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On the significance of radiation reaction

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Abstract. Radiation reaction has been a topic in physics for more than a century. The lack of a complete and consistent treatment in classical electrodynamics, and the appearance of unphysical solutions, has often postponed the discussion of reactive effects of radiation to late chapters in textbooks, with comprehensive discussion usually reserved for advanced texts. As a result, radiation reaction may appear to some mainly as a curiosity. This modest focus is in stark contrast to the fact that radiation reaction played a crucial role when Niels Bohr arrived at his postulates that became part of the foundation of quantum mechanics, and that it determines the collapse of binary astrophysical systems as well as the deceleration of high-energy electrons that penetrate matter. We discuss these cases and show how, for ultra-relativistic electrons penetrating single crystals, we have been able to achieve the at first glance bizarre scenario where the reaction force is many times greater than the interaction force between the electron and the crystal without which no radiation would appear.

1 Introduction

Radiation reaction, the back action on a particle when it emits radiation, has played an important role in physics on several occasions. While a consistent description has never surfaced within classical electrodynamics, radiation reaction is central to the birth of quantum mechanics, and it is crucial for the deceleration of highly relativistic electrons in matter as well as in the collapse of rotating binary astrophysical systems. In the first two cases, it is electromagnetic radiation that gives reaction; in the last case, it is gravitational radiation. Due to the great difference in the strengths of the two interactions, as well as in the roles the constituent masses play [1], the conditions are markedly different in the cases mentioned. For bound systems of atomic building blocks, the classical radiation response becomes so great that the systems would be highly unstable and collapse promptly if their behavior were controlled by classical physics. Effects central to quantum mechanics prevent the collapse, and one must instead look elsewhere to see the radiation reaction in classical electrodynamics. A good place is in the interaction of ultra-relativistic electrons with strong fields. As recent experiments have shown, one can achieve the very special situation where the radiation-attenuation force can become much larger than the electromagnetic forces that otherwise affect the electrons and ultimately are the reason why radiation can be emitted at all. This scenario may be obtained when the forces due to the external strong fields are perpendicular to the direction of main motion. The fate of the electrons is thus determined to a decisive degree by the radiation reaction. In the case of gravity, however, radiation reaction is less pronounced, and as demonstrated below, never becomes the dominant force. In order for gravitational radiation to have a significant effect on motion, it is necessary that very massive objects are involved. The radiation reaction for two such objects in rotation about their common center of mass can lead to collapse long before the binary system becomes relativistic. Since the system remains only mildly relativistic, with velocities up to around half the speed of light, $0.5c$, the radiation damping force will always be weaker than the gravitational force that binds the binary system together.

[1] In electrodynamics mass solely appears in Newton's second law (inertial mass). In gravity mass appears both in Newton's second law (inertial mass), as the constant of proportionality between force and acceleration, and in force itself (the gravitational mass). Hence, the Larmor formula for radiation in non-relativistic electrodynamics contains no mass, whereas the gravitational analog for a binary system will be proportional to mass squared.

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2 Bohr's considerations

Niels Bohr first postulated the existence of stationary states in his 1913 series of three papers on the structure of atoms [1]. The postulate is the first out of two on

Fig. 1 Bohr's sketch of an electron spiraling toward the atomic nucleus

which "the quantum theory of line-spectra rests". In a later formulation, it reads as follows [2]:

> That an atomic system can, and can only, exist permanently in a certain series of states corresponding to a discontinuous series of values for its energy, and that consequently any change of the energy of the system, including emission and absorption of electromagnetic radiation, must take place by a complete transition between two such states. These states will be denoted as the "stationary states" of the system.

Bohr's postulate was spurred by the question of why an atom can be stable when the orbiting electron particle is supposed to radiate according to classical electrodynamics.

The collected works of Niels Bohr include handwritten pages indicating Bohr's thoughts on the classical problem [3]. The sketch of the electron path expected according to classical electrodynamics reproduced in Fig. 1 appears on one of the pages.

In the handwritten pages, Bohr determines the amount of energy ΔW radiated away during one revolution according to classical electrodynamics under the assumption that the velocity is small in comparison with c. He gives a slightly long explanation, which reduces to the fractional change

$$\Delta W/W = (8\pi/3Z)(Z\alpha)^3 \qquad (1)$$

for a circular orbit in an atom with atomic number Z; $\alpha = e^2/\hbar c \simeq 1/137$ denotes the fine-structure constant. The ratio (1) is 3×10^{-6} for hydrogen, but for hydrogen-like lead or uranium (where a non-relativistic calculation is not that accurate) it amounts to a few percent.

The observation that the electron does not spiral in from what was later termed its ground state led Bohr to his radical postulate. While Bohr estimated the classical radiation reaction, the calculations are not included in the "trilogy" on the constitution of atoms and molecules. In vol. 2 of the collected works, two potential reasons are listed: "Rutherford's criticism of the length of his paper" and "considerations of the fact that his theory could not make any claim of completeness" [3].

3 Bremsstrahlung

At sufficiently high energies, typically beyond 10–100 MeV depending on the material, electrons lose energy primarily through emission of bremsstrahlung that potentially includes photons of essentially all energies up to the kinetic energy of the electron. This is perhaps the most well-known occurrence of radiation reaction. At a very high energy E_e, where screening is "complete," the electron on average slows down to $E_e/e = E_e \exp(-1)$ over the radiation length L_r. With a momentum change of order E_e/c in a time interval L_r/c, the radiative damping force is of order $F_r \approx E_e/L_r$. For a 200 GeV electron penetrating lead ($L_r = 5.6$ mm), this amounts to $F_r \approx 4 \times 10^{13}$ eV/m. This force is approximately four orders of magnitude below the force on the electron at a distance of the order of the ground-state radius from the nucleus. In addition, the force is approximately one order of magnitude below the force determined using the atomic radius from the statistical Thomas–Fermi model. While the moderation of the electron is dramatic, the radiative damping force is well below the typical forces experienced during the deflection in the atomic fields which is the origin of the emission.

4 Extreme radiation reaction

While the radiation-damping force is relatively weak compared to the force causing the radiation in the cases considered above, under certain conditions it is possible for the damping force to be stronger than the primary force. For that to occur, one needs highly relativistic electrons and a primary force that is essentially transverse to the electrons' direction of motion. With the radiative reaction dominating the dynamics, the century-old problem of how to incorporate it systematically in the equations of motion clearly appears.

Radiation reaction is traditionally described by the Lorentz–Abraham–Dirac (LAD) equation in classical electrodynamics [4–6]. The LAD equation, however, has nonphysical ("runaway") solutions with, for example, the acceleration of the radiating particle increasing exponentially even if no external field is present. Such features have rendered the LAD equation one of the most controversial equations in physics.

Provided the radiation–reaction force on an electron is much smaller than the Lorentz force in the instantaneous rest frame of the radiating particle, a "reduction of order" (a perturbation approach) may be applied with the electron's four-acceleration in the radiation–reaction four-force being replaced by the Lorentz four-force divided by the electron mass [7]. This results in the Landau–Lifshitz (LL) equation [7]:

$$m\frac{du^\mu}{ds} = eF^{\mu\nu}u_\nu + \frac{2}{3}e^2\left[\frac{e}{m}(\partial_\alpha F^{\mu\nu})u^\alpha u_\nu + \frac{e^2}{m^2}F^{\mu\nu}F_{\nu\alpha}u^\alpha + \frac{e^2}{m^2}(F^{\alpha\nu}u_\nu)(F_{\alpha\lambda}u^\lambda)u^\mu\right], \quad (2)$$

where $e < 0$ and m denote the electron charge and mass, respectively, $F^{\mu\nu}$ is the external electromagnetic field tensor, u^μ is the four-velocity of the electron, and s its proper time in units with $c = 1$. See Eq.(1) in [8] for an expression directly in terms of $\mathbf{E}$ when only a static electric field is active, as in the case of a crystal. The LL equation is free of the physical inconsistencies of the LAD equation, and it has been shown to feature all the physical solutions of the LAD equation [9].

The dynamics of the electron, and its emitted radiation, is sensitive to the magnitude of the strong-field parameter χ defined as:

$$\chi^2 = (F_{\mu\nu}u^\nu)^2/E_0^2, \quad E_0 = m^2/e\hbar\,, \quad (3)$$

where the critical field assumes a value of $E_0 \simeq 1.32 \times 10^{16}$ V/cm [10]. For an electron moving in a constant magnetic field, it is essentially $\hbar$ times the characteristic frequency for classical synchrotron radiation divided by the electron energy. This implies that quantum effects are decisive for χ approaching 1 and above. For an electron, or positron, moving in a purely transverse electric field (E) in the laboratory frame, as in the case of an aligned crystal, χ reduces to $\gamma E/E_0$, where γ is its Lorentz factor (total energy in units of m).

A perturbation approach can be used to derive the LL equation when the fields experienced by the radiating electron, or positron, in its rest frame are small compared with $m^2c^4/e^3 \equiv E_1$ [7]. We may write this condition as $\delta \ll 1$ with δ denoting the ratio field strength to E_1 $(= E_0/\alpha)$. It is convenient to express the classical parameter δ as the product $\chi\alpha$, but it should be noted that neither χ nor the fine-structure constant belong to classical physics. In the classical or near classical regime, where χ is order 1 or less, the perturbation condition

$$\delta = \chi\alpha \ll 1 \quad (4)$$

is clearly fulfilled.

The ratio of damping force to external force for a transverse electric field is given by the classical parameter η expressible as:

$$\eta = \alpha\gamma^2 E/E_0 = \alpha\gamma\chi = \gamma\delta\,. \quad (5)$$

For experimental investigations approaching the classical regime, i.e., for $\chi \ll 1$ where the perturbation condition (4) holds, it is then possible for the magnitude of the radiation damping force to be large compared to the Lorentz force for sufficiently large Lorentz factors ($\gamma \gg 1$). As emphasized by Landau and Lifshitz in a footnote [7], a large value of the ratio "does not in any way contradict" the application of the perturbation approach to the derivation of the LL equation. Since the damping force is longitudinal and the primary force due to the external field considered is transverse, their ratio is not Lorentz invariant.

The strong-damping regime $\eta \gg 1$ can be reached through the interaction of ultrarelativistic electrons with strong-field lasers or single crystals. Strong-field laser experiments offer a cleaner interaction compared to crystals, due to multiple scattering within the crystal, but they have a technical difficulty to overcome. Overlapping an electron beam with an intense ultrashort laser pulse, which have an inherent pulse-to-pulse fluctuations, implies that the exact conditions of the interaction are not as well known. Crystalline fields are by nature static and cannot be changed arbitrarily; one can only change the orientation, temperature, and the target material. Using lasers, instead of crystals, has the advantage of controlling the electric field. The relation of the parameters defined above to those used in the laser community may be found in [11].

The initial goal of the upcoming strong-field laser experiment E-320 at SLAC is to operate 13 GeV electrons in the regime of $\chi \approx 0.15$ which corresponds to a value of $\eta \approx 29$. In the crystal experiment reported in [8], a value of $\bar{\chi} \approx 0.06$ is obtained for axially aligned 80 GeV electrons in diamond crystal oriented along the $\langle 100\rangle$ axis, resulting in a value of $\eta \approx 69$. Comparing these two experiments, it is evident that the γ^2 scaling on η benefits the crystal experiments significantly for studying extreme classical radiation reaction, due to the higher electron energies and lower fields.

For η values in the range of 10–100, the radiation–reaction force dominates the particles dynamics, and since χ is on purpose sufficiently small, the influence of quantum effects is moderate. Quantum effects can be further reduced, but never entirely avoided, by using weaker fields, i.e. smaller χ. Unfortunately, the magnitude of the damping force, and thus η, would also decrease, making the radiation–reaction difficult to detect. The experimental conditions reported in the crystal experiment [8] therefore provide ideal conditions for testing the applicability of the Landau–Lifshitz equation. The experimental results verified the Landau–Lifshitz equations description of the classical radiation reaction to a high degree of precision.

An illustration of extreme radiation damping is a simulation for 80 GeV electrons penetrating a 1.5-mm-thick diamond crystal oriented along the $\langle 100\rangle$ axis as is in the experiment reported in [8]. The simulated energy distribution of the electrons upon exit of the crystal is shown in Fig. 2. Besides the damping, the motion of a charged particle incident at a small angle to a major crystallographic direction is effectively governed by the continuum potential obtained by smearing the atomic charges along the corresponding axis or plane, see [12–15]. Hence, the equation of motion used in the simulation is the Landau–Lifshitz equation with a static electric field corresponding to the continuum-string potential pertaining to the $\langle 100\rangle$ axis. The angle of incidence relative to the axis is assumed to be distributed evenly

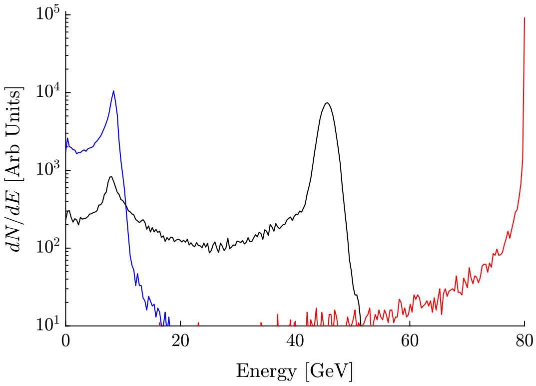

Fig. 2 Exit-energy distribution for 80 GeV electrons incident on a 1.5-mm-thick carbon target. The red curve is for an amorphous foil. The black curve is for a diamond crystal oriented along the ⟨100⟩ axis with incidence angles less than the critical channeling angle ψ_1 relative to the axis. The simulation in this case is performed exactly as the calculation that gives the red curve in Fig. 6, upper right, in [8]. The blue curve is for the same case as the black curve, but without multiple scattering to make the comparison to the astrophysical case clearer

with a maximum equal to the critical channeling angle ψ_1 (in this case 35 μrad). A major portion of the electrons will then initially be bound to move around a single string of atoms (channeling). A reduction factor to compensate for quantum effects under these conditions is included on the damping force in the Landau–Lifshitz equation (as described in [8]). For the purpose of illustration, multiple scattering is neglected in the blue curve in Fig. 2. This means that the only external force acting is that corresponding to an idealized "continuum crystal" where all charges are smeared out uniformly along the ⟨100⟩ direction. A very high energy loss results in an average of about 90 % of the original 80 GeV. In reality, the actual motion is perturbed by the discreteness of target constituents. Encounters with thermally displaced nuclei, that happen to be in the way, gradually steer the electrons away from the strings whereby they radiate less. The black curve represents the realistic case that includes this multiple scattering. It is determined from the simulation reported in [8] that closely reproduces the experimental radiation spectrum. Clearly, the radiative damping is very dramatic also in this case and much higher than that pertaining to an amorphous carbon foil of the same thickness (red curve). The extreme radiation–reaction scenario resulting from a damping force much higher than the primary force is evident.

5 The ratio between forces in the gravitational interaction

The term 'radiation reaction' is also used for phenomena based on the gravitational interaction, for instance in connection with the Hulse–Taylor binary pulsar B1913+16 [16]. As is well known, the cumulative shift of the B1913+16 periastron time observed is in beautiful agreement with general relativistic expectations based on the emission of gravitational waves [17]. In relation to the above discussion, on radiation reaction in electrodynamics, the question arises: Is the radiation reaction force due to the emission of gravitational waves smaller or larger than the force that keeps a binary system of stars gravitationally bound?

To address this question, we derive a simple estimate for the ratio between the radiation reaction force due to the emission of gravitational waves and the force that keeps a binary system of stars gravitationally bound. We assume the two stars to be identical, each with mass M, and to rotate in a circle of radius R around their common center of mass at velocity v. General relativity corrections to this simple estimate are of the order $(v/c)^2$ and diminish the radiation emission, see, e.g., [18], equation (1.4), whereby the result presents an upper limit to the ratio, which is shown not to exceed one.

We begin by estimating the maximum radiation emission, following [17], to be of the order $L_{\mathrm{GW},e} \sim c^5/G$, i.e. about one in geometrized units. Introducing an 'efficiency factor' ε which 'preliminary numerical simulations suggest' is around 10^{-2}, the maximum radiation emission is rewritten as $L_{\mathrm{GW},e} \sim \varepsilon c^5/G$. Similar to the electrodynamics case, the emitted power L_{GW} is assumed to originate from the work per time exerted by a radiation–reaction force F_r, i.e.,

$$F_r \cdot v = L_{\mathrm{GW}}/2 \,, \tag{6}$$

where the factor of 2 is included since both stars contribute equally to the emission. Using $L_{\mathrm{GW},e}$ for L_{GW} and setting $v \sim c$ leads to

$$F_{r,e} \sim \varepsilon c^4/2G. \tag{7}$$

The gravitational force is

$$F = \frac{GM^2}{(2R)^2} \,. \tag{8}$$

For R equal to the Schwarzschild radius $R_S = 2GM/c^2$, which sets the scale for collapse of the system, Eq. (8) reduces to $F = c^4/2^4G$. The ratio between the radiation reaction force (7) and the force that keeps a binary system of stars gravitationally bound is then given as:

$$\frac{F_{r,e}}{F} \sim 8\varepsilon \,, \tag{9}$$

which is roughly 10% using the suggested value of ε.

A slightly more accurate approach utilizes that the power emitted in gravitational radiation in a binary system of two identical stars (assuming non-relativistic motion and ignoring spin effects, etc.) is given by [17]

$$L_{\mathrm{GW}} = \frac{128 \cdot 4^{1/3}}{5} \frac{c^5}{G} \left(\frac{\pi G M}{c^3 P} \right)^{10/3}, \tag{10}$$

where P is the period. The acceleration of each star is given by

$$\frac{v^2}{R} = \frac{1}{R} \left(\frac{2\pi R}{P} \right)^2 = \frac{GM}{(2R)^2} \, . \tag{11}$$

Then, using the equations (6), (11), (8), and (10), we get the desired result of

$$\eta_g = \frac{F_r}{F} = \frac{8}{5} \left(\frac{GM}{c^2 R} \right)^{5/2} = \frac{1}{20} \left(\frac{2R_S}{R} \right)^{5/2}, \tag{12}$$

which is a compact expression considering the input from (10). The comparison of η from (5) and η_g from (12) is the primary objective of this paper, and since experiments using crystals have shown that η can be much larger than unity, we proceed with determining more precisely the possible numerical value for η_g for the most extreme gravitational case we can think of.

Experimental observations [19] show that maximum emission occurs at the brink of collapse when R approaches the Schwarzschild radius. As R becomes smaller than R_S, the emission drops rapidly due to the black holes merging when they enter each other's Schwarzschild radii. Near the maximum of emission power, at $R = R_S$, for the collision of two identical black holes, Eq. (12) gives $F_r/F \simeq 0.28$, which is comparable to the previous estimate (9). As a result, the radiation–reaction force due to the emission of gravitational waves is always smaller than the force that keeps a binary system of stars gravitationally bound. This is in contrast to electrodynamics where the radiation–reaction force can dominate the Lorentz force by a large factor as shown above. It is the combination of transverse external force and a very high Lorentz factor that allows for this. In the crystal experiments, γ assumes values of at least 1×10^5. In the binary astrophysical system collapsing due to emission of gravitational waves, the velocity of the rotating objects may approach the speed of light, for the case reported in [19] the maximum of emission occurs for velocities around $0.6c$. This corresponds to a very modest Lorentz factor, $\gamma = 1.25$.

Even though the ratio F_r/F is small in the gravitational case, it has drastic consequences for the inspiraling of the two stars. With the mechanical energy expressed as

$$E = -\Gamma P^{-2/3} \, , \tag{13}$$

where Γ is a constant depending on the type of interaction, the resulting $dE/dt = (-2/3) E P^{-1} dP/dt$ can be set equal to $-F_r \cdot v$. Multiplying by the period $P = 2\pi R/v$ and using that $-2E = FR$ leads to

$$\frac{dP}{dt} = -6\pi \frac{F_r}{F} \, . \tag{14}$$

This shows that even a 1% relative damping yields a dP/dt of -0.2, such that after a time $t = P(0) \equiv P_0$, the period has diminished by almost 20% (assuming that both forces, F_r and F, are time independent). From Eqs. (10) and (13), we can further infer that $dP/dt = -CP^{-5/3}$, which can be integrated to give

$$P(t) = \left(P_0^{8/3} - \frac{8}{3} C t \right)^{3/8} , \tag{15}$$

where the constant C is given by

$$C = \frac{96}{5 \cdot 2^{1/3}} (2\pi)^{8/3} \left(\frac{GM}{c^3} \right)^{5/3} , \tag{16}$$

see also [20]. With $E \propto 1/R$ and the scaling (13), we have $R \propto P^{2/3}$. Using Eq. (15), the phase of rotation at time t is determined to be

$$\phi(t) = \phi(0) + \frac{6\pi}{5C} \left(P_0^{5/3} - P(t)^{5/3} \right) \, . \tag{17}$$

With $R(t)$ and $\phi(t)$ given, it is straightforward to map the inspiraling directly. An example is shown in Fig. 3. Just after slightly more than 4.5 rotations, the emission of gravitational radiation has caused the system to shrink from $5R_s$ to R_s. In the course of the elapsed 219 ms, the ratio F_r/F of the damping force to the gravitational force has increased from just 0.005 to the value of 0.28 that was mentioned above. So despite starting out with less than a 1% damping force, it takes just a few turns before the system collapses. Figure 4 shows how F_r/F develops from the start at $R = 5R_S$ until collapse (blue curve). The ratio remains low until shortly before collapse. The figure also displays the development from start of the instantaneous rotation frequency Ω (green) and the total energy emitted into gravitational waves. The radiated energy is shown in units of the asymptotic value $Mc^2/8$ (red) as well as relative to the value of the total mechanical energy at start (black). The radiated energy increases dramatically when the system comes close to the final collapse.

6 Conclusion

Radiation reaction is directly linked to the birth of quantum mechanics, and it has drastic consequences in a variety of very different scenarios. Yet, no consistent closed-form description exists. In a previously published experiment [8], we have demonstrated that the Landau–Lifshitz equation, which rests on a perturbation expansion in the instantaneous rest frame, provides a very satisfactory description of the radiation–reaction phenomenon for the interaction of highly relativistic electrons with the strong fields of a single crystal. For such interactions, the radiation–reaction force is almost two orders of magnitude higher than the Lorentz force, and

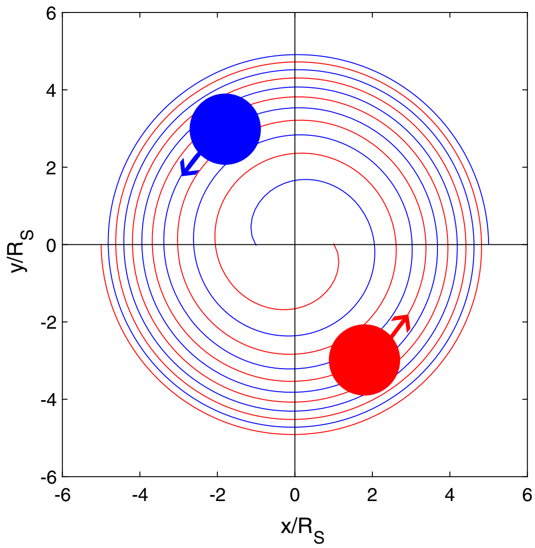

Fig. 3 Gravitational collapse of a binary astrophysical system. Two objects of 32 solar masses each are originally in circular motion around their common center of mass at a radius of $5R_S$

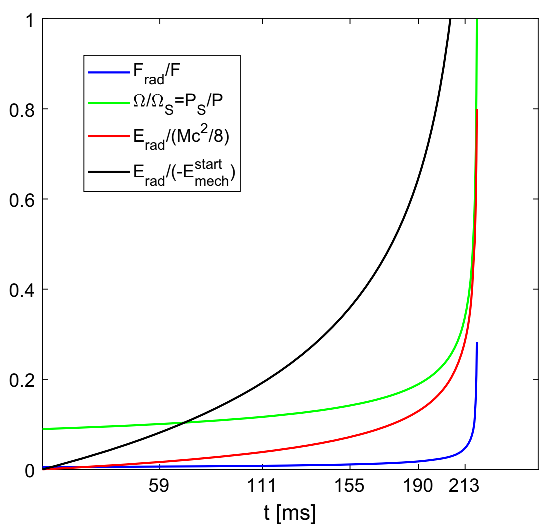

Fig. 4 Development of the relative damping force, the integrated radiated energy (in two different units), and the instantaneous rotation frequency in the gravitational collapse of the binary system shown in Fig. 3. The time axis is linear, and the time for each full rotation ($\phi(t_n) = n2\pi$) is marked. Index $_S$ refers to values at collapse, $R = R_S$

the dynamics of the particle is dominated by the radiation reaction.

The radiative collapse of a non-relativistic one-electron atom in classical electrodynamics studied by Niels Bohr in 1913, has close parallels to the collapse due to emission of gravitational radiation of rotating binary astrophysical systems. In the former case, quantum mechanics prevents a collapse; in the latter, the interaction is far too weak for quantum mechanics to get in the way. We have presented an, admittedly quite simplistic, evaluation of the radiation reaction in the gravitational case. This analysis shows that the binding force between two stellar objects will always be the dominant force in the case of gravity, even though the orbits are significantly affected by the radiation reaction which eventually drives the system to the brink of collapse.

Comparing the cases of gravity and electrodynamics, we conclude that radiation reaction may become completely dominating only for electrodynamics. It is the combination of a transverse external force and a very high Lorentz factor that allows for this dominance. It rests on the fact that the ratio of the longitudinal damping force due to radiation emission and the transverse external force is not Lorentz invariant. A modest damping force, relative to the external force, in the instantaneous rest frame of the radiating particle, may then transform to a damping force that by far dominates the external force in the laboratory frame for highly relativistic particles. This extreme phenomenon has been observed experimentally for electrons traveling through a crystal, and in the future, might be observed in a laser-electron interaction by several upcoming experiments.

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Author contributions

All authors contributed equally to the work presented in the paper.

Data Availability Statement This manuscript has associated data in a data repository. [Authors comment: The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request. This manuscript has no associated data or the data will not be deposited.]

References

1. N. Bohr, Phil. Mag. **26**(1), 476–857 (1913)
2. N. Bohr, D. Kgl. Danske Vidensk. Selsk. Skrifter, Naturvidensk. og Mathem. Afd. **8, IV**, 1 (1918), reproduced in: J. Rud Nielsen (ed.) Niels Bohr Collected Works, Vol. 3 (North-Holland, Amsterdam, 1976) with an interesting introduction by the editor. Further reprinted by Dover (2005) under Bohr's original On the Quantum Theory of Line-Spectra
3. U. Hoyer (ed.), Niels Bohr Collected Works, Vol. 2 (North-Holland, Amsterdam, 1981)
4. M. Abraham, Theorie der Elektrizität (Teubner, Leipzig, 1905)

5. H.A. Lorentz, The Theory of Electrons (Teubner, Leipzig, 1909)
6. P. A. M. Dirac, Proc. R. Soc. Lond. Ser. A **167**, 148 (1938)
7. L.D. Landau and E.M. Lifshitz, The Classical Theory of Fields (Elsevier, Oxford, 1975)
8. C. F. Nielsen, J. B. Justesen, A. H. Sørensen, U. I. Uggerhøj, R. Holtzapple, (CERN NA63 Collaboration) **102**, 052004 (2020). https://doi.org/10.1103/PhysRevD.102.052004
9. H. Spohn, Europhys. Lett. **50**, 287 (2000). https://doi.org/10.1209/epl/i2000-00268-x
10. V.B. Berestetskii, E.M. Lifshitz, and L.P. Pitaevskii, Quantum Electrodynamics (Pergamon, New York, 1989)
11. C. F. Nielsen, J. B. Justesen, A. H. Sørensen, U. I. Uggerhøj, R. Holtzapple, (CERN NA63) New J. Phys. **23**, 085001 (2021). https://doi.org/10.1088/1367-2630/ac1554
12. J. Lindhard, Kong. Danske Vidensk. Selsk, Mat.-Fys. Medd **34**(14), 1 (1965)
13. J.U. Andersen, Notes on channeling (2018), lecture notes, Aarhus University. https://phys.au.dk/publikationer/lecture-notes/
14. A.H. Sørensen, Nuclear Instruments and Methods in Physics Research Section B: Beam Interactions with Materials and Atoms **119**, 2 (1996) http://www.sciencedirect.com/science/article/pii/0168583X96003497
15. U.I. Uggerhøj, Rev. Mod. Phys. **77**, 1131 (2005) https://doi.org/10.1103/RevModPhys.77.1131
16. C.M. Will, Living Rev. Relativ. **17** (2014). https://doi.org/10.12942/lrr-2014-4
17. J.B. Hartle, Gravity: An Introduction to Einstein's General Relativity (Addison Wesley, San Francisco, 2003)
18. C. M. Will, A. G. Wiseman, Phys. Rev. D **54**, 4813 (1996). https://doi.org/10.1103/PhysRevD.54.4813
19. B. P. Abbott et al., (LIGO Scientific Collaboration and Virgo Collaboration). Phys. Rev. Lett. **116**, 061102 (2016). https://doi.org/10.1103/PhysRevLett.116.061102
20. L. J. Rubbo, S. L. Larson, M. B. Larson, D. R. Ingram, Am. J. Phys. **75**, 597 (2007). https://doi.org/10.1119/1.2721587