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LAGRANGIAN GEOMETRY OF MATROIDS

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1. Introduction

1.1. **Geometry of matroids.** A *matroid* M on a finite set E is a nonempty collection of subsets of E, called *flats* of M, that satisfies the following properties:

(1) The intersection of any two flats is a flat.
(2) For any flat F, any element in $E - F$ is contained in exactly one flat that is minimal among the flats strictly containing F.

The set $\mathcal{L}(\mathrm{M})$ of all flats of M is a geometric lattice, and all geometric lattices arise in this way from a matroid [Wel76, Chapter 3]. The theory of matroids captures the combinatorial essence shared by natural notions of independence in linear algebra, graph theory, matching theory, the theory of field extensions, and the theory of routings, among others.

Gian-Carlo Rota, who helped lay down the foundations of the field, was one of its most energetic ambassadors. He rejected the "ineffably cacophonous" name of matroids, preferring to call them combinatorial geometries instead [CR70]. This alternative name never really caught on, but the geometric roots of the field have since grown much deeper, bearing many new fruits. The geometric approach to matroid theory has recently led to solutions of long-standing conjectures, and to the development of fascinating mathematics at the intersection of combinatorics, algebra, and geometry.

There are at least three useful polyhedral models of a matroid M. For a short survey, see [Ard18]. The first one is the *basis polytope* of M introduced by Edmonds in optimization and Gelfand–Goresky-MacPherson-Serganova in algebraic geometry. It reveals an intricate relationship of matroids with the Grassmannian variety and the special linear group. The second model is the *Bergman fan* of M, introduced

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by Sturmfels and Ardila–Klivans in tropical geometry. It was used by Adiprasito–Huh–Katz to prove the log-concavity of the f-vectors of the independence complex and the broken circuit complex of M. The third model, which we call the *conormal fan* of M, is the main character of this paper. We use its intersection-theoretic and Hodge-theoretic properties to prove conjectures of Brylawski [Bry82], Dawson [Daw84], and Swartz [Swa03] on the h-vectors of the independence complex and the broken circuit complex of M.

1.2. **Conormal fans and their geometry.** Throughout the paper, we write $r+1$ for the rank of M, write $n+1$ for the cardinality of E, and suppose that n is positive.[1] Following [MS15], we define the *tropical projective torus* of E to be the n-dimensional vector space

$$\mathrm{N}_E = \mathbb{R}^E/\mathbb{R}\mathbf{e}_E, \qquad \mathbf{e}_E = \sum_{i\in E} \mathbf{e}_i.$$

The tropical projective torus is equipped with the functions

$$\alpha_j(z) = \max_{i\in E}(z_j - z_i), \text{ one for each element } j \text{ of } E.$$

These functions are equal to each other modulo global linear functions on N_E, and we write α for the common equivalence class of α_j. The *Bergman fan* of M, denoted Σ_{M}, is an r-dimensional fan in the n-dimensional vector space N_E whose underlying set is the *tropical linear space*

$$\mathrm{trop}(\mathrm{M}) = \Big\{ z \,|\, \min_{i\in C}(z_i) \text{ is achieved at least twice for every circuit } C \text{ of } \mathrm{M} \Big\} \subseteq \mathrm{N}_E\,.$$

It is a subfan of the *permutohedral fan* Σ_E cut out by the hyperplanes $x_i = x_j$ for each pair of distinct elements i and j in E. This is the normal fan of the *permutohedron* Π_E. The functions α_j are piecewise linear on the permutohedral fan, and hence piecewise linear on the Bergman fan of M.[2]

Tropical linear spaces are central objects in tropical geometry: For any linear subspace V of $\mathbb{C}^E$, the tropicalization of the intersection of $\mathbb{P}(V)$ with the torus of $\mathbb{P}(\mathbb{C}^E)$ is the tropical linear space of the linear matroid on E represented by V [Stu02]. Furthermore, tropical linear spaces are precisely the tropical fans of degree one with respect to α, that is, the tropical analogs of linear spaces [Fin13]. Tropical manifolds are thus defined to be spaces that locally look like Bergman fans of matroids [IKMZ19].

Adiprasito, Huh, and Katz showed that the Chow ring of the Bergman fan of M satisfies Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations [AHK18]. Furthermore, they interpreted the entries of the f-vector of the reduced broken circuit complex of M – an invariant of the matroid generalizing the chromatic polynomial for graphs – as intersection numbers in the Chow ring of Σ_{M}. The geometric interpretation then implied the log-concavity of the coefficients of the *characteristic polynomial* and the *reduced characteristic polynomial*

$$\chi_{\mathrm{M}}(q) := \sum_{F\in\mathcal{L}(\mathrm{M})} \mu(\varnothing, F) q^{\mathrm{corank}(F)}, \qquad \overline{\chi}_{\mathrm{M}}(q) := \chi_{\mathrm{M}}(q)/(q-1),$$

[1] There are exactly two matroids on a single element ground set, the *loop* and the *coloop*, which are dual to each other. These matroids will play exceptional roles in our inductive arguments.

[2] A continuous function f is said to be *piecewise linear* on a fan Σ if the restriction of f to any cone in Σ is linear. In this case, we say that the fan Σ *supports* the piecewise linear function f.

where μ is the *Möbius function* on the geometric lattice $\mathcal{L}(\mathrm{M})$ for a loopless matroid M.[3]

The conormal fan $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ is an alternative polyhedral model for M. Its construction uses the dual matroid $\mathrm{M}^\perp$, the matroid on E whose bases are the complements of bases of M. We refer to [Oxl11] for background on matroid duality and other general facts on matroids. A central role is played by the *addition map*

$$\mathrm{N}_{E,E} := \mathrm{N}_E \oplus \mathrm{N}_E \longrightarrow \mathrm{N}_E, \qquad (z, w) \longmapsto z + w.$$

The function α_j on N_E pulls back to a function δ_j on $\mathrm{N}_{E,E}$ under the addition map. Explicitly,

$$\delta_j(z, w) = \max_{i \in E}(z_j + w_j - z_i - w_i).$$

The function δ_j is piecewise linear on a fan that we construct, called the *bipermutohedral fan* $\Sigma_{E,E}$ (Section 2.3). This is the normal fan of a convex polytope $\Pi_{E,E}$ that we call the *bipermutohedron*. The functions δ_j for j in E are equal to each other modulo global linear functions on $\mathrm{N}_{E,E}$, and we write δ for their common equivalence class.

The *cotangent fan* Ω_E is the subfan of the bipermutohedral fan $\Sigma_{E,E}$ whose underlying set is the tropical hypersurface

$$\mathrm{trop}(\delta) = \Big\{(z, w) \mid \min_{i \in E}\{z_i + w_i\} \text{ is achieved at least twice}\Big\} \subseteq \mathrm{N}_{E,E}\,.$$

We show in Section 3.4 that, for any matroid M on E, we have

$$\mathrm{trop}(\mathrm{M}) \times \mathrm{trop}(\mathrm{M}^\perp) \subseteq \mathrm{trop}(\delta).$$

The *conormal fan* $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ is defined to be the subfan of the cotangent fan Ω_E that subdivides the product $\mathrm{trop}(\mathrm{M}) \times \mathrm{trop}(\mathrm{M}^\perp)$. For our purposes, it is necessary to work with the conormal fan of M instead of the product of the Bergman fans of M and $\mathrm{M}^\perp$, because the function δ_j need not be piecewise linear on the product of the Bergman fans.

The projections to the summands of $\mathrm{N}_{E,E}$ define morphisms of fans[4]

$$\pi \colon \Sigma_{\mathrm{M},\mathrm{M}^\perp} \longrightarrow \Sigma_{\mathrm{M}} \quad \text{and} \quad \overline{\pi} \colon \Sigma_{\mathrm{M},\mathrm{M}^\perp} \longrightarrow \Sigma_{\mathrm{M}^\perp}.$$

Thus, in addition to the functions δ_j, the conormal fan of M supports the pullbacks of α_j on Σ_{M} and $\overline{\alpha}_j$ on $\Sigma_{\mathrm{M}^\perp}$, which are the piecewise linear functions

$$\gamma_j(z, w) = \max_{i \in E}(z_j - z_i) \quad \text{and} \quad \overline{\gamma}_j(z, w) = \max_{i \in E}(w_j - w_i).$$

These define the equivalence classes γ and $\overline{\gamma}$ of functions on $\mathrm{N}_{E,E}$.

The conormal fan is a tropical analog of the incidence variety appearing in the classical theory of projective duality. For a subvariety X of a projective space $\mathbb{P}(V)$, the incidence variety $\mathcal{I}_X$ is a subvariety of the product of $\mathbb{P}(V)$ with the dual projective space $\mathbb{P}(V^\vee)$ that projects onto X and its dual $X^\vee$. Over the smooth locus of X, the incidence variety $\mathcal{I}_X$ is the total space of the projectivized conormal bundle of X and, over the smooth locus of $X^\vee$, it is the total space of

[3]If M has a loop, by definition, the characteristic polynomial and the reduced characteristic polynomial of M are zero.

[4]A *morphism* from a fan Σ_1 in $\mathrm{N}_1 = \mathbb{R} \otimes \mathrm{N}_{1,\mathbb{Z}}$ to a fan Σ_2 in $\mathrm{N}_2 = \mathbb{R} \otimes \mathrm{N}_{2,\mathbb{Z}}$ is an integral linear map from N_1 to N_2 such that the image of any cone in Σ_1 is a subset of a cone in Σ_2. In the context of toric geometry, a morphism from Σ_1 to Σ_2 can be identified with a toric morphism from the toric variety of Σ_1 to the toric variety of Σ_2 [CLS11, Chapter 3].

the projectivized conormal bundle of $X^\vee$.[5] It is the projectivization of a conic Lagrangian subvariety of $V \times V^\vee$, and any conic Lagrangian subvariety of $V \times V^\vee$ arises in this way. We refer to [GKZ94, Chapter 1] for a modern exposition of the theory of projective duality.

We use the conormal fan of M to give a geometric interpretation of the polynomial $\overline{\chi}_{\mathrm{M}}(q+1)$, whose coefficients form the h-vector of the broken circuit complex of M with alternating signs. In particular, we give a geometric formula for *Crapo's beta invariant*

$$\beta_{\mathrm{M}} := (-1)^r \ \overline{\chi}_{\mathrm{M}}(1).$$

This new tropical geometry is inspired by the Lagrangian geometry of conormal varieties in classical algebraic geometry, as we now explain.

Consider the category of complex algebraic varieties with proper morphisms. According to a conjecture of Deligne and Grothendieck, there is a unique natural transformation "csm" from the functor of constructible functions on complex algebraic varieties to the homology of complex algebraic varieties such that, for any smooth and complete variety X,

$$\begin{aligned}\mathrm{csm}(1_X) &= c(TX) \cap [X] \\ &= (\text{the total homology Chern class of the tangent bundle of } X).\end{aligned}$$

The conjecture was proved by MacPherson [Mac74], and it was recognized later in [BS81] that the class $\mathrm{csm}(1_X)$, for possibly singular X, coincides with a class constructed earlier by Schwartz [Sch65]. For any constructible subset X of Y, the k-th Chern–Schwartz–MacPherson class of X in Y is the homology class

$$\mathrm{csm}_k(1_X) \in H_{2k}(Y).$$

Aiming to introduce a tropical analog of this theory, López de Medrano, Rincón, and Shaw introduced the Chern–Schwartz–MacPherson cycle of the Bergman fan of M in [LdMRS20]: The *k-th Chern–Schwartz–MacPherson cycle* of M is, by definition, the weighted fan $\mathrm{csm}_k(\mathrm{M})$ supported on the k-dimensional skeleton of Σ_{M} with the weights

$$w(\sigma_{\mathcal{F}}) = (-1)^{r-k} \prod_{i=0}^{k} \beta_{\mathrm{M}(i)},$$

where $\sigma_{\mathcal{F}}$ is the k-dimensional cone corresponding to a flag of flats $\mathcal{F}$ of M and $\mathrm{M}(i)$ is the minor of M corresponding to the i-th interval in $\mathcal{F}$. This weighted fan behaves well combinatorially and geometrically. First, the weights satisfy the balancing condition in tropical geometry [LdMRS20, Theorem 1.1], so that we may view the Chern–Schwartz–MacPherson cycle as a Minkowski weight

$$\mathrm{csm}_k(\mathrm{M}) \in \mathrm{MW}_k(\Sigma_{\mathrm{M}}).$$

Second, when trop(M) is the tropicalization of the intersection $\mathbb{P}(V) \cap (\mathbb{C}^*)^E/\mathbb{C}^*$, the Minkowski weight can be identified with the k-th Chern–Schwartz–MacPherson class of $\mathbb{P}(V) \cap (\mathbb{C}^*)^E/\mathbb{C}^*$ in the toric variety of the permutohedron Π_E [LdMRS20, Theorem 1.2]. Third, the Chern-Schwartz-MacPherson cycles of M satisfy a

[5] Thus, to be precise, the conormal fan is a tropical analog of the projectivized conormal variety and the cotangent fan is a tropical analog of the projectivized cotangent space. We trust that the omission of the term "projectivized" will cause no confusion.

deletion-contraction formula, a matroid version of the inclusion-exclusion principle [LdMRS20, Proposition 5.2]. It follows that the degrees of these Minkowski weights determine the reduced characteristic polynomial of M by the formula

$$\overline{\chi}_{\mathrm{M}}(q+1) = \sum_{k=0}^{r} \deg(\mathrm{csm}_k(\mathrm{M}))q^k,$$

where the degrees are taken with respect to the class α [LdMRS20, Theorem 1.4]. Fourth, the Chern-Schwartz-MacPherson cycles of matroids can be used to define Chern classes of smooth tropical varieties. In codimension 1, the class agrees with the anticanonical divisor of a tropical variety defined by Mikhalkin in [Mik06]. For smooth tropical surfaces, these classes agree with the Chern classes of tropical surfaces introduced in [Car15] and [Sha15] to formulate Noether's formula for tropical surfaces.

Schwartz's and MacPherson's constructions of csm for complex algebraic varieties are rather subtle. Sabbah later observed that the Chern-Schwartz-MacPherson classes can be interpreted more simply as "shadows" of the characteristic cycles in the cotangent bundle. Sabbah summarizes the situation in the following quote from [Sab85]:

> *la théorie des classes de Chern de [Mac74] se ramène à une théorie de Chow sur T^*X, qui ne fait intervenir que des classes fondamentales.*

The functor of constructible functions is replaced with a functor of Lagrangian cycles of T^*X, which are exactly the linear combinations of the *conormal varieties* of the subvarieties of X. In the Lagrangian framework, key operations on constructible functions become more geometric.

Similarly, López de Medrano, Rincón, and Shaw's original definition of the Chern–Schwartz–MacPherson cycles of a matroid M is combinatorially intricate. We prove that they are "shadows" of much simpler cycles under the pushforward map

$$\pi_*\colon\ \mathrm{MW}_k(\Sigma_{\mathrm{M},\mathrm{M}^\perp}) \longrightarrow \mathrm{MW}_k(\Sigma_{\mathrm{M}}).$$

See Section 3.1 for a review of basic tropical intersection theory.

Theorem 1.1. *When* M *has no loops and no coloops, we have*

$$\mathrm{csm}_k(\mathrm{M}) = (-1)^{r-k}\pi_*(\delta^{n-k-1} \cap 1_{\mathrm{M},\mathrm{M}^\perp}) \text{ for } 0 \le k \le r,$$

where $1_{\mathrm{M},\mathrm{M}^\perp}$ *is the top-dimensional constant Minkowski weight* 1 *on the conormal fan of* M.

It follows from Theorem 1.1 and the projection formula that the reduced characteristic polynomial of M can be expressed in terms of the intersection theory of the conormal fan as follows:

Theorem 1.2. *When* M *has no loops and no coloops, we have*

$$\overline{\chi}_{\mathrm{M}}(q+1) = \sum_{k=0}^{r}(-1)^{r-k} \deg(\gamma^k\, \delta^{n-k-1})\, q^k,$$

where the degrees are taken with respect to the top-dimensional constant Minkowski weight $1_{\mathrm{M},\mathrm{M}^\perp}$ *on the conormal fan.*

When M is representable over $\mathbb{C}$,[6] the third author gave an algebro-geometric version of Theorem 1.1 in [Huh13]. The complex geometric version of the identity boils down to the general fact that the Chern–Schwartz–MacPherson class of a smooth variety X in its normal crossings compactification Y is the total Chern class of the logarithmic tangent bundle:

$$\mathrm{csm}(1_X) = c(\Omega^1_Y(\log Y - X)^\vee) \cap [Y].$$

In fact, the logarithmic formula can be used to construct the natural transformation csm [Alu06]. For precursors of the logarithmic viewpoint, see [Alu99] and [GP02]. The current paper demonstrates that a similar geometry exists for arbitrary tropical linear spaces.

1.3. **Inequalities for matroid invariants.** Let $a_0, a_1, \ldots, a_n$ be a sequence of nonnegative integers, and let d be the largest index with nonzero a_d.

- The sequence is said to be *unimodal* if

$$a_0 \leq a_1 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_n \text{ for some } 0 \leq k \leq n.$$

- The sequence is said to be *log-concave* if

$$a_{k-1}a_{k+1} \leq a_k^2 \text{ for all } 0 < k < n.$$

- The sequence is said to be *flawless* if

$$a_k \leq a_{d-k} \text{ for all } 0 \leq k \leq d/2.$$

Many enumerative sequences are conjectured to have these properties, but proving them often turns out to be difficult. Combinatorialists have been interested in these conjectures because their solution typically requires a fundamentally new construction or connection with a distant field, thus revealing hidden structural information about the objects in question. For surveys of known results and open problems, see [Bre94] and [Sta89, Sta00].

A *simplicial complex* Δ is a collection of subsets of a finite set, called *faces* of Δ, that is downward closed. The *face enumerator* of Δ and the *shelling polynomial* of Δ are the polynomials

$$f_\Delta(q) = \sum_{S \in \Delta} q^{d-|S|+1} = \sum_{k \geq 0} f_k(\Delta) q^{d-k+1}$$
$$\text{and} \quad h_\Delta(q) = f_\Delta(q-1) = \sum_{k \geq 0} h_k(\Delta) q^{d-k+1},$$

where d is the dimension of Δ. The *f-vector* of a simplicial complex is the sequence of coefficients of its face enumerator, and the *h-vector* of a simplicial complex is the sequence of coefficients of its shelling polynomial. When Δ is *shellable*,[7] the shelling polynomial of Δ enumerates the facets used in shelling Δ, and hence the h-vector of Δ is nonnegative.

[6] We say that M is *representable* over a field $\mathbb{F}$ if there exists a linear subspace $V \subseteq \mathbb{F}^E$ such that $S \subseteq E$ is independent in M if and only if the projection from V to $\mathbb{F}^S$ is surjective. Almost all matroids are not representable over any field [Nel18].

[7] An r-dimensional pure simplicial complex is said to be shellable if there is an ordering of its facets such that each facet intersects the simplicial complex generated by its predecessors in a pure $(r-1)$-dimensional complex.

We study the f-vectors and h-vectors of the following shellable simplicial complexes associated to M. For a gentle introduction, and for the proof of their shellability, see [Bjö92].

- The *independence complex* IN(M), the collection of subsets of E that are independent in M.
- The *broken circuit complex* BC(M), the collection of subsets of E which do not contain any broken circuit of M.

Here a *broken circuit* is a subset obtained from a circuit of M by deleting the least element relative to a fixed ordering of E. The notion was developed by Whitney [Whi32], Rota [Rot64], Wilf [Wil76], and Brylawski [Bry77], for the "chromatic" study of matroids. The f-vector and the h-vector of the broken circuit complex of M are determined by the characteristic polynomial of M, and in particular they do not depend on the chosen ordering of E:

$$\chi_{\mathrm{M}}(q) = \sum_{k=0}^{r+1} (-1)^k f_k(\mathrm{BC}(\mathrm{M})) q^{r-k+1}, \qquad \chi_{\mathrm{M}}(q+1) = \sum_{k=0}^{r+1} (-1)^k h_k(\mathrm{BC}(\mathrm{M})) q^{r-k+1}.$$

Conjecture 1.3. *The following hold for any matroid* M.

(1) *The f-vector of* IN(M) *is unimodal, log-concave, and flawless.*
(2) *The h-vector of* IN(M) *is unimodal, log-concave, and flawless.*
(3) *The f-vector of* BC(M) *is unimodal, log-concave, and flawless.*
(4) *The h-vector of* BC(M) *is unimodal, log-concave, and flawless.*

Welsh [Wel71] and Mason [Mas72] conjectured the log-concavity of the f-vector of the independence complex.[8] Dawson conjectured the log-concavity of the h-vector of the independence complex in [Daw84], and independently, Colbourn conjectured the same in [Col87] in the context of network reliability. Hibi conjectured that the h-vector of the independence complex must be flawless [Hib92]. The unimodality and the log-concavity conjectures for the f-vector of the broken circuit complex are due to Heron [Her72], Rota [Rot71], and Welsh [Wel76]. The same conjectures for the chromatic polynomials of graphs were given earlier by Read [Rea68] and Hoggar [Hog74]. We refer to [Whi87, Chapter 8] and [Oxl11, Chapter 15] for overviews and historical accounts. Brylawski [Bry82] conjectured the log-concavity of the h-vector of the broken circuit complex.[9] That the h-vector of the broken circuit complex is flawless was stated as an open problem in [Swa03] and reproduced in [JKL18] as a conjecture. We deduce all the above statements using the geometry of conormal fans.

[8] In [Mas72], Mason proposed a stronger conjecture that the f-vector of the independence complex of M satisfies

$$\frac{f_k^2}{\binom{n+1}{k}^2} \geq \frac{f_{k-1}}{\binom{n+1}{k-1}} \frac{f_{k+1}}{\binom{n+1}{k+1}} \text{ for all } k.$$

In [Bry82], Brylawski conjectures the same set of inequalities for the f-vector of the broken circuit complex of M. Mason's stronger conjecture was recently proved in [ALOGV18] and [BH18, BH20]. An extension of the same result to matroid quotients was obtained in [EH20].

[9] In [Bry82], Brylawski proposed a stronger conjecture that the h-vector of the broken circuit complex of M satisfies

$$\frac{h_k^2}{\binom{n-k}{n-r-1}^2} \geq \frac{h_{k-1}}{\binom{n-k+1}{n-r-1}} \frac{h_{k+1}}{\binom{n-k-1}{n-r-1}} \text{ for all } k.$$

Theorem 1.4. *Conjecture* 1.3 *holds.*

We prove the log-concavity of the h-vector of the broken circuit complex using Theorem 1.1. This log-concavity implies all other statements in Conjecture 1.3, thanks to the following known observations:

- For any simplicial complex Δ, the log-concavity of the h-vector implies the log-concavity of the f-vector [Bre94, Corollary 8.4].
- For any pure simplicial complex Δ, the f-vector of Δ is flawless. More generally, any pure O-sequence[10] is flawless [Hib89, Theorem 1.1].
- For any shellable simplicial complex Δ, the h-vector of Δ has no internal zeros, being an O-sequence [Sta77, Theorem 6]. Therefore, if the h-vector of Δ is log-concave, then it is unimodal.
- The broken circuit complex of M is the cone over the *reduced broken circuit complex* of M, and the two simplicial complexes share the same h-vector. The independence complex of M is the reduced broken circuit complex of another matroid, the *free dual extension* of M [Bry77, Theorem 4.2].
- If the h-vector of the broken circuit complex of M is unimodal for all M, then the h-vector of the broken circuit complex of M is flawless for all M [JKL18, Theorem 1.2].

Previous work. The log-concavity of the f-vector of the broken circuit complex was proved in [Huh12] for matroids representable over a field of characteristic 0. The result was extended to matroids representable over some field in [HK12] and to all matroids in [AHK18]. An alternative proof of the same fact using the volume polynomial of a matroid was obtained in [BES20]. It was observed in [Len13] that the log-concavity of the f-vector of the broken circuit complex implies that of the independence complex.

For matroids representable over a field of characteristic 0, the log-concavity of the h-vector of the broken circuit complex was proved in [Huh15]. The algebraic geometry behind the log-concavity of the h-vector, which became a model for the Lagrangian geometry of conormal fans in the present paper, was explored in [DGS12] and [Huh13]. In [JKL18], Juhnke-Kubitzke and Le used the result of [Huh15] to deduce that the h-vector of the broken circuit complex is flawless for matroids representable over a field of characteristic 0. The flawlessness of the h-vector of the independence complex was first proved by Chari using a combinatorial decomposition of the independence complex [Cha97]. The result was recovered by Swartz [Swa03] and Hausel [Hau05], who obtained stronger algebraic results. The other cases of Conjecture 1.3 remained open.

Our solution of Conjecture 1.3 was announced in [Ard18]. Shortly after this paper appeared on `arXiv:2004.13116`, Berget, Spink, and Tseng [BST20] announced an alternative proof of the log-concavity of the h-vector of the independence complex (Conjecture 1.3(2)). The relationship between our approach and theirs is still to be understood. The h-vector of the broken circuit complex (Conjecture 1.3(4)) is not currently accessible through the alternative method.

[10] A sequence of nonnegative integers $h_0, h_1, \ldots$ is an *O-sequence* if there is an order ideal of monomials $\mathcal{O}$ such that h_k is the number of degree k monomials in $\mathcal{O}$. The sequence is a *pure O-sequence* if the order ideal $\mathcal{O}$ can be chosen so that all the maximal monomials in $\mathcal{O}$ have the same degree. See [BMMR+12] for a comprehensive survey of pure O-sequences.

1.4. **Tropical Hodge theory.** Let us discuss in more detail the strategy of [AHK18] that led to the log-concavity of the f-vector of the broken circuit complex of M. For the moment, suppose that there is a linear subspace $V \subseteq \mathbb{C}^E$ representing M over $\mathbb{C}$, and consider the variety

$$Y_V = \text{the closure of } \mathbb{P}(V) \cap (\mathbb{C}^*)^E/\mathbb{C}^* \text{ in the toric variety of the permutohedron } X(\Sigma_E).^{11}$$

If nonempty, Y_V is an r-dimensional smooth projective complex variety which is, in fact, contained in the torus invariant open subset of $X(\Sigma_E)$ corresponding to the Bergman fan of M:

$$Y_V \subseteq X(\Sigma_{\mathrm{M}}) \subseteq X(\Sigma_E).$$

The work of Feichtner and Yuzvinsky [FY04], which builds upon the work of De Concini and Procesi [DCP95], reveals that the inclusion maps induce isomorphisms between integral cohomology and Chow rings:

$$H^{2\bullet}(Y_V, \mathbb{Z}) \simeq A^\bullet(Y_V, \mathbb{Z}) \simeq A^\bullet(X(\Sigma_{\mathrm{M}}), \mathbb{Z}).$$

As a result, the Chow ring of the n-dimensional variety $X(\Sigma_{\mathrm{M}})$ has the structure of the even part of the cohomology ring of an r-dimensional smooth projective variety. Remarkably, this structure on the Chow ring of $X(\Sigma_{\mathrm{M}})$ persists for any matroid M, even if M does not admit any representation over any field. In particular, the Chow ring of $X(\Sigma_{\mathrm{M}})$ satisfies the Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations [AHK18]. For a simpler proof of the three properties of the Chow ring, based on its semismall decomposition, see [BHM$^+$20].

For any simplicial fan Σ, let $A(\Sigma)$ be the ring of real-valued piecewise polynomial functions on Σ modulo the ideal of the linear functions on Σ, and let $\mathcal{K}(\Sigma)$ be the cone of *strictly convex* piecewise linear functions on Σ (Definition 5.1).

Definition 1.5. A d-dimensional simplicial fan Σ is *Lefschetz* if it satisfies the following.

(1) (Fundamental weight) The group of d-dimensional Minkowski weights on Σ is generated by a positive Minkowski weight w. We write deg for the corresponding linear isomorphism

$$\deg\colon A^d(\Sigma) \longrightarrow \mathbb{R}, \qquad \eta \longmapsto \eta \cap w.$$

(2) (Poincaré duality) For any $0 \le k \le d$, the bilinear map of the multiplication

$$A^k(\Sigma) \times A^{d-k}(\Sigma) \longrightarrow A^d(\Sigma) \xrightarrow{\deg} \mathbb{R}$$

is nondegenerate.

(3) (Hard Lefschetz property) For any $0 \le k \le \frac{d}{2}$ and any $\ell \in \mathcal{K}(\Sigma)$, the multiplication map

$$A^k(\Sigma) \longrightarrow A^{d-k}(\Sigma), \qquad \eta \longmapsto \ell^{d-2k}\eta$$

is a linear isomorphism.

(4) (Hodge–Riemann relations) For any $0 \le k \le \frac{d}{2}$ and any $\ell \in \mathcal{K}(\Sigma)$, the bilinear form

$$A^k(\Sigma) \times A^k(\Sigma) \longrightarrow \mathbb{R}, \qquad (\eta_1, \eta_2) \longmapsto (-1)^k \deg(\ell^{d-2k}\eta_1\eta_2)$$

is positive definite when restricted to the kernel of the multiplication map ℓ^{d-2k+1}.

(5) (Hereditary property) For any $0 < k \leq d$ and any k-dimensional cone σ in Σ, the star of σ in Σ is a Lefschetz fan of dimension $d - k$.

The Hodge–Riemann relations give analogs of the Alexandrov–Fenchel inequality among degrees of products of convex piecewise linear functions $\ell_1, \ell_2, \ldots, \ell_d$ on Σ:

$$\deg(\ell_1\ell_2\ell_3\cdots\ell_d)^2 \geq \deg(\ell_1\ell_1\ell_3\cdots\ell_d)\deg(\ell_2\ell_2\ell_3\cdots\ell_d).$$

The Bergman fan of a matroid M is Lefschetz, and the log-concavity of the f-vector of the broken circuit complex of M follows from the Hodge–Riemann relations for the Bergman fan of M [AHK18].

We establish the log-concavity of the h-vector of the broken circuit complex of M in the same way, using the conormal fan of M in place of the Bergman fan of M. Theorem 1.2 relates the intersection theory of the conormal fan of M to the h-vector of the broken circuit complex of M via the Chern-Schwartz-MacPherson cycles of M. In order to proceed, we need to show that the conormal fan of M is Lefschetz. We obtain this from the following general result.

Theorem 1.6. *Let Σ_1 and Σ_2 be simplicial fans that have the same support $|\Sigma_1| = |\Sigma_2|$. If $\mathcal{K}(\Sigma_1)$ and $\mathcal{K}(\Sigma_2)$ are nonempty, then Σ_1 is Lefschetz if and only if Σ_2 is Lefschetz.*

Theorem 1.6 implies, for example, that the reduced normal fan of any simple polytope is Lefschetz, because the reduced normal fan of the standard simplex is Lefschetz.[12] In the context of matroid theory, Theorem 1.6 implies that the conormal fan of M is Lefschetz, because the Bergman fans of M and $\mathrm{M}^\perp$ are Lefschetz and the product of Lefschetz fans is Lefschetz. When $\mathcal{K}(\Sigma)$ is empty, the hard Lefschetz property and the Hodge–Riemann relations for Σ hold vacuously. The proof of Theorem 1.6 shows that, if two simplicial fans Σ_1 and Σ_2 have the same support $|\Sigma_1| = |\Sigma_2|$, then Σ_1 satisfies Poincaré duality if and only if Σ_2 satisfies Poincaré duality.

2. The bipermutohedral fan

Let E be a finite set of cardinality $n + 1$. For notational convenience, we often identify E with the set of nonnegative integers at most n. As before, we let N_E be the n-dimensional space

$$\mathrm{N}_E = \mathbb{R}^E/\mathbb{R}\mathbf{e}_E, \qquad \mathbf{e}_E = \sum_{i\in E}\mathbf{e}_i.$$

We write $\mathrm{N}_{E,E}$ for the $2n$-dimensional space $\mathrm{N}_E \oplus \mathrm{N}_E$, and μ for the addition map

$$\mu\colon \mathrm{N}_{E,E} \longrightarrow \mathrm{N}_E, \qquad (z, w) \longmapsto z + w.$$

Throughout the paper, all fans in N_E will be rational with respect to the lattice $\mathbb{Z}^E/\mathbb{Z}\mathbf{e}_E$, and all fans in $\mathrm{N}_{E,E}$ will be rational with respect to the lattice $\mathbb{Z}^E/\mathbb{Z}\mathbf{e}_E \oplus \mathbb{Z}^E/\mathbb{Z}\mathbf{e}_E$. We follow [CLS11] when using the terms *fan* and *generalized fan*: A generalized fan is a fan if and only if each of its cones is strongly convex. The

[12] McMullen gave an elementary proof of this fact in [McM93]. See [Tim99] and [FK10] for alternative presentations. Our proof of Theorem 1.6 is modeled on these arguments. Theorem 1.6 gives another proof of the necessity of McMullen's bounds [McM93] on the face numbers of simplicial polytopes. In the context of matroid theory, the authors of [GS21] used a similar argument to show that any unimodular fan whose support is a tropical linear space satisfies Poincaré duality.

notion of morphism of fans is extended to morphism of generalized fans in the obvious way. For any subset S of E, we write $\mathbf{e}_S$ and $\mathbf{f}_S$ for the vectors

$$\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i, \qquad \mathbf{f}_S = \sum_{i \in S} \mathbf{f}_i,$$

where $\mathbf{e}_i$ are the standard basis vectors of $\mathbb{R}^E$ defining the first summand of $\mathrm{N}_{E,E}$ and $\mathbf{f}_i$ are the standard basis vectors of $\mathbb{R}^E$ defining the second summand of $\mathrm{N}_{E,E}$.

In this section, we construct a complete simplicial fan $\Sigma_{E,E}$ in $\mathrm{N}_{E,E}$. We offer five equivalent descriptions; each one of them will play a role for us. We call it the *bipermutohedral fan* because it is the normal fan of a polytope which we call the *bipermutohedron*. Before we begin defining the bipermutohedral fan $\Sigma_{E,E}$ in $\mathrm{N}_{E,E}$, we recall some basic facts on the permutohedral fan Σ_E in N_E.

2.1. **The normal fan of the simplex.** Consider the standard n-dimensional simplex

$$\mathrm{conv}\{\mathbf{e}_i\}_{i \in E} \subseteq \mathbb{R}^E.$$

Its normal fan in $\mathbb{R}^E$ has a lineality space spanned by $\mathbf{e}_E$. For any convex polytope, we call the quotient of the normal fan by its lineality space the *reduced normal fan* of the polytope.[13] For example, the reduced normal fan of the standard simplex, denoted Γ_E, is the complete fan in N_E with the cones

$$\sigma_S := \mathrm{cone}\{\mathbf{e}_i\}_{i \in S} \subseteq \mathrm{N}_E, \ \text{for every proper subset } S \text{ of } E.$$

The cone σ_S consists of the points $z \in \mathrm{N}_E$ such that $\min_{i \in E} z_i = z_s$ for all s not in S. For each element j of E, the function $\alpha_j = \max_{i \in E}\{z_j - z_i\}$ is piecewise linear on the fan Γ_E. These piecewise linear functions are equal to each other modulo global linear functions on N_E, and we write α for the common equivalence class of α_j.

2.2. **The normal fan of the permutohedron.** Let Π_E be the n-dimensional permutohedron

$$\mathrm{conv}\Big\{(x_0, x_1, \ldots, x_n) \mid x_0, x_1, \ldots, x_n \text{ is a permutation of } 0, 1, \ldots, n\Big\} \subseteq \mathbb{R}^E.$$

The *permutohedral fan* Σ_E, also known as the *braid fan* or the *type A Coxeter complex*, is the reduced normal fan of the permutohedron Π_E. It is the complete simplicial fan in N_E whose chambers are separated by the n-dimensional *braid arrangement*, the real hyperplane arrangement in N_E consisting of the $\binom{n+1}{2}$ hyperplanes

$$z_i = z_j, \ \text{for distinct elements } i \text{ and } j \text{ of } E.$$

The face of the permutohedral fan containing a given point z in its relative interior is determined by the relative order of its homogeneous coordinates $(z_0, \ldots, z_n)$. Therefore, the faces of the permutohedral fan correspond to the ordered set partitions

$$\mathcal{P} = (E = P_1 \sqcup \cdots \sqcup P_{k+1}),$$

[13]The normal fan of a convex polytope in a real vector space is a generalized fan in the dual space whose face poset is anti-isomorphic to the face poset of the polytope. Unlike the reduced normal fan, the normal fan of a polytope need not be a fan. We trust that the use of the term "normal fan" will cause no confusion.

which are in bijection with the strictly increasing sequences of nonempty proper subsets

$$\mathcal{S} = (\varnothing \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E), \qquad S_m = \bigcup_{\ell=1}^{m} P_\ell.$$

The collection of ordered set partitions of E forms a poset under *adjacent refinement*, where $\mathcal{P} \leq \mathcal{P}'$ if $\mathcal{P}$ can be obtained from $\mathcal{P}'$ by merging adjacent parts.

Proposition 2.1. *The face poset of the permutohedral fan Σ_E is isomorphic to the poset of ordered set partitions of E.*

Thus the permutohedral fan has $2(2^n - 1)$ rays corresponding to the nonempty proper subsets of E and $(n+1)!$ chambers corresponding to the permutations of E.

We now describe the permutohedral fan in terms of its rays. Two subsets S and S' of E are said to be *comparable* if

$$S \subseteq S' \text{ or } S \supseteq S'.$$

A *flag* in E is a set of pairwise comparable subsets of E. For any flag $\mathcal{S}$ of subsets of E, we define

$$\sigma_{\mathcal{S}} = \text{cone}\{\mathbf{e}_S\}_{S \in \mathcal{S}} \subseteq \mathrm{N}_E\,.$$

We identify a flag in E with the strictly increasing sequence obtained by ordering the subsets in the flag.

Proposition 2.2. *The permutohedral fan Σ_E is the complete fan in N_E with the cones*

$$\sigma_{\mathcal{S}} = \text{cone}\{\mathbf{e}_S\}_{S \in \mathcal{S}}, \ \textit{where } \mathcal{S} \textit{ is a flag of nonempty proper subsets of } E.$$

For example, the cone corresponding to the ordered set partition 25|013|4 is

$$\text{cone}(\mathbf{e}_{25}, \mathbf{e}_{01235}) = \{z \in \mathrm{N}_E \mid z_2 = z_5 \geq z_0 = z_1 = z_3 \geq z_4\}.$$

Proposition 2.2 shows that the permutohedral fan is a *unimodular fan*: The set of primitive ray generators in any cone in Σ_E is a subset of a basis of the free abelian group $\mathbb{Z}^E/\mathbb{Z}$. It also shows that the permutohedral fan is a refinement of the fan Γ_E in Section 2.1.

It will be useful to view the permutohedral fan as a configuration space as follows. Regard N_E as the space of E-tuples of points $(p_0, \ldots, p_n)$ moving in the real line, modulo simultaneous translation:

$$p = (p_0, \ldots, p_n) = (p_0 + \lambda, \ldots, p_n + \lambda) \text{ for any } \lambda \in \mathbb{R}.$$

The *ordered set partition of* p, denoted $\pi(p)$, is obtained by reading the labels of the points in the real line from right to left, as shown in Figure 1. This model gives the permutohedral fan Σ_E the following geometric interpretation.

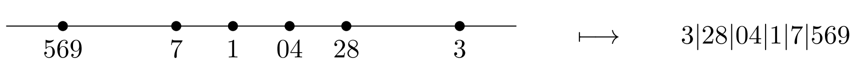


FIGURE 1. An E-tuple of points p and its ordered set partition $\pi(p) = 3|28|04|1|7|569$

Proposition 2.3. *The permutohedral fan Σ_E is the configuration space of E-tuples of points in the real line modulo simultaneous translation, stratified according to their ordered set partition.*

In Section 2.4, we give an analogous description of the bipermutohedral fan $\Sigma_{E,E}$ as a configuration space of E-tuples of points in the real plane.

2.3. **The bipermutohedral fan as a subdivision.** Denote a point in $\mathrm{N}_{E,E}$ by (z, w). We construct the bipermutohedral fan $\Sigma_{E,E}$ in $\mathrm{N}_{E,E}$ as follows.

First, we subdivide $\mathrm{N}_{E,E}$ into the *charts* $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_n$, where $\mathcal{C}_k$ is the cone

$$\mathcal{C}_k = \Big\{(z,w) \mid \min_{i\in E}(z_i + w_i) = z_k + w_k\Big\}.$$

These form the chambers of a complete generalized fan in $\mathrm{N}_{E,E}$, denoted Δ_E. The chamber $\mathcal{C}_k$ is the inverse image of the cone σ_{E-k} under the addition map, and hence Δ_E is the coarsest complete generalized fan in $\mathrm{N}_{E,E}$ for which the addition map is a morphism to the fan Γ_E in Section 2.1. To each chart $\mathcal{C}_k$ we associate the linear functions

$$Z_i = z_i - z_k, \quad W_i = -w_i + w_k, \text{ for every } i \text{ in } E.$$

Omitting the zero function $Z_k = W_k$, we obtain a coordinate system (Z, W) for $\mathrm{N}_{E,E}$ such that

$$\mathcal{C}_k = \Big\{(Z,W) \mid Z_i \geq W_i \text{ for every } i \text{ in } E\Big\}.$$

This coordinate system depends on k, but we will drop k from the notation for better readability.

Second, we consider the subdivision Σ_k of the cone $\mathcal{C}_k$ obtained from the braid arrangement of $\binom{2n+1}{2}$ hyperplanes

$$Z_a = Z_b, \quad W_a = W_b, \quad Z_a = W_b, \quad \text{for all } a \text{ and } b \text{ in } E.$$

Note that the arrangement contains the n hyperplanes that cut out $\mathcal{C}_k$ in $\mathrm{N}_{E,E}$. One may view the subdivision Σ_k of $\mathcal{C}_k$ as a copy of $1/2^n$-th of the $2n$-dimensional permutohedral fan, namely, the part of the permutohedral fan in $2n+1$ variables $Z_0, W_0, \ldots, Z_k = W_k, \ldots, Z_n, W_n$ where $Z_i \geq W_i$ for every $i \neq k$. Figure 4 illustrates Σ_0 and Σ_1 when $n = 1$.

Proposition 2.4. *The union of the fans Σ_i for $i \in E$ is a fan in $\mathrm{N}_{E,E}$. We call it the* bipermutohedral fan *$\Sigma_{E,E}$.*

Proof. To check that $\Sigma_{E,E}$ is indeed a fan, we need to check that the fans Σ_i glue compatibly along the boundaries of $\mathcal{C}_i$. For this, we verify that Σ_i and Σ_j induce the same subdivision on $\mathcal{C}_i \cap \mathcal{C}_j$ for all $i \neq j$.

Consider the system of linear functions (Z, W) for $\mathcal{C}_i$ and the system of linear functions (Z', W') for $\mathcal{C}_j$. It is straightforward to check that, for any point in $\mathrm{N}_{E,E}$, we have

$$Z_a - Z_b = Z'_a - Z'_b \text{ and } W_a - W_b = W'_a - W'_b \text{ for all } a \text{ and } b \text{ in } E.$$

Furthermore, on the intersection of $\mathcal{C}_i$ and $\mathcal{C}_j$, where $z_i + w_i = z_j + w_j$, we have

$$Z_a - W_b = (z_a - z_i) - (w_i - w_b) = (z_a - z_j) - (w_j - w_b) = Z'_a - W'_b.$$

Thus the hyperplanes separating the chambers of Σ_i and Σ_j have the same intersections with $\mathcal{C}_i \cap \mathcal{C}_j$. □

The following subfan of the bipermutohedral fan will serve as a guide toward Theorem 1.1.

Definition 2.5. The *cotangent fan* Ω_E is the union of the fans $\Sigma_i \cap \Sigma_j$ for $i \neq j \in E$.

In other words, Ω_E is the subfan of $\Sigma_{E,E}$ whose support is the tropical hypersurface

$$\text{trop}(\delta) = \left\{ (z, w) \mid \min_{i \in E}(z_i + w_i) \text{ is achieved at least twice} \right\} \subseteq \mathrm{N}_{E,E} .$$

In Section 3.4, we show that the cotangent fan contains the conormal fan of any matroid on E.

2.4. **The bipermutohedral fan as a configuration space.** It will be useful to view the bipermutohedral fan $\Sigma_{E,E}$ as a configuration space as follows. Regard $\mathrm{N}_{E,E}$ as the space of E-tuples of points $(p_0, \dots, p_n)$ moving in the real plane, modulo simultaneous translation:

$$(p_0, \dots, p_n) = (p_0 + \lambda, \dots, p_n + \lambda) \text{ for any } \lambda \in \mathbb{R}^2.$$

The point (z, w) in $\mathrm{N}_{E,E}$ corresponds to the points $p_i = (z_i, w_i)$ in $\mathbb{R}^2$ for i in E.

Definition 2.6. A *bisequence* on E is a sequence $\mathcal{B}$ of nonempty subsets of E, called the *parts* of $\mathcal{B}$, such that

(1) every element of E appears in at least one part of $\mathcal{B}$,
(2) every element of E appears in at most two parts of $\mathcal{B}$, and
(3) some element of E appears in exactly one part of $\mathcal{B}$.

The *trivial bisequence* on E is the bisequence with exactly one part E. A *bisubset* of E is a nontrivial bisequence on E of minimal length 2. A *bipermutation* of E is a bisequence on E of maximal length $2n + 1$.

We will write bisequences by listing the elements of its parts, separated by vertical bars. For example, the bisequence $\{2\}, \{0, 1\}, \{1\}, \{2\}$ on $\{0, 1, 2\}$ will be written $2|01|1|2$.

Definition 2.7. Let $p = (p_0, \dots, p_n)$ be an E-tuple of points in $\mathbb{R}^2$.

(1) The *supporting line* of p, denoted $\ell(p)$, is the lowest line of slope -1 containing a point in p.
(2) For each point p_i, the vertical and horizontal projections of p_i onto $\ell(p)$ will be labeled i.
(3) The *bisequence of* p, denoted $\mathcal{B}(p)$, is obtained by reading the labels on $\ell(p)$ from right to left.

See Figure 2 for an illustration of Definition 2.7.

Remark 2.8. One can recover any configuration p from their projections onto the supporting line $\ell(p)$ and their labels. Therefore, modulo translations, we may also consider p as a configuration of $2n + 2$ points on the real line labeled 0, 0, 1, 1, $\dots$, n, n such that at least one pair of points with the same label coincide. This is illustrated at the bottom of Figure 2.

This model gives the bipermutohedral fan $\Sigma_{E,E}$ the following geometric interpretation.

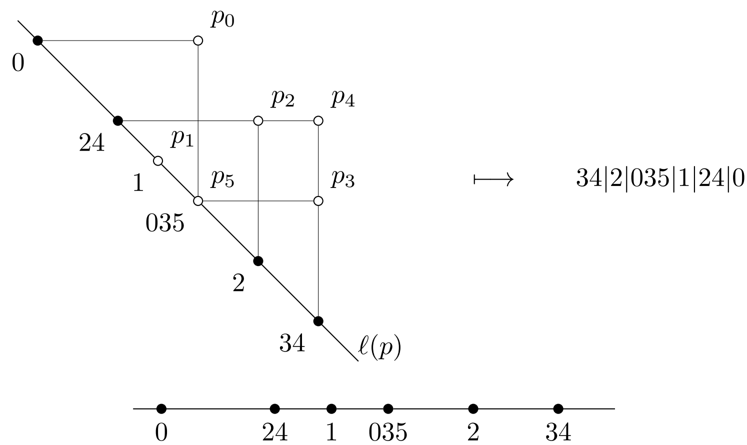


FIGURE 2. An E-tuple of points $p = (p_0, \ldots, p_5)$ in the plane, their vertical and horizontal projections onto the supporting line $\ell(p)$, and the bisequence $\mathcal{B}(p)$

Proposition 2.9. *The bipermutohedral fan $\Sigma_{E,E}$ is the configuration space of E-tuples of points in the real plane modulo simultaneous translation, stratified according to their bisequence.*

Proof. Consider a point (z, w) in $\mathrm{N}_{E,E}$ and the associated configuration of points p_i in the plane. The chart $\mathcal{C}_k$ consists of configurations p where k appears exactly once in the bisequence $\mathcal{B}(p)$. In other words, p is in $\mathcal{C}_k$ if and only if p_k is on the supporting line $\ell(p)$. We consider the system of linear functions (Z, W) for $\mathcal{C}_k$ discussed in Section 2.3. The cones in the subdivision Σ_k of $\mathcal{C}_k$ encode the relative order of $Z_0, \ldots, Z_n, W_0, \ldots, W_n$, where

$$Z_k = W_k = 0 \text{ and } Z_i \geq W_i \text{ for every } i \text{ in } E.$$

On the other hand, the bisequence $\mathcal{B}(p)$ keeps track of the relative order of the vertical and horizontal projections of p_i onto $\ell(p)$. As shown in Figure 3, after the translation by $(-z_k, -w_k)$, the vertical and horizontal projections of p_i onto $\ell(p)$ are

$$(z_i, z_k+w_k-z_i)-(z_k, w_k) = (Z_i, -Z_i) \text{ and } (z_k+w_k-w_i, w_i)-(z_k, w_k) = (W_i, -W_i).$$

Their relative order along $\ell(p)$ is given by the relative order of $Z_0, \ldots, Z_n, W_0, \ldots, W_n$. □

The collection of bisequences on E forms a poset under *adjacent refinement*, where $\mathcal{B} \leq \mathcal{B}'$ if $\mathcal{B}$ can be obtained from $\mathcal{B}'$ by merging adjacent parts. The poset of bisequences on E is a graded poset. Its k-th level consists of the bisequences of $k+1$ nonempty subsets of E, and the top level consists of the bipermutations of E.

Proposition 2.10. *The face poset of the bipermutohedral fan $\Sigma_{E,E}$ is isomorphic to the poset of bisequences on E.*

Proof. Remark 2.8 shows that, given any bisequence $\mathcal{B}$ on E, there is a configuration p with $\mathcal{B}(p) = \mathcal{B}$. Thus, by Proposition 2.9, the cones in $\Sigma_{E,E}$ are in bijection with

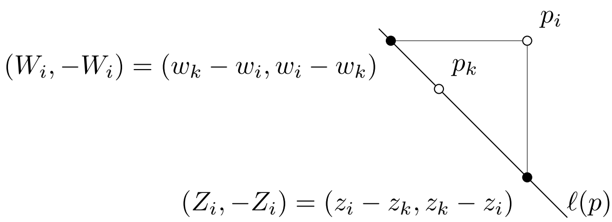


FIGURE 3. The vertical and horizontal projections of p_i onto the supporting line $\ell(p)$, after the translation by $(-z_k, -w_k)$

the bisequences on E. If a configuration p moves into more special position, then some adjacent parts of $\mathcal{B}(p)$ merge. □

For a bisequence $\mathcal{B}$ on E, we write $\sigma_{\mathcal{B}}$ for the corresponding cone defined by

$$\sigma_{\mathcal{B}} = \text{closure}\Big\{\text{configurations } p \text{ satisfying } \mathcal{B}(p) = \mathcal{B}\Big\} \subseteq \mathrm{N}_{E,E}\,.$$

In terms of the cones $\sigma_{\mathcal{B}}$, the fan Σ_i subdividing the chart $\mathcal{C}_i$ can be described as the subfan

$$\Sigma_i = \{\sigma_{\mathcal{B}} \mid i \text{ appears exactly once in the bisequence } \mathcal{B}\} \subseteq \Sigma_{E,E}.$$

See Figure 4 for an illustration of Proposition 2.10 when $n = 1$.

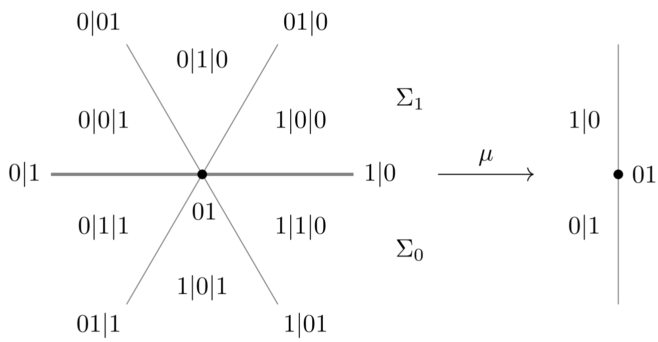


FIGURE 4. The map $\mu\colon \Sigma_{\{0,1\},\{0,1\}} \to \Sigma_{\{0,1\}}$ from the bipermutohedral fan to the permutohedral fan, and the labeling of their cones with bisequences on $\{0,1\}$ and ordered set partitions on $\{0,1\}$

2.5. **The bipermutohedral fan as a common refinement.** The importance of the bipermutohedral fan $\Sigma_{E,E}$ stems from its relationship with the normal fan Γ_E of the standard simplex and the permutohedral fan Σ_E described in Sections 2.1 and 2.2. Recall that a *morphism* from a fan Σ_1 in N_1 to a fan Σ_2 in N_2 is an integral linear map from N_1 to N_2 that maps any cone in Σ_1 into a cone in Σ_2.

Proposition 2.11. *The bipermutohedral fan $\Sigma_{E,E}$ has the following properties.*

(1) *The projections $\pi(z,w) = z$ and $\overline{\pi}(z,w) = w$ are morphisms of fans from $\Sigma_{E,E}$ to Σ_E.*

(2) *The addition map* $\mu(z,w) = z+w$ *is a morphism of fans from* $\Sigma_{E,E}$ *to* Γ_E.

Proof. That $\Sigma_{E,E}$ has the stated properties follows from the interpretation of Σ_E and $\Sigma_{E,E}$ as configuration spaces, as we now explain. Suppose (z,w) is a point in $\mathrm{N}_{E,E}$ and p is the corresponding E-tuple of points in $\mathbb{R}^2$ modulo simultaneous translation, with corresponding bisequence $\mathcal{B}(p)$. Then the smallest cone of Γ_E containing $z+w$ is given by the entries that appear twice in $\mathcal{B}(p)$. The ordered set partition of z in N_E is given by the first occurrence of each i in $\mathcal{B}(p)$. Similarly, the ordered set partition of w in N_E is given by the order of the last occurrence of each i in $\mathcal{B}(p)$. For example, if a point (z,w) has the bisequence $34|2|035|1|24|0$, as in Figure 2, then the sum $z+w$ is in the cone of 0234 in Γ_E, the first projection z is in the cone of $34|2|05|1$ in Σ_E, and the second projection w is in the cone of $0|24|1|35$ in Σ_E. □

We note, however, that the bipermutohedral fan is not the coarsest fan structure for which projections and addition are morphisms: the reader is referred to the discussion in Section 2.8.

2.6. **The bipermutohedral fan in terms of its rays and cones.** The rays of the bipermutohedral fan $\Sigma_{E,E}$ correspond to the bisubsets of E. In other words, the rays of $\Sigma_{E,E}$ correspond to the ordered pairs of nonempty subsets $S|T$ of E such that

$$S \cup T = E \text{ and } S \cap T \neq E.$$

Proposition 2.12. *The* $3(3^n-1)$ *rays of the bipermutohedral fan* $\Sigma_{E,E}$ *are generated by*

$$\mathbf{e}_{S|T} := \mathbf{e}_S + \mathbf{f}_T, \text{ where } S|T \text{ is a bisubset of } E.$$

Proof. The configuration p corresponding to $\mathbf{e}_{S|T}$ has points with labels in $S\cap T$ located at $(1,1)$, the points with labels in $S-T$ located at $(1,0)$, and the points with labels in $T-S$ located at $(0,1)$. The bisequence of p is indeed $S|T$, and hence the conclusion follows from Proposition 2.9. □

Proposition 2.13. *The bipermutohedral fan* $\Sigma_{E,E}$ *has* $(2n+2)!/2^{n+1}$ *chambers.*

Proof. By Proposition 2.10, the chambers correspond to the bipermutations. These are obtained bijectively from the $(2n+2)!/2^{n+1}$ permutations of the multiset $\{0,0,\dots,n,n\}$ by dropping the last letter in the one-line notation for permutations. For example, the bipermutation $1|0|1|2|3|0|3$ corresponds to the permutation 10123032 of $\{0,0,1,1,2,2,3,3\}$. □

It is worth understanding Proposition 2.13 in a different way. Recall that the bipermutohedral fan is obtained by gluing copies of $1/2^n$-th of the $2n$-dimensional permutohedral fan. There are $(n+1)$ such copies, and each copy contains $(2n+1)!/2^n$ chambers, producing the total of $(2n+2)!/2^{n+1}$ chambers. This viewpoint explains why Figure 4 deceivingly looks like a permutohedral fan: For $n=1$, the bipermutohedral fan consists of two glued copies of half of the permutohedral fan.

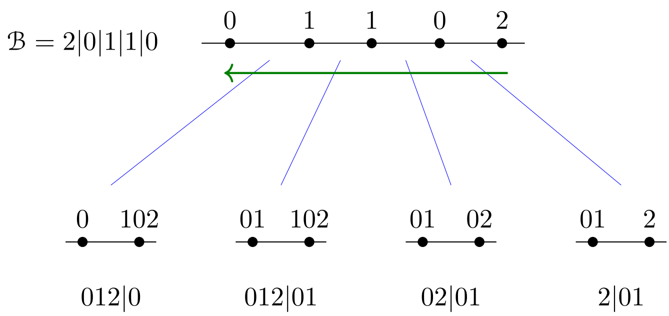
We now describe the cones in the bipermutohedral fan in terms of their generating rays. Let $\mathcal{B} = B_0|B_1|\cdots|B_k$ be a bisequence on E. Propositions 2.10 and 2.12 show that the rays of the k-dimensional cone $\sigma_{\mathcal{B}}$ are generated by the vectors

$$\mathbf{e}_{S_1|T_1},\dots,\mathbf{e}_{S_k|T_k}, \text{ where } S_i = \bigcup_{j=0}^{i-1} B_j \text{ and } T_i = \bigcup_{j=i}^{k} B_j.$$

See Figure 5 for an illustration. We use the following table to record the rays of $\sigma_{\mathcal{B}}$:

$\varnothing$	$\subsetneq$	S_1	$\subseteq$	S_2	$\subseteq$	$\cdots$	$\subseteq$	S_k	$\subseteq$	E
E	$\supseteq$	T_1	$\supseteq$	T_2	$\supseteq$	$\cdots$	$\supseteq$	T_k	$\supsetneq$	$\varnothing$

For each index j such that $S_j \subsetneq S_{j+1}$ and $T_j \supsetneq T_{j+1}$, we mark those two strict inclusions in bold. We write $\mathcal{S}(\mathcal{B})|\mathcal{T}(\mathcal{B})$ for the collection of bisubsets $S_i|T_i$ constructed from $\mathcal{B}$ as above by merging adjacent parts. For convenience, we also refer to the pairs n$S_0|T_0 = \varnothing|E$ and $S_{k+1}|T_{k+1} = E|\varnothing$.



$\varnothing$	$\subsetneq$	2	$\subseteq$	02	$\subseteq$	012	$\subseteq$	012	$\subseteq$	E
E	$\supseteq$	01	$\supseteq$	01	$\supseteq$	01	$\supseteq$	0	$\supsetneq$	$\varnothing$

FIGURE 5. The cone of $2|0|1|1|0$ has the rays generated by $\mathbf{e}_{2|01}, \mathbf{e}_{02|01}, \mathbf{e}_{012|01}, \mathbf{e}_{012|0}$

Conversely, we may ask which subsets of k rays in $\Sigma_{E,E}$ generate a k-dimensional cone in $\Sigma_{E,E}$. To answer this question, we introduce the notion of a flag of bisubsets.

Definition 2.14. We say that two bisubsets $S|T$ and $S'|T'$ of E are *comparable* if

$$(S \subseteq S' \text{ and } T \supseteq T') \text{ or } (S \supseteq S' \text{ and } T \subseteq T').$$

A *flag of bisubsets* in E, or a *biflag* in E, is a set $\mathcal{S}|\mathcal{T}$ of pairwise comparable bisubsets of E satisfying

$$\bigcup_{S|T \in \mathcal{S}|\mathcal{T}} S \cap T \neq E.$$

The *length* of a biflag is the number of bisubsets in it.

We have the following useful alternative characterization of biflags in E.

Proposition 2.15. *Let $\mathcal{S}$ be an increasing sequence of k nonempty subsets of E, say*

$$\mathcal{S} = (\varnothing \subsetneq S_1 \subseteq \cdots \subseteq S_k \subseteq E),$$

and let $\mathcal{T}$ be a decreasing sequence of k nonempty subsets of E, say

$$\mathcal{T} = (E \supseteq T_1 \supseteq \cdots \supseteq T_k \supsetneq \varnothing).$$

Then the set $\mathcal{S}|\mathcal{T}$ consisting of the pairs $S_1|T_1, \dots, S_k|T_k$ is a flag of bisubsets if and only if

$$S_j \cup T_j = E \text{ for every } 1 \leq j \leq k \text{ and } S_j \cup T_{j+1} \neq E \text{ for some } 0 \leq j \leq k.$$

Proof. If $\mathcal{S}|\mathcal{T}$ is a biflag in E, then each $S_j|T_j$ is a bisubset of E, and hence $S_j \cup T_j = E$ for all j. Now let e be an element not in the union of all $S_j \cap T_j$, and consider the largest index i for which $e \notin S_i$. Then $e \in S_{i+1}$, which implies $e \notin T_{i+1}$ by the definition of e. Therefore, $S_i \cup T_{i+1} \neq E$.

Conversely, if $\mathcal{S}$ and $\mathcal{T}$ satisfy the stated conditions, then the pairs $S_j|T_j$ form a set of pairwise comparable bisubsets of E. If e is an element not in $S_j \cup T_{j+1}$ for some index j, then e is not in S_k for all indices $k \leq j$ and e is not in T_k for all indices $k > j$. Therefore, e is not in the union of all $S_k \cap T_k$, as desired. □

Since $S_j \cup T_{j+1} \neq E$ implies $S_j \subsetneq S_{j+1}$ and $T_j \supsetneq T_{j+1}$, Proposition 2.15 shows that the table of any biflag has at least one pair of strict inclusions marked in bold.

For a biflag $\mathcal{S}|\mathcal{T}$ of length k, we write $\mathcal{S}$ for the increasing sequence of k nonempty subsets

$$\mathcal{S} = (\varnothing \subsetneq S_1 \subseteq \cdots \subseteq S_k \subseteq E), \text{ where } S_j \text{ are the first parts of the bisubsets in } \mathcal{S}|\mathcal{T},$$

and write $\mathcal{T}$ for the decreasing sequence of k nonempty subsets

$$\mathcal{T} = (E \supseteq T_1 \supseteq \cdots \supseteq T_k \supsetneq \varnothing), \text{ where } T_j \text{ are the second parts of the bisubsets in } \mathcal{S}|\mathcal{T}.$$

We use $\mathcal{S}$ and $\mathcal{T}$ to define $\mathcal{B}(\mathcal{S}|\mathcal{T})$ as the sequence of $k+1$ nonempty sets

$$B_0|B_1|\cdots|B_k, \text{ where } B_j = (S_{j+1} - S_j) \cup (T_j - T_{j+1}).$$

The above construction gives an isomorphism between the poset of bisequences under adjacent refinement and the poset of biflags under inclusion.

Proposition 2.16. *The bisequences on E are in bijection with the biflags in E. More precisely,*

(1) *if $\mathcal{B}$ is a bisequence on E, then $\mathcal{S}(\mathcal{B})|\mathcal{T}(\mathcal{B})$ is a biflag in E,*
(2) *if $\mathcal{S}|\mathcal{T}$ is a biflag in E, then $\mathcal{B}(\mathcal{S}|\mathcal{T})$ is a bisequence on E, and*
(3) *the constructions $\mathcal{S}(\mathcal{B})|\mathcal{T}(\mathcal{B})$ and $\mathcal{B}(\mathcal{S}|\mathcal{T})$ are inverses to each other.*

Note that a bisubset $S|T$ corresponds to the biflag $\{S|T\}$ under the above bijection. For simplicity, we use the two symbols interchangeably.

Proof. Let $\mathcal{B}$ be a bisequence on E. Since every element of E appears at least once in $\mathcal{B}$, the increasing flag $\mathcal{S}(\mathcal{B})$ and the decreasing flag $\mathcal{T}(\mathcal{B})$ satisfy $S_j \cup T_j = E$ for all j. In addition, since some element of E appears exactly once in $\mathcal{B}$, say in B_j, we have $S_j \cup T_{j+1} \neq E$ for some j. Therefore, by Proposition 2.15, the pair $\mathcal{S}(\mathcal{B})|\mathcal{T}(\mathcal{B})$ is a biflag in E.

Conversely, let $\mathcal{S}|\mathcal{T}$ be a biflag in E. Since $S_1|T_1, \ldots, S_k|T_k$ are pairwise distinct, B_j must be nonempty for all j. Clearly, every element in E must appear in B_j for some j. In addition, each element e in E can occur at most twice in $\mathcal{B}(\mathcal{S}|\mathcal{T})$, namely, in the parts B_a and B_b whose indices satisfy $e \in S_{a+1} - S_a$ and $e \in T_b - T_{b+1}$. Furthermore, by Proposition 2.15, there is an element e not in $S_c \cup T_{c+1}$ for some index c, and in this case we must have $a = b = c$. That element e can occur only in the part B_a of $\mathcal{B}(\mathcal{S}|\mathcal{T})$, and hence $\mathcal{B}(\mathcal{S}|\mathcal{T})$ indeed is a bisequence.

It is straightforward to check that the constructions $\mathcal{S}(\mathcal{B})|\mathcal{T}(\mathcal{B})$ and $\mathcal{B}(\mathcal{S}|\mathcal{T})$ are inverses to each other. □

We identify a biflag $\mathcal{S}|\mathcal{T}$ in E with the sequence of bisubsets of E obtained by ordering the bisubsets in $\mathcal{S}|\mathcal{T}$ as above. For any sequence $\mathcal{S}|\mathcal{T}$ of bisubsets of E, we define

$$\sigma_{\mathcal{S}|\mathcal{T}} = \operatorname{cone}\{\mathbf{e}_{S|T}\}_{S|T \in \mathcal{S}|\mathcal{T}} \subseteq \mathrm{N}_{E,E}\,.$$

Thus, for any bisequence $\mathcal{B}$ on E, we have $\sigma_{\mathcal{B}} = \sigma_{\mathcal{S}(\mathcal{B})|\mathcal{T}(\mathcal{B})}$.

Corollary 2.17. *The bipermutohedral fan $\Sigma_{E,E}$ is the complete fan in $\mathrm{N}_{E,E}$ with the cones*

$$\sigma_{\mathcal{S}|\mathcal{T}} = \mathrm{cone}\{\mathbf{e}_{S|T}\}_{S|T \in \mathcal{S}|\mathcal{T}}, \text{ for flags of bisubsets } \mathcal{S}|\mathcal{T} \text{ of } E.$$

Proof. The statement is straightforward, given Propositions 2.10 and 2.16. □

Corollary 2.17 can be used to show that the bipermutohedral fan is a unimodular fan.[14]

Proposition 2.18. *The set of primitive ray generators of any chamber of $\Sigma_{E,E}$ is a basis of the free abelian group $\mathbb{Z}^E/\mathbb{Z}\mathbf{e}_E \oplus \mathbb{Z}^E/\mathbb{Z}\mathbf{f}_E$. In particular, $\Sigma_{E,E}$ is simplicial.*

Proof. Let $\mathcal{S} = \mathcal{S}(\mathcal{B})$ and $\mathcal{T} = \mathcal{T}(\mathcal{B})$ for a bipermutation $\mathcal{B}$ of E. If 0 is the unique element of E that appears exactly once in $\mathcal{B}$, then

$$\Big\{\mathbf{e}_{S_{j+1}|T_{j+1}} - \mathbf{e}_{S_j|T_j} \mid 0 \text{ is contained in } S_j \cup T_{j+1}\Big\} = \Big\{\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{f}_1, \ldots, \mathbf{f}_n\Big\}.$$

Therefore, the set of $2n$ primitive ray generators of $\sigma_{\mathcal{B}}$ generates $\mathbb{Z}^E/\mathbb{Z}\mathbf{e}_E \oplus \mathbb{Z}^E/\mathbb{Z}\mathbf{f}_E$. □

2.7. **The bipermutohedral fan as the normal fan of the bipermutohedron.** We construct a polytope $\Pi_{E,E}$, called the *bipermutohedron*, whose reduced normal fan is $\Sigma_{E,E}$. The reader may skip this subsection without interrupting the main logical flow of the paper (see Remark 5.30).

For each bipermutation $\mathcal{B}$ of E, we construct a vertex $\mathbf{v}_{\mathcal{B}}$ in $\mathbb{R}^E \oplus \mathbb{R}^E$ as follows. Let $k = k_{\mathcal{B}}$ be the element appearing exactly once in $\mathcal{B}$. Consider the word $\pi_{\mathcal{B}}$ obtained from $\mathcal{B}$ by replacing k with $k\overline{k}$ and replacing the first and the second occurrences of each $j \neq k$ with j and $\overline{j}$. We identify this word with the bijection

$$\pi_{\mathcal{B}} \colon E \cup \overline{E} \longrightarrow \{-2n-1, \ldots, -3, -1, 1, 3, \ldots, 2n+1\}$$

that sends the letters of the word to the odd integers in increasing order. For example,

$$\pi_{1|2|3|1|3|0|0} = \begin{pmatrix} 1 & 2 & \overline{2} & 3 & \overline{1} & \overline{3} & 0 & \overline{0} \\ -7 & -5 & -3 & -1 & 1 & 3 & 5 & 7 \end{pmatrix}.$$

Let $\mathbf{u}_{\mathcal{B}} = (x, y)$ be the vector in $\mathbb{R}^E \oplus \mathbb{R}^E$ with $x_j = \pi_{\mathcal{B}}(j)$ and $y_j = -\pi_{\mathcal{B}}(\overline{j})$. We define

$$\mathbf{v}_{\mathcal{B}} = \mathbf{u}_{\mathcal{B}} - s_{\mathcal{B}}(\mathbf{e}_k + \mathbf{f}_k), \text{ where } s_{\mathcal{B}} = \sum_{i \in E} x_i = \sum_{i \in E} y_i.$$

For example, writing (x, y) as a matrix whose top and bottom rows are x and y respectively,

$$\mathbf{v}_{1|2|3|1|3|0|0} = \begin{pmatrix} 5 & -7 & -5 & -1 \\ -7 & -1 & 3 & -3 \end{pmatrix} + 8 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The row sums of $\mathbf{v}_{\mathcal{B}}$ are both equal to 0, so $\mathbf{v}_{\mathcal{B}}$ is in $\mathrm{M}_E \oplus \mathrm{M}_E$, where M_E is the vector space dual to N_E.

[14] Alternatively, one may appeal to the unimodularity of the $2n$-dimensional braid arrangement fan in (Z, W)-coordinates discussed in Section 2.3.

Definition 2.19. The *bipermutohedron* of E is

$$\Pi_{E,E} := \operatorname{conv}\{\mathbf{v}_{\mathcal{B}} \mid \mathcal{B} \text{ is a bipermutation of } E\} \subseteq \mathrm{M}_E \oplus \mathrm{M}_E\,.$$

We refer to [Ard22] for a detailed study of this remarkable polytope. In [Ard22, Proposition 2.8], it is shown that the bipermutohedron in $\mathrm{M}_E \oplus \mathrm{M}_E$ is defined by the inequalities

$$\sum_{i\in S} x_i + \sum_{i \in T} y_i \geq -(|S| + |S-T|)\cdot(|T|+|T-S|), \text{ for each bisubset } S|T \text{ of } E.$$

The inequality description is reminiscent of that of the permutohedron in M_E, which reads

$$\sum_{i\in S} x_i \geq -|S|\cdot|E-S|, \text{ for each nonempty proper subset } S \text{ of } E.$$

The automorphism group of the permutohedron is the symmetric group $\mathfrak{S}_E$, and the automorphism group of the bipermutohedron is the product $\mathfrak{S}_E \times \mathbb{Z}/2\mathbb{Z}$ [Ard22, Proposition 7.2].

Proposition 2.20. *The bipermutohedral fan $\Sigma_{E,E}$ is the normal fan of $\Pi_{E,E}$.*

Proof. Let $\mathcal{B}$ be a bipermutation on E. We claim that the cone of the normal fan of $\Pi_{E,E}$ corresponding to $\mathbf{v}_{\mathcal{B}}$ is precisely the maximal cone $\sigma_{\mathcal{B}}$ of the bipermutohedral fan $\Sigma_{E,E}$:

$$\mathcal{N}_{\Pi_{E,E}}(\mathbf{v}_{\mathcal{B}}) = \sigma_{\mathcal{B}}.$$

This will also show that each $\mathbf{v}_{\mathcal{B}}$ is indeed a vertex of the bipermutohedron $\Pi_{E,E}$.

It is enough to show that the left-hand side is contained in the right-hand side, as the two fans we are comparing have the same support. Let $\varphi = (z, w)$ be a linear functional such that the φ-minimal face of $\Pi_{E,E}$ contains $\mathbf{v}_{\mathcal{B}}$, and let k be the letter that is not repeated in $\mathcal{B}$. We use the description of $\Sigma_{E,E}$ in Section 2.3. We need to show that (z, w) is in the chart $\mathcal{C}_k$, and that when we rewrite (z, w) in the coordinate system

$$Z_i = z_i - z_k, \qquad W_i = -w_i + w_k,$$

the relative order of $Z_0, \ldots, Z_n, W_0, \ldots, W_n$ agrees with the order of $0, \ldots, n, \overline{0}, \ldots, \overline{n}$ in $\pi_{\mathcal{B}}$.

Let i and j be any two adjacent letters in $\mathcal{B}$ appearing in that order. When $i \neq j$, we write $\mathcal{B}'$ for the bipermutation obtained from $\mathcal{B}$ by swapping i and j:

$$\mathcal{B} = \cdots|i|j|\cdots, \qquad \mathcal{B}' = \cdots|j|i|\cdots.$$

When $i = j$, we write $\mathcal{B}'$ for the bipermutation obtained from $\mathcal{B}$ by making i occur only once and k occur twice consecutively, as follows:

$$\mathcal{B} = \cdots|i|i|\cdots|k|\cdots, \qquad \mathcal{B}' = \cdots|i|\cdots|k|k|\cdots.$$

We use the inequality $\varphi(\mathbf{v}_{\mathcal{B}}) \leq \varphi(\mathbf{v}_{\mathcal{B}'})$ to determine the relative order of $Z_0, \ldots, Z_n, W_0, \ldots, W_n$. In what follows, we consider $\mathbf{v}_{\mathcal{B}}$ and $\mathbf{v}_{\mathcal{B}'}$ as matrices with two rows whose columns are labeled by E. Since $\mathbf{v}_{\mathcal{B}}$ and $\mathbf{v}_{\mathcal{B}'}$ can only differ in columns labeled by i, j, or k, we only display those columns in the computations below.

First, we consider the case $i \neq j$ and $i, j \neq k$. There are four subcases.

(1-1) Suppose both i and j are their first occurrences in $\mathcal{B}$. In this case, we have $s_{\mathcal{B}} = s_{\mathcal{B}'}$, and

$$\pi_{\mathcal{B}}(i) = a-1, \quad \pi_{\mathcal{B}}(j) = a+1, \quad \pi_{\mathcal{B}'}(i) = a+1, \quad \pi_{\mathcal{B}'}(j) = a-1 \quad \text{for some } a.$$

Therefore, the condition that the φ-minimal face of $\Pi_{E,E}$ contains $\mathbf{v}_{\mathcal{B}}$ implies

$$\varphi\begin{pmatrix} a-1 & a+1 & -s \\ b & c & -s \end{pmatrix} \le \varphi\begin{pmatrix} a+1 & a-1 & -s \\ b & c & -s \end{pmatrix} \text{ for some } b \text{ and } c.$$

We thus have $(a-1)z_i + (a+1)z_j \le (a+1)z_i + (a-1)z_j$, and hence $Z_j \le Z_i$.

(1-2) Suppose both i and j are their second occurrences in $\mathcal{B}$. Similarly to the previous case,

$$\varphi\begin{pmatrix} b & c & -s \\ -a+1 & -a-1 & -s \end{pmatrix} \le \varphi\begin{pmatrix} b & c & -s \\ -a-1 & -a+1 & -s \end{pmatrix} \text{ for some } b \text{ and } c.$$

We thus have $-(a-1)w_i - (a+1)w_j \le -(a+1)w_i - (a-1)w_j$, and hence $W_j \le W_i$.

(1-3) Suppose i is its first occurrence in $\mathcal{B}$ and j is its second occurrence in $\mathcal{B}$. We have

$$\pi_{\mathcal{B}}(i) = a-1, \quad \pi_{\mathcal{B}}(\bar{j}) = a+1, \quad \pi_{\mathcal{B}'}(i) = a+1, \quad \pi_{\mathcal{B}'}(\bar{j}) = a-1 \quad \text{for some } a,$$

and hence $s_{\mathcal{B}'} = s_{\mathcal{B}} + 2$. The condition that the φ-minimal face of $\Pi_{E,E}$ contains $\mathbf{v}_{\mathcal{B}}$ implies

$$\varphi\begin{pmatrix} a-1 & b & -s \\ c & -a-1 & -s \end{pmatrix} \le \varphi\begin{pmatrix} a+1 & b & -s-2 \\ c & -a+1 & -s-2 \end{pmatrix} \text{ for some } b \text{ and } c.$$

We thus have $(a-1)z_i - (a+1)w_j \le (a+1)z_i - (a-1)w_j - 2z_k - 2w_k$, and hence $W_j \le Z_i$.

(1-4) Suppose i is its second occurrence in $\mathcal{B}$ and j is its first occurrence in $\mathcal{B}$. Computing as in the previous case, we get $Z_j \le W_i$.

Second, we consider the case $i \ne j$ and $i = k$. There are two subcases.

(2-1) Suppose j is its first occurrence in $\mathcal{B}$. In this case, for some a, we have

$$\pi_{\mathcal{B}}(k) = a-2, \quad \pi_{\mathcal{B}}(\bar{k}) = a, \quad \pi_{\mathcal{B}}(j) = a+2, \quad \pi_{\mathcal{B}'}(k) = a,$$
$$\pi_{\mathcal{B}'}(\bar{k}) = a+2, \quad \pi_{\mathcal{B}'}(j) = a-2,$$

and hence $s_{\mathcal{B}'} = s_{\mathcal{B}} - 2$. The condition that the φ-minimal face of $\Pi_{E,E}$ contains $\mathbf{v}_{\mathcal{B}}$ implies

$$\varphi\begin{pmatrix} a-s-2 & a+2 \\ -a-s & b \end{pmatrix} \le \varphi\begin{pmatrix} a-s+2 & a-2 \\ -a-s & b \end{pmatrix} \text{ for some } b.$$

We thus have $(a-s-2)z_i + (a+2)z_j \le (a-s+2)z_i + (a-2)z_j$, and hence $Z_j \le Z_i$.

(2-2) Suppose j is its second occurrence in $\mathcal{B}$. Computing as above, we get $W_j \le W_i$.

Third, we consider the case $i \ne j$ and $j = k$. There are two subcases.

(3-1) Suppose i is its first occurrence in $\mathcal{B}$. Computing as in (2-1), we get $Z_j \le Z_i$.

(3-2) Suppose i is its second occurrence in $\mathcal{B}$. Computing as in (2-1), we get $W_j \le W_i$.

Last, we consider the case $i = j$. In this case, we have $\pi_{\mathcal{B}} = \pi_{\mathcal{B}'}$, and hence

$$\mathbf{u}_{\mathcal{B}} = \mathbf{u}_{\mathcal{B}'} \quad \text{and} \quad s_{\mathcal{B}} = s_{\mathcal{B}'}.$$

Notice that, since ℓ precedes $\overline{\ell}$ in $\pi_{\mathcal{B}}$ for all ℓ, we have

$$s_{\mathcal{B}} \leq -(2n-1) - (2n-5) - \cdots + (2n-7) + (2n-3) < 0.$$

Therefore, $\varphi\big(\mathbf{u}_{\mathcal{B}} - s_{\mathcal{B}}(\mathbf{e}_k + \mathbf{f}_k)\big) \leq \varphi\big(\mathbf{u}_{\mathcal{B}} - s_{\mathcal{B}}(\mathbf{e}_i + \mathbf{f}_i)\big)$ implies $W_i \leq Z_i$.

Applying the above analysis to each pair of adjacent letters of $\mathcal{B}$, we conclude that, given k, the relative order of $Z_0, \ldots, Z_n, W_0, \ldots, W_n$ is determined by $\pi_{\mathcal{B}}$. In particular, since i precedes $\overline{i}$ in $\pi_{\mathcal{B}}$ for all i, we have that $Z_i \geq W_i$ for all i, that is, φ belongs to the chart $\mathcal{C}_k$. □

2.8. **The bipermutohedral fan: An origin story.** The bipermutohedral fan can be approached from several different points of view, and it has many favorable properties, as the previous sections show. However, it may not yet be clear where this fan comes from, or why it is a good setting for the Lagrangian geometry of matroids. In this section we explain the geometric motivation for its construction, and the role its various properties play in the theory.

When M is the matroid of a subspace V of $\mathbb{C}^E$, the conormal fan $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ is a tropical model of the projectivized conormal bundle of V. Since $\mathrm{M}^\perp$ is the matroid of the orthogonal complement of V, we expect the conormal fan to be supported on $\mathrm{trop}(\mathrm{M}) \times \mathrm{trop}(\mathrm{M}^\perp)$. A desirable fan structure Σ on this support should have the following properties:

(1) The classes γ and δ can be defined in its Chow ring, so we can state Theorems 1.1 and 1.2.
(2) The Chow ring is tractable for computations, so we can prove Theorems 1.1 and 1.2.
(3) The fan is a subfan of the normal fan of a polytope, so its ample cone is nonempty.
(4) The fan is Lefschetz, so we can derive Conjecture 1.3 in Theorem 1.4.

We resolve requirement (4) by showing in Theorem 1.6 that being Lefschetz only depends on the support $\mathrm{trop}(\mathrm{M}) \times \mathrm{trop}(\mathrm{M}^\perp)$ – which is the support of a Lefschetz fan by [AHK18] – and not on the fan structure that we choose. Thus we can focus on the first three.

Requirement (2) is stated imprecisely, but a very desirable initial property is that our fan Σ is simplicial. When this is the case, the Chow ring $A(\Sigma)$ of the toric variety $X(\Sigma)$ has an algebraic combinatorial presentation due to Brion, and an interpretation in terms of piecewise polynomial functions due to Billera; see Section 3.1. This will allow us to carry out intersection-theoretic computations in this Chow ring. Thus the first fan structure on $\mathrm{trop}(\mathrm{M}) \times \mathrm{trop}(\mathrm{M}^\perp)$ that one might try is the product of Bergman fans $\Sigma_{\mathrm{M}} \times \Sigma_{\mathrm{M}^\perp}$, which is simplicial and does have a nice combinatorial structure. It is also a subfan of the normal fan of the product of permutohedra $\Pi_E \times \Pi_E$.

To address requirement (1), we rely on the geometry of the representable case, as studied in [DGS12, Huh13], which tells us what the classes γ and δ of Theorems 1.1 and 1.2 should be. If α is the piecewise linear function on the tropical projective torus defined in Section 1, then γ and δ should be the pullbacks of α along the following maps from $\mathrm{N}_E \times \mathrm{N}_E$ to N_E:

$$\pi\colon \Sigma \longrightarrow \Gamma_E, \quad (z,w) \longmapsto z \quad \text{and} \quad \mu\colon \Sigma \longrightarrow \Gamma_E, \quad (z,w) \longmapsto z+w,$$

where Γ_E is the reduced normal fan of the standard simplex. If $\Sigma = \Sigma_{\mathrm{M}} \times \Sigma_{\mathrm{M}^\perp}$ or any refinement of it, the first map is a map of fans, and γ *is* well-defined. However, the second map is *not* a map of fans for $\Sigma = \Sigma_{\mathrm{M}} \times \Sigma_{\mathrm{M}^\perp}$, as we will see in Example 3.8. Thus the product fan structure will not serve our purposes; we need to subdivide it further. How might we do this simultaneously for all M?

At this point, it is instructive to return to the case of tropical linear spaces, as used by Adiprasito, Huh, and Katz in [AHK18]. In that case, one wants a similarly convenient fan structure for the tropical linear space trop(M). Fortunately, one can do this for all matroids on E at once, by intersecting trop(M) with the permutohedral fan Σ_E. The result is the *Bergman fan* Σ_{M} of M introduced by Ardila and Klivans in [AK06], where it is called the *fine subdivision*.

Similarly, we might try to find a suitable fan structure of trop(M) $\times$ trop($\mathrm{M}^\perp$) for all matroids M on E simultaneously, by intersecting them with an appropriate complete fan. There is a natural candidate: the coarsest common refinement of the product of permutohedral fans $\Sigma_E \times \Sigma_E$, which induces the fan structure $\Sigma_{\mathrm{M}} \times \Sigma_{\mathrm{M}^\perp}$, and $\mu^{-1}(\Gamma_E)$, the coarsest fan that guarantees that the class δ is well-defined. This is the reduced normal fan of a polytope

$$H_{E,E} := (\Pi_E \times \Pi_E) + D_E,$$

the Minkowski sum of the product of two permutohedra and the diagonal simplex $D_E = \operatorname{conv}\{\mathbf{e}_i + \mathbf{f}_i\}_{i \in E}$. The polytope $H_{E,E}$ does have an elegant combinatorial structure, as shown in [AE21]. They call it the *harmonic polytope* because its number of vertices is

$$|E|! \cdot |E|! \cdot \left(1 + \frac{1}{2} + \cdots + \frac{1}{|E|}\right).$$

However, this polytope has a drawback for our purposes: it is not simple, so the resulting fan structure on trop(M) $\times$ trop($\mathrm{M}^\perp$) is not simplicial. Thus we need to find a simplicial refinement of the corresponding *harmonic fan*, with simple enough combinatorial structure that we can carry out computations.

The bipermutohedral fan $\Sigma_{E,E}$ is our answer to those requirements. It refines the harmonic fan by Proposition 2.11, so γ and δ are well-defined. It is simplicial by Proposition 2.18, and it has an elegant combinatorial structure that makes explicit computations possible. It is the normal fan of the bipermutohedron, thanks to Proposition 2.20.

For the above reasons, we define the conormal fan $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ to be the fan on trop(M) $\times$ trop(M)$^\perp$ obtained by intersecting with the bipermutohedral fan $\Sigma_{E,E}$. The construction of $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ is a Lagrangian analog of the construction of Σ_{M} in [AK06]. What remains is to understand the resulting combinatorial structure and carry out the necessary intersection-theoretic computations to prove Theorems 1.1 and 1.2 – which is the goal of Sections 3 and 4 – and to prove Theorem 1.6 – which we do in Section 5. We believe that the bipermutohedral fan will have applications beyond those presented in this paper. For example, the bipermutohedral perspective could be a guide in finding useful tropical models for Lagrangian matroids studied in [BGW03], of which the conormal fan of a matroid will be a particular case.

3. The conormal intersection theory of a matroid

In this section, we construct the *conormal fan* of a matroid M on E, and describe its Chow ring. Our running example will be the graphic matroid $\mathrm{M}(G)$ of the graph G of the square pyramid, whose dual is the graphic matroid of the dual graph $G^\perp$ shown in Figure 6.

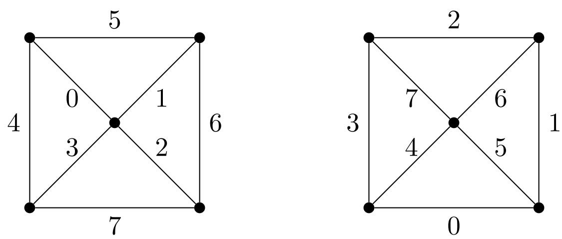


Figure 6. The graph G of the square pyramid and its dual graph $G^\perp$

3.1. **Homology and cohomology.** Throughout this section, we fix a simplicial rational fan Σ in $\mathrm{N} = \mathbb{R} \otimes \mathrm{N}_\mathbb{Z}$. For each ray ρ in Σ, we write $\mathbf{e}_\rho$ for the primitive generator of ρ in $\mathrm{N}_\mathbb{Z}$, and introduce a variable x_ρ.

- Let $S(\Sigma)$ be the polynomial ring with real coefficients that has x_ρ as its variables, one for each ray ρ of Σ.
- Let $I(\Sigma)$ be the *Stanley-Reisner* ideal of $S(\Sigma)$, generated by the square-free monomials indexing the subsets of rays of Σ which do not generate a cone in Σ.
- Let $J(\Sigma)$ be the ideal of $S(\Sigma)$ generated by the linear forms $\sum_\rho \ell(\mathbf{e}_\rho)x_\rho$, where ℓ is any linear function on N and the sum is over all the rays in Σ.

Definition 3.1. The *Chow ring* of Σ, denoted $A(\Sigma)$, is the graded algebra $S(\Sigma)/(I(\Sigma)+J(\Sigma))$.

Billera [Bil89] constructed an isomorphism from the monomial quotient $S(\Sigma)/I(\Sigma)$ to the algebra of continuous piecewise polynomial functions on Σ by identifying the variable x_ρ with the piecewise linear *Courant function* on Σ determined by the condition

$$x_\rho(\mathbf{e}_{\rho'}) = \begin{cases} 1, & \text{if } \rho \text{ is equal to } \rho', \\ 0, & \text{if } \rho \text{ is not equal to } \rho'. \end{cases}$$

Thus, under this isomorphism, a piecewise linear function ℓ on Σ is identified with the linear form

$$\ell = \sum_\rho \ell(\mathbf{e}_\rho)x_\rho.$$

We regard the elements of $A(\Sigma)$ as equivalence classes of piecewise polynomial functions on Σ, modulo the restrictions of global linear functions to Σ.

Brion [Bri97, Theorem 6.7] showed that the Chow ring of the toric variety $X(\Sigma)$ of Σ with real coefficients is isomorphic to $A(\Sigma)$.[15]

[15]In [Bri97], Brion identifies $A(\Sigma)$ with the Chow group of $X(\Sigma)$ with real coefficients. For the existence of the ring structure and the pullback, see [Vis89].

Under this isomorphism, the class of the torus orbit closure of a cone σ in Σ is identified with $\mathrm{mult}(\sigma)\, x_\sigma$, where x_σ is the monomial $\prod_{\rho \subseteq \sigma} x_\rho$ and $\mathrm{mult}(\sigma)$ is the index of the subgroup

$$\Big(\sum_{\rho\subseteq\sigma} \mathbb{Z}\mathbf{e}_\rho\Big) \subseteq \mathrm{N}_\mathbb{Z} \cap \Big(\sum_{\rho\subseteq\sigma} \mathbb{R}\mathbf{e}_\rho\Big).$$

All the fans appearing in this section will be unimodular, so $\mathrm{mult}(\sigma) = 1$ for every σ in Σ.

We write $\Sigma(k)$ for the set of k-dimensional cones in Σ.

A *k-dimensional Minkowski weight* on Σ is a real-valued function ω on $\Sigma(k)$ that satisfies the *balancing condition*: For every $(k-1)$-dimensional cone τ in Σ,

$$\sum_{\tau\subset\sigma} \omega(\sigma)\mathbf{e}_{\sigma/\tau} = 0 \text{ in the quotient space } \mathrm{N}\,/\,\mathrm{span}(\tau),$$

where $\mathbf{e}_{\sigma/\tau}$ is the primitive generator of the ray $(\sigma + \mathrm{span}(\tau))/\,\mathrm{span}(\tau)$. We say that w is *positive* if $w(\sigma)$ is positive for every σ in $\Sigma(k)$. We write $\mathrm{MW}_k(\Sigma)$ for the space of k-dimensional Minkowski weights on Σ, and set

$$\mathrm{MW}(\Sigma) = \bigoplus_{k\geq 0} \mathrm{MW}_k(\Sigma).$$

We will make use of the basic fact that the Chow group of a toric variety is generated by the classes of torus orbit closures [CLS11, Lemma 12.5.1].

Thus, there is an injective linear map from the dual of $A^k(\Sigma)$ to the space of k-dimensional weights on Σ, whose image turns out to be $\mathrm{MW}_k(\Sigma)$, as noted in [AHK18, Section 5].[16] Explicitly, the inverse isomorphism from the image is

$$\mathrm{MW}_k(\Sigma) \longrightarrow \mathrm{Hom}(A^k(\Sigma), \mathbb{R}), \qquad w \longmapsto (\mathrm{mult}(\sigma)x_\sigma \longmapsto w(\sigma)).$$

Following [AHK18], we define the *cap product*, denoted $\eta \cap w$, using the composition

$$A^\ell(\Sigma) \longrightarrow \mathrm{Hom}(A^{k-\ell}(\Sigma), A^k(\Sigma)) \longrightarrow \mathrm{Hom}(\mathrm{MW}_k(\Sigma), \mathrm{MW}_{k-\ell}(\Sigma)),$$
$$\eta \longmapsto (w \longmapsto \eta \cap w),$$

where the first map is given by the multiplication in the Chow ring of Σ. In short, $\mathrm{MW}(\Sigma)$ has the structure of a graded $A(\Sigma)$-module given by the isomorphism $\mathrm{MW}(\Sigma) \simeq \mathrm{Hom}(A(\Sigma), \mathbb{R})$.

Let $f\colon \Sigma \to \Sigma'$ be a morphism of simplicial fans. The pullback of functions defines the *pullback homomorphism* between the Chow rings

$$f^*\colon A(\Sigma') \longrightarrow A(\Sigma),$$

whose dual is the *pushforward homomorphism* between the space of Minkowski weights

$$f_*\colon \mathrm{MW}(\Sigma) \longrightarrow \mathrm{MW}(\Sigma').$$

Since f^* is a homomorphism of graded rings, f_* is a homomorphism of graded modules.[17] In other words, the pullback and the pushforward homomorphisms

[16] In [AHK18, Proposition 5.6], all fans are unimodular and their Chow rings have integral coefficients. The same argument works for simplicial fans and their Chow rings with real or rational coefficients.

[17] To see that the pullback between the Chow rings is determined by the pullback of piecewise linear functions, note that every divisor on simplicial toric variety is $\mathbb{Q}$-Cartier [CLS11, Proposition 4.2.7] and that the pullback of Chern classes of line bundles corresponds to the pullback of piecewise linear functions [CLS11, Proposition 6.2.7].

satisfy the *projection formula*

$$\eta \cap f_* w = f_*(f^*\eta \cap w).$$

3.2. **The Bergman fan of a matroid.** The *Bergman fan* of a matroid M on E, denoted Σ_{M}, is the r-dimensional subfan of the n-dimensional permutohedral fan Σ_E whose underlying set is the *tropical linear space*

$$\mathrm{trop}(\mathrm{M}) = \Big\{ z \,|\, \min_{i\in C}(z_i) \text{ is achieved at least twice for every circuit } C \text{ of M} \Big\} \subseteq \mathrm{N}_E\,.$$

The Bergman fan of M is equipped with the piecewise linear functions

$$\alpha_j = \max_{i\in E}(z_j - z_i),$$

and the space of linear functions on the Bergman fan is spanned by the differences

$$\alpha_i - \alpha_j = z_i - z_j.$$

Note that trop(M) is nonempty if and only if M is *loopless*. In the remainder of this section, we suppose that M has no loops. In this case, the Bergman fan of M is the induced subfan of Σ_E generated by the rays corresponding to the nonempty proper flats of M [AK06].

Proposition 3.2. *The Bergman fan of* M *is the unimodular fan in* N_E *with the cones*

$$\sigma_{\mathcal{F}} = \mathrm{cone}\{\mathbf{e}_F\}_{F\in\mathcal{F}}, \textit{ for flags of flats } \mathcal{F} \textit{ of } \mathrm{M}.$$

The most important geometric property of Σ_{M} is the following description of its top-dimensional Minkowski weights. For a proof, see, for example, [AHK18, Proposition 5.2].

Proposition 3.3. *An r-dimensional weight on Σ_{M} is balanced if and only if it is constant.*

We write 1_{M} for the *fundamental weight* on Σ_{M}, the r-dimensional Minkowski weight on the Bergman fan that has the constant value 1.

3.3. **The Chow ring of the Bergman fan.** In the context of matroids, for simplicity, we set

$$S_{\mathrm{M}} = S(\Sigma_{\mathrm{M}}), \quad I_{\mathrm{M}} = I(\Sigma_{\mathrm{M}}), \quad J_{\mathrm{M}} = J(\Sigma_{\mathrm{M}}), \quad A_{\mathrm{M}} = A(\Sigma_{\mathrm{M}}).$$

We identify the elements of $S_{\mathrm{M}}/I_{\mathrm{M}}$ with the piecewise linear functions on Σ_{M} as before.

Let x_F be the variable of the polynomial ring corresponding to the ray generated by $\mathbf{e}_F$ in the Bergman fan. For any set $\mathcal{F}$ of nonempty proper flats of M, we write $x_{\mathcal{F}}$ for the monomial

$$x_{\mathcal{F}} = \prod_{F\in\mathcal{F}} x_F.$$

The variable x_F, viewed as a piecewise linear function on the Bergman fan, is given by

$$x_F(\mathbf{e}_{F'}) = \begin{cases} 1, & \text{if } F \text{ is equal to } F', \\ 0, & \text{if } F \text{ is not equal to } F', \end{cases}$$

and hence the piecewise linear function α_j on the Bergman fan satisfies the identity

$$\alpha_j = \sum_F \alpha_j(\mathbf{e}_F) x_F = \sum_{j\in F} x_F.$$

Thus, in the above notation,

- S_{M} is the ring of polynomials in the variables x_F, where F is a nonempty proper flat of M,
- I_{M} is the ideal generated by the monomials $x_{\mathcal{F}}$, where $\mathcal{F}$ is not a flag, and
- J_{M} is the ideal generated by the linear forms $\alpha_i - \alpha_j$, for any i and j in E.

We write α for the common equivalence class of α_j in the Chow ring of the Bergman fan.

Definition 3.4. The fundamental weight 1_{M} defines the *degree map*

$$\deg\colon A^r_{\mathrm{M}} \longrightarrow \mathbb{R}, \quad x_{\mathcal{F}} \longmapsto x_{\mathcal{F}} \cap 1_{\mathrm{M}} = \begin{cases} 1 & \text{if } \mathcal{F} \text{ is a flag,} \\ 0 & \text{if } \mathcal{F} \text{ is not a flag.} \end{cases}$$

By Proposition 3.3, the degree map is an isomorphism. In other words, for any maximal flag $\mathcal{F}$ of nonempty proper flats of M, the class of the monomial $x_{\mathcal{F}}$ in the Chow ring of the Bergman fan of M is nonzero and does not depend on $\mathcal{F}$.

3.4. **The conormal fan of a matroid.** The *conormal fan* of a matroid M on E, denoted $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$, is the $(n-1)$-dimensional subfan of the $2n$-dimensional bipermutohedral fan $\Sigma_{E,E}$ whose support is the product of tropical linear spaces

$$|\Sigma_{\mathrm{M},\mathrm{M}^\perp}| = \mathrm{trop}(\mathrm{M}) \times \mathrm{trop}(\mathrm{M}^\perp).$$

Equivalently, the conormal fan is the largest subfan of the bipermutohedral fan for which the projections to the factors are morphisms of fans

$$\pi\colon \Sigma_{\mathrm{M},\mathrm{M}^\perp} \longrightarrow \Sigma_{\mathrm{M}} \quad \text{and} \quad \overline{\pi}\colon \Sigma_{\mathrm{M},\mathrm{M}^\perp} \longrightarrow \Sigma_{\mathrm{M}^\perp}.$$

The addition map $(z, w) \mapsto z + w$ is also a morphism of fans $\Sigma_{\mathrm{M},\mathrm{M}^\perp} \to \Gamma_E$.

The conormal fan of M is equipped with the piecewise linear functions

$$\gamma_j = \max_{i \in E}(z_j - z_i), \qquad \overline{\gamma}_j = \max_{i \in E}(w_j - w_i), \qquad \delta_j = \max_{i \in E}(z_j + w_j - z_i - w_i),$$

which are the pullbacks of α_j under the projections π and π' and the addition map, respectively. The space of linear functions on the conormal fan is spanned by the differences

$$\gamma_i - \gamma_j = z_i - z_j \quad \text{and} \quad \overline{\gamma}_i - \overline{\gamma}_j = w_i - w_j.$$

Note that the support of the conormal fan of M is nonempty if and only if M is *loopless* and *coloopless*. In the remainder of this section, we suppose that M has no loops and no coloops.

Definition 3.5. A *biflat* $F|G$ of M consists of a flat F of M and a flat G of $\mathrm{M}^\perp$ that form a bisubset; that is, they are nonempty, they are not both equal to E, and their union is E. A *biflag* of M is a flag of biflats.

We give an analog of Proposition 3.2 for conormal fans in terms of biflats.

Proposition 3.6. *The conormal fan of* M *is the unimodular fan in* $\mathrm{N}_{E,E}$ *with the cones*

$$\sigma_{\mathcal{F}|\mathcal{G}} = \mathrm{cone}\{\mathbf{e}_{F|G}\}_{F|G \in \mathcal{F}|\mathcal{G}}, \text{ for flags of biflats } \mathcal{F}|\mathcal{G} \text{ of } \mathrm{M}.$$

Proof. The proof is straightforward, given Corollary 2.17 and Proposition 3.2: If $\mathcal{F}|\mathcal{G}$ is a flag of biflats of M, then $\mathcal{F}$ is an increasing sequence of flats of M and $\mathcal{G}$ is a decreasing sequence of flats of $\mathrm{M}^\perp$, and hence

$$\sigma_{\mathcal{F}|\mathcal{G}} \subseteq \sigma_{\mathcal{F}} \times \sigma_{\mathcal{G}} \in \Sigma_{\mathrm{M}} \times \Sigma_{\mathrm{M}^\perp}.$$

Therefore, the conormal fan of M contains the induced subfan of $\Sigma_{E,E}$ generated by the rays corresponding to the biflats of M. The other inclusion follows from the easy implication

$$\mathbf{e}_{F|G} \text{ is in the support of the conormal fan of M} \Longrightarrow F|G \text{ is a biflat of M.} \qquad \square$$

We also have the following analog of Proposition 3.3 for conormal fans.

Proposition 3.7. *An $(n-1)$-dimensional weight on $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ is balanced if and only if it is constant.*

We write $1_{\mathrm{M},\mathrm{M}^\perp}$ for the *fundamental weight* on $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$, the top-dimensional Minkowski weight on the conormal fan that has the constant value 1.

Proof. Proposition 3.3 applied to M and $\mathrm{M}^\perp$ shows that a top-dimensional weight on $\Sigma_\mathrm{M} \times \Sigma_{\mathrm{M}^\perp}$ satisfies the balancing condition if and only if it is constant. This property of the fan remains invariant under any subdivision of its support, as shown in [GKM09, Section 2]. $\square$

For our purposes, the product of the Bergman fans of M and $\mathrm{M}^\perp$ has a shortcoming: The addition map need not be a morphism from the product to the fan Γ_E. Thus, in general, we cannot define the class of δ_j in the Chow ring of the product. This is our motivation for subdividing it further, to obtain the conormal fan $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$.

Example 3.8. Let M and $\mathrm{M}^\perp$ be the graphic matroids of the graphs in Figure 6. Consider the cone $\sigma_\mathcal{F} \times \sigma_\mathcal{G}$ in the product of Bergman fans of M and $\mathrm{M}^\perp$, where

$$\mathcal{F} = (\varnothing \subsetneq 1 \subsetneq 015 \subsetneq 01345 \subsetneq E) \quad \text{and} \quad \mathcal{G} = (\varnothing \subsetneq 2 \subsetneq 267 \subsetneq 12567 \subsetneq E).$$

This cone is subdivided into the chambers of $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ corresponding to the biflags

$$\left|\begin{array}{cc|ccccccccc|cc} \varnothing & \subsetneq & 1 & \subseteq & 015 & \subseteq & 01345 & \subseteq & 01345 & \subseteq & 01345 & \subseteq & E \\ E & \supseteq & E & \supseteq & E & \supseteq & E & \supseteq & 12567 & \supseteq & 267 & \supseteq & 2 \end{array}\right| \begin{array}{cc} \subseteq & E \\ \supsetneq & \varnothing \end{array}\Bigg|,$$

$$\left|\begin{array}{cc|ccccccccc|cc} \varnothing & \subsetneq & 1 & \subseteq & 015 & \subseteq & 01345 & \subseteq & 01345 & \subseteq & E & \subseteq & E \\ E & \supseteq & E & \supseteq & E & \supseteq & 12567 & \supseteq & 267 & \supseteq & 267 & \supseteq & 2 \end{array}\right| \begin{array}{cc} \subseteq & E \\ \supsetneq & \varnothing \end{array}\Bigg|,$$

$$\left|\begin{array}{cc|ccccccccc|cc} \varnothing & \subsetneq & 1 & \subseteq & 015 & \subseteq & 01345 & \subseteq & E & \subseteq & E & \subseteq & E \\ E & \supseteq & E & \supseteq & E & \supseteq & 12567 & \supseteq & 12567 & \supseteq & 267 & \supseteq & 2 \end{array}\right| \begin{array}{cc} \subseteq & E \\ \supsetneq & \varnothing \end{array}\Bigg|.$$

If (z, w) is inside the first chamber, then the minimum of $z_i + w_i$ is attained by $z_6+w_6 = z_7+w_7$, and hence $z+w$ is in the cone σ_{012345}. If (z, w) is inside the second or the third chamber, then the minimum of z_i+w_i is attained by $z_3+w_3 = z_4+w_4$, and hence $z + w$ is in the cone σ_{012567}. Thus, the product cone does not map into a cone in Γ_E under the addition map.

Recall from Definition 2.5 that the cotangent fan Ω_E is the subfan of $\Sigma_{E,E}$ with support

$$\mathrm{trop}(\delta) = \Big\{(z, w) \mid \min_{i\in E}(z_i + w_i) \text{ is achieved at least twice}\Big\} \subseteq \mathrm{N}_{E,E}\,.$$

In other words, the cotangent fan is the collection of cones $\sigma_\mathcal{B}$ for bisequences $\mathcal{B}$ on E, where at least two elements of E appear exactly once in $\mathcal{B}$. We show that the cotangent fan contains all the conormal fans of matroids on E.

Proposition 3.9. *For any matroid* M *on* E*, we have* $\mathrm{trop}(\mathrm{M}) \times \mathrm{trop}(\mathrm{M}^\perp) \subseteq \mathrm{trop}(\delta)$.

In other words, if the minimum of $(z_i)_{i\in C}$ is achieved at least twice for every circuit C of M and the minimum of $(w_i)_{i\in C^\perp}$ is achieved at least twice for every circuit $C^\perp$ of $\mathrm{M}^\perp$, then the minimum of $(z_i+w_i)_{i\in E}$ is achieved at least twice. We deduce Proposition 3.9 from Proposition 3.15, a stronger statement on the flags of biflats of M. The notion of gaps introduced here for Proposition 3.15 will be useful in Section 4.

Let $\mathcal{F}|\mathcal{G}$ be a flag of biflats of M. As before, we write $\mathcal{F}$ and $\mathcal{G}$ for the sequences

$$\mathcal{F}=(\varnothing\subsetneq F_1\subseteq\cdots\subseteq F_k\subseteq E),\ \text{where } F_j \text{ are the first parts of the biflats in } \mathcal{F}|\mathcal{G},$$
$$\mathcal{G}=(E\supseteq G_1\supseteq\cdots\supseteq G_k\supsetneq\varnothing),\ \text{where } G_j \text{ are the second parts of the biflats in } \mathcal{F}|\mathcal{G},$$

where k is the length of $\mathcal{F}|\mathcal{G}$. Thus, the bisequence $\mathcal{B}(\mathcal{F}|\mathcal{G})$ from Proposition 2.16 can be written

$$B_0|B_1|\cdots|B_k,\ \text{where } B_j=(F_{j+1}-F_j)\cup(G_j-G_{j+1}).$$

Definition 3.10. The *gap sequence* of $\mathcal{F}|\mathcal{G}$, denoted $\mathcal{D}(\mathcal{F}|\mathcal{G})$, is the sequence of *gaps*

$$D_0|D_1|\cdots|D_k,\ \text{where } D_j=(F_{j+1}-F_j)\cap(G_j-G_{j+1}).$$

Note that D_j consists of the elements of B_j that appear exactly once in the bisequence $\mathcal{B}(\mathcal{F}|\mathcal{G})$.

Example 3.11. The three maximal flags of biflats shown in Example 3.8 have the gap sequences

$$\varnothing|\varnothing|\varnothing|\varnothing|\varnothing|67|\varnothing,\quad \varnothing|\varnothing|34|\varnothing|\varnothing|\varnothing|\varnothing,\quad \varnothing|\varnothing|34|\varnothing|\varnothing|\varnothing|\varnothing.$$

We show in Proposition 3.17 that any maximal flag of biflats has a unique nonempty gap.

Lemma 3.12. *The complement of the gap D_j in E is the union of F_j and G_{j+1}.*

Therefore, by Proposition 2.15, at least one of the gaps of $\mathcal{F}|\mathcal{G}$ must be nonempty.

Proof. Since $F_j|G_j$ and $F_{j+1}|G_{j+1}$ are bisubsets, we have $G_j^c\subseteq F_j$ and $F_{j+1}^c\subseteq G_{j+1}$. Thus,

$$D_j^c=(F_{j+1}\cap F_j^c\cap G_j\cap G_{j+1}^c)^c=F_{j+1}^c\cup F_j\cup G_j^c\cup G_{j+1}=F_j\cup G_{j+1}. \qquad \square$$

Lemma 3.13. *Let $e\in E$. There exists an index i for which $e\in F_i\cap G_i$ if and only if e is not in any gap. In symbols, the union of the gaps of $\mathcal{F}|\mathcal{G}$ is*

$$\bigsqcup_{j=0}^{k}D_j=E-\bigcup_{i=1}^{k}(F_i\cap G_i)\,.$$

Proof. First suppose $e\in F_i\cap G_i$. Then $e\in F_j$ for all $j\geq i$, which means $e\notin D_j$ for $i\leq j\leq k$. Dually, $e\in G_j$ for all $j\leq i$, so $e\notin D_j$ for all $0\leq i\leq j-1$.

Now suppose e is not in any gap, and consider the index $1\leq i\leq k+1$ for which $e\in F_i-F_{i-1}$. Since $e\in F_{i-1}\cup G_i$, we must have $e\in G_i$ and hence $e\in F_i\cap G_i$. $\square$

We will often use the following basic result. Recall that $|E|=n+1$.

Lemma 3.14. *The union of a flat and a coflat cannot have exactly n elements.*

Proof. Let F be a flat and G be a coflat. Recall that, for any matroid, the complement of any hyperplane is a cocircuit [Oxl11, Proposition 2.1.6] and that any flat is an intersection of hyperplanes [Oxl11, Proposition 1.7.8]. So we may write the complement of $F \cup G$ as

$$\Big(\bigcup_{C \in \mathcal{C}} C\Big) \cap \Big(\bigcup_{C^\perp \in \mathcal{C}^\perp} C^\perp\Big),$$

where $\mathcal{C}$ is a collection of circuits and $\mathcal{C}^\perp$ is a collection of cocircuits. Thus, if the complement is nonempty, there are $C \in \mathcal{C}$ and $C^\perp \in \mathcal{C}^\perp$ that intersect nontrivially. Now the conclusion follows from the classical fact that the intersection of a circuit and a cocircuit is either empty or contains at least two elements [Oxl11, Proposition 2.11]. □

Proposition 3.15. *Every nonempty gap of a biflag* $\mathcal{F}|\mathcal{G}$ *of* M *has at least two distinct elements.*

Proof. Since the complement of a gap of $\mathcal{F}|\mathcal{G}$ is the union of a flat and a coflat by Lemma 3.12, the claim follows from Lemma 3.14. □

For any biflag $\mathcal{F}|\mathcal{G}$, there are at least two elements of E that appear exactly once in the bisequence $\mathcal{B}(\mathcal{F}|\mathcal{G})$; therefore

$$\mathrm{trop}(\mathrm{M}) \times \mathrm{trop}(\mathrm{M}^\perp) \subseteq \mathrm{trop}(\delta),$$

proving Proposition 3.9.

For later use, we record here another elementary property of the flags of biflats of a matroid.

Definition 3.16. The *jump sets* of $\mathcal{F}$ and $\mathcal{G}$ are the sets of indices

$$\mathsf{J}(\mathcal{F}) = \{j \mid 0 \le j \le k \text{ and } F_j \ne F_{j+1}\}$$
$$\mathsf{J}(\mathcal{G}) = \{j \mid 0 \le j \le k \text{ and } G_j \ne G_{j+1}\}.$$

The elements of $\mathsf{J}(\mathcal{F}) \cap \mathsf{J}(\mathcal{G})$ are called the *double jumps* of $\mathcal{F}|\mathcal{G}$.

The double jumps are colored bold in the table of $\mathcal{F}|\mathcal{G}$, as shown in Example 3.8. Clearly, j is a double jump whenever the corresponding gap D_j is nonempty. We show that the converse holds when $\mathcal{F}|\mathcal{G}$ is maximal.

Proposition 3.17. *Every maximal flag of biflats* $\mathcal{F}|\mathcal{G}$ *of* M *has a unique double jump. Ignoring repetitions,* $\mathcal{F}$ *and* $\mathcal{G}$ *are complete flags of nonempty flats in* M *and* $\mathrm{M}^\perp$*, respectively.*

In particular, every maximal flag of biflats $\mathcal{F}|\mathcal{G}$ of M has a unique nonempty gap.

Proof. Recall that at least one of the gaps of $\mathcal{F}|\mathcal{G}$ is nonempty. In addition, since tropical linear spaces are pure-dimensional, the length of any maximal flag of biflats must be $n-1$. Thus,

$$|\mathsf{J}(\mathcal{F}) \cap \mathsf{J}(\mathcal{G})| \ge 1 \quad \text{and} \quad |\mathsf{J}(\mathcal{F}) \cup \mathsf{J}(\mathcal{G})| = n.$$

On the other hand, writing $r+1$ for the rank of M as before, we have

$$|\mathsf{J}(\mathcal{F})| \le r+1 \quad \text{and} \quad |\mathsf{J}(\mathcal{G})| \le n-r.$$

Therefore, $n+1 \le |\mathsf{J}(\mathcal{F}) \cup \mathsf{J}(\mathcal{G})| + |\mathsf{J}(\mathcal{F}) \cap \mathsf{J}(\mathcal{G})| = |\mathsf{J}(\mathcal{F})| + |\mathsf{J}(\mathcal{G})| \le n+1$, and hence

$$|\mathsf{J}(\mathcal{F})| = r+1, \qquad |\mathsf{J}(\mathcal{G})| = n-r \quad \text{and} \quad |\mathsf{J}(\mathcal{F}) \cap \mathsf{J}(\mathcal{G})| = 1$$

which imply the desired results. □

By way of contrast, nonmaximal biflags have several double jumps, and they can have a double jump whose corresponding gap is empty:

Example 3.18. For the graphic matroids of Figure 6 again, consider the biflag

$$\mathcal{F}|\mathcal{G} := \left|\begin{array}{cc|ccccc|cc} \varnothing & \subsetneq & 1 & \subseteq & 01345 & \subseteq & E & \subseteq & E \\ E & \supseteq & E & \supseteq & 12567 & \supseteq & 267 & \supsetneq & \varnothing \end{array}\right|.$$

We see that $\mathcal{F}|\mathcal{G}$ has two double jumps, with gaps 034 and $\varnothing$, respectively. In view of Proposition 3.17, any maximal biflag containing $\mathcal{F}|\mathcal{G}$ would necessarily contain another biflat $F|G$ satisfying $01345 \subsetneq F \subsetneq E$ and $12567 \supsetneq G \supsetneq 267$.

3.5. **The Chow ring of the conormal fan.** For notational simplicity, we set

$$S_{\mathrm{M},\mathrm{M}^\perp} = S(\Sigma_{\mathrm{M},\mathrm{M}^\perp}), \quad I_{\mathrm{M},\mathrm{M}^\perp} = I(\Sigma_{\mathrm{M},\mathrm{M}^\perp}),$$
$$J_{\mathrm{M},\mathrm{M}^\perp} = J(\Sigma_{\mathrm{M},\mathrm{M}^\perp}), \quad A_{\mathrm{M},\mathrm{M}^\perp} = A(\Sigma_{\mathrm{M},\mathrm{M}^\perp}).$$

We identify the elements of $S_{\mathrm{M},\mathrm{M}^\perp}/I_{\mathrm{M},\mathrm{M}^\perp}$ with the piecewise linear functions on the conormal fan.

Let $x_{F|G}$ be the variable of the polynomial ring corresponding to the ray generated by $\mathbf{e}_{F|G}$ in the conormal fan. For any set $\mathcal{F}|\mathcal{G}$ of biflats of M, we write $x_{\mathcal{F}|\mathcal{G}}$ for the monomial

$$x_{\mathcal{F}|\mathcal{G}} = \prod_{F|G \in \mathcal{F}|\mathcal{G}} x_{F|G}.$$

We note that the piecewise linear function δ_j on the conormal fan satisfies the identity

$$\delta_j = \sum_{F|G} \delta_j(\mathbf{e}_{F|G}) x_{F|G} = \sum_{j \in F \cap G} x_{F|G}.$$

Similarly, the piecewise linear functions γ_j and $\overline{\gamma}_j$ satisfy the identities

$$\gamma_j = \sum_{j \in F \neq E} x_{F|G} \quad \text{and} \quad \overline{\gamma}_j = \sum_{j \in G \neq E} x_{F|G}.$$

Thus, in the above notation,

- $S_{\mathrm{M},\mathrm{M}^\perp}$ is the ring of polynomials in the variables $x_{F|G}$, where $F|G$ is a biflat of M,
- $I_{\mathrm{M},\mathrm{M}^\perp}$ is the ideal generated by the monomials $x_{\mathcal{F}|\mathcal{G}}$, where $\mathcal{F}|\mathcal{G}$ is not a biflag, and
- $J_{\mathrm{M},\mathrm{M}^\perp}$ is the ideal generated by the linear forms $\gamma_i - \gamma_j$ and $\overline{\gamma}_i - \overline{\gamma}_j$, for any i and j in E.

We write γ, $\overline{\gamma}$, and δ, respectively, for the equivalence classes of γ_j, $\overline{\gamma}_j$, and δ_j in the Chow ring of the conormal fan.

Definition 3.19. The fundamental weight $1_{\mathrm{M},\mathrm{M}^\perp}$ of the conormal fan defines the *degree map*

$$\deg\colon A^{n-1}_{\mathrm{M},\mathrm{M}^\perp} \longrightarrow \mathbb{R}, \quad x_{\mathcal{F}|\mathcal{G}} \longmapsto x_{\mathcal{F}|\mathcal{G}} \cap 1_{\mathrm{M},\mathrm{M}^\perp} = \begin{cases} 1 & \text{if } \mathcal{F}|\mathcal{G} \text{ is a biflag,} \\ 0 & \text{if } \mathcal{F}|\mathcal{G} \text{ is not a biflag.} \end{cases}$$

By Proposition 3.7, the degree map is a linear isomorphism. In other words, for maximal flag of biflats $\mathcal{F}|\mathcal{G}$ of M, the class of the monomial $x_{\mathcal{F}|\mathcal{G}}$ in the Chow ring of the conormal fan of M is nonzero and does not depend on $\mathcal{F}|\mathcal{G}$.

Recall that the projection π is a morphism from the conormal fan of M to the Bergman fan of M. The projection has the special property that the image of a cone in the conormal fan is a cone in the Bergman fan (and not just contained in one). This property leads to the following simple description of the pullback $\pi^*\colon A_{\mathrm{M}} \to A_{\mathrm{M},\mathrm{M}^\perp}$.

Proposition 3.20. *For any flag of nonempty proper flats $\mathcal{F}$ of* M*,*

$$\pi^*(x_{\mathcal{F}}) = \sum_{\mathcal{G}} x_{\mathcal{F}|\mathcal{G}},$$

where the sum is over all decreasing sequences $\mathcal{G}$ such that $\mathcal{F}|\mathcal{G}$ is a flag of biflats of M*.*

Dually, the pushforward of any Minkowski weight w on the conormal fan is given by

$$\pi_*(w)(\sigma_{\mathcal{F}}) = \sum_{\mathcal{G}} w(\sigma_{\mathcal{F}|\mathcal{G}}),$$

where the sum is over all decreasing sequences $\mathcal{G}$ such that $\mathcal{F}|\mathcal{G}$ is a flag of biflats of M.

Proof. Since $\pi(\mathbf{e}_{F|G}) = \mathbf{e}_F$, the pullback of the piecewise linear function x_F satisfies

$$\pi^*(x_F) = \sum_{G} x_{F|G},$$

where the sum is over all G such that $F|G$ is a biflat of M. Thus, for any given $\mathcal{F}$,

$$\pi^*(x_{\mathcal{F}}) = \prod_{F\in\mathcal{F}} \pi^*(x_F) = \sum_{\mathcal{G}} x_{\mathcal{F}|\mathcal{G}},$$

where the sum is over all decreasing sequences $\mathcal{G}$ such that $\mathcal{F}|\mathcal{G}$ is a flag of biflats of M. □

4. Degree computations in the conormal fan

Throughout this section, we fix a matroid M of rank $r+1$ on the ground set $E = \{0, 1, \dots, n\}$. We fix the usual ordering on the ground set

$$0 < 1 < \cdots < n.$$

We aim to evaluate various elements of $A(\Sigma_{\mathrm{M},\mathrm{M}^\perp})$ under the degree map (Definition 3.19).

Let $\mathcal{F} = (F_1 \subsetneq \cdots \subsetneq F_k)$ be a flag of nonempty proper flats of M. The *beta invariant of* $\mathcal{F}$ is

$$\beta_{\mathrm{M}[\mathcal{F}]} = \prod_{i=0}^{k} \beta_{\mathrm{M}[F_i, F_{i+1}]},$$

where $\beta_{\mathrm{M}[F_{i-1}, F_i]}$ is the beta invariant of the minor $\mathrm{M}[F_{i-1}, F_i] = M|F_i/F_{i-1}$.[18] The main goal of this section is to prove Proposition 4.19 in Section 4.4, which states the identity

$$\deg(\pi^*(x_{\mathcal{F}})\delta^{n-k-1}) = \beta_{\mathrm{M}[\mathcal{F}]}.$$

[18]We continue to use the convention that $F_0 = \varnothing$ and $F_{k+1} = E$.

In particular, the degree of the conormal fan with respect to δ is the beta invariant:

$$\deg(\delta^{n-1}) = \beta_{\mathrm{M}}.$$

The result will be used in Section 4.5 to prove Theorems 1.1 and 1.2 stated in Section 1.

Since $\pi^*(x_{\mathcal{F}}) = \sum x_{\mathcal{F}|\mathcal{G}}$ by Proposition 3.20, it is enough to compute the degree of $x_{\mathcal{F}|\mathcal{G}}\, \delta^{n-k-1}$ for all possible $\mathcal{G}$. We will show in Lemma 4.15 that, in fact, the degree is nonzero for at most one $\mathcal{G}$.

The degree computation will require us to study more closely the combinatorial structure of conormal fans, and develop algebraic combinatorial techniques for computing in their Chow rings.

4.1. **Canonical expansions in the conormal fan.** In order to compute the degree of $x_{\mathcal{F}|\mathcal{G}}\, \delta^{n-k-1}$, we seek to express it as a sum of square-free monomials, each of which has degree one. One fundamental feature of this computation, which is simultaneously an advantage and a difficulty, is that there are many ways to carry it out. We may choose any one of the different expressions for δ to compute, namely $\delta = \delta_i$ for each i in E. To have control over the computation, we require some structure amidst that freedom.

For every nonnegative integer m, we prescribe a canonical way of expressing $x_{\mathcal{F}|\mathcal{G}}\, \delta^m$ as a sum of square-free monomials. Let $e = e(\mathcal{F}|\mathcal{G})$ be the largest gap element of $\mathcal{F}|\mathcal{G}$, which exists by Lemma 3.13. In the notation of that lemma, we have

$$e = \max\Big(\bigsqcup_{j=0}^{k} D_j\Big) = \max\Big(E - \bigcup_{i=1}^{k}(F_i \cap G_i)\Big),$$

where $D_0, \ldots, D_k$ is the gap sequence of $\mathcal{F}|\mathcal{G}$ defined in Definition 3.10.

Definition 4.1. The *canonical expansion of* $x_{\mathcal{F}|\mathcal{G}}\, \delta$ is the expression

$$x_{\mathcal{F}|\mathcal{G}}\, \delta = x_{\mathcal{F}|\mathcal{G}}\, \delta_e = \sum_{e \in F \cap G} x_{\mathcal{F}|\mathcal{G}} x_{F|G},$$

where the sum is over all biflats $F|G$ such that $e \in F \cap G$.

We recursively obtain the *canonical expansion of* $x_{\mathcal{F}|\mathcal{G}}\, \delta^m$ by multiplying each monomial in the canonical expansion of $x_{\mathcal{F}|\mathcal{G}}\, \delta^{m-1}$ by δ, again using the canonical expansion.

The canonical expansions are sums of square-free monomials in $A_{\mathrm{M},\mathrm{M}^\perp}$. Note that some or all of its summands may be zero in the Chow ring. Lemma 4.2 describes the nonzero terms.

Lemma 4.2. *If a summand* $x_{\mathcal{F}|\mathcal{G}} x_{F|G}$ *of the canonical expansion of* $x_{\mathcal{F}|\mathcal{G}}\, \delta$ *is nonzero and* e *is in* D_j*, then* $F_j \subseteq F \subseteq F_{j+1}$ *and* $G_j \supseteq G \supseteq G_{j+1}$*.*

Proof. If $\sigma_{\mathcal{F}\cup F|\mathcal{G}\cup G}$ is a cone in the conormal fan with $e \in F \cap G$, then $e \notin F_j$ and $e \notin G_{j+1}$. Thus, the biflat $F|G$ must be added to $\mathcal{F}|\mathcal{G}$ in between the indices j and $j+1$. $\square$

We may think of the canonical expansion of δ^m as a recursive procedure to produce a list of m-dimensional cones in the conormal fan $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$, where each cone is built up one ray at a time according to the rules prescribed in Lemma 4.2.

Example 4.3. For the graph G of the square pyramid in Figure 6, the canonical expansion of the highest nonzero power of δ in $A_{\mathrm{M},\mathrm{M}^\perp}$ is

$$\begin{aligned}\delta^6 =& x_{6|E}\, x_{56|E}\, x_{4567|E}\, x_{E|23467}\, x_{E|347}\, x_{E|7} \\ &+ x_{7|E}\, x_{67|E}\, x_{4567|E}\, x_{E|235}\, x_{E|35}\, x_{E|5} \\ &+ x_{7|E}\, x_{57|E}\, x_{4567|E}\, x_{E|23467}\, x_{E|36}\, x_{E|6}.\end{aligned}$$

This expression is deceivingly short. Carrying out this seemingly simple computation by hand is very tedious; if one were to do it by brute force, one would find that the number of terms of the canonical expansions of $\delta^0, \ldots, \delta^6$ is the following:

	δ^0	δ^1	δ^2	δ^3	δ^4	δ^5	δ^6
number of monomials counted with multiplicities	1	29	352	658	383	69	3
number of distinct monomials	1	29	333	621	370	68	3

Example 4.3 shows typical behavior: for small k the number of cones in the expansion of δ^k increases with k, but as k approaches $n-1$, increasingly many products $x_{\mathcal{F}|\mathcal{G}}\,\delta$ are zero, and the canonical expansions become shorter.

We record an explicit description of the canonical expansion of powers of δ in Proposition 4.4.

Proposition 4.4. *For each m, the canonical expansion of δ^m is given by*

$$\delta^m = \sum_{(\mathcal{F}|\mathcal{G},\mathbf{e})} x_{\mathcal{F}|\mathcal{G}},$$

where the sum is over all pairs $\mathcal{F}|\mathcal{G} = (F_1|G_1, \ldots, F_m|G_m)$ and $\mathbf{e} = (e_1, \ldots, e_m)$ satisfying

$$e_i \in F_i \cap G_i \quad \text{and} \quad e_i = \max\Big(E - \bigcup_{e_j > e_i} (F_j \cap G_j)\Big) \quad \text{for all } 1 \le i \le m.$$

Proof. The displayed formula for δ^m is an immediate consequence of the construction of the canonical decomposition. ☐

It will be convenient to encode each summand of the canonical expansion in a table:

$\varnothing$	$\subsetneq$	F_1	$\subseteq$	$\cdots$	$\subseteq$	F_i	$\subseteq$	$\cdots$	$\subseteq$	F_m	$\subseteq$	E
E	$\supseteq$	G_1	$\supseteq$	$\cdots$	$\supseteq$	G_i	$\supseteq$	$\cdots$	$\supseteq$	G_m	$\supsetneq$	$\varnothing$
		e_1		$\cdots$		e_i		$\cdots$		e_m		

The canonical expansion of δ^m may contain repeated terms $x_{\mathcal{F}|\mathcal{G}}$ coming from tables that have the same biflag $\mathcal{F}|\mathcal{G}$ but different sequences $\mathbf{e}$, as the numerics for the canonical expansions of $\delta^2, \delta^3, \delta^4, \delta^5$ in Example 4.3 show. On the other hand, we will see in Proposition 4.9 that the canonical expansion of δ^{n-1} does not contain repeated terms.

Example 4.5. We revisit the canonical expansion of δ^6 in Example 4.3. The first monomial arises from the following table:

$\varnothing$	$\subset$	6	$\subsetneq$	56	$\subsetneq$	4567	$\subsetneq$	E	$=$	E	$=$	E	$=$	E
E	$=$	E	$=$	E	$=$	E	$\supsetneq$	23467	$\supsetneq$	347	$\supsetneq$	7	$\supset$	$\varnothing$
		6		5		4		2		3		7		

The variables $x_{F_i|G_i}$ arrive at the monomial in the order

$$x_{E|7} x_{6|E} x_{56|E} x_{4567|E} x_{E|347} x_{E|23467},$$

in decreasing order of the e_is. The two other monomials are

$$x_{7|E}\, x_{67|E}\, x_{E|5}\, x_{4567|E}\, x_{E|35}\, x_{E|235}$$

and

$$x_{7|E}\, x_{E|6}\, x_{57|E}\, x_{4567|E}\, x_{E|36}\, x_{E|23467},$$

where the terms are again listed in their order of arrival.

4.2. **The degree of the conormal fan.** We now prove Proposition 4.9, which shows that the degree of δ^{n-1} in the Chow ring of the conormal fan of M is the beta invariant of M. Proposition 4.9 will be used later to obtain the more general Proposition 4.19.

We write cl and $\mathrm{cl}^\perp$ for the closure operators of M and $\mathrm{M}^\perp$. For each basis B of M, denote the corresponding basis of $\mathrm{M}^\perp$ by $B^\perp := E - B$.

Definition 4.6. A *broken circuit* of M is a set of the form $C - \min C$ where C is a circuit of M.

(1) An *nbc-basis* of M is a basis of M that contains no broken circuits of M.
(2) A *β-nbc-basis* of M is an *nbc*-basis B of M such that $B^\perp \cup \{0\} - \{1\}$ is an *nbc*-basis of $\mathrm{M}^\perp$.

The number of *nbc*-bases of M is the Möbius number $|\mu_{\mathrm{M}}|$, whereas the number of β-*nbc*-bases of M is the beta invariant β_{M}. The independence complex IN(M) and the reduced broken circuit complex $\overline{\mathrm{BC}}$(M) of M are shellable, and hence homotopy equivalent to wedges of spheres. The *nbc*-bases and β-*nbc*-bases of M naturally index the spheres in the lexicographic shellings of IN(M) and $\overline{\mathrm{BC}}$(M), respectively. For the *nbc* and β-*nbc* facts stated in this paragraph, we refer to [Bjö92] and [Zie92].

Definition 4.7. For a β-*nbc*-basis B of M, we define a sequence $e_1, \ldots, e_{n-1}$ by setting

$$B - 0 = \{e_1 > \cdots > e_r\} \quad \text{and} \quad B^\perp - 1 = \{e_{r+1} < \cdots < e_{n-1}\}.$$

We write β-cone(B) for the maximal cone in $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ corresponding to the table

$\mathrm{cl}(e_1)$	$\subsetneq$	$\cdots$	$\subsetneq$	$\mathrm{cl}(e_1,\ldots,e_r)$	$\subsetneq$	E	$=$	$\cdots$	$=$	E
E	$=$	$\cdots$	$=$	E	$\supsetneq$	$\mathrm{cl}^\perp(e_{r+1},\ldots,e_{n-1})$	$\supsetneq$	$\cdots$	$\supsetneq$	$\mathrm{cl}^\perp(e_{n-1})$

The unique double jump of the displayed biflag is r, one less than the rank of M.

To see that the displayed table indeed defines a biflag, we verify

$$\mathrm{cl}(B-0) \cup \mathrm{cl}^\perp(B^\perp - 1) \neq E.$$

Since $B^\perp$ is a basis of $\mathrm{M}^\perp$, we have $1 \notin \mathrm{cl}^\perp(B^\perp - 1)$; and if we had $1 \in \mathrm{cl}(B-0)$, then $B - 0 \cup 1$ would contain a circuit C whose minimum element is 1, and hence B would contain the broken circuit $C - 1$, contradicting that B is *nbc*.

Example 4.8. The matroid of Figure 6 has three β-*nbc*-bases, namely

$$B_1 = 0456, \quad B_2 = 0457, \quad B_3 = 0467.$$

The corresponding maximal biflags are precisely the ones in the expansion of Example 4.3.

We show that this is a general phenomenon.

Proposition 4.9. *Let* M *be a loopless and coloopless matroid on* E*. Then, in the Chow ring of the conormal fan of* M*, we have the canonical expansion*

$$\delta^{n-1} = \sum_{B \in \beta\text{-}nbc(\mathrm{M})} x_{\beta\text{-cone}(B)},$$

where the sum is over the β*-nbc-bases of* M*.*

Thus, the degree of δ^{n-1} *in the Chow ring of the conormal fan of* M *is the* β*-invariant of* M*.*

We proceed with a series of elementary lemmas. Proposition 4.4 describes the canonical expansion of δ^{n-1} in terms of pairs $(\mathcal{F}|\mathcal{G}, \mathbf{e})$ of the form

$\varnothing$	$\subsetneq$	F_1	$\subseteq$	$\cdots$	$\subseteq$	F_d	$\subsetneq$	F_{d+1}	$\subseteq$	$\cdots$	$\subseteq$	F_{n-1}	$\subseteq$	E
E	$\supseteq$	G_1	$\supseteq$	$\cdots$	$\supseteq$	G_d	$\supsetneq$	G_{d+1}	$\supseteq$	$\cdots$	$\supseteq$	G_{n-1}	$\supsetneq$	$\varnothing$
		e_1		$\cdots$		e_d		e_{d+1}		$\cdots$		e_{n-1}		

,

which have a unique double jump d by Proposition 3.17. A priori, the double jump could occur at any d. We will show that, in fact, the double jump must occur at $d = r$. In the remainder of this section, we fix a pair $(\mathcal{F}|\mathcal{G}, \mathbf{e})$ that gives a nonzero summand of the canonical expansion of δ^{n-1}, and write e_n and e_{n+1} for the two elements of E missing from the sequence $\mathbf{e}$.

Lemma 4.10. *We consider the jump sets of* $\mathcal{F}$ *and* $\mathcal{G}$ *in Definition* 3.16*.*

(1) *If* i *is in the jump set of* $\mathcal{F}$ *but not in the jump set of* $\mathcal{G}$*, then* $e_i > e_{i+1}$*.*
(2) *If* i *is in the jump set of* $\mathcal{G}$ *but not in the jump set of* $\mathcal{F}$*, then* $e_i < e_{i+1}$*.*
(3) *If* $i < j$ *and* $e_i < e_j$*, then* $e_i \notin G_j$*.*
(4) *If* $i < j$ *and* $e_i > e_j$*, then* $e_j \notin F_i$*.*

Proof. We prove the first statement. By way of contradiction, suppose i is in the jump set of $\mathcal{F}$, not in the jump set of $\mathcal{G}$, and $e_i < e_{i+1}$.

Then, in the canonical expansion of δ^{n-1}, the variable $x_{F_i|G_i}$ arrives after the variable $x_{F_{i+1}|G_{i+1}}$ to the monomial of $(\mathcal{F}|\mathcal{G}, \mathbf{e})$. It follows that e_i is not in the intersection $F_{i+1} \cap G_{i+1}$, and this contradicts $e_i \in F_i \cap G_i \subseteq F_{i+1} \cap G_i = F_{i+1} \cap G_{i+1}$.

We prove the third statement. Suppose that $i < j$ and $e_i < e_j$. Then, in the canonical expansion of δ^{n-1}, the variable $x_{F_i|G_i}$ arrives after the variable $x_{F_j|G_j}$ to the monomial of $(\mathcal{F}|\mathcal{G}, \mathbf{e})$. It follows that $e_i \notin F_j \cap G_j$. Since $e_i \in F_i \subseteq F_j$, we have $e_i \notin G_j$. □

Lemma 4.11. *The unique double jump is at* $d = r$*, and the table of* $(\mathcal{F}|\mathcal{G}, \mathbf{e})$ *is of the form*

$\varnothing$	$\subsetneq$	F_1	$\subsetneq$	$\cdots$	$\subsetneq$	F_r	$\subsetneq$	E	$=$	$\cdots$	$=$	E	$=$	E
E	$=$	E	$=$	$\cdots$	$=$	E	$\supsetneq$	G_{r+1}	$\supsetneq$	$\cdots$	$\supsetneq$	G_{n-1}	$\supsetneq$	$\varnothing$
		e_1	$>$	$\cdots$	$>$	e_r		e_{r+1}	$<$	$\cdots$	$<$	e_{n-1}		

.

Proof. Since $\{e_1, \ldots, e_d\} \subseteq F_d$ and $\{e_{d+1}, \ldots, e_{n-1}\} \subseteq G_{d+1}$, Lemma 3.14 gives $F_d \cup G_{d+1} = \{e_1, \ldots, e_{n-1}\}$. Therefore, the unique nonempty gap $D_d = E - (F_d \cup G_{d+1})$ is equal to $\{e_n, e_{n+1}\}$.

We next show $e_1 > e_2 > \cdots > e_d$ and $e_{d+1} < \cdots < e_{n-2} < e_{n-1}$.

By symmetry, it suffices to show the first set of inequalities. For contradiction, suppose $e_j < e_{j+1}$ for a minimal choice of $j < d$. If $j > 1$, then $e_{j-1} > e_j$ implies $e_j \notin F_{j-1}$ by Lemma 4.10(4); if $j = 1$ this holds trivially. On the other hand,

$e_j < e_{j+1}$ implies $e_j \notin G_{j+1}$ by Lemma 4.10(3). However, we have

$$\{e_1, \ldots, e_{j-1}\} \subseteq F_{j-1}, \quad \{e_{j+1}, \ldots, e_{n-1}\} \subseteq G_{j+1},$$
$$\text{and} \quad \{e_n, e_{n+1}\} \subseteq G_d \subseteq G_{j+1}.$$

It follows that $F_{j-1} \cup G_{j+1} = E - e_j$, contradicting Lemma 3.14.

For $1 \leq j < d$, the inequality $e_j > e_{j+1}$ implies $e_{j+1} \in F_{j+1} - F_j$, and hence j is in the jump set of $\mathcal{F}$. It follows that the jump set of $\mathcal{F}$ is $\{0, \ldots, d\}$, and similarly, the jump set of $\mathcal{G}$ is $\{d, \ldots, n-1\}$. □

Lemma 4.12. *We have* $\{e_1, \ldots, e_{n-1}\} = \{2, 3, \ldots, n\}$ *and* $\{e_n, e_{n+1}\} = \{0, 1\}$. *Moreover,*

$$e_i = \min F_i \quad \textit{and} \quad F_i = \mathrm{cl}(e_1, \ldots, e_i), \quad \textit{for } 1 \leq i \leq r,$$
$$e_i = \min G_i \quad \textit{and} \quad G_i = \mathrm{cl}^{\perp}(e_i, \ldots, e_{n-1}) \quad \textit{for } r < i < n.$$

In particular, the biflag $\mathcal{F}|\mathcal{G}$ *and the sequence* $\mathbf{e}$ *determine each other.*

Proof. We may assume without loss of generality that $e_r < e_{r+1}$, so that the variable x_{F_r, G_r} is the last term to arrive in the monomial corresponding to $(\mathcal{F}|\mathcal{G}, \mathbf{e})$. By definition, we have

$$e_r = \max\Big(E - \bigcup_{\substack{1 \leq j \leq n-1 \\ j \neq r}} (F_j \cap G_j)\Big) = \max\Big(E - (F_{r-1} \cup G_{r+1})\Big).$$

If we had $e_r \leq 1$, then $|F_{r-1} \cup G_{r+1}| \geq n-1$, which would imply $|F_r \cup G_{r+1}| = n$, contradicting Lemma 3.14. Thus $e_r = 2$ and $F_r \cup G_{r+1} = E - \{0, 1\}$.

We now show that $e_i = \min F_i$ for $1 \leq i \leq r$. If this was not the case, then we would have $\min F_i = e_j < e_i$ for some $j \neq i$, because 0 and 1 are not in F_i. Since $e_1 > \cdots > e_i$, this would imply $i < j$, and Lemma 4.10(4) would tell us that $e_j \notin F_i$, contradicting $e_j = \min F_i$. Similarly, we have $e_i = \min G_i$ for $r < i < n$.

Finally, since F_i has rank i and $e_i \in F_i - F_{i-1}$ for $1 \leq i \leq r$ by Lemma 4.11, the list $e_1, \ldots, e_i$ must be a basis of F_i for $1 \leq i \leq r$. The analogous statement holds for G_i as well. □

Lemma 4.13. *The set* $\{0, e_1, \ldots, e_r\}$ *is a* β*-nbc-basis of* M.

Proof. Since $e_r = \min F_r$ by Lemma 4.12, we have $0 \notin F_r$, and hence $B = \{0, e_1, \ldots, e_r\}$ indeed is a basis of M. We prove by contradiction that B is *nbc*. Assume that B contains a broken circuit $C - \min C$. Since $\min C$ is not in B, one of the following must hold:

(i) $\min C = 1$. Let $C = \{1, e_{a(1)}, \ldots, e_{a(k)}\}$ where $1 \leq a(1) < \ldots < a(k) \leq r$. Then
$$1 \in \mathrm{cl}(e_{a(1)}, \ldots, e_{a(k)}) \subseteq F_{a(k)} \subseteq F_r.$$
This contradicts Lemma 4.12, which shows that the unique nonempty gap is $D_r = \{0, 1\}$.

(ii) $\min C = e_s$ for $s \geq r+1$. Let $C = \{e_s, e_{a(1)}, \ldots, e_{a(k)}\}$ where $1 \leq a(1) < \ldots < a(k) \leq r$. Then
$$e_s \in \mathrm{cl}(e_{a(1)}, \ldots, e_{a(k)}) \subseteq F_{a(k)}.$$
This contradicts Lemma 4.10(4), since $a(k) \leq r < s$ and $e_{a(k)} > e_s$.

The same argument for $\mathrm{M}^{\perp}$ shows that $B^{\perp} - \{0\} \cup \{1\}$ is an *nbc*-basis of $\mathrm{M}^{\perp}$. We conclude that B is a β-*nbc* basis of M, as desired. □

We now have all the ingredients to complete the proof of Proposition 4.9.

Proof of Proposition 4.9. Lemma 4.12 tells us that each monomial $x_{\mathcal{F}|\mathcal{G}}$ that appears in the canonical expansion of δ^{n-1} has coefficient 1. Combined with Lemma 4.13, it also tells us that every term that appears is of the form $x_{\beta\text{-cone}(B)}$ for a β-*nbc*-basis B.

Conversely, if $\mathcal{F}|\mathcal{G}$ is the biflag corresponding to the β-cone of a β-*nbc*-basis B, and if we define $\mathbf{e}$ by setting

$$B = \{e_1 > \cdots > e_r > 0\}$$

and

$$E - B = \{e_{n-1} > \cdots > e_{r+1} > 1\},$$

then $(\mathcal{F}|\mathcal{G}, \mathbf{e})$ satisfies the conditions of Proposition 4.4, so it does arise in the canonical expansion of δ^{n-1}. □

4.3. **A vanishing lemma for the conormal fan.** Throughout the remainder of this section, we fix a flag of k nonempty proper flats

$$\mathcal{F} = \{F_1 \subsetneq \cdots \subsetneq F_k\}, \text{ keeping the convention that } F_0 = \varnothing \text{ and } F_{k+1} = E.$$

The interval $\mathrm{M}(i) = [F_i, F_{i+1}]$ is said to be *short* if $|F_{i+1} - F_i| = 1$ and *long* if $|F_{i+1} - F_i| > 1$.

Definition 4.14. We define the *orthogonal flag* $\mathcal{F}^\perp$ of $\mathcal{F}$ to be the flag of coflats

$$\mathcal{F}^\perp = \{F_1^\perp \supseteq \cdots \supseteq F_k^\perp\}, \quad \text{where} \quad F_i^\perp = \mathrm{cl}^\perp(E - F_i) \text{ for } 1 \le i \le k.$$

The orthogonal flag may contain repeated coflats, and it may contain the trivial coflat E.[19]

Our goal this section is to prove Lemma 4.15, which shows that many monomials in the Chow ring of the conormal fan vanish when multiplied by the highest possible power of δ.

Lemma 4.15 (Vanishing lemma). *Suppose $\mathcal{F}|\mathcal{G}$ is a biflag of length k satisfying the condition*

$$x_{\mathcal{F}|\mathcal{G}}\, \delta^{n-k-1} \textit{ is nonzero in the Chow ring of the conormal fan of } \mathrm{M}.$$

Then $\mathcal{G}$ must be the orthogonal flag $\mathcal{F}^\perp$. Furthermore, the interval $\mathrm{M}(i)$ *is either short or loopless and coloopless for all* $0 \le i \le k$.

Let $x_{\mathcal{F}^+|\mathcal{G}^+}$ be a nonzero summand of the canonical expansion of $x_{\mathcal{F}|\mathcal{G}}\, \delta^{n-k-1}$, and let

$$\mathcal{F}|\mathcal{G} = \mathcal{F}_k|\mathcal{G}_k,\ \mathcal{F}_{k+1}|\mathcal{G}_{k+1},\ \ldots,\ \mathcal{F}_{n-1}|\mathcal{G}_{n-1} = \mathcal{F}^+|\mathcal{G}^+$$

be some sequence of biflags obtained by recursively applying Lemma 4.2 in the expansion. We write $D_{i,0}|\cdots|D_{i,i}$ for the gap sequence of $\mathcal{F}_i|\mathcal{G}_i$ in Definition 3.10. With Lemma 3.13 in mind, we set

$$Y_i = \bigsqcup_{j=0}^{i} D_{i,j} = E - \bigcup_{F|G \in \mathcal{F}_i|\mathcal{G}_i} (F \cap G). \tag{4.3.1}$$

We write $D_0|\cdots|D_k$ and Y for the gap sequence and the union of the gaps of the initial flag $\mathcal{F}|\mathcal{G}$. To prove the Vanishing Lemma 4.15, we need a preliminary result.

[19] Strictly speaking, the notation $F^\perp$ conflicts with the notation $B^\perp$ used in Section 4.2 for the dual basis of a basis B. We trust that no confusion will arise within a given context.

Lemma 4.16. *Suppose that the assumption of Lemma* 4.15 *holds for* $\mathcal{F}|\mathcal{G}$*.*

(1) *If* $\mathcal{F}|\mathcal{G}$ *has* m *empty gaps, then the union of its gaps has size* $|Y| = n+1-m$*.*
(2) *For each empty gap* D_j*, we have* $F_{j+1} - F_j = \{e_j\}$ *for some* $e_j \in E$*. Furthermore,*
$$Y = E - \{e_j \mid D_j = \varnothing\}.$$
(3) *For all* $0 \le i \le k$*, setting* $r_i = \mathrm{rank}_{\mathrm{M}}(F_i)$ *and* $r_i^\perp = \mathrm{rank}_{\mathrm{M}^\perp}(G_i)$*, we have*
$$|F_{i+1} - F_i| = (r_{i+1} - r_i) + (r_i^\perp - r_{i+1}^\perp).$$

Proof of Lemma 4.16. Let m be the number of empty gaps of $\mathcal{F}|\mathcal{G}$. We first prove the inequality

$$|Y| \le n+1-m. \tag{4.3.2}$$

For each empty gap D_j, choose an element $e_j \in F_{j+1} - F_j$. Since $e_j \notin D_j = E - (F_j \cup G_{j+1})$, we must have $e_j \in G_{j+1}$. This implies that $e_j \in F_{j+1} \cap G_{j+1}$, so the second equality in (4.3.1) gives $e_j \notin Y$. There are m such elements e_j, which are all distinct by construction; this implies (4.3.2).

To prove the first statement, it remains to show the opposite inequality

$$|Y| \ge n+1-m. \tag{4.3.3}$$

We obtained $\mathcal{F}_{i+1}|\mathcal{G}_{i+1}$ from $\mathcal{F}_i|\mathcal{G}_i$ by choosing the largest gap element $e = \max Y_i$, finding the unique gap $D_{i,j}$ of $\mathcal{F}_i|\mathcal{G}_i$ containing e, and inserting a new pair $F|G$ with $e \in F \cap G$ between the j-th and $(j+1)$-th biflats of $\mathcal{F}_i|\mathcal{G}_i$:

$$\mathcal{F}_{i+1}|\mathcal{G}_{i+1} = \boxed{\begin{array}{ccccccccc} \cdots & \subseteq & F_{i,j} & \subseteq & F & \subseteq & F_{i,j+1} & \subseteq & \cdots \\ \cdots & \supseteq & G_{i,j} & \supseteq & G & \supseteq & G_{i,j+1} & \supseteq & \cdots \end{array}}$$

Thus the only difference between the gaps of $\mathcal{F}_i|\mathcal{G}_i$ and the gaps of $\mathcal{F}_{i+1}|\mathcal{G}_{i+1}$ is that we are replacing the gap $D_{i,j}$ with two smaller disjoint gaps $D_{i+1,j}$ and $D_{i+1,j+1}$ that do not contain e:

$$D_{i,j} \supseteq D_{i+1,j} \sqcup D_{i+1,j+1} \sqcup e. \tag{4.3.4}$$

It is helpful to visualize this data as a graded forest of levels from k to $n-1$. The vertices of the bottom level k are the gaps $D_0, \ldots, D_k$ of the original biflag $\mathcal{F}|\mathcal{G}$; they are the roots of the trees in the forest. The vertices of the i-th level are the gaps of $\mathcal{F}_i|\mathcal{G}_i$. To go from level i to level $i+1$, we connect the split gap $D_{i,j}$ with the gaps $D_{i+1,j}$ and $D_{i+1,j+1}$ that replace it. Every other gap $D_{i,k}$ is connected to the gap in the next level that is equal to it; this is $D_{i+1,k}$ if $k < j$ and $D_{i+1,k+1}$ if $k > j$. The final biflag $\mathcal{F}^+|\mathcal{G}^+$ has n gaps, of which $n-1$ are empty and one of them, say D, has size at least 2. Each gap of $\mathcal{F}^+|\mathcal{G}^+$ originates from one of the original gaps of $\mathcal{F}|\mathcal{G}$ through successive gap replacements. For each $0 \le i \le k$, we set

$$d_i = \text{number of gaps of } \mathcal{F}^+|\mathcal{G}^+ \text{ that descend from the initial gap } D_i \text{ of } \mathcal{F}|\mathcal{G}.$$

We give an upper bound of d_i in terms of $|D_i|$, in each of the following three cases:

Case 1 ($D_i = \varnothing$). In this case, the gap D_i eventually becomes a single empty gap in $\mathcal{F}^+|\mathcal{G}^+$, so $d_i = 1$.

Case 2 ($D_i \ne \varnothing$ is the progenitor of the unique nonempty gap D of $\mathcal{F}^+|\mathcal{G}^+$). Consider the gaps that descend from D_i throughout the process. By (4.3.4), every time one such gap gets replaced by two smaller ones, the size of the union of the gaps strictly decreases. In the end, this union has size $|D| \geq 2$. Therefore these gaps were split at most $|D_i| - 2$ times, so $d_i \leq |D_i| - 1$.

Case 3 ($D_i \neq \varnothing$ is not the progenitor of the unique nonempty gap D of $\mathcal{F}^+|\mathcal{G}^+$). Again, every time a descendant of D_i gets replaced by two smaller ones, the size of their union decreases. Furthermore, their union can never have size 1 by Proposition 3.15. Thus $d_i \leq |D_i|$. Since the final number of gaps is n, we conclude that

$$n = \sum_{i=0}^{k} d_i \leq m + \left(\sum_{i:D_i \neq \varnothing} |D_i| \right) - 1 = m + |Y| - 1.$$

This proves the opposite inequality (4.3.3), and hence the first statement of the lemma. Furthermore, every inequality we applied along the way must in fact have been an equality. We record these facts:

(a) For (4.3.2) to be an equality, we must have $F_{j+1} - F_j = \{e_j\}$ for each empty gap D_j, and

$$Y = E - \{e_j \mid D_j = \varnothing\}.$$

This proves the second statement of the lemma.

(b) For (4.3.3) to be an equality, we must have

$$d_i = 1 \text{ in Case 1}, \quad d_i = |D_i| - 1 \text{ in Case 2}, \quad \text{and} \quad d_i = |D_i| \text{ in Case 3}.$$

We use (a) and (b) to prove the third statement of the lemma, in two steps. First, we show

$$(4.3.5) \qquad d_i = \begin{cases} |F_{i+1} - F_i| & \text{in Case 1 and Case 3,} \\ |F_{i+1} - F_i| - 1 & \text{in Case 2.} \end{cases}$$

If D_i is empty, then $d_i = 1$ and $|F_{i+1} - F_i| = 1$ by (a). If D_i is nonempty, we claim that

$$D_i = F_{i+1} - F_i.$$

The forward inclusion holds by definition. For the backward inclusion, let e be an element of $F_{i+1} - F_i$. By (a), we must have $e \in Y$, and since D_i is the only gap intersecting $F_{i+1} - F_i$, we must have $e \in D_i$. Thus (b) implies (4.3.5). Second, we show

$$(4.3.6) \qquad d_i = \begin{cases} (r_{i+1} - r_i) + (r_i^\perp - r_{i+1}^\perp) & \text{in Case 1 and Case 3,} \\ (r_{i+1} - r_i) + (r_i^\perp - r_{i+1}^\perp) - 1 & \text{in Case 2.} \end{cases}$$

If D_i is not the progenitor of D, then the part of $\mathcal{F}^+|\mathcal{G}^+$ between $F_i|G_i$ and $F_{i+1}|G_{i+1}$ contains no double jumps. In each of the d_i single jumps, either the rank increases by 1 or the corank decreases by 1, but not both. Therefore d_i must equal the sum of the rank increase $r_{i+1} - r_i$ and the corank decrease $r_i^\perp - r_{i+1}^\perp$. If D_i is the progenitor of D, then the part of $\mathcal{F}^+|\mathcal{G}^+$ between $F_i|G_i$ and $F_{i+1}|G_{i+1}$ contains one double jump. In each of the $d_i - 1$ single jumps, either the rank increases by 1 or the corank decreases by 1, but not both. In the double jump, both changes occur. Therefore $d_i + 1$ must equal the sum of the rank increase $r_{i+1} - r_i$ and the corank decrease $r_i^\perp - r_{i+1}^\perp$. Combining (4.3.5) and (4.3.6), we get the third statement of the lemma. ☐

Proof of the Vanishing Lemma 4.15. First, we prove that $\mathcal{G}$ must be the orthogonal flag $\mathcal{F}^\perp$. We write rk and $\mathrm{rk}^\perp$ for the rank functions of M and $\mathrm{M}^\perp$. One readily verifies that

$$(\mathrm{rk}(F_{i+1})-\mathrm{rk}(F_i))+(\mathrm{rk}^\perp(E-F_i)-\mathrm{rk}^\perp(E-F_{i+1})) = |F_{i+1}-F_i| \qquad \text{for } 0 \le i \le k.$$

By the third statement of Lemma 4.16, the sequences $\mathrm{rk}^\perp(E - F_i)$ and $\mathrm{rk}^\perp(G_i)$ satisfy the same recurrence; they also have the same initial value, so

$$\mathrm{rk}^\perp(E - F_i) = \mathrm{rk}^\perp(G_i) \quad \text{for } 0 \le i \le k.$$

Now $F_i \cup G_i = E$ implies $G_i \supseteq E - F_i$. Since G_i is a coflat, we have $G_i \supseteq \mathrm{cl}^\perp(E - F_i) = F_i^\perp$. It follows that $G_i \supseteq F_i^\perp$ are flats of $\mathrm{M}^\perp$ of the same rank, and hence $G_i = F_i^\perp$ for all i.

Next, we prove that every long interval $\mathrm{M}(i)$ must be loopless and coloopless. We argue by contradiction.

First assume that $\mathrm{M}(i) = (\mathrm{M}/F_i)|(F_{i+1} - F_i)$ has a loop l. Since restriction cannot create new loops, the element l must also be a loop of M/F_i. This contradicts the fact that F_i is a flat.

Now assume that $\mathrm{M}(i) = (\mathrm{M}|F_{i+1})/F_i$ has a coloop c. Since contraction cannot create new coloops, the element c must also be a coloop of $\mathrm{M}|F_{i+1}$. Thus $\mathrm{rk}(F_{i+1} - c) = \mathrm{rk}(F_{i+1}) - 1$, which implies that $\mathrm{rk}^\perp((E - F_{i+1}) \cup c) = \mathrm{rk}^\perp(E - F_{i+1})$. This means that $c \in \mathrm{cl}^\perp(E-F_{i+1}) = F_{i+1}^\perp$. Now, since $\mathrm{M}(i)$ is long, the second statement of Lemma 4.16 implies that D_{i+1} is nonempty and that $c \in Y$. But then we must have $c \in D_{i+1} = (F_{i+1} - F_i) \cap (F_i^\perp - F_{i+1}^\perp)$, contradicting that $c \in F_{i+1}^\perp$. □

4.4. **The beta invariant of a flag and the conormal intersection theory.** In this section, we complete the proof that the degree of $\pi^*(x_\mathcal{F})\delta^{n-k-1}$ is equal to the β-invariant $\beta_{\mathrm{M}[\mathcal{F}]}$. To prove by induction, we need a lemma relating the conormal fan of M with that of the contraction M/i, where i is any fixed element of E that has no parallel elements in M.

We may assume that M has no loops and no coloops. Thus, we have $i^\perp$ is the ground set E and $i|E$ is a biflat of M. We consider the simplicial fan

$$\mathrm{st}_{i|E}\, \Sigma_{\mathrm{M},\mathrm{M}^\perp} \subseteq (\mathrm{N}_E/\mathbf{e}_i) \oplus \mathrm{N}_E\,.$$

We write $\overline{\mathbf{e}}_S$ for the image of $\mathbf{e}_S$ in $\mathrm{N}_E/\mathbf{e}_i$, and $\overline{x}_{F|G}$ for the variable in the Chow ring of the star corresponding to a biflat $F|G$; we set it equal to 0 if $F|G$ does not correspond to a ray in this star.

Lemma 4.17. *Consider the natural projection* $\psi\colon (\mathrm{N}_E/\mathbf{e}_i)\oplus\mathrm{N}_E \longrightarrow \mathrm{N}_{E-i} \oplus \mathrm{N}_{E-i}$.

(1) *The projection ψ induces a morphism of fans*

$$\psi\colon\ \mathrm{st}_{i|E}\, \Sigma_{\mathrm{M},\mathrm{M}^\perp} \longrightarrow \Sigma_{\mathrm{M}/i,(\mathrm{M}/i)^\perp}.$$

(2) *The pullback ψ^* between the Chow rings is given by*

$$\psi^*(x_{P|Q}) = \overline{x}_{(P\cup i)|Q} + \overline{x}_{(P\cup i)|(Q\cup i)},$$

where at least one of the terms on the right-hand side is nonzero.

(3) *For any element j of E, the pullback ψ^* maps the class δ to the class*

$$\overline{\delta} = \overline{\delta}_j := \sum_{i\in F,\, j\in F\cap G} \overline{x}_{F|G}.$$

(4) *The pullback* ψ^* *commutes with the degree maps of the star and the conormal fan* $\Sigma_{\mathrm{M}/i,(\mathrm{M}/i)^\perp}$*:*

$$\deg_{\mathrm{M}/i} x_{\mathcal{P}|\mathcal{Q}} = \deg_{\mathrm{M}}(x_{i|E}\, \psi^*(x_{\mathcal{P}|\mathcal{Q}})).$$

Proof. Let $F|G$ be a biflat of M with $i \in F$. The image of a ray corresponding to $F|G$ in the star is

$$\psi(\overline{\mathbf{e}}_F + \mathbf{f}_G) = \mathbf{e}_{F-i} + \mathbf{f}_{G-i},$$

which is a ray of the conormal fan $\Sigma_{\mathrm{M}/i,(\mathrm{M}/i)^\perp}$ because $(F-i)|(G-i)$ is a biflat of M/i:

$$\begin{aligned} \mathrm{cl}_{\mathrm{M}/i}(F-i) &= \mathrm{cl}_{\mathrm{M}}(F) - i = F - i, \text{ and} \\ \mathrm{cl}_{\mathrm{M}^\perp - i}(G-i) &= \mathrm{cl}_{\mathrm{M}^\perp}(G-i) - i \subseteq \mathrm{cl}_{\mathrm{M}^\perp}(G) - i = G - i. \end{aligned}$$

Furthermore, if $i|E \cup \mathcal{F}|\mathcal{G}$ is a biflag of M, its gaps occur to the right of $i|E$, and there will also be gaps in the corresponding positions of the biflag

$$(\mathcal{F}-i)|(\mathcal{G}-i) := \Big\{(F-i)|(G-i)\Big\}_{F|G\in\mathcal{F}|\mathcal{G}} \text{ of } \mathrm{M}/i.$$

Therefore, the projection ψ maps cones to cones. This proves the first statement.

The value of the piecewise linear function $\psi^* x_{P|Q}$ on a ray $\overline{\mathbf{e}}_F + \mathbf{f}_G$ of the star is

$$\psi^* x_{P|Q}(\overline{\mathbf{e}}_F + \mathbf{f}_G) = x_{P|Q}(\mathbf{e}_{F-i} + \mathbf{f}_{G-i})) = \begin{cases} 1 & \text{if } F = P \cup i \text{ and } G \in \{Q, Q \cup i\}, \\ 0 & \text{if otherwise.} \end{cases}$$

Since Q is a flat of M/i, at least one of Q and $Q \cup i$ is a flat of M, and we have the second statement. Given the second statement, the third statement is a straightforward computation

$$\psi^*(\delta_j) = \sum_{j\in P\cap Q} \Big(\overline{x}_{(P\cup i)|Q} + \overline{x}_{(P\cup i)|(Q\cup i)}\Big) = \sum_{i\in F,\, j\in F\cap G} \overline{x}_{F|G} = \overline{\delta}_j,$$

where the first sum is over the biflats $P|Q$ of M/i and the second sum over the biflats $F|G$ of M.

For the last statement, we need to verify that, for any maximal biflag $\mathcal{P}|\mathcal{Q}$ of M/i,

$$\deg_{\mathrm{M}/i} x_{\mathcal{P}|\mathcal{Q}} = \deg_{\mathrm{M}}(x_{i|E}\, \psi^*(x_{\mathcal{P}|\mathcal{Q}})).$$

Applying the second statement to each $P|Q$ in $\mathcal{P}|\mathcal{Q}$, we may express $x_{i|E}\, \psi^*(x_{\mathcal{P}|\mathcal{Q}})$ as a sum of square-free monomials. One of the terms in this expression is $x_{i|E}\, x_{(\mathcal{P}\cup i)|\,\mathrm{cl}^\perp(\mathcal{Q})}$, where

$$(\mathcal{P}\cup i)|\,\mathrm{cl}^\perp(\mathcal{Q}) := \Big\{(P\cup i)|\,\mathrm{cl}^\perp(Q)\Big\}_{P|Q\in\mathcal{P}|\mathcal{Q}}.$$

We need to prove that this is the only nonzero term.

Consider any term $x_{i|E} x_{\mathcal{F}|\mathcal{G}}$ that arises in the expression for $x_{i|E}\, \psi^*(x_{\mathcal{P}|\mathcal{Q}})$. We automatically have $F_j = P_j \cup i$ for all j, so it remains to prove that G_j is the closure of Q_j in $\mathrm{M}^\perp$ for all j. Let k be the largest index satisfying $i \in \mathrm{cl}^\perp(Q_k)$, so that

$$\mathrm{cl}^\perp(Q_j) = Q_j \cup i \quad \text{for } j \le k \quad \text{and} \quad \mathrm{cl}^\perp(Q_j) = Q_j \quad \text{for } j > k.$$

For $j \le k$, the set Q_j is not a flat in $\mathrm{M}^\perp$, and hence $G_j = Q_j \cup i = \mathrm{cl}^\perp(Q_j)$. Since Q_k and Q_{k+1} are flats of consecutive ranks in $(\mathrm{M}/i)^\perp = \mathrm{M}^\perp - i$, the flats $Q_k \cup i = \mathrm{cl}^\perp(Q_k)$ and $Q_{k+1} = \mathrm{cl}^\perp(Q_{k+1})$ of $\mathrm{M}^\perp$ also have consecutive ranks. Since $Q_{k+1} \cup i$ is strictly between the two, it cannot be a flat of $\mathrm{M}^\perp$. Thus, we must have $G_{k+1} = Q_{k+1}$, and hence $G_j = Q_j = \mathrm{cl}^\perp(Q_j)$ for $j > k$. We conclude that $\mathcal{F}|\mathcal{G} = (\mathcal{P} \cup i)|\,\mathrm{cl}^\perp(\mathcal{Q})$ as desired. □

We can now give an intersection-theoretic interpretation of the beta invariant of a flag. Together with the Vanishing Lemma 4.15, it gives the identity $\deg(\pi^*(x_\mathcal{F})\,\delta^{n-k-1}) = \beta_{\mathrm{M}[\mathcal{F}]}$.

Proposition 4.18. *For any strictly increasing flag $\mathcal{F}$ of k nonempty proper flats of* M, *we have*

$$\deg(x_{\mathcal{F}|\mathcal{F}^\perp}\,\delta^{n-k-1}) = \beta_{\mathrm{M}[\mathcal{F}]}.$$

Proof. We proceed by induction on k. The base case $k = 0$ follows from Proposition 4.9:

$$\deg(\delta^{n-1}) = \deg\Big(\sum_{B \in \beta\text{-}nbc(\mathrm{M})} x_{\beta\text{-cone}(B)}\Big) = \beta_\mathrm{M}.$$

When k is positive, write $F|F^\perp$ for the first biflat in $\mathcal{F}|\mathcal{F}^\perp$, and write $\mathcal{F}|\mathcal{F}^\perp$ as the disjoint union $F|F^\perp \sqcup \mathcal{G}|\mathcal{G}^\perp$. We consider the contraction M/F, its flag of flats $\mathcal{G} - F := \{G - F\}_{G \in \mathcal{G}}$, and the corresponding biflag

$$(\mathcal{G} - F)|(\mathcal{G} - F)^\perp := \Big\{(G - F)|(G^\perp - F)\Big\}_{G \in \mathcal{G}}.$$

The displayed description is justified by $G^\perp - F = \mathrm{cl}_{(\mathrm{M}/F)^\perp}((E - F) - (G - F))$. Clearly,

$$\beta_{\mathrm{M}[\mathcal{F}]} = \beta_{\mathrm{M}|F} \cdot \beta_{(\mathrm{M}/F)[\mathcal{G}-F]}.$$

We separately consider two cases, depending on the shortness of the first interval $[\varnothing, F]$:

Case 1. The flat F contains exactly one element $i \in E$.

Recall that the beta invariant of $[\varnothing, i]$ is equal to 1. Therefore, $\beta_{\mathrm{M}[\mathcal{F}]}$ is equal to

$$\begin{aligned}
&\beta_{(\mathrm{M}/i)[\mathcal{G}-i]}\\
&= \deg_{\mathrm{M}/i}\big(x_{\mathcal{G}-i|(\mathcal{G}-i)^\perp}\,\delta_{\mathrm{M}/i}^{(n-1)-(k-1)-1}\big) && \text{by the inductive hypothesis,}\\
&= \deg_\mathrm{M}\big(x_{i|E}\,\psi^*(x_{\mathcal{G}-i|(\mathcal{G}-i)^\perp})\,\delta_\mathrm{M}^{n-k-1})\big) && \text{by Lemma 4.17(3) and 4.17(4),}\\
&= \deg_\mathrm{M}\big(x_{i|E}\prod_{G\in\mathcal{G}}(x_{G|(G^\perp-i)} + x_{G|G^\perp})\,\delta_\mathrm{M}^{n-k-1}\big) && \text{by Lemma 4.17(2).}
\end{aligned}$$

By the Vanishing Lemma 4.15, the right-hand side simplifies to

$$\deg_\mathrm{M}\big(x_{i|E}\,x_{\mathcal{G}|\mathcal{G}^\perp}\,\delta_\mathrm{M}^{n-k-1}\big) = \deg_\mathrm{M}\big(x_{\mathcal{F}|\mathcal{F}^\perp}\,\delta_\mathrm{M}^{n-k-1}\big).$$

Case 2. The flat F contains more than one element.

We may assume the interval $[\varnothing, F]$ is coloopless by Lemma 4.15. This means that the flat F is *cyclic*, that is, $F^\perp = E - F$. We then have the natural bijections

$$\phi_1\colon \Big\{\text{biflats of } \mathrm{M}|F\Big\} \longrightarrow \Big\{\text{biflats } F'|G' \text{ of M with } F' \subseteq F \text{ and } G' \supseteq E - F\Big\} \quad \text{and}$$

$$\phi_2\colon \Big\{\text{biflats of } \mathrm{M}/F\Big\} \longrightarrow \Big\{\text{biflats } F'|G' \text{ of M with } F' \supseteq F \text{ and } G' \subseteq E - F\Big\},$$

where $\phi_1(A|B) = A|(B \cup (E - F))$ and $\phi_2(A|B) = (A \cup F)|B$. The bijection ϕ_1 extends to the bijection between the biflags of $\mathrm{M}|F$ and the biflags of M that are supported on the corresponding set of biflats, and have a gap to the left of $F|(E - F)$. Similarly, the bijection ϕ_2 extends to the bijection between the biflags of M/F and the biflags of M that are supported on the corresponding set of biflats, and have a gap to the right of $F|(E-F)$.

We now compute the degree of $x_{\mathcal{F}|\mathcal{F}^\perp}\,\delta^{n-k-1}$ using the following variant of the canonical expansion of Definition 4.1, which proceeds in two stages:

Stage 1. At each step, choose e to be the largest gap element that is in F, if there is one.

Stage 2. At each step, choose e to be the largest gap element in $E-F$.

The first $|F|-2$ steps of this computation will give $x_{\mathcal{F}|\mathcal{F}^\perp}$ times the image under ϕ_1 of the canonical expansion of $\delta_{M|F}^{|F|-2}$. By Proposition 4.9, there will be $\beta_{\mathrm{M}|F}$ square-free monomials.

Each such monomial will have a unique nonempty gap before F; say it is D_j, between biflats $F_j|G_j$ and $F_{j+1}|G_{j+1}$ of M. The flats F_j and F_{j+1} have consecutive ranks, and the coflats G_j and G_{j+1} have consecutive coranks. In step $|F|-1$ of the computation, this gap D_j will be filled in a unique way by the biflat $F_{j+1}|G_j$. There will no longer be gap elements in F.

In step $|F|$, the computation will enter Stage 2 for each of the resulting $\beta_{\mathrm{M}|F}$ monomials. The following $(|E-F|-1)-(k-1)-1$ steps will compute the image under ϕ_2 of the canonical expansion of $x_{(\mathcal{G}-F)|(\mathcal{G}-F)^\perp}\delta_{M/F}^{|F|-2}$. This expansion has $\beta_{(\mathrm{M}/F)[\mathcal{G}-F]}$ square-free monomials, by the inductive hypothesis.

This concludes the computation of $x_{\mathcal{F}|\mathcal{F}^\perp}\delta^{n-k-1}$. The result will be the sum of $\beta_{\mathrm{M}[\mathcal{F}]}$ square-free monomials, as we wished to prove. □

Proposition 4.19. *Let $\mathcal{F}=\{F_1\subsetneq\cdots\subsetneq F_k\}$ be a strictly increasing flag of flats of* M. *We have*

$$\deg(\pi^*(x_{\mathcal{F}})\,\delta^{n-k-1})=\beta_{\mathrm{M}[\mathcal{F}]}.$$

Proof. Since $\pi^*(x_{\mathcal{F}})=\sum_{\mathcal{F}|\mathcal{G}\text{ biflag}}x_{\mathcal{F}|\mathcal{G}}$, this follows from Lemma 4.15 and Proposition 4.18. □

4.5. **A conormal interpretation of the Chern–Schwartz–MacPherson cycles.** Recall that the k-dimensional Chern–Schwartz–MacPherson cycle of M is the Minkowski weight $\mathrm{csm}_k(\mathrm{M})$ on the Bergman fan of M defined by the formula

$$\mathrm{csm}_k(\mathrm{M})(\sigma_{\mathcal{F}})=(-1)^{r-k}\beta_{\mathrm{M}[\mathcal{F}]},$$

where $\sigma_{\mathcal{F}}$ is the k-dimensional cone corresponding to a flag of flats $\mathcal{F}$ of M.

Theorem 1.1. *When* M *has no loops and no coloops, we have*

$$\mathrm{csm}_k(\mathrm{M})=(-1)^{r-k}\pi_*(\delta^{n-k-1}\cap 1_{\mathrm{M},\mathrm{M}^\perp})\ \textit{for}\ 0\le k\le r.$$

Proof. By Proposition 4.19, Definition 3.4, and the projection formula, we have

$$\beta_{\mathrm{M}[\mathcal{F}]}=\deg(\pi^*(x_{\mathcal{F}})\delta^{n-k-1})=\pi^*(x_{\mathcal{F}})\delta^{n-k-1}\cap 1_{\mathrm{M},\mathrm{M}^\perp}=\pi_*(\delta^{n-k-1}\cap 1_{\mathrm{M},\mathrm{M}^\perp})(\sigma_{\mathcal{F}}).$$

The result then follows from the definition of the Chern–Schwartz-MacPherson cycle of M. □

Theorem 1.2. *When* M *has no loops and no coloops, we have*

$$\overline{\chi}_{\mathrm{M}}(q+1)=\sum_{k=0}^{r}(-1)^{r-k}\deg(\gamma^k\,\delta^{n-k-1})\,q^k.$$

Proof. We use [LdMRS20, Theorem 1.4], which states that

$$\overline{\chi}_{\mathrm{M}}(q+1) = \sum_{k=0}^{r} \alpha^k \cap \mathrm{csm}_k(\mathrm{M})\, q^k.$$

The authors of [AB20] give a nonrecursive proof of the identity using tropical intersection theory. For representable matroids, the identity was given earlier in [Alu13, Theorem 1.2]. By Theorem 1.1 and the projection formula, the k-th coefficient of the displayed polynomial is

$$\alpha^k \cap \pi_*(\delta^{n-k-1} \cap 1_{\mathrm{M},\mathrm{M}^\perp}) = \pi_*\big(\pi^*\alpha^k \cap (\delta^{n-k-1} \cap 1_{\mathrm{M},\mathrm{M}^\perp})\big) = \pi_*\big(\gamma^k \delta^{n-k-1} \cap 1_{\mathrm{M},\mathrm{M}^\perp}\big).$$

This proves the desired formula for the reduced characteristic polynomial. □

5. Tropical Hodge theory

Throughout this section, we fix a rational simplicial fan Σ in $\mathrm{N} = \mathbb{R} \otimes \mathrm{N}_{\mathbb{Z}}$. Our goal is to prove Theorem 1.6, which says that the property of Σ being Lefschetz only depends on the support of Σ. We deduce the Lefschetz property of $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ from the Lefschetz property of $\Sigma_{\mathrm{M}} \times \Sigma_{\mathrm{M}^\perp}$, and use it to prove Theorem 1.4.

5.1. **Convexity of piecewise linear functions.**

A piecewise linear function $\phi\colon \Sigma \to \mathbb{R}$ is said to be *positive* on Σ if $\phi(x)$ is positive for all nonzero $x \in |\Sigma|$. A class in $A^1(\Sigma)$ is said to be *positive* if it has a positive representative. We write $\mathrm{Eff}^\circ(\Sigma) \subseteq A^1(\Sigma)$ for the open cone of positive classes.

For each cone σ of Σ, the projection $\mathrm{N} \to \mathrm{N}\,/\,\mathrm{span}(\sigma)$ defines a morphism from the closed star

$$\pi_\sigma\colon\ \overline{\mathrm{st}}_\Sigma(\sigma) \longrightarrow \mathrm{st}_\Sigma(\sigma).$$

It is straightforward to check that the associated pullback map π_σ^* between their Chow rings is an isomorphism, and that for a ray ν of $\mathrm{st}_\Sigma(\sigma)$, we have

$$\pi_\sigma^*(x_\nu) = \frac{\mathrm{mult}(\sigma \cup \{\nu\})}{\mathrm{mult}(\sigma)} x_\nu.$$

Thus, the inclusion of fans $i_\sigma\colon \overline{\mathrm{st}}_\Sigma(\sigma) \to \Sigma$ defines a ring homomorphism

$$i_\sigma^*\colon A(\Sigma) \longrightarrow A(\overline{\mathrm{st}}_\Sigma(\sigma)) \simeq A(\mathrm{st}_\Sigma(\sigma)),$$

where the first factor is given by the restriction of piecewise linear functions and the second factor is the inverse of π_σ^*.

We use the pullback homomorphism i_σ^* to define *strict convexity* of piecewise linear functions on Σ. The notion agrees with the one used in [AHK18, Section 4].

Definition 5.1. The cone $\mathcal{K}(\Sigma) \subseteq A^1(\Sigma)$ is defined by the following conditions:

(1) If Σ is at most 1-dimensional, $\mathcal{K}(\Sigma) = \mathrm{Eff}^\circ(\Sigma)$.
(2) If otherwise,

$$\mathcal{K}(\Sigma) = \{\ell \in A^1(\Sigma)\colon \ell \in \mathrm{Eff}^\circ(\Sigma) \text{ and } i_\sigma^*(\ell) \in \mathcal{K}(\mathrm{st}_\Sigma(\sigma)) \text{ for all nonzero } \sigma \in \Sigma\}.$$

The piecewise linear functions on Σ whose classes are in $\mathcal{K}(\Sigma)$ are said to be *strictly convex*.

Clearly, ℓ belongs to $\mathcal{K}(\Sigma)$ if and only if $i_\sigma^*(\ell)$ belongs to $\mathrm{Eff}^\circ(\mathrm{st}_\Sigma(\sigma))$ for all $\sigma \in \Sigma$.[20] Geometrically, ℓ belongs to $\mathcal{K}(\Sigma)$ if and only if, for each cone σ, the class ℓ has a piecewise linear representative which is zero on σ and positive on the cones containing σ. When Σ has convex support of full dimension, the notion coincides with the usual notion of strict convexity of piecewise linear functions [CLS11, Section 6.1]. In general, $\mathcal{K}(\Sigma)$ is an open polyhedral cone, and $i_\sigma^* \mathcal{K}(\Sigma) \subseteq \mathcal{K}(\mathrm{st}_\Sigma(\sigma))$ for all $\sigma \in \Sigma$.

Remark 5.2. A fan Σ is *quasiprojective* if it is a subfan of the normal fan of a convex polytope. When Σ is quasiprojective, the cone $\mathcal{K}(\Sigma)$ is nonempty. More generally, for simplicial fans $\Sigma_1 \subseteq \Sigma_2$, the restriction of piecewise linear functions maps $\mathcal{K}(\Sigma_2)$ to $\mathcal{K}(\Sigma_1)$.

A map of fans $f\colon \Sigma \to \Sigma'$ is said to be *projective* if the induced map of toric varieties $X_\Sigma \to X_{\Sigma'}$ is projective in the sense of Grothendieck [Gro61, Définition 5.5.2]. According to [CLS11, Theorem 7.2.12], the map f is projective if and only if f is proper and there exists a piecewise linear function η on Σ for which η is strictly convex on $\left|f^{-1}(\sigma')\right|$ for each cone σ' of Σ', where $f^{-1}(\sigma')$ denotes the subfan of Σ consisting of cones mapping into σ' under f. If f is induced by the identity map $\mathrm{N} \to \mathrm{N}$, then f is proper if and only if Σ subdivides Σ'. In this case, we will call Σ a *projective refinement* of Σ' if f is moreover projective.

Proposition 5.3. *Let $f\colon \Sigma \to \Sigma'$ be a projective map of simplicial fans. If $\mathcal{K}(\Sigma')$ is nonempty, then $\mathcal{K}(\Sigma)$ is nonempty.*

Proof. We proceed by induction on the dimension of Σ. If $\dim \Sigma = 1$, then $\mathcal{K}(\Sigma) \neq \emptyset$. Otherwise, we choose any $\ell \in \mathcal{K}(\Sigma')$, and let η be a piecewise linear function given by the projectivity of f.

First, since ℓ is strictly convex, ℓ has a representative which is positive on $|\Sigma'| - \{0\}$. Thus $f^*\ell$ is nonnegative on $|\Sigma|$, and positive outside of $\left|f^{-1}(0)\right|$. Modulo a global linear function, we may choose η to be positive on $\left|f^{-1}(0)\right| - \{0\}$. It follows that $f^*\ell + \epsilon_0 \cdot \eta$ is positive on $|\Sigma| - \{0\}$ for sufficiently small $\epsilon_0 > 0$.

For a nonzero cone σ, let σ' be the smallest cone of Σ' containing $f(\sigma)$. A fortiori, the restriction $f\colon\ \mathrm{st}_\Sigma(\sigma) \to \mathrm{st}_{\Sigma'}(\sigma')$ is projective, and $\dim \mathrm{st}_\Sigma(\sigma) < \dim \Sigma$. Since ℓ is strictly convex, we have $i_{\sigma'}^*\ell \in \mathcal{K}(\mathrm{st}_{\Sigma'}(\sigma'))$. Therefore the restriction of $f^*\ell + \epsilon_\sigma \cdot \eta$ is in $\mathcal{K}(\mathrm{st}_\Sigma(\sigma))$ for sufficiently small ϵ_σ, by the induction hypothesis. We conclude that $f^*\ell + \epsilon \cdot \eta \in \mathcal{K}(\Sigma)$ for all positive $\epsilon \leq \min\{\epsilon_\sigma\}$. ☐

We now focus on how subdividing a cone by adding a ray affects the Chow ring. For $\sigma \in \Sigma$, let ρ denote a new ray spanned by a primitive vector

$$\mathbf{e}_\rho := \sum_{\nu \in \sigma(1)} a_\nu \mathbf{e}_\nu,$$

for some positive rational coefficients $\{a_\nu\}$. The *stellar subdivision* of Σ by ρ, denoted $\mathrm{stellar}_\rho \Sigma$, is obtained from Σ by setting

$$\mathrm{stellar}_\rho \Sigma := \big(\Sigma - \{\tau \in \Sigma\colon \tau \supseteq \sigma\}\big) \cup \big(\rho + \partial\sigma + \mathrm{link}_\Sigma(\sigma)\big),$$

[20] When $\mathcal{K}(\Sigma)$ is nonempty, its closure in $A^1(\Sigma)$ is the cone $\mathcal{L}(\Sigma)$ introduced in [GM12, Definition 2.5]. The cone $\mathcal{L}(\Sigma)$ consists of divisor classes on the toric variety X_Σ of Σ whose pullback to any torus orbit closure is effective. When Σ is complete, $\mathcal{K}(\Sigma)$ is the ample cone of X_Σ [CLS11, Theorem 6.1.14].

where the right-hand $+$ is an internal direct sum of fans.[21] In the special case when X_Σ is smooth and each $a_\nu = 1$, the toric variety of the stellar subdivision is the blowup of X_Σ along the torus orbit closure $V(\sigma)$ [CLS11, Section 3.3]. In the remainder of this section, we write $\widetilde{\Sigma}$ for the stellar subdivision of Σ by ρ, and write $p\colon \widetilde{\Sigma} \to \Sigma$ for the map of fans given by the identity map of N.

Any stellar subdivision is projective. In fact, the function $\eta = -x_\rho$ is strictly convex on the preimage of each cone of Σ [CLS11, Proposition 11.1.6]. The proof of Proposition 5.3 then shows the following.

Proposition 5.4. *If $\ell \in p^*(\mathcal{K}(\Sigma))$, then $\ell - \epsilon \cdot x_\rho \in \mathcal{K}(\widetilde{\Sigma})$ for sufficiently small $\epsilon > 0$.*

We will distinguish two cases of stellar subdivisions. The first is the case when every closed orbit in X_Σ meets $V(\sigma)$. In terms of fans, this means that Σ is the closed star of σ and $\widetilde{\Sigma}$ is the closed star of the new ray ρ. In this case, we have

$$A(\Sigma) \simeq A(\mathrm{st}_\Sigma(\sigma)) \quad \text{and} \quad A(\widetilde{\Sigma}) \simeq A(\mathrm{st}_{\widetilde{\Sigma}}(\rho)),$$

which are Chow rings of fans of dimensions $\dim(\Sigma) - \dim(\sigma)$ and $\dim(\Sigma) - 1$. We will call this a *star-shaped subdivision*. If otherwise, we will call the stellar subdivision *ordinary*.

Remark 5.5. In general, the quotient map $\mathrm{N} / \mathrm{span}(\rho) \to \mathrm{N} / \mathrm{span}(\sigma)$ induces a map between the stars $\mathrm{st}_{\widetilde{\Sigma}}(\rho) \to \mathrm{st}_\Sigma(\sigma)$, and the corresponding map of toric varieties is a projective bundle. If the stellar subdivision is star-shaped, then Σ and $\widetilde{\Sigma}$ cannot be Lefschetz, as their Chow rings vanish in degree $\dim \Sigma$. However, the smaller-dimensional fans $\mathrm{st}_\Sigma(\sigma)$ and $\mathrm{st}_{\widetilde{\Sigma}}(\rho)$, whose Chow rings are isomorphic to the Chow rings of Σ and $\widetilde{\Sigma}$ respectively, can be Lefschetz.

In the star-shaped case, we will freely use the isomorphisms $A(\Sigma) \simeq A(\mathrm{st}_\Sigma(\sigma))$ and $A(\widetilde{\Sigma}) \simeq A(\mathrm{st}_{\widetilde{\Sigma}}(\rho))$ in the arguments that follow. This allows us to think of the bundle map $\mathrm{st}_{\widetilde{\Sigma}}(\rho) \to \mathrm{st}_\Sigma(\sigma)$ as the stellar subdivision $\widetilde{\Sigma} \to \Sigma$.

5.2. **Lefschetz fans.** Recall from Definition 1.5 that a d-dimensional Lefschetz fan Σ has a d-dimensional fundamental weight w which induces *Poincaré duality*. We shall abbreviate this statement by $\mathrm{PD}(\Sigma)$. A Lefschetz fan also satisfies the *hard Lefschetz property* and *Hodge–Riemann relations* in Definition 1.5. We will call these statements $\mathrm{HL}^k(\Sigma, \ell)$ and $\mathrm{HR}^k(\Sigma, \ell)$, respectively, for $0 \le k \le \frac{d}{2}$ and $\ell \in \mathcal{K}(\Sigma)$. We say that $\mathrm{HL}^k(\Sigma)$ holds if $\mathrm{HL}^k(\Sigma, \ell)$ holds for all $\ell \in \mathcal{K}(\Sigma)$, and that $\mathrm{HL}(\Sigma)$ holds if $\mathrm{HL}^k(\Sigma)$ holds for all k. We will use the symbols $\mathrm{HR}^k(\Sigma)$ and $\mathrm{HR}^k(\Sigma, \ell)$ analogously.

Definition 5.6. Let Σ be a rational simplicial fan.

(1) The fan Σ satisfies the *mixed hard Lefschetz property* if, for $0 \le k \le \frac{d}{2}$ and all $\ell_1, \ldots, \ell_{d-2k} \in \mathcal{K}(\Sigma)$, the multiplication map

$$A^k(\Sigma) \longrightarrow A^{d-k}(\Sigma), \qquad \eta \longmapsto \left(\prod_{i=1}^{d-2k} \ell_i \right) \eta$$

is a linear isomorphism.

[21] If $\Sigma \subseteq \mathrm{N}$ and $\Sigma' \subseteq \mathrm{N}'$ are fans for which $\mathrm{N} \cap \mathrm{N}' = \{0\}$, the internal direct sum, by definition, consists of cones $\sigma + \sigma'$ for all $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$, where $+$ denotes Minkowski sum.

(2) The fan Σ satisfies the *mixed Hodge–Riemann relations* if, for all $0 \le k \le \frac{d}{2}$ and $\ell_0, \ell_1, \ldots, \ell_{d-2k} \in \mathcal{K}(\Sigma)$, the bilinear form

$$A^k(\Sigma) \times A^k(\Sigma) \longrightarrow \mathbb{R}, \qquad (\eta_1, \eta_2) \longmapsto (-1)^k \deg \left(\prod_{i=1}^{d-2k} \ell_i \right) \eta_1 \eta_2$$

is positive definite when restricted to the kernel of the multiplication map $\prod_{i=0}^{d-2k} \ell_i$.

Clearly, the mixed properties imply the ordinary ones. Using results from [CKS87], Cattani showed that the converse is true as well [Cat08]. Since the mixed HR property is particularly convenient for applications such as Theorem 1.4, we include a self-contained proof that Lefschetz fans also possess the "mixed" properties; see Theorem 5.20.

Example 5.7. We remark that any complete simplicial fan Σ is Lefschetz. In this case, $\mathcal{K}(\Sigma)$ is the cone of Kähler classes on the compact complex variety X_Σ, and there are isomorphisms

$$A^k(\Sigma) \simeq H^{2k}(X_\Sigma, \mathbb{R}) \simeq IH^{2k}(X_\Sigma, \mathbb{R}).$$

The Lefschetz property of Σ follows from Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations for the intersection cohomology of X_Σ [CLS11, Section 12.5]. Alternatively, one may use Theorem 1.6 to deduce the Lefschetz property of Σ.

5.3. **The weak factorization theorem.** Alexander proved that any subdivision of a simplicial complex can be expressed as a sequence of stellar subdivisions of edges and their inverses [Ale30]. We will use a refined version of his result for simplicial fans. We continue to write $\widetilde{\Sigma}$ for the stellar subdivision of Σ by ρ. For brevity, we adopt the language of simplicial complexes and call $\widetilde{\Sigma}$ an *edge subdivision* of Σ if the cone σ containing ρ in its relative interior is two-dimensional.

Lemma 5.8. *There exists a sequence of simplicial fans* $(\Sigma_0, \Sigma_1, \ldots, \Sigma_n)$ *such that*

(1) *the initial entry is the fan* $\widetilde{\Sigma}$, *the final entry is the fan* Σ, *and,*
(2) *for each* i, *either* Σ_i *is an edge subdivision of* Σ_{i+1} *or* Σ_{i+1} *is an edge subdivision of* Σ_i.

Moreover, if Σ *is a projective refinement of some fan* Δ, *then so is* Σ_i *for every* i.

Proof. We use induction on the dimension of σ. The composition of projective maps $\widetilde{\Sigma} \to \Sigma \to \Delta$ is projective, so the statement when $\dim \sigma = 2$ is trivial.

Suppose $\dim \sigma > 2$ and ρ is a ray in the relative interior of σ. Let σ' be any maximal cone of the boundary $\partial\sigma$, and let μ be the unique ray in $\sigma(1) - \sigma'(1)$. The span of $\{\rho, \mu\}$ intersects σ' along a ray which we call ρ', and we let $\Sigma' = \mathrm{stellar}_{\rho'} \Sigma$. Let σ'' be the 2-dimensional cone spanned by μ and ρ'. The ray ρ lies in σ'', and we let $\Sigma'' = \mathrm{stellar}_\rho \Sigma'$. We claim that $\Sigma'' = \mathrm{stellar}_{\rho'} \widetilde{\Sigma}$.

By construction, the fans have the same rays. To compare the remaining cones, it is sufficient to characterize those subsets of rays $\Sigma''(1)$ which fail to span a cone. We recall notation from Section 3: If Δ is a simplicial fan, $I(\Delta)$ is the Stanley-Reisner ideal of Δ in a polynomial ring $S(\Delta)$, generated by square-free monomials x_A, where $A \subseteq \Delta(1)$ runs over subsets of rays that do not span a cone. For cones $\sigma \in \Delta$, we will continue to write x_σ for the monomial indexed by the rays of σ. If we subdivide a cone τ of a fan Δ by a ray ρ, some cones of Δ are unchanged. We denote that set by

$$U(\Delta, \tau) := \Delta - \overline{\mathrm{st}}_\Delta(\tau).$$

By definition of the stellar subdivision, in the polynomial ring $S(\mathrm{stellar}_\tau(\Delta))$, we have

$$I(\mathrm{stellar}_\rho(\Delta)) = I(\Delta) + (x_\tau) + (x_\rho x_\upsilon \colon \upsilon \in U(\Delta, \tau)).$$

Thus, it follows that

$$\begin{aligned} I(\Sigma'') &= I(\Sigma') + (x_{\sigma''}) + (x_\rho x_\upsilon \colon \upsilon \in U(\Sigma', \sigma'')) \\ &= I(\Sigma) + (x_{\sigma'}, x_{\rho'} x_\mu) + (x_{\rho'} x_\upsilon \colon \upsilon \in U(\Sigma, \sigma')) + (x_\rho x_\upsilon \colon \upsilon \in U(\Sigma', \sigma'')). \end{aligned}$$

On the other hand, we have

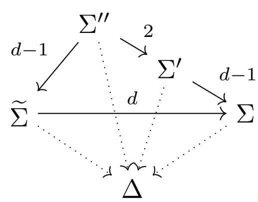
$$\begin{aligned} I(\mathrm{stellar}_{\rho'} \widetilde{\Sigma}) &= I(\widetilde{\Sigma}) + (x_{\sigma'}) + (x_{\rho'} x_\upsilon \colon \upsilon \in U(\widetilde{\Sigma}, \sigma')) \\ &= I(\Sigma) + (x_{\sigma'}, x_\sigma) + (x_\rho x_\upsilon \colon \upsilon \in U(\Sigma, \sigma)) + (x_{\rho'} x_\upsilon \colon \upsilon \in U(\widetilde{\Sigma}, \sigma')) \\ &= I(\Sigma) + (x_{\sigma'}) + (x_{\rho'} x_\upsilon \colon \upsilon \in U(\widetilde{\Sigma}, \sigma')) + (x_\rho x_\upsilon \colon \upsilon \in U(\Sigma, \sigma)), \end{aligned}$$

noting that $x_\sigma = x_\mu x_{\sigma'}$. To conclude that $\Sigma'' = \mathrm{stellar}_{\rho'}(\widetilde{\Sigma})$, we check that their ideals have the same generators, using two observations:

- There is a bijection $\overline{\mathrm{st}}_\Sigma(\sigma) \simeq \overline{\mathrm{st}}_{\Sigma'}(\sigma'')$: if $\tau \in \overline{\mathrm{st}}_\Sigma(\sigma)$, we can write $\tau = \tau' + \sigma' + \mu$ for some cone τ': then $\tau' + \rho' + \mu$ is a cone of $\overline{\mathrm{st}}_{\Sigma'}(\sigma'')$ (because $\sigma'' = \rho' + \mu$). This map is easily seen to be invertible. It follows that $U(\Sigma', \sigma'') = U(\Sigma, \sigma)$.
- There is a bijection $\overline{\mathrm{st}}_\Sigma(\sigma') \simeq \overline{\mathrm{st}}_{\widetilde{\Sigma}}(\sigma')$: suppose a cone $\tau \in \Sigma$ contains σ'. If $\tau \not\supseteq \mu$, then $\tau \in \widetilde{\Sigma}$. Otherwise, $\tau = \tau' + \mu$ for some τ', and $\tau' + \rho$ is in $\widetilde{\Sigma}$. It follows that

$$U(\widetilde{\Sigma}, \sigma') = U(\Sigma, \sigma') \cup \left\{ \tau \in \widetilde{\Sigma} \colon \tau \supseteq \mu \right\}.$$

This gives a commuting diagram of refinements of Δ, as illustrated in Figure 7.



Since stellar subdivisions are projective, so are the refinements $\Sigma' \to \Delta$ and $\Sigma'' \to \Delta$. The stellar subdivisions $\widetilde{\Sigma} \leftarrow \Sigma'' \to \Sigma' \to \Sigma$ take place over cones of dimension $< d$, so by induction there are sequences of fans from $\widetilde{\Sigma}$ to Σ'' and from Σ' to Σ so that each step is an edge subdivision, and each fan is a projective refinement of Δ. □

Theorem 5.9. *If Σ and Σ' are simplicial fans with the same support, there exists a sequence of simplicial fans $\Sigma_0, \Sigma_1, \ldots, \Sigma_n$ such that*

(1) *the initial entry is the fan Σ, the final entry is the fan Σ', and,*
(2) *for each i, either Σ_i is an edge subdivision of Σ_{i+1} or Σ_{i+1} is an edge subdivision of Σ_i.*

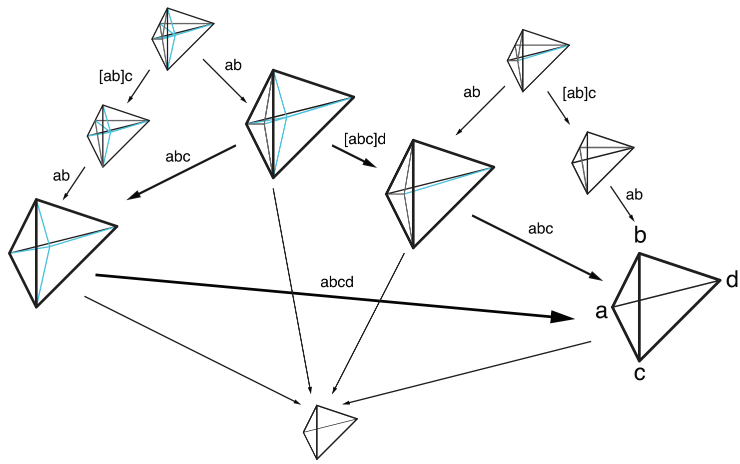


FIGURE 7. Factoring a codimension-4 subdivision into edge subdivisions

Furthermore, the entries can be chosen in such a way that there is an index i_0 for which Σ_i is a projective refinement of Σ for all $i \leq i_0$, and Σ_i is a projective refinement of Σ' for all $i \geq i_0$.

Proof. By [CLS11, Theorem 11.1.9], there exists a sequence of stellar subdivisions of both Σ and Σ' that refine those fans, respectively, to unimodular fans. Thus, by Proposition 5.4 and Lemma 5.8, we can reduce to the case where Σ and Σ' are both unimodular fans.

By [Wło97, Theorem A], there is a sequence of simplicial fans for which Σ_i and Σ_{i-1} differ by a stellar subdivision, for each i. The assertion that these can be chosen to have the second property follows from [AKMW02, Theorem 2.7.1]. As stated, [AKMW02, Theorem 2.7.1] applies to the more general setting of toroidal embeddings and polyhedral complexes, and the proof given in [AKMW02] specializes to the case of toric varieties and fans.[22]

Suppose now that $i \leq i_0$ and $\Sigma_i = \text{stellar}_\rho(\Sigma_{i-1})$. Since Σ_{i-1} is a projective refinement of Σ, we use Lemma 5.8 to obtain a sequence of fans between Σ_i and Σ_{i-1} in which consecutive fans differ by edge subdivisions, and which are also projective refinements of Σ. The remaining cases are treated by exchanging i with $i-1$, and Σ with Σ'. □

In terms of toric varieties, edge subdivisions correspond to morphisms that are *semismall* in the sense of Goresky–MacPherson. In projective geometry, the semismallness is particularly convenient for transferring the Lefschetz property, as pullbacks of ample line bundles by semismall maps satisfy the hard Lefschetz property and the Hodge–Riemann bilinear relations [dCM02]. We will use Theorem 5.9 to prove Theorem 1.6 by establishing an analogous property in the context of Lefschetz fans.

5.4. **Chow rings of stellar subdivisions.** We continue to write $\widetilde{\Sigma}$ for the stellar subdivision of Σ by ρ, a ray in the relative interior of σ. The primitive ray generator

[22]The birational cobordism used in the proof can be chosen to be toric, using a toric resolution of singularities.

of ρ is given by

$$\mathbf{e}_\rho := \sum_{\nu \in \sigma(1)} a_\nu \mathbf{e}_\nu,$$

for some positive rational coefficients $\{a_\nu\}$. We relate the Chow rings of Σ and $\widetilde{\Sigma}$. For economy of notation, we abbreviate $E := \overline{\mathrm{st}}_{\widetilde{\Sigma}}(\rho)$ and $Z := \overline{\mathrm{st}}_\Sigma(\sigma)$. We let $j\colon E \to \widetilde{\Sigma}$ denote the inclusion of fans, and let $q\colon E \to Z$ denote the restriction of $p\colon \widetilde{\Sigma} \to \Sigma$. We will write i in place of i_σ through the rest of this section:

$$\begin{array}{ccc} E & \xhookrightarrow{\ j\ } & \widetilde{\Sigma} \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle p} \\ Z & \xhookrightarrow{\ i\ } & \Sigma \end{array}$$

A straightforward calculation shows that the pullback homomorphism $p^*\colon A(\Sigma) \to A(\widetilde{\Sigma})$ is determined by the formula

$$p^*(x_\nu) = \begin{cases} x_\nu & \text{if } \nu \notin \sigma(1), \\ x_\nu + a_\nu x_\rho & \text{if } \nu \in \sigma(1). \end{cases}$$

Since p is a proper map of fans [CLS11, Theorem 3.4.11], there is a pushforward $p_*\colon A(\widetilde{\Sigma}) \to A(\Sigma)$, which is a homomorphism of $A(\Sigma)$-modules. By definition [Ful98, Section 1.4], for any $\widetilde{\tau} \in \widetilde{\Sigma}$, we have

$$p_*(x_{\widetilde{\tau}}) = \begin{cases} x_\tau & \text{if } p(\widetilde{\tau}) \subseteq \tau \text{ and } \dim \widetilde{\tau} = \dim \tau, \\ 0 & \text{if otherwise.} \end{cases}$$

Proposition 5.10. *If σ is two-dimensional, then the map*

$$p_* \oplus q_* j^*\colon A(\widetilde{\Sigma}) \to A(\Sigma) \oplus A(\mathrm{st}_\Sigma(\sigma))[-1]$$

is an isomorphism of graded $A(\Sigma)$-modules.

The proof, given below, uses some preliminary calculations. Let ν_1, ν_2 be the rays of σ and $\mathbf{e}_1$, $\mathbf{e}_2$ the primitive ray generators. Then $\mathbf{e}_\rho = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ for some positive rational coefficients a_1, a_2. For $i, j \in \{1, 2, \rho\}$, we let $m_{i,j}$ denote the index of $\mathbb{Z}\{\mathbf{e}_i, \mathbf{e}_j\}$ inside $\mathbb{R}\{\mathbf{e}_i, \mathbf{e}_j\} \cap \mathrm{N}_\mathbb{Z}$. Computing determinants, we see that

$$a_1 = m_{2,\rho}/m_{1,2} \quad \text{and} \quad a_2 = m_{1,\rho}/m_{1,2}.$$

Lemma 5.11. *We have*

$$q_* j^* j_* q^* = -\frac{m_{1,2}}{m_{1,\rho} m_{2,\rho}}.$$

Proof. Consider an element $v \in A(Z)$. Since j^* is surjective, we have $q^*(v) = j^*(u)$ for some $u \in A(\widetilde{\Sigma})$. Then, by the projection formula,

$$\begin{aligned} q_* j^* j_* q^*(v) &= q_* j^* j_* q^*(1_Z \cdot v) \\ &= q_* j^* j_* \big(q^*(1_Z) \cdot q^*(v)\big) \\ &= q_* j^* j_* \big(q^*(1_Z) \cdot j^*(u)\big) \\ &= q_* j^* \big(j_* q^*(1_Z) \cdot u\big) \\ &= q_* \big(j^* j_* q^*(1_Z) \cdot j^*(u)\big) \\ &= q_* \big(j^* j_* q^*(v)\big) \\ &= q_* j^* j_* q^*(1_Z) \cdot v, \end{aligned}$$

so it is enough to verify the claim on the fundamental class 1_Z.

Let $h = j^*(x_\rho) \in A^1(E)$. By definition, we have

$$(q_* j^* j_* q^*)(1_Z) = q_* j^* j_*(1_E) = q_* j^*(x_\rho) = q_*(h).$$

We extend $\{\mathbf{e}_1, \mathbf{e}_2\}$ to a basis for N, and write $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \ldots\}$ for the dual basis. The piecewise-linear function $x_\rho - a_1^{-1}\mathbf{e}_1^*$ is equivalent to x_ρ, and its values on $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_\rho$ are $-a_1^{-1}$, 0, and 0, respectively. This is to say that $h = j^*(-a_1^{-1}x_1 + g)$ for some piecewise linear function g on $\widetilde{\Sigma}$ which is zero on $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_\rho$, and hence

$$h = -a_1^{-1} m_{1,\rho}^{-1} x_1 + j^*(g).$$

As $j^*(g)$ is a linear combination of Courant functions x_ν for rays ν in $\mathrm{st}_{\widetilde{\Sigma}}(\rho)$ not contained in the support of σ, we have $q_* j^*(g) = 0$. On the other hand, $q_*(x_1) = 1_Z$, so

$$q_*(h) = -a_1^{-1} m_{1,\rho}^{-1} 1_Z = -m_{1,2}/(m_{1,\rho} m_{2,\rho}) 1_Z. \qquad \square$$

Proof of Proposition 5.10. Let $\psi(u) = (p_*(u), q_* j^*(u))$ for $u \in A(\widetilde{\Sigma})$, and let $\phi(u,v) = p^*(u) + j_* q^*(v)$ for $(u,v) \in A(\Sigma) \oplus A(Z)[-1]$. We first check that $\psi \circ \phi$ is an isomorphism.

Observe that $p_* p^* = 1$, because p is birational, and $q_* q^* = 0$, because q has positive relative dimension. Therefore, we have

$$\begin{aligned}
\psi \circ \phi(u,v) &= p_*\big(p^*(u) + j_* q^*(v)\big) + q_* j^*\big(p^*(u) + j_* q^*(v)\big) \\
&= p_* p^*(u) + p_* j_* q^*(v) + q_* j^* p^*(u) + q_* j^* j_* q^*(v) \\
&= p_* p^*(u) + i_* q_* q^*(v) + q_* q^* i^*(u) + q_* j^* j_* q^*(v) \\
&= u + q_* j^* j_* q^*(v),
\end{aligned}$$

which is invertible by Lemma 5.11. It follows that ϕ is injective, and we now argue that it is also surjective.

Since squarefree monomials span $A(\widetilde{\Sigma})$, it is enough to show that each monomial x_τ is of the form $p^*(u) + j_* q^*(v)$ for suitable $u \in A(\Sigma)$ and $v \in A(Z)$. If none of ν_1, ν_2 or ρ is contained in τ, clearly $x_\tau = p^*(x_\tau)$. Noting that no cone of $\widetilde{\Sigma}$ contains both ν_1 and ν_2, it remains to consider the following three cases.

Case 1. Suppose $\{\nu_1, \nu_2, \rho\} \cap \tau(1) = \{\rho\}$. If we set $\tau' = \tau - \{\rho\}$, then

$$j_* q^*(x_{\tau'}) = j_*(x_{\tau'}) = x_\rho x_{\tau'},$$

so we may take $u = 0$ and $v = x_{\tau'}$.

Case 2. Suppose $\{\nu_1, \nu_2, \rho\} \cap \tau(1) = \{\nu_1\}$. If we set $\tau' = \tau - \{\nu_1\}$, then

$$x_1 x_{\tau'} = (x_1 + a_1 x_\rho) x_{\tau'} - a_1 x_\rho x_{\tau'}.$$

The first summand equals $p^*(x_1 x_{\tau'})$, and the second summand is in the image of ϕ in view of Case 1.

Case 3. Suppose $\{\nu_1, \nu_2, \rho\} \cap \tau(1) = \{\nu_1, \rho\}$. We set $\tau' = \tau - \{\nu_1, \rho\}$, and extend the vectors $\mathbf{e}_1, \mathbf{e}_2$ to a basis for N. Then the linear function $\mathbf{e}_2^*$ may be written

$$\mathbf{e}_2^* = x_2 + a_2 x_\rho + g,$$

where g is a piecewise linear function vanishing on the rays $\{\nu_1, \nu_2, \rho\}$. Since the class of $\mathbf{e}_2^*$ is zero in the Chow ring of $\widetilde{\Sigma}$, multiplying it by $x_1 x_{\tau'}$ gives

$$0 = 0 \cdot x_1 x_{\tau'} = 0 + a_2 x_1 x_\rho x_{\tau'} + g x_1 x_{\tau'},$$

because $x_1x_2 = 0$. Thus $a_2x_\tau = -gx_1x_{\tau'}$, which is in the image of ϕ by Case 2.

This completes the proof of Proposition 5.10. □

Corollary 5.12. *The pullback homomorphism p^* is injective, and it restricts to an isomorphism in degree $d = \dim \Sigma$.*

Proof. The isomorphism ϕ restricts to p^* on $A(\Sigma)$, so p^* is injective. Since $\mathrm{st}_\Sigma(\sigma)$ is $(d-2)$-dimensional, $A^{d-1}(\mathrm{st}_\Sigma(\sigma)) = 0$, and hence the isomorphism ϕ agrees with p^* in degree d. □

5.5. **Hodge–Riemann forms and their signatures.** Our goal in the next few pages is to understand how the Lefschetz property behaves under edge subdivisions. In this subsection, we fix a d-dimensional simplicial fan Σ that satisfies Poincaré duality (Definition 1.5) and $k \le \frac{d}{2}$. Suppose that the multiplication by $L \in A^{d-2k}(\Sigma)$ is an isomorphism in degree k. Using Poincaré duality, one can check directly that the multiplication by $L \in A^{d-2k}(\Sigma)$ is an isomorphism in degree k if and only if the corresponding *Hodge–Riemann form*

$$\mathrm{hr}^k(\Sigma, L)\colon A^k(\Sigma) \times A^k(\Sigma) \longrightarrow \mathbb{R}, \qquad (\eta_1, \eta_2) \longmapsto (-1)^k \deg(L\eta_1\eta_2)$$

is nondegenerate. Thus, in this case, $\mathrm{hr}^k(\Sigma, L)$ has b_k^+ positive eigenvalues and b_k^- negative eigenvalues, where $b_k^+ + b_k^-$ is the dimension of $A^k(\Sigma)$. We use its signature $b_k^+ - b_k^-$ can be used to characterize the HR property. This characterization appears as [AHK18, Proposition 7.6] and [McM93, Theorem 8.6] in the case when $L = \ell^{d-2k}$ for $\ell \in A^1(\Sigma)$.

In what follows, we write $b_k(\Sigma)$ for the dimension of $A^k(\Sigma)$. Given $L \in A^{d-2k}(\Sigma)$ and $\ell_0 \in A^1(\Sigma)$, we define the *primitive part* of $A^k(\Sigma)$ to be the subspace

$$PA^k(\Sigma, \ell_0, L) := \Big\{ \eta \in A^k(\Sigma) \mid \ell_0 \cdot L \cdot \eta = 0 \Big\}.$$

For simplicity, when $L = \ell_0^{d-2k}$ and there is no possibility of confusion, we write $\mathrm{hr}^k(\Sigma, \ell_0)$ for $\mathrm{hr}^k(\Sigma, L)$ and $PA^k(\Sigma, \ell_0)$ for $PA^k(\Sigma, \ell_0, L)$.

Proposition 5.13. *If $U \subseteq A^{d-2k}(\Sigma)$ is a connected subset in the Euclidean topology and if the Hodge–Riemann form $\mathrm{hr}^k(\Sigma, L)$ is nondegenerate for all $L \in U$, then the signature of $\mathrm{hr}^k(\Sigma, L)$ is constant for all $L \in U$.*

Proof. The eigenvalues of $\mathrm{hr}^k(\Sigma, L)$ are real, and they vary continuously with L. By hypothesis, they are all nonzero for $L \in U$, so their signs are constant on U, because U is connected. □

We write $\mathrm{Sym}^{d-2k}\,\mathcal{K}(\Sigma) \subseteq A^{d-2k}(\Sigma)$ for the subset of products of elements of $\mathcal{K}(\Sigma) \subseteq A^1(\Sigma)$.

Proposition 5.14 (HR signature test)**.** *Suppose that Σ satisfies the conditions*

1. *$\mathrm{hr}^i(\Sigma, L)$ is nondegenerate for all $0 \le i \le k$ and all $L \in \mathrm{Sym}^{d-2i}\,\mathcal{K}(\Sigma)$, and*
2. *$\mathrm{hr}^i(\Sigma, L)$ is positive definite on the kernel of the multiplication by $\ell_0 L$ for all $\ell_0 \in \mathcal{K}(\Sigma)$, all $L \in \mathrm{Sym}^{d-2i}\,\mathcal{K}(\Sigma)$, and all $i < k$.*

Then $\mathrm{hr}^k(\Sigma, L)$ *is positive definite on the kernel of the multiplication by* $\ell_0 L$ *for all* $\ell_0 \in \mathcal{K}(\Sigma)$ *and all* $L \in \mathrm{Sym}^{d-2k}\,\mathcal{K}(\Sigma)$ *if and only if its signature equals*

$$\sum_{i=0}^{k}(-1)^{k-i}\big(b_i(\Sigma) - b_{i-1}(\Sigma)\big).$$

Proof. The proof is the same as the one given in [AHK18, Proposition 7.6] for the special case $L = \ell_0^{d-2k}$. The result follows from the induction on k and the Lefschetz decomposition

$$A^k(\Sigma) = PA^k(\Sigma, \ell_0, L) \oplus \ell_0 A^{k-1}(\Sigma),$$

which is orthogonal for the Hodge–Riemann form $\mathrm{hr}^k(\Sigma, L)$. □

Corollary 5.15. *If* Σ *satisfies mixed* HR^i *for all* $i < k$*, mixed* HL^k*, as well as* $\mathrm{HR}^k(L')$ *for some* $L' \in \mathrm{Sym}^{d-2k}\,\mathcal{K}(\Sigma)$*, then* Σ *satisfies* HR^k*.*

Proof. Let $L \in \mathrm{Sym}^{d-2k}\,\mathcal{K}(\Sigma)$ be any element. By the hypothesis mixed HL^k, the Hodge–Riemann form $\mathrm{hr}^k(\Sigma, L)$ is nondegenerate. By Proposition 5.13, it has the same signature as $\mathrm{hr}^k(\Sigma, L')$. Since Σ satisfies mixed HR^i for $i < k$, Proposition 5.14 shows that $\mathrm{HR}^k(L)$ and $\mathrm{HR}^k(L')$ are equivalent. □

Let $\widetilde{\Delta} = \mathrm{st}_{\widetilde{\Sigma}}(\rho)$ and $\Delta = \mathrm{st}_\Sigma(\sigma)$. In the case of a star-shaped blowup, the signature test simplifies slightly. In this case, by Propositions 5.10 and 5.14, the signature of $\mathrm{hr}^k(\widetilde{\Delta}, L)$ satisfying the Hodge–Riemann relations is $b_k(\Delta) - b_{k-1}(\Delta)$ for $k < \frac{d}{2}$.

5.6. **The Lefschetz property and edge subdivisions.** With these preparations, we now set out to show that the Lefschetz property of a fan is unaffected by edge subdivisions and their inverses. The precise statements and their proofs appear in Section 5.7 as Theorems 5.25 and 5.26. Here, we first consider Poincaré duality, and we first do so for star-shaped subdivisions.

Let $\widetilde{\Sigma}$ be the stellar subdivision of a d-dimensional simplicial fan Σ by a ray ρ in a two-dimensional cone σ. As before, we set $\widetilde{\Delta} = \mathrm{st}_{\widetilde{\Sigma}}(\rho)$ and $\Delta = \mathrm{st}_\Sigma(\sigma)$.

Proposition 5.16. *Poincaré duality holds for* $\widetilde{\Delta}$ *if and only if it holds for* Δ*.*

Proof. Assume that PD holds for at least one of $\widetilde{\Delta}$ and Δ. By Proposition 5.10, for all positive i,

$$A^i(\widetilde{\Delta}) \simeq A^i(\Delta) \oplus x_\rho A^{i-1}(\Delta).$$

We see that $A^{d-2}(\Delta) \simeq A^{d-1}(\widetilde{\Delta})$, so if one of Δ or $\widetilde{\Delta}$ has a fundamental weight, they both do. By inspection, $b_i(\Delta) = b_{d-2-i}(\Delta)$ for all i if and only if $b_i(\widetilde{\Delta}) = b_{d-1-i}(\widetilde{\Delta})$ for all i. So we may assume both sets of equalities hold.

For any $u \in A^i(\widetilde{\Delta})$ and $v \in A^{d-1-i}(\widetilde{\Delta})$, we write $u = u_0 + u_1 x_\rho$ and $v = v_0 + v_1 x_\rho$ where u_0, u_1, v_0, v_1 are elements of $A(\Delta)$ of degrees i, $i-1$, $d-1-i$, and $d-2-i$, respectively. Then $u_0 v_0 \in A^d(\Delta) = 0$, and $x_\rho^2 = c_1 \cdot x_\rho + c_2$ for some $c_1, c_2 \in A(\Delta)$. With respect to the decomposition above, the matrix of the multiplication pairing has the form

$$M^i(\widetilde{\Delta}) = \left(\begin{array}{c:c} 0 & -M^{i-1}(\Delta) \\ \hdashline -M^i(\Delta) & * \end{array}\right),$$

where $M^i(\Delta)$ denotes the matrix of the pairing $A^i(\Delta) \times A^{d-2-i}(\Delta) \to \mathbb{R}$. Thus if each matrix $M^i(\widetilde{\Delta})$ is invertible, so is each matrix $M^i(\Delta)$, and conversely. Therefore, if either Δ or $\widetilde{\Delta}$ has PD, then they both do. □

Proposition 5.17. *Suppose that Poincaré duality holds for Δ. Then Poincaré duality holds for $\widetilde{\Sigma}$ if and only if it holds for Σ.*

Proof. Let us assume that at least one of Σ and $\widetilde{\Sigma}$ has Poincaré duality, then show that they both do. For dimensional reasons, $\widetilde{\Sigma}$ must be an ordinary subdivision. By Corollary 5.12, we have $A^d(\widetilde{\Sigma}) \simeq A^d(\Sigma)$, and they have the common degree map. By Proposition 5.10 and Poincaré duality for $\mathrm{st}_\Sigma(\sigma)$, we have $b_i(\Sigma) = b_{d-i}(\Sigma)$ and $b_i(\widetilde{\Sigma}) = b_{d-i}(\widetilde{\Sigma})$ for all $0 \le i \le d$. Since $A^s(\Sigma) \times A^t(\mathrm{st}_\Sigma(\sigma)) \to A^{s+t}(\mathrm{st}_\Sigma(\sigma))$ is the zero map when $s+t > d-2$, ordering bases compatibly with the decomposition in Proposition 5.10 gives a block-diagonal matrix:

$$M^i(\widetilde{\Sigma}) = \left(\begin{array}{c|c} M^i(\Sigma) & 0 \\ \hline 0 & M^{i-1}(\Delta) \end{array}\right).$$

Clearly $M^i(\widetilde{\Sigma})$ has full rank if and only if $M^i(\Sigma)$ and $M^{i-1}(\Delta)$ both do as well, which completes the proof. □

Lemma 5.18. *Suppose that Poincaré duality holds for Σ. If $I \subseteq \Sigma(1)$ is a subset of rays for which $\{x_\nu\}_{\nu \in I}$ spans $A^1(\Sigma)$, then $\oplus i_\nu^*\colon A^i(\Sigma) \to \bigoplus_{\nu \in S} A^i(\mathrm{st}_\Sigma(\nu))$ is injective for all $0 \le i < d$.*

Proof. Suppose $i_\nu^*(u) = 0$ for each ray ν. Then $i_*^\nu i_\nu^*(u) = x_\nu u = 0$ for a set of generators x_ν of $A(\Sigma)$. Since $A(\Sigma)$ has no nonzero socle in degree $< d$ by Poincaré duality, the element u must be zero. □

Proposition 5.19. *Suppose that Poincaré duality holds for Σ. If $\mathrm{st}_\Sigma(\nu)$ satisfies mixed* HR *for each ray $\nu \in \Sigma(1)$, then Σ satisfies mixed* HL.

Proof. Let $L := \ell_1 \cdots \ell_{d-2k}$ be an element of $\mathrm{Sym}^{d-2k}\,\mathcal{K}(\Sigma)$, and consider the map $L\,\cdot\colon A^k(\Sigma) \to A^{d-k}(\Sigma)$. By Poincaré duality, we know that the domain and the target have the same dimension, so it is enough to show that $L\,\cdot$ is injective. Suppose, then, that $L \cdot u = 0$ for some $u \in A^k(\Sigma)$.

Let $L' := \ell_2 \cdots \ell_{d-2k}$. Note that, for each index i and each ray ν in Σ, the pullback $i_\nu^*(\ell_i)$ belongs to $\mathcal{K}(\mathrm{st}_\Sigma(\nu))$. Furthermore, since $L \cdot u = 0$, the pullback of u around ν is primitive:

$$i_\nu^*(u) \in PA^k(\mathrm{st}_\Sigma(\nu), i_\nu^*(\ell_1), i_\nu^*(L')).$$

We may write $\ell_1 = \sum_{\nu \in \Sigma(1)} c_\nu x_\nu$ where each coefficient $c_\nu > 0$, since we can represent ℓ_1 by a piecewise linear function which is strictly positive on each ray. We have

$$\begin{aligned} 0 = \deg_\Sigma(L \cdot u \cdot u) &= \deg_\Sigma\Big(\sum_{\nu \in \Sigma(1)} c_\nu x_\nu L' \cdot u \cdot u\Big) \\ &= \sum_\nu c_\nu \deg_{\mathrm{st}_\Sigma(\nu)}(i_\nu^*(L') \cdot i_\nu^*(u) \cdot i_\nu^*(u)) \\ &= (-1)^{k-1} \sum_{\nu \in \Sigma(1)} c_\nu \left\langle i_\nu^*(u), i_\nu^*(u) \right\rangle_{i_\nu^*(L')}, \end{aligned}$$

where the last summands are the Hodge–Riemann forms for $i_\nu^*(L')$. Since the c_ν's are strictly positive, each summand is zero, and the mixed HR property in $\mathrm{st}_\Sigma(\nu)$ implies $i_\nu^*(u) = 0$, for each ν. By Lemma 5.18, we have $u = 0$, and $L\cdot$ is injective. □

As an application, we see that the mixed Lefschetz properties in Definition 5.6 are actually no stronger than the pure ones. See [Cat08] for a discussion in a more general context.

Theorem 5.20. *If Σ is a Lefschetz fan, then it also has the mixed* HL *and mixed* HR *properties.*

Proof. We use induction on dimension. If $\dim \Sigma = 1$, the mixed and pure properties are identical, so let us suppose the claim is true for all Lefschetz fans of dimension less than d, for some $d > 1$. Let Σ be a Lefschetz fan of dimension d. By induction, $\mathrm{st}_\Sigma(\nu)$ satisfies mixed HR for all rays $\nu \in \Sigma(1)$. By Proposition 5.19, then Σ satisfies mixed HL.

Now we establish mixed HR for Σ. For any $\ell \in \mathcal{K}(\Sigma)$ and $0 \le k \le \frac{d}{2}$, the "pure" property $\mathrm{HR}^k(L')$ holds for $L' = \ell^{d-2k}$. Corollary 5.15 states that mixed HL and mixed HR^i for $i < k$ implies mixed HR^k. Setting $k = 0$, we see Σ has the mixed HR^0 property. Arguing by induction on k, we obtain mixed HR^k for all $k \le \frac{d}{2}$. □

We now examine how the Hodge–Riemann forms fare under stellar subdivisions. As before, we write $p\colon \widetilde{\Sigma} \to \Sigma$ for the map of fans given by the edge subdivision under consideration, write $x_\rho = a_1x_1 + a_2x_2$ for some positive scalars a_1, a_2, and consider the diagram

$$\begin{array}{ccc} E & \xhookrightarrow{\ j\ } & \widetilde{\Sigma} \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle p} \\ Z & \xhookrightarrow{\ i\ } & \Sigma. \end{array}$$

Lemma 5.21. *We have $p_*(x_\rho) = 0$ and $p_*(x_\rho^2) = -a_1a_2x_\sigma$.*

Proof. The first identity follows from the definition pushforward p_* between the Chow groups. Now let x_1, x_2 be the Courant functions for the rays ν_1, ν_2 of the cone σ, so $x_\sigma = x_1x_2$. For $i = 1, 2$, we have

$$0 = p_*(x_\rho)x_i = p_*\big(x_\rho(x_i + a_ix_\rho)\big),$$

so $p_*(x_\rho x_i) = -a_ip_*(x_\rho^2)$. Since $\{\nu_1, \nu_2\}$ is not contained in a cone of $\widetilde{\Sigma}$, we have

$$x_\sigma = p_*p^*(x_1x_2) = p_*\big((x_1 + a_1x_\rho)(x_2 + a_2x_\rho)\big) = (0 - 2a_1a_2 + a_1a_2)p_*(x_\rho^2). \qquad \square$$

Lemma 5.22. *Suppose that Poincaré duality holds for Σ. If $\widetilde{\Sigma}$ is an ordinary edge subdivision of Σ, then, for all $0 \le k \le \frac{d}{2}$ and all $L \in \mathrm{Sym}^{d-2k}\,\mathcal{K}(\Sigma)$, we have the orthogonal direct sum*

$$\mathrm{hr}^k(\widetilde{\Sigma}, p^*L) \cong \mathrm{hr}^k(\Sigma, L) \oplus \mathrm{hr}^{k-1}(\mathrm{st}_\Sigma(\sigma), i_\sigma^*(L)).$$

Proof. We consider $\mathrm{hr}^k(\widetilde{\Sigma}, p^*L)$ under the isomorphism $\phi\colon A^k(\Sigma) \oplus A^{k-1}(\mathrm{st}_\Sigma(\sigma)) \cong A^k(\widetilde{\Sigma})$ from Proposition 5.10. Recall that $\phi(u, v) = p^*(u) + j_*q^*(v)$. Let $\hat{v}$ be any preimage of v through the surjective map i_σ^*: then $j_*q^*(v) = j_*j^*p^*(\hat{v}) = x_\rho p^*(\hat{v})$.

We use the notation $\langle -, - \rangle$ to pair elements under the various Hodge–Riemann forms, and check first that the two summands are indeed orthogonal. We calculate:

$$(-1)^k \langle (u,0),(0,v)\rangle = \deg_{\widetilde{\Sigma}} \left(p^*(L)\cdot p^*(u)\cdot x_\rho p^*_\sigma(\hat v)\right)$$
$$= \deg_\Sigma \left(L \cdot u\hat v \cdot p_*(x_\rho)\right) = 0,$$

using the projection formula and the fact that $p_*(x_\rho) = 0$.

If $u, v \in A^k(\Sigma)$, the equality $\langle (u,0),(v,0)\rangle_{p^*(L)} = \langle u,v\rangle_L$ is straightforward. If $u, v \in A^{k-1}(\mathrm{st}_\Sigma(\sigma))$, as before write $u = i^*_\sigma(\hat u)$ and $v = i^*_\sigma(\hat v)$ for some $\hat u, \hat v \in A^{k-1}(\Sigma)$. Then, calculating as above,

$$\begin{aligned}\langle (0,u),(0,v)\rangle &= (-1)^k \deg_{\widetilde{\Sigma}} \left(p^*(L)\cdot p^*(\hat u)p^*(\hat v)\cdot x_\rho^2\right)\\ &= (-1)^k \deg_\Sigma \left(L \cdot \hat u\hat v\cdot p_*(x_\rho^2)\right)\\ &= -(-1)^k a_1a_2 \deg_\Sigma \left(L\cdot \hat u\hat v\cdot x_\sigma\right) \quad \text{by Lemma 5.21;}\\ &= (-1)^{k-1}a_1a_2 \deg_{\mathrm{st}_\Sigma(\sigma)} \left(i^*_\sigma(L)\cdot i^*_\sigma(\hat u) i^*_\sigma(\hat v)\right) = a_1a_2 \langle u,v\rangle_{i^*_\sigma(L)}\,.\end{aligned}$$

The conclusion follows, since $a_1, a_2 > 0$. ☐

Next we address star-shaped subdivisions. Set $e := \dim \mathrm{st}_\Sigma(\sigma) = \dim \Sigma - 2$.

Lemma 5.23. *Suppose P and Q are $n\times n$ matrices with real entries and $Q = Q^T$. Let*

$$M := \begin{pmatrix} 0 & P \\ P^T & Q\end{pmatrix}.$$

If P is nonsingular, then M has signature zero.

Proof. Assume first that Q is invertible, and let $S = -PQ^{-1}P^T$ (the Schur complement). Then it is easily seen that M is congruent to a block-diagonal matrix:

$$M = \begin{pmatrix} I_n & PQ^{-1} \\ 0 & I_n\end{pmatrix}\begin{pmatrix} S & 0 \\ 0 & Q\end{pmatrix}\begin{pmatrix} I_n & 0 \\ Q^{-1}P^T & I_n\end{pmatrix},$$

and the signature of S is the negative of the signature of Q. It follows that M has signature zero.

Now suppose Q is singular. We replace Q by $Q(\epsilon)$ to define $M(\epsilon)$ as above, for some real, invertible symmetric matrices $Q(\epsilon)$ with $\lim_{\epsilon\to 0} Q(\epsilon) = Q$. Then $\det(M(\epsilon)) = (-1)^n \det(P)^2 \neq 0$, regardless of ϵ, so the argument above shows $M(\epsilon)$ has n positive eigenvalues and n negative eigenvalues. By continuity, so does M. ☐

The last result in this section relates HL and HR along an edge subdivision.

Proposition 5.24. *Suppose that at least one of $\mathrm{st}_\Sigma(\sigma)$ and $\mathrm{st}_{\widetilde{\Sigma}}(\rho)$ satisfies Poincaré duality, and that $\ell \in \mathcal{K}(\mathrm{st}_\Sigma(\sigma))$ has the hard Lefschetz property. Then*

1. *$\ell_\epsilon := \ell - \epsilon\cdot x_\rho \in \mathcal{K}(\mathrm{st}_{\widetilde{\Sigma}}(\rho))$ has the HL property for sufficiently small $\epsilon > 0$, and*
2. *for such ϵ, the fan $\mathrm{st}_{\widetilde{\Sigma}}(\rho)$ satisfies $\mathrm{HR}(\ell_\epsilon)$ if $\mathrm{st}_\Sigma(\sigma)$ satisfies $\mathrm{HR}(\ell)$.*

Proof. Let $\Delta = \mathrm{st}_\Sigma(\sigma)$ and $\widetilde{\Delta} = \mathrm{st}_{\widetilde{\Sigma}}(\rho)$. By Proposition 5.17, we may assume both Δ and $\widetilde{\Delta}$ have Poincaré duality. By Proposition 5.4, we have $\ell_\epsilon \in \mathcal{K}(\widetilde{\Delta})$ for small enough positive ϵ.

If $k < (e+1)/2$, we use the HR property of $\ell \in \mathcal{K}(\Delta)$ and Proposition 5.10 to obtain a decomposition

$$A^k(\widetilde{\Delta}) = PA^k(\Delta,\ell) \oplus \ell A^{k-1}(\Delta) \oplus x_\rho A^{k-1}(\Delta),$$

with respect to which $\mathrm{hr}^k(\widetilde{\Delta}, \ell_\epsilon)$ is represented by a block matrix

$$\mathrm{hr}^k(\Delta, \ell_\epsilon) = \begin{pmatrix} H_{11}(\epsilon) & H_{12}(\epsilon) & H_{13}(\epsilon) \\ H_{21}(\epsilon) & H_{22}(\epsilon) & H_{23}(\epsilon) \\ H_{31}(\epsilon) & H_{32}(\epsilon) & H_{33}(\epsilon) \end{pmatrix}.$$

For any $\epsilon > 0$, the matrix above is congruent to the matrix

$$\overline{\mathrm{hr}}^k(\epsilon) := \begin{pmatrix} \epsilon^{-1}H_{11}(\epsilon) & \epsilon^{-1}H_{12}(\epsilon) & H_{13}(\epsilon) \\ \epsilon^{-1}H_{21}(\epsilon) & \epsilon^{-1}H_{22}(\epsilon) & H_{23}(\epsilon) \\ H_{31}(\epsilon) & H_{32}(\epsilon) & \epsilon H_{33}(\epsilon) \end{pmatrix},$$

the entries of which we will see are polynomial in ϵ. For elements $p_1, p_2 \in PA^k(\Delta, \ell)$, we have

$$\begin{aligned} \langle p_1, p_2 \rangle_{\ell_\epsilon} &= (-1)^k \deg_{\widetilde{\Delta}} \left((\ell - \epsilon x_\rho)^{e+1-2k} p_1 p_2\right) \\ &= -(-1)^k \cdot \epsilon \cdot \deg_{\widetilde{\Delta}} \left(\ell^{e-2k}(e+1-2k) p_1 p_2 x_\rho\right) + O(\epsilon^2) \\ &= (-1)^k \epsilon (e+1-2k) \deg_{\Delta} \left(\ell^{e-2k} p_1 p_2\right) + O(\epsilon^2) \\ &= (e+1-2k)\epsilon \cdot \langle p_1, p_2 \rangle_\ell + O(\epsilon^2), \end{aligned}$$

so the block $H_{11}(\epsilon)$ represents a positive multiple of the pairing $\mathrm{hr}^k(\Delta, \ell)$, modulo ϵ^2.

Similar computations show that the block $H_{22}(\epsilon)$ is the matrix of the pairing $(e+1-2k)\epsilon \cdot \mathrm{hr}^{k-1}(\Delta, \ell)$, modulo ϵ^2, and $H_{23}(\epsilon) = H_{32}(\epsilon) = -\mathrm{hr}^{k-1}(\Delta, \ell)$ modulo ϵ. Along the same lines, we see $H_{12}(\epsilon) = H_{21}(\epsilon)$ are divisible by ϵ^2, and $H_{13}(\epsilon) = H_{31}(\epsilon)$ is divisible by ϵ. Returning to the matrix for $\overline{\mathrm{hr}}^k(\epsilon)$, we have

$$\overline{\mathrm{hr}}^k(\epsilon) = \begin{pmatrix} (d+1-2k)\,\mathrm{hr}^k(\Delta, \ell)\,|_{PA^k} & 0 & 0 \\ 0 & -(d+1-2k)\,\mathrm{hr}^{k-1}(\Delta, \ell) & -\mathrm{hr}^{k-1}(\Delta, \ell) \\ 0 & -\mathrm{hr}^{k-1}(\Delta, \ell) & 0 \end{pmatrix} + O(\epsilon).$$

Given our assumption that $k < (e+1)/2$, the matrix $\overline{\mathrm{hr}}^k(0)$ is invertible, because each nonzero block is nondegenerate (since ℓ has the HL property). It follows that ℓ_ϵ has the HL^k property for all $0 \le k < (e+1)/2$, for some sufficiently small $\epsilon > 0$. Using Lemma 5.23, we see the signature of $\overline{\mathrm{hr}}^k(\epsilon)$ agrees with that of the top-left block. By hypothesis, $\mathrm{hr}^k(\Delta, \ell)$ is positive definite on $PA^k(\Delta, \ell)$. Now $\dim PA^k(\Delta, \ell) = b_k(\Delta) - b_{k-1}(\Delta)$, which by Propositions 5.10 and 5.14 is the expected signature for $\overline{\mathrm{hr}}^k(\epsilon)$; that is, $\mathrm{HR}^k(\ell_\epsilon)$ holds for sufficiently small ϵ.

It remains to consider the case where e is odd and $k = (e+1)/2$. In this case we have $A^k(\widetilde{\Delta}) = A^{k-1}(\Delta) \oplus x_\rho A^{k-1}(\Delta)$, and, up to a sign, the pairing is equal to the Poincaré pairing $M^k(\widetilde{\Delta})$. In the middle dimension, $M^k(\Delta) = M^{k-1}(\Delta)$, so we have a block decomposition

$$M^k(\widetilde{\Delta}) = \begin{pmatrix} 0 & -M^k(\Delta) \\ -M^k(\Delta) & Q \end{pmatrix}$$

for some square matrix Q. The matrix $M^k(\Delta)$ is nonsingular, by HL^k, so $M^k(\widetilde{\Delta})$ has signature zero by Lemma 5.23, which shows ℓ_ϵ has HR^k for any ϵ by Propositions 5.10 and 5.14. ∎

5.7. **Proofs of the main results.** We are now ready to prove the main result of this section. We will treat the star-shaped and ordinary cases separately, beginning with the former. As before, let $\widetilde{\Sigma}$ be a subdivision of a d-dimensional simplicial fan Σ by a ray ρ contained in a two-dimensional cone σ, and set $\widetilde{\Delta} = \mathrm{st}_{\widetilde{\Sigma}}(\rho)$ and $\Delta = \mathrm{st}_{\Sigma}(\sigma)$.

Theorem 5.25. *The fan Δ is Lefschetz if and only if the fan $\widetilde{\Delta}$ is Lefschetz.*

Proof. First, suppose that Δ is Lefschetz, and let ν_1, ν_2 denote the two extreme rays of σ. First, we check that the star of each cone $\tau \in \widetilde{\Delta}$ is Lefschetz. This is easy if τ does not contain ν_1 or ν_2, since then τ is a cone of Δ. Otherwise, τ contains (exactly) one such ray, say ν_1. The remaining rays of τ span a cone τ' of Δ, and by inspection, $\mathrm{st}_{\widetilde{\Delta}}(\tau) = \mathrm{st}_{\Delta}(\tau')$, which is again Lefschetz by hypothesis.

Poincaré duality for $\widetilde{\Delta}$ follows from Proposition 5.17. To establish HL, we use Proposition 5.19. For this, we need to know that the star of each ray satisfies mixed HR, but the star of a ray of $\widetilde{\Delta}$ is also a star in Δ, so HL for $\widetilde{\Delta}$ follows. Finally, we use Proposition 5.24: for any $\ell \in \mathcal{K}(\Delta)$, there exists some $\ell_\epsilon \in \mathcal{K}(\widetilde{\Delta})$ with the HR property. By Corollary 5.15, $\widetilde{\Delta}$ has HR.

Conversely, if $\widetilde{\Delta}$ is Lefschetz, then $\mathrm{st}_{\widetilde{\Delta}}(\nu_1) = \Delta$, so Δ is Lefschetz too. □

We note that $\mathcal{K}(\Delta)$ is nonempty if and only if $\mathcal{K}(\widetilde{\Delta})$ is nonempty. The forward implication follows immediately from Proposition 5.4. The converse holds because Δ is a star in $\widetilde{\Delta}$. However, $\mathcal{K}(\widetilde{\Sigma})$ can be nonempty while $\mathcal{K}(\Sigma)$ is empty.

Theorem 5.26. *If Σ is a Lefschetz fan with nonempty $\mathcal{K}(\Sigma)$, then $\widetilde{\Sigma}$ is a Lefschetz fan. Conversely, if $\widetilde{\Sigma}$ is a Lefschetz fan, then Σ is a Lefschetz fan.*

Proof. We prove the first statement by induction on the dimension d. The statement is vacuously true if $d = 1$, so let us assume it holds for all Lefschetz fans of dimension less than d.

First we check that the star of every cone $\tau \in \widetilde{\Sigma}$ is Lefschetz, for which we consider two cases. First suppose $\tau \in \Sigma$. If $\sigma \notin \mathrm{st}_{\Sigma}(\tau)$, then $\mathrm{st}_{\widetilde{\Sigma}}(\tau) = \mathrm{st}_{\Sigma}(\tau)$, which is Lefschetz. If, on the other hand, $\sigma \in \mathrm{st}_{\Sigma}(\tau)$, then $\mathrm{st}_{\widetilde{\Sigma}}(\tau) = \mathrm{stellar}_{\rho}(\mathrm{st}_{\Sigma}(\tau))$, which is a star-shaped subdivision. Since $\mathrm{st}_{\Sigma}(\tau)$ is Lefschetz, so is $\mathrm{st}_{\widetilde{\Sigma}}(\tau)$, by Theorem 5.25.

Now suppose $\tau \notin \Sigma$. Then $\rho \in \tau$, so $\mathrm{st}_{\widetilde{\Sigma}}(\tau) \subseteq \mathrm{st}_{\widetilde{\Sigma}}(\rho)$: in fact, $\mathrm{st}_{\widetilde{\Sigma}}(\tau) = \mathrm{st}_{\Sigma'}(\tau)$, where $\Sigma' = \mathrm{st}_{\widetilde{\Sigma}}(\rho)$. Since $\Sigma' = \mathrm{stellar}_{\rho}(\mathrm{st}_{\Sigma}(\sigma))$, a star-shaped subdivision, Σ' is Lefschetz by Theorem 5.25, and it follows that $\mathrm{st}_{\widetilde{\Sigma}}(\tau)$ is Lefschetz too.

By Propositions 5.17 and 5.19, respectively, the fan $\widetilde{\Sigma}$ satisfies PD and HL. It remains to check that $\widetilde{\Sigma}$ satisfies HR as well.

Consider any $0 \le k \le d/2$ and $\ell \in \mathcal{K}(\Sigma)$. By Lemma 5.22, we have $\mathrm{hr}^k(\widetilde{\Sigma}, p^*\ell) = \mathrm{hr}^k(\Sigma, \ell) \oplus \mathrm{hr}^{k-1}(\mathrm{st}_{\Sigma}(\sigma), i_\sigma^*(\ell))$. The summands are nondegenerate, because Σ and $\mathrm{st}_{\Sigma}(\sigma)$ satisfy $\mathrm{HL}(\ell)$ and $\mathrm{HL}(i_\sigma^*\ell)$, respectively, so $\mathrm{hr}^k(\widetilde{\Sigma}, p^*\ell)$ is nondegenerate as well.

By the HR signature test (Proposition 5.14) the signature of $\mathrm{hr}^k(\widetilde{\Sigma}, p^*\ell)$ equals

$$\begin{aligned}&\sum_{i=0}^{k}(-1)^{k-i}\big(b_i(\Sigma)-b_{i-1}(\Sigma)\big)+\sum_{i=0}^{k-1}(-1)^{k-(i-1)}\big(b_{i-1}(\mathrm{st}_\Sigma(\sigma))-b_{i-2}(\mathrm{st}_\Sigma(\sigma))\big)\\ &=\sum_{i=0}^{k}(-1)^{k-i}\big(b_i(\Sigma)+b_i(\mathrm{st}_\Sigma(\sigma))-b_{i-1}(\Sigma)-b_{i-1}(\mathrm{st}_\Sigma(\sigma))\big)\\ &=\sum_{i=0}^{k}(-1)^{k-i}\big(b_i(\widetilde{\Sigma})-b_{i-1}(\widetilde{\Sigma})\big).\end{aligned}$$

Proposition 5.4 states $p^*\ell$ is in the closure of $\mathcal{K}(\widetilde{\Sigma})$. Then there exists an open ball $U \subseteq A^1(\widetilde{\Sigma})$ containing $p^*\ell$ on which $\mathrm{hr}^k(\widetilde{\Sigma}, -)$ is nondegenerate. Choosing any $\ell' \in U \cap \mathcal{K}(\widetilde{\Sigma})$, we can use Corollary 5.15 to conclude that $\widetilde{\Sigma}$ satisfies HR^k.

The converse is similar in spirit. Again, we argue by induction on the dimension d. The base case being trivial, we assume that, if $\widetilde{\Sigma}$ is Lefschetz and has dimension less than d, then Σ is Lefschetz as well. Now assume $\widetilde{\Sigma}$ is a Lefschetz fan of dimension d, and we show Σ is as well.

PD for Σ follows from Proposition 5.17. Next, consider a ray $\nu \in \Sigma(1)$. If $\nu \notin \overline{\mathrm{st}}_\Sigma(\sigma)(1)$, then $\mathrm{st}_\Sigma(\nu) = \mathrm{st}_{\widetilde{\Sigma}}(\nu)$, which is Lefschetz. If, on the other hand, $\nu \in \overline{\mathrm{st}}_\Sigma(\sigma)(1)$, then $\sigma \in \overline{\mathrm{st}}_\Sigma(\nu)(2)$, and $\overline{\mathrm{st}}_{\widetilde{\Sigma}}(\nu) = \mathrm{stellar}_\rho(\overline{\mathrm{st}}_\Sigma(\nu))$. Since $\mathrm{st}_{\widetilde{\Sigma}}(\nu)$ is Lefschetz, so is $\mathrm{st}_\Sigma(\nu)$, by Theorem 5.25. Either way, $\mathrm{st}_\Sigma(\nu)$ has the HR property for each ray ν, so Σ has the HL property (by Proposition 5.19).

A similar argument shows that $\mathrm{st}_\Sigma(\tau)$ is Lefschetz for all cones τ of Σ: if the star remains a star in $\widetilde{\Sigma}$, it is Lefschetz by hypothesis. Otherwise, a subdivision of it is a star in $\widetilde{\Sigma}$. If $\tau = \sigma$, the subdivided edge, we invoke Theorem 5.25. Otherwise, we note the dimension is less than d, so $\mathrm{st}_\Sigma(\tau)$ is Lefschetz by induction.

It remains to establish HR^k for Σ, for $0 \le k \le d/2$. The condition is vacuous if $\mathcal{K}(\Sigma) = \varnothing$. Otherwise, choose any $\ell \in \mathcal{K}(\Sigma)$. By Lemma 5.22,

$$\mathrm{hr}^k(\widetilde{\Sigma}, p^*\ell) = \mathrm{hr}^k(\Sigma, \ell) \oplus \mathrm{hr}^{k-1}(\mathrm{st}_\Sigma(\sigma), i_\sigma^*(\ell)).$$

Since the second factor is the blowdown of $\mathrm{st}_{\widetilde{\Sigma}}(\rho)$, it is Lefschetz by Theorem 5.25, and the first factor is Lefschetz by the argument above. So both summands are nondegenerate, and so is $\mathrm{hr}^k(\widetilde{\Sigma}, p^*\ell)$.

By HR, the bilinear form $\mathrm{hr}^k(\widetilde{\Sigma}, \widetilde{\ell})$ has the expected signature for all $\widetilde{\ell} \in \mathcal{K}(\widetilde{\Sigma})$. It follows by Proposition 5.13 that $\mathrm{hr}^k(\widetilde{\Sigma}, p^*\ell)$ also has that signature, since it is nondegenerate and $p^*\ell$ lies in the boundary of $\mathcal{K}(\widetilde{\Sigma})$.

The HR property for $\mathrm{st}_\Sigma(\sigma)$ determines the signature of $\mathrm{hr}^{k-1}(\mathrm{st}_\Sigma(\sigma), i_\sigma^*(\ell))$, and we obtain the signature of $\mathrm{hr}^k(\Sigma, \ell)$ by subtraction. By the HR signature test again, we find that it equals $\sum_{i=0}^{k}(-1)^{k-i}\big(b_i(\Sigma) - b_{i-1}(\Sigma)\big)$, and we conclude Σ has the HR^k property. □

Putting the pieces together gives a proof that the Lefschetz property is an invariant of the support of a fan.

Theorem 1.6. *Let Σ_1 and Σ_2 be simplicial fans that have the same support $|\Sigma_1| = |\Sigma_2|$. If $\mathcal{K}(\Sigma_1)$ and $\mathcal{K}(\Sigma_2)$ are nonempty, then Σ_1 is Lefschetz if and only if Σ_2 is Lefschetz.*

Proof of Theorem 1.6. Suppose $|\Sigma| = |\Sigma'|$. According to Theorem 5.9, there is a sequence of fans $(\Sigma_0, \Sigma_1, \cdots, \Sigma_N)$ with $\Sigma = \Sigma_0$, $\Sigma_N = \Sigma'$, and for which either $\Sigma_i \to \Sigma_{i+1}$ or $\Sigma_{i+1} \to \Sigma_i$ is an edge subdivision, for each i. Furthermore, there is some i_0 for which $\Sigma_i \to \Sigma$ is a projective map of fans for each $i \leq i_0$, and $\Sigma_i \to \Sigma'$ is a projective map of fans for each $i \geq i_0$. By Proposition 5.3, we see that the cone $\mathcal{K}(\Sigma_i)$ is nonempty for each i. By Theorem 5.26, if any one of these fans is Lefschetz, then they all are. □

In our terminology, the main result of [AHK18] says that the Bergman fan of M is Lefschetz. We use the result to show that the conormal fan of M is Lefschetz.

Lemma 5.27. *If Σ_1 and Σ_2 are Lefschetz fans, then so is $\Sigma_1 \times \Sigma_2$.*

Proof. It was shown in [AHK18, Section 7.2] that, if Σ_1 and Σ_2 have PD, HL, and HR, then so does $\Sigma_1 \times \Sigma_2$. Since stars of cones in a product are products of stars in the factors, we conclude that $\Sigma_1 \times \Sigma_2$ is a Lefschetz fan, by induction on dimension. □

Theorem 5.28. *For any matroid* M, *the conormal fan $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ is Lefschetz.*

Proof. We may assume that M is loopless and coloopless. Since the Bergman fan is Lefschetz, from Lemma 5.27 we see the fan $\Sigma_{\mathrm{M}} \times \Sigma_{\mathrm{M}^\perp}$ is Lefschetz. Moreover, its support is equal to that of $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$. Bergman fans are quasiprojective, since they are subfans of the permutohedral fan, so $\mathcal{K}(\Sigma_{\mathrm{M}} \times \Sigma_{\mathrm{M}^\perp})$ is nonempty. We saw that the bipermutohedral fan $\Sigma_{E,\overline{E}}$ is the normal fan of the bipermutohedron, so the conormal fan is also quasiprojective, and $\mathcal{K}(\Sigma_{\mathrm{M},\mathrm{M}^\perp})$ is nonempty as well. By Theorem 1.6, then, $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ is Lefschetz. □

The extra structure present in the Chow rings of Lefschetz fans leads easily to an Aleksandrov–Fenchel-type inequality.

Theorem 5.29. *Let Σ be a Lefschetz fan of dimension d, and $\ell_2, \ell_3, \ldots, \ell_d$ elements in the closure of $\mathcal{K}(\Sigma)$. Then for any $\ell_1 \in A^1(\Sigma)$,*

$$\deg(\ell_1\ell_2\cdots\ell_d)^2 \geq \deg(\ell_1\ell_1\ell_3\cdots\ell_d)\cdot\deg(\ell_2\ell_2\ell_3\cdots\ell_d).$$

Proof. We first verify the inequality when $\ell_i \in \mathcal{K}(\Sigma)$ for each $2 \leq i \leq d$. For this, let $L = \ell_3 \cdots \ell_d$, a Lefschetz element, and consider $\langle -,- \rangle := \langle -,- \rangle_L$ on $A^1(\Sigma)$.

If $\langle \ell_2, \ell_2 \rangle \neq 0$, let $\ell_1' = \ell_1 - \frac{\langle \ell_1, \ell_2 \rangle}{\langle \ell_2, \ell_2 \rangle}\ell_2$, so that $\langle \ell_1', \ell_2 \rangle = 0$. This means $\ell_1' \in PA^1(\Sigma, \ell_2)$, so by HR,

$$0 \leq \langle \ell_1', \ell_1' \rangle = \langle \ell_1, \ell_1' \rangle = \langle \ell_1, \ell_1 \rangle - \frac{\langle \ell_1, \ell_2 \rangle}{\langle \ell_2, \ell_2 \rangle}\langle \ell_1, \ell_2 \rangle .$$

By the signature test, $\langle -,- \rangle$ is negative-definite on the orthogonal complement of ℓ_1'. Therefore $\langle \ell_2, \ell_2 \rangle < 0$, and we see that

$$\langle \ell_1, \ell_2 \rangle^2 \geq \langle \ell_1, \ell_1 \rangle \cdot \langle \ell_2, \ell_2 \rangle .$$

If, on the other hand, $\langle \ell_2, \ell_2 \rangle = 0$, then the displayed inequality is obvious.

If we relax the hypothesis to consider $\ell_2, \ldots, \ell_d$ in the closure of $\mathcal{K}(\Sigma)$, then the desired inequality continues to hold by continuity, as in [AHK18, Theorem 8.8]. □

Theorem 1.4. *For any matroid* M, *the h-vector of the broken circuit complex of* M *is log-concave.*

Proof. It suffices to assume that M is loopless and coloopless. The classes $\gamma = \gamma_i$ and $\delta = \delta_i$ are pullbacks of the nef classes $\alpha = \alpha_i \in A^1(\Sigma_{\mathrm{M}})$ and $\alpha = \alpha_i \in A^1(\Delta_E)$, along the two maps $\pi\colon \Sigma_{\mathrm{M},\mathrm{M}^\perp} \to \Sigma_{\mathrm{M}}$ and $\mu\colon \Sigma_{\mathrm{M},\mathrm{M}^\perp} \to \Delta_E$, respectively. The pullback of a convex function on a fan is convex, so both γ and δ are represented by convex functions on the conormal fan. Since $\mathcal{K}(\Sigma_{\mathrm{M},\mathrm{M}^\perp})$ is nonempty by Proposition 2.20, we see that γ and δ are in the closure of $\mathcal{K}(\Sigma_{\mathrm{M},\mathrm{M}^\perp})$, following the discussion at the end of Section 5.1. By Theorem 1.2, we have

$$h_{r-k}(\mathrm{BC}(\mathrm{M})) = \deg_{\Sigma_{\mathrm{M},\mathrm{M}^\perp}}(\gamma^k\delta^{n-k-1}) = \langle \gamma, \delta \rangle_L\,,$$

where $L = \gamma^{k-1}\delta^{n-k-2}$. Since $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ is Lefschetz by Theorem 5.28, the log-concave inequalities follow from Theorem 5.29. □

Remark 5.30. In the above proof of Theorem 1.4, our use of the existence of the bipermutohedron (Proposition 2.20) can be avoided. The toric Chow lemma [CLS11, Theorem 6.1.18] guarantees that the conormal fan has a refinement that is the normal fan of a polytope, and we may apply the same argument to that refinement.

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