_W(f^\dagger) = \overline{t_W(f)}.$$

Proof By Proposition 4.16 the dagger functor commutes with the partial trace of morphisms. It follows that the family of linear maps $\left(\mathsf{t}_V^\dagger\right)_{V\in\mathsf{Proj}}$ given by:

$$\begin{aligned}\mathsf{t}_V^\dagger : \mathrm{End}_{\mathscr{C}}(V) &\to \mathbb{C}\\ f &\mapsto \mathsf{t}_V^\dagger(f) := \overline{\mathsf{t}_V(f^\dagger)}\end{aligned}$$

is a modified trace on Proj. By unicity of the modified trace, $\mathsf{t}^\dagger = \lambda\mathsf{t}$ for some $\lambda \in \mathbb{C}$. Now for any $\alpha \in (\mathbb{C}\setminus\mathbb{Z}) \cup r\mathbb{Z}$, $\mathsf{t}(\mathrm{Id}_{V_\alpha}) = \mathsf{t}^\dagger(\mathrm{Id}_{V_\alpha}) = \mathsf{d}(V_\alpha) \in \mathbb{R}$ since we choose $\mathsf{d}_0 \in \mathbb{R}$. □

Lemma 4.20 *For any objects V, W of $\mathscr{D}^{\mathrm{Herm}}$ with V or W projective, the pairing*

$$\langle f, g\rangle \mapsto t_V(f^\dagger g) = t_W(gf^\dagger) : \mathrm{Hom}_{\mathscr{D}}(V, W) \times \mathrm{Hom}_{\mathscr{D}}(V, W) \to \mathbb{C}$$

(when both make sense) is a non-degenerate Hermitian pairing.

Proof Since the mapping $\mathrm{Hom}_{\mathscr{D}}(V, W) \to \mathrm{Hom}_{\mathscr{D}}(W, V)$ given by $f \mapsto f^\dagger$ is an involution, the pairing of the lemma is non-degenerate. The Hermitian symmetry follows from

$$\langle g, f\rangle = \mathsf{t}_V(g^\dagger f) = \mathsf{t}_V((f^\dagger g)^\dagger) = \overline{\mathsf{t}_V(f^\dagger g)} = \overline{\langle f, g\rangle}$$

where the second to last equality follows from Lemma 4.19. □

Example 4.21 Let $\alpha \in \mathbb{R}$ then

$$\left\langle \overrightarrow{\mathrm{ev}}_{V_\alpha}, \overrightarrow{\mathrm{ev}}_{V_\alpha} \right\rangle = \mathsf{t}_{V_\alpha}(\mathrm{Id}_{V_\alpha}) = \mathsf{d}_0 \frac{\sin(\alpha\pi/r)}{\sin(\alpha\pi)}$$

which is not positive for all simples labelled by $(\alpha + 1 - r, \alpha + 3 - r, ..., \alpha + r - 1)$. These simples all lie in the same graded piece of the category.

4.3 Non-uniqueness of the half twist

In this subsection, we show that up to isomorphism, $\mathscr{D}^{\mathrm{Herm}}$ does not depend on the choice of the half twist.

The tensor product on $\mathscr{D}^{\mathrm{Herm}}$ relies on the isomorphism X, which itself depends on the choice of the half twist $\sqrt{\theta}$ that we fixed in (28). Suppose that $\sqrt{\theta}^{(1)}$ and $\sqrt{\theta}^{(2)}$ are two different half twists leading to two different tensor structures on $\mathscr{D}^{\mathrm{Herm}}$ that we denote by $\mathscr{D}^{(1)}$ and $\mathscr{D}^{(2)}$. Let $\mathcal{I}$ be the class of indecomposable objects of $\mathscr{D}$ and define $\varepsilon : \mathcal{I} \to \{\pm 1\}$ by $\varepsilon(V)\sqrt{\theta}_V^{(1)} = \sqrt{\theta}_V^{(2)}$. Note that $\varepsilon(\mathbb{I}) = 1$ and $\varepsilon(V) = \varepsilon(V^*)$ only depend on the block of the category in which V belongs. Then, there exist inverse functors, both called $\mathcal{E}$, defined on indecomposable objects (and extended by additivity) by

$$\begin{aligned} \mathcal{E} : \mathscr{D}^{(i)} &\to \mathscr{D}^{(j)} \\ (V, (\cdot,\cdot)) &\mapsto (V, \varepsilon(V)(\cdot,\cdot)) \end{aligned}$$

where $\{i, j\} = \{1, 2\}$ and $\mathcal{E}$ is the identity on morphisms.

Proposition 4.22 *$\mathcal{E}$ is a tensor functor.*

Proof Let $(W_1, (\cdot,\cdot)_{W_1})$, $(W_2, (\cdot,\cdot)_{W_2})$ be indecomposable objects of $\mathscr{D}^{(1)}$ and using Lemma 4.10, let us split in $\mathscr{D}^{(1)}$ the module $W_1 \otimes W_2$ into a direct sum of orthogonal Hermitian modules $W_1 \otimes W_2 = \bigoplus_i V_i$. Then for $v, v' \in V_i$, we have $\mathsf{X}^{(2)}(v') = \varepsilon(W_1)\varepsilon(W_2)\varepsilon(V_i)\mathsf{X}^{(1)}(v')$ and we can compute in $\mathscr{D}^{(1)}$ that $(v, v')_{W_1\otimes W_2} = (v, \mathsf{X}^{(1)}(v'))_p$, where $(\cdot,\cdot)_p$ is the product of $(\cdot,\cdot)_{W_1}$ and $(\cdot,\cdot)_{W_2}$. The corresponding computation in $\mathscr{D}^{(2)}$ is

$$(v, v')_{\mathcal{E}(W_1)\otimes\mathcal{E}(W_2)} = \varepsilon(W_1)\varepsilon(W_2)(v, \mathsf{X}^{(2)}(v'))_p = \varepsilon(V_i)(v, \mathsf{X}^{(1)}(v'))_p = \varepsilon(V_i)(v, v')_{W_1\otimes W_2},$$

and this last scalar is the evaluation of $\mathcal{E}((\cdot,\cdot)_{W_1\otimes W_2})$ on (v, v'). Hence, $\mathcal{E}((W_1, (\cdot,\cdot)_{W_1}) \otimes (W_2, (\cdot,\cdot)_{W_2})) = \mathcal{E}((W_1, (\cdot,\cdot)_{W_1})) \otimes \mathcal{E}((W_2, (\cdot,\cdot)_{W_2})$. □

Proposition 4.23 *$\mathcal{E}$ is an isomorphism of Hermitian ribbon categories and the dagger map on morphisms in $\mathscr{D}^{\mathrm{Herm}}$ is independent of the choice of the half twist.*

Proof We need to show that the dagger map on morphisms in $\mathscr{D}^{(1)}$ and $\mathscr{D}^{(2)}$ coincide. Once again, we can restrict our attention to the case of a morphism $f : V \to W$ where V and W are indecomposable modules. But then if f is not zero, the two modules belong to the same block of $\mathscr{D}$ and thus $\varepsilon(V) = \varepsilon(W)$. Then,

$$(f(\cdot),\cdot)_W = (\cdot, f^\dagger(\cdot))_V \implies \varepsilon(W)(f(\cdot),\cdot)_W = \varepsilon(V)(\cdot, f^\dagger(\cdot))_V,$$

so $\mathcal{E}(f^\dagger) = \mathcal{E}(f)^\dagger$. □

5 Implications for non-semisimple TQFT

In this section, we assume $r \notin 4\mathbb{Z}$ (see [2]) so that $\mathscr{C}$ is a relative $\mathbb{C}^*$-modular category. We will show that the TQFT constructed in [2] has a Hermitian structure.

5.1 Hermitian structure on decorated cobordisms

We follow Turaev [24, I.II.5.1] and define the dagger of a $\mathscr{D}^{\mathrm{Herm}}$-colored ribbon graph T as the following transformation: invert the orientation in the surface of T, reverse directions of bands and annuli of T, exchange bottom and top bases of coupons and replace the color f of a coupons with $f^\dagger$. The colors of edges do not change.

The boundary of a $\mathscr{D}^{\mathrm{Herm}}$-colored ribbon graph is a set of $\mathscr{D}^{\mathrm{Herm}}$-colored framed points where a framed point $p = (V, \varepsilon)$ is a point equipped with a sign ε, a nonzero vector (its framing) tangent to the surface T and to the direction of the band, and a color which is an object $V \in \mathscr{D}^{\mathrm{Herm}}$. Let $F(p) = V$ if $\varepsilon = +$ and $F(p) = V^*$ if $\varepsilon = -$. The conjugate $\overline{p}$ of a $\mathscr{D}^{\mathrm{Herm}}$-colored framed point p is obtained by changing the sign and framing to their opposites. One easily check that $\partial(T^\dagger) = \overline{\partial T}$.

Recall the category of decorated cobordism introduced in [2]. Here, we replace the ribbon category with $\mathscr{D}^{\mathrm{Herm}}$ where we restrict to $\mathbb{R}/2\mathbb{Z}$ valued cohomology classes (instead of $\mathbb{C}/2\mathbb{Z}$) and we only admit as objects *admissible* decorated surfaces defined as follows:

Definition 5.1 (Objects of $\mathcal{C}ob$) A *decorated surface* is a 4-tuple $\widetilde{\Sigma} = (\Sigma, \{p_i\}, \omega, \mathscr{L})$ where

- Σ is a closed, oriented surface which is an ordered disjoint union of connected surfaces each having a distinguished base point $*$;
- $\{p_i\}$ is a finite (possibly empty) set of $\mathscr{D}^{\mathrm{Herm}}$-colored framed points with framing tangent to the surface Σ;
- $\omega \in H^1(\Sigma \setminus \{p_1, \ldots, p_k\}, *; \mathbb{R}/2\mathbb{Z})$ is a cohomology class;
- *compatibility condition*: letting $\omega(m_i) = a_i \in \mathbb{R}/2\mathbb{Z}$ where m_i is a positively oriented circle around p_i, then we require $F(p_i) \in \mathscr{D}^{\mathrm{Herm}}_{a_i}$;
- $\mathscr{L}$ is a Lagrangian subspace of $H_1(\Sigma; \mathbb{R})$.

In what follows we will restrict our attention to admissible surfaces that have the additional property that each component of $\widetilde{\Sigma}$ has either a $\mathscr{D}^{\mathrm{Herm}}$-colored framed point p with $F(p) \in \mathsf{Proj}$ or it contains a closed curve γ such that $\omega(\gamma) \notin \mathbb{Z}/2\mathbb{Z}$.

Remark 5.2 Note that this restriction is not just a cosmetic simplification. The non admissible surfaces considered in [2] would not lead to a Hilbert space valued TQFT.

Definition 5.3 (Morphisms of $\mathcal{C}ob$) Let $\widetilde{\Sigma}_\pm = (\Sigma_\pm, \{p_i^\pm\}, \omega_\pm, \mathscr{L}_\pm)$ be admissible, decorated surfaces. A *decorated cobordism* from $\widetilde{\Sigma}_-$ to $\widetilde{\Sigma}_+$ is a 5-tuple $\widetilde{M} = (M, T, f, \omega, n)$ where

- M is an oriented 3-manifold with boundary ∂M;
- $f : \overline{\Sigma_-} \sqcup \Sigma_+ \to \partial M$ is a diffeomorphism preserving the orientation, and the image under f of the base points of $\overline{\Sigma_-} \sqcup \Sigma_+$ is denoted by $*$;

- T is a $\mathscr{D}^{\mathrm{Herm}}$-colored ribbon graph in M such that $\partial T = \{\overline{f(p_i^-)}\} \cup \{f(p_i^+)\}$;
- $\omega \in H^1(M \setminus T, *; \mathbb{R}/2\mathbb{Z})$ is a cohomology class relative to the base points on ∂M, such that the restriction of ω to $(\partial M \setminus \partial T) \cap \Sigma_\pm$ is $(f^{-1})^*(\omega_\pm)$;
- the coloring of T is compatible with ω, i.e. each oriented edge e of T is colored by an object in $\mathscr{D}^{\mathrm{Herm}}_{\omega(m_e)}$ where m_e is the oriented meridian of e;
- n is an arbitrary integer called the *signature-defect* of $\widetilde{M}$.

We can summarize the first four items by saying that $\partial\widetilde{M} = \widetilde{\Sigma}_-^* \sqcup \widetilde{\Sigma}_+$ where the dual of a decorated surface $\widetilde{\Sigma} = (\Sigma, \{p_i\}, \omega, \mathscr{L})$ is defined to be $\widetilde{\Sigma}^* = (\overline{\Sigma}, \overline{\{p_i\}}, \omega, \mathscr{L})$.

For cobordisms, we only consider admissible, decorated cobordisms that have the additional property that each component of M contains either a component of T with a Proj-colored edge or it contains a closed curve γ such that $\omega(\gamma) \notin \mathbb{Z}/2\mathbb{Z}$. This condition is automatically satisfied by components of $\widetilde{M}$ with non-empty boundary. This restriction is also in [2]. Hence, morphisms of $\mathcal{C}ob$ are orientation preserving diffeomorphism classes of admissible decorated cobordisms.

Proposition 5.4 *Let $\widetilde{M} = (M, T, f, \omega, n) : \widetilde{\Sigma}_- \to \widetilde{\Sigma}_+$ be a cobordism in $\mathcal{C}ob$. Then, the following defines a cobordism $\widetilde{M}^\dagger : \widetilde{\Sigma}_+ \to \widetilde{\Sigma}_-$ in $\mathcal{C}ob$:*

$$\widetilde{M}^\dagger = (\overline{M}, T^\dagger, \overline{f}, \omega, -n),$$

where $\overline{M}$ is M with opposite orientation, and $\overline{f} = \overline{\Sigma_+} \sqcup \Sigma_- \to \partial\overline{M}$ is the same set-theoretic map as f. Furthermore, with the above assignment, $\mathcal{C}ob$ is a Hermitian ribbon category.

Proof The first statement follows from the fact that $\partial\widetilde{M} = \widetilde{\Sigma}_-^* \sqcup \widetilde{\Sigma}_+$ implies $\partial\big(\widetilde{M}^\dagger\big) = \widetilde{\Sigma}_+^* \sqcup \widetilde{\Sigma}_-$. The category $\mathcal{C}ob$ has the ordered disjoint union as tensor product and it is obvious that $(\widetilde{M}_1 \circ \widetilde{M}_2)^\dagger = \widetilde{M}_2^\dagger \circ \widetilde{M}_1^\dagger$, and $(\widetilde{M}_1 \sqcup \widetilde{M}_2)^\dagger = \widetilde{M}_1^\dagger \sqcup \widetilde{M}_2^\dagger$. The braiding in $\mathcal{C}ob$ is symmetric. The pivotal structure is given by the cylinder $\widetilde{\Sigma} \times [0, 1]$ that gives four morphisms $\emptyset \to \widetilde{\Sigma} \otimes \widetilde{\Sigma}^*, \emptyset \to \widetilde{\Sigma}^* \otimes \widetilde{\Sigma}, \widetilde{\Sigma}^* \otimes \widetilde{\Sigma} \to \emptyset$ or $\widetilde{\Sigma} \otimes \widetilde{\Sigma}^* \to \emptyset$ which satisfy the zig-zag relations. Finally, (12) follows because the twist is trivial in $\mathcal{C}ob$ and the cylinder $\widetilde{\Sigma} \times [0, 1]$ is positively diffeomorphic to itself with opposite orientation by the diffeomorphism $\mathrm{Id}_\Sigma \times (t \mapsto 1 - t)$. □

Note that the last diffeomorphism of cylinders in the proof generalizes to the non-compact case of a $\mathscr{D}^{\mathrm{Herm}}$-colored ribbon graph T in $\mathbb{R}^2 \times [0, 1]$. In this case, the dagger of this 3-manifold is identified via the reflection through the plane $\mathbb{R}^2 \times \left\{\frac{1}{2}\right\}$ with the graph $T^\dagger$ embedded upside-down in $\mathbb{R}^2 \times [0, 1]$ with standard orientation. This transformation is compatible with the Reshetikhin-Turaev functor F in the sense that $F(T^\dagger) = F(T)^\dagger$.

5.2 The 3-manifold invariant

If $\widetilde{M} = (M, T, f, \omega, n)$ is a closed decorated manifold (i.e $\widetilde{M} \in \mathrm{End}_{\mathcal{C}ob}(\emptyset)$) which is connected, a surgery presentation of $\widetilde{M}$ is a $\mathscr{C}$-colored ribbon graph $T \cup L \subset S^3$ such that M is obtained from S^3 by surgery on L and each component of L is colored by

a Kirby color $\Omega_\alpha = \sum_{k=0}^{r-1} \mathsf{d}(\alpha+1-r+2k)V_{\alpha+1-r+2k}$ of degree equal to the value of the cohomology class on its meridian. Then, the invariant of $\widetilde{M}$ is given by

$$\mathsf{Z}(\widetilde{M}) = \eta\lambda^{m+n}\delta^{-\sigma}F'(L\cup T) \tag{33}$$

where $m \in \mathbb{N}$ is the number of components of L, $\sigma \in \mathbb{Z}$ is the signature of the linking matrix $\mathrm{lk}(L)$ and

$$\delta = q^{-\frac{3}{2}}\mathrm{e}^{-i(s+1)\pi/4} \text{ where } s \text{ is in } \{1,2,3\} \text{ with } s \equiv r \bmod 4, \tag{34}$$

$$\lambda = \frac{\sqrt{r'}}{r^2}, \qquad \eta = \frac{1}{r\sqrt{r'}}. \tag{35}$$

The invariant is extended multiplicatively for disjoint unions.

Lemma 5.5 *Let $\widetilde{M}$ be a closed decorated manifold then*

$$Z(\widetilde{M}^\dagger) = \overline{Z(\widetilde{M})}$$

Proof If a framed link L gives rise via surgery to the manifold M, then the manifold $\overline{M}$ may be constructed from $\overline{L}$ where $\overline{L}$ is the mirror image of L. It follows from the conjugation on the underlying Hermitian category that $F'(\overline{L}\cup T^\dagger) = \overline{F'(L\cup T)}$.

If the linking matrix of L has signature σ, then the signature of the linking matrix of $\overline{L}$ is $-\sigma$. Since λ and η are real and the modulus of δ is 1, the lemma follows. □

Corollary 5.6 *The pairing $\mathcal{C}ob(\emptyset,\widetilde{\Sigma})\times\mathcal{C}ob(\emptyset,\widetilde{\Sigma})\to\mathbb{C}$, given by*

$$(\widetilde{M}_1,\widetilde{M}_2)\mapsto Z(\widetilde{M}_1^\dagger\circ\widetilde{M}_2)\in\mathbb{C}$$

has Hermitian symmetry.

Proof This follows directly from Lemma 5.5. □

5.3 The (2 + 1)-TQFT

Recall V_0 is the simple projective module with highest weight $r-1$ and $\mathbb{C}^H_{kr}$ (for $k\in\mathbb{Z}$) are the one-dimensional modules described in Sect. 2. Let $\widehat{S}_k$ be the decorated sphere defined in [2] colored with points U, where

$$U = \begin{cases} ((V_0,1),(\mathbb{C}^H_{kr},1),(V_0,-1)) & \text{if } k\neq 0,\\ ((V_0,1),(V_0,-1)) & \text{if } k=0. \end{cases} \tag{36}$$

Here, all modules are enhanced with their preferred Hermitian structure (see Sect. 4).

Now we define the state space associated to a decorated surface in the following way:

$$\mathsf{V}(\widetilde{\Sigma}) = \mathrm{Span}_{\mathbb{C}}\left\{\mathrm{Hom}_{\mathcal{C}ob}(\emptyset,\widetilde{\Sigma})\right\}/K_{\widetilde{\Sigma}}$$

where $K_{\widetilde{\Sigma}}$ is the right kernel of the bilinear pairing given on generators

$$\begin{aligned} \mathrm{Span}_{\mathbb{C}}\left\{\mathcal{C}ob(\widetilde{\Sigma},\emptyset)\right\} \otimes \mathrm{Span}_{\mathbb{C}}\left\{\mathcal{C}ob(\emptyset,\widetilde{\Sigma})\right\} &\to \mathbb{C} \\ [\widetilde{M}_1]\otimes[\widetilde{M}_2] &\mapsto \mathsf{Z}(\widetilde{M}_1\circ\widetilde{M}_2) \end{aligned}$$

$$\mathbb{V}(\widetilde{\Sigma}) = \bigoplus_{k\in\mathbb{Z}} \mathbb{V}_k(\widetilde{\Sigma}) \text{ where } \mathbb{V}_k(\widetilde{\Sigma}) = \mathsf{V}(\widetilde{\Sigma}\sqcup\widehat{S}_k).$$

These state spaces are part of a $(2+1)$-TQFT constructed in [2]. Since $\dagger : \mathcal{C}ob(\emptyset,\widetilde{\Sigma}) \to \mathcal{C}ob(\widetilde{\Sigma},\emptyset)$ is bijective, the pairing described in Corollary 5.6 descends to a non-degenerate Hermitian pairing on the state spaces $\mathbb{V}(\widetilde{\Sigma})$ of the TQFT. For details, see [2, Proposition 4.28, Definition 5.3].

Theorem 5.7 *The TQFT $(\mathbb{V}, Z)$ is Hermitian. More specifically, for any decorated surface $\widetilde{\Sigma}$, there is a non-degenerate Hermitian pairing*

$$\langle\cdot,\cdot\rangle_{\mathbb{V}(\widetilde{\Sigma})}$$

and for any $y\in\mathbb{V}(\widetilde{\Sigma}_-)$ and for any $x\in\mathbb{V}(\widetilde{\Sigma}_+)$, and for any decorated cobordism $\widetilde{M}:\widetilde{\Sigma}_-\to\widetilde{\Sigma}_+$ between decorated surfaces, there is an equality

$$\left\langle x, \mathbb{V}(\widetilde{M})(y)\right\rangle_{\mathbb{V}(\widetilde{\Sigma}_+)} = \left\langle \mathbb{V}(\widetilde{M}^\dagger)(x), y\right\rangle_{\mathbb{V}(\widetilde{\Sigma}_-)}. \tag{37}$$

Proof The fact that the pairing is non-degenerate and Hermitian follows from the discussion above.

The equality (37) follows from the functoriality of the TQFT and Lemma 5.5. The details follow as in [24, Theorem III.5.3]. □

The mapping class group action in $\mathcal{C}ob$ is given through mapping cylinders: if $f:\widetilde{\Sigma}_1\to\widetilde{\Sigma}_2$ is a diffeomorphism, the mapping cylinder of f is the decorated cobordism from Σ_1 to Σ_2 given by $M_f = (\Sigma_2\times[0,1], \{p_i^2\}\times[0,1], f\times\{0\}\sqcup\mathrm{Id}\times\{1\}, \pi^*(\omega_2), 0)$. The mapping cylinder construction is functorial: $M_f\circ M_g = M_{fg}$.

Proposition 5.8 *If f, M_f are as above, $M_f^\dagger = M_{f^{-1}}$. In particular, the TQFT $\mathbb{V}$ induces projective representations of the mapping class group in the group of indefinite unitary matrices.*

Proof The map $f\times(t\mapsto 1-t) : M_{f^{-1}}\to M_f^\dagger$ is an isomorphism, and thus, the two cobordisms are equal in $\mathcal{C}ob$. □

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Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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