

The tropical critical points of an affine matroid

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Abstract. We prove that the maximum likelihood degree of a matroid M equals its beta invariant $\beta(M)$. For an element e of M that is neither a loop nor a coloop, this is defined to be the degree of the intersection of the Bergman fan of (M, e) and the inverted Bergman fan of $N = (M/e)^\perp$. Equivalently, for a generic vector $w \in \mathbb{R}^{E-e}$, this is the number of ways to find weights $(0, x)$ on M and y on N with $x + y = w$ such that on each circuit of M (resp. N), the minimum x -weight (resp. y -weight) occurs at least twice.

Keywords: matroid, Bergman fan, maximum likelihood degree

1 Introduction

During the Workshop on Nonlinear Algebra and Combinatorics from Physics at the Center for the Mathematical Sciences and Applications at Harvard University in April 2022, Sturmfels [15] posed one of those combinatorial problems that is deceptively simple to state, but whose answer requires a deeper understanding of the objects at hand.

Problem 1.1. [15] (Combinatorial version) Let M be a matroid on ground set E . Let e be an element that is neither a loop nor a coloop. Let M/e be the contraction of M by e and let $N = (M/e)^\perp$ be its dual matroid. Given a vector $w \in \mathbb{R}^{E-e}$, find weight vectors $(0, x) \in \mathbb{R}^E$ on M (where e has weight 0) and $y \in \mathbb{R}^{E-e}$ on N such that

- on each circuit of M , the minimum x -weight occurs at least twice,
- on each circuit of N , the minimum y -weight occurs at least twice, and
- $w = x + y$.

Can this always be done? What is the number of solutions for generic w ?

Theorem 1.2. (Geometric Version) *Problem 1.1 can always be solved. If $w \in \mathbb{R}^{E-e}$ is generic, the number of solutions equals the beta invariant $\beta(M)$ of the matroid.*

We now restate [Theorem 1.2](#) in tropical terms; see relevant definitions in [Section 2.3](#).

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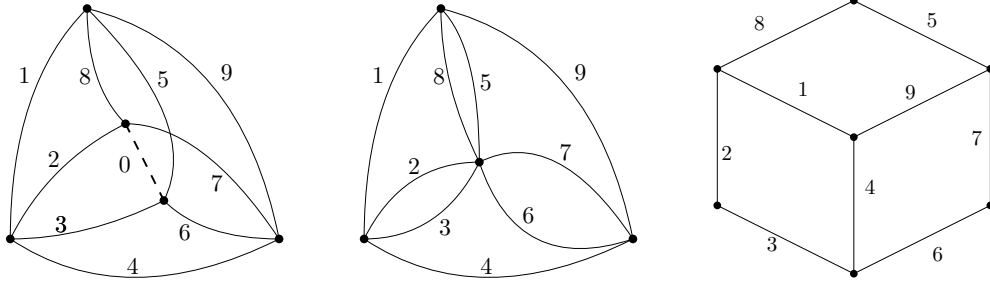


Figure 1: A graph G , its contraction $G/0$, and its dual $H = (G/0)^\perp$.

Theorem 1.3. *Let M be a matroid on E , and let $e \in E$ be an element that is neither a loop nor a coloop. Let (M, e) be the affine matroid of M with respect to e , and let $N = (M/e)^\perp$. Then the degree of the intersection of the Bergman fan $\Sigma_{(M,e)}$ and the inverted Bergman fan $-\Sigma_N$ is*

$$\deg(\Sigma_{(M,e)} \cdot -\Sigma_N) = \beta(M).$$

Agostini, Brysiewicz, Fevola, Kühne, Sturmfels, and Telen [1] first encountered (a special case of) [Problem 1.1](#) in their study of the maximum likelihood estimation for linear discrete models. Using algebro-geometric results of Huh and Sturmfels [10], which built on work of Varchenko [16], they proved [Theorem 1.2](#) and [Theorem 1.3](#) for matroids realizable over the real numbers.

We prove the equivalent [Theorem 1.2](#) and [Theorem 1.3](#) for all matroids. Following the original motivation, we call the answer to [Problem 1.1](#) the maximum likelihood degree of a matroid; our main result is that it equals the beta invariant.

We first prove [Theorem 1.2](#) combinatorially, relying on the tropical geometric fact that all generic w give the same intersection degree. We show that when the entries of w are super-increasing with respect to some order $<$ on E , the solutions to [Problem 1.1](#) are naturally in bijection with the β -nbc bases of the matroid with respect to $<$. We then sketch a proof of [Theorem 1.3](#) that relies on the theory of tautological classes of matroids of Berget, Eur, Spink, and Tseng [6]. This is an extended abstract of our results in [5].

2 Notation and preliminaries

2.1 The lattice of set partitions

A set partition λ of a set E is a collection of subsets, called blocks, of E , say $\lambda = \{\lambda_1, \dots, \lambda_\ell\}$, that cover E and the pairwise intersection is empty. We write $\lambda \models E$. We let $|\lambda| = \ell$ be the number of blocks of λ . If $e \in E$ and $\lambda \models E$, we write $\lambda(e)$ for the block of λ that contains e .

We define the *linear space of a set partition* $\lambda = \{\lambda_1, \dots, \lambda_\ell\} \models E$ to be

$$\begin{aligned} L(\lambda) &:= \text{span}\{e_{\lambda_1}, \dots, e_{\lambda_\ell}\} \subseteq \mathbb{R}^E \\ &= \{x \in \mathbb{R}^E \mid x_i = x_j \text{ whenever } i, j \text{ are in the same block of } \lambda\}, \end{aligned}$$

where $\{e_i : i \in E\}$ is the standard basis of $\mathbb{R}^E$, $e_S = \sum_{s \in S} e_s$ for $S \subseteq E$. Notice that $\dim L(\lambda) = |\lambda|$. The map $\lambda \mapsto L(\lambda)$ is a bijection between the set partitions of E and the flats of the *braid arrangement*, which is the hyperplane arrangement in $\mathbb{R}^E$ given by the hyperplanes $x_i = x_j$ for $i \neq j$ in E .

If $\lambda \models 0 \sqcup E$ then we write $L(\lambda)|_{x_0=0} = \{x \in \mathbb{R}^E : (0, x) \in L(\lambda) \subseteq \mathbb{R}^{0 \sqcup E}\}$.

2.2 The intersection graph of two set partitions

The following construction from [3] will play an important role.

Definition 2.1. Let $\lambda \models 0 \sqcup E$ and $\mu \models E$ be set partitions. The intersection graph $\Gamma = \Gamma_{\lambda, \mu}$ is the bipartite graph with vertex set $\lambda \sqcup \mu$ and edge set E , where the edge e connects the parts $\lambda(e)$ of λ and $\mu(e)$ of μ containing e . On this graph, the vertex corresponding to $\lambda(0)$ is marked with a hollow point.

The intersection graph may have several parallel edges connecting the same pair of vertices. Notice that the label of a vertex in Γ is just the set of labels of the edges incident to it. Therefore we can remove the vertex labels, and simply think of Γ as a bipartite multigraph on edge set E . This is illustrated in Figure 2.

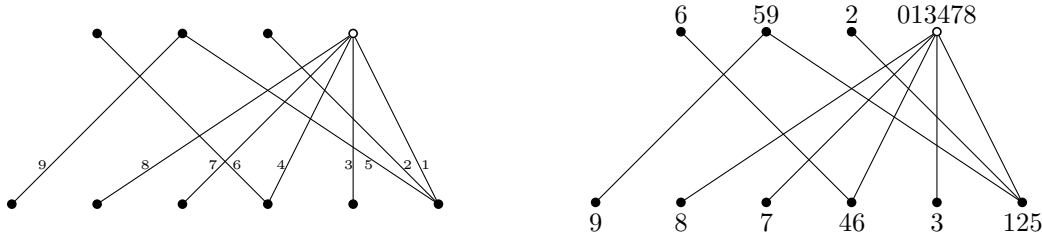


Figure 2: The intersection graph of $\{6, 59, 2, 013478\} \models [0, 9]$ and $\{9, 8, 7, 46, 3, 125\} \models [9]$, omitting brackets for easier legibility. **Left:** The elements of $[0, 9]$ are labelling the edges. **Right:** the vertices are labelled by parts of the set partitions.

Lemma 2.2. Let $\lambda \models 0 \sqcup E$ and $\mu \models E$ be set partitions and $\Gamma_{\lambda, \mu}$ be their intersection graph.

1. If $\Gamma_{\lambda, \mu}$ has a cycle, then $L(\lambda)|_{x_0=0} \cap (w - L(\mu)) = \emptyset$ for generic¹ $w \in \mathbb{R}^E$.

¹This means that this property holds for all w outside of a set of measure 0.

2. If $\Gamma_{\lambda, \mu}$ is disconnected, then $L(\lambda)|_{x_0=0} \cap (w - L(\mu))$ is not a point for any $w \in \mathbb{R}^E$.

3. If $\Gamma_{\lambda, \mu}$ is a tree, then $L(\lambda)|_{x_0=0} \cap (w - L(\mu))$ is a point for any $w \in \mathbb{R}^E$.

Proof. Let $x \in L(\lambda)$ and $y \in L(\mu)$ such that $x + y = w$. Write $x_{\lambda(i)} := x_i$ and $y_{\mu(j)} := y_j$ for simplicity. The subspace $L(\lambda)|_{x_0=0} \cap (w - L(\mu))$ is cut out by the equalities

$$\begin{aligned} x_{\lambda(i)} + y_{\mu(i)} &= w_i & \text{for } i \in E, \\ x_{\lambda(0)} &= 0. \end{aligned}$$

This system has $|E| + 1$ equations and $|\lambda| + |\mu|$ unknowns. The linear dependences among these equations are controlled by the cycles of the graph $\Gamma_{\lambda, \mu}$. More precisely, the first $|E|$ linear functionals $\{x_{\lambda(i)} + y_{\mu(i)} : i \in E\}$ gives a realization of the graphical matroid of $\Gamma_{\lambda, \mu}$. The last equation is clearly linearly independent from the others.

If $\Gamma_{\lambda, \mu}$ has a cycle with edges $i_1, i_2, \dots, i_{2k}$ in that order, then the above equalities imply that $w_{i_1} - w_{i_2} + w_{i_3} - \dots - w_{i_{2k}} = 0$. For a generic w , this equation does not hold, so $L(\lambda)|_{x_0=0} \cap (w - L(\mu)) = \emptyset$.

If $\Gamma_{\lambda, \mu}$ is disconnected, let A be the set of edges in a connected component not containing the vertex $\lambda(0)$. If $x \in L(\lambda)$ and $y \in L(\mu)$ satisfy $x + y = w$ and $x_0 = 0$, then $x + re_A \in L(\lambda)$ and $y - re_A \in L(\mu)$ also satisfy those equations for any real number r . Therefore $L(\lambda)|_{x_0=0} \cap (w - L(\mu))$ is not a point.

Finally, if $\Gamma_{\lambda, \mu}$ is a tree, then its number of vertices is one more than the number of edges, that is, $|E| + 1 = |\lambda| + |\mu|$, so the system of equations has equally many equations and unknowns. Also, these equations are linearly independent since $\Gamma_{\lambda, \mu}$ is a tree. It follows that the system has a unique solution. $\square$

When $\Gamma_{\lambda, \mu}$ is a tree, we call λ and μ an *arboreal pair*.

Lemma 2.3. *Let $\lambda \models 0 \sqcup E$ and $\mu \models E$ be an arboreal pair of set partitions and let $\Gamma_{\lambda, \mu}$ be their intersection tree. Let $w \in \mathbb{R}^E$. The unique vectors $x \in L(\lambda)$ and $y \in L(\mu)$ such that $x + y = w$ and $x_0 = 0$ are given by*

$$\begin{aligned} x_{\lambda_i} &= w_{e_1} - w_{e_2} + \dots \pm w_{e_k} & \text{where } e_1 e_2 \dots e_k \text{ is the unique path from } \lambda_i \text{ to } \lambda(0) \\ y_{\mu_j} &= w_{f_1} - w_{f_2} + \dots \pm w_{f_l} & \text{where } f_1 f_2 \dots f_l \text{ is the unique path from } \mu_j \text{ to } \lambda(0) \end{aligned}$$

for any i and j .

Proof. This follows readily from the fact that, for each $1 \leq i \leq k$, the values of $x_{\lambda(e_i)}$ and $y_{\mu(e_i)}$ on the vertices incident to edge i have to add up to w_{e_i} . $\square$

Example 2.4. *The set partitions $\lambda = \{6, 59, 2, 013478\} \models [0, 9]$, $\mu = \{9, 8, 7, 46, 3, 125\} \models [9]$ form an arboreal pair, whose intersection tree is shown in Figure 2. We have, for example,*

$y_9 = w_9 - w_5 + w_1$ because the path from $\mu(9) = \{9\}$ to $\lambda(0) = \{013478\}$ uses edges 9, 5, 1 in that order. The remaining values are:

$$\begin{aligned} x_6 &= w_6 - w_4, & x_{59} &= w_5 - w_1, & x_2 &= w_2 - w_1, & x_{13478} &= 0, \\ y_9 &= w_9 - w_5 + w_1, & y_8 &= w_8, & y_7 &= w_7, & y_{46} &= w_4, & y_3 &= w_3, & y_{1235} &= w_1. \end{aligned}$$

Definition 2.5. A vector $w \in \mathbb{R}^{n+1}$ is *super-increasing* if $w_{i+1} > 3w_i > 0$ for $1 \leq i \leq n$.

Lemma 2.6. Let w be super-increasing. For any $1 \leq a < b \leq n+1$ and any choice of ϵ_i s and δ_j s in $\{-1, 0, 1\}$, we have $w_a + \sum_{i=1}^{a-1} \epsilon_i w_i < w_b + \sum_{j=1}^{b-1} \delta_j w_j$.

Proof. After verifying inductively that $w_c > 2 \sum_{i=1}^{c-1} w_i$ for all c , we see that

$$w_a + \sum_{i=1}^{a-1} \epsilon_i w_i \leq w_a + \sum_{i=1}^{a-1} w_i < \frac{3}{2} w_a < \frac{1}{2} w_b < w_b - \sum_{j=1}^{b-1} w_j \leq w_b + \sum_{j=1}^{b-1} \delta_j w_j$$

as desired. □

Definition 2.7. Given a super-increasing vector $w \in \mathbb{R}^{n+1}$ and a real number x , we will say x is near w_i and write $x \approx w_i$ if $w_i - (w_1 + \cdots + w_{i-1}) \leq x \leq w_i + (w_1 + \cdots + w_i)$. Note that if $x \approx w_i$ and $y \approx w_j$ for $i < j$ then $x < y$.

2.3 Matroids, Bergman fans, and tropical geometry

We assume familiarity with basic notions in matroid theory; for definitions and proofs, see [13, 17]. We also state here some facts from tropical geometry that we will need; see [11, 12] for a thorough introduction.

Let M be a matroid on E . The *dual matroid* $M^\perp$ is the matroid on E whose set of bases is $\{B^\perp \mid B \text{ is a basis of } M\}$, where $B^\perp := E - B$.

The following lemma is useful to how M and $M^\perp$ interact; see [13, Proposition 2.1.11].

Lemma 2.8. [2, Lemma 3.14] Let M be a matroid, and let F be a flat of M and G be a flat of $M^\perp$. Then $|F \cup G| \neq |E| - 1$.

Definition 2.9. Fix a linear order $<$ on M . A *broken circuit* is a set of the form $C - \min_{<} C$ where C is a circuit of M . An *nbc-basis* of M is a basis of M that contains no broken circuits. A β nbc-basis of M is an nbc-basis B such that $B^\perp \cup 0 \setminus 1$ is an nbc-basis of $M^\perp$.

Theorem 2.10. [7] The number of β nbc-bases of M is the beta invariant $\beta(M)$ which is given by $\beta(M) = |\chi'_M(1)|$, where χ_M is the characteristic polynomial of M :

$$\chi_M(t) = \sum_{X \subseteq E} (-1)^{|X|} t^{r(M) - r(X)}.$$

Assume that $B = \{b_1 > \dots > b_r\}$ is a basis of the matroid M . We define the flats $F_i := \text{cl}_M\{b_1, \dots, b_i\}$ and the flag $\mathcal{F}_M(B) := \{F_i\}$. The following characterization of nbc-basis will be useful.

Lemma 2.11. *Let M be a matroid of size $n + 1$ and rank $r + 1$, and B a basis of M . Then B is an nbc-basis of M if and only if $b_i = \min F_i$ for $i = 1, \dots, r + 1$.*

Proof. Omitted. □

An *affine matroid* (M, e) on E is a matroid M on E with a chosen element $e \in E$.

Definition 2.12. [14] *The Bergman fan of a matroid M on E is*

$$\Sigma_M = \{x \in \mathbb{R}^E \mid \min_{c \in C} x_c \text{ is attained at least twice for any circuit } C \text{ of } M\}.$$

The Bergman fan of an affine matroid (M, e) on E is

$$\Sigma_{(M, e)} = \{x \in \mathbb{R}^{E-e} \mid (0, x) \in \Sigma_M\}.$$

The motivation for this definition comes from tropical geometry. A subspace $V \subset \mathbb{R}^E$ determines a matroid M_V , and the tropicalization of V is precisely the Bergman fan of M_V . Similarly, an affine subspace $W \subset \mathbb{R}^{E-e}$ determines an affine matroid (M_W, e) consisting of a matroid M_W on E and a special element e , which represents the hyperplane at infinity. The tropicalization of W is the Bergman fan $\Sigma_{(M_W, e)}$.

Theorem 2.13. [4] *The Bergman fan of a matroid M is a tropical fan equal to the union of the cones*

$$\begin{aligned} \sigma_{\mathcal{F}} &= \text{cone}(e_{F_1}, \dots, e_{F_{r+1}}) \\ &= \{x \in \mathbb{R}^E \mid x_a > x_b \text{ whenever } a \in F_i \text{ and } b \notin F_i \text{ for some } 1 \leq i \leq r + 1\} \end{aligned}$$

for the complete flags $\mathcal{F} = \{\emptyset = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_r \subsetneq F_{r+1} = E\}$ of flats of M .

If Σ_1 and Σ_2 are tropical fans of complementary dimensions, then Σ_1 and $\Sigma_2 + v$ intersect transversally at a finite set of points for generic vectors $v \in \mathbb{R}^n$. Each intersection point p is equipped with a weight $w(p)$ that depends on the respective intersecting cones. It is a general fact that the quantity

$$\deg(\Sigma_1 \cdot \Sigma_2) := \sum_{p \in \Sigma_1 \cap (v + \Sigma_2)} w(p)$$

is constant for generic v ; this is the degree of the intersection [12, Proposition 4.3.3, 4.3.6].

In all the intersections that arise in this paper, one can verify that the weight $w(p)$ is equal to 1 for every intersection point p . Therefore the degree of the intersection will be simply the number of intersection points:

$$\deg(\Sigma_{(M, e)} \cdot -\Sigma_N) := |\Sigma_{(M, e)} \cap (v - \Sigma_N)|$$

for generic $v \in \mathbb{R}^{E-e}$.

3 A combinatorial proof of the main theorem

Let M be a matroid on $[0, n]$ of rank $r + 1$ such that 0 is not a loop nor a coloop. Then $M/0$ has rank r , and $N = (M/0)^\perp$ has rank $n - r$. For any basis B of M containing 0 , $B^\perp = [0, n] - B$ is a basis of $N = (M/0)^\perp$. Conversely, every basis of N equals $B^\perp$ for a basis B of M containing 0 .

The following Lemma constructs an intersection point in $\Sigma_M|_{x_0=0} \cap (w - \Sigma_N)$ for each β -nbc basis B of M .

Lemma 3.1. *Let M be a matroid on $E = [0, n]$ of rank $r + 1$ such that 0 is not a coloop, and let $N = (M/0)^\perp$. Let $w \in \mathbb{R}^n$ be super-increasing. For any β -nbc basis B of M , there exist unique vectors $(0, x) \in \sigma_{\mathcal{F}_M(B)}$ and $y \in \sigma_{\mathcal{F}_N(B^\perp)}$ such that $x + y = w$.*

Proof. First we show that the set partitions π of $\mathcal{F}_M(B)$ and $\pi^\perp$ of $\mathcal{F}_N(B^\perp)$ form an arboreal pair. Since they have sizes $|B| = r + 1$ and $|B^\perp| = n - r$, respectively, their intersection graph has $n + 1$ vertices and n edges. Therefore it is sufficient to prove that the intersection graph $\Gamma_{\pi, \pi^\perp}$ is connected.

Assume contrariwise, and let A be a connected component not containing the edge 1 . Let $a > 1$ be the smallest edge in A . Then a is the smallest element of its part $\pi(a)$ in π , and since B is nbc in M , this implies $a \in B$. Similarly, since $B^\perp$ is nbc in N , this also implies $a \in B^\perp$. This is a contradiction.

It follows from Lemma 2.2 that there exist unique $(0, x) \in L(\pi)$ and $y \in L(\pi^\perp)$ such that $x + y = w$. It remains to show that $(0, x) \in \sigma_{\mathcal{F}}$ and $y \in \sigma_{\mathcal{F}^\perp}$.

Lemma 2.3 provides formulas for x and y in terms of the paths from the various vertices of the tree of $\Gamma_{\pi, \pi^\perp}$ to $\pi(0)$. To understand those paths, let us give each edge e an orientation as follows:

$$\begin{aligned} \pi(e) &\longrightarrow \pi^\perp(e) & \text{if } \min \pi(e) > \min \pi^\perp(e), \\ \pi(e) &\longleftarrow \pi^\perp(e) & \text{if } \min \pi(e) < \min \pi^\perp(e). \end{aligned}$$

We never have $\min \pi(e) = \min \pi^\perp(e)$, because as above, that would imply $e \in B \cap B^\perp$.

We claim that every vertex other than $\pi(0)$ has an outgoing edge under this orientation. Consider a part $\pi_i \neq \pi(0)$ of π ; let $\min \pi_i = b$. Edge b connects $\pi_i = \pi(b)$ to $\pi^\perp(b) \ni b$, and we cannot have $\min \pi^\perp(b) > b = \min \pi(b)$, so we must have $\pi_i \rightarrow \pi^\perp(b)$. The same argument works for any part $\pi_j^\perp$ of $\pi^\perp$.

Now, every element $b \in B$ is minimum in $\pi(b)$, so there is a directed path that starts at $\pi(b)$ and can only end at $\pi(0)$ whose first edge is b . Furthermore, by the definition of the orientation, the labels of the edges decrease along this path. Thus in the alternating sum $x_b = w_b \pm \dots$ given by Lemma 2.3, the first term dominates, and $x_b \approx w_b$. Similarly, $y_c \approx w_c$ for all $c \in B^\perp$.

Therefore, if we write $B = \{b_1 > \dots > b_r > b_{r+1} = 0\}$, since w is super-increasing, it follows that $x_{b_1} > x_{b_2} > \dots > x_{b_r} > x_{b_{r+1}} = 0$, so indeed $(0, x) \in \sigma_{\mathcal{F}}$. Similarly, if we

write $B^\perp = E - B = \{c_1 > \cdots > c_{n-r} > c_{n-r+1} = 1\}$, then $y_{c_1} > y_{c_2} > \cdots > y_{c_{n-r+1}}$, so $y \in \sigma_{\mathcal{F}^\perp}$. The desired result follows. $\square$

Example 3.2. The graphical matroid M of the graph G in Figure 1 has six β -nbc bases: 0256, 0257, 0259, 0368, 0378, and 0379. Let us compute the intersection point in $\Sigma_{(M,0)} \cap (w - \Sigma_N)$ associated to 0257 for the super-increasing vector $w = (10^0, 10^1, \dots, 10^8) \in \mathbb{R}^9$.

For $B = 0257$, we have $B^\perp = 134689$. Then

$$\begin{aligned}\mathcal{F}_M(B) &= \{\emptyset \subsetneq 7 \subsetneq 57 \subsetneq 2457 \subsetneq 0123456789\} \\ \mathcal{F}_N(B^\perp) &= \{\emptyset \subsetneq 9 \subsetneq 89 \subsetneq 689 \subsetneq 46789 \subsetneq 346789 \subsetneq 123456789\},\end{aligned}$$

give rise to the corresponding set compositions

$$\pi = 7|5|24|013689, \quad \pi^\perp = 9|8|6|47|3|125.$$

This is indeed an arboreal pair, as evidenced by their intersection graph in Figure 3.

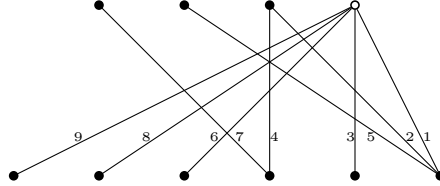


Figure 3: The intersection graph of $\pi = 7|5|24|013689$ and $\pi^\perp = 9|8|6|47|3|125$.

Lemma 3.1 gives us the unique points $(0, x) \in \mathcal{F}_\pi$ and $y \in \mathcal{F}_{\pi^\perp}$ such that $x + y = w$; they are given by the paths to the special vertex $\pi(0)$ in the intersection tree $\Gamma_{\pi, \pi^\perp}$. For example $x_7 = 10^6 - 10^3 + 10^1 - 10^0 = 999009$ and $y_4 = 10^3 - 10^1 + 10^0 = 991$ are given by the paths 7421 and 421 from $\pi(7) = \pi_1$ and $\pi^\perp(7) = \pi_4^\perp$ to $\pi(0)$, respectively. In this way we obtain:

$$\begin{array}{cccccccccc} x = & 0 & 9 & 0 & 9 & 9999 & 0 & 999009 & 0 & 0 \\ y = & 1 & 1 & 100 & 991 & 1 & 100000 & 991 & 10000000 & 100000000 \\ w = & 1 & 10 & 100 & 1000 & 10000 & 100000 & 1000000 & 10000000 & 100000000 \end{array}$$

and x is in the intersection $\Sigma_{(M,0)} \cap (w - \Sigma_N)$.

The following lemma indicates that any intersection point between $\Sigma_{(E,e)}$ and $v - \Sigma_N$ is an intersection between the cones described above. That is, cones corresponding to a β -nbc basis.

Lemma 3.3. Let M be a matroid on $E = [0, n]$ of rank $r + 1$, such that 0 is not a loop nor a coloop, and $N = (M/0)^\perp$. Let $w \in \mathbb{R}^n$ be generic and super-increasing. Let

$$\begin{aligned}\mathcal{F} &= \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_r \subsetneq F_{r+1} = E\} \\ \mathcal{G} &= \{\emptyset = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_{n-r-1} \subsetneq G_{n-r} = E - 0\}\end{aligned}$$

be complete flags of the matroids M and N , respectively, such that $\Sigma_{(M,0)}$ and $w - \Sigma_N$ intersect at $\sigma_{\mathcal{F}}$ and $w - \sigma_{\mathcal{G}}$. Then there exists a β -nbc basis B such that $\mathcal{F} = \mathcal{F}_M(B)$ and $\mathcal{G} = \mathcal{F}_N(B^\perp)$.

Proof. By Lemma 2.2, the set compositions π and τ of $\mathcal{F}$ and $\mathcal{G}$ form an arboreal pair. In particular, $\pi_a \cap \tau_b = (F_a - F_{a-1}) \cap (G_b - G_{b-1})$ cannot have more than one element for any a and b . We proceed in several steps.

1. Our first step will be to show that in the intersection tree $\Gamma_{\pi,\tau}$, the top right vertex π_{r+1} contains 0 and 1, the bottom right vertex τ_{n-r} contains 1, and thus the edge 1 connects these two rightmost vertices.

Each G_i is a flat of $N = M^\perp - 0$, so $G_i^\bullet := \text{cl}_{M^\perp}(G_i) \in \{G_i, G_i \cup 0\}$ is a flat of $M^\perp$. Consider the flag of flats of $M^\perp$

$$\mathcal{G}^\bullet := \{\emptyset = G_0^\bullet \subsetneq G_1^\bullet \subsetneq \cdots \subsetneq G_{n-r-1}^\bullet \subsetneq G_{n-r}^\bullet = E\},$$

where $G_{n-r}^\bullet = E$ because 0 is not a coloop of $M^\perp$ and $G_0^\bullet = \emptyset$ because 0 is not a loop of $M^\perp$. Let M be the minimal index such that $0 \in G_M^\bullet$, so

$$\mathcal{G}^\bullet := \{\emptyset = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_{M-1} \subsetneq G_M \cup 0 \subsetneq \cdots \subsetneq G_{n-r-1} \cup 0 \subsetneq G_{n-r} \cup 0 = E\},$$

Consider the unions of the flat F_r with the coflats in $\mathcal{G}^\bullet$; let j be the index such that

$$F_r \cup G_{j-1}^\bullet \neq E, \quad F_r \cup G_j^\bullet = E$$

Since it is the union of a flat and a coflat, the former cannot have size $|E| - 1$, so $(F_r \cup G_j^\bullet) - (F_r \cup G_{j-1}^\bullet) = (E - F_r) \cap (G_j^\bullet - G_{j-1}^\bullet)$ has size at least 2. But $\mathcal{F}$ and $\mathcal{G}$ are arboreal so $\pi_{r+1} \cap \tau_j = (E - F_r) \cap (G_j - G_{j-1})$ has size at most 1. This has two consequences:

- a) $G_j^\bullet = G_j \cup 0$ and $G_{j-1}^\bullet = G_{j-1}$, that is, $j = M$.
- b) $0 \in E - F_r = \pi_{r+1}$.

Similarly, consider the unions of the coflat $G_{n-r-1}^\bullet$ with the flats in $\mathcal{F}$; let i be the index such that

$$F_{i-1} \cup G_{n-r-1}^\bullet \neq E, \quad F_i \cup G_{n-r-1}^\bullet = E.$$

Analogously, we get that $(F_i - F_{i-1}) \cap (E - G_{n-r-1}^\bullet)$ has size at least 2, whereas $\pi_i \cap \tau_{n-r} = (F_i - F_{i-1}) \cap (E - 0 - G_{n-r-1})$ has size at most 1. This has three consequences:

- c) $G_{n-r-1}^\bullet = G_{n-r-1}$, that is, $M = n - r$.
- d) $0 \in F_i - F_{i-1}$, which in light of b) implies that $i = r + 1$.

e) $(F_i - F_{i-1}) \cap (E - 0 - G_{n-r-1}) = \pi_{r+1} \cap \tau_{n-r} = \{e\}$ for some element $e \in E - 0$. But $e \in \pi_{r+1}$ means that $x_e = 0$ is minimum among all x_i s for any $(0, x) \in \sigma_{\mathcal{F}}$, and $e \in \tau_{n-r}$ means that y_e is minimum among all y_i s for any $y \in \sigma_{\mathcal{G}}$. Since $w = x + y$ for some such x and y , $w_e = x_e + y_e$ is minimum among all w_i s, and since w is super-increasing, $e = 1$.

It follows that in the intersection tree $\Gamma_{\pi,\tau}$, the top right vertex π_{r+1} contains 0 and 1 by d) and e), the bottom right vertex τ_{n-r} contains 1 by e), and thus 1 connects them.

2. Next we claim that for any path in the tree $\Gamma_{\pi,\tau}$ that ends with the edge 1, the first edge has the largest label.² Assume contrariwise, and consider a containment-minimal path P that does not satisfy this property; its edges must be labelled $e < f > f_2 > \dots > f_k$ sequentially. If edge e goes from $\pi(e)$ to $\tau(e)$, [Lemma 2.3](#) gives $x_e = w_e - w_f \pm$ (terms smaller than w_f) $\approx -w_f < 0 = x_1$, contradicting that $(0, x) \in \sigma_{\mathcal{F}}$. If e goes from bottom to top, we get $y_e = w_e - w_f \pm$ (terms smaller than w_f) $\approx -w_f < w_1 = y_1$, contradicting that $y \in \sigma_{\mathcal{G}}$.

3. Now define

$$\begin{aligned} b_i &:= \min(F_i - F_{i-1}) \quad \text{for } i = 1, \dots, r+1, \\ c_j &:= \min(G_j - G_{j-1}) \quad \text{for } j = 1, \dots, n-r. \end{aligned}$$

Then $B := \{b_1, \dots, b_{r+1}\}$ and $C := \{c_1, \dots, c_{n-r}\}$ are bases of M and N , and $\mathcal{F} = \mathcal{F}_M(B)$ and $\mathcal{G} = \mathcal{F}_N(C)$. We claim that B is β -nbc and $C = B^\perp$.

To do so, we first notice that the path from vertex $\pi_i = F_i - F_{i-1}$ (resp. $\tau_j = G_j - G_{j-1}$) to edge 1 must start with edge b_i (resp. c_j): if it started with some larger edge $b' \in F_i - F_{i-1}$, then the path from edge b_i to edge 1 would not start with the largest edge. This has two consequences:

f) The sets B and C are disjoint. If we had $b_i = c_j = e$, then edge e , which connects vertices $\pi_i = F_i - F_{i-1}$ and $\tau_j = G_j - G_{j-1}$, would have to be the first edge in the paths from both of these vertices to edge 1; this is impossible in a tree. We conclude that B and C are disjoint. Since $|B| = r+1$ and $|C| = n-r$, we have $C = B^\perp$.

g) For each i we have $x_{b_i} \approx w_{b_i}$, because the path from τ_i to vertex 0 – which is the path from τ_i to edge 1, with edge 1 possibly removed – starts with the largest edge b_i , so [Lemma 2.3](#) gives $x_{b_i} = w_{b_i} \pm$ (smaller terms) $\approx w_{b_i}$. Similarly $y_{c_i} \approx w_{c_i}$. Now, $(0, x) \in \sigma_{\mathcal{F}}$ gives $x_{b_1} > \dots > x_{b_{r+1}}$, which implies $w_{b_1} > \dots > w_{b_{r+1}}$, which in turn gives

$$b_1 > \dots > b_r > b_{r+1}; \quad \text{and analogously,} \quad c_1 > \dots > c_{n-r-1} > c_{n-r} = 1.$$

The former implies that B is nbc in M by [Lemma 2.11](#). The latter, combined with c), implies that $c_1 > \dots > c_{n-r-1} > 0$ respectively are the minimum elements of the flats $G_1^\bullet, \dots, G_{n-r-1}^\bullet, G_{n-r}^\bullet = E$ that they sequentially generate, so $C \cup 0 \setminus 1 = B^\perp \cup 0 \setminus 1$ is nbc in $M^\perp$. It follows that B is β -nbc in M .

We conclude that B is β -nbc in M , $\mathcal{F} = \mathcal{F}_M(B)$, and $\mathcal{G} = \mathcal{F}_N(B^\perp)$, as desired. $\square$

Combinatorial proof of [Theorem 1.2](#). This follows from the previous two lemmas. $\square$

²It follows that the edge labels decrease along any such path, but we will not use this in the proof.

4 A proof via torus-equivariant geometry

We sketch a proof of [Theorem 1.3](#) using the framework of *tautological classes* of matroids of Berget, Eur, Spink, and Tseng. See [\[5\]](#) for details on what follows.

In this framework, one works with the permutohedral fan Σ_E , which is the Bergman fan of the Boolean matroid on E . Toric geometry equips Σ_E with two “cohomology” rings: The torus-equivariant Chow ring $A_T^\bullet(\Sigma_E)$, and the non-equivariant Chow ring $A^\bullet(\Sigma_E)$, which is a quotient of $A_T^\bullet(\Sigma_E)$ by an explicitly described ideal I .

First, one defines certain elements $[\Sigma_{(M,e)}]$ and $[-\Sigma_{(M/e)^\perp}]$ of $A^\bullet(\Sigma_E)$ by using the fact that the fans $\Sigma_{(M,e)}$ and $-\Sigma_{(M/e)^\perp}$ are subfans of Σ_E that satisfy a balancing condition in the sense of Minkowski weights [\[8\]](#). These elements have the following property: The ring $A^\bullet(\Sigma_E)$ is equipped with a degree map $\deg_{\Sigma_E} : A^\bullet(\Sigma_E) \rightarrow \mathbb{Z}$, which agrees with the map $\deg$ in [Theorem 1.3](#) in the sense that

$$\deg(\Sigma_{(M,e)} \cap -\Sigma_{(M/e)^\perp}) = \deg_{\Sigma_E}([\Sigma_{(M,e)}] \cdot [-\Sigma_{(M/e)^\perp}]).$$

For a survey of these facts, see [\[9, Section 4\]](#) or [\[6, Section 7.1\]](#).

Second, one notes that [\[6\]](#) provided a distinguished representative in $A_T^\bullet(\Sigma_E)$ of the class $[\Sigma_{(M,e)}] \in A^\bullet(\Sigma_E) = A_T^\bullet(\Sigma_E)/I$, and similarly for the class $[-\Sigma_{(M/e)^\perp}]$. In more detail, for a matroid M of rank $r + 1$ on a ground set E of size $n + 1$, [\[6, Definition 3.9\]](#) defined *tautological Chern classes* of M as certain elements $\{c_i^T(\mathcal{S}_M^\vee)\}_{i=0,\dots,r+1}$ and $\{c_j^T(\mathcal{Q}_M)\}_{j=0,\dots,n-r}$ in the equivariant Chow ring $A_T^\bullet(\Sigma_E)$, and established the following properties.

Lemma 4.1. [\[6, Theorem 7.6, Propositions 5.11, 5.13\]](#) *Let M be a matroid of rank $r + 1$ on a ground set E of size $n + 1$. Define elements $[\Sigma_{(M,e)}]^T$ and $[-\Sigma_{(M/e)^\perp}]^T$ in $A_T^\bullet(\Sigma_E)$ by $[\Sigma_{(M,e)}]^T = c_{n-r}^T(\mathcal{Q}_M)$ and $[-\Sigma_{(M/e)^\perp}]^T = c_r^T(\mathcal{S}_{M/e \oplus U_{0,e}}^\vee)$. Then, their images in the quotient $A^\bullet(\Sigma_E)$ are exactly $[\Sigma_{(M,e)}]$ and $[-\Sigma_{(M/e)^\perp}]$, respectively.*

Finally, to prove [Theorem 1.3](#), one begins with [\[6, Theorem 6.2\]](#) which states that

$$\deg_{\Sigma_E}([\Sigma_{(M,e)}] \cdot c_r(\mathcal{S}_M^\vee)) = \beta(M).$$

Thus, the desired statement $\deg_{\Sigma_E}([\Sigma_{(M,e)}] \cdot [-\Sigma_{(M/e)^\perp}]) = \beta(M)$ will follow once one shows that $[\Sigma_{(M,e)}] \cdot (c_r(\mathcal{S}_M^\vee) - [-\Sigma_{(M/e)^\perp}]) = 0$ in $A^\bullet(\Sigma_E)$. For this end, one considers the distinguished representative $[\Sigma_{(M,e)}]^T \cdot (c_r^T(\mathcal{S}_M^\vee) - [-\Sigma_{(M/e)^\perp}]^T)$ of this product in $A_T^\bullet(\Sigma_E)$. An explicit description of this representative straightforwardly displays that it belongs to the ideal I .

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