# ON THE SPECTRUM OF THE KRONIG–PENNEY MODEL IN A CONSTANT ELECTRIC FIELD

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ABSTRACT. We are interested in the nature of the spectrum of the one-dimensional Schrödinger operator

$$-\frac{d^2}{dx^2} - Fx + \sum_{n\in\mathbb{Z}} g_n\delta(x-n) \qquad \text{in } L^2(\mathbb{R})$$

with $F > 0$ and two different choices of the coupling constants $\{g_n\}_{n\in\mathbb{Z}}$. In the first model $g_n \equiv \lambda$ and we prove that if $F \in \pi^2\mathbb{Q}$ then the spectrum is $\mathbb{R}$ and is furthermore absolutely continuous away from an explicit discrete set of points. In the second model $g_n$ are independent random variables with mean zero and variance $\lambda^2$. Under certain assumptions on the distribution of these random variables we prove that almost surely the spectrum is $\mathbb{R}$ and it is dense pure point if $F < \lambda^2/2$ and purely singular continuous if $F > \lambda^2/2$.

## 1. Introduction and main results

In this paper we investigate spectral properties of Schrödinger operators defined through the formal differential expression

$$-\frac{d^2}{dx^2} - Fx + \lambda\sum_{n\in\mathbb{Z}} \delta(x-n) \tag{1}$$

where $F \geq 0$ and $\lambda \in \mathbb{R}$ (a precise definition is given in Section 2). In particular, we study the absence or presence of absolutely continuous, singular continuous, and pure point spectrum.

When $F = 0$ or $\lambda = 0$ the operators defined by (1) are completely understood. Indeed, in these cases the operator is the Laplace operator ($F = \lambda = 0$), the Stark operator ($F \neq 0, \lambda = 0$), or that of the Kronig–Penney model ($F = 0, \lambda \neq 0$). While the spectrum in these three cases differs rather drastically, it is always purely absolutely continuous. When neither of $F$ and $\lambda$ is zero, the structure of the spectrum is less clear. Starting with the work of Berezhkovskii and Ovchinnikov [7] the model (1) was frequently studied in the physics literature, as we will review below. Berezhkovskii and Ovchinnikov arrived at the conclusion that localization prevails (in their words, 'the width of the electron levels

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is exactly zero') independently of the parameters $F, \lambda \neq 0$. Later, Ao [1] suggested that the nature of the spectrum (or rather dynamical localization and delocalization properties) depend both on the size of $F/\lambda^2$ and on whether a certain resonance condition on $F$ is satisfied. In [1, Eq. (9)] the resonance condition is stated explicitly as an integer condition, but other parts of the paper rather suggest a rationality condition. Borysowicz [9] arrives at a similar size condition on $F/\lambda^2$ determining the nature of the spectrum, but not at Ao's resonance condition. Through more quantitative, but still non-rigorous arguments Buslaev [10] notices that a special role is played by a rationality condition which, when satisfied, would lead to absolutely continuous spectrum.

While we are still rather far from fully understanding the spectrum in the general case, in this paper we provide evidence for the validity of Buslaev's analysis and almost completely solve the problem under a rationality assumption. In particular, Ao's integer condition [1, Eq. (9)] needs to be replaced by a rationality condition and Borysowicz's predictions can at most be correct in the irrational case. We would also like to emphasize that, while the rationality assumption is required for our definitive result, the bulk of our analysis is valid without it and we expect this part to play an important role in any future advances in the irrational case.

Our main result on the spectral structure for this model is summarized in the following theorem.

**Theorem 1.1.** *Fix $F \in \pi^2\mathbb{Q}_+$, $\lambda \in \mathbb{R}$, and write $F = \frac{\pi^2 q}{3p}$ with $p, q \in \mathbb{N}$. Let $L_{F,\lambda}$ be the self-adjoint realization of* (1) *in $L^2(\mathbb{R})$, then*

$$\sigma_{ac}(L_{F,\lambda}) = \mathbb{R}\,, \quad \sigma_{sc}(L_{F,\lambda}) = \emptyset\,, \quad \textit{and} \quad \sigma_{pp}(L_{F,\lambda}) \subseteq \Big\{ \frac{\pi^2}{3p} m + \lambda : m \in \mathbb{Z} \Big\}\,.$$

*Remark* 1.2. A few remarks are in order:

(1) by a unit translation $L_{F,\lambda}$ is unitarily equivalent to $L_{F,\lambda} + F$ and, in particular, the spectrum of $L_{F,\lambda}$ is $F$ periodic. Therefore, the existence of an eigenvalue at $E = \frac{\pi^2}{3p} m + \lambda$ only depends on $m \mod q$. Consequently, to fully determine the spectrum it remains to understand $q$ additional values of $E$.
(2) the assumption $F \geq 0$ is purely for convenience. Indeed, by the change of variables $x \mapsto -x$ the operator $L_{F,\lambda}$ is unitarily equivalent to $L_{-F,\lambda}$.

We will discuss the strategy of the proof of Theorem 1.1 later in this introduction. There we will also comment on its relation to the works of Buslaev [10], whose predictions we prove rigorously, and Perelman [36], whose techniques we adapt to do so.

### 1.1. **A related random model.**

In addition to the operators $L_{F,\lambda}$ we shall also consider the following closely related random model, which was suggested in [43]. Our results refine those obtained by Delyon, Simon, and Souillard in [14, 15].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $g\colon \Omega \to \mathbb{R}^{\mathbb{Z}}$, $\omega \mapsto \{g_n(\omega)\}_{n\in\mathbb{Z}}$ a measurable function, and consider the random differential expression

$$-\frac{d^2}{dx^2} - Fx + \sum_{n\in\mathbb{Z}} g_n(\omega)\delta(x-n)\,. \tag{2}$$

Throughout we assume that the random sequence of coupling constants $\{g_n(\omega)\}_{n\in\mathbb{Z}}$ satisfies

(i) $g_n$ are independent for different $n \in \mathbb{Z}$,
(ii) $\mathbb{E}_\omega[g_n] = 0$,
(iii) $\mathbb{E}_\omega[g_n^2] = \lambda^2$, for some $\lambda \neq 0$ and all $n$,
(iv) $\sum_{n\geq 1} \mathbb{E}_\omega[g_n^4] n^{-2} < \infty$,
(v) $\mathbb{P}_\omega\big[\lim_{n\to\infty} n^{-1/4} g_n = 0\big] = \mathbb{P}_\omega\big[\lim_{n\to-\infty} |n|^{-1/2} g_n = 0\big] = 1$, and
(vi) there exists $n_0 \in \mathbb{Z}$ such that the distribution of $g_{n_0}$ is absolutely continuous with respect to Lebesgue measure.

We note that if $g_n$ are i.i.d. copies of a random variable $X$, the assumptions (iv)-(v) are satisfied if and only if $\mathbb{E}_\omega[X^4] < \infty$; for details, see Remark 4.6. In particular, the assumptions are valid with $g_n$ chosen as independent $\mathcal{N}(0, \lambda^2)$. If the $g_n$ are not identically distributed, assumptions (iv)-(v) are valid if $\mathbb{E}_\omega[|g_n|^\alpha]$ are uniformly bounded for some $\alpha > 4$ (see Remark 4.6).

**Theorem 1.3.** *Let $\{g_n(\omega)\}_{n\in\mathbb{Z}}$ satisfy assumptions (i)-(vi). Then* (2) *almost surely defines a self-adjoint operator $L^\omega_{F,\lambda}$ in $L^2(\mathbb{R})$. Moreover, $\sigma_{ess}(L^\omega_{F,\lambda}) = \mathbb{R}$ almost surely and the spectrum is almost surely purely*

- *singular continuous if $F > \lambda^2/2$,*
- *dense pure point if $F < \lambda^2/2$.*

As mentioned above, this theorem refines a result obtained in [14, 15]. What the authors of these papers prove is that if $F$ is sufficiently small (depending on $\lambda$) the spectrum is almost surely pure point, while if $F$ is sufficiently large (depending on $\lambda$) the spectrum is almost surely purely continuous. Our result refines this in two aspects; first, we obtain a precise transition point $F = \lambda^2/2$, which separates the two regimes, and secondly we classify the continuous spectrum above the transition point as being singular continuous. That the latter aspect was left open is explicitly mentioned in [14]. Analogous sharp spectral transitions have been shown to occur by Minami [34] in a model related to that studied here and by Kiselev, Last, and Simon [26] for certain Schrödinger operators in $l^2(\mathbb{Z})$ with decaying random potential. In particular, the techniques in the latter paper will be important for us.

We wish to point out that the transition from singular continuous to pure point spectrum in Theorem 1.3 might appear somewhat more drastic than it actually is. As we shall see, the spectral transition can be traced to the asymptotic behavior of generalized eigenfunctions. In what follows we prove that generalized eigenfunctions have power-like decay given by $x^{-1/4-\lambda^2/(8F)}$ as $x \to \infty$ (up to oscillations and small corrections). The nature of the spectrum is classified depending on whether these generalized eigenfunctions are square integrable or not. So while the spectral nature changes instantaneously when $F/\lambda^2$ crosses the point 1/2, the change at the level of behavior of eigenfunctions is continuous as a function of $F/\lambda^2$.

1.2. **Strategy of proof.** While our main interest is towards the spectral theoretic results in Theorems 1.1 and 1.3, the bulk of the paper will concern understanding the asymptotic behavior of solutions of the generalized eigenvalue equation

$$\begin{cases} -\psi''(x) - Fx\psi(x) = E\psi(x) & \text{for } x \in \mathbb{R}\setminus\mathbb{Z}\,, \\ \lim_{\varepsilon\to 0^+}(\psi(n+\varepsilon)-\psi(n-\varepsilon)) = 0 & \text{for } n\in\mathbb{Z}\,, \\ \lim_{\varepsilon\to 0^+}(\psi'(n+\varepsilon)-\psi'(n-\varepsilon)) = g_n\psi(n) & \text{for } n\in\mathbb{Z}\,, \end{cases} \tag{3}$$

where $g_n = \lambda$ and $g_n = g_n(\omega)$ in the first and second models, respectively. Passing from precise asymptotics of the solutions $\psi$ to spectral properties of $L_{F,\lambda}$ and $L^\omega_{F,\lambda}$ can be accomplished through arguments which are fairly standard in the field. Indeed, in the case of Theorem 1.1 it consists of an application of Gilbert–Pearson subordinacy theory [20, 21] while Theorem 1.3 follows from rank-one perturbation theory and, in particular, a spectral averaging argument (see [42] and references therein). Moreover, the refinements of Gilbert–Pearson theory developed by Jitomirskaya and Last [23, 24] and furthered by Damanik, Killip, and Lenz [13] imply that for $F > \frac{\lambda^2}{2}$ the spectral measure of $L^\omega_{F,\lambda}$ almost surely vanishes on sets of Hausdorff dimension less than $1-\frac{\lambda^2}{2F}$; see Proposition 4.16.

The key in both arguments will be to study whether there exist solutions of (3) which are subordinate at $\infty$ and whether these solutions are square integrable or not. We recall that a non-trivial solution $\psi$ of (3) is subordinate at $\infty$ if for any linearly independent solution $\varphi$ it holds that

$$\lim_{x\to\infty}\frac{\int_0^x|\psi(t)|^2\,dt}{\int_0^x|\varphi(t)|^2\,dt} = 0\,.$$

What we shall eventually prove for the ODE in the first and second models differs significantly; in the deterministic case subordinate solutions do not exist while the converse is almost surely true in the random setting. Nonetheless, the overall strategy we employ to prove these ODE results is the same. However, the analysis required to understand the deterministic model is considerably more intricate. In both cases the key part of the proof is to understand the nature of certain exponential sums with amplitude given in terms of the coupling constants. In the random case, the assumption that $\mathbb{E}_\omega[g_n]=0$ leads to large cancellations in these sums with high probability. In the deterministic model, no such probabilistic cancellations take place and it becomes necessary to understand to a precise degree how cancellations arise from the oscillatory nature of the summand.

1.3. **Reduction to a discrete system.** We wish to emphasize that it is only in the very last step of our proof of Theorem 1.1 that the rationality assumption $F\in\pi^2\mathbb{Q}_+$ enters. In fact, what we believe to be our main contribution to the analysis of the operators $L_{F,\lambda}$ is not that stated in the theorem, but rather a reduction of the understanding of their spectral properties to the understanding of a certain discrete system. The discrete system we arrive at resembles those studied in the theory of orthogonal polynomials on the unit circle (OPUC), for which we refer the reader to [40, 41]. (As an aside we mention that the

$F$ periodicity of the spectrum of $L_{F,\lambda}$ mentioned in Remark 1.2 is reflected in the fact that the spectrum of CMV matrices underlying OPUC is a subset of the unit circle.)

The idea of a reduction to a discrete system goes back to Buslaev [10] and the discrete system that we arrive at is essentially equivalent to his. Buslaev, however, is very clear that his arguments are not rigorous (see, in particular, the last paragraph in Section 1 in [10]). Our approach to rigorously providing this reduction follows closely that of Perelman in [36] who performed a similar analysis for less singular periodic perturbations of the Stark operator; see also [38]. While our overall strategy is essentially the same as Perelman's, there are significant technical differences and, in particular, subtle exponential sum estimates play a much more pronounced role in our paper. In this regard we also refer to the discussion at the beginning of Section 5.

While it should be emphasized that neither the Stark nor the Kronig–Penney part of the potential is in any proper sense weaker than the other, the essence of our analysis is to relate the solutions of (3) to the corresponding solutions of the Stark equation. Note that this is somewhat different from Ao's tilted band picture, which rather emphasizes the solutions of the Kronig–Penney equation. A distinguished role in this analysis will be played by a particular solution of

$$-\zeta''(x) - Fx\zeta(x) = E\zeta(x)\,, \quad \text{for } x \in \mathbb{R}\,, \tag{4}$$

namely

$$\zeta(x) = \Big(\frac{\pi}{F^{1/3}}\Big)^{1/2}\big(i\mathrm{Ai}(-F^{1/3}(x+E/F)) + \mathrm{Bi}(-F^{1/3}(x+E/F))\big)\,, \tag{5}$$

where Ai, Bi denote the standard Airy functions [16, §9].

**Theorem 1.4.** *Fix $F > 0, E \in \mathbb{R}$, and $\lambda \in \mathbb{R}$. Let $\psi$ be a real-valued solution of (3) then there are functions $\mathcal{R}\colon \mathbb{N} \to \mathbb{R}_+$ and $\Lambda\colon \mathbb{N} \to \mathbb{R}$ such that, for $x \in \big(\frac{\pi^2}{F}(l-\frac12)^2, \frac{\pi^2}{F}(l+\frac12)^2\big]$,*

$$\psi(x) = \mathcal{R}(l)\frac{e^{i(\Lambda(l)-\lambda\sqrt{\lceil x\rceil/F})}\zeta(x) - e^{-i(\Lambda(l)-\lambda\sqrt{\lceil x\rceil/F})}\bar{\zeta}(x)}{2i} + O\big(|\zeta(x)|\mathcal{R}(l)l^{-1/2}\big)\,, \tag{6}$$

*and*

$$\int_{\frac{\pi^2}{F}(l-\frac12)^2}^{\frac{\pi^2}{F}(l+\frac12)^2} |\psi(x)|^2\,dx = \frac{\pi}{F}\,\mathcal{R}(l)^2\Big(1 + O(l^{-1/2})\Big)\,. \tag{7}$$

*Moreover, the functions $\mathcal{R}$ and $\Lambda$ satisfy*

$$\begin{aligned} \log\Big(\frac{\mathcal{R}(l+1)}{\mathcal{R}(l)}\Big) &= \frac{\lambda}{\sqrt{2Fl}}\sin(2\Theta(l)) + \frac{\lambda^2}{4Fl}\big(1+\cos(4\Theta(l))\big) + O(l^{-5/4})\,, \\ \Lambda(l+1) - \Lambda(l) &= \frac{\lambda}{\sqrt{2Fl}}\cos(2\Theta(l)) + \lambda^2\mathcal{S}(l) + O(l^{-5/4}) \end{aligned} \tag{8}$$

*where*

$$\Theta(l) = \Lambda(l) + \Gamma(l)\,, \qquad \Gamma(l) = -\frac{\pi^3 l^3}{3F} + \frac{\pi l}{F}(E-\lambda) + \frac{5\pi}{8}\,,$$

*and $\mathcal{S}$ is independent of $\mathcal{R}, \Lambda$, and satisfies*

$$\mathcal{S}(l) = O(l^{-3/4}) .$$

*Remark* 1.5. A couple of remarks:

(1) As we shall see below, $|\zeta(x)| = |Fx|^{-1/4}(1 + o(1))$, so the error term in (6) can be written as $O(\mathcal{R}(l)/l)$.
(2) Since $\psi, \zeta$, and $\bar{\zeta}$ all solve the ODE in (4) between consecutive integers, the same is true for both the main term and the remainder in (6).
(3) While the approximate equation for $\mathcal{R}$ does not fully determine $\mathcal{R}$, it is sufficient to determine the asymptotic behavior of $\mathcal{R}$ up to a constant factor. Similarly, the equation for $\Lambda$ suffices to determine its asymptotic behavior up to a bounded additive error.
(4) In the proof of Theorem 1.4 we will provide an explicit expression for $\mathcal{S}$. However, this expression is somewhat complicated and not particularly illuminating, so we refrain from writing it out here. Furthermore, one can probably show that $\mathcal{S}$ is substantially smaller than stated (in fact, $O(l^{-1-\varepsilon})$ for some $\varepsilon > 0$), but the rough bound that we state is sufficient for our purposes.

1.4. **Historical remarks & related work.** While a complete mathematical treatment of the spectral properties of the operator (1) is still missing, both this model and random variants thereof was a topic of some discussion in theoretical solid state physics in the 80's [7, 43, 6, 31, 30, 29, 32, 19, 8, 1, 9] (see also [25] for a review of mathematical results on a different, but related model). We note that the Hamiltonian (1) arises in two different physical models, namely, in that of a crystal in an electric field and in that of a conducting ring threaded by a magnetic flux which increases linearly in time.

If the periodic $\delta$-array in the definition of the operator is replaced by a smoother periodic potential the mathematical literature is more extensive. In particular, it has been shown that under appropriate assumptions the absolutely continuous spectrum is all of $\mathbb{R}$, see for instance [46, 4, 5, 37, 38, 35, 12, 39]. In the opposite direction, it has been shown that if the $\delta$-array is replaced by the even more singular potential given by a $\delta'$-array, the absolutely continuous spectrum is empty [3, 33, 17, 2]. As such, the case that we consider here is in some sense critical and the spectral properties of (1) could very well depend in a very subtle manner on the parameters $F$ and $\lambda$.

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## 2. Preliminaries & Notation

Throughout the paper we shall frequently make use the following standard asymptotics notation $\lesssim, \gtrsim, \sim, o$, and $O$. When we use this notation the dependence of the implicit constants on the parameters of the problem varies from time to time. In particularly, certain implicit constants in our analysis of the random model are allowed to depend on the realization $\omega$ and may only be finite with probability one.

We write $\mathbb{R}_+, \mathbb{R}_-$ for $(0,\infty)$ and $(-\infty,0)$, respectively. Similarly we let $\mathbb{Q}_\pm = \mathbb{Q}\cap\mathbb{R}_\pm$.

For a function $u \in L^2(\mathbb{R})$ such that $u, u'$ are locally absolutely continuous in $\mathbb{R}\setminus\mathbb{Z}$ we define the jumps of $u$ and $u'$ at $n$ by

$$\begin{aligned} Ju(n) &= \lim_{\varepsilon\to 0^+}\left(u(n+\varepsilon)-u(n-\varepsilon)\right),\\ Ju'(n) &= \lim_{\varepsilon\to 0^+}\left(u'(n+\varepsilon)-u'(n-\varepsilon)\right), \end{aligned}$$

whenever these limits exist.

To unify our notation we shall for $F \geq 0$ and $g = \{g_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ write $L_{F,g}$ for the Schrödinger operator in $L^2(\mathbb{R})$ defined by the differential expression

$$-\frac{d^2}{dx^2} - Fx$$

with domain

$$\begin{aligned} D(L_{F,g}) = \{u \in L^2(\mathbb{R}) : u, u' \text{ locally abs. cont. in } \mathbb{R}\setminus\mathbb{Z},\ -u''-Fxu \in L^2(\mathbb{R}\setminus\mathbb{Z})\\ Ju(n)=0,\ Ju'(n) = g_n u(n)\}\,. \end{aligned}$$

That is, in the deterministic model $g$ is the constant sequence, $g_n \equiv \lambda$, while in the random case $g = \{g_n(\omega)\}_{n\in\mathbb{Z}}$. Analogously we can define corresponding Schrödinger operators in $L^2(I)$ for any $I \subseteq \mathbb{R}$.

The following theorem ensures that at least under certain assumptions on the coupling constants the Schrödinger operator $L_{F,g}$ is well-defined [11]. In particular, we emphasize that these assumptions are valid for the operators considered here (almost surely in the setting of Theorem 1.3).

**Lemma 2.1.** *Let $L_{F,g}$ be the Schrödinger operator defined as above. Then provided there exists constants $C_1, C_2$ such that $g_n \geq -C_1|n| - C_2$ the operator $L_{F,g}$ is self-adjoint and the differential equation* (3) *is limit point at $\pm\infty$.*

As noted above, the assumption $F > 0$ is only for definiteness. The reason for this assumption is made clear by the following lemma, which essentially tells us all that we need about the eigenvalue equation corresponding to our operators on the negative half-line.

**Lemma 2.2.** *Let $F > 0$ and $\{g_n\}_{n<0}$ satisfy $\liminf_{n<0}(|n|^{-1/2}g_n) > -\sqrt{F}$. For any $E \in \mathbb{R}$ there exists a non-trivial $\psi \in L^2(\mathbb{R}_-)$ which solves*

$$\begin{cases} -\psi''(x) - Fx\psi(x) = E\psi(x) & \text{in } \mathbb{R}_-\setminus\mathbb{Z}\,,\\ J\psi(n) = 0 & \text{for } n \in \mathbb{Z}\cap\mathbb{R}_-\,,\\ J\psi'(n) = g_n\psi(n) & \text{for } n\in\mathbb{Z}\cap\mathbb{R}_-\,. \end{cases}$$

*Proof.* Let $L_\theta$ be the corresponding operator in $L^2(\mathbb{R}_-)$ defined by the boundary conditions

$$\psi(0)\sin(\theta) + \psi'(0)\cos(\theta) = 0\,.$$

Let also $Q_\theta$ denote the corresponding quadratic form. For $\theta \neq 0$ $Q_\theta$ is defined by taking the closure of the quadratic form

$$L^2 \cap C^\infty(\mathbb{R}_-) \ni \psi \mapsto \|\psi'\|^2_{L^2(\mathbb{R}_-)} + F\|\sqrt{|x|}\psi\|^2_{L^2(\mathbb{R}_-)} + \sum_{n<0} g_n|\psi(n)|^2 + \cot(\theta)|\psi(0)|^2 \,,$$

while for $\theta = 0$ it is the closure of

$$C_0^\infty(\mathbb{R}_+) \ni \psi \mapsto \|\psi'\|^2_{L^2(\mathbb{R}_+)} + F\|\sqrt{x}\psi\|^2_{L^2(\mathbb{R}_+)} + \sum_{n<0} g_n|\psi(n)|^2 \,.$$

By the Sobolev inequality $\psi$ in the domain of $Q_\theta$ is continuous and for any $x \in \mathbb{R}_-, \varepsilon > 0$, and bounded interval $I \subset \mathbb{R}_-$ containing $x$,

$$|u(x)|^2 \leq \varepsilon\|u'\|^2_{L^2(I)} + (|I|^{-1} + \varepsilon^{-1})\|u\|^2_{L^2(I)} \,. \tag{9}$$

To avoid cumbersome notation, the term $\cot(\theta)|\psi(0)|^2$ should in what follows be interpreted as zero in the case $\theta = 0$. By assumption there exist $\delta \in (0,1)$ and $N$ so that $g_n|n|^{-1/2} \geq -\sqrt{F}(1-\delta)$ for $n < -N$. Using (9), with $I = (-n-1, -n]$ and $\varepsilon = (\sqrt{F}n^{1/2})^{-1}$ to bound $|\psi(-n)|^2$, we find for $\psi$ in the quadratic form domain

$$\begin{aligned}
Q_\theta(\psi) \geq\ & \|\psi'\|^2_{L^2(\mathbb{R}_-)} + F\|\sqrt{|x|}\psi\|^2_{L^2(\mathbb{R}_-)} + \sum_{n=1}^N g_{-n}|\psi(-n)|^2 + \cot(\theta)|\psi(0)|^2 \\
& - \sum_{n>N} \sqrt{F}(1-\delta)n^{1/2}|\psi(-n)|^2 \\
\geq\ & \delta\|\psi'\|^2_{L^2(\mathbb{R}_-)} + \sum_{n=1}^N g_{-n}|\psi(-n)|^2 + \cot(\theta)|\psi(0)|^2 \\
& + \int_{\mathbb{R}_-} |\psi(x)|^2 \Big[F|x| - \sum_{n>N} \mathbb{1}_{(-n-1,-n]}(x)\sqrt{F}(1-\delta)n^{1/2}(1+\sqrt{F}n^{1/2})\Big]\,dx \,.
\end{aligned}$$

Since $\delta > 0$, the 'effective potential'

$$F|x| - \sum_{n>N} \mathbb{1}_{(-n-1,-n]}(x)\sqrt{F}(1-\delta)n^{1/2}(1+\sqrt{F}n^{1/2}) \,.$$

tends to infinity at $-\infty$. Consequently, the domain of $Q_\theta$ is compactly embedded in $L^2(\mathbb{R}_-)$ and the spectrum of $L_\theta$ is discrete. Consequently at all $E$ which are in the spectrum of $L_\theta$ for some $\theta$ we have an $L^2$-solution of the eigenvalue equation. It remains to prove that $\cup_{\theta\in[0,\pi)}\sigma(L_\theta) = \mathbb{R}$.

Let $\lambda_k(\theta)$ denote the $k$-th smallest eigenvalue of $L_\theta$. By the variational principle, for each $k$ the function $\theta \mapsto \lambda_k(\theta)$ is continuous and strictly decreasing. Thus the image of $[0,\pi)$ under the map $\lambda_k$ is a half-open interval, and these intervals are disjoint for different $k$. To prove that the eigenvalues cover all of $\mathbb{R}$ it remains to prove that there are no gaps between these intervals.

We first note that $\lambda_1(\theta) \to -\infty$ as $\theta \to \pi$. Indeed, this follows by taking as a test function in the Rayleigh quotient any smooth and $L^2$-normalized function which vanishes outside $(-1/2,0]$ and is non-zero at 0.

By the continuity of $\theta \mapsto \lambda_k(\theta)$ it remains to show that $\lambda_k(0) = \lim_{\varepsilon\to 0^+} \lambda_{k+1}(\pi-\varepsilon)$ for all $k \geq 1$. Assume for contradiction that this is not the case, and let $k_0$ be the smallest $k$ for which the statement fails. We aim to construct an $L^2$-normalized function $\tilde{\phi}$ in the form domain of $L_0$ that is orthogonal to the first $k_0$ eigenfunctions of $L_0$ and satisfies $Q_0(\tilde{\phi}) < \lambda_{k_0+1}(0)$. By the variational characterisation of the eigenvalues this is a contradiction.

Let $\phi_{k,\theta}$ be an $L^2$-normalized eigenfunction of $L_\theta$ corresponding to $\lambda_k(\theta)$. We abbreviate $\phi_j = \phi_{j,0}$ and $\psi_\varepsilon = \phi_{k_0+1,\pi-\varepsilon}$. Fix $\eta \in C^\infty(\mathbb{R}_-)$ with $0 \leq \eta \leq 1$ with $\eta(x) = 0$ for $x \geq -1$ and $\eta(x) = 1$ for $x \leq -2$. Let $\psi_{\varepsilon,\delta}(x) = \eta(x/\delta)\psi_\varepsilon(x)$. By an integration by parts using that $\psi_\varepsilon, \phi_j$ solve the eigenvalue equations we find

$$\begin{aligned}\int_{\mathbb{R}_-} \phi_j(x)\psi_{\varepsilon,\delta}(x)\,dx &= \frac{\lambda_{k_0+1}(\pi-\varepsilon)}{\lambda_j(0)} \int_{\mathbb{R}_-} \phi_j(x)\psi_{\varepsilon,\delta}(x)\,dx \\ &\quad - \frac{1}{\lambda_j(0)} \int_{-2\delta}^{-\delta} \left(2\delta^{-1}\psi_\varepsilon'(x)\eta'(x/\delta) + \delta^{-2}\psi_\varepsilon(x)\eta''(x/\delta)\right)\phi_j(x)\,dx\,.\end{aligned} \tag{10}$$

For the second integral on the right side Cauchy–Schwarz and $\eta \in C^\infty$ implies

$$\begin{aligned}\delta^{-1}\left|\int_{-2\delta}^{-\delta} \psi_\varepsilon'(x)\eta'(x/\delta)\phi_j(x)\,dx\right| &\lesssim \delta^{-1}\left(\int_{-2\delta}^{-\delta} |\psi_\varepsilon'(x)|^2\,dx\right)^{1/2}\left(\int_{-2\delta}^{-\delta} |\phi_j(x)|^2\,dx\right)^{1/2}, \\ \delta^{-2}\left|\int_{-2\delta}^{-\delta} \psi_\varepsilon(x)\eta''(x/\delta)\phi_j(x)\,dx\right| &\lesssim \delta^{-2}\left(\int_{-2\delta}^{-\delta} |\psi_\varepsilon(x)|^2\,dx\right)^{1/2}\left(\int_{-2\delta}^{-\delta} |\phi_j(x)|^2\,dx\right)^{1/2}.\end{aligned} \tag{11}$$

Since $\phi_j \in H_0^1(\mathbb{R}_-)$ we can bound

$$\begin{aligned}\int_{-2\delta}^{-\delta} |\phi_j(x)|^2\,dx &= \int_{-2\delta}^{-\delta}\left|\int_x^0 \phi_j'(t)\,dt\right|^2 dx \\ &\leq \int_{-2\delta}^{-\delta}\left(\int_x^0 1\,dt\right)\left(\int_x^0 |\phi_j'(t)|^2\,dt\right)dx \\ &\leq \frac{3}{2}\,\delta^2 \int_{-2\delta}^0 |\phi_j'(x)|^2\,dx\,.\end{aligned}$$

Since $\phi_j' \in L^2(\mathbb{R}_-)$, the second integral on both right sides of (11) tends to zero. This together with the fact that $\psi_\varepsilon'$ is bounded in $L^2(\mathbb{R}_-)$ uniformly in $\varepsilon$, as is easily checked, implies that the left side in the first equation in (11) tends to zero as $\delta \to 0$, uniformly in $j = 1, \ldots, k_0$ and $\varepsilon > 0$.

For the second equation in (11) the same argument as above yields

$$\begin{aligned}\int_{-2\delta}^{-\delta} |\psi_\varepsilon(x)|^2\,dx &= \int_{-2\delta}^{-\delta}\left|\psi_\varepsilon(0) - \int_x^0 \psi_\varepsilon'(t)\,dt\right|^2 dx\\ &\lesssim \delta|\psi_\varepsilon(0)|^2 + \int_{-2\delta}^{-\delta}\left(\int_x^0 1\,dt\right)\left(\int_x^0 |\psi_\varepsilon'(t)|^2\,dt\right)dx\\ &\leq \delta|\psi_\varepsilon(0)|^2 + \frac{3}{2}\,\delta^2\int_{-2\delta}^0 |\psi_\varepsilon'(x)|^2\,dx\,.\end{aligned}$$

Since $\lambda_j(\theta)$ are ordered and bounded for all $\theta \in [0,\pi)$, we conclude that $\phi_{j,\theta}(0) \to 0$ as $\theta \to \pi$ for all $j \geq 2$. In particular, for given $\varepsilon > 0$ we can choose $\delta = |\psi_\varepsilon(0)|^2$. Again by the boundedness of $\psi_\varepsilon'$ in $L^2(\mathbb{R}_-)$ we conclude that the left side in the second equation in (11) tends to zero as $\varepsilon \to 0$, uniformly in $j = 1,\dots,k_0$ and with $\delta = |\psi_\varepsilon(0)|$.

Since by assumption $\lim_{\varepsilon\to 0}\lambda_{k_0+1}(\pi-\varepsilon) > \lambda_{k_0}(0)$, we therefore deduce from (10) that the functions $\phi_j$ and $\psi_{\varepsilon,\delta}$ are almost orthogonal. By choosing $\varepsilon$ small enough, the function

$$\tilde\phi = \frac{\psi_{\varepsilon,\delta} - \sum_{j=1}^{k_0}\langle\psi_{\varepsilon,\delta},\phi_j\rangle\phi_j}{\|\psi_{\varepsilon,\delta} - \sum_{j=1}^{k_0}\langle\psi_{\varepsilon,\delta},\phi_j\rangle\phi_j\|_{L^2}}\,,$$

is well-defined, $L^2$-normalized, orthogonal to $\phi_j$, $j = 1,\dots,k_0$ and in the form domain of $L_0$. The estimate $Q_0(\tilde\phi) < \lambda_{k_0+1}(0)$ follows by the triangle inequality, $\lambda_{k_0+1}(\pi-\varepsilon) < \lambda_{k_0+1}(0)$, and choosing $\varepsilon$ appropriately small. We have arrived at the desired contradiction, thus completing the proof of Lemma 2.2. □

## 3. Relative Prüfer variables

**3.1. Derivation of the Prüfer equations.** As mentioned in the previous subsection our main results will follow from standard techniques once we understand the asymptotic behavior of solutions of the ODE

$$\begin{cases}-\psi''(x) - Fx\psi(x) = E\psi(x) & \text{for } x \in \mathbb{R}\setminus\mathbb{Z}\,,\\ J\psi(n) = 0 & \text{for } n \in \mathbb{Z}\,,\\ J\psi'(n) = g_n\psi(n) & \text{for } n\in\mathbb{Z}\,.\end{cases} \tag{12}$$

In what we introduce certain modified Prüfer coordinates for $\psi$ solving (12). This modified Prüfer set-up has been used in several instances before. In particular, we follow the notation used by Kiselev, Remling, and Simon [27] where they set up such coordinates in a general framework and successfully use them to derive a variety of spectral theoretic results. While our operators are far from those considered in [27], the algebra involve in setting up the coordinates and deriving the associated equations is unchanged. It is perhaps worth noting that the exact same change of coordinates is a key step in Minami's analysis [34].

The Prüfer coordinates are defined relative to a reference solution of an unperturbed ODE, in our case the Stark equation (4). The reference solution $\zeta$ should be chosen such that the Wronskian

$$\{\zeta,\bar\zeta\}(x) = \zeta(x)\bar\zeta'(x) - \zeta'(x)\bar\zeta(x) = \zeta(0)\bar\zeta'(0) - \zeta'(0)\bar\zeta(0) \neq 0\,. \tag{13}$$

Here we shall to choose our reference solution to be that defined in (5). Since Ai and Bi are linearly independent, our choice of reference solution satisfies (13).

By the assumption that $\{\zeta,\bar\zeta\}\neq 0$ any $\psi$ solving (4) on an interval $I\subset\mathbb{R}$ can be written as $\psi(x)=\alpha\zeta(x)+\beta\bar\zeta(x)$, $x\in I$, for uniquely determined constants $\alpha,\beta\in\mathbb{C}$. In particular, this applies to any solution of (12) restricted to an interval of the form $(n-1,n)$, with $n\in\mathbb{Z}$. Given a generalized eigenfunction we wish to understand the change of the coefficients $\alpha,\beta$ when going from the interval $(n-1,n)$ to $(n,n+1)$.

Fix $\psi$ solving (12). Define $\alpha(n),\beta(n)\in\mathbb{C}$ so that for all $x\in(n-1,n)$

$$\begin{cases}\psi(x)=\alpha(n)\zeta(x)+\beta(n)\bar\zeta(x)\\ \psi'(x)=\alpha(n)\zeta'(x)+\beta(n)\bar\zeta'(x)\,.\end{cases}$$

By the continuity of $\psi$ and $\zeta$ we have, for any $n\in\mathbb{Z}$,

$$\alpha(n)\zeta(n)+\beta(n)\bar\zeta(n)=\alpha(n+1)\zeta(n)+\beta(n+1)\bar\zeta(n)\,. \tag{14}$$

Using the fact that $\zeta\in C^\infty(\mathbb{R})$ the jump condition in (12) can be written equivalently as

$$g_n\big[\alpha(n)\zeta(n)+\beta(n)\bar\zeta(n)\big]=\alpha(n+1)\zeta'(n)+\beta(n+1)\bar\zeta'(n)-\alpha(n)\zeta'(n)-\beta(n)\bar\zeta'(n)\,. \tag{15}$$

Writing equations (14) and (15) in matrix form we have proved

$$\begin{pmatrix}\zeta(n) & \bar\zeta(n)\\ g_n\zeta(n)+\zeta'(n) & g_n\bar\zeta(n)+\bar\zeta'(n)\end{pmatrix}\begin{pmatrix}\alpha(n)\\ \beta(n)\end{pmatrix}=\begin{pmatrix}\zeta(n) & \bar\zeta(n)\\ \zeta'(n) & \bar\zeta'(n)\end{pmatrix}\begin{pmatrix}\alpha(n+1)\\ \beta(n+1)\end{pmatrix}.$$

Since $\det\left(\begin{smallmatrix}\zeta & \bar\zeta\\ \zeta' & \bar\zeta'\end{smallmatrix}\right)=\{\zeta,\bar\zeta\}\neq 0$ we thus have the recursion

$$\begin{pmatrix}\zeta(n) & \bar\zeta(n)\\ \zeta'(n) & \bar\zeta'(n)\end{pmatrix}^{-1}\begin{pmatrix}\zeta(n) & \bar\zeta(n)\\ g_n\zeta(n)+\zeta'(n) & g_n\bar\zeta(n)+\bar\zeta'(n)\end{pmatrix}\begin{pmatrix}\alpha(n)\\ \beta(n)\end{pmatrix}=\begin{pmatrix}\alpha(n+1)\\ \beta(n+1)\end{pmatrix}. \tag{16}$$

A direct computation yields

$$\begin{pmatrix}\zeta(n) & \bar\zeta(n)\\ \zeta'(n) & \bar\zeta'(n)\end{pmatrix}^{-1}\begin{pmatrix}\zeta(n) & \bar\zeta(n)\\ g_n\zeta(n)+\zeta'(n) & g_n\bar\zeta(n)+\bar\zeta'(n)\end{pmatrix}=\mathbb{1}+\frac{g_n}{\{\zeta,\bar\zeta\}}\begin{pmatrix}-|\zeta(n)|^2 & -\bar\zeta(n)^2\\ \zeta(n)^2 & |\zeta(n)|^2\end{pmatrix}.$$

As we shall see in the next subsection we can write $\zeta(x)=|\zeta(x)|e^{i\gamma(x)}$ for a smooth increasing function $\gamma\colon\mathbb{R}\to\mathbb{R}$. In particular, by Lemma 3.3 below we can write the one-step transfer matrices as

$$A_n=\mathbb{1}+\frac{g_n}{\{\zeta,\bar\zeta\}}\begin{pmatrix}-|\zeta(n)|^2 & -\bar\zeta(n)^2\\ \zeta(n)^2 & |\zeta(n)|^2\end{pmatrix}=\mathbb{1}+\frac{g_n}{2i\gamma'(n)}\begin{pmatrix}1 & e^{-2i\gamma(n)}\\ -e^{2i\gamma(n)} & -1\end{pmatrix}.$$

We note that the transfer matrices $A_n$ are elements of the group $\mathbb{SU}(1,1)$, i.e. for each $n$

$$A_n^*\sigma_3A_n=\sigma_3\quad\text{and}\quad\det(A_n)=1\,,$$

where $\sigma_3=\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ is the third Pauli matrix.

Since the coefficients in the ODE are all real it is sufficient to study real-valued solutions $\psi$ of (12). For such $\psi$ the vectors $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)\in\mathbb{C}^2$ in the above representation are of the form $\beta=\bar\alpha$. *From here on we restrict ourselves to studying real-valued generalized eigenfunctions.*

Following [27] we represent and study our real-valued solution $\psi$ in terms of the complex Prüfer coordinate

$$\rho(n) = 2i\alpha(n)$$

and the real-valued Prüfer radius $R\colon \mathbb{N} \to (0,\infty)$ and angle $\eta\colon \mathbb{N} \to \mathbb{R}$ defined by

$$\rho(n) = R(n)e^{i\eta(n)}\,,$$

with $\eta(1) \in (-\pi, \pi]$ and $\eta(n+1) - \eta(n) \in (-\pi, \pi]$. For notational convenience we also define $\theta\colon \mathbb{N} \to \mathbb{R}$ by

$$\theta(n) = \eta(n) + \gamma(n)\,.$$

To simplify notation we extend the functions $\rho, R, \eta, \theta$ to $\mathbb{R}$ as left continuous step functions by setting $\rho(x) = \rho(\lceil x \rceil)$ and similarly for $R, \eta, \theta$. With these definitions

$$\psi(x) = \frac{1}{2i}\Big(\rho(x)\zeta(x) - \bar\rho(x)\bar\zeta(x)\Big) \quad \text{for all } x \in (0,\infty).$$

Our understanding of the asymptotic behavior of $\psi$ will be based on studying the recursion equations satisfied by the Prüfer coordinates.

**Lemma 3.1** ([27, Theorem 3.3])**.** *Set*

$$U(n) = \frac{g_n}{\gamma'(n)}\,. \tag{17}$$

*Then*

$$\rho(n+1) - \rho(n) = U(n)\rho(n)\sin(\theta(n))e^{-i\theta(n)} \tag{18}$$

$$R(n+1)^2 = R(n)^2\big[1 + U(n)\sin(2\theta(n)) + U(n)^2\sin(\theta(n))^2\big] \tag{19}$$

$$\cot(\eta(n+1) + \gamma(n)) = \cot(\eta(n) + \gamma(n)) + U(n)\,. \tag{20}$$

*If* $|U(n)| \lesssim 1$ *then*

$$\begin{aligned}\log\Big(\frac{R(n+1)}{R(n)}\Big) &= \frac{U(n)}{2}\sin(2\theta(n)) + \frac{U(n)^2}{8}\\ &\quad - \frac{U(n)^2}{8}\Big(2\cos(2\theta(n)) - \cos(4\theta(n))\Big) + O(|U(n)|^3)\end{aligned} \tag{21}$$

*and*

$$\begin{aligned}\eta(n+1) - \eta(n) &= -\frac{U(n)}{2} + \frac{U(n)}{2}\cos(2\theta(n))\\ &\quad + \frac{U(n)^2}{4}\Big(\sin(2\theta(n)) - \frac{1}{2}\sin(4\theta(n))\Big) + O(|U(n)|^3)\,.\end{aligned} \tag{22}$$

*Moreover, if* $|U(n)| \le 1$ *then*

$$|\eta(n+1) - \eta(n)| \le \frac{\pi}{2}|U(n)|\,. \tag{23}$$

*Remark* 3.2. The assumption $|U(n)| \lesssim 1$ is true uniformly for $n \geq 1$ in the case of our deterministic model since $|\gamma'(x)| \sim |x|^{1/2}$ as $x \to \infty$ (see Lemma 3.5). Similarly, in the random model by assumption there exists almost surely $C_\omega < \infty$ such that $|U(n)| = \frac{|g_n(\omega)|}{\gamma'(n)} \leq \frac{C_\omega}{n^{1/4}}$ for all $n \geq 1$.

*Proof of Lemma 3.1.* Equation (18) follows from equation (16) together with the identities in Lemma 3.3, and the fact that $\rho(n) = 2i\alpha(n)$. Equations (19) and (20) along with the bound (23) can be deduced form (18) precisely as in the proof in [27].

What remains is to prove (21) and (22). This is simply a question of appropriate Taylor expansions. By (19) we have

$$\log\Big(\frac{R(n+1)}{R(n)}\Big) = \frac{1}{2}\log\Big[1 + U(n)\sin(2\theta(n)) + U(n)^2\sin^2(\theta(n))\Big]$$

so using the Taylor expansion

$$\log(1+x) = x - \frac{x^2}{2} + O(x^3)$$

one finds

$$\begin{aligned}\log\Big(\frac{R(n+1)}{R(n)}\Big) &= \frac{1}{2}\Big[U(n)\sin(2\theta(n)) + U(n)^2\sin^2(\theta(n)) \\ &\quad - \frac{1}{2}\Big(U(n)\sin(2\theta(n)) + U(n)^2\sin^2(\theta(n))\Big)^2 \\ &\quad + O\Big(\big|U(n)\sin(2\theta(n)) + U(n)^2\sin^2(\theta(n))\big|^3\Big)\Big] \\ &= \frac{U(n)}{2}\sin(2\theta(n)) + \frac{U(n)^2}{8} - \frac{U(n)^2}{8}\Big(2\cos(2\theta(n)) - \cos(4\theta(n))\Big) \\ &\quad + O(|U(n)|^3)\,,\end{aligned}$$

where we made used of the identity

$$\sin^2(x) - \frac{1}{2}\sin^2(2x) = \frac{1}{4} + \frac{1}{4}\cos(4x) - \frac{1}{2}\cos(2x)\,.$$

Similarly, using the fact that

$$\arg(1+z) = \Im(z) - \Im(z)\Re(z) + O(|z|^3)$$

and elementary trigonometric identities

$$\begin{aligned}
&\eta(n+1)-\eta(n)\\
&\quad=\arg\Big(\frac{\rho(n+1)}{\rho(n)}\Big)\\
&\quad=\arg\big(1+U(n)\sin(\theta(n))e^{-i\theta(n)}\big)\\
&\quad=-U(n)\sin(\theta(n))^2+U(n)^2\cos(\theta(n))\sin(\theta(n))^3+O(|U(n)|^3)\\
&\quad=-\frac{U(n)}{2}+\frac{U(n)}{2}\cos(2\theta(n))+\frac{U(n)^2}{4}\Big(\sin(2\theta(n))-\frac{1}{2}\sin(4\theta(n))\Big)+O(|U(n)|^3)\,.
\end{aligned}$$

This completes the proof of Lemma 3.1. □

3.2. **Properties of a particular solution to the Stark equation.** In the construction that follows we shall need to use certain properties of our reference solution

$$\zeta(x)=\Big(\frac{\pi}{F^{1/3}}\Big)^{1/2}\big(i\mathrm{Ai}(-F^{1/3}(x+E/F))+\mathrm{Bi}(-F^{1/3}(x+E/F))\big)\,. \tag{24}$$

Since $\mathrm{Ai}, \mathrm{Bi}$ solve the Airy equation it is easily checked that $\zeta$ is indeed a solution of (4). Since $\mathrm{Ai}, \mathrm{Bi}$ are linearly independent their real zeroes are distinct. Therefore $|\zeta(x)|^2>0$ for all $x$ and $\arg(\zeta)$ is well-defined. We can thus define $\gamma\in C^\infty(\mathbb{R})$ by

$$\zeta(x)=|\zeta(x)|e^{i\gamma(x)}\quad\text{with } \gamma(0)\in(-\pi,\pi]\,.$$

Most of the properties of $\zeta$ which are of interest to us concern $\gamma$ and its derivatives.

While we do not reflect this in our notation, we note that both $\zeta,\gamma$ implicitly depend on the values of $F, E$. It should however be emphasized that $\zeta$ is independent of the coupling constants $\{g_n\}_{n\in\mathbb{Z}}$. We also note that the dependence on the parameters $F, E$ is very simple. Indeed, from the explicit expression for $\zeta$ we have that

$$\zeta_{F,E}(x)=F^{-1/6}\zeta_{1,0}(F^{1/3}(x+E/F))\quad\text{and}\quad\gamma_{F,E}(x)=\gamma_{1,0}(F^{1/3}(x+E/F))\,.$$

In particular, all quantitative properties of $\zeta,\gamma$ which we shall discuss are uniform for $F, E$ in compact subsets of their respective domains.

For future reference in the next lemmas we collect some properties of $\zeta$.

**Lemma 3.3.** *For any $F>0, E\in\mathbb{R}$ and with $\zeta,\gamma$ as above it holds that*

$$\{\zeta,\bar\zeta\}=-2i\quad\textit{and}\quad|\zeta(x)|^2=\frac{1}{\gamma'(x)}\ \textit{for all } x\in\mathbb{R}\,.$$

For the record we note the following corollary of the second identity in Lemma 3.3 when combined with the fact that $\mathrm{Ai}, \mathrm{Bi}$ are continuous and tend to zero as $x\to-\infty$.

**Corollary 3.4.** *For any $F>0, E\in\mathbb{R}$ there exists a $\delta>0$ such that $\gamma'(x)>\delta$ for all $x>0$. Moreover, $\delta$ can be chosen uniform for $F, E$ in compact subsets of $\mathbb{R}_+$ and $\mathbb{R}$, respectively.*

**Lemma 3.5.** *As $x \to \infty$ it holds that*

$$\gamma(x) = \frac{2\sqrt{F}}{3} x^{3/2} + \frac{E}{\sqrt{F}} x^{1/2} + \frac{\pi}{2} + O(x^{-1/2}),$$
$$\gamma'(x) = \sqrt{F} x^{1/2} + O(x^{-1/2}),$$
$$\gamma''(x) = \frac{\sqrt{F}}{2} x^{-1/2} + O(x^{-3/2}).$$

*In particular, $\gamma$ is asymptotically increasing and convex. Corresponding asymptotic expansions can be proved for higher derivatives, here we shall only need that $|\partial_x^k \gamma(x)| \lesssim |x|^{3/2-k}$ for $k = 3, \dots, 7$. Here all the implicit constants are uniform for $F, E$ in compact subsets of $\mathbb{R}_+$ and $\mathbb{R}$, respectively.*

*Proof of Lemma 3.3.* By the definition of $\gamma$ there exists a branch of the logarithm such that $\gamma(x) = \Im(\log(\zeta(x)))$. By differentiating we find

$$\gamma'(x) = \Im\Big(\frac{\zeta'(x)}{\zeta(x)}\Big) = \frac{\Im(\zeta'(x)\bar{\zeta}(x))}{|\zeta(x)|^2} = -\frac{\{\zeta, \bar{\zeta}\}(x)}{2i|\zeta(x)|^2}$$

and thus

$$|\zeta(x)|^2 = -\frac{\{\zeta, \bar{\zeta}\}(x)}{2i\gamma'(x)}.$$

Since $\zeta, \bar{\zeta}$ solve the same ODE the Wronskian is constant and it suffices to compute its value at any given point. By setting $x = -E/F$ the value claimed in the lemma follows from the fact that (see [16, §9.2(ii)])

$$\mathrm{Ai}(0) = \frac{1}{3^{2/3}\Gamma(2/3)}, \quad \mathrm{Ai}'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)}, \quad \mathrm{Bi}(0) = \frac{1}{3^{1/6}\Gamma(2/3)}, \quad \mathrm{Bi}'(0) = \frac{3^{1/6}}{\Gamma(1/3)},$$

□

*Proof Lemma 3.5.* The asymptotic expansion for $\gamma$ can be deduced from that in [16, 9.8.22] by an appropriate change of variables. Asymptotic expansions for the derivatives of $\gamma$ are obtained by justifying term-by-term differentiation. Alternatively it can be reduced to asymptotic formulas for Bessel functions which are somewhat more standard. Indeed, for $x > -E/F$, by the definition of $\gamma$ combined with [16, 9.8.9 & 9.8.14] we have the identity, with $y = F^{1/3}(x + E/F)$,

$$\begin{aligned}\gamma'(x) &= \frac{3}{\pi(x+E/F)(J_{1/3}^2(\frac{2}{3}F^{1/2}(x+E/F)^{3/2}) + Y_{1/3}^2(\frac{2}{3}F^{1/2}(x+E/F)^{3/2}))} \\ &= \frac{3F^{1/3}}{\pi y(J_{1/3}^2(\frac{2}{3}y^{3/2}) + Y_{1/3}^2(\frac{2}{3}y^{3/2}))}.\end{aligned}$$

By differentiating the identity and using the asymptotic formulae for $J_\nu, Y_\nu$ in [16, 10.17.3 & 10.17.4] one arrives at the desired expansions. Alternatively, one can use Nicholson's integral representation for $J_\nu^2 + Y_\nu^2$ [16, 10.9.30] to obtain asymptotics for the denominator directly. □

3.3. **Comparability of asymptotics.** In our analysis we need to understand how the asymptotic behavior of $R$ relates to the growth or decay of the generalized eigenfunction $\psi$. Specifically we need to understand the behavior of

$$\int_0^x |\psi(t)|^2\,dt \quad \text{as } x\to\infty\,.$$

That this asymptotic behavior can be understood in terms of that of $R$ is the content of the following lemma.

**Lemma 3.6.** *Fix $F>0, E\in\mathbb{R}$, and a real-valued $\psi$ which solves (4) for $x\in(n-1,n)$ and let $R(n)$ be the associated Prüfer radius. Then*

$$\int_{n-1}^n |\psi(x)|^2\,dx = \frac{R(n)^2}{2\sqrt{Fn}}(1+O(n^{-1/2}))\,,$$

*and*

$$\int_{n-1}^n |\psi'(x)|^2\,dx = \frac{\sqrt{Fn}R(n)^2}{2}(1+O(n^{-1/2}))\,.$$

*Moreover, the implicit constants can be chosen uniform for $F, E$ in compact subsets of their respective domains.*

*Proof of Lemma 3.6.* Since the proofs of the two bounds are analogous we write out only the first in detail. The additional ingredient needed for the proof of the second bound is the identity

$$\zeta'(x) = \zeta(x)\Bigl(-\frac{\gamma''(x)}{2\gamma'(x)}+i\gamma'(x)\Bigr)\,,$$

which follows from the second identity in Lemma 3.3.

For any $x\in(n-1,n)$ we have

$$|\psi(x)|^2 = \frac14|\rho(n)\zeta(x)-\bar\rho(n)\bar\zeta(x)|^2 = \frac{R(n)^2}{2}|\zeta(x)|^2\Bigl(1-\cos(2\eta(n)+2\gamma(x))\Bigr)\,.$$

By the mean value theorem there exists an $x_0\in[n-1,n]$ so that

$$\int_{n-1}^n |\zeta(x)|^2(1-\cos(2\gamma(x)+2\eta(n))\,dx = |\zeta(x_0)|^2\biggl(1-\int_{n-1}^n \cos(2\gamma(x)+2\eta(n))\,dx\biggr).$$

We claim that the remaining integral is $O(n^{-1/2})$ uniformly in $\eta(n)$. Indeed, by Corollary 3.4 $\gamma$ is increasing, therefore a change of variables yields

$$\int_{n-1}^n \cos(2\gamma(x)+2\eta(n))\,dx = \int_{\gamma(n-1)}^{\gamma(n)} \frac{\cos(2y+2\eta(n))}{\gamma'(\gamma^{-1}(y))}\,dy\,.$$

Moreover, by Lemma 3.5 we conclude that

$$\gamma'(x) = \sqrt{Fn}\,(1+O(n^{-1})) \quad \text{for } x\in[n-1,n]\,,$$
$$|\gamma(n)-\gamma(n-1)| \lesssim \sqrt{n}\,.$$

Consequently,

$$\int_{\gamma(n-1)}^{\gamma(n)} \frac{\cos(2y+2\eta(n))}{\gamma'(\gamma^{-1}(y))}\,dy = \frac{1}{\sqrt{Fn}}\int_{\gamma(n-1)}^{\gamma(n)} \cos(2y+2\eta(n))\,dy + O(n^{-1})\,,$$

the claim follows since the modulus of the cosine integral is at most 2.

Therefore, there exists $x_0 \in [n-1,n]$ such that

$$\int_{n-1}^{n} |\psi(x)|^2\,dx = \frac{1}{2}R(n)^2|\zeta(x_0)|^2\big(1+O(n^{-1/2})\big)\,.$$

By Lemmas 3.3 and 3.5

$$|\zeta(x_0)|^2 = \frac{1}{\sqrt{Fn}}(1+O(n^{-1})) \quad \text{as } n\to\infty\,,$$

and hence we conclude that

$$\int_{n-1}^{n} |\psi(x)|^2\,dx = \frac{R(n)^2}{2\sqrt{Fn}}\big(1+O(n^{-1/2})\big)\,,$$

which completes the proof of the lemma. □

## 4. Analysis of the random model

In this section we analyse the random model with the goal of proving Theorem 1.3. The reader who is only interested in the deterministic model can with the exception of Section 4.1 jump directly to Section 5.

4.1. **Exponential sums with phase $\gamma$.** A central part of the analysis which is to follow will be understanding the behavior of certain exponential sums of the form

$$\sum_{a<n\le b} u(n)e^{i\varphi(n)}\,.$$

Specifically, we will frequently encounter such sums where the amplitude $u$ is a negative power of $\gamma'$ and the phase $\varphi$ given in terms of $\gamma(n)$. In this section we collect a number of bounds for such sums which we shall frequently use later on.

We begin by recalling the classical bound of van der Corput (see for instance [45, Theorem 5.9]).

**Lemma 4.1.** *Let $a<b$ with $b-a\ge 1$ and let $f\in C^2[a,b]$ be a real function satisfying*

$$\kappa \le |f''(x)| \le h\kappa \quad \text{for all } x\in[a,b]\,.$$

*Then*

$$\Big|\sum_{a<n\le b} e^{2\pi i f(n)}\Big| \lesssim h(b-a)\kappa^{1/2} + \kappa^{-1/2}\,.$$

In the setting considered here the following almost direct corollary of Lemma 4.1 will be most relevant.

**Corollary 4.2.** *Fix $\mu, F > 0$ and $E \in \mathbb{R}$. Then for any $0 < a < b$,*

$$\Big|\sum_{a<j\leq b} e^{\mu i\gamma(j)}\Big| \lesssim \frac{b^{1/4}}{a^{1/2}}(b-a+a^{1/2})\,.$$

*Moreover, the implicit constant can be taken uniform for $\mu, F, E$ in compact subsets of their respective domains.*

*Proof.* By the asymptotic behavior of $\gamma'''$ there exists an $A > 0$ large enough so that $\gamma''$ is monotone decreasing on $[A, \infty)$.

If $a < b \leq A$ we bound the sum trivially by $b - a$, which is less than the right-hand side of the claimed bound provided the implicit constant is sufficiently large. If $a < A < b$ we split the sum into two pieces: one with $a < j \leq A$ and one with $A < j \leq b$. The first of the two we bound by $A - a$, which again is bounded by the right-hand side of the claimed inequality. Since the right-hand side of the inequality is increasing with respect $b$ and decreasing with respect to $a$ it only remains to prove the bound under the assumption $a \geq A$.

If $a \geq A$, the monotonicity of $\gamma''$ implies

$$|\gamma''(b)| \leq |\gamma''(x)| \leq |\gamma''(a)| \quad \text{for all } x \in (a, b]\,.$$

Thus with $\kappa = \mu|\gamma''(b)|/(2\pi)$ and $h = \frac{|\gamma''(a)|}{|\gamma''(b)|} \lesssim \big(\frac{b}{a}\big)^{1/2}$ the conditions of Lemma 4.1 hold and we find

$$\begin{aligned}\Big|\sum_{a<j\leq b} e^{i\mu\gamma(j)}\Big| &\lesssim (b/a)^{1/2}(b-a)|\gamma''(b)|^{1/2} + |\gamma''(b)|^{-1/2}\\ &\lesssim \frac{b^{1/4}}{a^{1/2}}(b-a+a^{1/2})\,,\end{aligned}$$

which completes the proof of the lemma. □

For our proof of Theorems 1.1 we shall require very precise estimates for exponential sums with amplitude given in terms of $\gamma'$ and phase function $\gamma$. These estimates, which become rather technical, are the topic of Section 5. However, in our analysis of the random model it shall suffice to have the following bound which is similar in spirit but much cruder than what we shall prove later on.

**Theorem 4.3.** *Let $\mu > 0$ and $h\colon \mathbb{N} \to \mathbb{R}$ satisfy*

$$\lim_{j\to\infty} |h(j+1) - h(j)| j^{1/4} = 0\,. \tag{25}$$

*Then,*

$$\sum_{j=1}^{N} \frac{e^{i(\mu\gamma(j)+h(j))}}{\gamma'(j)^2} = o(\log(N))$$

*where the error term can be quantified in terms of that in* (25) *and chosen uniform for $\mu, F, E$ in compact subsets of their respective domains.*

*Remark* 4.4. If the convergence rate in (25) is sufficiently rapid, for instance $j^{-\varepsilon}$ for some $\varepsilon > 0$, then a straight-forward modification of the proof below implies the convergence of the series

$$\sum_{j=1}^{\infty} \frac{e^{i(\mu\gamma(j)+h(j))}}{\gamma'(j)^2} \, .$$

*Proof.* By assumption there exists a function $\epsilon\colon \mathbb{R}_+ \to [0,\infty)$ with $\lim_{j\to\infty} \epsilon(j) = 0$ such that

$$|h(j+1) - h(j)| j^{1/4} \le \epsilon(j) \, .$$

Without loss of generality we may assume that $\epsilon$ is bounded and monotonically decreasing.

We decompose the partial sum into sums with $(l-1)^2 < j \le l^2$. For $N_0$ large enough we can write

$$\sum_{j=1}^{N} \frac{e^{i(\mu\gamma(j)+h(j))}}{\gamma'(j)^2} = \sum_{1\le l\le\sqrt{N}} \left[ \sum_{(l-1)^2<j\le l^2} \frac{e^{i(\mu\gamma(j)+h(j))}}{\gamma'(j)^2} \right] + \sum_{\lfloor\sqrt{N}\rfloor^2<j\le N} \frac{e^{i(\mu\gamma(j)+h(j))}}{\gamma'(j)^2} \, .$$

For the second sum the asymptotic behavior of $\gamma'$ in Lemma 3.5 implies

$$\Big| \sum_{\lfloor\sqrt{N}\rfloor^2<j\le N} \frac{e^{i(\mu\gamma(j)+h(j))}}{\gamma'(j)^2} \Big| \lesssim \frac{N - \lfloor\sqrt{N}\rfloor^2}{N} \lesssim \frac{1}{\sqrt{N}} \, .$$

For fixed $l$ a summation by parts yields

$$\begin{aligned} \sum_{(l-1)^2<j\le l^2} \frac{e^{i(\mu\gamma(j)+h(j))}}{\gamma'(j)^2} &= \sum_{(l-1)^2<j\le l^2-1} \left[ \frac{e^{ih(j)}}{\gamma'(j)^2} - \frac{e^{ih(j+1)}}{\gamma'(j+1)^2} \right] \sum_{(l-1)^2<k\le j} e^{i\mu\gamma(k)} \\ &\quad + \frac{e^{ih(l^2)}}{\gamma'(l^2)^2} \sum_{(l-1)^2<j\le l^2} e^{i\mu\gamma(j)} \, . \end{aligned}$$

Using the fundamental theorem of calculus, the asymptotics of Lemma 3.5, and the assumption on $h$ we can estimate

$$\begin{aligned} \Big| \frac{e^{ih(j)}}{\gamma'(j)^2} - \frac{e^{ih(j+1)}}{\gamma'(j+1)^2} \Big| &\le \Big| \frac{1}{\gamma'(j)^2} - \frac{1}{\gamma'(j+1)^2} \Big| + \frac{1}{\gamma'(j+1)^2} \Big| e^{ih(j)} - e^{ih(j+1)} \Big| \\ &\lesssim \frac{1}{\gamma'(j)^2\gamma'(j+1)^2} \Big| \int_j^{j+1} \gamma'(t)\gamma''(t)\, dt \Big| + \frac{\epsilon(j)}{j^{1/4}\gamma'(j+1)^2} \\ &\lesssim j^{-5/4}(j^{-3/4} + \epsilon(j)) \, . \end{aligned}$$

Inserting this into the above we find

$$\begin{aligned} \Big| \sum_{(l-1)^2<j\le l^2} \frac{e^{i(\mu\gamma(j)+h(j))}}{\gamma'(j)^2} \Big| &\lesssim l^{-5/2}(l^{-3/2} + \epsilon(l^2)) \sum_{(l-1)^2<j\le l^2} \Big| \sum_{(l-1)^2<k\le j} e^{i\mu\gamma(k)} \Big| \\ &\quad + \frac{1}{l^2} \Big| \sum_{(l-1)^2<j\le l^2} e^{i\mu\gamma(j)} \Big| \, . \end{aligned}$$

By Corollary 4.2 the remaining exponential sums are $\lesssim l^{1/2}$ so that

$$\Big|\sum_{(l-1)^2<j\le l^2}\frac{e^{i(\mu\gamma(j)+h(j))}}{\gamma'(j)^2}\Big|\lesssim l^{-5/2}(l^{-3/2}+\epsilon(l^2))\sum_{(l-1)^2<j\le l^2}l^{1/2}+l^{-3/2}\lesssim l^{-3/2}(1+l^{1/2}\epsilon(l^2))\,.$$

Since $l^{-3/2}$ is summable and

$$\begin{aligned}\sum_{1\le l\le\sqrt{N}}l^{-1}\epsilon(l^2)&\le\|\epsilon\|_{L^\infty}\sum_{1\le l\le e^{\sqrt{\log(N)}}}l^{-1}+\epsilon(e^{2\sqrt{\log(N)}})\sum_{e^{\sqrt{\log(N)}}<l\le\sqrt{N}}l^{-1}\\&\lesssim\|\epsilon\|_{L^\infty}\sqrt{\log(N)}+\epsilon(e^{2\sqrt{\log(N)}})\log(N)\,,\end{aligned}$$

this concludes the proof. □

4.2. **Asymptotics of $R$ for non-subordinate solutions.** The aim of this section is to analyse the behavior of non-subordinate solutions of (12). Since the subordinate solutions constitute a one-dimensional subspace of the two-dimensional solution space, all non-subordinate solutions have essentially the same asymptotic behavior. Describing the asymptotic behavior is precisely the content of our next theorem, which is a direct analogue of [26, Theorem 8.2] and our proof follows that given there.

**Theorem 4.5.** *Suppose that $\{g_n(\omega)\}_{n\in\mathbb{Z}}$ are independent random variables satisfying*

*(i)* $\mathbb{E}_\omega[g_n]=0$,
*(ii)* $\mathbb{E}_\omega[g_n^2]=\lambda^2$,
*(iii)* $\sum_{n\ge1}\mathbb{E}_\omega[g_n^4]n^{-2}<\infty$, *and*
*(iv)* $\mathbb{P}_\omega\big[\lim_{n\to\infty}g_nn^{-1/4}=0\big]=1$.

*Fix $F>0, E\in\mathbb{R}$, and $\theta_0\in[0,\pi)$. Then, almost surely, the Prüfer radius $R$ associated to the solution $\psi$ of* (12) *with*

$$\psi(0)=\sin(\theta_0)\quad\textit{and}\quad\psi'(0)=\cos(\theta_0)$$

*satisfies*

$$\lim_{n\to\infty}\frac{\log(R(n))}{\log(n)}=\frac{\lambda^2}{8F}\,.$$

*Remark* 4.6. As noted earlier, the assumptions are valid for $g_n$ chosen as i.i.d. copies of a random variable $X$ with mean zero, variance $\lambda^2$, if and only if $\mathbb{E}_\omega[|X|^4]<\infty$. Let us show this.

By the dominated convergence theorem

$$\mathbb{P}_\omega\Big[\lim_{n\to\infty}g_nn^{-1/4}=0\Big]=\lim_{\substack{N\to\infty\\ \varepsilon\to0^+}}\mathbb{P}_\omega\Big[\sup_{n\ge N}|g_n|n^{-1/4}<\varepsilon\Big]\,.$$

Assume that $N\in\mathbb{N}$ is chosen so large that $\mathbb{P}_\omega[|g_n|<\varepsilon n^{1/4}]>1/2$ for all $n\ge N$ (since $\mathbb{E}_\omega[|g_n|^2]=\lambda^2$ this is possible by Chebyshev's inequality). By the independence of the $g_n$ and the inequality $-2x\le\log(1-x)\le-x$, for $x\in[0,1/2]$, we find

$$e^{-2\sum_{n=N}^\infty\mathbb{P}_\omega[|g_n|\ge\varepsilon n^{1/4}]}\le\mathbb{P}_\omega\Big[\sup_{n\ge N}|g_n|n^{-1/4}<\varepsilon\Big]\le e^{-\sum_{n=N}^\infty\mathbb{P}_\omega[|g_n|\ge\varepsilon n^{1/4}]}\,.$$

If for some $\varepsilon > 0$ the series in the exponents diverges then the probability that the limit vanishes is zero. Conversely, the probability that the limit vanishes is positive if for any $\varepsilon > 0$ there exists an $N$ so that the series converges. If for any $\varepsilon > 0$ such an $N$ exists the monotone convergence theorem allows us to send $N$ to infinity and conclude that the probability is in fact 1. We also note that if for some $\varepsilon > 0, N > 0$ the series is convergent, then since $\{|g_n|n^{1/4}\}_{n=1}^N$ are uniformly bounded with probability 1 the monotone convergence theorem this time applied as $\varepsilon \to \infty$ implies

$$\mathbb{P}_\omega\Big[\sup_{n\geq 1}|g_n|n^{-1/4} < \infty\Big] = 1\,.$$

If the $g_n$ are i.i.d. copies of a random variable $X$ the layer cake representation implies that for every non-negative function $f$, $\mathbb{P}_\omega[f(|X|) \geq n] \in l^1$ is equivalent to $\mathbb{E}_\omega[f(|X|)] < \infty$. Since $\mathbb{E}_\omega[(|X|/\varepsilon)^4] = \varepsilon^{-4}\mathbb{E}_\omega[|X|^4]$ for all $\varepsilon > 0$ the series above is convergent if and only if $\mathbb{E}_\omega[|X|^4] < \infty$. By the above argument (iv) is valid if and only if $\mathbb{E}_\omega[|X|^4] < \infty$.

For non-identically distributed $g_n$ we can use essentially the same argument with an additional application of Markov's inequality $\mathbb{P}_\omega[|g_n| \geq x] \leq \frac{\mathbb{E}_\omega[f(|g_n|)]}{f(x)}$, for a non-negative increasing function $f$, to deduce the validity of (iv) if $\frac{\mathbb{E}_\omega[f(|g_n|)]}{f(\varepsilon n^{1/4})} \in l^1$ for all $\varepsilon > 0$. In particular, this can be applied to show the validity of our assumptions if $\mathbb{E}_\omega[|g_n|^\alpha]$ is uniformly bounded for some $\alpha > 4$.

*Proof of Theorem 4.5.* Fix a typical realization $\{g_n\}_{n\in\mathbb{Z}} = \{g_n(\omega)\}_{n\in\mathbb{Z}}$ and let $R, \theta$ denote the Prüfer coordinates corresponding to $\psi$. Recall that $U(n) = \frac{g_n}{\gamma'(n)}$. Since $|U(n)| \to 0$ as $n \to \infty$ (a.s.), equation (21) yields, for $n$ large enough,

$$\begin{aligned}\log\Big(\frac{R(n+1)}{R(n)}\Big) = \frac{U(n)}{2}\sin(2\theta(n)) + \frac{U(n)^2}{8}\\ - \frac{U(n)^2}{8}\Big(2\cos(2\theta(n)) - \cos(4\theta(n))\Big) + O(|U(n)|^3)\,.\end{aligned} \tag{26}$$

By (iv) almost surely there exists $C_\omega < \infty$ such that $|g_n(\omega)| \leq C_\omega n^{1/4}$ for all $n \geq 1$. Therefore, by Lemma 3.5 and [26, Lemma 8.4] almost surely

$$\sum_{n\geq 1}|U(n)|^3 \lesssim C_\omega \sum_{n\geq 1}\frac{g_n(\omega)^2}{n^{5/4}} = C_\omega\sum_{n\geq 1}\frac{\lambda^2}{n^{5/4}} + C_\omega\sum_{n\geq 1}\frac{g_n(\omega)^2-\lambda^2}{n^{5/4}} \lesssim 1\,,$$

so the error term in (26) is summable. Therefore, repeated use of equation (26) and rearranging yields

$$\begin{aligned}\log\Big(\frac{R(n+1)}{R(1)}\Big) &= \frac{1}{8}\sum_{j=1}^{n}\mathbb{E}_\omega[U(j)^2] + \frac{1}{2}\sum_{j=1}^{n}U(j)\sin(2\theta(j))\\ &\quad + \frac{1}{8}\sum_{j=1}^{n}\big[U(j)^2-\mathbb{E}_\omega[U(j)^2]\big]\Big[1-2\cos(2\theta(j))+\cos(4\theta(j))\Big]\\ &\quad - \frac{1}{8}\sum_{j=1}^{n}\mathbb{E}_\omega[U(j)^2]\Big[2\cos(2\theta(j))-\cos(4\theta(j))\Big]+O(1)\,.\end{aligned}$$

Define

$$\begin{aligned}C_1(n,\omega) &= \frac{1}{2}\sum_{j=1}^{n}U(j)\sin(2\theta(j))\,,\\ C_2(n,\omega) &= \frac{1}{8}\sum_{j=1}^{n}\big[U(j)^2-\mathbb{E}_\omega[U(j)^2]\big]\Big[1-2\cos(2\theta(j))+\cos(4\theta(j)))\Big]\,,\\ C_3(n,\omega) &= \frac{1}{8}\sum_{j=1}^{n}\mathbb{E}_\omega[U(j)^2]\Big[2\cos(2\theta(j))-\cos(4\theta(j))\Big]\,,\end{aligned}$$

so that

$$\log\Big(\frac{R(n+1)}{R(1)}\Big) = \frac{1}{8}\sum_{j=1}^{n}\mathbb{E}_\omega[U(j)^2]+\sum_{m=1}^{3}C_m(n,\omega)+O(1)\,.$$

By application of [26, Lemma 8.4] we find, almost surely, $|C_1(n,\omega)| \lesssim o(\log(n))$ and $|C_3(n,\omega)| \lesssim 1$ where we used $\gamma'(j)\sim j^{1/2}$ and our assumptions on $g_j$.

By Lemma 3.1 and (iv) we have almost surely that $|\eta(j+1)-\eta(j)| \lesssim |U(j)| \lesssim o(j^{-1/4})$. Thus we can apply Theorem 4.3, with $\mu = 2, 4$ and $h(j)=\mu\eta(j)$, to deduce

$$\Big|\sum_{j=1}^{n}\mathbb{E}_\omega[U(j)^2]\cos(\mu\theta(j))\Big| \le \lambda^2\Big|\sum_{j=1}^{n}\frac{e^{i(\mu\gamma(j)+\mu\eta(j))}}{\gamma'(j)^2}\Big| = o(\log(n))\,.$$

Consequently, the term $C_3(n,\omega)$ does not contribute to the limit in the theorem.

By Lemma 3.5, $\gamma'(x) = F^{1/2}x^{1/2}+O(x^{-1/2})$ so that

$$\frac{1}{8}\sum_{j=1}^{n}\mathbb{E}_\omega[U(j)^2] = \frac{\lambda^2}{8}\sum_{j=1}^{n}\frac{1}{\gamma'(j)^2} = \frac{\lambda^2}{8F}\log(n)+O(1)\,,$$

which completes the proof of Theorem 4.5. □

To turn our knowledge of the behavior of non-subordinate solutions into information about subordinate solutions we shall need the following theorem which tells us that the asymptotic behavior of two linearly independent solutions typically differs only by a constant factor.

**Theorem 4.7.** *Fix $F > 0, E \in \mathbb{R}$, and $\theta_+, \theta_- \in [0,\pi)$ with $\theta_+ \neq \theta_-$. Assume that $\{g_n(\omega)\}_{n\geq 1}$ satisfy the hypothesis of Theorem 4.5. Let $\psi_+, \psi_-$ be the two solutions of* (12) *corresponding to the boundary condition*

$$\psi_\pm(0) = \sin(\theta_\pm) \quad \text{and} \quad \psi'_\pm(0) = \cos(\theta_\pm)$$

*and let $R_+, R_-$ denote the corresponding Prüfer radii. Then, almost surely, there exists $\varrho_\infty \in (0,\infty)$ such that*

$$\limsup_{n\to\infty} \frac{\log\left|\frac{R_+(n)}{R_-(n)} - \varrho_\infty\right|}{\log(n)} \leq -\frac{\lambda^2}{4F}.$$

*Proof of Theorem 4.7.* Since by assumption $\theta_+ \neq \theta_-$ we have

$$\{\psi_+, \psi_-\}(x) = \{\psi_+, \psi_-\}(0) = \sin(\theta_+ - \theta_-) \neq 0\,,$$

so the solutions $\psi_+$ and $\psi_-$ are linearly independent.

By expressing the Wronskian in terms of our Prüfer coordinates we also have for $x \in (n-1, n)$

$$\begin{aligned}
\{\psi_+, \psi_-\}(x) &= \Big\{\frac{R_+e^{i\eta_+}}{2i}\zeta - \frac{R_+e^{-i\eta_+}}{2i}\bar{\zeta}, \frac{R_-e^{i\eta_-}}{2i}\zeta - \frac{R_-e^{-i\eta_-}}{2i}\bar{\zeta}\Big\}(x) \\
&= \frac{i}{2}R_+(n)R_-(n)\sin(\eta_+(n) - \eta_-(n))\{\zeta, \bar{\zeta}\}(x) \\
&= R_+(n)R_-(n)\sin(\eta_+(n) - \eta_-(n))
\end{aligned}$$

and so, since the Wronskian is constant and $R_+, R_-$ non-zero,

$$\sin(\eta_+(n) - \eta_-(n)) = \frac{\{\psi_+, \psi_-\}(0)}{R_+(n)R_-(n)}\,. \tag{27}$$

By Theorem 4.5, almost surely

$$\lim_{n\to\infty} \frac{\log(R_\pm(n))}{\log(n)} = \frac{\lambda^2}{8F}\,.$$

Thus by (27) with $\theta_\pm(n) = \eta_\pm(n) + \gamma(n)$, almost surely

$$\lim_{n\to\infty} \frac{\log|\sin(\theta_+(n) - \theta_-(n))|}{\log(n)} = \lim_{n\to\infty} \frac{\log|\sin(\eta_+(n) - \eta_-(n))|}{\log(n)} = -\frac{\lambda^2}{4F}\,, \tag{28}$$

which implies that there exists a sequence of integers $k_n$ such that

$$\lim_{n\to\infty} \frac{\log|\theta_+(n) - \theta_-(n) - k_n\pi|}{\log(n)} = -\frac{\lambda^2}{4F}\,. \tag{29}$$

Set $\varrho(n) = \frac{R_+(n)}{R_-(n)}$. We wish to prove that $\varrho(n)$ converges sufficiently fast to a limit different from 0 and $\infty$.

To this end let $M(n) = \log(\varrho(n+1)) - \log(\varrho(n))$ and use Lemma 3.1 to deduce

$$\begin{aligned} M(n) &= \log(\varrho(n+1)) - \log(\varrho(n)) \\ &= \frac{1}{2}\log\Big(\frac{R_+(n+1)^2}{R_+(n)^2}\Big) - \frac{1}{2}\log\Big(\frac{R_-(n+1)^2}{R_-(n)^2}\Big) \\ &= \frac{1}{2}\Big[\log\Big(1 + U(n)\sin(2\theta_+(n)) + U(n)^2\sin^2(\theta_+(n))\Big) \\ &\qquad - \log\Big(1 + U(n)\sin(2\theta_-(n)) + U(n)^2\sin^2(\theta_-(n))\Big)\Big] \\ &= \frac{1}{2}\Big[\log\Big(1 + U(n)\sin(2\theta_+(n) - 2k_n\pi) + U(n)^2\sin^2(\theta_+(n) - k_n\pi)\Big) \\ &\qquad - \log\Big(1 + U(n)\sin(2\theta_-(n)) + U(n)^2\sin^2(\theta_-(n))\Big)\Big]\,. \end{aligned}$$

Set

$$K(a,\theta) = \log(1 + a\sin(2\theta) + a^2\sin^2(\theta))\,.$$

By a Taylor expansion with respect to $a$,

$$K(a,\theta) = \sum_{j=1}^{J-1} a^j P_j(\theta) + O(a^J)$$

with $P_1(\theta) = \sin(2\theta)$ and each $P_j \in C^\infty(\mathbb{R})$. By (29) we know that $|\theta_+(n) - \theta_-(n) - k_n\pi| = O(n^{-\lambda^2/(4F)+\varepsilon})$ for any $\varepsilon > 0$, and thus almost surely

$$\begin{aligned} &K(U(n), \theta_+(n) - k_n\pi) - K(U(n), \theta_-(n)) \\ &\quad = \sum_{j=1}^{J-1} U(n)^j\big[P_j(\theta_+(n) - k_n\pi) - P_j(\theta_-(n))\big] + O(|U(n)|^J) \\ &\quad = U(n)\Big[\sin(2\theta_+(n) - 2k_n\pi) - \sin(2\theta_-(n))\Big] \\ &\qquad + \sum_{j=2}^{J-1} O\Big(|U(n)|^j|\theta_+(n) - \theta_-(n) - k_n\pi|\Big) + O(|U(n)|^J) \\ &\quad = U(n)\Big[\sin(2\theta_+(n)) - \sin(2\theta_-(n))\Big] \\ &\qquad + O\Big(U(n)^2 n^{-\lambda^2/(4F)+\varepsilon}\Big) + O\Big(U(n)^2 n^{-(J-2)/4}\Big)\,. \end{aligned}$$

Here we used the fact that by (iv) and Lemma 3.5 almost surely $|U(n)| \le C_\omega n^{-1/4}$.

By choosing $J$ sufficiently large (any $J \ge \frac{\lambda^2}{F} + 2$ suffices), and since $\theta_\pm(n)$ only depend on $\{g_k\}_{k\le n-1}$ one can use the bounds of [26, Lemma 8.3] to deduce that almost surely

$$\lim_{N\to\infty} \log(\varrho(N)) = \log(\varrho(1)) + \lim_{n\to\infty}\sum_{n=1}^{N-1} M(n)$$

exists and is finite and

$$\left|\sum_{n=N}^{\infty} M(n)\right| \lesssim N^{-\lambda^2/(4F)+2\varepsilon}\,.$$

Consequently, setting

$$\varrho_\infty = \lim_{n\to\infty} \varrho(n)$$

we conclude that, almost surely,

$$\frac{\log\left|\frac{R_+(n)}{R_-(n)} - \varrho_\infty\right|}{\log(n)} = \frac{\log\left|1 - \frac{\varrho_\infty}{\varrho(n)}\right|}{\log(n)} + o(1) \leq -\frac{\lambda^2}{4F} + 2\varepsilon + o(1)\,.$$

Since $\varepsilon > 0$ is arbitrary this concludes the proof. □

4.3. **Existence of subordinate solution.** In this subsection we turn the statements concerning the behavior of $R$ studied in the previous section into information about the existence of solutions which fail to behave as the typical ones. Specifically, we are interested in the existence of subordinate solutions and whether or not they are square integrable. How the properties of subordinate solutions of (12) relate to the spectral theory of the corresponding differential operator is the content of the Gilbert–Pearson subordinacy theory originally developed in [21, 20]. While our equation is not a standard differential equation on $\mathbb{R}$, but rather an infinite a system of coupled equations on unit intervals, the subordinacy theory goes through with obvious changes. Before we begin we recall the precise definition of subordinate solutions.

Let $L$ be a second order differential expression and assume that $L$ is limit point at $\infty$. A non-trivial solution to the equation $L\psi = E\psi$ with $R \in \mathbb{R}$ on $(0,\infty)$ is called subordinate at $\infty$ if

$$\lim_{x\to\infty} \frac{\int_0^x |\psi(t)|^2\,dt}{\int_0^x |\varphi(t)|^2\,dt} = 0$$

for any solution of the equation $L\varphi = E\varphi$ which linearly independent of $\psi$. The subordinacy of a solution at $-\infty$ is defined analogously. Since the solution space of $L\varphi = E\varphi$ is two-dimensional it follows that subordinate solutions are unique up to a multiplicative constant.

By Lemma 2.2, and the fact that there can only be one $L^2$ solution for the differential equation corresponding to the operator $L_{F,g}$ we always have a non-trivial solution that is subordinate at $-\infty$.

When it comes to subordinate solutions at $+\infty$ our aim is to prove the following theorem.

**Theorem 4.8.** *Let $F > 0, E \in \mathbb{R}$ be fixed and assume that $\{g_n(\omega)\}_{n\geq 1}$ satisfy the hypothesis of Theorem 4.5. Then almost surely there exists a $\theta_0 = \theta_0(\omega) \in [0,\pi)$ such that the solution of* (12) *with boundary conditions*

$$\psi'(0) = \cos(\theta_0)\,, \quad \psi(0) = \sin(\theta_0)$$

*is subordinate at $\infty$ and the associated Prüfer radius $R$ satisfies*

$$\lim_{n\to\infty} \frac{\log(R(n))}{\log(n)} = -\frac{\lambda^2}{8F}\,. \tag{30}$$

For our proof of Theorem 1.3 we shall need the following corollary which follows from Theorem 4.8 by an application of Fubini's theorem.

**Corollary 4.9.** *Suppose the $\{g_n(\omega)\}_{n\geq 1}$ satisfy the hypothesis of Theorem 4.5 and $F>0$ is fixed. Then almost surely for almost every $E\in\mathbb{R}$ there exists a $\theta_0=\theta_0(E,\omega)\in[0,\pi)$ such that the solution of* (12) *with boundary conditions*

$$\psi'(0)=\cos(\theta_0)\,,\quad \psi(0)=\sin(\theta_0)$$

*is subordinate at $\infty$ and the associated Prüfer radius $R$ satisfies* (30)*.*

To prove Theorem 4.8 we begin by turn the information provided by Theorems 4.5 and 4.7 into statements about transfer matrices. After we have accomplished this we can apply results from the OPUC literature to extract the desired existence and properties of the subordinate solution. The overall argument closely follows that of Kiselev–Last–Simon [26] but formulated in slightly different terms since we work with transfer matrices in $\mathbb{SU}(1,1)$ instead of $\mathbb{SL}(2,\mathbb{R})$. The analogue of the argument in [26] for $\mathbb{SU}(1,1)$ matrices can be found in [41, §10.5].

Recall that transfer matrices in the complex Prüfer coordinates are given by

$$A_n=\mathbb{1}+\frac{U(n)}{2i}\begin{pmatrix}1 & e^{-2i\gamma(n)}\\ -e^{2i\gamma(n)} & -1\end{pmatrix}. \tag{31}$$

We note that by the assumptions on $\{g_n(\omega)\}_{n\in\mathbb{Z}}$ the norms $\|A_n\|$ are almost surely uniformly bounded. Define the $n$-step transfer matrix

$$T_n=A_n\cdots A_1\,.$$

Since $A_n\in\mathbb{SU}(1,1)$ we have $T_n\in\mathbb{SU}(1,1)$.

Recall that for $A\in\mathbb{SU}(1,1)$ the matrix $|A|=(A^*A)^{1/2}$ has two eigenvalues, $\|A\|$ and $\|A\|^{-1}$. If $\|A\|>1$ in what follows we let $P_-(A)$ denote the orthogonal projection onto the one-dimensional eigenspace corresponding to the eigenvalue $\|A\|^{-1}$, if $\|A\|=1$ then $|A|=\mathbb{1}$ and for notational convenience we let $P_-(A)=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$.

We begin by proving the analogue of Theorem 4.5 for the growth of the norm of $\|T_n\|$.

**Lemma 4.10.** *Under the assumptions of Theorem 4.5 almost surely*

$$\lim_{n\to\infty}\frac{\log\|T_n\|}{\log(n)}=\frac{\lambda^2}{8F}\,.$$

*Proof.* If $u\in\mathbb{C}^2$ has the form $u=(u_1,\bar u_1)$ with $u_1\in\mathbb{C}\setminus\{0\}$, then $|T_nu|=R(n)/\sqrt{2}$ with $R$ corresponding to the real valued solution $\psi$ of (12) defined by the boundary conditions

$$\psi(0)=u_1\zeta(0)+\bar u_1\bar\zeta(0)\quad\text{and}\quad\psi'(0)=u_1\zeta'(0)+\bar u_1\bar\zeta'(0)\,.$$

Therefore, $\|T_n\|=\sup_{u\in\mathbb{C}^2,|u|=1}|T_nu|\geq R(n)/\sqrt{2}$ where $R$ is the Prüfer radius associated to any fixed choice of boundary conditions. By Theorem 4.5, $\log\|T_n\|\geq\frac{\lambda^2}{8F}\log(n)(1+o(1))$.

It remains to prove a matching upper bound. To this end note that for any $u\in\mathbb{C}^2$ and any $A\in\mathbb{SU}(1,1)$

$$|Au|^2=\|A\|^2|(\mathbb{1}-P_-(A))u|^2+\|A\|^{-2}|P_-(A)u|^2\,.$$

For $A = T_n$ and by dropping the non-negative second term and applying Theorem 4.5 with two initial conditions corresponding to $u = (u_1, \bar{u}_1), v = (v_1, \bar{v}_1)$ such that $(u, v)_{\mathbb{C}^2} = 0$ and $|u| = |v| = 1$ and denoting the corresponding Prüfer radii by $R_u, R_v$ we deduce that, almost surely,

$$\begin{aligned}\log \|T_n\| &\leq \frac{1}{2} \log\Big(\frac{|T_n u|^2 + |T_n v|^2}{|(\mathbb{1} - P_-(T_n))u|^2 + |(\mathbb{1} - P_-(T_n))v|^2}\Big) \\ &= \frac{1}{2} \log\Big(\frac{R_u(n)^2 + R_v(n)^2}{2}\Big) \\ &= \frac{\lambda^2}{8F} \log(n)(1 + o(1)) \,,\end{aligned}$$

where we used

$$|(\mathbb{1} - P_-(T_n))u|^2 + |(\mathbb{1} - P_-(T_n))v|^2 = \mathrm{Tr}(\mathbb{1} - P_-(T_n))^2 = 1$$

since $(u, v)_{\mathbb{C}^2} = 0$ and $|u| = |v| = 1$. This completes the proof of the lemma. □

A key step in proving Theorem 4.8 is provided by the following result which can be found in [41] (see also Kotani–Ushiroya [28]).

**Theorem 4.11** ([41, Theorem 10.5.34])**.** *Let $A_1, A_2, \ldots$ be elements of $\mathbb{U}(1, 1)$. Let $T_n = A_n \cdots A_1$. Define*

$$\varrho(T_n) = \frac{|T_n\binom{1}{1}|}{|T_n\binom{1}{-1}|} \,,$$

*and assume that*

$$\lim_{n\to\infty} \|T_n\| = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\|T_n^{-1} T_{n+1}\|}{\|T_n\| \, \|T_{n+1}\|} = 0 \,. \tag{32}$$

*Then*

1. *$P_-(T_n)$ has a limit $P_\infty$ if and only if $\lim_{n\to\infty} \varrho(T_n) \equiv \varrho_\infty$ exists. That $\varrho_\infty = \infty$ is allowed, and holds if $\lim_{n\to\infty} P_-(T_n)$ is the projection onto $\binom{1}{-1}$.*
2. *If $\varrho(T_n)$ has a limit $\varrho_\infty \neq 0, \infty$, then, with $u_\infty \in \operatorname{ran} P_\infty$,*

$$\lim_{n\to\infty} \frac{\log |T_n u_\infty|}{\log \|T_n\|} = -1$$

*if and only if*

$$\limsup_{n\to\infty} \frac{\log |\varrho(T_n) - \varrho_\infty|}{\log \|T_n\|} \leq -2 \,. \tag{33}$$

With these transfer matrix results on hand we are ready to prove Theorem 4.8.

*Proof of Theorem 4.8.* Note that almost surely $T_n^{-1} T_{n+1} = A_{n+1}$ is bounded uniformly in $n$, and $\|T_n\| \to \infty$ by Lemma 4.10. Therefore, the assumptions in (32) are almost surely satisfied.

Moreover, since $|T_n\binom{1}{-1}| = |T_n\binom{i}{-i}|$ the quantity $\varrho(T_n)$ in Theorem 4.11 is precisely the quantity $\varrho(n)$ of Theorem 4.7 with $\psi_+, \psi_-$ chosen to satisfy the boundary conditions corresponding to $\alpha(1) = 1$ and $\alpha(1) = i$, respectively. Consequently, Theorem 4.7 and Lemma 4.10 imply the almost sure validity of (33).

By Theorem 4.11 there almost surely exist a non-zero $u_\infty \in \mathbb{C}^2$ such that

$$\log |T_n u_\infty| = -\frac{\lambda^2}{8F} \log(n)(1 + o(1)) .$$

By Theorem 4.5 and 3.6 the corresponding solution is subordinate at $\infty$. However, the provided solution need not be real-valued, i.e. $u_\infty$ might not have the form $(z, \bar{z})$. But since the real and imaginary part of this solution cannot simultaneously be zero, one of the two provides us with a non-trivial and real-valued solution which is subordinate at $\infty$. This completes the proof of Theorem 4.8. □

4.4. **Proof of Theorem 1.3.** With the results of the previous subsections in place we are ready to prove our main theorem concerning the random model, that is Theorem 1.3. However, in addition to the ODE results proved above we need to recall two results both based on a spectral averaging argument.

While we have not found these precise statements in the literature they can be proved by following the arguments in [42, §11–12]. The arguments in [42] are formulated under the assumption that the operator subject to the rank-one perturbation is bounded from below and the associated spectral measure decays sufficiently to have a well-defined Borel transform. However, for what we are interested in both of these assumptions can be removed without substantially changing the arguments (one needs to work with general Nevanlinna functions instead of Borel transforms, but this causes no substantial issues). Furthermore, there is an a priori cyclicity assumption in many of the relevant statements in [42] which need not be valid in our setting. Specifically, to utilize the full power of rank-one perturbation theory requires that $H = \{(L^\omega_{F,\lambda} - z)^{-1}\delta : z \in \mathbb{C} \setminus \mathbb{R}\}$ is a total set in $L^2(\mathbb{R})$. The subspace $H$ is reducing for $L^\omega_{F,\lambda}$ and by the generalised resolvent identity the operators for different values of $g_0$ coincide on $H^\perp$. Without, this totality assumption the spectral averaging arguments only yield information concerning the spectrum of the restriction of $L^\omega_{F,\lambda}$ to $H$.

In the following lemmas we write $L_0$ for the differential expression

$$L_0 = -\frac{d^2}{dx^2} - Fx + \sum_{n \in \mathbb{Z}\setminus\{0\}} g_n \delta(x - n)$$

where the coupling constants $\{g_n\}_{n\neq 0}$ are such that $L_0$ is limit point at $\pm\infty$ (which is satisfied almost surely under the assumptions of our main result).

**Lemma 4.12.** *Fix an open interval $I \subseteq \mathbb{R}$. Consider $L_\alpha$ the family of Schrödinger operators defined by the differential expression $L_0 + \alpha\delta$ in $L^2(\mathbb{R})$. Assume that:*

*(1) For almost every $E \in I$ there exists a non-trivial solution of the equation $L_0 u(x) = Eu(x)$ which is in $L^2(\mathbb{R}_-)$.*
*(2) For almost every $E \in I$ there exists a non-trivial solution of $L_0 u(x) = Eu(x)$ on $\mathbb{R}_+$ which is in $L^2(\mathbb{R}_+)$.*

*(3) The set $\{(L_0-z)^{-1}\delta : z\in\mathbb{C}\setminus\mathbb{R}\}$ is a total set in $L^2(\mathbb{R})$.*

*Then $L_\alpha$ has only pure point spectrum in $I$ for a.e. $\alpha$.*

**Lemma 4.13.** *Fix an open interval $I\subseteq\mathbb{R}$. Consider $L_\alpha$ the family of Schrödinger operators defined by the differential expression $L_0+\alpha\delta$ in $L^2(\mathbb{R})$. Assume that:*

*(1) The Schrödinger $L_\theta^-$ operator in $L^2(\mathbb{R}_-)$ defined by the differential expression $L_0$ and boundary condition*
$$\cos(\theta)u(0)+\sin(\theta)u'(0)=0$$
*is such that $\sigma_{ac}(L_\theta^-)\cap I=\emptyset$ for all $\theta\in[0,\pi)$.*

*(2) For almost every $E\in I$ there exists a solution of $L_0u(x)=Eu(x)$ on $\mathbb{R}_+$ which is subordinate at $+\infty$ but which is not in $L^2(\mathbb{R}_+)$.*

*(3) The set $\{(L_0-z)^{-1}\delta : z\in\mathbb{C}\setminus\mathbb{R}\}$ is a total set in $L^2(\mathbb{R})$.*

*Then $L_\alpha$ has only singular continuous spectrum in $I$ for a.e. $\alpha$.*

Before turning to the proof of classifying the spectral nature of $L^\omega_{F,\lambda}$ we prove two preparatory results. The first results proves that almost surely $\sigma_{ess}(L^\omega_{F,\lambda})=\mathbb{R}$ while the second verifies that the third assumption of Lemmas 4.13 and 4.13 is almost surely valid.

**Proposition 4.14.** *Fix $F,\lambda>0$ and let $\{g_n(\omega)\}_{n\in\mathbb{Z}}$ satisfy the assumptions of Theorem 4.5. Then almost surely $\sigma_{ess}(L^\omega_{F,\lambda})=\mathbb{R}$.*

*Proof.* Fix a $E\in\mathbb{R}$ and let $\psi$ be the solution of (12) with $\psi(0)=0$ and $\lim_{\varepsilon\to0^+}\psi'(0+\varepsilon)=1$. Let $R$ be the Prüfer radius associated to $\psi$. By Theorem 4.5 and Lemma 3.6 almost surely
$$\int_{n-1}^n|\psi(x)|^2\,dx=n^{-1/2+\lambda^2/(4F)+o(1)}\quad\text{and}\quad\int_{n-1}^n|\psi'(x)|^2\,dx=n^{1/2+\lambda^2/(4F)+o(1)}\,. \tag{34}$$

Let $\varphi\in C_0^\infty(0,1)$ with $0\le\varphi\le1$ and $\varphi\equiv1$ in $(1/4,3/4)$. Define $\varphi_k(x)=\varphi(\varepsilon_kx-1)$ for $\varepsilon_k>0$ with $\lim_{k\to\infty}\varepsilon_k\to0$ to be determined below. Define $\psi_k=\|\varphi_k\psi\|^{-1}_{L^2(\mathbb{R})}\varphi_k\psi$ so that $\|\psi_k\|_{L^2(\mathbb{R})}=1$ and $\psi_k\rightharpoonup0$. Since $\varphi\in C^\infty(\mathbb{R})$ it is easily checked that $\psi_k\in D(L^\omega_{F,\lambda})$ and, for $x\in\mathbb{R}\setminus\mathbb{Z}$,
$$L^\omega_{F,\lambda}(\psi_k(x))-E\psi_k(x)=-\frac{\varphi_k''(x)\psi(x)+2\varphi_k'(x)\psi'(x)}{\|\varphi_k\psi\|_{L^2(\mathbb{R})}}\,.$$

Thus (34) implies, for any $0<\delta<1/2$ and since $\varepsilon_k\to0$,
$$\begin{aligned}\int_0^\infty|(L^\omega_{F,\lambda}-E)\psi_k|^2\,dx&\le\|\varphi_k\psi\|^{-2}_{L^2(\mathbb{R}_+)}\int_0^\infty\Big(|\varphi_k''|^2|\psi|^2+4|\varphi_k'|^2|\psi'|^2\Big)\,dx\\&\lesssim\|\varphi_k\psi\|^{-2}_{L^2(\mathbb{R}_+)}\int_{1/\varepsilon_k}^{2/\varepsilon_k}\Big(\varepsilon_k^4|\psi|^2+\varepsilon_k^2|\psi'|^2\Big)\,dx\\&\lesssim\varepsilon_k^{1-2\delta}(\varepsilon_k^3+1)\,.\end{aligned}$$

Since $\varepsilon_k \to 0$ and $\delta > 0$ is arbitrary, we conclude that $\psi_k$ forms a Weyl sequence at energy $E$ and therefore $E \in \sigma_{ess}(L^\omega_{F,\lambda})$. By repeating the argument for a countable dense set of $E$'s we conclude that almost surely $\sigma_{ess}(L^\omega_{F,\lambda}) = \mathbb{R}$. □

**Lemma 4.15.** *Fix $F, \lambda > 0$ and let $g_n = \{g_n(\omega)\}_{n\in\mathbb{Z}\setminus\{0\}}$ satisfy assumptions (i)–(v) of Theorem 1.3. Then almost surely $\{(L_0 - z)^{-1}\delta : z \in \mathbb{C} \setminus \mathbb{R}\}$ is a total set in $L^2(\mathbb{R})$.*

*Proof.* Let $H = \{(L_0 - \zeta)^{-1}\delta : \zeta \in \mathbb{C}\}$. As noted, $H$ is a reducing subspace for $L_0$ and on $H^\perp$ all the operators $L_\alpha = L_0 + \alpha\delta$ coincide. In particular, $\sigma(L_\alpha|_{H^\perp})$ is independent of $\alpha$. We aim to prove that almost surely $\sigma(L_0|_{H^\perp}) = \emptyset$ which is impossible unless $H^\perp$ is trivial, i.e. $H$ is a total set in $L^2(\mathbb{R})$.

By Lemma 2.2, Theorem 4.8, and Gilbert–Pearson subordinacy theory, almost surely $\sigma_{ac}(L_\alpha) = \emptyset$ for all $\alpha$. In particular, almost surely $\sigma_{ac}(L_0|_{H^\perp}) = \emptyset$.

By Gilbert–Pearson subordinacy theory an essential support for the singular part of the spectral measure of $L_\alpha$ is given by

$$\Sigma_\alpha = \{E \in \mathbb{R} : \exists \psi \not\equiv 0 \text{ solving } L_\alpha\psi = E\psi \text{ subordinate at } +\infty \text{ and } -\infty\}.$$

Since subordinate solutions (in either direction) are unique up to multiplication by constants, $E \in \Sigma_\alpha \cap \Sigma_{\alpha'}$ with $\alpha \neq \alpha'$ if and only if there exists a solution $\psi \not\equiv 0$ subordinate in both directions with $\psi(0) = 0$. Conversely, if for some $\alpha$ and $E \in \Sigma_\alpha$ the corresponding solution $\psi$ vanishes at 0, then $E \in \cap_{\alpha\in\mathbb{R}}\Sigma_\alpha$. Write the spectral measure of $L_\alpha$ as $\mu_\alpha = \mu^H_\alpha + \mu^\perp$, where $\mu^H_\alpha$ is the spectral measure of $L_\alpha|_H$ and $\mu^\perp$ that of $L_\alpha|_{H^\perp}$. Note that $\mu^\perp$ is independent of $\alpha$. By the before mentioned subordinacy result, any essential support of $\mu^\perp$ is up to $\mu^\perp$-negligible sets contained in $\Sigma_\alpha$ for all $\alpha$. We conclude that an essential support of $\mu^\perp$ is contained in

$$\begin{aligned}\Sigma^\perp &:= \cap_{\alpha\in\mathbb{R}}\Sigma_\alpha \\ &= \{E \in \mathbb{R} : \exists \psi \not\equiv 0 \text{ solving } L_\alpha\psi = E\psi \text{ subordinate at } +\infty \text{ and } -\infty \text{ with } \psi(0) = 0\}.\end{aligned}$$

Let $L^-$ denote the realisation of $L_0$ in $L^2(\mathbb{R}_-)$ with Dirichlet boundary condition at 0. By Lemma 2.2 (and its proof) the spectrum of $L^-$ is almost surely discrete. By Gilbert–Pearson subordinacy theory, if $E \notin \sigma(L^-)$ the solution of $L_0\psi = E\psi$ on $\mathbb{R}_-$ with $\psi(0) = 0, \psi'(0) = 1$ is not subordinate at $-\infty$. Consequently, it holds that $\Sigma^\perp \subseteq \sigma(L^-)$. (As an aside, we mention that at this point we could already conclude that $\sigma_{sc}(L_0|_{H^\perp}) = \emptyset$, since an essential support of a singular continuous measure cannot be contained in the discrete set $\sigma(L^-)$.)

Clearly, the spectrum of $L^-$ depends only on $\{g_n(\omega)\}_{n<0}$. Fix a typical realization of $\{g_n(\omega)\}_{n<0}$. Since the coupling constants are independent, this does not alter the probabilistic properties of $\{g_n(\omega)\}_{n\geq 1}$. We apply Theorem 4.5 and Lemma 3.6 for a fixed $E \in \sigma(L^-)$ and infer that the extension to $\mathbb{R}$ of the eigenfunction of $L^-$ corresponding to eigenvalue $E$ fails to be subordinate at $+\infty$ almost surely depending only on $\{g_n(\omega)\}_{n\geq 1}$. Since $\sigma(L^-)$ is a countable set and the intersection of countably many almost sure events is almost sure, this property holds simultaneously for all $E \in \sigma(L^-)$ almost surely depending only on $\{g_n(\omega)\}_{n\geq 1}$. We conclude that almost surely $\Sigma^\perp = \emptyset$, implying that the spectral measure of $L_0|_{H^\perp}$ is trivial and we have arrive at the desired contradiction. □

*Proof of Theorem 1.3.* The claim that $\sigma_{ess}(L^{\omega}_{F,\lambda}) = \mathbb{R}$ almost surely is the content of Proposition 4.14. By translation we may without loss of generality assume that the distribution of $g_0$ is absolutely continuous with respect to Lebesgue measure. The idea is to apply Theorem 4.5 and Corollary 4.9 in conjunction with Lemmas 4.12 and 4.13. Since none of our statements depend on the precise value of $E$ we shall in what follows apply the spectral averaging results for all of the spectrum simultaneously, i.e. when applying Lemmas 4.12 and 4.13 we always set $I = \mathbb{R}$. Note that the almost sure validity of (3) in Lemmas 4.12 and 4.13 it the content of Lemma 4.15.

If $F < \lambda^2/2$ then Lemma 2.2, Corollary 4.9, and Lemma 3.6 imply that almost surely the restrictions of $L^{\omega}_{F,\lambda}$ to the positive and negative half-lines satisfy the assumptions of Lemma 4.12. Consequently almost surely and for Lebesgue almost every $g_0$ the spectrum of $L^{\omega}_{F,\lambda}$ is pure point. Since the distribution of $g_0$ was assumed absolutely continuous with respect to Lebesgue measure, this implies the almost sure statement in the theorem and completes the proof in the case $F < \lambda^2/2$.

If $F > \lambda^2/2$ then Lemma 2.2, Corollary 4.9, and Lemma 3.6 imply that almost surely the restrictions of $L^{\omega}_{F,\lambda}$ to the positive and negative half-line satisfy the first and second assumptions of Lemma 4.13. Consequently, almost surely and for Lebesgue almost every $g_0$ the operator $L^{\omega}_{F,\lambda}$ has only singular continuous spectrum. Again by the absolute continuity of the the distribution of $g_0$ this implies the claim of the theorem and therefore completes the proof. $\square$

As mentioned in the introduction the refinements of Gilbert–Pearson subordinacy theory developed in [23, 24, 13] imply almost sure continuity properties of the spectral measure in the case of singular continuous spectrum.

**Proposition 4.16.** *Fix $F, \lambda > 0$ satisfying $F > \frac{\lambda^2}{2}$. If $\{g_n(\omega)\}_{n\in\mathbb{Z}}$ satisfy the assumptions of Theorem 1.3 then almost surely the spectral measure of $L^{\omega}_{F,\lambda}$ vanishes on sets of Hausdorff dimension less than $1 - \frac{\lambda^2}{2F}$.*

Here spectral measure refers to the measure associated with the Titchmarsh–Weyl $m$-function of $L^{\omega}_{F,\lambda}$ through its integral representation as a Nevanlinna (or Herglotz) function. In view of Lemma 4.15 this measure is almost surely the spectral measure of the pair $(L^{\omega}_{F,\lambda}, \delta)$.

*Proof.* For almost every $E \in \mathbb{R}$, Theorem 4.5, Corollary 4.9, and Lemma 3.6 imply that there exists two linearly independent $\eta$ and $\psi$ solving (12) such that

$$\|\eta\|^2_{L^2(0,L)} = L^{1/2-\lambda^2/(4F)+o(1)} \quad \text{and} \quad \|\psi\|^2_{L^2(0,L)} = L^{1/2+\lambda^2/(4F)+o(1)} \quad \text{as } L \to \infty\,.$$

By Lemma 4.15 and rank-one perturbation theory one concludes that these asymptotics remain true almost surely for almost every $E$ with respect to the spectral measure of $L^{\omega}_{F,\lambda}$. The proof is completed by applying the continuum analogue of [13, Theorem 1] (see their Remark 2). $\square$

## 5. Refined bounds for exponential sums

To perform the corresponding analysis for the deterministic model, we shall need to replace the use of Martingale theory with a precise understanding of the exponential sums appearing in the argument. In particular, much of the difficulty in the analysis to follow will arise from the term linear in $U(j)$ which could almost immediately be discarded in the random case as a consequence of $\mathbb{E}[g_n] = 0$. To understand this sum we follow arguments of Perelman [36]. However, at this stage of the argument Perelman has oscillatory integrals in place of our exponential sums. So while the overall structure of the analysis is similar, there are different technical difficulties needed to be dealt with.

In the same spirit as stationary points of the phase function play an important role in the study of oscillatory integrals, the regions where the derivative of the phase is close to $2\pi\mathbb{Z}$ play a distinguished role in the corresponding exponential sums. In our case, $n \in \mathbb{Z}$ for which $\min_{\nu\in\mathbb{Z}} |\gamma'(n) - \pi\nu|$ is small will be of particular importance. As we shall see, most of the contribution to our sums come from such regions. Indeed, in the sum over a range of $n$ where $\gamma'(n)$ is far from an integer multiple of $\pi$ the cancellation effects are strong, and the contribution to the asymptotic behavior of the full sum comparatively small.

For $l$ sufficiently large define $X_l$ as the unique solution to the equation

$$\gamma'(X_l) = \pi l\,. \tag{35}$$

Define also $x_l$ by

$$x_l = \Big\lceil \frac{\pi^2}{F}\Big(l - \frac{1}{2}\Big)^2 \Big\rceil - \frac{1}{2}\,. \tag{36}$$

Note that the step functions appearing in our construction are defined in such a way that their value at $x_l$ coincides with that at $\frac{\pi^2}{F}(l - \frac{1}{2})^2$. For technical reasons it is somewhat convenient to ensure that the $x_l$ are half-integers.

By the asymptotics of $\gamma'$,

$$X_l = \frac{\pi^2}{F} l^2 + O(1)\,, \quad \text{and} \quad \gamma'(x_l) \;= \pi l - \frac{\pi}{2} + O(l^{-1})\,, \tag{37}$$

where the implicit constants can be taken uniform for $F, E$ in compact subsets of their respective domains. The importance of the $X_l$, which we shall refer to as resonant points, can be motivated in several manners. Above we arrived at the relevance of these points by their connection to regions with small cancellations in our exponential sums. However, they can be argued to play an important role for solutions of our generalized eigenvalue equation directly via what is known as the tilted band picture. Namely, one can write the eigenvalue equation as

$$H_\lambda^{\mathrm{KP}}\psi(x) = (E + Fx)\psi(x)\,,$$

where $H_\lambda^{\mathrm{KP}}$ is the Kronig–Penney operator, and, in the spirit of the adiabatic theorem, think of $E + Fx$ as an effective energy. For $x$ close to $X_l$ this effective energy falls exactly into the $l$-th spectral gap of $H_\lambda^{\mathrm{KP}}$.

While the definition of the points $X_l$ is uniquely determined by the phase, the precise definition of the $x_l$'s is not as important. Indeed, any choice of points $x_l'$ interlacing the $X_l$'s for which $\min_{\nu\in\mathbb{Z}} |\gamma'(x_l') - \pi\nu|$ is uniformly bounded away from zero would likely work just as well. However, the choice of $x_l$'s above is in a sense a natural one and their particular structure will lead to certain simplifications in formulas to come.

In what follows our aim is to understand to high precision the exponential sums analogous to those appearing in our treatment of the random model when the summation index $n$ ranges from $x_l$ to $x_{l+1}$ or some subset thereof. When we return in Section 6 to studying the solutions of (12), understanding these partial sums will allow us to describe how the Prüfer coordinates $R, \eta$ change when we move from $x_l$ to $x_{l+1}$, i.e. when we transition over the resonant point $X_l$.

The first bound we prove is in the same spirit as Theorem 4.3 but capturing the strong cancellations present in regions away from the resonant points $X_l$.

**Theorem 5.1.** *Fix $F > 0, E \in \mathbb{R}, \alpha \geq 0, \beta > 0, \sigma \in [1/2, 1]$, and $h \colon \mathbb{N} \to \mathbb{R}$ satisfying*

$$|h(n+1) - h(n)| \leq Cn^{-\beta} \quad \text{for all } n \in [X_l, X_{l+1}] .$$

*Then, for $l$ sufficiently large and all $X_l + Cl^\sigma \leq a < b \leq X_{l+1} - Cl^\sigma$,*

$$\Big| \sum_{a<n\leq b} \frac{e^{i(2\gamma(n)+h(n))}}{\gamma'(n)^\alpha} \Big| \lesssim l^{-\alpha-\sigma+1}(1 + l^{1-2\beta})$$

*The implicit constants are uniformly bounded for $E, F, \alpha, \beta, \sigma, C$ in compact subsets of their respective domains.*

The above bound captures the behavior of our exponential sums to a sufficient precision away from the resonant points $X_l$. However, the main contribution to the sums we are interested in comes from small neighbourhoods of $X_l$ and to carry out our analysis we need to understand this contribution in greater detail. This is accomplished by the next two theorems.

The first theorem provides an order-sharp bound for the sum over the range $(x_l, x_{l+1}]$.

**Theorem 5.2.** *Fix $F > 0, E \in \mathbb{R}, \alpha \geq 0, \beta > 0, \mu > 0$, and $h \colon \mathbb{N} \to \mathbb{R}$ satisfying*

$$|h(n+1) - h(n)| \leq Cn^{-\beta} \quad \text{for all } n \in [x_l, x_{l+1}] .$$

*Then, for $l$ sufficiently large and all $x_l \leq a < b \leq x_{l+1}$,*

$$\Big| \sum_{a<n\leq b} \frac{e^{i(\mu\gamma(n)+h(n))}}{\gamma'(n)^\alpha} \Big| \lesssim l^{-\alpha+1/2}(1 + l^{1-2\beta}) .$$

*The implicit constant is uniformly bounded for $E, F, \alpha, \beta, \mu, C$ in compact subsets of their respective domains.*

While the previous theorem is often good enough for our purposes when it comes to estimating error terms, we shall need more precise knowledge to analyse the leading order terms of our equations.

**Theorem 5.3.** *Fix $F > 0$ and $E \in \mathbb{R}$. Let $h \in C^7(0,\infty)$ satisfy $|\partial_x^j h(x)| \le Cx^{1/2-j}$ for $j = 0,\dots,7$. Then*

$$\sum_{x_l < n \le x_{l+1}} \frac{e^{i(2\gamma(n)+h(n))}}{\gamma'(n)} = \Big(\frac{2}{Fl}\Big)^{1/2} e^{2i\Gamma_h(l)} - \frac{e^{i(2\gamma(x_{l+1})+h(x_{l+1})-\pi/2-\pi(l+1))}}{2\pi(l+1)} + \frac{e^{i(2\gamma(x_l)+h(x_l)-\pi/2-\pi l)}}{2\pi l} + O(l^{-3/2})\,,$$

*with*

$$\Gamma_h(l) = -\frac{\pi^3 l^3}{3F} + \frac{\pi E}{F} l + \frac{5\pi}{8} + \frac{1}{2} h\Big(\frac{\pi^2}{F} l^2\Big)\,. \tag{38}$$

*Moreover, the constant in the error term can be chosen uniform for $F, E, C$ in compact subsets of their respective domains.*

The proofs of Theorems 5.1, 5.2, and 5.3 occupy what remains of this section. Since the proofs get rather technical and do not reveal anything crucial to our main results, we advise the reader who is more interested in the spectral theory to skip ahead to Section 6 where our focus returns to understanding the asymptotic behavior of generalized eigenfunctions.

### 5.1. **Proof of Theorems 5.1 and 5.2.**

The proofs of Theorems 5.1 and 5.2 will follow the same strategy as that of Theorem 4.3. Indeed the bounds in the theorems follow along the same line of reasoning. When we are close to the resonant points $X_l$ we get an order-sharp bound by applying Corollary 4.2. However, if we are far from the resonant points in order to capture more of the cancellations we instead apply the following classical inequality due to Kuzmin and Landau (see for instance [45, Lemma 4.19]).

**Lemma 5.4.** *Let $f \in C^1[a,b]$ be a real function that is convex or concave, and let $\min_{\nu\in\mathbb{Z}} |f'(x) - \nu| \ge \kappa > 0$ for all $x \in [a,b]$. Then*

$$\Big| \sum_{a<n\le b} e^{2\pi i f(n)} \Big| \lesssim \kappa^{-1}\,.$$

*Proof of Theorems 5.1 and 5.2.* Let $a$ and $b$ be the left resp. right endpoints of one of the intervals of summation in the theorems. By a summation by parts

$$\begin{aligned}\sum_{a<n\le b} \frac{e^{i(\mu\gamma(n)+h(n))}}{\gamma'(n)^\alpha} &= \sum_{a<n\le b-1} \Big[\frac{e^{ih(n)}}{\gamma'(n)^\alpha} - \frac{e^{ih(n+1)}}{\gamma'(n+1)^\alpha}\Big] \sum_{a<j\le n} e^{i\mu\gamma(j)} \\ &\quad + \frac{e^{ih(\lfloor b\rfloor)}}{\gamma'(\lfloor b\rfloor)^\alpha} \sum_{a<n\le b} e^{i\mu\gamma(n)}\,.\end{aligned} \tag{39}$$

By Lemma 3.5, for $l$ sufficiently large, $\gamma'(n)^{-\alpha} \lesssim l^{-\alpha}$ for $a < n \le b$. Similarly, by Lemma 3.5 and using the fundamental theorem of calculus as in the proof of Theorem 4.2, we can estimate

$$\begin{aligned}\Big|\frac{e^{ih(n)}}{\gamma'(n)^\alpha}-\frac{e^{ih(n+1)}}{\gamma'(n+1)^\alpha}\Big| &\le \Big|\frac{1}{\gamma'(n)^\alpha}-\frac{1}{\gamma'(n+1)^\alpha}\Big| \\ &\quad+\frac{}{\gamma'(n+1)^\alpha}|e^{ih(n)}-e^{ih(n+1)}| \\ &\lesssim l^{-\alpha-2}+l^{-\alpha}|h(n)-h(n+1)| \\ &\lesssim l^{-\alpha}(l^{-2}+l^{-2\beta})\,.\end{aligned}$$

Therefore from (39) we have the bound

$$\begin{aligned}\Big|\sum_{a<n\le b}\frac{e^{i(\mu\gamma(n)+h(n))}}{\gamma'(n)^\alpha}\Big| &\lesssim l^{-\alpha}(l^{-2}+l^{-2\beta})\sum_{a<n\le b-1}\Big|\sum_{a<j\le n}e^{i\mu\gamma(j)}\Big| \\ &\quad+l^{-\alpha}\Big|\sum_{a<n\le b}e^{i\mu\gamma(n)}\Big|\,.\end{aligned} \tag{40}$$

At this point the proofs of Theorems 5.2 and 5.1 diverge.

In the case of Theorem 5.2 we apply Corollary 4.2 to bound the exponential sums on the right-hand side of (40). Combined with the fact that $|b-a|\le|x_{l+1}-x_l|\lesssim l$ this completes the proof of Theorem 5.2.

To prove Theorem 5.1 we instead argue as follows. Recall that in this theorem $\mu=2$. By the definition of the $x_l$'s and the asymptotics of $\gamma'$ we have for any $y\in(x_l,x_{l+1}]$

$$\min_{\nu\in\mathbb{Z}}\Big|\frac{\gamma'(y)}{\pi}-\nu\Big|\ge\frac{1}{2}\Big|\frac{\gamma'(y)}{\pi}-l\Big|\,.$$

Moreover, for any $y\in(x_l,x_{l+1}]$ a Taylor expansion around $X_l$ implies that

$$\frac{\gamma'(y)}{\pi}-l=\frac{1}{\pi}\gamma''(\xi)(y-X_l)$$

for some $\xi\in(x_l,x_{l+1}]$. Since $\gamma''(x)\sim x^{-1/2}$ for $l$ large enough we have

$$\min_{\nu\in\mathbb{Z}}\Big|\frac{\gamma'(y)}{\pi}-\nu\Big|\gtrsim\frac{|y-X_l|}{l}\,. \tag{41}$$

By Lemma 3.5 $\gamma$ is convex for $l$ large enough. Therefore, Lemma 5.4 and (41) imply that, for $X_l+Cl^\sigma\le x<y\le X_{l+1}-Cl^\sigma$,

$$\Big|\sum_{x<n\le y}e^{i2\gamma(n)}\Big|\lesssim l^{1-\sigma}\,. \tag{42}$$

The proof of Theorem 5.1 is completed by using (42) to bound the exponential sums on the right-hand side of (40). □

5.2. **Proof of Theorem 5.3.** Our proof of Theorem 5.3 will combine two classical results: the Poisson summation formula and the method of stationary phase. The particular form of the Poisson summation formula that we shall use is the following which can be found in [44, Theorem 45].

**Lemma 5.5.** *For $f \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$ it holds that*

$$\sum_{n\in\mathbb{Z}} \tilde{f}(n) = \hat{f}(0) + \sum_{\nu=1}^{\infty}\bigl(\hat{f}(\nu) + \hat{f}(-\nu)\bigr)$$

*where*

$$\tilde{f}(x) = \lim_{\varepsilon\to 0}\frac{f(x+\varepsilon)+f(x-\varepsilon)}{2} \quad \text{and} \quad \hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}\,dx\,.$$

*Remark* 5.6. It should be emphasized that while the series in the right-hand side is absolutely convergent this is generally not the case for $\sum_{\nu\in\mathbb{Z}}\hat{f}(\nu)$. Indeed, to see this it suffices to consider $f = \mathbb{1}_{[a,b]}$ for suitably chosen $a, b \in \mathbb{R}$.

*Proof of Theorem 5.3.* We wish to apply the Poisson summation formula in Lemma 5.5 to the function

$$f(x) = \frac{e^{i(2\gamma(x)+h(x))}}{\gamma'(x)}\mathbb{1}_{(x_l,x_{l+1}]}(x)\,.$$

Since this function has compact support and is smooth away from the jumps at $x_l, x_{l+1}$ it satisfies the assumptions of Lemma 5.5. By construction $x_l, x_{l+1} \notin \mathbb{Z}$ so the function $f$ is continuous at the integers. Therefore, the lemma implies

$$\begin{aligned}\sum_{x_l<n\leq x_{l+1}}\frac{e^{i(2\gamma(n)+h(n))}}{\gamma'(n)} &= \int_{x_l}^{x_{l+1}}\frac{e^{i(2\gamma(x)+h(x))}}{\gamma'(x)}dx\\ &\quad+\sum_{\nu=1}^{\infty}\biggl[\int_{x_l}^{x_{l+1}}\frac{e^{i(2\gamma(x)+h(x)-2\pi\nu x)}}{\gamma'(x)}dx+\int_{x_l}^{x_{l+1}}\frac{e^{i(2\gamma(x)+h(x)+2\pi\nu x)}}{\gamma'(x)}dx\biggr].\end{aligned} \tag{43}$$

We wish to apply the method of stationary phase to find an asymptotic expansion of the integrals in the sum on the right-hand side.

By Lemma 3.5, the assumptions on $h$, and a Taylor expansion around $X_l$ one finds that, for all $x \in (X_{l-1}, X_{l+1})$,

$$\begin{aligned}2\gamma'(x)+h'(x) &= 2\pi l + h'(X_l) + (2\gamma''(X_l)+h''(X_l))(x-X_l) + O(l^{-1})\\ &= 2\pi l + 2\gamma''(X_l)(x-X_l) + O(l^{-1})\\ &= 2\pi l - \sqrt{F}\Bigl(\frac{\pi^2}{F}l^2 + O(1)\Bigr)^{-1/2}(x-X_l) + O(l^{-1})\\ &= 2\pi l - F\frac{x-X_l}{\pi l} + O(l^{-1})\,.\end{aligned} \tag{44}$$

Consequently, the only $\nu \in \mathbb{Z}$ for which the phase $2\gamma(x)+h(x)-2\pi\nu x$ has a stationary point in $[x_l, x_{l+1}]$ is $\nu = l$. Furthermore, by asymptotic convexity of the phase when $\nu = l$, and $l$ is sufficiently large, there exists a unique stationary point in $(x_l, x_{l+1})$, we denote it by $\tilde{X}_l$. By the Taylor expansion (44) it holds that

$$|X_l - \tilde{X}_l| \lesssim 1\,.$$

To apply the method of stationary phase we first rescale the integrals as follows. Let $x(y) = x_l + (x_{l+1} - x_l)y$ and define

$$\begin{aligned} u_l(y) &= \frac{l}{\gamma'(x(y))}\,,\\ \phi_{l,\nu}(y) &= 2\gamma(x(y)) + h(x(y)) - 2\pi\nu x(y)\,,\\ \omega_{l,\nu} &= \|\phi'_{l,\nu}\|_{L^\infty(0,1)}\,,\\ \Phi_{l,\nu}(y) &= \frac{\phi_{l,\nu}(y) - \phi_{l,\nu}(0)}{\omega_{l,\nu}}\,. \end{aligned}$$

Note that the normalization has been chosen so that

$$\Phi_{l,\nu}(0) = 0 \quad \text{for all } l, \nu\,.$$

By a change of variables we find

$$\int_{x_l}^{x_{l+1}} \frac{e^{i(2\gamma(x)+h(x)-2\pi\nu x)}}{\gamma'(x)}\,dx = (x_{l+1} - x_l)l^{-1}e^{i\phi_{l,\nu}(0)} \int_0^1 u_l(y)e^{i\omega_{l,\nu}\Phi_{l,\nu}(y)}\,dy\,.$$

Note that

$$\omega_{l,\nu} \gtrsim l|l - \nu + \delta_{l,\nu}| \gg 1\,,$$

where $\delta_{l,\nu}$ denotes the Kronecker delta.

By Lemma 3.5 it readily follows that

$$\|\partial_y^k u_l\|_{L^\infty(0,1)} \lesssim l^{-k} \quad \text{for } k = 0, \ldots, 6\,.$$

Similarly, for $k = 2, \ldots, 7$,

$$\|\partial_y^k \Phi_{l,\nu}\|_{L^\infty(0,1)} \lesssim \omega_{l,\nu}^{-1} l^{3-k} \lesssim |l - \nu + \delta_{l,\nu}|^{-1} l^{2-k}\,,$$

where the constants are independent of $l, \nu$. The definition of $\Phi_{l,\nu}$ and (44) ensures that for $l$ sufficiently large, $\nu \neq l$, and all $y \in [0, 1]$

$$|\Phi'_{l,\nu}(y)| \gtrsim 1\,,$$

in particular these phase functions have no stationary points in $[0, 1]$. For $\Phi_{l,l}$ we have a unique stationary point at $\tilde{y}_l = \frac{\tilde{X}_l - x_l}{x_{l+1} - x_l} = \frac{1}{2} + O(l^{-1})$. Moreover, for all $y \in [0, 1]$, by a Taylor expansion

$$\frac{|y - \tilde{y}_l|}{|\Phi'_{l,l}(y)|} = \frac{|y - \tilde{y}_l|}{|\Phi''_{l,l}(\tilde{y}_l)(y - \tilde{y}_l) + \Phi'''_{l,l}(z_l)(y - \tilde{y}_l)^2|} = \frac{1}{|\Phi''_{l,l}(\tilde{y}_l) + \Phi'''_{l,l}(z_l)(y - \tilde{y}_l)|} = \frac{1}{2} + O(l^{-1})\,,$$

where we use $\Phi''_{l,l}(\tilde{y}_l) = 2 + O(l^{-1})$, $|y - \tilde{y}_l| \lesssim 1$, and $\|\Phi'''_{l,l}\|_{L^\infty} \lesssim l^{-1}$. Indeed, by Lemma 3.5

$$\omega_{l,l} = (x_{l+1} - x_l)\max\{|2\gamma'(x_{l+1}) + h'(x_{l+1}) - 2\pi l|, |2\gamma'(x_l) + h'(x_l) - 2\pi l|\} = \frac{2\pi^3 l}{F} + O(1)\,,$$

and

$$\Phi''_{l,l}(\tilde y_l) = \omega_{l,l}^{-1}(x_{l+1}-x_l)^2(2\gamma''(\tilde X_l)+h''(\tilde X_l)) = 2 + O(l^{-1})\,.$$

We conclude that the phase functions $\Phi_{l,\nu}$ all belong to a bounded subset of $C^7(0,1)$ and the stationary point of $\Phi_{l,l}$ is uniformly non-degenerate in $l$. As such we can apply Lemmas A.1 and A.2 to compute the integrals with uniform error estimates.

By Lemma A.2 when $\nu = l$ and Lemma A.1 when $\nu \neq l$, we conclude that for $l$ sufficiently large

$$\begin{aligned}\int_0^1 u_l(y)e^{i\omega_{l,\nu}\Phi_{l,\nu}(y)}\,dy &= \delta_{l,\nu}\frac{(2\pi)^{1/2}e^{i(\omega_{l,l}\Phi_{l,l}(\tilde y_l)+\pi/4)}}{\omega_{l,l}^{1/2}\Phi''_{l,l}(\tilde y_l)^{1/2}}\sum_{j=0}^{1} l^{-j}\mathcal{L}_j u_l(\tilde y_l)\\ &\quad + \frac{iu_l(0)e^{i\omega_{l,\nu}\Phi_{l,\nu}(0)}}{\omega_{l,\nu}\Phi'_{l,\nu}(0)} - \frac{iu_l(1)e^{i\omega_{l,\nu}\Phi_{l,\nu}(1)}}{\omega_{l,\nu}\Phi'_{l,\nu}(1)} + O(\omega_{l,\nu}^{-2})\,,\end{aligned}$$

where $\mathcal{L}_j$ are as in the lemma and the implicit constant is independent of $l,\nu$. In the particular cases $j = 0, 1$,

$$\begin{aligned}&\mathcal{L}_0 v(\tilde y_l) = v(\tilde y_l)\,,\\ &\mathcal{L}_1 v(\tilde y_l) = -i\Big(\frac{v''(\tilde y_l)}{2\Phi''_{l,l}(\tilde y_l)} - \frac{4v'(\tilde y_l)\Phi'''_{l,l}(\tilde y_l)+v(\tilde y_l)\Phi''''_{l,l}(\tilde y_l)}{8\Phi''_{l,l}(\tilde y_l)^2} + \frac{5v(\tilde y_l)\Phi'''_{l,l}(\tilde y_l)^2}{24\Phi''_{l,l}(\tilde y_l)^3}\Big)\,.\end{aligned}$$

By the definitions of $u_l, \Phi_{l,\nu}$, Lemma 3.5 combined with the assumptions on $h$ imply that

$$\mathcal{L}_0 u_l(\tilde y_l) = \frac{l}{\gamma'(\tilde X_l)} = \frac{1}{\pi} + O(l^{-2}) \quad \text{and} \quad \mathcal{L}_1 u_l(\tilde y_l) = O(l^{-2})\,.$$

By inserting the above into (43) and using the definition of $\omega_{l,\nu}$ and $\Phi_{l,\nu}$ we have arrived at

$$\begin{aligned}&\sum_{x_l<n\le x_{l+1}}\frac{e^{i(2\gamma(n)+h(n))}}{\gamma'(n)}\\ &\quad= \Big(\frac{2}{\pi l^2(2\gamma''(\tilde X_l)+h''(\tilde X_l))}\Big)^{1/2}e^{i(2\gamma(\tilde X_l)+h(\tilde X_l)-2\pi l\tilde X_l+\pi/4)}\\ &\qquad+\frac{ie^{i(2\gamma(x_l)+h(x_l))}}{\gamma'(x_l)}\frac{1}{2\gamma'(x_l)+h'(x_l)} - \frac{ie^{i(2\gamma(x_{l+1})+h(x_{l+1}))}}{\gamma'(x_{l+1})}\frac{1}{2\gamma'(x_{l+1})+h'(x_{l+1})}\\ &\qquad+\frac{ie^{i(2\gamma(x_l)+h(x_l))}}{\gamma'(x_l)}\sum_{\nu=1}^\infty\Big[\frac{e^{-i2\pi\nu x_l}}{2\gamma'(x_l)+h'(x_l)-2\pi\nu}+\frac{e^{i2\pi\nu x_l}}{2\gamma'(x_l)+h'(x_l)+2\pi\nu}\Big]\\ &\qquad-\frac{ie^{i(2\gamma(x_{l+1})+h(x_{l+1}))}}{\gamma'(x_{l+1})}\sum_{\nu=1}^\infty\Big[\frac{e^{-i2\pi\nu x_{l+1}}}{2\gamma'(x_{l+1})+h'(x_{l+1})-2\pi\nu}+\frac{e^{i2\pi\nu x_{l+1}}}{2\gamma'(x_{l+1})+h'(x_{l+1})+2\pi\nu}\Big]\\ &\qquad+O(l^{-2})\,,\end{aligned}$$

where we used $\sum_{\nu\neq l}|\nu-l|^{-2}\lesssim 1$.

To simplify the boundary contributions we shall use the identity

$$\frac{1}{y}+\sum_{\nu=1}^{\infty}(-1)^{\nu}\Big[\frac{1}{y-\nu}+\frac{1}{y+\nu}\Big]=\frac{\pi}{\sin(\pi y)}\quad\text{for } y\notin\mathbb{Z}\,.$$

Recalling that the $x_l$'s are half-integers and using Lemma 3.5 to deduce

$$2\gamma'(x_l)+h'(x_l)-2\pi\nu=2\pi(l-\nu)-\pi+O(l^{-1})\notin 2\pi\mathbb{Z}$$

we conclude that

$$\begin{aligned}\sum_{x_l<n\le x_{l+1}}\frac{e^{i(2\gamma(n)+h(n))}}{\gamma'(n)}&=\Big(\frac{2}{\pi l^2(2\gamma''(\tilde X_l)+h''(\tilde X_l))}\Big)^{1/2}e^{i(2\gamma(\tilde X_l)+h(\tilde X_l)-2\pi l\tilde X_l+\pi/4)}\\&-\frac{e^{i(2\gamma(x_l)+h(x_l)-\pi/2)}}{2\gamma'(x_l)\sin(\gamma'(x_l)+h'(x_l)/2)}\\&+\frac{e^{i(2\gamma(x_{l+1})+h(x_{l+1})-\pi/2)}}{2\gamma'(x_{l+1})\sin(\gamma'(x_{l+1})+h'(x_{l+1})/2)}+O(l^{-2})\end{aligned}$$

What remains is to simplify the terms by using Taylor expansion and Lemma 3.5.

By Lemma 3.5, the assumptions on $h$, and since $\tilde X_l=\frac{\pi^2}{F}l^2+O(1)$ we find

$$\frac{1}{2\gamma''(\tilde X_l)+h''(\tilde X_l)}=\frac{\pi l}{F}+O(l^{-1})\,.$$

Similarly, Lemma 3.5, the assumptions on $h$, and $x_l=\frac{\pi^2}{F}(l-1/2)^2+O(1)$ imply

$$\frac{1}{\gamma'(x_l)\sin(\gamma'(x_l)+h'(x_l)/2)}=-\frac{(-1)^l}{\pi l}+O(l^{-2})\,.$$

Finally, for the effective phase in the main term the expansions $2\gamma'\big(\frac{\pi^2}{F}l^2\big)=2\pi l+O(l^{-1})$ and $\tilde X_l-\frac{\pi^2}{F}l^2=O(1)$ imply that

$$\begin{aligned}2\gamma(\tilde X_l)+h(\tilde X_l)-2\pi l\tilde X_l&=2\gamma\Big(\frac{\pi^2}{F}l^2\Big)+h\Big(\frac{\pi^2}{F}l^2\Big)-\frac{2\pi^3}{F}l^3\\&\quad+\Big[2\gamma'\Big(\frac{\pi^2}{F}l^2\Big)+h'\Big(\frac{\pi^2}{F}l^2\Big)-2\pi l\Big]\Big(\tilde X_l-\frac{\pi^2}{F}l^2\Big)+O(l^{-1})\\&=-\frac{2\pi^3l^3}{3F}+\frac{2\pi E}{F}l+\pi+h\Big(\frac{\pi^2}{F}l^2\Big)+O(l^{-1})\,.\end{aligned}$$

Consequently,

$$e^{i(2\gamma(\tilde X_l)+h(\tilde X_l)-2\pi l\tilde X_l+\pi/4)}=e^{2i\Gamma_h(l)}+O(l^{-1})\,,$$

with $\Gamma_h(l)$ as in the theorem. Combining the last three estimates concludes the proof of Theorem 5.3. □

*Remark* 5.7. Even though in the end the only non-negligible contribution comes from the $j=0$ term in the stationary phase expansion we need to compute the expansion to second order for the error term to be sufficiently precise. Indeed, directly applying Lemma A.2 with $k = 1$ instead of 2 would only yield the error estimate

$$l^{-1}\|u_l\|_{C^2(0,1)} \sim l^{-1}$$

which is not good enough. We also emphasize that with the exception of needing higher order terms in the asymptotic expansion for $\gamma$ there is in theory nothing that prevents us from using the above argument to obtain asymptotic expansions to much higher precision.

## 6. Coarse-graining the Prüfer equations

We return to the analysing the asymptotic behavior of solutions of the equation

$$\begin{cases} -\psi''(x) - Fx\psi(x) = E\psi(x) & \text{in } \mathbb{R}\setminus\mathbb{Z}\,, \\ J\psi(n) = 0 & \text{for } n \in \mathbb{Z}\,, \\ J\psi'(n) = \lambda\psi(n) & \text{for } n \in \mathbb{Z}\,. \end{cases} \tag{45}$$

for fixed $F > 0, E \in \mathbb{R}$, and $\lambda \in \mathbb{R}$.

As in the case of the random model, the proof of our main results concerning the deterministic operator relies on understanding the Prüfer equations of Lemma 3.1. The main difference between the two models is that we can no longer use martingale techniques. In particular, our main challenge lies in understanding the behavior of

$$\sum_{n=1}^{N} U(n)\sin(2\theta(n)),$$

which could be shown to be sub-leading when the $U(n)$ where independent random variables with mean zero. As we shall see, this is not the case in our current setting.

In this section our main goal is to show that we can coarse-grain our system by first understanding the partial sums with $n \in (x_l, x_{l+1}]$. This is where the results of the previous section come into play. Before we can apply the expansion of Theorem 5.3 we need to remove the dependence of the phase on the Prüfer angle $\eta$. However, $\eta$ varies too much over the interval $[x_l, x_{l+1}]$ to simply pull it out of the sum. To remedy this it is convenient to work with a modification of $\eta$ obtained by extracting its leading-order behavior. To that end we define

$$\tilde\eta(x) = \eta(x) + \lambda\sqrt{\frac{\lceil x\rceil}{F}} \quad \text{and} \quad \tilde\gamma(x) = \gamma(x) - \lambda\sqrt{\frac{x}{F}}\,, \tag{46}$$

so that $\theta(n) = \gamma(n) + \eta(n) = \tilde\gamma(n) + \tilde\eta(n)$ for $n \in \mathbb{Z}$. Note that this correction of the phase $\gamma$ is sufficiently small to be covered by the assumptions in Theorems 5.2, 5.1, and 5.3.

The main result of this section is to provide asymptotic equations for how the (modified) Prüfer variables $R$ and $\tilde\eta$ evolve along the points $\{x_l\}_{l\in\mathbb{N}}$, that is when we pass from one of the regions where we have strong cancellations to the next and in the process pick up the contribution from the region close to the resonant point $X_l$. The precise statement we shall prove is the content of Theorem 6.1.

**Theorem 6.1.** *Fix $F > 0, E \in \mathbb{R}$, and $\lambda \in \mathbb{R}$. Then the (modified) Prüfer coordinates $R, \tilde{\eta}$ corresponding to a real-valued solution of* (45) *satisfy*

$$\log\Big(\frac{R(x_{l+1})}{R(x_l)}\Big) = \frac{\lambda}{\sqrt{2Fl}}\sin(2\Gamma(l) + 2\tilde{\eta}(x_l)) + \frac{\lambda^2}{4Fl}\big(1 + \cos(4\Gamma(l) + 4\tilde{\eta}(x_l))\big) + \frac{(-1)^{l+1}\lambda}{4\pi(l+1)}\cos(2\theta(x_{l+1})) - \frac{(-1)^l\lambda}{4\pi l}\cos(2\theta(x_l)) + O(l^{-5/4}) .$$

*and*

$$\tilde{\eta}(x_{l+1}) - \tilde{\eta}(x_l) = \frac{\lambda}{\sqrt{2Fl}}\cos(2\Gamma(l) + 2\tilde{\eta}(x_l)) - \frac{\lambda^2}{4Fl}\sin(4\Gamma(l) + 4\tilde{\eta}(x_l)) + \frac{\lambda^2}{4}\Im\Big[\sum_{x_l < n \le x_{l+1}}\sum_{n < j \le x_{l+1}}\frac{e^{2i(\tilde{\gamma}(n) - \tilde{\gamma}(j))}}{\gamma'(n)\gamma'(j)}\Big] - \frac{(-1)^{l+1}\lambda}{4\pi(l+1)}\sin(2\theta(x_{l+1})) + \frac{(-1)^l\lambda}{4\pi l}\sin(2\theta(x_l)) + O(l^{-5/4})$$

*where*

$$\Gamma(l) = -\frac{\pi^3 l^3}{3F} + \frac{\pi l}{F}(E - \lambda) + \frac{5\pi}{8} .$$

*Remark* 6.2. A couple of remarks:

(1) We emphasize that Theorem 6.1 holds without the assumption $F \in \pi^2\mathbb{Q}_+$. In fact, this will be the case with all results proved in the current section. The rationality assumption enters only once we wish to understand the behavior on even larger scales than that determined by the resonant points (see Section 7).
(2) We note that $\Gamma$ in Theorem 6.1 is $\Gamma_h$ from Theorem 5.3 with $h(x) = -2\lambda\sqrt{x/F}$.
(3) While it is not obvious from the equations in the theorem, the leading terms consist almost entirely of the contribution coming from a small region around the point $X_l$.

Before we move on to proving Theorem 6.1 we justify the introduction of the modified Prüfer angle $\tilde{\eta}$ which was chosen in such a manner that it varies little on the scale of the intervals $[x_l, x_{l+1}]$ (which is not the case for $\eta$). Simultaneously, we prove that understanding the asymptotic behavior of $R$ can be reduced to understanding $R$ evaluated at $\{x_l\}_{l\in\mathbb{N}}$. Without this knowledge Theorem 6.1 tells us very little about the asymptotic behavior of $R$. Indeed, a priori it could be that the values of $R$ along the sequence $\{x_l\}_{l\in\mathbb{N}}$ do not capture its general behavior. To show that this is not the case we need to prove that also $R$ does not vary too much over the intervals $[x_l, x_{l+1}]$.

**Theorem 6.3.** *Fix $F > 0, E \in \mathbb{R}$, $\lambda \in \mathbb{R}$, and let $R, \tilde{\eta}$ be the (modified) Prüfer coordinates associated to a real-valued solution of* (45)*. Then*

$$\tilde{\eta}(n+1) - \tilde{\eta}(n) = \frac{U(n)}{2}\cos(2\theta(n)) + \frac{U(n)^2}{4}\Big(\sin(2\theta(n)) - \frac{1}{2}\sin(4\theta(n))\Big) + O(n^{-3/2}) .$$

*Moreover, for $x_l \le x \le x_{l+1}$,*

$$\Big|\log\Big(\frac{R(x)}{R(x_l)}\Big)\Big| \lesssim l^{-1/2} \quad \textit{and} \quad |\tilde\eta(x) - \tilde\eta(x_l)| \lesssim l^{-1/2}\,.$$

*Proof of Theorem 6.3.* From Lemma 3.5 it follows that

$$\frac{1}{\gamma'(n)} = 2\sqrt{\frac{n+1}{F}} - 2\sqrt{\frac{n}{F}} + O(n^{-3/2})\,.$$

Therefore, by the definition of $\tilde\eta$,

$$\begin{aligned}\tilde\eta(n+1) - \tilde\eta(n) &= \eta(n+1) + \lambda\sqrt{\frac{n+1}{F}} - \eta(n) - \lambda\sqrt{\frac{n}{F}}\\ &= \eta(n+1) - \eta(n) + \frac{\lambda}{2\gamma'(n)} + O(n^{-3/2})\,.\end{aligned}$$

Consequently, the equation for $\eta$ in Lemma 3.1 implies

$$\begin{aligned}\tilde\eta(n+1) - \tilde\eta(n) = \frac{U(n)}{2}\cos(2\theta(n)) &+ \frac{U(n)^2}{4}\sin(2\theta(n))\\ &- \frac{U(n)^2}{8}\sin(4\theta(n)) + O(n^{-3/2})\,.\end{aligned}$$

This proves the first part of the lemma.

By Lemma 3.5, $U(n) = \frac{\lambda}{\sqrt{Fn}}(1+O(n^{-1}))$. Therefore, equation (21) of Lemma 3.1 yields

$$\log\Big(\frac{R(x)}{R(x_l)}\Big) = \sum_{x_l<n<x}\log\Big(\frac{R(n+1)}{R(n)}\Big) = \frac{\lambda}{2}\sum_{x_l<n<x}\frac{\sin(2\theta(n))}{\gamma'(n)} + O(l^{-1})\,.$$

The desired bound is obtained by an application of Theorem 5.2 with $\alpha = 1, \mu = 2, \beta = 1/2$, and $h(n) = 2\eta(n)$ which, by (23), satisfies the assumptions of the theorem. Similarly,

$$\begin{aligned}|\tilde\eta(x) - \tilde\eta(x_l)| &= \Big|\sum_{x_l<n<x}(\tilde\eta(n+1) - \tilde\eta(n))\Big|\\ &= \Big|\sum_{x_l<n<x}\Big[\frac{U(n)}{2}\cos(2\theta(n)) + O(n^{-1})\Big]\Big| \lesssim l^{-1/2}\,.\end{aligned}$$

This completes the proof of Theorem 6.3. □

6.1. **Proof of Theorem 6.1.** We shall split the proof of Theorem 6.1 into a number of lemmas. A key role in the proof of Theorem 6.1 is played by the following result.

**Lemma 6.4.** *Fix $F > 0, E \in \mathbb{R}$, and $\lambda \in \mathbb{R}$. Let $\psi$ be a real-valued solution of* (45) *and $\theta, \tilde{\eta}$ be the corresponding Prüfer variables, then*

$$\begin{aligned}\sum_{x_l < n \le x_{l+1}} U(n) e^{2i\theta(n)} = e^{2i\tilde{\eta}(x_l)} &\sum_{x_l < n \le x_{l+1}} U(n) e^{2i\tilde{\gamma}(n)} + \frac{i}{4}\Big| \sum_{x_l < n \le x_{l+1}} U(n) e^{2i\tilde{\gamma}(n)} \Big|^2 \\ &- \frac{i}{4} \sum_{x_l < n \le x_{l+1}} U(n)^2 + \frac{1}{2}\Im \Big[ \sum_{x_l < n \le x_{l+1}} \sum_{n < j \le x_{l+1}} U(n) U(j) e^{2i(\tilde{\gamma}(n) - \tilde{\gamma}(j))} \Big] \\ &+ \frac{i e^{4i\tilde{\eta}(x_l)}}{4} \Big[ \sum_{x_l < n \le x_{l+1}} U(n) e^{2i\tilde{\gamma}(n)} \Big]^2 + O(l^{-5/4}) \,.\end{aligned}$$

While at first glance the left-hand side of the identity in Lemma 6.4 might look simpler than the right, this is not the case. The crucial point is that the exponential sums on the right-hand side only depend on the explicit functions $\gamma, \tilde{\gamma}$ and not the unknown function $\tilde{\eta}$. This fact allows us to compute the sums to high precision. Indeed, apart from the double sum, this is what was done in Theorem 5.3. Later, in the proof of Theorem 6.1, we will see that by Theorem 5.3 the second and third terms in Lemma 6.4 cancel up to small remainders. While it might very well be possible to prove a precise asymptotic expansion also for the double sum in Lemma 6.4, for our purposes it will be sufficient to have the following rough bound.

**Lemma 6.5.** *For any $F > 0$ and $E \in \mathbb{R}$,*

$$\Big| \sum_{x_l < n \le x_{l+1}} \sum_{n < j \le x_{l+1}} \frac{e^{2i(\tilde{\gamma}(n) - \tilde{\gamma}(j))}}{\gamma'(n)\gamma'(j)} \Big| \lesssim l^{-3/4} \,.$$

*Remark* 6.6. While the bound in Lemma 6.5 is good enough for us, we do not believe it to be sharp. In fact, we believe the sum to be $\lesssim l^{-1-\varepsilon}$, for some $\varepsilon > 0$.

Since the proofs of Lemmas 6.4 and 6.5 are almost entirely technical exercises in summation by parts, we postpone them until after the proof that they indeed imply Theorem 6.1. However, before we give the proof of Theorem 6.1 we need the following simple result concerning the non-oscillatory sum on the right-hand side in Lemma 6.4.

**Lemma 6.7.** *For any $F > 0$ and $E \in \mathbb{R}$,*

$$\sum_{x_l < n \le x_{l+1}} \frac{1}{\gamma'(n)^2} = \frac{2}{Fl} + O(l^{-2})$$

*Proof of Lemma 6.7.* By Lemma 3.5 and since $x_l = \frac{\pi^2}{F}(l - 1/2)^2 + O(1)$,

$$\sum_{x_l < n \le x_{l+1}} \frac{1}{\gamma'(n)^2} = \sum_{x_l < n \le x_{l+1}} \frac{1}{Fn} + O(l^{-3}) \,.$$

With $\gamma_E$ denoting the Euler–Mascheroni constant we have

$$\sum_{j=1}^{N} \frac{1}{n} = \log(N) + \gamma_E + O(N^{-1})\,.$$

Therefore, since by definition $x_l \in \mathbb{N} + 1/2$ and the fact that $x_l = \frac{\pi^2}{F}(l-1/2)^2 + O(1)$,

$$\sum_{x_l < n \le x_{l+1}} \frac{1}{n} = \log\Bigl(\frac{x_{l+1} - 1/2}{x_l - 1/2}\Bigr) + O(x_l^{-1}) = \frac{2}{l} + O(l^{-2})\,.$$

This conclude the proof of the lemma. □

With these technical lemmas on hand we are ready to prove Theorem 6.1.

*Proof of Theorem 6.1.* Since $U(n) \sim n^{-1/2}$, equation (21) in Lemma 3.1 yields that

$$\begin{aligned}\log\Bigl(\frac{R(x_{l+1})}{R(x_l)}\Bigr) &= \sum_{x_l < n \le x_{l+1}} \log\Bigl(\frac{R(n+1)}{R(n)}\Bigr)\\ &= \sum_{x_l < n \le x_{l+1}} \biggl[\frac{U(n)}{2}\sin(2\theta(n)) + \frac{U(n)^2}{8}\\ &\qquad - \frac{U(n)^2}{8}\Bigl(2\cos(2\theta(n)) - \cos(4\theta(n))\Bigr) + O(|U(n)|^3)\biggr]\,.\end{aligned}$$

Applying Theorem 5.2, (23), Lemma 6.7, and using the asymptotic behavior of $\gamma'(x)$ and $x_l$, we deduce

$$\log\Bigl(\frac{R(x_{l+1})}{R(x_l)}\Bigr) = \frac{1}{2}\sum_{x_l < n \le x_{l+1}} U(n)\sin(2\theta(n)) + \frac{\lambda^2}{4Fl} + O(l^{-3/2})\,. \tag{47}$$

Similarly for $\tilde\eta$ we conclude from Theorem 6.3 and Theorem 5.2 combined with (23), that

$$\tilde\eta(x_{l+1}) - \tilde\eta(x_l) = \frac{1}{2}\sum_{x_l < n \le x_{l+1}} U(n)\cos(2\theta(n)) + O(l^{-3/2})\,. \tag{48}$$

Thus what needs to be understood is the exponential sum

$$\sum_{x_l < n \le x_{l+1}} U(n) e^{2i\theta(n)}\,.$$

By Lemma 6.4, this sum can be written as a combination of five terms plus an error. To the first, second, and third term in the right-hand side of Lemma 6.4 we apply Theorem 5.3 with $\Gamma = \Gamma_h$ for

$$h(x) = 2\tilde\gamma(x) - 2\gamma(x) = -2\lambda\sqrt{x/F}\,.$$

To the third term we apply Lemma 6.7. In this way, we obtain

$$\sum_{x_l<n\le x_{l+1}} U(n)e^{2i\theta(n)} = \lambda\left(\frac{2}{Fl}\right)^{1/2} e^{2i\Gamma(l)+2i\tilde{\eta}(x_l)} - \frac{\lambda}{2\pi(l+1)} e^{i(2\theta(x_{l+1})-\pi/2-\pi(l+1))} + \frac{\lambda}{2\pi l} e^{i(2\theta(x_l)-\pi/2-\pi l)} + \frac{\lambda^2}{2}\Im\Big[\sum_{x_l<n\le x_{l+1}}\sum_{n<j\le x_{l+1}} \frac{e^{2i(\tilde{\gamma}(n)-\tilde{\gamma}(j))}}{\gamma'(n)\gamma'(j)}\Big] + \frac{i\lambda^2}{2Fl} e^{4i\Gamma(l)+4i\tilde{\eta}(x_l)} + O(l^{-5/4})\,.$$

Here we also used Theorem 6.3 to estimate $e^{2i(\tilde{\eta}(x_l)-\tilde{\eta}(x_{l+1}))} = 1+O(l^{-1/2})$ to write $\tilde{\eta}(x_{l+1})$ instead of $\tilde{\eta}(x_l)$ in the phase of the third term. Taking the real and imaginary parts and combining with (47) and (48) completes the proof of Theorem 6.1. □

*Proof of Lemma 6.4.* By a summation by parts we can write the sum as

$$\sum_{x_l<n\le x_{l+1}} U(n)e^{2i\theta(n)} = e^{2i\tilde{\eta}(x_l)} \sum_{x_l<n\le x_{l+1}} U(n)e^{2i\tilde{\gamma}(n)} + \sum_{x_l<n\le x_{l+1}} \Big[e^{2i\tilde{\eta}(n+1)} - e^{2i\tilde{\eta}(n)}\Big] \sum_{n<j\le x_{l+1}} U(j)e^{2i\tilde{\gamma}(j)}\,.$$

By Theorem 6.3 and the expansion $e^{2ix}-1 = 2ix+O(x^2)$,

$$e^{2i\tilde{\eta}(n+1)} - e^{2i\tilde{\eta}(n)} = iU(n)\cos(2\theta(n))e^{2i\tilde{\eta}(n)} + O(n^{-1})\,.$$

Therefore, using Theorem 5.2 to estimate the contribution from the error term,

$$\begin{aligned}\sum_{x_l<n\le x_{l+1}} &\Big[e^{2i\tilde{\eta}(n+1)} - e^{2i\tilde{\eta}(n)}\Big] \sum_{n<j\le x_{l+1}} U(j)e^{2i\tilde{\gamma}(j)} \\ &= i\sum_{x_l<n\le x_{l+1}}\sum_{n<j\le x_{l+1}} U(n)U(j)\cos(2\theta(n))e^{2i(\tilde{\eta}(n)+\tilde{\gamma}(j))} + O(l^{-3/2}) \\ &= \frac{i}{2}\sum_{x_l<n\le x_{l+1}}\sum_{n<j\le x_{l+1}} U(n)U(j)e^{-2i(\tilde{\gamma}(n)-\tilde{\gamma}(j))} \\ &\quad + \frac{i}{2}\sum_{x_l<n\le x_{l+1}}\sum_{n<j\le x_{l+1}} U(n)U(j)e^{2i(\tilde{\gamma}(n)+\tilde{\gamma}(j)+2\tilde{\eta}(n))} + O(l^{-3/2})\,.\end{aligned}$$

Observing that

$$\Big|\sum_{x_l<n\le x_{l+1}} U(n)e^{2i\tilde{\gamma}(n)}\Big|^2 = 2\Re\Big[\sum_{x_l<n\le x_{l+1}}\sum_{n<j\le x_{l+1}} U(n)U(j)e^{-2i(\tilde{\gamma}(n)-\tilde{\gamma}(j))}\Big] + \sum_{x_l<n\le x_{l+1}} U(n)^2\,,$$

the real part of the first sum can be written as

$$\Re\Big[\sum_{x_l<n\le x_{l+1}}\sum_{n<j\le x_{l+1}} U(n)U(j)e^{-2i(\tilde\gamma(n)-\tilde\gamma(j))}\Big] = \frac{1}{2}\Big|\sum_{x_l<n\le x_{l+1}} U(n)e^{2i\tilde\gamma(n)}\Big|^2 - \frac{1}{2}\sum_{x_l<n\le x_{l+1}} U(n)^2\,.$$

Since

$$\Im\Big[\sum_{x_l<n\le x_{l+1}}\sum_{n<j\le x_{l+1}} U(n)U(j)e^{-2i(\tilde\gamma(n)-\tilde\gamma(j))}\Big] = -\Im\Big[\sum_{x_l<n\le x_{l+1}}\sum_{n<j\le x_{l+1}} U(n)U(j)e^{2i(\tilde\gamma(n)-\tilde\gamma(j))}\Big]$$

we have extracted the first four terms in the expression of the lemma.

What remains is to analyse the sum

$$\sum_{x_l<n\le x_{l+1}}\sum_{n<j\le x_{l+1}} U(n)U(j)e^{2i(\tilde\gamma(n)+\tilde\gamma(j)+2\tilde\eta(n))}\,. \tag{49}$$

Our aim is to use a similar symmetry as for the real part of the previous sum. However, in order to do this we must first show that up to an acceptable error we can replace $\tilde\eta(n)$ by $\tilde\eta(x_l)$. We claim that

$$\Big|\sum_{x_l<n\le x_{l+1}}\sum_{n<j\le x_{l+1}} U(n)U(j)e^{2i(\tilde\gamma(n)+\tilde\gamma(j))}\Big[e^{4i\tilde\eta(n)}-e^{4i\tilde\eta(x_l)}\Big]\Big| \lesssim l^{-5/4}\,. \tag{50}$$

Thus we can replace $\tilde\eta(n)$ by $\tilde\eta(x_l)$ in (49) at the cost of an error $O(l^{-5/4})$ (which is sufficient for our purposes). It is in the estimate (50) that the switch from the original Prüfer angle $\eta$ to the slowly varying $\tilde\eta$ comes into play.

Assuming that we have proved (50) one can write (49) as follows

$$\begin{aligned}
&\sum_{x_l<n\le x_{l+1}}\sum_{n<j\le x_{l+1}} U(n)U(j)e^{2i(\tilde\gamma(n)+\tilde\gamma(j)+2\tilde\eta(n))}\\
&\quad= e^{4i\tilde\eta(x_l)}\sum_{x_l<n\le x_{l+1}}\sum_{n<j\le x_{l+1}} U(n)U(j)e^{2i(\tilde\gamma(n)+\tilde\gamma(j))}+O(l^{-5/4})\\
&\quad= \frac{e^{4i\tilde\eta(x_l)}}{2}\Big[\sum_{x_l<n\le x_{l+1}} U(n)e^{2i\tilde\gamma(n)}\Big]^2 - \frac{e^{4i\tilde\eta(x_l)}}{2}\sum_{x_l<n\le x_{l+1}} U(n)^2e^{4i\tilde\gamma(n)}+O(l^{-5/4})\\
&\quad= \frac{e^{4i\tilde\eta(x_l)}}{2}\Big[\sum_{x_l<n\le x_{l+1}} U(n)e^{2i\tilde\gamma(n)}\Big]^2 + O(l^{-5/4})\,,
\end{aligned}$$

where in the final equality we used Theorem 5.2 to bound the contribution from the diagonal $n=j$. This is the last term in the statement of the lemma. Thus the only thing remaining to complete the proof of Lemma 6.4 is a proof of (50).

To prove (50) we split the sum into several regions depending on the distance of $n, j$ to the resonant point $X_l$. For $\sigma \in [1/2, 1]$ to be determined and some $c > 0$, we write

$$\sum_{x_l < n \le x_{l+1}} \sum_{n < j \le x_{l+1}} U(n)U(j)e^{2i(\tilde\gamma(n)+\tilde\gamma(j))}\Big[e^{4i\tilde\eta(n)} - e^{4i\tilde\eta(x_l)}\Big] = S_1 - S_2 + S_3 + S_4\,,$$

with

$$S_1 = \sum_{x_l < n \le X_l - cl^\sigma} \sum_{x_l < j \le x_{l+1}} U(n)U(j)e^{2i(\tilde\gamma(n)+\tilde\gamma(j))}\Big[e^{4i\tilde\eta(n)} - e^{4i\tilde\eta(x_l)}\Big]\,, \tag{51}$$

$$S_2 = \sum_{x_l < n \le X_l - cl^\sigma} \sum_{x_l < j \le n} U(n)U(j)e^{2i(\tilde\gamma(n)+\tilde\gamma(j))}\Big[e^{4i\tilde\eta(n)} - e^{4i\tilde\eta(x_l)}\Big]\,, \tag{52}$$

$$S_3 = \sum_{X_l - cl^\sigma < n \le X_l + cl^\sigma} \sum_{n < j \le x_{l+1}} U(n)U(j)e^{2i(\tilde\gamma(n)+\tilde\gamma(j))}\Big[e^{4i\tilde\eta(n)} - e^{4i\tilde\eta(x_l)}\Big]\,, \tag{53}$$

$$S_4 = \sum_{X_l + cl^\sigma < n \le x_{l+1}} \sum_{n < j \le x_{l+1}} U(n)U(j)e^{2i(\tilde\gamma(n)+\tilde\gamma(j))}\Big[e^{4i\tilde\eta(n)} - e^{4i\tilde\eta(x_l)}\Big]\,. \tag{54}$$

We shall bound each of the sums (51)-(54) by using Theorems 5.1 and 5.2 combined with the bounds $|U(n)| \lesssim l^{-1}$ and $|e^{4i\tilde\eta(n)} - e^{4i\tilde\eta(x_l)}| \lesssim |\tilde\eta(n) - \tilde\eta(x_l)| \lesssim l^{-1/2}$ (by Theorem 6.3).

For $S_1$ the key is that the double sum decouples so that Theorem 5.2 followed by Theorem 5.1 yields

$$\begin{aligned}
|S_1| &= \Bigg|\sum_{x_l < n \le X_l - cl^\sigma} \sum_{x_l < j \le x_{l+1}} U(n)U(j)e^{2i(\tilde\gamma(n)+\tilde\gamma(j))}\Big[e^{4i\tilde\eta(n)} - e^{4i\tilde\eta(x_l)}\Big]\Bigg| \\
&= \Bigg|\sum_{x_l < j \le x_{l+1}} U(j)e^{2i\tilde\gamma(j)}\Bigg|\Bigg|\sum_{x_l < n \le X_l - cl^\sigma} U(n)e^{2i\tilde\gamma(n)}\Big[e^{4i\tilde\eta(n)} - e^{4i\tilde\eta(x_l)}\Big]\Bigg| \\
&\lesssim l^{-1/2}\Bigg|\sum_{x_l < n \le X_l - cl^\sigma} U(n)e^{2i\tilde\gamma(n)}\Big[e^{4i\tilde\eta(n)} - e^{4i\tilde\eta(x_l)}\Big]\Bigg| \\
&\lesssim l^{-1/2}\Bigg(\Bigg|\sum_{x_l < n \le X_l - cl^\sigma} U(n)e^{2i\tilde\gamma(n)+4i\tilde\eta(n)}\Bigg| + \Bigg|\sum_{x_l < n \le X_l - cl^\sigma} U(n)e^{2i\tilde\gamma(n)}\Bigg|\Bigg) \\
&\lesssim l^{-1/2-\sigma}\,.
\end{aligned}$$

The sum $S_2$ we can estimate brutally since we are away from the resonant point. By Theorem 5.1

$$\begin{aligned}
|S_2| &= \Bigg|\sum_{x_l < n \le X_l - cl^\sigma} \sum_{x_l < j \le n} U(n)U(j)e^{2i(\tilde\gamma(n)+\tilde\gamma(j))}\Big[e^{4i\tilde\eta(n)} - e^{4i\tilde\eta(x_l)}\Big]\Bigg| \\
&\le \sum_{x_l < n \le X_l - cl^\sigma} \underbrace{|U(n)|}_{\lesssim l^{-1}} \underbrace{\Big|e^{4i\tilde\eta(n)} - e^{4i\tilde\eta(x_l)}\Big|}_{\lesssim l^{-1/2}} \underbrace{\Bigg|\sum_{x_l < j \le n} U(j)e^{2i\tilde\gamma(j)}\Bigg|}_{\lesssim l^{-\sigma}} \lesssim l^{-1/2-\sigma}\,.
\end{aligned}$$

In an analogous manner the same bound can be proved for $S_4$.

Finally, to bound $S_3$ instead of the precise bounds in Theorem 5.1 valid away from $X_l$ we use the cruder bound of Theorem 5.2 valid over the full range,

$$\Big|\sum_{X_l-cl^\sigma<n\le X_l+cl^\sigma}\ \sum_{n<j\le x_{l+1}} U(n)U(j)e^{2i(\tilde\gamma(n)+\tilde\gamma(j))}\Big[e^{4i\tilde\eta(n)}-e^{4i\tilde\eta(x_l)}\Big]\Big|$$
$$\le \sum_{X_l-cl^\sigma<n\le X_l+cl^\sigma} \underbrace{|U(n)|}_{\lesssim l^{-1}}\underbrace{\Big|e^{4i\tilde\eta(n)}-e^{4i\tilde\eta(x_l)}\Big|}_{\lesssim l^{-1/2}}\underbrace{\Big|\sum_{n<j\le x_{l+1}} U(j)e^{2i\tilde\gamma(j)}\Big|}_{\lesssim l^{-1/2}} \lesssim l^{\sigma-2}\,,$$

where we use that the number of terms in the $n$-sum is $\lesssim l^\sigma$.

We have arrived at the bound

$$\Big|\sum_{x_l<n\le x_{l+1}}\ \sum_{n<j\le x_{l+1}} U(n)U(j)e^{2i(\tilde\gamma(n)+\tilde\gamma(j))}\Big[e^{4i\tilde\eta(n)}-e^{4i\tilde\eta(x_l)}\Big]\Big| \lesssim l^{\sigma-2}+l^{-1/2-\sigma}\,.$$

Choosing $\sigma=3/4$ completes the proof of (50), and hence the proof of Lemma 6.4. □

The proof of Lemma 6.5 is similar to that of (50) above.

*Proof of Lemma 6.5.* For $\sigma\in[1/2,1]$ to be chosen and some $c>0$ we write the sum as

$$\sum_{x_l<n\le x_{l+1}}\ \sum_{n<j\le x_{l+1}} \frac{e^{2i(\tilde\gamma(n)-\tilde\gamma(j))}}{\gamma'(n)\gamma'(j)} = S_1-S_2+S_3+S_4\,,$$

where this time

$$S_1=\sum_{x_l<n\le X_l-cl^\sigma}\ \sum_{x_l<j\le x_{l+1}} \frac{e^{2i(\tilde\gamma(n)-\tilde\gamma(j))}}{\gamma'(n)\gamma'(j)}\,, \tag{55}$$

$$S_2=\sum_{x_l<n\le X_l-cl^\sigma}\ \sum_{x_l<j\le n} \frac{e^{2i(\tilde\gamma(n)-\tilde\gamma(j))}}{\gamma'(n)\gamma'(j)}\,, \tag{56}$$

$$S_3=\sum_{X_l-cl^\sigma<n\le X_l+cl^\sigma}\ \sum_{n<j\le x_{l+1}} \frac{e^{2i(\tilde\gamma(n)-\tilde\gamma(j))}}{\gamma'(n)\gamma'(j)}\,, \tag{57}$$

$$S_4=\sum_{X_l+cl^\sigma<n\le x_{l+1}}\ \sum_{n<j\le x_{l+1}} \frac{e^{2i(\tilde\gamma(n)-\tilde\gamma(j))}}{\gamma'(n)\gamma'(j)}\,. \tag{58}$$

Again we shall bound each of these terms by combined application of Theorems 5.1 and 5.2.

For $S_1$ Theorems 5.1 and 5.2 applied as before yields

$$|S_1|=\Big|\sum_{x_l<n\le X_l-cl^\sigma} \frac{e^{2i\tilde\gamma(n)}}{\gamma'(n)}\Big|\Big|\sum_{x_l<j\le x_{l+1}} \frac{e^{-2i\tilde\gamma(j)}}{\gamma'(j)}\Big| \lesssim l^{-1/2-\sigma}\,.$$

For $S_2$ we are again far from the resonant points and Theorem 5.1 implies

$$|S_2| = \Big|\sum_{x_l<n\le X_l-cl^\sigma}\sum_{x_l<j\le n}\frac{e^{2i(\tilde\gamma(n)-\tilde\gamma(j))}}{\gamma'(n)\gamma'(j)}\Big| \le \sum_{x_l<n\le X_l-cl^\sigma}\frac{1}{\gamma'(n)}\Big|\sum_{x_l<j\le n}\frac{e^{-2i\tilde\gamma(j))}}{\gamma'(j)}\Big| \lesssim l^{-\sigma}$$

Analogously the same bound holds for $S_4$.

For $S_3$ we use Theorem 5.2 to deduce

$$|S_3| \le \sum_{X_l-cl^\sigma<n\le X_l+cl^\sigma}\frac{1}{\gamma'(n)}\Big|\sum_{n<j\le x_{l+1}}\frac{e^{-2i\tilde\gamma(j)}}{\gamma'(j)}\Big| \lesssim l^{\sigma-3/2}\,.$$

Collecting the four bounds we have proved that

$$\Big|\sum_{x_l<n\le x_{l+1}}\sum_{n<j\le x_{l+1}}\frac{e^{2i(\tilde\gamma(n)-\tilde\gamma(j))}}{\gamma'(n)\gamma'(j)}\Big| \lesssim l^{-\sigma}+l^{\sigma-3/2}\,.$$

Choosing $\sigma = 3/4$ completes the proof Lemma 6.5. □

6.2. **Proof of Theorem 1.4.** We end this section by proving Theorem 1.4. Most of what is needed for the proof has been accomplished above. What remains is to make one final modification of our Prüfer variables to obtain the simplified equations stated in Theorem 1.4 and interpret what has been done in terms of the asymptotic representation of $\psi$ in (6).

*Proof of Theorem 1.4.* Let $\psi$ be real-valued solution of (45). By the definition of our Prüfer variables

$$\psi(x) = \frac{R(x)}{2i}\Big(e^{i(\tilde\eta(x)-\lambda\sqrt{\lceil x\rceil/F})}\zeta(x) + e^{-i(\tilde\eta(x)-\lambda\sqrt{\lceil x\rceil/F})}\bar\zeta(x)\Big)\,.$$

Our aim is to prove that, up to a relatively small error, we have a corresponding representation with $R, \tilde\eta$ replaced by functions which are constant on each of the intervals $(\frac{\pi^2}{F}(l-\frac12)^2, \frac{\pi^2}{F}(l+\frac12)^2]$.

By Theorem 6.3 we have, for any $x \in (\frac{\pi^2}{F}(l-\frac12)^2, \frac{\pi^2}{F}(l+\frac12)^2]$,

$$\Big|\log\Big(\frac{R(x)}{R(x_l)}\Big)\Big| \lesssim l^{-1/2} \quad\text{and}\quad |\tilde\eta(x)-\tilde\eta(x_l)| \lesssim l^{-1/2}\,.$$

Consequently

$$R(x) = R(x_l)(1+O(l^{-1/2})) \quad\text{and}\quad e^{i\tilde\eta(x)} = e^{i\tilde\eta(x_l)}(1+O(l^{-1/2})) \tag{59}$$

and

$$\psi(x) = \frac{R(x_l)}{2i}\Big(e^{i(\tilde\eta(x_l)-\lambda\sqrt{\lceil x\rceil/F})}\zeta(x)+e^{-i(\tilde\eta(x_l)-\lambda\sqrt{\lceil x\rceil/F})}\bar\zeta(x)\Big)+O(|\zeta(x)|R(x_l)l^{-1/2})\,. \tag{60}$$

While this accomplishes our main goal of finding the desired representation, by making slight modifications of $R(x_l), \tilde\eta(x_l)$ we can simplify the equations that the functions satisfy.

To this end define

$$\mathcal{R}(l) = R(x_l)e^{\frac{(-1)^{l+1}\lambda}{4\pi l}\cos(2\theta(x_l))} \quad\text{and}\quad \Lambda(l) = \tilde\eta(x_l)+\frac{(-1)^l\lambda}{4\pi l}\sin(2\theta(x_l))\,.$$

It is an immediate consequence of the definitions that

$$\mathcal{R}(l) = R(x_l)(1+O(l^{-1})) \quad \text{and} \quad \Lambda(l) = \tilde\eta(x_l) + O(l^{-1}) \tag{61}$$

so we can without changing the error estimate replace $R(x_l)$ by $\mathcal{R}(l)$ and $\tilde\eta(x_l)$ by $\Lambda(l)$ in (60). Moreover, from Theorem 6.1 we see that $\mathcal{R}, \Lambda$ satisfy

$$\begin{aligned}\log\Big(\frac{\mathcal{R}(l+1)}{\mathcal{R}(l)}\Big) &= \frac{\lambda}{\sqrt{2Fl}}\sin(2\Gamma(l)+2\tilde\eta(x_l)) + \frac{\lambda^2}{4Fl}\big(1+\cos(4\Gamma(l)+4\tilde\eta(x_l))\big) + O(l^{-5/4})\,,\\ \Lambda(l+1)-\Lambda(l) &= \frac{\lambda}{\sqrt{2Fl}}\cos(2\Gamma(l)+2\tilde\eta(x_l)) - \frac{\lambda^2}{4Fl}\sin(4\Gamma(l)+4\tilde\eta(x_l))\\ &\quad + \frac{\lambda^2}{4}\Im\Big[\sum_{x_l<n\le x_{l+1}}\sum_{n<j\le x_{l+1}}\frac{e^{2i(\tilde\gamma(n)-\tilde\gamma(j))}}{\gamma'(n)\gamma'(j)}\Big] + O(l^{-5/4})\,.\end{aligned}$$

Indeed, the choice of $\mathcal{R}, \Lambda$ was made precisely to absorb the terms in the equations of Theorem 6.1 which are no longer present.

Defining, as in Theorem 1.4, $\Theta(l) = \Gamma(l) + \Lambda(l)$, and combining the second estimate in (61) with Taylor expansions of the cos and sin terms we arrive at

$$\log\Big(\frac{\mathcal{R}(l+1)}{\mathcal{R}(l)}\Big) = \frac{\lambda}{\sqrt{2Fl}}\sin(2\Theta(l)) + \frac{\lambda^2}{4Fl}\big(1+\cos(4\Theta(l))\big) + O(l^{-5/4})\,, \tag{62}$$

and

$$\begin{aligned}\Lambda(l+1)-\Lambda(l) &= \frac{\lambda}{\sqrt{2Fl}}\cos(2\Theta(l)) - \frac{\lambda^2}{4Fl}\sin(4\Theta(l))\\ &\quad + \frac{\lambda^2}{4}\Im\Big[\sum_{x_l<n\le x_{l+1}}\sum_{n<j\le x_{l+1}}\frac{e^{2i(\tilde\gamma(n)-\tilde\gamma(j))}}{\gamma'(n)\gamma'(j)}\Big] + O(l^{-5/4})\,.\end{aligned} \tag{63}$$

Writing

$$\mathcal{S}(l) = \frac{1}{4}\Im\Big[\sum_{x_l<n\le x_{l+1}}\sum_{n<j\le x_{l+1}}\frac{e^{2i(\tilde\gamma(n)-\tilde\gamma(j))}}{\gamma'(n)\gamma'(j)}\Big]$$

these are the equations claimed in Theorem 1.4. Clearly $\mathcal{S}$ is independent of both $\mathcal{R}, \Lambda$, and $|\mathcal{S}(l)| \lesssim l^{-3/4}$ by Lemma 6.5.

Finally, by (59) and (61) it holds that

$$\sum_{x_l<n\le x_{l+1}}\frac{R(n)^2}{2\sqrt{Fn}} = \frac{\mathcal{R}(l)^2}{2\sqrt{F}}(1+O(l^{-1/2}))\sum_{x_l<n\le x_{l+1}}\frac{1}{\sqrt{n}} = \frac{\pi}{F}\mathcal{R}(l)^2(1+O(l^{-1/2}))\,.$$

Therefore, Lemma 3.6 implies

$$\int_{\frac{\pi^2}{F}(l-\frac12)^2}^{\frac{\pi^2}{F}(l+\frac12)^2}|\psi(x)|^2\,dx = \frac{\pi}{F}\mathcal{R}(l)^2(1+O(l^{-1/2})\,.$$

This completes the proof of Theorem 1.4. ∎

*Remark* 6.8. While substantial improvements of the error term in the representation (6) for $x$ in the full range $(\frac{\pi^2}{F}(l-\frac{1}{2})^2, \frac{\pi^2}{F}(l+\frac{1}{2})^2)$ are unlikely to be possible, it is possible to improve the error on large subsets of these intervals. Indeed, by using the bounds of Theorem 5.1 in place of Theorem 5.2 in the proof of Theorem 6.3 one finds that, for $x \in (X_{l-1}+Cl^\sigma, X_l - Cl^\sigma)$ with $\sigma \in [1/2, 1]$,

$$\Big|\log\Big(\frac{R(x)}{\mathcal{R}(l)}\Big)\Big| \lesssim l^{-\sigma} \quad \text{and} \quad |\tilde{\eta}(x) - \Lambda(l)| \lesssim l^{-\sigma} \, .$$

Plugging these refinements into the proof of Theorem 1.4, one obtains improved error estimates in (6) as soon as the distance from $x$ to the resonant points is large enough. Note that Buslaev [10] predicted a representation of the form (6) whenever $|X_l - x| \gtrsim l^\sigma$ with $\sigma = 1/2$ and Pozharskiĭ [38] proved such a representation in the case of a more regular periodic potential with $\sigma = 2/3$.

## 7. The deterministic model with rationality

We now specialize to the case when $F \in \pi^2\mathbb{Q}_+$ and as in Theorem 1.3 fix $p, q \in \mathbb{N}$ so that $\frac{\pi^2}{3F} = \frac{p}{q}$. While it is not strictly necessary, the statement we prove is strongest when $p, q$ are chosen so that $\gcd(p,q) = 1$.

### 7.1. A second coarse graining.

We begin by noting that since $q$ is fixed (and finite) the equation for $\mathcal{R}$ in Theorem 1.4 yields, for any $qk \le l \le q(k+1)$,

$$\Big|\log\Big(\frac{\mathcal{R}(l)}{\mathcal{R}(qk)}\Big)\Big| \lesssim k^{-1/2}. \tag{64}$$

Consequently, if $\lim_{k\to\infty}\mathcal{R}(qk)$ exists then $\lim_{l\to\infty}\mathcal{R}(l)$ exists, and by Theorem 6.3 so does $\lim_{n\to\infty} R(n)$. Moreover, all three limits coincide.

In a manner similar to that in the previous section we wish to compute asymptotic equations for $\mathcal{R}, \Lambda$ when we transition from $l = qk$ to $l = q(k+1)$. The idea of the proof is almost identical to what we have done before, but as we shall see the assumption $\pi^2 F \in \mathbb{Q}_+$ will lead to crucial simplifications in the effective model that arises.

**Theorem 7.1.** *Let $F = \frac{\pi^2 q}{3p}$ with $p, q \in \mathbb{N}$ and set*

$$\Omega(k) = 3p\frac{E-\lambda}{\pi}k + \frac{5\pi}{8} \quad \textit{and} \quad w(E,\lambda,q,p) = \sum_{j=0}^{q-1} e^{-2\pi i \frac{p}{q} j^3 + 6ip\frac{E-\lambda}{q\pi} j} \, .$$

*Let $\psi$ be a real-valued solution of* (45) *and $\mathcal{R}, \Lambda$ the associated Prüfer coordinates as in Theorem 1.4, then*

$$\log\Big(\frac{\mathcal{R}(q(k+1))}{\mathcal{R}(qk)}\Big) = \frac{\lambda}{\sqrt{2Fqk}}\Im\Big[e^{2i(\Omega(k)+\Lambda(qk))}w(E,\lambda,q,p)\Big] + \frac{\lambda^2}{4Fqk}|w(E,\lambda,q,p)|^2 + \frac{\lambda^2}{4Fqk}\Re\Big[e^{4i(\Omega(k)+\Lambda(qk))}w(E,\lambda,q,p)^2\Big] + O(k^{-5/4})$$

*and*

$$\Lambda(q(k+1)) - \Lambda(qk) = \frac{\lambda}{\sqrt{2Fqk}}\Re\Big[e^{2i(\Omega(k)+\Lambda(qk))}w(E,\lambda,q,p)\Big] + O(k^{-3/4})\,.$$

*Proof of Theorem 7.1.* By the equation for $\mathcal{R}$ in Theorem 1.4

$$\begin{aligned}\log\Big(\frac{\mathcal{R}(q(k+1))}{\mathcal{R}(qk)}\Big) &= \sum_{j=0}^{q-1}\log\Big(\frac{\mathcal{R}(qk+j+1)}{\mathcal{R}(qk+j)}\Big)\\ &= \frac{\lambda}{\sqrt{2F}}\sum_{j=0}^{q-1}\frac{\sin(2\Theta(qk+j))}{\sqrt{qk+j}} + \frac{\lambda^2}{4F}\sum_{j=0}^{q-1}\frac{1+\cos(4\Theta(qk+j))}{qk+j} + O(k^{-5/4})\,.\end{aligned} \tag{65}$$

We begin by analysing the sum of terms with decay $\sim k^{-1/2}$,

$$\sum_{j=0}^{q-1}\frac{\sin(2\Theta(qk+j))}{\sqrt{qk+j}} = \Im\Big[\sum_{j=0}^{q-1}\frac{e^{2i\Theta(qk+j))}}{\sqrt{qk+j}}\Big]\,.$$

By the equation for $\Lambda$ in Theorem 1.4 and since $q$ is fixed and finite we have for all $qk \leq l \leq q(k+1)$ the estimate

$$|\Lambda(qk) - \Lambda(l)| \lesssim k^{-1/2}\,. \tag{66}$$

Combining the equation for $\Lambda$ in Theorem 1.4, the estimate (66), and the expansion $e^x = 1 + x + O(x^2)$, we can write

$$\begin{aligned}\sum_{j=0}^{q-1}\frac{e^{2i\Theta(qk+j))}}{\sqrt{qk+j}} &= e^{2i\Lambda(qk)}\sum_{j=0}^{q-1}\frac{e^{2i\Gamma(qk+j)}}{\sqrt{qk+j}}e^{2i(\Lambda(qk+j)-\Lambda(qk))}\\ &= e^{2i\Lambda(qk)}\sum_{j=0}^{q-1}\frac{e^{2i\Gamma(qk+j)}}{\sqrt{qk+j}}\Big[1 + 2i(\Lambda(qk+j)-\Lambda(qk)) + O(k^{-1})\Big]\\ &= e^{2i\Lambda(qk)}\sum_{j=0}^{q-1}\frac{e^{2i\Gamma(qk+j)}}{\sqrt{qk+j}}\\ &\quad + i\lambda e^{2i\Lambda(qk)}\sqrt{\frac{2}{F}}\sum_{j=0}^{q-1}\frac{e^{2i\Gamma(qk+j)}}{\sqrt{qk+j}}\sum_{r=0}^{j-1}\frac{\cos(2\Theta(qk+r))}{\sqrt{qk+r}} + O(k^{-5/4})\,.\end{aligned} \tag{67}$$

Since $F = \frac{\pi^2 q}{3p}$ we have

$$\begin{aligned}\Gamma(qk) &= -\pi p q^2 k^3 + \frac{\pi qk}{F}(E-\lambda) + \frac{5\pi}{8}\,,\\ \Gamma(qk+j) - \Gamma(qk) &= -\frac{\pi p}{q}j^3 - 3\pi j^2 kp - 3\pi jk^2 pq + \frac{\pi j}{F}(E-\lambda)\end{aligned}$$

so that

$$e^{2i\Gamma(qk)} = e^{2i(\frac{\pi qk}{F}(E-\lambda)+\frac{5\pi}{8})} = e^{2i\Omega(k)}\,,$$
$$e^{2i(\Gamma(qk+j)-\Gamma(qk))} = e^{-2\pi i\frac{p}{q}j^3+2i\frac{\pi j}{F}(E-\lambda)} = e^{-2\pi i\frac{p}{q}j^3+6ip\frac{E-\lambda}{q\pi}j}\,.$$

Therefore, since

$$\frac{1}{\sqrt{qk+j}} - \frac{1}{\sqrt{qk}} = O(k^{-3/2})$$

we can rewrite the first sum in the right-hand side of (67) as

$$\begin{aligned}\sum_{j=0}^{q-1}\frac{e^{2i\Gamma(qk+j)}}{\sqrt{qk+j}} &= \frac{e^{2i\Gamma(qk)}}{\sqrt{qk}}\sum_{j=0}^{q-1}e^{2i(\Gamma(qk+j)-\Gamma(qk))} + O(k^{-3/2})\\ &= \frac{e^{2i\Omega(k)}}{\sqrt{qk}}w(E,\lambda,q,p) + O(k^{-3/2})\,.\end{aligned} \tag{68}$$

This is exactly the term with $\sim k^{-1/2}$ decay in the equation for $\mathcal{R}$ in Theorem 7.1.

Using the same idea as in the proof of Theorem 6.1 we can write parts of the double sum in the right-hand side of (67) as squares of sums. We argue as follows,

$$\begin{aligned}&\sum_{j=0}^{q-1}\frac{e^{2i\Gamma(qk+j)}}{\sqrt{qk+j}}\sum_{r=0}^{j-1}\frac{\cos(2\Theta(qk+r))}{\sqrt{qk+r}}\\ &\quad= \frac{1}{2}\sum_{j=0}^{q-1}\sum_{r=0}^{j-1}\frac{e^{2i(\Gamma(qk+j)+\Theta(qk+r))}}{\sqrt{qk+j}\sqrt{qk+r}} + \frac{1}{2}\sum_{j=0}^{q-1}\sum_{r=0}^{j-1}\frac{e^{2i(\Gamma(qk+j)-\Theta(qk+r))}}{\sqrt{qk+j}\sqrt{qk+r}}\,.\end{aligned}$$

Using (66) and the fact that, for $0 \le r, j \le q-1$,

$$\frac{1}{\sqrt{qk+r}\sqrt{qk+j}} - \frac{1}{qk} = O(k^{-2})$$

we find

$$\begin{aligned}\sum_{j=0}^{q-1}\frac{e^{2i\Gamma(qk+j)}}{\sqrt{qk+j}}\sum_{r=0}^{j-1}\frac{\cos(2\Theta(qk+r))}{\sqrt{qk+r}} &= \frac{e^{2i\Lambda(qk)}}{2qk}\sum_{j=0}^{q-1}\sum_{r=0}^{j-1}e^{2i(\Gamma(qk+j)+\Gamma(qk+r))}\\ &\quad+ \frac{e^{-2i\Lambda(qk)}}{2qk}\sum_{j=0}^{q-1}\sum_{r=0}^{j-1}e^{2i(\Gamma(qk+j)-\Gamma(qk+r))} + O(k^{-3/2})\,.\end{aligned} \tag{69}$$

We next observe that

$$\begin{aligned}\sum_{j=0}^{q-1}\sum_{r=0}^{j-1} e^{2i(\Gamma(qk+j)+\Gamma(qk+r))} &= \frac{1}{2}\Big(\sum_{j=0}^{q-1} e^{2i\Gamma(qk+j)}\Big)^2 - \frac{1}{2}\sum_{j=0}^{q-1} e^{4i\Gamma(qk+j)} \\ &= \frac{e^{4i\Omega(k)}}{2}\Big[\Big(\sum_{j=0}^{q-1} e^{2i(\Gamma(qk+j)-\Gamma(qk))}\Big)^2 - \sum_{j=0}^{q-1} e^{4i(\Gamma(qk+j)-\Gamma(qk))}\Big] \\ &= \frac{e^{4i\Omega(k)}}{2}\Big[w(E,\lambda,q,p)^2 - w(E,\lambda,q,2p)\Big]\,,\end{aligned} \tag{70}$$

and similarly

$$\begin{aligned}\sum_{j=0}^{q-1}\sum_{r=0}^{j-1} e^{2i(\Gamma(qk+j)-\Gamma(qk+r))} &= \frac{1}{2}\Big|\sum_{j=0}^{q-1} e^{2i\Gamma(qk+j)}\Big|^2 - \frac{q}{2} + i\Im\Big[\sum_{j=0}^{q-1}\sum_{r=0}^{j-1} e^{2i(\Gamma(qk+j)-\Gamma(qk+r))}\Big] \\ &= \frac{1}{2}\Big|\sum_{j=0}^{q-1} e^{2i(\Gamma(qk+j)-\Gamma(qk))}\Big|^2 - \frac{q}{2} + i\Im\Big[\sum_{j=0}^{q-1}\sum_{r=0}^{j-1} e^{2i(\Gamma(qk+j)-\Gamma(qk+r))}\Big] \\ &= \frac{1}{2}|w(E,\lambda,q,p)|^2 - \frac{q}{2} + i\Im\Big[\sum_{j=0}^{q-1}\sum_{r=0}^{j-1} e^{2i(\Gamma(qk+j)-\Gamma(qk+r))}\Big]\end{aligned} \tag{71}$$

Collecting (68)–(71) and inserting them into (67) we arrive at

$$\begin{aligned}&\sum_{j=0}^{q-1}\frac{e^{2i\Theta(qk+j))}}{\sqrt{qk+j}} \\ &\quad= \frac{1}{\sqrt{qk}}e^{2i(\Omega(k)+\Lambda(qk))}w(E,\lambda,q,p) + \frac{i\lambda e^{4i(\Omega(k)+\Lambda(qk))}}{\sqrt{8F}qk}\Big[w(E,\lambda,q,p)^2 - w(E,\lambda,q,2p)\Big] \\ &\qquad+ \frac{i\lambda}{\sqrt{8F}qk}\Bigg[|w(E,\lambda,q,p)|^2 - q + 2i\Im\Big[\sum_{j=0}^{q-1}\sum_{r=0}^{j-1} e^{2i(\Gamma(qk+j)-\Gamma(qk+r))}\Big]\Bigg] + O(k^{-5/4})\,.\end{aligned} \tag{72}$$

Therefore, for the first sum in (65) we have arrived at

$$\begin{aligned}&\frac{\lambda}{\sqrt{2F}}\sum_{j=0}^{q-1}\frac{\sin(2\Theta(qk+j))}{\sqrt{qk+j}} \\ &\quad= \frac{\lambda}{\sqrt{2Fqk}}\Im\Big[e^{2i(\Omega(k)+\Lambda(qk))}w(E,\lambda,q,p)\Big] + \frac{\lambda^2}{4Fqk}\Big[|w(E,\lambda,q,p)|^2 - q\Big] \\ &\qquad+ \frac{\lambda^2}{4Fqk}\Re\Big[e^{4i(\Omega(k)+\Lambda(qk))}\Big[w(E,\lambda,q,p)^2 - w(E,\lambda,q,2p)\Big]\Big] + O(k^{-5/4})\,.\end{aligned}$$

For the second sum in (65) the analogous calculation leads to

$$\frac{\lambda^2}{4F}\sum_{j=0}^{q-1}\frac{1+\cos(4\Theta(qk+j))}{qk+j}=\frac{\lambda^2}{4Fqk}\Big[q+\Re\Big[e^{4i(\Omega(k)+\Lambda(qk))}w(E,\lambda,q,2p)\Big]\Big]+O(k^{-3/2})\,.$$

We note that with the decay $\sim k^{-1}$ present in this sum the expansion is of sufficiently high precision already if we bound the double sum that arises in the analogue of (67) trivially.

Inserting all of the above into (65) proves the equation for $\mathcal{R}$ claimed in Theorem 7.1.

For $\Lambda$ the equation in Theorem 1.4 combined with (72) yields

$$\begin{aligned}\Lambda(q(k+1))-\Lambda(qk)&=\sum_{j=0}^{q-1}(\Lambda(qk+j+1)-\Lambda(qk+j))\\&=\frac{\lambda}{\sqrt{2F}}\sum_{j=0}^{q-1}\frac{\cos(2\Theta(qk+j))}{\sqrt{qk+j}}+O(k^{-3/4})\\&=\frac{\lambda}{\sqrt{2Fqk}}\Re\Big[e^{2i(\Omega(k)+\Lambda(qk))}w(E,\lambda,q,p)\Big]+O(k^{-3/4})\,.\end{aligned}$$

This completes the proof of Theorem 7.1. □

7.2. **Asymptotics of $R$.** The main goal of this section is to prove the following theorem.

**Theorem 7.2.** *Fix $\lambda\in\mathbb{R}$ and $F\in\pi^2\mathbb{Q}_+$ which write as $F=\frac{\pi^2 q}{3p}$ with $q,p\in\mathbb{N}$. Then, if $R$ is the Prüfer radius of a solution $\psi$ of (45) with $E\in\mathbb{R}\setminus\{\frac{\pi^2}{3p}m+\lambda : m\in\mathbb{Z}\}$, then*

$$\lim_{n\to\infty}\log(R(n))$$

*exists and is finite. Moreover, if $p,q,m$ are such that*

$$\sum_{j=0}^{q-1}e^{-2\pi i\frac{pj^3-jm}{q}}=0 \tag{73}$$

*then the same limit exists also when $E=\frac{\pi^2}{3p}m+\lambda$.*

The sum in the left-hand side of (73) is $w(\frac{\pi^2}{4p}m+\lambda,\lambda,q,p)$. In addition to appearing also in Perelman's argument [36], this sum appears in work of Fedotov and Klopp [18]. Specifically, Fedotov and Klopp where interested in resonances of the Schrödinger operator

$$-\frac{d^2}{dx^2}-Fx+2\cos(2\pi x)\quad\text{in } L^2(\mathbb{R})\,.$$

Of particular interest for us are their Lemmas 1 and 2 which state that, for co-prime $p,q\in\mathbb{N}$,

$$\sum_{m=0}^{q-1}\Big|\sum_{j=0}^{q-1}e^{-2\pi i\frac{pj^3-jm}{q}}\Big|^2=q^2\quad\text{and}\quad\#\Big\{0\le m<q:\sum_{j=0}^{q-1}e^{-2\pi i\frac{pj^3-jm}{q}}\neq 0\Big\}\gtrsim q^{2/3}\,.$$

In particular, the sum cannot vanish for all $m$. However, there are cases when the sum vanishes for all but one $m$ (for instance, when $q = 2, 3$, or $6$).

*Proof of Theorem 7.2.* We wish to sum the equations of Theorem 7.1 in order to conclude that $\log(\mathcal{R}(qk))$ has a limit as $k$ tends to infinity. By (64) and Theorem 6.3 this implies that the claim of the theorem. Our aim is to show that $\log(\mathcal{R}(qk))$ forms a Cauchy sequence in $k$. That is we want to prove that for any $\varepsilon > 0$ there exists $K_0$ such that $\left|\log\left(\frac{\mathcal{R}(qK_1)}{\mathcal{R}(qK_2)}\right)\right| < \varepsilon$ for all $K_0 \leq K_1 < K_2$.

By the equation for $\mathcal{R}$ in Theorem 7.1 and since $k^{-5/4}$ is summable,

$$\begin{aligned}
\log\Big(\frac{\mathcal{R}(qK_2)}{\mathcal{R}(qK_1)}\Big) &= \sum_{K_1 \leq k < K_2} \log\Big(\frac{\mathcal{R}(q(k+1))}{\mathcal{R}(qk)}\Big)\\
&= \frac{\lambda}{\sqrt{2Fq}} \Im\Bigg[w(E,\lambda,q,p) \sum_{K_1 \leq k < K_2} \frac{e^{2i(\Omega(k)+\Lambda(qk))}}{\sqrt{k}}\Bigg]\\
&\quad + \frac{\lambda^2}{4Fq}|w(E,\lambda,q,p)|^2 \sum_{K_1 \leq k < K_2} \frac{1}{k}\\
&\quad + \frac{\lambda^2}{4Fq} \Re\Bigg[w(E,\lambda,q,p)^2 \sum_{K_1 \leq k < K_2} \frac{e^{4i(\Omega(k)+\Lambda(qk))}}{k}\Bigg] + O(K_1^{-1/4})\,.
\end{aligned} \tag{74}$$

We begin by considering the first sum. By summation by parts

$$\begin{aligned}
\sum_{K_1 \leq k < K_2} \frac{e^{2i(\Omega(k)+\Lambda(qk))}}{\sqrt{k}} &= \sum_{K_1 \leq k < K_2 - 1} \Big(\frac{e^{2i\Lambda(q(k+1))}}{\sqrt{k+1}} - \frac{e^{2i\Lambda(qk)}}{\sqrt{k}}\Big) \sum_{k < j < K_2} e^{2i\Omega(j)}\\
&\quad + \frac{e^{2i\Lambda(qK_1)}}{\sqrt{K_1}} \sum_{K_1 \leq k < K_2} e^{2i\Omega(k)}\,.
\end{aligned}$$

By assumption $E \notin \left\{\frac{\pi^2}{3p}m + \lambda : m \in \mathbb{Z}\right\}$ and thus $\Omega'(k) = 3p\frac{E-\lambda}{\pi} \notin \pi\mathbb{Z}$. Therefore, the inner sums can be explicitly computed and remain uniformly bounded

$$\sum_{k<j<K_2} e^{2i\Omega(j)} = \frac{e^{2i\Omega(k+1)}}{1 - e^{2i\pi q\frac{E-\lambda}{F}}} - \frac{e^{2i\Omega(K_2)}}{1 - e^{2i\pi q\frac{E-\lambda}{F}}} = \frac{e^{2i\Omega(k)}}{e^{-2i\pi q\frac{E-\lambda}{F}} - 1} - \frac{e^{2i\Omega(K_2)}}{1 - e^{2i\pi q\frac{E-\lambda}{F}}}\,.$$

Consequently, we can estimate

$$\begin{aligned}
&\sum_{K_1\le k<K_2} \frac{e^{2i(\Omega(k)+\Lambda(qk))}}{\sqrt{k}} \\
&\quad= \sum_{K_1\le k<K_2-1}\left(\frac{e^{2i\Lambda(q(k+1))}}{\sqrt{k+1}} - \frac{e^{2i\Lambda(qk)}}{\sqrt{k}}\right)\left[\frac{e^{2i\Omega(k)}}{e^{-2i\pi q\frac{E-\lambda}{F}}-1} - \frac{e^{2i\Omega(K_2)}}{1-e^{2i\pi q\frac{E-\lambda}{F}}}\right] \\
&\qquad + \frac{e^{2i\Lambda(qK_1)}}{\sqrt{K_1}}\left[\frac{e^{2i\Omega(K_1)}}{1-e^{2i\pi q\frac{E-\lambda}{F}}} - \frac{e^{2i\Omega(K_2)}}{1-e^{2i\pi q\frac{E-\lambda}{F}}}\right] \\
&\quad= \frac{1}{e^{-2i\pi q\frac{E-\lambda}{F}}-1}\sum_{K_1\le k<K_2-1}\left(\frac{e^{2i\Lambda(q(k+1))}}{\sqrt{k+1}} - \frac{e^{2i\Lambda(qk)}}{\sqrt{k}}\right)e^{2i\Omega(k)} + O(K_1^{-1/2})\,,
\end{aligned}$$

where in the last step we used the fact that the terms in the sum which get multiplied by $e^{2i\Omega(K_2)}$ telescope.

To understand the remaining sum we again appeal to Theorem 7.1, however, this time the equation for $\Lambda$. Combined with

$$\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k}} = O(k^{-3/2})\,,$$

the Taylor expansion $e^x = 1 + x + O(x^2)$, and (66) we find

$$\begin{aligned}
&\sum_{K_1\le k<K_2-1}\left(\frac{e^{2i\Lambda(q(k+1))}}{\sqrt{k+1}} - \frac{e^{2i\Lambda(qk)}}{\sqrt{k}}\right)e^{2i\Omega(k)} \\
&\quad= i\lambda\sqrt{\frac{2}{Fq}}\sum_{K_1\le k<K_2-1}\frac{e^{2i(\Omega(k)+\Lambda(qk))}}{k}\Re\Big[e^{2i(\Omega(k)+\Lambda(qk))}w(E,\lambda,q,p)\Big] + O(K_1^{-1/4})\,.
\end{aligned}$$

Therefore, writing $w = w(E,\lambda,q,p)$,

$$\begin{aligned}
&\frac{\lambda}{\sqrt{2Fq}}\Im\left[w\sum_{K_1\le k<K_2}\frac{e^{2i(\Omega(k)+\Lambda(qk))}}{\sqrt{k}}\right] \\
&= \frac{\lambda^2}{Fq}\Im\left[\frac{i}{e^{-2i\pi q\frac{E-\lambda}{F}}-1}\sum_{K_1\le k<K_2-1}\frac{e^{2i(\Omega(k)+\Lambda(qk))}w\,\Re\big[e^{2i(\Omega(k)+\Lambda(qk))}w\big]}{k}\right] + O(K_1^{-1/4})\,.
\end{aligned}$$

Using the fact that for any $z \in \mathbb{C}$

$$z\Re(z) = \frac{|z|^2}{2} + \frac{z^2}{2}$$

and, for $\phi \in \mathbb{R}\setminus\mathbb{Z}$,

$$\frac{i}{e^{2\pi i\phi}-1} = -\frac{\cot(\pi\phi)}{2} - \frac{i}{2}$$

we furthermore deduce that

$$\begin{aligned}
&\frac{\lambda}{\sqrt{2Fq}}\Im\Big[w\sum_{K_1\le k<K_2}\frac{e^{2i(\Omega(k)+\Lambda(qk))}}{\sqrt{k}}\Big]\\
&=\frac{\lambda^2}{2Fq}\Im\Big[\frac{i}{e^{-2i\pi q\frac{E-\lambda}{F}}-1}\sum_{K_1\le k<K_2-1}\frac{|w|^2+w^2e^{4i(\Omega(k)+\Lambda(qk))}}{k}\Big]+O(K_1^{-1/4})\\
&=-\frac{\lambda^2}{4Fq}|w|^2\sum_{K_1\le k<K_2-1}\frac{1}{k}-\frac{\lambda^2}{4Fq}\Re\Big[w^2\sum_{K_1\le k<K_2-1}\frac{e^{4i(\Omega(k)+\Lambda(qk))}}{k}\Big]\\
&\quad+\frac{\lambda^2}{4Fq}\cot\Big(\pi q\frac{E-\lambda}{F}\Big)\Im\Big[w^2\sum_{K_1\le k<K_2-1}\frac{e^{4i(\Omega(k)+\Lambda(qk))}}{k}\Big]+O(K_1^{-1/4})\,.
\end{aligned}$$

Inserting, this back into (74) several of the terms cancel, yielding

$$\begin{aligned}
&\log\Big(\frac{\mathcal{R}(qK_2)}{\mathcal{R}(qK_1)}\Big)\\
&=\frac{\lambda^2}{4Fq}\cot\Big(\pi q\frac{E-\lambda}{F}\Big)\Im\Big[w(E,\lambda,q,p)^2\sum_{K_1\le k<K_2}\frac{e^{4i(\Omega(k)+\Lambda(qk))}}{k}\Big]+O(K_1^{-1/4})\,.
\end{aligned}$$

Repeating the summation by parts argument used in analysing the first sum yields, due to the additional decay, that

$$\sum_{K_1\le k<K_2}\frac{e^{4i(\Omega(k)+\Lambda(qk))}}{k}=O(K_1^{-1/2})\,.$$

Therefore, we finally conclude that $\{\log(\mathcal{R}(qk))\}_{k\ge1}$ forms a Cauchy sequence and hence converges. By (64) and Theorem 6.3 this implies that the full sequence $\{\log(R(n))\}_{n\ge1}$ converges which completes the proof of the first statement of Theorem 7.2.

Let $E$ be one of the exceptional energies, i.e. $E=\frac{\pi^2}{3p}m+\lambda$ for some $m\in\mathbb{Z}$. Then

$$w(E,\lambda,p,q)=\sum_{j=0}^{q-1}e^{-2\pi i\frac{pj^3-mj}{q}}\,.$$

Therefore, if this sum is 0, the equation (74) implies that $\log(\mathcal{R}(qk))$ forms a Cauchy sequence. The same argument as above implies that $\log(R(n))$ has a limit. This completes the proof of Theorem 7.2. □

7.3. **Proof of Theorem 1.1.** With Theorem 7.2 in hand deducing Theorems 1.1 is a direct consequence of Gilbert–Pearson subordinacy theory. Specifically we shall use the following result which is a special case of Proposition 7 in [11]:

**Proposition 7.3** ([11, Proposition 7]). *Fix an open interval $I\subseteq\mathbb{R}$. Let $L_{F,\lambda}$ be as in Theorem 1.1 and assume that for all $E\in I$ there is no subordinate solution of* (3). *Then $I\subseteq\sigma(L_{F,\lambda})$ and $\sigma(L_{F,\lambda})\cap I$ is purely absolutely continuous.*

*Proof of Theorem 1.1.* For $E \in \mathbb{R} \setminus \{\frac{\pi^2}{3p}m + \lambda : m \in \mathbb{Z}\}$ Theorem 7.2 implies that for any real-valued solution $\psi$ of (45) the associated Prüfer radius $R_\psi$ has a limit as $n \to \infty$ which is different from 0. We denote the limit by $R_\psi(\infty)$. By Theorem 1.4

$$\frac{1}{L}\int_0^L |\psi(x)|^2\, dx = \frac{\pi}{F} R_\psi(\infty)^2(1 + o(1)),$$

and therefore no solution of (45) is subordinate at $\infty$. Proposition 7.3 implies $\sigma_{ac}(L_{F,\lambda}) = \mathbb{R}$ and that away from the exceptional energies $E = \frac{\pi^2}{3p}m + \lambda$ the spectrum is purely absolutely continuous. Since a discrete set of points cannot support singular continuous spectrum this completes the proof of Theorem 1.1. □

## Appendix A. The one dimensional method of stationary phase with boundary contributions

In this appendix we recall the asymptotic expansion of an oscillatory integral in one-dimension given by the method of stationary phase. The following is certainly well-known to experts but we include since have been unable to find the statement with the desired uniformity with respect to the involved functions taking into account the contributions from boundary points.

We begin by first analysing the case when phase function has no stationary point in our interval. The proof we provide follows very closely that of [22, Theorem 7.7.1].

**Lemma A.1.** *Let $I = (a, b) \subset \mathbb{R}$ be a bounded interval. Let $u \in C^k(I), \phi \in C^{k+1}(I)$ and assume that $\phi$ is real-valued and $\inf_{x\in I} |\phi'(y)| \geq \delta > 0$. Then for $\omega > 0$*

$$\left|\int_I u(x)e^{i\omega\phi(x)}\, dx - \sum_{j=0}^{k-1}\Big(\frac{i}{\omega}\Big)^{j+1}\left[\frac{\mathcal{B}^j u(a)}{\phi'(a)}e^{i\omega\phi(a)} - \frac{\mathcal{B}^j u(b)}{\phi'(b)}e^{i\omega\phi(b)}\right]\right| \lesssim |I|\omega^{-k}\sum_{m=0}^{k}\delta^{m-2k}\|\partial_y^m u\|_{L^\infty(I)}$$

*where*

$$\mathcal{B}^0 v(y) = v(y), \quad \text{and} \quad \mathcal{B}^j v(y) = \partial_y((\phi')^{-1}\mathcal{B}^{j-1}v)(y).$$

*Moreover, the implicit constant is uniform for $\phi$ in bounded subsets of $C^{k+1}(I)$.*

By combining the more classical statement of the stationary phase expansion, with compactly supported smooth amplitude, and the previous lemma one readily deduces the following result.

**Lemma A.2.** *Let $I = (a, b) \subset \mathbb{R}$ be a bounded interval and $k \in \mathbb{N}$. Assume that $u \in C^{2k}(I), \phi \in C^{3k+1}(I)$ with $\phi$ real-valued and such that there exists a unique $x_0 \in I$ with $\operatorname{dist}(x, \partial I) > \kappa|I|$ such that $\phi'(x_0) = 0$, $\phi''(x_0) \neq 0$, and for all $x \in I$*

$$\frac{|x - x_0|}{|\phi'(x)|} \leq K. \tag{75}$$

*Then, for* $\omega > 0$,

$$\bigg| \int_I u(x) e^{i\omega\phi(x)}\,dx - \frac{(2\pi)^{1/2} e^{i\omega\phi(x_0)+i\pi/4}}{\omega^{1/2}\phi''(x_0)^{1/2}} \sum_{j=0}^{k-1} \omega^{-j}\mathcal{L}_j u(x_0) - \sum_{j=0}^{k-2}\Big(\frac{i}{\omega}\Big)^{j+1}\Big[\frac{\mathcal{B}^j u(a)}{\phi'(a)} e^{i\omega\phi(a)} - \frac{\mathcal{B}^j u(b)}{\phi'(b)} e^{i\omega\phi(b)}\Big]\bigg| \lesssim \omega^{-k} \sum_{j\le 2k} \|\partial_y^j u\|_{L^\infty(I)}\,.$$

*where*

$$\mathcal{L}_j v(y) = \sum_{\nu-\mu=j}\sum_{2\nu\ge 3\mu} \frac{1}{i^j 2^\nu \nu!\mu!\phi''(x_0)^\nu}(-i\partial_y)^{2\nu}(f_l^\mu v)(y)\,,$$

$$\mathcal{B}^0 v(y) = v(y)\,,\quad \mathcal{B}^j v(y) = \partial_y((\phi')^{-1}\mathcal{B}^{j-1}v)(y)\,,$$

*and*

$$f_l(y) = \phi(y) - \phi''(x_0)(y-x_0)^2/2\,.$$

*Moreover, the implicit constant is uniform for* $\phi$ *in bounded subsets of* $C^{3k+1}(I)$ *and* $\kappa, K, |I|$ *in compact subsets of* $(0,1/2)$, $(0,\infty)$, *and* $(0,\infty)$, *respectively,*

*Proof of Lemma A.1.* Set

$$\mathcal{I}_{\phi,\omega}[u] = \int_I u(x)e^{i\omega\phi(x)}\,dx$$

then for $\omega > 0$ an integration by parts yields

$$\begin{aligned}\mathcal{I}_{\phi,\omega}[u] &= -\frac{i}{\omega}\int_I \frac{u(x)}{\phi'(x)}(e^{i\omega\phi(x)})'\,dx\\ &= \frac{i}{\omega}\Big[\frac{u(a)}{\phi'(a)}e^{i\omega\phi(a)} - \frac{u(b)}{\phi'(b)}e^{i\omega\phi(b)}\Big] + \frac{i}{\omega}\mathcal{I}_{\phi,\omega}[\mathcal{B}^1 u]\,.\end{aligned}$$

Iterating, we arrive at

$$\mathcal{I}_{\phi,\omega}[u] = \sum_{j=0}^{k-1}\Big(\frac{i}{\omega}\Big)^{j+1}\Big[\frac{\mathcal{B}^j u(a)}{\phi'(a)}e^{i\omega\phi(a)} - \frac{\mathcal{B}^j u(b)}{\phi'(b)}e^{i\omega\phi(b)}\Big] + \Big(\frac{i}{\omega}\Big)^k \mathcal{I}_{\phi,\omega}[\mathcal{B}^k u]\,.$$

Since trivially

$$|\mathcal{I}_{\phi,\omega}[v]| \le |I|\|v\|_{L^\infty(I)}$$

it suffices to prove that, with a constant $C_\phi$ which can be taken uniform for $\phi$ in bounded subsets of $C^{k+1}(I)$,

$$\|\mathcal{B}^k u\|_{L^\infty(I)} \le C_\phi \sum_{m=0}^{k}\delta^{m-2k}\|\partial_y^m u\|_{L^\infty(I)}$$

We shall argue by induction to prove that for $r = 0,\dots,k$ and $j = 0,\dots,k-r$ we have

$$\|\partial_y^j\mathcal{B}^r u\|_{L^\infty} \le C_\phi\sum_{m=0}^{r+j}\delta^{-j+m-2r}\|\partial_y^m u\|_{L^\infty(I)},$$

which when $r = k, j = 0$ this is the desired bound.

For $r = 0$ we by definition have for $j = 0, \dots, k$

$$\|\partial_y^j \mathcal{B}^0 u\|_{L^\infty(I)} = \|\partial_y^j u\|_{L^\infty(I)}\,,$$

since $\delta$ is bounded this implies the claim when $r = 0$ and all $j \le k$.

For $r > 0$, $0 \le j \le k - r$ assume that the statement is known for all smaller $r$. By repeated use of the product rule

$$\begin{aligned}\partial_y^j \mathcal{B}^r v = \partial_y^{j+1}((\phi')^{-1}\mathcal{B}^{r-1} v) &= \sum_{m=0}^{j+1} \binom{j+1}{m} (\partial_y^m \mathcal{B}^{r-1} v)\partial_y^{j+1-m}(\phi')^{-1} \\ &\le C_\phi \sum_{m=0}^{j+1} \delta^{-j+m-2} |\partial_y^m \mathcal{B}^{r-1} v|\end{aligned}$$

where $C_\phi$ is uniformly bounded for $\phi$ in bounded subsets of $C^{k+1}(I)$ (or rather $C^{j+2}(I)$ but $j + 2 \le k + 1$ since $r > 0$). Therefore, by the induction hypothesis

$$\begin{aligned}\|\partial_y^j \mathcal{B}^r v\|_{L^\infty(I)} &\le C_\phi \sum_{m=0}^{j+1} \delta^{-j+m-2} \sum_{m'=0}^{r-1+m} \delta^{-m+m'-2(r-1)} \|\partial_y^{m'} u\|_{L^\infty(I)} \\ &= C_\phi \sum_{m=0}^{j+1} \sum_{m'=0}^{r-1+m} \delta^{-j+m'-2r} \|\partial_y^{m'} u\|_{L^\infty(I)} \\ &\le C_\phi \sum_{m'=0}^{r+j} \delta^{-j+m'-2r} \|\partial_y^{m'} u\|_{L^\infty(I)}\end{aligned}$$

which completes the proof of the lemma. □

*Proof of Lemma A.2.* By translating we may without loss of generality assume that $I = (-\frac{|I|}{2}, \frac{|I|}{2})$. Fix $\chi \in C_0^\infty(I)$ with $0 \le \chi \le 1$ and such that $\chi \equiv 1$ in $(-\frac{|I|(1-\kappa)}{2}, \frac{|I|(1-\kappa)}{2})$ which ensures that $\chi = 1$ in a neighbourhood of the stationary point of $\phi$.

Write

$$\begin{aligned}\int_I u(x)e^{i\omega\phi(x)}\,dx = &\int_{\mathbb{R}} \chi(x)u(x)e^{i\omega\phi(x)}\,dx \\ &+ \int_I (1-\chi(x))u(x)e^{i\omega\phi(x)}\,dx\,.\end{aligned}$$

By the non-degeneracy assumption of the phase (75) $|\phi'|$ is bounded from below uniformly in the support of $1-\chi$. As such we can apply Lemma A.1 to the latter integral by splitting it into two parts; one close to $\frac{|I|}{2}$ and one close to $-\frac{|I|}{2}$. The proof of the lemma is completed by applying the classical stationary phase method (with compactly supported amplitude) as stated in [22, Theorem 7.7.5] to the first integral. □

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