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Integral Equations and Operator Theory

Integral Operators on Fock–Sobolev Spaces via Multipliers on Gauss–Sobolev Spaces

Brett D. Wick and Shengkun Wu

Abstract. In this paper, we obtain an isometry between the Fock–Sobolev space and the Gauss–Sobolev space with the same order. As an application, we use multipliers on the Gauss–Sobolev space to characterize the boundedness of an integral operator on the Fock–Sobolev space.

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1. Introduction

In this paper, we study the Fock space and Fock–Sobolev spaces. First, we set some notations and recall the necessary objects. Let $\mathbb{C}^n$ be the complex n dimensional space and dv be the ordinary volume measure on $\mathbb{C}^n$. If $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ are points in $\mathbb{C}^n$, we write

$$z \cdot \overline{w} = \sum_{j=1}^{n} z_j \overline{w}_j, \quad |z| = (z \cdot \overline{z})^{1/2}.$$

Let

$$d\lambda(z) = \pi^{-n} e^{-|z|^2} dv(z)$$

be the Gaussian measure on $\mathbb{C}^n$. Denote by $L^2(\mathbb{C}^n, d\lambda)$ the set of square integrable functions with respect to $d\lambda$. The Fock space $F^2 := F^2(\mathbb{C}^n)$ consists of all entire functions f on the complex Euclidean space $\mathbb{C}^n$ such that

$$\|f\|_{F^2} = \left(\int_{\mathbb{C}^n} |f(z)|^2 d\lambda(z) \right)^{\frac{1}{2}} < \infty.$$

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F^2 is a closed subspace of the Hilbert space $L^2(\mathbb{C}^n, d\lambda)$ with inner product

$$\langle f, g\rangle_{F^2} = \int_{\mathbb{C}^n} f(z)\overline{g(z)}d\lambda(z).$$

The orthogonal projection $P : L^2(\mathbb{C}^n, d\lambda) \to F^2$ is given by

$$Pf(z) = \frac{1}{\pi^n}\int_{\mathbb{C}^n} f(w)K(z,w)e^{-|w|^2}dv(w),$$

where $K(z,w) = e^{z\cdot\overline{w}}$ is the reproducing kernel of F^2.

Next, we introduce Fock–Sobolev spaces. In what follows we use some standard multi-index notations. For an n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of non-negative integers, we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \dots \alpha_n!.$$

If $z = (z_1, \dots, z_n)$, then $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$ and $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$, where ∂_j denotes the partial differentiation with respect to the j-th component. For any positive integer m we consider the space $F^{2,m}$, called the Fock–Sobolev space, consisting of entire functions f on $\mathbb{C}^n$ such that

$$\|f\|_{F^{2,m}} := \sum_{|\alpha|\le m} \|\partial^\alpha f\|_{F^2} < \infty,$$

where $\|\cdot\|_{F^2}$ is the norm in F^2.

One reason that we need to study Fock–Sobolev spaces is to study Creation and annihilation operators. Creation and annihilation operators on the Fock space are important operators in quantum field theory. However, these two operators are unbounded operators on the Fock space. In general, it is important to understand the domain of the definition of an unbounded operator. In the particular case of creation and annihilation operators, because they involved differentiation, they are bounded from Fock–Sobolev spaces to the Fock space.

A useful tool for the analysis on the Fock space is the Bargmann transform which acts as an isometry between $L^2(\mathbb{R}^n)$ and the Fock space $F^2(\mathbb{C}^n)$. By connecting these spaces through the Bargmann transform, tools from one side can be transported to the other for analysis. A natural question arises: Is the Bargmann transform an isomorphism between the Fock–Sobolev space $F^{2,m}$ and the classical Sobolev space $W^{2,m}(\mathbb{R}^n)$? Addressing this question is one of the goals of this paper. To address this question we will explore the connection with Gauss–Sobolev spaces in Gaussian harmonic analysis.

Next, we introduce the Gauss Sobolev space. Let Gaussian measure $d\gamma$ on $\mathbb{R}^n$ be given by

$$d\gamma(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}e^{-\frac{|x|^2}{2}}dx.$$

For any positive integer m, the Gauss–Sobolev space $W^{2,m}(\gamma)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{W^{2,m}(\gamma)} = \sum_{0\le|\alpha|\le m}\left[\int_{\mathbb{R}^n}|\partial^\alpha f(x)|^2 d\gamma(x)\right]^{\frac{1}{2}}.$$

In [1], some properties of the Gauss–Sobolev space $W^{2,m}(\gamma)$ are discussed.

The study of Gaussian harmonic analysis arises from probability theory, quantum mechanics, and differential geometry. The Riesz transforms associated with the Gaussian measure, and a key operator in the theory of Gaussian harmonic analysis is the Ornstein–Uhlenbeck operator. From the Ornstein–Uhlenbeck operator, we can define the Gaussian Bessel potential which is important to our proof, see [8] and [5]. In Sect. 2, we will obtain an isometry between the Fock–Sobolev space $F^{2,m}$ and the Gauss–Sobolev space $W^{2,m}(\gamma)$. Because of the isometry between the Fock–Sobolev space $F^{2,m}$ and the Gauss–Sobolev space $W^{2,m}(\gamma)$, we will connect questions on these two spaces together.

As an application of the results we obtained, we will study a class of integral operators. For $\varphi \in F^2$, we consider the integral operator

$$S_\varphi f(z) = \int_{\mathbb{C}^n} f(w) e^{z\cdot\overline{w}} \varphi(z-\overline{w}) d\lambda(w),$$

for any $f \in F^{2,m}$. In [11], Zhu used the Bargmann transform to transfer some singular integral operators to S_φ and proposed an open question about the boundedness of S_φ. In [3], the authors gave a necessary and sufficient condition for S_φ to be bounded on F^2. In this paper, we consider the same problem in Fock–Sobolev spaces.

In Sect. 3, we will study multipliers on Gauss–Sobolev spaces. Then, in Sect. 4, we will obtain an isomorphism between multipliers on the Gauss–Sobolev space $W^{2,m}(\gamma)$ and the set of bounded S_φ on $F^{2,m}$. Then we use the conclusion on the Gauss–Sobolev space to characterize the boundedness of the integral operator on the Fock–Sobolev space and study other properties.

Multipliers on Sobolev spaces has been studied in [6]. In [4], the authors studied the Gaussian Capacity theory in the Gauss–Sobolev space with order 1. In this paper, we will use the idea in [6] and some operators in Gaussian Harmonic analysis to obtain the boundedness of multiplication operators between two Gauss–Sobolev spaces. Then we can apply conclusions in Gauss–Sobolev spaces to Fock–Sobolev spaces.

2. Gauss–Sobolev Spaces

In this section, we introduce the Gauss–Bargmann transform and show that the Gauss–Bargmann transform is an isometry that maps the Gauss–Sobolev space to the Fock–Sobolev space. On the other hand, we show that the Bargmann transform is not an isomorphism between the Fock–Sobolev space and the Sobolev space.

For any multi-index $\beta = (\beta_1, \dots, \beta_n)$, the Hermite function is defined to be

$$H_\beta(x) = \prod_{i=1}^{n} (-1)^{\beta_i} e^{x_i^2} \frac{\partial^{\beta_i}}{\partial x_i^{\beta_i}} \left(e^{-x_i^2}\right).$$

Then the normalized Hermite function with respect to the Gaussian measure is given by

$$h_\beta(x) = \frac{1}{(2^{|\beta|}\beta!)^{1/2}} H_\beta\left(\frac{x}{\sqrt{2}}\right).$$

That is to say

$$\int_{\mathbb{R}^n} h_\beta(x) h_\alpha(x) d\gamma(x) = \delta_{\alpha\beta},$$

where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha\beta} = 0$ if $\alpha \neq \beta$.

For any multi-index α, one easily computes that

$$\partial^\alpha h_\beta(x) = \begin{cases} \left(\prod_{j=1}^n \beta_j(\beta_j - 1)\dots(\beta_j - \alpha_j + 1)\right)^{1/2} h_{\beta-\alpha}(x), & \text{if } \alpha_j \leq \beta_j, \\ 0, & \text{otherwise.} \end{cases}$$

By [1, Proposition 1.5.4], we know that the linear space generated by Hermite polynomials is dense in $W^{2,m}(\gamma)$.

For $z \in \mathbb{C}$, let $e_\beta(z) = \frac{z^\beta}{\sqrt{\beta!}}$ be the basis of the Fock space, we know that

$$\partial^\alpha e_\beta(z) = \begin{cases} \left(\prod_{j=1}^n \beta_j(\beta_j - 1)\dots(\beta_j - \alpha_j + 1)\right)^{1/2} e_{\beta-\alpha}(z), & \text{if } \alpha_j \leq \beta_j, \\ 0, & \text{otherwise.} \end{cases}$$

From these two observations, we know that

$$\|e_\beta\|_{F^{2,m}} = \|h_\beta\|_{W^{2,m}(\gamma)}, \tag{2.1}$$

for any β. We define the Gauss–Bargmann transform G mapping the linear span of $\{h_\beta\}$ to $F^{2,m}$ such that

$$Gh_\beta = e_\beta.$$

Theorem 2.1. *Let m be a non-negative integer. The Gauss–Bargmann transform G is an isometry from the Gauss–Sobolev space $W^{2,m}(\gamma)$ to the Fock–Sobolev space $F^{2,m}$.*

Proof. We know that $\{e_\beta\}$ and $\{h_\beta\}$ are complete orthogonal sets in $F^{2,m}$ and $W^{2,m}(\gamma)$ respectively. The statement then follows from (2.1). □

We want to contrast this new transform with the more well-known Bargmann transform. Recall that the Bargmann transform is an isometry from $L^2(\mathbb{R}^n, dx)$ to F^2 such that

$$Bf(z) = \left(\frac{2}{\pi}\right)^{\frac{n}{4}} \int_{\mathbb{R}^n} f(x) e^{2x\cdot z - x^2 - \frac{z^2}{2}} dx,$$

where $z^2 = z_1^2 + z_2^2 + \cdots + z_n^2$, $x^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $x \cdot z = x_1 z_1 + x_2 z_2 + \cdots + x_n z_n$. Let

$$\widetilde{h}_\beta = \left(\frac{2}{\pi}\right)^{\frac{n}{4}} \frac{1}{\sqrt{2^\beta \beta!}} e^{-|x|^2} H_\beta(\sqrt{2}x),$$

we know that $B\widetilde{h}_\beta = e_\beta$, see [10, Theorem 6.8]. That is to say

$$\begin{aligned}
e_\beta &= B\widetilde{h}_\beta(z) \\
&= \left(\frac{2}{\pi}\right)^{\frac{n}{4}} \int_{\mathbb{R}^n} \left(\frac{2}{\pi}\right)^{\frac{n}{4}} \frac{1}{\sqrt{2^\beta \beta!}} e^{-|x|^2} H_\beta(\sqrt{2}x) e^{2x\cdot z - x^2 - \frac{z^2}{2}}\, dx \\
&= \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \frac{1}{\sqrt{2^\beta \beta!}} e^{\frac{-|x|^2}{4}} H_\beta\left(\frac{x}{\sqrt{2}}\right) e^{x\cdot z - \frac{x^2}{4} - \frac{z^2}{2}} \frac{1}{2^n} dx \\
&= \int_{\mathbb{R}^n} \frac{1}{\sqrt{2^\beta \beta!}} H_\beta\left(\frac{x}{\sqrt{2}}\right) e^{x\cdot z - \frac{z^2}{2}}\, d\gamma(x) \\
&= \int_{\mathbb{R}^n} h_\beta(x) e^{x\cdot z - \frac{z^2}{2}}\, d\gamma(x).
\end{aligned}$$

By the argument above, we know that for any $f \in W^{2,m}(\gamma)$, we have

$$Gf(z) = \int_{\mathbb{R}^n} f(x) e^{x\cdot z - \frac{z^2}{2}}\, d\gamma(x).$$

Similarly, for any $g \in F^{2,m}$, we have

$$G^{-1}g(x) = \int_{\mathbb{C}^n} g(z) e^{x\cdot \overline{z} - \frac{\overline{z}^2}{2}}\, d\lambda(z).$$

Next, we will discuss the relationship between the Gauss–Bargmann transform and the Bargmann transform. The key point will be that the order of smoothness matters for these operators.

Let $C_{\frac{1}{2}}$ be the composition operator from $L^2(\mathbb{R}^n, dx)$ to $L^2(\mathbb{R}^n, dx)$ such that $C_{\frac{1}{2}} f(x) = f(\frac{x}{2})$, for any $f \in L^2(\mathbb{R}^n, dx)$. Let $M_{(\frac{\pi}{2})^{\frac{n}{4}} \exp(\frac{|x|^2}{4})}$ be the multiplication operator from $L^2(\mathbb{R}^n, dx)$ to $L^2(\mathbb{R}^n, d\gamma)$ such that

$$M_{(\frac{\pi}{2})^{\frac{n}{4}} \exp(\frac{|x|^2}{4})} f(x) = \left(\frac{\pi}{2}\right)^{\frac{n}{4}} \exp\left(\frac{|x|^2}{4}\right) f(x).$$

For simplicity of notation, we denote $M_{(\frac{\pi}{2})^{\frac{n}{4}} \exp(\frac{|x|^2}{4})}$ with M.

Proposition 2.2. *The relationship between the Bargmann transform B and the Gauss–Bargmann transform G is given by*

$$B = GMC_{\frac{1}{2}}.$$

Proof. This is simply a computation from the definitions of the operators involved. For any $f \in L^2(\mathbb{R}^n, dx)$, we have

$$\begin{aligned}
GMC_{\frac{1}{2}} f(z) &= \int_{\mathbb{R}^n} \left(\frac{\pi}{2}\right)^{\frac{n}{4}} \exp\left(\frac{|x|^2}{4}\right) f\left(\frac{x}{2}\right) e^{x\cdot z - \frac{z^2}{2}}\, d\gamma(x) \\
&= \int_{\mathbb{R}^n} \left(\frac{\pi}{2}\right)^{\frac{n}{4}} \exp\left(\frac{|x|^2}{4}\right) f\left(\frac{x}{2}\right) e^{x\cdot z - \frac{z^2}{2}} \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2}}\, dx \\
&= Bf(z)
\end{aligned}$$

to complete the proof. □

To discuss the relationship between Sobolev spaces, Gauss–Sobolev spaces and Fock–Sobolev spaces, we need some basic facts about Fock–Sobolev spaces. The following theorem is a special case of [2, Theorem 11].

Theorem 2.3. *Suppose m is a non-negative integer, and f is an entire function on $\mathbb{C}^n$. Then $f \in F^{2,m}$ if and only if every function $z^\alpha f(z)$ is in F^2, where $|\alpha| = m$. Moreover, there is a positive constant c such that*

$$c^{-1}\big\||z|^m f\big\|_{F^2} \leq \|f\|_{F^{2,m}} \leq c\big\||z|^m f\big\|_{F^2}$$

for all $f \in F^{2,m}$.

Let A_j and A_j^* be two unbounded operators on F^2 such that $A_j f(z) = \partial_{z_j} f(z)$ and $A_j^* f(z) = z_j f(z)$. By [10, Lemma 6.13], we have

$$B\partial_{x_j}B^{-1} = A_j - A_j^* \text{ and } BM_{x_j}B^{-1} = \frac{1}{2}(A_j + A_j^*). \tag{2.2}$$

For any $f \in F^{2,m}$, by Theorem 2.3, we have

$$\|A_j^* f\|_{F^{2,m-1}} = \|z_j f\|_{F^{2,m-1}} \lesssim \||z|^{m-1} z_j f\|_{F^2} \lesssim \|f\|_{F^{2,m}}.$$

We obtain that A_j^* is bounded from $F^{2,m}$ to $F^{2,m-1}$. That A_j is bounded from $F^{2,m}$ to $F^{2,m-1}$ follows from the definition of Fock–Sobolev spaces.

We also need a theorem about Sobolev spaces. We define the (p,m)-capacity of a compact set $K \subset \mathbb{R}^n$ by

$$C_{p,m}(K) = \inf\left\{\|f\|^p_{L^p(\mathbb{R}^n)} :\ f \in L^p(\mathbb{R}^n),\ f \geq 0,\ B_m f \geq 1 \text{ on } K\right\},$$

where B_m is the Bessel potential of order m. By [6, p. 16], we have

$$C_{p,m}(K) \approx \inf\left\{\|u\|^p_{W^{p,m}(dx)} :\ u \in C_0^\infty(\mathbb{R}^n),\ u \geq 1 \text{ on } K\right\}. \tag{2.3}$$

Recall that $C_0^\infty(\mathbb{R}^n)$ is the set of smooth functions on $\mathbb{R}^n$ with compact support.

Theorem 2.4. ([6, Theorem 1.2.2]) *Let $p \in (1,\infty), m \in \mathbb{N}$ and let μ be a measure in $\mathbb{R}^n$. Then the best constant C in*

$$\int_{\mathbb{R}^n} |u(x)|^p d\mu(x) \leq C\|u\|^p_{W^{p,m}(dx)}, \quad u \in C_0^\infty(\mathbb{R}^n),$$

is equivalent to

$$\sup_K \frac{\mu(K)}{C_{p,m}(K)},$$

where K is an arbitrary compact set in $\mathbb{R}^n$.

The following proposition tells us the property of the Bargmann transform on Sobolev spaces.

Proposition 2.5. *The inverse of the Bargmann transform is bounded from the Fock–Sobolev space $F^{2,m}$ to the Sobolev space $W^{2,m}(dx)$. However, if $m \geq 1$, the image of the Bargmann transform B on $W^{2,m}(dx)$ is not contained in $F^{2,m}$.*

Proof. Suppose $f \in F^{2,m}$, we have $B^{-1}f = C_{\frac{1}{2}}^{-1} M^{-1} G^{-1} f$. We only need to prove that $M^{-1}G^{-1}f \in W^{2,m}(dx)$. For any $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| \le m$, there is a set of constants $\{c_\beta : \beta = (\beta_1, \beta_2, \ldots, \beta_n)\}$ such that

$$\begin{aligned}\left\|\partial^\alpha M^{-1}G^{-1}f\right\|_{L^2(\mathbb{R}^n,dx)} &= \left\|\sum_{\beta\le\alpha} c_\beta x^\beta M^{-1}\partial^{\alpha-\beta}(G^{-1}f)\right\|_{L^2(\mathbb{R}^n,dx)} \\ &\lesssim \sum_{\beta\le\alpha} \left\|x^\beta M^{-1}\partial^{\alpha-\beta}(G^{-1}f)\right\|_{L^2(\mathbb{R}^n,dx)} \\ &\lesssim \sum_{\beta\le\alpha} \left\|x^\beta \partial^{\alpha-\beta}(G^{-1}f)\right\|_{L^2(\mathbb{R}^n,d\gamma)} \\ &\lesssim \sum_{\beta\le\alpha} \left\|Gx^\beta G^{-1} G\partial^{\alpha-\beta}(G^{-1}f)\right\|_{F^2}.\end{aligned}$$

By direct computation, we know that

$$M_{x^\beta} = 2^\beta M C_{\frac{1}{2}} M_{x^\beta} C_{\frac{1}{2}}^{-1} M \text{ and } \partial^{\alpha-\beta} = \frac{1}{2^{\alpha-\beta}} M C_{\frac{1}{2}} \partial^{\alpha-\beta} C_{\frac{1}{2}}^{-1} M.$$

Then

$$\|\partial^\alpha M^{-1}G^{-1}f\|_{L^2(\mathbb{R}^n,dx)} \lesssim \sum_{\beta\le\alpha} \|Bx^\beta B^{-1} B\partial^{\alpha-\beta} B^{-1} f\|_{F^2}.$$

By (2.2), we have $\|\partial^\alpha M^{-1}G^{-1}f\|_{L^2(\mathbb{R}^n,dx)} \lesssim \|f\|_{F^{2,|\alpha|}}$, which means that

$$\|B^{-1}f\|_{W^{2,m}(dx)} \lesssim \|f\|_{F^{2,m}}.$$

Next, we prove the second part of this theorem by contradiction. Suppose $Bg \in F^{2,m}$ for any $g \in W^{2,m}(dx)$, that is to say $GMC_{\frac{1}{2}} g \in F^{2,m}(\gamma)$. Then, for any $g \in W^{2,m}(dx)$, we have $Mg \in W^{2,m}(\gamma)$. Since $m \ge 1$, we have $\|\partial_{x_1} Mg\|_{L^2(\mathbb{R}^n,d\gamma)} < \infty$. Since

$$\|\partial_{x_1} Mg\|_{L^2(\mathbb{R}^n,d\gamma)} = \|M\partial_{x_1} g + \frac{x_1}{2} Mg\|_{L^2(\mathbb{R}^n,d\gamma)}$$

and $\|M\partial_{x_1} g\|_{L^2(\mathbb{R}^n,d\gamma)} = \|\partial_{x_1} g\|_{L^2(\mathbb{R}^n,dx)} \le \|g\|_{W^{2,m}(dx)}$, we have

$$\|x_1 g\|_{L^2(\mathbb{R}^n,dx)} = \|x_1 Mg\|_{L^2(\mathbb{R}^n,d\gamma)} < \infty.$$

We have proved that $M_{x_1} g \in L^2(\mathbb{R}^n, dx)$ for any $g \in W^{2,m}(\gamma)$. Since M_{x_1} is a closed operator, we know that M_{x_1} is a bounded operator from $W^{2,m}(dx)$ to $L^2(\mathbb{R}^n, dx)$.

Let $d\mu = |x_1|^2 dx$. For any positive N, let $K_N = \overline{B(0,N)}$, there is a $u_N \in C_0^\infty(\mathbb{R}^n)$ with $u_N = 1$ on K_N and $u_N = 0$ on $B^c(0, N+1)$ such that

$$\sup_{|\alpha|\le m} \sup_x |\partial^\alpha u_N(x)| \le c < \infty,$$

where c is independent of N. Thus we have

$$\|u_N\|^2_{W^{2,m}(dx)} \lesssim |B(0, N+1)| \approx (N+1)^n.$$

By (2.3), we have

$$C_{2,m}(K_N) \lesssim (N+1)^n.$$

Then

$$\sup_K \frac{\mu(K)}{C_{2,m}(K)} \geq \frac{\mu(K_N)}{C_{2,m}(K_N)} \gtrsim \frac{\int_{K_N} |x_1|^2 dx}{(N+1)^n}.$$

Since $[-\frac{N}{\sqrt{n}}, \frac{N}{\sqrt{n}}]^n \subset K_N$, we have

$$\int_{K_N} |x_1|^2 dx \geq \int_{[-\frac{N}{\sqrt{n}}, \frac{N}{\sqrt{n}}]^n} |x_1|^2 dx \gtrsim N^{n+2}.$$

That is to say $\sup_K \frac{\mu(K)}{C_{p,m}(K)} = \infty$, which is a contradiction by Theorem 2.4. □

3. Multipliers on Gauss–Sobolev Spaces

In this section, we study multipliers on Gauss–Sobolev spaces. First, we recall the definition of Gauss–Bessel potentials. Some similar conclusions about the multipliers for classical Sobolev spaces have been proved in [6]. However, in the Gauss–Sobolev spaces, we need some properties of the Ornstein–Uhlenbeck differential operator.

The Ornstein–Uhlenbeck differential operator is defined as

$$L = \sum_{j=1}^{n} \partial_{x_j}^2 - \sum_{j=1}^{n} x_j \partial_{x_j}.$$

Let C_n be the closed subspace of $L^2(\gamma)$ generated by the linear combinations of $\{h_\beta : |\beta| = n\}$. For any $s \geq 0$, we consider the Gaussian-Bessel potentials defined by

$$(I-L)^{-s/2} f = \sum_{n=0}^{\infty} (1+n)^{-s/2} J_n f, \quad \text{for } f \in L^2(\gamma), \tag{3.1}$$

where J_n is the orthogonal projection from $L^2(\gamma)$ to C_n. The Gauss–Bessel potential space with order s is

$$L^{2,s}(\gamma) = \{f \in L^2(\gamma) : f = (I-L)^{-\frac{s}{2}} u \text{ for some } u \in L^2(\gamma)\}.$$

The norm is defined as

$$\|f\|_{L^{2,s}(\gamma)} = \|u\|_{L^2(\gamma)}, \text{ if } f = (I-L)^{-\frac{s}{2}} u.$$

Theorem 3.1. ([5]) *If s is a non-negative integer, then*

$$W^{2,s}(\gamma) = L^{2,s}(\gamma).$$

We also need a theorem of interpolation for Gauss–Sobolev spaces. Let

$$S = \{w \in \mathbb{C} : 0 \leq \mathrm{Re}(w) \leq 1\}.$$

Given a compatible pair of Banach spaces X_0 and X_1, let $\mathcal{F}(X_0, X_1)$ be the space of all functions F from $\bar{S}$ into $X_0 + X_1$ with the following properties:

1. F is bounded and continuous on $\bar{S}$ and analytic in S;
2. $y \to F(k+iy)$ with $k = 0, 1$ are continuous from the real line into X_k.

$\mathcal{F}(X_0, X_1)$ is clearly a vector space. We provide $\mathcal{F} = \mathcal{F}(X_0, X_1)$ with the norm

$$\|F\|_{\mathcal{F}} = \max\left\{\sup_{y\in\mathbb{R}}\|F(iy)\|_{X_0}, \sup_{y\in\mathbb{R}}\|F(1+iy)\|_{X_1}\right\}.$$

Given $0 \le \theta \le 1$, let X_θ be the space of vectors v in $X_0 + X_1$ such that $v = f(\theta)$ for some f in $\mathcal{F}(X_0, X_1)$. We norm X_θ with $\|v\|_\theta = \inf\{\|f\|_{\mathcal{F}} : v = f(\theta)\}$.

Theorem 3.2. *Let $0 \le \theta \le 1$ and $m_0 \le m_\theta \le m_1$ be three non-negative constants with*

$$m_\theta = m_0(1-\theta) + m_1\theta,$$

then

$$\left[L^{2,m_0}(\gamma), L^{2,m_1}(\gamma)\right]_\theta = L^{2,m_\theta}(\gamma),$$

where $\left[L^{2,m_0}(\gamma), L^{2,m_1}(\gamma)\right]_\theta$ is the interpolation space between $L^{2,m_0}(\gamma)$ and $L^{2,m_1}(\gamma)$.

Proof. Since $L^{2,m_1} \subset L^{2,m_0}$, we know that $L^{2,m_1} + L^{2,m_0} = L^{2,m_0}$. If $u \in L^{2,m_\theta}(\gamma)$, then there is $f \in L^2(\gamma)$ such that

$$u = (I-L)^{-m_\theta/2}f.$$

For any $z \in \{w : 0 \le \mathrm{Re}(w) \le 1\}$, we define

$$F(z) = \sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{1+n}}\right)^{m_0(1-z)+m_1z} J_n f.$$

It is easy to check that $F(z)$ is a vector-valued function from $\{w : 0 \le \mathrm{Re}(w) \le 1\}$ to $L^{2,m_0}(\gamma)$ which is continuous on $\{w : 0 \le \mathrm{Re}(w) \le 1\}$ and analytic on $\{w : 0 < \mathrm{Re}(w) < 1\}$. We know that

$$F(\theta) = u,$$

Then we have

$$\|u\|_\theta \le \|F\|_{\mathcal{F}} \le \|f\|_{L^2(\gamma)} = \|u\|_{L^{2,m_\theta}(\gamma)}.$$

Conversely, if $u \in \left[L^{2,m_0}(\gamma), L^{2,m_1}(\gamma)\right]_\theta$, then for any $\epsilon > 0$, there is a

$$F_\epsilon \in \mathcal{F}(L^{2,m_0}(\gamma), L^{2,m_1}(\gamma)).$$

with $F_\epsilon(\theta) = u$ such that

$$\|F_\epsilon\|_{\mathcal{F}} \le \|u\|_\theta + \epsilon.$$

For any $g \in L^2(\gamma)$, $l \in \mathbb{N}$ and $z \in S$, we define

$$H(z) = \sum_{n=0}^{l}(\sqrt{1+n})^{m_0(1-z)+m_1z}\langle F_\epsilon(z), J_n g\rangle_{L^2(\gamma)}.$$

It is easy to show that $H(z)$ is bounded and continuous on $\overline{S}$ and analytic in S. We consider

$$|H(ix)| = \left| \left\langle \sum_{n=0}^{l} (\sqrt{1+n})^{m_0(1-ix)+m_1 ix} J_n F_\epsilon(ix), g \right\rangle_{L^2(\gamma)} \right|.$$

Since $F_\epsilon(ix) \in L^{2,m_0}(\gamma)$ for any $x \in \mathbb{R}^n$, we know that there is $f_x \in L^2(\gamma)$ such that

$$F_\epsilon(ix) = (I-L)^{-\frac{m_0}{2}} f_x = \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{1+n}} \right)^{m_0} J_n f_x.$$

Then, we have

$$\begin{aligned} &\sup_{x\in\mathbb{R}} \left\| \sum_{n=0}^{l} (\sqrt{1+n})^{m_0(1-ix)+m_1 ix} J_n F_\epsilon(ix) \right\|_{L^2(\gamma)} \\ &= \sup_{x\in\mathbb{R}} \left\| \sum_{n=0}^{l} (\sqrt{1+n})^{-ixm_0+m_1 ix} J_n f_x \right\|_{L^2(\gamma)} \\ &\leq \sup_{x\in\mathbb{R}} \|f_x\|_{L^2(\gamma)} = \sup_{x\in\mathbb{R}} \|F_\epsilon(ix)\|_{W^{2,m_0}(\gamma)} \\ &\leq \|F_\epsilon\|_{\mathcal{F}}. \end{aligned}$$

Then $\sup_{x\in\mathbb{R}} |H(ix)| \leq \|F_\epsilon\|_{\mathcal{F}} \|g\|_{L^2}$. Similarly, we can obtain

$$\sup_{x\in\mathbb{R}} |H(1+ix)| \leq \|F_\epsilon\|_{\mathcal{F}} \|g\|_{L^2(\gamma)}.$$

By the Three Lines Lemma, see [9, p. 28], we have $|H(\theta)| \leq \|F_\epsilon\|_{\mathcal{F}} \|g\|_{L^2(\gamma)}$. That is to say

$$\left| \left\langle \sum_{n=0}^{l} (\sqrt{1+n})^{m_\theta} J_n u, g \right\rangle_{L^2(\gamma)} \right| \leq \|F_\epsilon\|_{\mathcal{F}} \|g\|_{L^2(\gamma)},$$

for any $l \in \mathbb{N}$ and $g \in L^2(\gamma)$. We obtain $\sum_{n=0}^{\infty} (\sqrt{1+n})^{m_\theta} J_n u \in L^2(\gamma)$ and

$$\left\| \sum_{n=0}^{\infty} (\sqrt{1+n})^{m_\theta} J_n u \right\|_{L^2(\gamma)} \leq \|F_\epsilon\|_{\mathcal{F}}.$$

Since $u = (I-L)^{-\frac{m_\theta}{2}} \left[\sum_{n=0}^{\infty} (\sqrt{1+n})^{m_\theta} J_n u \right]$, we have

$$\|u\|_{L^{2,m_\theta}} \leq \left\| \sum_{n=0}^{\infty} (\sqrt{1+n})^{m_\theta} J_n u \right\|_{L^2(\gamma)} \leq \|u\|_\theta + \epsilon$$

to complete the proof. □

Before proving the next lemma, we need some additional notations. For two multi-indexes $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, if for all $k = 1, \ldots, n$ we have $\alpha_k \leq \beta_k$, then we write

$$\alpha \leq \beta.$$

For any $u \in L^1_{loc}$, let M_u denote the multiplication operator on $W^{2,m}(\gamma)$. Then u is called a multiplier on $W^{2,m}(\gamma)$ if M_u is bounded on $W^{2,m}(\gamma)$. Let $M\left(W^{2,m}(\gamma) \to W^{2,m'}(\gamma)\right)$ denote the set of bounded multiplication operators from $W^{2,m}(\gamma)$ to $W^{2,m'}(\gamma)$. If $m = m'$, we write $M\left(W^{2,m}(\gamma) \to W^{2,m}(\gamma)\right)$ as $MW^{2,m}(\gamma)$. We have following simple lemma.

Lemma 3.3. *For any $u \in C^\infty(\mathbb{R}^n)$, we have*

$$\|u\|_{MW^{2,m}(\gamma)} \lesssim \sum_{|\alpha|\le m} \sup_x |\partial^\alpha u(x)|.$$

Proof. The proof is obvious as it follows from the definition of the norm of $W^{2,m}(\gamma)$, the product rule for differentiation and immediate estimates. □

Lemma 3.4. *Suppose that*

$$u \in MW^{2,m}(\gamma) \cap ML^2(\gamma).$$

Then, for any multi-index α of order $|\alpha| \le m$, we have

$$\partial^\alpha u \in M\left(W^{2,m}(\gamma) \to W^{2,m-|\alpha|}(\gamma)\right).$$

Furthermore, for any ϵ, there is a constant $c(\epsilon)$ such that

$$\|\partial^\alpha u\|_{M(W^{2,m}(\gamma)\to W^{2,m-|\alpha|}(\gamma))} \le \varepsilon \|u\|_{ML^2(\gamma)} + c(\varepsilon)\|u\|_{MW^{2,m}(\gamma)}.$$

Proof. If $\alpha = 0$, the conclusion is obvious. We suppose that $\alpha \ne 0$. By [6, p. 39], for any $g \in W^{2,m}(\gamma)$, just using the product rule applied to ug and rearranging, we have

$$g\partial^\alpha u = \sum_{\{\beta:\alpha\ge\beta\ge 0\}} \frac{\alpha!}{\beta!(\alpha-\beta)!} \partial^\beta (u(-\partial)^{\alpha-\beta} g).$$

Then

$$\begin{aligned}
\|g\partial^\alpha u\|_{W^{2,m-|\alpha|}(\gamma)} &\lesssim \sum_{\{\beta:\alpha\ge\beta\ge 0\}} \|u\partial^{\alpha-\beta} g\|_{W^{2,m-|\alpha|+|\beta|}(\gamma)} \\
&\le \sum_{\{\beta:\alpha\ge\beta\ge 0\}} \|u\|_{MW^{2,m-|\alpha|+|\beta|}(\gamma)} \|\partial^{\alpha-\beta} g\|_{W^{2,m-|\alpha|+|\beta|}(\gamma)} \\
&\le \sum_{\{\beta:\alpha\ge\beta\ge 0\}} \|u\|_{MW^{2,m-|\alpha|+|\beta|}(\gamma)} \|g\|_{W^{2,m}(\gamma)}.
\end{aligned}$$

Thus, by Theorems 3.2 and 3.1, we have

$$\begin{aligned}
&\|\partial^\alpha u\|_{M(W^{2,m}(\gamma)\to W^{2,m-|\alpha|}(\gamma))} \\
&\quad\le \sum_{\{\beta:\alpha\ge\beta\ge 0\}} \|u\|_{MW^{2,m-|\alpha|+|\beta|}(\gamma)} \\
&\quad\le \sum_{\{\beta:\alpha\ge\beta\ge 0\}} \|u\|_{MW^{2,m}(\gamma)}^{\frac{m-|\alpha|+|\beta|}{m}} \|u\|_{ML^2(\gamma)}^{\frac{|\alpha|-|\beta|}{m}} \\
&\quad\le \sum_{\{\beta:\alpha>\beta\ge 0\}} \|u\|_{MW^{2,m}(\gamma)}^{\frac{m-|\alpha|+|\beta|}{m}} \|u\|_{ML^2(\gamma)}^{\frac{|\alpha|-|\beta|}{m}} + \|u\|_{MW^{2,m}(\gamma)}.
\end{aligned} \tag{3.2}$$

For any $\epsilon > 0$, by Young's inequality, we have

$$\begin{aligned}
&\sum_{\{\beta:\alpha>\beta\geq 0\}} \|u\|_{MW^{2,m}(\gamma)}^{\frac{m-|\alpha|+|\beta|}{m}} \|u\|_{ML^2(\gamma)}^{\frac{|\alpha|-|\beta|}{m}} \\
&\quad = \sum_{\{\beta:\alpha>\beta\geq 0\}} \epsilon^{\frac{|\beta|-|\alpha|}{m}} \|u\|_{MW^{2,m}(\gamma)}^{\frac{m-|\alpha|+|\beta|}{m}} (\epsilon\|u\|)_{ML^2(\gamma)}^{\frac{|\alpha|-|\beta|}{m}} \\
&\quad \lesssim \sum_{\{\beta:\alpha>\beta\geq 0\}} \Big[\frac{m+|\beta|-|\alpha|}{m} \epsilon^{\frac{|\alpha|-|\beta|}{m-|\alpha|+|\beta|}} \|u\|_{MW^{2,m}(\gamma)} + \frac{|\alpha|-|\beta|}{m} \epsilon \|u\|_{ML^2(\gamma)}\Big]
\end{aligned}$$

to complete the proof. □

Lemma 3.5. *For any non-negative integer m and $g \in L^2(\gamma)$, there is a set of functions $\{g_\alpha : |\alpha| \leq m\}$ such that*

$$g = \sum_{|\alpha|\leq m} \partial^\alpha g_\alpha \quad \text{and} \quad \|g_\alpha\|_{W^{2,m}(\gamma)} \lesssim \|g\|_{L^2(\gamma)}.$$

Proof. If $m = 0$, then the conclusion is true. Suppose that the conclusion is true for $m = k$, we will prove that the conclusion is true for $m = k + 1$. For any $g \in L^2(\gamma)$, we know that

$$g = \sum_{|\beta|\leq k} \partial^\beta g_\beta,$$

where $g \in W^{2,k}(\gamma)$ and $\|g_\beta\|_{W^{2,k}(\gamma)} \lesssim \|g\|_{L^2(\gamma)}$. Then $g_\beta = (I - L)(I - L)^{-1} g_\beta$. Since

$$I - L = \sum_{j=1}^{n} \partial_{x_j}(M_{x_j} - \partial_{x_j}) - (n-1)I,$$

we have

$$g_\beta = \sum_{j=1}^{n} \partial_{x_j}(M_{x_j} - \partial_{x_j})(I - L)^{-1} g_\beta - (n-1)(I - L)^{-1} g_\beta.$$

By Theorem 3.1, we know that $(I - L)^{-1}$ is bounded from $W^{2,k}(\gamma)$ to $W^{2,k+2}(\gamma)$, then $(I-L)^{-1} g_\beta \in W^{2,k+2}(\gamma)$. By (2.2), we know that $(M_{x_j} - \partial_{x_j})$ is bounded from $W^{2,k+2}(\gamma)$ to $W^{2,k+1}(\gamma)$. We then obtain

$$g = \sum_{|\beta|\leq k} \partial^\beta \left[\sum_{j=1}^{n} \partial_{x_j}(M_{x_j} - \partial_{x_j})(I - L)^{-1} g_\beta - (n-1)(I - L)^{-1} g_\beta\right],$$

where

$$\|(M_{x_j} - \partial_{x_j})(I - L)^{-1} g_\beta\|_{W^{2,k+1}(\gamma)} \lesssim \|g_\beta\|_{W^{2,k}(\gamma)} \lesssim \|g\|_{L^2(\gamma)}$$

and

$$\|(n-1)(I - L)^{-1} g_\beta\|_{W^{2,k+1}(\gamma)} \lesssim \|(I - L)^{-1} g_\beta\|_{W^{2,k+2}(\gamma)} \lesssim \|g\|_{L^2(\gamma)}.$$

We have completed the proof. □

For any $b \in \mathbb{R}^n$, let W_b be an operator on F^2 such that

$$W_b h(z) = h(z-b)e^{z\cdot b - \frac{b^2}{2}},$$

for any $h \in F^2$. This operator is the analogue of translation in the Fock space setting.

Lemma 3.6. *For any $b \in \mathbb{R}^n$, W_b is a bounded operator on $F^{2,m}$ and*

$$\|W_b\|_{F^{2,m}} \le c_{m,n}\left(\sum_{j=0}^{m} |b|^{2j}\right),$$

where $c_{m,n}$ is a constant that depends only on m and n.

Proof. For any $h \in F^{2,m}$, we have

$$\begin{aligned}
\|W_b h\|_{F^{2,m}} &\lesssim \||z|^m W_b h\|_{F^2} \\
&= \left[\int_{\mathbb{C}^n} |z|^{2m}|h(z-b)|^2 e^{2z\cdot b - b^2} d\lambda(z)\right]^{1/2} \\
&= \left[\int_{\mathbb{C}^n} |z+b|^{2m}|h(z)|^2 d\lambda(z)\right]^{1/2} \\
&\le \left[\int_{\mathbb{C}^n} 2^m(|z|^2+|b|^2)^m|h(z)|^2 d\lambda(z)\right]^{1/2} \\
&\lesssim \left(\sum_{j=0}^{m} |b|^{2j}\right) \max_{0\le k\le m}\{\||z|^k h\|_{F^2}\} \\
&\lesssim \left(\sum_{j=0}^{m} |b|^{2j}\right) \|h\|_{F^{2,m}},
\end{aligned}$$

where the last inequality is due to Theorem 2.3. □

Lemma 3.7. *Suppose that $u \in MW^{2,m}(\gamma)$ for some $m \ge 0$, let*

$$u_r(x) = \int_{\mathbb{R}^n} r^{-n}K(r^{-1}t)u(x-t)dt,$$

where $K \in C_c^\infty(\mathbb{B}^n)$, $K \ge 0$ and $0 \le r \le 1$. Then we have

$$\sup_{0<r\le 1} \|u_r\|_{MW^{2,m}(\gamma)} \le c_{m,n}\|u\|_{MW^{2,m}(\gamma)}$$

and

$$\sup_{0<r\le 1} \|\partial^\alpha u_r\|_{M(W^{2,m}(\gamma)\to L^2(\gamma))} \le c'_{m,n}\|\partial^\alpha u\|_{M(W^{2,m}(\gamma)\to L^2(\gamma))}$$

for any α with $|\alpha| \le m$, where $c_{m,n}$ and $c'_{m,n}$ are constants that depend only on m and n.

Proof. For any $g \in W^{2,m}(\gamma)$, by Minkowski's inequality, we have

$$\|u_r g\|_{W^{2,m}(\gamma)} = \sum_{|\alpha|\le m} \|\partial^\alpha(u_r g)\|_{L^2(\gamma)}$$

$$= \sum_{|\alpha|\le m} \left[\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} r^{-n} K(r^{-1}t)\partial^\alpha\big(u(x-t)g(x)\big)dt\right|^2 d\gamma(x)\right]^{\frac{1}{2}}$$
$$\le \sum_{|\alpha|\le m} \int_{\mathbb{R}^n} r^{-n}K(r^{-1}t)\left[\int_{\mathbb{R}^n} |\partial^\alpha\big(u(x-t)g(x)\big)|^2 d\gamma(x)\right]^{\frac{1}{2}} dt.$$

Let τ_t be the translation operator such that $\tau_t u(x) = u(x-t)$ and $M_{\tau_t u}$ be the multiplication operator, then

$$\|u_r g\|_{W^{2,m}(\gamma)} \le \sum_{|\alpha|\le m} \int_{\mathbb{R}^n} r^{-n}K(r^{-1}t)\|M_{\tau_t u} g\|_{W^{2,m}(\gamma)}dt$$
$$\le c_m \|g\|_{W^{2,m}(\gamma)} \int_{|t|\le r} r^{-n}K(r^{-1}t)\|M_{\tau_t u}\|_{MW^{2,m}(\gamma)}dt.$$

We claim that $M_{\tau_t u} = G^{-1}W_{\frac{t}{2}}GM_u G^{-1}W_{\frac{-t}{2}}G$, then

$$\|M_{\tau_t u}\|_{MW^{2,m}(\gamma)} \le \|W_{\frac{t}{2}}\|_{F^{2,m}}\|u\|_{MW^{2,m}(\gamma)}\|W_{\frac{-t}{2}}\|_{F^{2,m}}.$$

By Lemma 3.6, we have

$$\sup_{0<r\le 1} \|u_r\|_{MW^{2,m}(\gamma)}$$
$$\le \sup_{0<r\le 1} c_m \int_{|t|\le r} r^{-n}K(r^{-1}t)\|W_{\frac{t}{2}}\|_{F^{2,m}}\|W_{\frac{-t}{2}}\|_{F^{2,m}}dt\|u\|_{MW^{2,m}(\gamma)}$$
$$\le c_{m,n}\|u\|_{MW^{2,m}(\gamma)}$$

for some constant $c_{m,n}$.

Next, we prove the claim $M_{\tau_t u} = G^{-1}W_{\frac{t}{2}}GM_u G^{-1}W_{\frac{-t}{2}}G$. First, we show that $G^{-1}W_{\frac{t}{2}}G = M_{\exp[x\cdot\frac{t}{2}-\frac{t^2}{4}]}\tau_t$. For any $g \in W^{2,m}(\gamma)$, we have

$$(W_{\frac{t}{2}}Gg)(z) = e^{z\cdot\frac{t}{2}-\frac{t^2}{8}} \int_{\mathbb{R}^n} g(x)e^{x\cdot(z-\frac{t}{2})-\frac{(z-\frac{t}{2})^2}{2}} \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2}}dx$$
$$= e^{z\cdot\frac{t}{2}-\frac{t^2}{8}} \int_{\mathbb{R}^n} g(x-t)e^{(x-t)\cdot(z-\frac{t}{2})-\frac{(z-\frac{t}{2})^2}{2}} \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x-t|^2}{2}}dx$$
$$= e^{-\frac{t^2}{4}} \int_{\mathbb{R}^n} g(x-t)e^{x\cdot\frac{t}{2}}e^{x\cdot z-\frac{z^2}{2}}d\gamma$$
$$= e^{-\frac{t^2}{4}} G[g(x-t)e^{x\cdot\frac{t}{2}}](z).$$

Thus, we have

$$(G^{-1}W_{\frac{t}{2}}Gg)(x) = e^{x\cdot\frac{t}{2}-\frac{t^2}{4}}g(x-t).$$

Direct computation shows that

$$M_{\tau_t u} = G^{-1}W_{\frac{t}{2}}GM_u G^{-1}W_{\frac{-t}{2}}G,$$

which completes the proof of the claim.

Similarly, for any α with $|\alpha| \le m$ and $g \in L^2(\gamma)$, we have

$$\begin{aligned}
&\|(\partial^\alpha u_r)g\|_{L^2(\gamma)} \\
&\quad\le \int_{\mathbb{R}^n} r^{-n}K(r^{-1}t)\|M_{\tau_t\partial^\alpha u}g\|_{L^2(\gamma)}dt \\
&\quad\le c_m\|g\|_{W^{2,m}(\gamma)}\int_{|t|\le r} r^{-n}K(r^{-1}t)\|M_{\tau_t\partial^\alpha u}\|_{M\left(W^{2,m}(\gamma)\to L^2(\gamma)\right)}dt.
\end{aligned}$$

By the argument above, for any α with $|\alpha| \le m$, we have

$$\begin{aligned}
&\|M_{\tau_t\partial^\alpha u}\|_{M\left(W^{2,m}(\gamma)\to L^2(\gamma)\right)} \\
&\quad= \|G^{-1}W_{\frac{t}{2}}GM_{\partial^\alpha u}G^{-1}W_{\frac{-t}{2}}G\|_{M\left(W^{2,m}(\gamma)\to L^2(\gamma)\right)} \\
&\quad\le \|G^{-1}W_{\frac{t}{2}}G\|_{M\left(L^2(\gamma)\to L^2(\gamma)\right)}\|M_{\partial^\alpha u}\|_{M\left(W^{2,m}(\gamma)\to L^2(\gamma)\right)} \\
&\qquad\times \|G^{-1}W_{\frac{-t}{2}}G\|_{M\left(W^{2,m}(\gamma)\to W^{2,m}(\gamma)\right)} \\
&\quad\le c'_{m,n}\|\partial^\alpha u\|_{M\left(W^{2,m}(\gamma)\to L^2(\gamma)\right)}
\end{aligned}$$

for some constant $c'_{m,n}$, which completes the proof. □

Proposition 3.8. *If $u \in MW^{2,m}(\gamma)\bigcap u \in ML^2(\gamma)$ and $|\alpha| = m$ then $\partial^\alpha u \in M\big(W^{2,|\alpha|}(\gamma) \to L^2(\gamma)\big)$. Moreover, we have*

$$\sum_{|\alpha|=m}\|\partial^\alpha u\|_{M(W^{2,|\alpha|}(\gamma)\to L^2(\gamma))} + \|u\|_{M(L^2(\gamma))} \lesssim \|u\|_{MW^{2,m}(\gamma)}.$$

Proof. First, we suppose that $u \in ML^2(\gamma)$. For any $g \in W^{2,m}(\gamma)$ and multi-index α with $|\alpha| = m$, we have

$$\begin{aligned}
&\|(\partial^\alpha u)g\|_{L^2(\gamma)} \\
&\quad= \|\partial^\alpha(ug) - \sum_{\beta:0\le\beta<\alpha}\partial^\beta u\partial^{\alpha-\beta}g\|_{L^2(\gamma)} \\
&\quad\le \|ug\|_{W^{2,|\alpha|}(\gamma)} + \|\sum_{\beta:0\le\beta<\alpha}\partial^\beta u\partial^{\alpha-\beta}g\|_{L^2(\gamma)} \\
&\quad\le \|u\|_{MW^{2,|\alpha|}(\gamma)}\|g\|_{W^{2,|\alpha|}(\gamma)} + \sum_{\beta:0\le\beta<\alpha}\|\partial^\beta u\partial^{\alpha-\beta}g\|_{L^2(\gamma)} \\
&\quad\le \|u\|_{MW^{2,|\alpha|}(\gamma)}\|g\|_{W^{2,|\alpha|}(\gamma)} + \sum_{\beta:0\le\beta<\alpha}\|\partial^\beta u\|_{M(W^{2,|\beta|}(\gamma)\to L^2(\gamma))}\|\partial^{\alpha-\beta}g\|_{W^{2,|\beta|}(\gamma)} \\
&\quad\le \left[\|u\|_{MW^{2,|\alpha|}(\gamma)} + \sum_{\beta:0\le\beta<\alpha}\|\partial^\beta u\|_{M(W^{2,|\beta|}(\gamma)\to L^2(\gamma))}\right]\|g\|_{W^{2,m}(\gamma)}.
\end{aligned}$$

By Lemma 3.4, for any $\epsilon > 0$ there is a constant $c(\epsilon)$ such that

$$\|\partial^\beta u\|_{M(W^{2,|\beta|}(\gamma)\to L^2(\gamma))} \le \varepsilon\|u\|_{ML^2(\gamma)} + c(\varepsilon)\|u\|_{MW^{2,|\beta|}(\gamma)}.$$

Further, by Theorem 3.2, we have

$$\|u\|_{MW^{2,|\alpha|}(\gamma)} \lesssim \|u\|_{ML^2(\gamma)} + \|u\|_{MW^{2,m}(\gamma)}.$$

Thus, we obtain

$$\sum_{|\alpha|=m} \|\partial^\alpha u\|_{M(W^{2,m}(\gamma)\to L^2(\gamma))} \lesssim \|u\|_{ML^2(\gamma)} + \|u\|_{MW^{2,m}(\gamma)}.$$

Next, we will prove that $\|u\|_{ML^2(\gamma)} \lesssim \|u\|_{MW^{2,m}(\gamma)}$, which implies the conclusion.

For any $g \in L^2(\gamma)$, we have the decomposition $g = \sum_{|\alpha|\le m} \partial^\alpha g_\alpha$ in Lemma 3.5. Then

$$\begin{aligned}
\|ug\|_{L^2(\gamma)} &\le \sum_{|\alpha|\le m} \|u\partial^\alpha g_\alpha\|_{L^2(\gamma)} \\
&= \sum_{|\alpha|\le m} \Big\| \sum_{\{\beta:\alpha\ge\beta\ge 0\}} \frac{\alpha!}{\beta!(\alpha-\beta)!} \partial^\beta (g_\alpha(-\partial)^{\alpha-\beta} u)\Big\|_{L^2(\gamma)} \\
&\lesssim \sum_{|\alpha|\le m} \sum_{\{\beta:\alpha\ge\beta\ge 0\}} \big\| \partial^\beta (g_\alpha(-\partial)^{\alpha-\beta} u)\big\|_{L^2(\gamma)} \\
&\lesssim \sum_{|\alpha|\le m} \sum_{\{\beta:\alpha\ge\beta\ge 0\}} \big\| g_\alpha(-\partial)^{\alpha-\beta} u\big\|_{W^{2,m-|\alpha|+|\beta|}(\gamma)} \\
&\lesssim \sum_{|\alpha|\le m} \sum_{\{\beta:\alpha\ge\beta\ge 0\}} \big\|\partial^{\alpha-\beta} u\big\|_{M\left(W^{2,m}(\gamma)\to W^{2,m-|\alpha|+|\beta|}(\gamma)\right)} \|g_\alpha\|_{W^{2,m}(\gamma)} \\
&\lesssim \sum_{|\alpha|\le m} \sum_{\{\beta:\alpha\ge\beta\ge 0\}} \big\|\partial^{\alpha-\beta} u\big\|_{M\left(W^{2,m}(\gamma)\to W^{2,m-|\alpha|+|\beta|}(\gamma)\right)} \|g\|_{L^2(\gamma)}.
\end{aligned}$$

By Lemma 3.4 and the inequality above, for any $0 < \epsilon < 1$, there is a constant $c(\epsilon)$ such that

$$\|u\|_{ML^2(\gamma)} \lesssim \varepsilon \|u\|_{ML^2(\gamma)} + c(\varepsilon)\|u\|_{MW^{2,m}(\gamma)}.$$

Then, we have $\|u\|_{ML^2(\gamma)} \lesssim \|u\|_{MW^{2,m}(\gamma)}$.

Next, we remove the hypothesis. For any $r > 0$, let u_r be the function in Lemma 3.7. Thus u_r is in $C^\infty(\mathbb{R}^n)$. We can choose a set of smooth function ϕ_r such that $\phi_r(x) = 1$ when $|x| \le \frac{1}{r}$, $\phi_r(x) = 0$ when $|x| > \frac{1}{r} + 1$ and

$$\sum_{|\alpha|\le m} \sup_x |\partial^\alpha \phi_r(x)| \le c,$$

where c is independent with r. We know that $\phi_r u_r$ is bounded, thus $\phi_r u_r \in ML^2(\gamma)$. By the conclusion above we know that

$$\|\phi_r u_r\|_{ML^2(\gamma)} \le c' \|\phi_r u_r\|_{MW^{2,m}(\gamma)},$$

where c' is an absolute constant. Since $\lim_{r\to 0} \phi_r u_r = u$ almost everywhere. Thus for any $g \in L^2(\gamma)$, we have

$$\|ug\|_{L^2(\gamma)} \le \liminf_{r\to 0} \|\phi_r u_r g\|_{L^2(\gamma)}.$$

Then by Lemmas 3.7 and 3.3, we have

$$\begin{aligned}
\|u\|_{L^2(\gamma)} &\le \liminf_{r\to 0} \|\phi_r u_r\|_{L^2(\gamma)} \lesssim \liminf_{r\to 0} \|\phi_r u_r\|_{MW^{2,m}(\gamma)} \\
&\le \liminf_{r\to 0} \|\phi_r\|_{MW^{2,m}(\gamma)} \|u_r\|_{MW^{2,m}(\gamma)} \le c_{m,n} c \|u\|_{MW^{2,m}(\gamma)}
\end{aligned}$$

to complete the proof of the claim. □

To prove our main theorem in the next section, we need the following theorem about multipliers on the Gauss–Sobolev space $W^{2,m}(\gamma)$.

Theorem 3.9. *If $|\alpha| = m$ and $u \in ML^2(\gamma)$, then $u \in MW^{2,m}(\gamma)$ if and only if $\partial^\alpha u \in M\big(W^{2,|\alpha|}(\gamma) \to L^2(\gamma)\big)$. In this case, we have*

$$\|u\|_{MW^{2,m}(\gamma)} \simeq \sum_{|\alpha|=m} \|\partial^\alpha u\|_{M(W^{2,|\alpha|}(\gamma)\to L^2(\gamma))} + \|u\|_{ML^2(\gamma)}.$$

Proof. If $\partial^\alpha u \in M\big(W^{2,|\alpha|}(\gamma) \to L^2(\gamma)\big)$ for any $|\alpha| = m$ and $u \in ML^2(\gamma)$. Let u_r be the function corresponding to u as in Lemma 3.7. Since $u \in ML^2(\gamma)$, we know that u is bounded. It is easy to prove that

$$\sum_{|\alpha|\le m} \sup_x |\partial^\alpha u_r(x)| < \infty$$

for any $r > 0$, thus $\|u_r\|_{MW^{2,m}(\gamma)} < \infty$. Then for any $g \in W^{2,m}(\gamma)$, we have

$$\begin{aligned}
&\|u_r g\|_{W^{2,m}(\gamma)} \\
&\quad= \sum_{|\alpha|\le m} \|\partial^\alpha (u_r g)\|_{L^2(\gamma)} \\
&\quad\le \sum_{|\alpha|\le m} \sum_{0\le\beta\le\alpha} \|\partial^\beta u_r \partial^{\alpha-\beta} g\|_{L^2(\gamma)} \\
&\quad= \sum_{|\alpha|\le m} \sum_{0\le\beta\le\alpha} \|\partial^\beta u_r\|_{M\left(W^{2,|\beta|}(\gamma)\to L^2(\gamma)\right)} \|\partial^{\alpha-\beta} g\|_{W^{2,|\beta|}(\gamma)} \\
&\quad\lesssim \|g\|_{W^{2,m}(\gamma)} \sum_{0\le|\beta|<m} \|\partial^\beta u_r\|_{M\left(W^{2,|\beta|}(\gamma)\to L^2(\gamma)\right)} \\
&\qquad+ \|g\|_{W^{2,m}(\gamma)} \sum_{|\beta|=m} \|\partial^\beta u_r\|_{M\left(W^{2,|\beta|}(\gamma)\to L^2(\gamma)\right)}.
\end{aligned}$$

By Lemma 3.4 and Theorem 3.2, for any $\epsilon > 0$, there is a $c(\epsilon)$ such that

$$\begin{aligned}
\sum_{0\le|\beta|<m} \|\partial^\beta u_r\|_{M(W^{2,|\beta|}(\gamma)\to L^2(\gamma))} &\lesssim \sum_{\{\beta:0\le|\beta|<m\}} \|u_r\|_{MW^{2,|\beta|}(\gamma)} \\
&\lesssim \epsilon\|u_r\|_{MW^{2,m}(\gamma)} + c(\epsilon)\|u_r\|_{ML^2(\gamma)}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&\|u_r\|_{MW^{2,m}(\gamma)} \\
&\quad\lesssim \epsilon\|u_r\|_{MW^{2,m}(\gamma)} + c(\epsilon)\|u_r\|_{ML^2(\gamma)} + \sum_{|\beta|=m} \|\partial^\beta u_r\|_{M\left(W^{2,|\beta|}(\gamma)\to L^2(\gamma)\right)}.
\end{aligned}$$

Let ϵ be small enough, then we get

$$\|u_r\|_{MW^{2,m}(\gamma)} \lesssim \|u_r\|_{ML^2(\gamma)} + \sum_{|\beta|=m} \|\partial^\beta u_r\|_{M\left(W^{2,|\beta|}(\gamma)\to L^2(\gamma)\right)}.$$

By Lemma 3.7, we have

$$\begin{aligned}\|u\|_{MW^{2,m}(\gamma)} &\leq \liminf_{r\to 0}\|u_r\|_{MW^{2,m}(\gamma)}\\ &\lesssim \liminf_{r\to 0}\|u_r\|_{ML^2(\gamma)} + \liminf_{r\to 0}\sum_{|\beta|=m}\|\partial^\beta u_r\|_{M\left(W^{2,|\beta|}(\gamma)\to L^2(\gamma)\right)}\\ &\lesssim \|u\|_{ML^2(\gamma)} + \sum_{|\beta|=m}\|\partial^\beta u\|_{M\left(W^{2,|\beta|}(\gamma)\to L^2(\gamma)\right)}.\end{aligned}$$

The converse is due to Proposition 3.8. □

4. Applications to Certain Operators on the Fock–Sobolev Space

In this section, we study the boundedness of S_φ. We need several lemmas. Let C_i and C_{-i} be composition operators on F^2 such that for any $f \in F^2$

$$C_i f(z) = f(iz) \quad \text{and} \quad C_{-i} f(z) = f(-iz).$$

It is easy to show that C_i and C_{-i} are isometries on $F^{2,m}$ for any $m \in \mathbb{N}$.

Lemma 4.1. *For any $a \in \mathbb{R}^n$, let $M_{e^{-ia\cdot x}}$ be the multiplication operator on $W^{2,m}(\gamma)$. If S_φ is bounded on $F^{2,m}$, then $G^{-1}C_{-i}S_\varphi C_i G$ commutes with $M_{e^{-ia\cdot x}}$.*

Proof. By [3, Lemma 3.3], we know that S_φ commutes with W_a on F^2. Since W_a is bounded on $F^{2,m}$, we know that S_φ commutes with W_a on $F^{2,m}$. Then $G^{-1}C_{-i}S_\varphi C_i G$ commutes with $G^{-1}C_{-i}W_a C_i G$. We only need to show that

$$G^{-1}C_{-i}W_a C_i G = M_{e^{-ia\cdot x}}.$$

For any $f \in F^{2,m}$ and $z \in \mathbb{C}^n$, we have

$$C_{-i}W_a C_i f(z) = f(z - ia)e^{-iz\cdot a - \frac{a^2}{2}}.$$

On the other hand

$$\begin{aligned}M_{e^{-ia\cdot x}}G^{-1}f(x) &= e^{-ia\cdot x}\int_{\mathbb{C}^n} f(z)e^{x\cdot\overline{z} - \frac{\overline{z}^2}{2}}\,d\lambda(z)\\ &= e^{-ia\cdot x}\int_{\mathbb{C}^n} f(z)e^{x\cdot\overline{z} - \frac{\overline{z}^2}{2}}\pi^{-n}e^{-|z|^2}\,dv(z)\\ &= e^{-ia\cdot x}\int_{\mathbb{C}^n} f(z-ia)e^{x\cdot\overline{(z-ia)} - \frac{\overline{(z-ia)}^2}{2}}\pi^{-n}e^{-|z-ia|^2}\,dv(z)\\ &= \int_{\mathbb{C}^n} f(z-ia)e^{-iz\cdot a - \frac{a^2}{2}}e^{x\cdot\overline{z} - \frac{\overline{z}^2}{2}}\,d\lambda(z)\\ &= G^{-1}[f(z-ia)e^{-iz\cdot a - \frac{a^2}{2}}](x).\end{aligned}$$

Then

$$GM_{e^{-ia\cdot x}}G^{-1}f(z) = g(z-ia)e^{-iz\cdot a - \frac{a^2}{2}} = C_{-i}W_a C_i f(z),$$

which completes the proof. □

Let $C_p^\infty(\mathbb{R}^n)$ denote the set of smooth function f such that there is a positive number $N = N_f$, such that

$$f(x + 2Ny) = f(x)$$

for any $x \in [-N, N]^n$ and $y \in \mathbb{Z}^n$, moreover, $f(x) = 0$ when $x \in [-N, N]^n \setminus \left[-\frac{N}{2\sqrt{n}}, \frac{N}{2\sqrt{n}}\right]^n$. We call N_f the period of f.

Lemma 4.2. *For any $f \in C_p^\infty(\mathbb{R}^n)$, there is a sequence $f_n \in \operatorname{span}\{e^{ia\cdot x} : a \in \mathbb{R}^n\}$ such that*

$$\lim_{n\to\infty} \|M_{f_n} - M_f\|_{MW^{2,m}(\gamma)} = 0.$$

Proof. By [7, Theorem 2.11 and Corollary 1.9, Chapter 7], there is a sequence of functions $\{f_n\} \subset \operatorname{span}\{e^{ia\cdot x} : a \in \mathbb{R}^n\}$ such that

$$\lim_{n\to\infty} \sup_x |\partial^\alpha f(x) - \partial^\alpha f_n(x)| = 0,$$

for any $\alpha \in \mathbb{R}^n$ with $|\alpha| \le m$. By Lemma 3.3, we obtain the conclusion. $\square$

Lemma 4.3. *$C_p^\infty(\mathbb{R}^n)$ is a dense subset of $W^{2,m}(\gamma)$.*

Proof. First, we show that $C_p^\infty(\mathbb{R}^n)$ is contained in $W^{2,m}(\gamma)$. For any $f \in C_p^\infty(\mathbb{R}^n)$ and any $\alpha \in \mathbb{N}^n$, let $N = N_f$ be the period of f, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\partial^\alpha f(x)|^2 d\gamma(x)\\
&\quad = \sum_{y\in\mathbb{Z}^n} \int_{[-N,N]^n+2Ny} |\partial^\alpha f(x)|^2 d\gamma(x)\\
&\quad = \sum_{y\in\mathbb{Z}^n} \int_{[-N,N]^n} |\partial^\alpha f(x)|^2 \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x+2Ny|^2}{2}} dx\\
&\quad = \sum_{y\in\mathbb{Z}^n\setminus\{0\}} \int_{[-N,N]^n} |\partial^\alpha f(x)|^2 \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x+2Ny|^2}{2}} dx + \int_{[-N,N]^n} |\partial^\alpha f(x)|^2 d\gamma(x)\\
&\quad = \sum_{y\in\mathbb{Z}^n\setminus\{0\}} \int_{\left[-\frac{N}{2\sqrt{n}}, \frac{N}{2\sqrt{n}}\right]^n} |\partial^\alpha f(x)|^2 \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x+2Ny|^2}{2}} dx\\
&\qquad + \int_{\left[-\frac{N}{2\sqrt{n}}, \frac{N}{2\sqrt{n}}\right]^n} |\partial^\alpha f(x)|^2 d\gamma(x).
\end{aligned}$$

When $x \in \left[-\frac{N}{2\sqrt{n}}, \frac{N}{2\sqrt{n}}\right]^n$ and $y \in \mathbb{Z}^n \setminus \{0\}$, we have

$$|x| \le \frac{N}{2} \le \frac{N|y|}{2}.$$

Then

$$e^{-\frac{|x+2Ny|^2}{2}} \le e^{-\frac{|x|^2}{2} + 2N\frac{N|y|}{2}|y| - 2N^2|y|^2} \le e^{-\frac{|x|^2}{2} - N^2|y|^2}.$$

That is to say

$$\sum_{y\in\mathbb{Z}^n\setminus\{0\}}\int_{\left[-\frac{N}{2\sqrt{n}},\frac{N}{2\sqrt{n}}\right]^n}|\partial^\alpha f(x)|^2\frac{1}{(2\pi)^{\frac{n}{2}}}e^{-\frac{|x+2Ny|^2}{2}}dx$$
$$\le\sum_{y\in\mathbb{Z}^n\setminus\{0\}}e^{-N^2|y|^2}\int_{\left[-\frac{N}{2\sqrt{n}},\frac{N}{2\sqrt{n}}\right]^n}|\partial^\alpha f(x)|^2d\gamma(x).$$

Since

$$\begin{aligned}\sum_{y\in\mathbb{Z}^n\setminus\{0\}}e^{-N^2|y|^2}&\le\sum_{j=0}^{n}\sum_{y\in\mathbb{Z}^n,y_j\neq 0}e^{-N^2|y|^2}\\&=n\sum_{y_1=1}^{\infty}\sum_{y_2=0}^{\infty}\cdots\sum_{y_n=0}^{\infty}e^{-N^2|y|^2}\\&=n(\sum_{y_1=1}^{\infty}e^{-N^2|y_1|^2})(\sum_{y_2=0}^{\infty}e^{-N^2|y_2|^2})\ldots(\sum_{y_n=0}^{\infty}e^{-N^2|y_n|^2})\\&\le\frac{ne^{-N^2}}{(1-e^{-N^2})^n},\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^n}|\partial^\alpha f(x)|^2d\gamma(x)\le\left(\frac{ne^{-N^2}}{(1-e^{-N^2})^n}+1\right)\int_{\left[-\frac{N}{2\sqrt{n}},\frac{N}{2\sqrt{n}}\right]^n}|\partial^\alpha f(x)|^2d\gamma(x)<\infty.$$

On the other hand, since $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{2,m}(\gamma)$, we only need to approximate any $g\in C_0^\infty(\mathbb{R}^n)$. For any $\epsilon>0$, there is an positive integer N such that

$$g(x)=0,\ \text{when } x\in\mathbb{R}^n\setminus\left[-\frac{N}{2\sqrt{n}},\frac{N}{2\sqrt{n}}\right]^n$$

and

$$\sum_{y\in\mathbb{Z}^n\setminus\{0\}}e^{-N^2|y|^2}\int_{\mathbb{R}^n}|\partial^\alpha g(x)|^2d\gamma(x)\le\epsilon^2,$$

for any $\alpha\in\mathbb{N}^n$ with $|\alpha|\le m$. Let

$$f(x)=\sum_{y\in\mathbb{Z}^n}g(x+2Ny).$$

Then, we know that $f\in C_p^\infty(\mathbb{R}^n)$ and

$$f(x)=g(x),\ \text{when } x\in\left[-\frac{N}{2\sqrt{n}},\frac{N}{2\sqrt{n}}\right]^n.$$

Then

$$\|g-f\|_{W^{2,m}(\gamma)} = \sum_{|\alpha|\le m}\left[\int_{\mathbb{R}^n}|\partial^\alpha g(x)-\partial^\alpha f(x)|^2 d\gamma(x)\right]^{1/2}$$
$$= \sum_{|\alpha|\le m}\left[\int_{\mathbb{R}^n\setminus[-N,N]^n}|\partial^\alpha f(x)|^2 d\gamma(x)\right]^{1/2}.$$

By the argument above, we know that

$$\int_{\mathbb{R}^n\setminus[-N,N]^n}|\partial^\alpha f(x)|^2 d\gamma(x)$$
$$\le \sum_{y\in\mathbb{Z}^n\setminus\{0\}} e^{-N^2|y|^2}\int_{\left[-\frac{N}{2\sqrt{n}},\frac{N}{2\sqrt{n}}\right]^n}|\partial^\alpha f(x)|^2 d\gamma(x)$$
$$= \sum_{y\in\mathbb{Z}^n\setminus\{0\}} e^{-N^2|y|^2}\int_{[-\frac{N}{2\sqrt{n}},\frac{N}{2\sqrt{n}}]^n}|\partial^\alpha g(x)|^2 d\gamma(x)$$
$$\le \sum_{y\in\mathbb{Z}^n\setminus\{0\}} e^{-N^2|y|^2}\left[\int_{\mathbb{R}^n}|\partial^\alpha g(x)|^2 d\gamma(x)\right]\le \epsilon^2.$$

Then, we have

$$\|g-f\|_{W^{2,m}(\gamma)}\le c_m\epsilon,$$

where $c_m=\mathrm{card}\{\alpha:|\alpha|\le m\}$. We have completed the proof. □

We can now give a characterization of the boundedness of S_φ on $F^{2,m}$. This is the analogue of the result in [3] obtained for the Fock space F^2.

Theorem 4.4. *Let m be a positive integer, then S_φ is bounded on $F^{2,m}$ if and only if*

$$S_\varphi = C_i G M_u G^{-1} C_{-i},$$

where u is a multiplier on $W^{2,m}(\gamma)$. In this case, we have

$$\varphi(z)=\int_{\mathbb{R}^n} u(2x)e^{-2\left(x-\frac{i}{2}z\right)\cdot\left(x-\frac{i}{2}z\right)}dx.$$

Proof. Recall that G and C_i are isometries. If $S_\varphi = C_i G M_u G^{-1} C_{-i}$, where u is a multiplier, then S_φ is bounded.

On the other hand, suppose that S_φ is bounded. By Lemmas 4.1 and 4.2, we know that for any $h\in C_p^\infty(\mathbb{R}^n)$, $G^{-1}C_{-i}S_\varphi C_i G$ commutes with M_h. Let

$$u = G^{-1}C_{-i}S_\varphi C_i G1.$$

Then

$$G^{-1}C_{-i}S_\varphi C_i G h = G^{-1}C_{-i}S_\varphi C_i G M_h 1 = M_h u = M_u h.$$

Since $C_p^\infty(\mathbb{R}^n)$ is a dense subset of $W^{2,m}(\gamma)$, by Lemma 4.3, we know that

$$G^{-1}C_{-i}S_\varphi C_i G = M_u.$$

That is to say $S_\varphi = C_i G M_u G^{-1} C_{-i}$, where u is a multiplier on $W^{2,m}(\gamma)$.

Next, we prove the second part. By Theorem 3.9, we know that u is in $ML^2(\gamma) = L^\infty$. Thus $S_\varphi = C_i G M_u G^{-1} C_{-i}$ is bounded on F^2. Then, by [3, Proposition 3.6 and Theorem 1.1], we have

$$S_\varphi = B\mathcal{F}^{-1} M_v \mathcal{F} B^{-1} \text{ and } \varphi(z) = \int_{\mathbb{R}^n} v(x) e^{-2\left(x-\frac{i}{2}z\right)\cdot\left(x-\frac{i}{2}z\right)} dx,$$

where $\mathcal{F}$ is the Fourier transform and M_v is a multiplication operator with $v \in L^\infty(\mathbb{R}^n)$. On the Fock space, by [3, Lemma 2.3], we have $C_i = B\mathcal{F}^{-1}B^{-1}$. By Proposition 2.2, we have

$$\begin{aligned} S_\varphi &= C_i G M_u G^{-1} C_{-i} \\ &= B\mathcal{F}^{-1} B^{-1} G M_u G^{-1} B \mathcal{F} B^{-1} \\ &= B\mathcal{F}^{-1} C_{\frac{1}{2}}^{-1} M^{-1} G^{-1} G M_u G^{-1} G M C_{\frac{1}{2}} \mathcal{F} B^{-1} \\ &= B\mathcal{F}^{-1} C_{\frac{1}{2}}^{-1} M_u C_{\frac{1}{2}} \mathcal{F} B^{-1} \\ &= B\mathcal{F}^{-1} M_{C_{\frac{1}{2}}^{-1} u} \mathcal{F} B^{-1}. \end{aligned}$$

By the argument above we obtain $v(x) = C_{\frac{1}{2}}^{-1} u = u(2x)$. □

4.1. Other Operator Theoretic Properties

According to the theorems above, we can obtain some properties of S_φ on Fock–Sobolev spaces.

Corollary 4.5. *For any $m > 0$, if S_φ is bounded on $F^{2,m}$, we have following conclusions.*

1. *The set of operators $\{S_\varphi : S_\varphi$ is bounded$\}$ is a commutative algebra.*
2. *S_φ is compact on $F^{2,m}$ if and only if $S_\varphi = 0$.*
3. *S_φ is invertible on $F^{2,m}$ if and only if $\frac{1}{u}$ is essentially bounded, where u is the multiplier on $W^{2,m}(\gamma)$ corresponding to S_φ in Theorem 4.4.*

Proof. (1) follows from Theorem 4.4 and the fact that the set of multiplication operators is a commutative algebra.

To prove (2), we need a fact. For any smooth function η with compact support, there is a sequence of functions f_n such that

$$f_n \to 0 \text{ weakly and } \|f_n\|_{W^{2,m}(\gamma)} = \|\eta\|_{L^2(\gamma)} + O(n^{-1}).$$

Moreover, if $u \in MW^{2,m}(\gamma)$, then we have

$$\|u f_n\|_{W^{2,m}(\gamma)} = \|u\eta\|_{L^2(\gamma)} + O(n^{-1}).$$

For the construction see [6, p. 270]. Although the construction is made for the Sobolev space, the proof is also valid for the Gauss–Sobolev space. If $u \in MW^{2,m}(\gamma)$ is compact, then

$$\lim_{n\to\infty} \|u f_n\|_{W^{2,m}(\gamma)} = 0.$$

That is to say $\|u\eta\|_{L^2(\gamma)} = 0$ for any η, which implies that $u = 0$. By Theorem 4.4, we get the conclusion.

Next we prove (3). If $\frac{1}{u}$ is essentially bounded, we claim that $\frac{1}{u}$ is also a multiplier on $W^{2,m}(\gamma)$. For any α with $|\alpha| = m$, we have

$$\partial^\alpha \frac{1}{u} = \sum_{\beta^1+\cdots+\beta^m \leq \alpha} c_{\beta^1,\ldots,\beta^m,\alpha} \frac{\partial^{\beta^1} u \ldots \partial^{\beta^m} u}{u^{m+1}},$$

where $\{c_{\beta^1,\ldots,\beta^m,\alpha}\}$ are some constants. By Lemma 3.4, we have $\partial^{\beta^1} u \ldots \partial^{\beta^m} u$ is a multiplier from $W^{2,m}(\gamma)$ to $L^2(\gamma)$ for any $\beta^1, \ldots, \beta^m$ with $\beta^1 + \cdots + \beta^m \leq \alpha$, which implies that $\partial^\alpha \frac{1}{u}$ is a multiplier from $W^{2,m}(\gamma)$ to $L^2(\gamma)$. By Theorem 3.9, we obtain that $\frac{1}{u}$ is a multiplier on $W^{2,m}(\gamma)$. Then $M_{\frac{1}{u}}$ is the inverse operator of M_u, which implies that S_φ is invertible.

On the other hand, if S_φ is invertible on $F^{2,m}$, then M_u is invertible on $W^{2,m}(\gamma)$. For any $g \in W^{2,m}(\gamma)$, there is a $f \in W^{2,m}(\gamma)$ such that $g = uf$. Then $\frac{1}{u}g = f \in W^{2,m}(\gamma)$. Since $M_{\frac{1}{u}}$ is a closed operator, we have $M_{\frac{1}{u}}$ is bounded on $W^{2,m}(\gamma)$. By Theorem 3.9, we know that $M_{\frac{1}{u}}$ is bounded on $L^2(\gamma)$. That is to say $\frac{1}{u}$ is essentially bounded. □

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Brett D. Wick
Department of Mathematics and Statistics
Washington University in St. Louis
St. Louis MO 63130
USA
e-mail: wick@math.wustl.edu

Shengkun Wu (✉)
College of Mathematics and Statistics
Chongqing University
Chongqing 401331
People's Republic of China
e-mail: shengkunwu@foxmail.com

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