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# Hardy Factorization in Terms of Multilinear CalderÓN–Zygmund Operators using Morrey Spaces

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**Abstract**
In this paper, we provide a constructive proof of $\mathbf{H}^1(\mathbb{R}^n)$ factorization in terms of multilinear Calderón–Zygmund operators in Morrey spaces. As a direct application, we obtain a characterization of functions in $\mathrm{BMO}(\mathbb{R}^n)$ via commutators of multilinear Calderón–Zygmund operators. Furthermore, we prove a Morrey compactness characterization of $[b, T]_l$, the commutator in the $l$-th entry.

**Keywords** Hardy space · BMO space · Morrey space · Block space · multilinear operators of homogeneous type

**Mathematics Subject Classification (2010)** 42B35 · 42B20

## 1 Introduction and main results

Our main purpose of this paper is to study the Hardy factorization in terms of commutators of multilinear Calderón–Zygmund operators in Morrey spaces. The theory of Hardy spaces has been studied and developed extensively in harmonic analysis. Particularly, the real-variable Hardy space theory on $n$-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 1$, plays an important role in harmonic analysis and has been systematically developed in [6, 15]. A

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celebrated result of 1976, by Coifman–Rochberg–Weiss [6], is that, every $f \in \mathbf{H}^1(\mathbb{R}^n)$ can be written as

$$f = \sum_{k\geq 1}\sum_{j=1}^{n} g_{k,j}\mathcal{R}_j(h_{k,j}) + h_{k,j}\mathcal{R}_j(g_{k,j})$$

with

$$\sum_{k\geq 1}\sum_{j=1}^{n} \left\|g_{k,j}\right\|_{L^2} \left\|h_{k,j}\right\|_{L^2} \leq c\|f\|_{\mathbf{H}^1}$$

where $\mathcal{R}_j$ are the Riesz transforms, for $j = 1, \dots, n$.

As a consequence, a characterization of functions $b$ in BMO($\mathbb{R}^{\mathrm{n})}$ can be obtained via the boundedness of $[b, \mathcal{R}_j]$. After that, the theory of $\mathbf{H}^1(\mathbb{R}^n)$ space has been developed by many authors, see, e.g., [12, 20–22, 28, 30, 31] and the references therein. In [31], Uchiyama extended the Hardy factorization to $\mathbf{H}^p$ on the space of homogeneous type, for any $p \in (\frac{1}{1+\gamma}, 1)$, where $\gamma > 0$ refers to the $\gamma$-Hölder smoothness of singular kernels. Moreover, Komori and Mizuhara [20] obtained a factorization of functions in $\mathbf{H}^1(\mathbb{R}^n)$ in generalized Morrey spaces. Recently, Tao et al. [28] obtained a result of $\mathbf{H}^1(\mathbb{R}^n)$ factorization via $[b, C_\Gamma]$ in Morrey spaces, where $C_\Gamma$ is the Cauchy integral (see [10, 11] for the Morrey boundedness and compactness characterization of $[b, T]$ on spaces of homogeneous type). It is known that the Morrey spaces $L^{p,\alpha}(\mathbb{R}^n)$ (see Definition 1.7) are generalizations of $L^p$ spaces, and they have many important applications to the PDEs (see e.g. [4, 14, 18, 25, 27, 29]).

On the other hand, the multilinear Calderón–Zygmund theory was introduced and studied in the pioneering papers by Coifman and Meyer in [7–9]. The study of multilinear singular integrals was motivated not only as generalizations of the theory of linear ones but also its natural appearance in harmonic analysis. In recent years, this topic has received increasing attentions and well development, such as the systemic treatment of multilinear Calderón–Zygmund operators by Grafakos and Torres in [16, 17], by Christ and Journé in [5], and multilinear fractional integrals by Kenig and Stein in [19]. Weighted estimates and commutators in this multilinear setting were then studied by the authors in [17, 23, 26].

Before we formulate our results, let us first recall the definition of a standard $m$-linear Calderón–Zygmund kernel, $m \geq 1$.

**Definition 1.1** *Let $K(y_0, y_1, \dots, y_m)$, $y_i \in \mathbb{R}^n$, $i = 0, 1, \dots, m$, be a locally integrable function, defined away from the diagonal $\{y_0 = y_1 = \cdots = y_m\}$. Then, we say that $K$ is an $m$-linear Calderón–Zygmund kernel if it satisfies the following size and smoothness conditions:*

$$|K(y_0, y_1, \dots, y_m)| \leq \frac{C_0}{\left(\sum_{k,l=0}^{m} |y_k - y_l|\right)^{mn}},$$

*for some constant $C_0 > 0$ and for all $(y_0, y_1, ..., y_m) \in (\mathbb{R}^n)^{m+1}$ away from the diagonal; and*

$$|K(y_0, y_1, \dots, y_j, \dots, y_m) - K(y_0, y_1, \dots, y_j', \dots, y_m)| \leq \frac{C_0|y_j - y_j'|^\eta}{\left(\sum_{k,l=0}^{m} |y_k - y_l|\right)^{mn+\eta}} \tag{1.1}$$

*for some $\eta > 0$, whenever $0 \leq j \leq m$, and $|y_j - y_j'| \leq \frac{1}{2}\max_{0\leq k\leq m} |y_j - y_k|$.*

Suppose $T$ is an $m$-linear operator mapping from $\mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)\times\cdots\times\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, where we denote by $\mathcal{S}(\mathbb{R}^n)$ the spaces of all Schwartz functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ its dual space. We further assume that $T$ is associated with the $m$-linear Calderón–Zygmund kernel $K$, defined as above, i.e.,

$$T(f_1,\ldots,f_m)(x)=\int_{\mathbb{R}^{mn}} K(x,y_1,\ldots,y_m)\prod_{j=1}^{m} f_j(y_j)\,dy_1\cdots dy_m\,, \tag{1.2}$$

whenever $f_1,\ldots,f_m\in\mathcal{S}(\mathbb{R}^n)$ with compact support and $x\notin\cap_{j=1}^m\operatorname{supp}(f_j)$. We also recall the $j$-th transpose $T_j^*$ of $T$, defined via

$$\langle T_j^*(f_1,\ldots,f_m),h\rangle=\langle T(f_1,\ldots,f_{j-1},h,f_{j+1},\ldots,f_m),f_j\rangle \tag{1.3}$$

for all $f_1,\ldots,f_m,h\in\mathcal{S}(\mathbb{R}^n)$ (see [16, pp. 127–128]). It is easy to verify that the kernel $K_j^*$ of $T_j^*$ is related to the kernel $K$ of $T$ via

$$K_j^*(x,y_1,\ldots,y_{j-1},y_j,y_{j+1},\ldots,y_m)=K(y_j,y_1,\ldots,y_{j-1},x,y_{j+1},\ldots,y_m)\,. \tag{1.4}$$

Now, it suffices to define an $m$-linear operator of Calderón–Zygmund type.

**Definition 1.2** *Let $T$ satisfy (1.2). If $T: L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$ for some*

$$1<p_1,\ldots,p_m<\infty,\ \text{and}\ \frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}\,, \tag{1.5}$$

*then $T$ is called an $m$-linear Calderón–Zygmund operator.*

According to [16, Theorem 3], $T$ can be extended to a bounded operator from $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Moreover, we also have the following pointwise estimate for $T$ (see e.g. [2, 23]).

**Lemma 1.3** *Let $p,p_1,\ldots,p_m>1$ satisfy (1.5). Suppose that $T$ is an $m$-linear Calderón–Zygmund operator. Then, for any $1<q<p$, there exists a constant $C>0$ such that for any vector function $\vec{f}=(f_1,\ldots,f_m)$, where each component is smooth and with compact support, the following inequality holds true*

$$\mathbf{M}^\sharp\left(T(\vec{f})\right)(x)\le C\prod_{j=1}^{m}\mathbf{M}_{q_j}(f_j)(x)\,, \tag{1.6}$$

*with $q_j=\frac{qp_j}{p}$, and where $\mathbf{M}^\sharp$ is denoted by the sharp maximal function.*

A typical example of $m$-linear Calderón–Zygmund operator is the $m$-linear $i$-th Riesz transform, defined by

$$\mathcal{R}_i(f_1,\ldots,f_m)(x)=p.v.\int_{(\mathbb{R}^n)^m}\frac{\sum_{j=1}^m(x_i-(y_j)_i)}{\left(\sum_{j=1}^m|x-y_j|^2\right)^{\frac{mn+1}{2}}}\prod_{j=1}^{m}f_j(y_j)\,dy_1\cdots dy_m\,,$$

where $(y_j)_i$ denotes the $i$-th coordinate of $y_j$.

Next, we denote $\mathcal{T}$ the corresponding maximal operator of $T$, defined as

$$\mathcal{T}(f_1,\ldots,f_m)(x)=\sup_{\delta>0}|T_\delta(f_1,\ldots,f_m)(x)|\,, \tag{1.7}$$

where $T_\delta$, the truncated operator of $T$, is

$$T_\delta(f_1,\ldots,f_m)(x)=\int_{\{\sum_{j=1}^m |x-y_j|^2>\delta^2\}} K(x,y_1,\ldots,y_m)\prod_{j=1}^m f_j(y_j)\,dy_1\cdots dy_m\,. \quad (1.8)$$

Similar to the linear setting, we also have Cotlar's inequality (see [17, Theorem 2.1]). That is for all $r>0$,

$$\mathcal{T}(f_1,\ldots,f_m)(x)\le C\left(\mathbf{M}_r\left(T(f_1,\ldots,f_m)\right)(x)+\prod_{j=1}^m \mathbf{M}f_j(x)\right), \quad (1.9)$$

where

$$\mathbf{M}_r(f)(x)=\sup_{x\in Q}\left(|Q|^{-1}\int_Q |f|^r\,dx\right)^{1/r}$$

and the *supremum* is taken over all cube $Q$ containing $x$. As a consequence, $\mathcal{T}$ maps $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$, where $p_1,p_2,\ldots,p_m$ satisfy (1.5).

In analogy with the linear case, we define the $l$-th partial multilinear commutators of the $m$-linear Calderón–Zygmund operator $T$ as follows.

**Definition 1.4** *Suppose $T$ is an $m$-linear Calderón–Zygmund operator as defined above. For $l=1,2,\ldots,m$, we set*

$$[b,T]_l(f_1,\ldots,f_m)(x):=T(f_1,\ldots,bf_l,\ldots,f_m)(x)-bT(f_1,\ldots,f_m)(x)\,.$$

This is simply measuring the commutation properties in each linear coordinate separately. Dual to the multilinear commutator, in both language and via a formal computation, we define the multilinear "multiplication" operators $\Pi_l$:

**Definition 1.5** *Let $T$ be an $m$-linear Calderón–Zygmund operator. For $l=1,2,\ldots,m$, associate with $T$ the operator*

$$\Pi_l(g,h_1,\ldots,h_m)(x):=h_l(x)T_l^*(h_1,\ldots,h_{l-1},g,h_{l+1},\ldots,h_m)(x)-g(x)T(h_1,\ldots,h_m)(x).$$

**Definition 1.6** *We say that $T$ is $mn$-homogeneous if $T$ satisfies*

$$|T(\mathbf{1}_{B_1},\ldots,\mathbf{1}_{B_m})(x)|\ge\frac{1}{M^{mn}},\qquad \forall x\in B_0(x_0,r) \quad (1.10)$$

*for $(m+1)$ pairwise disjoint balls $B_0=B_0(x_0,r),\ldots,B_m=B_m(x_m,r)$ satisfying the condition that $|y_0-y_l|\approx Mr$ for all $y_0\in B_0$, and $y_l\in B_l$, $l=1,2,\ldots,m$, where $r>0$ and $M>100$.*

Recently, Li and Wick [21] obtained $\mathbf{H}^1(\mathbb{R}^n)$ factorization in terms of multilinear Calderón–Zygmund operators in Lebesgue spaces. As a direct application, they obtained a characterization of $\mathrm{BMO}(\mathbb{R}^n)$ via commutators of the multilinear Riesz transforms.

Inspired by the above works, we want to provide the constructive proof of the weak factorization $\mathbf{H}^1(\mathbb{R}^n)$ in terms of multilinear operators of Calderón–Zygmund type on Morrey spaces.

For convenience, we recall its definition here.

**Definition 1.7** *Let $\alpha\in[0,n)$, and $1<p<\infty$. The Morrey space $L^{p,\alpha}(\mathbb{R}^n)$ is defined by*

$$L^{p,\alpha}(\mathbb{R}^n)=\left\{f\in L^p_{loc}(\mathbb{R}^n):\|f\|_{L^{p,\alpha}}<\infty\right\},$$

*with*

$$\|f\|_{L^{p,\alpha}} = \sup_{B(x,r)} \left( r^{-\alpha} \int_{B(x,r)} |f(y)|^p \, dy \right)^{1/p},$$

*where the supremum is taken over all balls* $B(x,r)$ *in* $\mathbb{R}^n$.

Then, our main results are as follows.

**Theorem 1.8** *Let* $1 \le l \le m$, *and let* $p_1, \ldots, p_m, p > 1$ *satisfy* (1.5). *Suppose that* $T$ *is an* $m$*-linear Calderón–Zygmund operator, satisfying* $mn$*-homogeneous condition* (1.10). *Then, for every function* $f \in \mathbf{H}^1(\mathbb{R}^n)$, *there exist sequences* $\{\lambda_j^k\} \in l^1$ *and functions* $g_j^k, h_{j,1}^k, \ldots, h_{j,m}^k \in L_c^\infty(\mathbb{R}^n)$ *(the space of bounded functions with compact support), such that*

$$f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_j^k \, \Pi_l \left( g_j^k, h_{j,1}^k, \ldots, h_{j,m}^k \right) \tag{1.11}$$

*in the sense of* $\mathbf{H}^1(\mathbb{R}^n)$ *(see the formula of* $\Pi_l(\cdots)$ *in Definition* 1.5*). Moreover, we have that*

$$\|f\|_{\mathbf{H}^1} \approx \inf \left\{ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_j^k| \, \|g_j^k\|_{\mathcal{B}^{p',\alpha}} \, \|h_{j,1}^k\|_{L^{p_1,\alpha_1}} \ldots \|h_{j,m}^k\|_{L^{p_m,\alpha_m}} \right\},$$

*with*

$$0 \le \alpha, \alpha_1, \ldots, \alpha_m < n, \ \text{ and } \ \frac{\alpha}{p} = \sum_{j=1}^{m} \frac{\alpha_j}{p_j}, \tag{1.12}$$

*and where the infimum above is taken over all possible representations of* $f$ *that satisfy* (1.11).

Our next result is a characterization of BMO($\mathbb{R}^n$) in terms of the commutators with the multilinear operators in Morrey spaces.

**Theorem 1.9** *Let* $1 \le l \le m$. *Suppose that* $T$ *is an* $m$*-linear Calderón–Zygmund operator. If* $b \in BMO(\mathbb{R}^n)$, *then the commutator*

$$[b, T]_l : L^{p_1,\alpha_1}(\mathbb{R}^n) \times \cdots \times L^{p_m,\alpha_m}(\mathbb{R}^n) \to L^{p,\alpha}(\mathbb{R}^n)$$

*for* $p_1, ..., p_m, p > 1$ *satisfy* (1.5), *and for* $0 \le \alpha, \alpha_1, ..., \alpha_m < n$ *satisfy* (1.12). *Moreover, there holds true*

$$\|[b, T]_l\|_{L^{p_1,\alpha_1} \times \cdots \times L^{p_m,\alpha_m} \to L^{p,\alpha}} \le C \|b\|_{BMO}.$$

*Conversely, for* $b \in L^1_{loc}(\mathbb{R}^n)$, *if* $T$ *is* $mn$*-homogeneous, and* $[b,T]_l$ *maps* $L^{p_1,\alpha_1}(\mathbb{R}^n) \times \cdots \times L^{p_m,\alpha_m}(\mathbb{R}^n) \to L^{p,\alpha}(\mathbb{R}^n)$, *then* $b \in BMO(\mathbb{R}^n)$ *and*

$$\|b\|_{BMO} \le C \, \|[b, T]_l\|_{L^{p_1,\alpha_1} \times \cdots \times L^{p_m,\alpha_m} \to L^{p,\alpha}}.$$

Finally, we prove a Morrey compactness characterization of $[b, T]_l$ in the following theorem.

**Theorem 1.10** *Same hypotheses as in Theorem 1.9. Then, the commutator*

$$[b, T]_l : L^{p_1,\alpha_1}(\mathbb{R}^n) \times \cdots \times L^{p_m,\alpha_m}(\mathbb{R}^n) \to L^{p,\alpha}(\mathbb{R}^n)$$

*is compact if provided that* $b \in CMO(\mathbb{R}^n)$.

*Conversely, for* $b \in L^1_{loc}(\mathbb{R}^n)$*, if* $T$ *is* $mn$*-homogeneous, and* $[b, T]_l$ *is a compact operator on* $L^{p_1,\alpha_1}(\mathbb{R}^n) \times \cdots \times L^{p_m,\alpha_m}(\mathbb{R}^n)$*, then* $b \in CMO(\mathbb{R}^n)$.

Our paper is organized as follows. In the next section, we give definition of some functional spaces and preliminary results. Section 3 is devoted to the study of Hardy factorization in terms of commutator $[b, T]_l$ on Morrey spaces. As a consequence, we obtain Theorem 1.9. In the last Section, we provide the proof of Theorem 1.10.

**Notation** Through this paper, we denote by $C$ constant which can change from line to line. Next, we denote $A \lesssim B$ if there exists a constant $c > 0$ such that $A \leq cB$. Moreover, we denote $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

## 2 Functional Setting and Preliminary Results

### 2.1 Block Spaces

Following Blasco, Ruiz and Vega [1], we define the function called a *block*.

**Definition 2.1** *Let* $\alpha \in [0, n)$, $1 < q < \infty$, *and* $1/q + 1/q' = 1$. *A function* $b(x)$ *is called a* $(q, \alpha)$*-block, if there exists a ball* $B(x_0, r)$ *such that*

$$\text{supp}(b) \subset B(x_0, r), \quad \|b\|_{L^q} \leq r^{-\frac{\alpha}{q'}}.$$

We further recall the definition of $\mathcal{B}^{q,\alpha}(\mathbb{R}^n)$ via $(q, \alpha)$-blocks from [1].

**Definition 2.2** *Let* $q \in (1, \infty)$ *and* $\alpha \in (0, n)$. *The space* $\mathcal{B}^{q,\alpha}(\mathbb{R}^n)$ *is defined by setting*

$$\mathcal{B}^{q,\alpha}(\mathbb{R}^n) = \left\{ g \in L^1_{loc}(\mathbb{R}^n) : g = \sum_{j=1}^{\infty} m_j b_j, \ \{b_j\}_{j\geq 1} \text{ are } (q,\alpha)\text{-block, and } \sum_{j=1}^{\infty} |m_j| < \infty \right\}.$$

*Furthermore, for every* $g \in \mathcal{B}^{q,\alpha}(\mathbb{R}^n)$, *let*

$$\|g\|_{\mathcal{B}^{q,\alpha}} = \inf \left\{ \sum_{j=1}^{\infty} |m_j| \right\},$$

*where the infimum is taken over all possible decompositions of* $g$ *as above.*

*Remark 2.3* It was showed in [1] that $\mathcal{B}^{q,\alpha}(\mathbb{R}^n)$ is a Banach space, and the dual space of $\mathcal{B}^{q,\alpha}(\mathbb{R}^n)$ is $L^{q',\alpha}(\mathbb{R}^n)$.

Next, we denote $\mathbf{1}_A$, by the characteristic function of $A$. Then, we recall fundamental results, repeatedly used in the following.

**Lemma 2.4** *Let* $\alpha \in [0, n)$, *and* $1 < p < \infty$. *Then, for any ball* $B(x, r)$ *in* $\mathbb{R}^n$, *we have*

$$\left\|\mathbf{1}_{B(x,r)}\right\|_{L^{p,\alpha}} \approx r^{\frac{n-\alpha}{p}}, \tag{2.1}$$

*and*

$$\left\|\mathbf{1}_{B(x,r)}\right\|_{\mathcal{B}^{p',\alpha}} \leq C(n, p)\, r^{n+\frac{\alpha-n}{p}}. \tag{2.2}$$

*Proof* The proof of Eq. 2.1 (respectively (2.2)) is straightforward from the definition of Morrey spaces (respectively $\mathcal{B}^{p',\alpha}$), so we leave it to the reader. □

The next result is a dual inequality between $L^{p,\alpha}(\mathbb{R}^n)$ and $\mathcal{B}^{p',\alpha}(\mathbb{R}^n)$.

**Lemma 2.5** *Let* $1 < p < \infty$, *and* $\alpha \in (0, n)$. *If* $f \in L^{p,\alpha}(\mathbb{R}^n)$, *and* $g \in \mathcal{B}^{p',\alpha}(\mathbb{R}^n)$, *then*

$$\left| \int f(x)g(x)\,dx \right| \leq \|f\|_{L^{p,\alpha}} \|g\|_{\mathcal{B}^{p',\alpha}}\,.$$

*Proof* Since $g \in \mathcal{B}^{p',\alpha}(\mathbb{R}^n)$, then we have

$$g(x) = \sum_{j=1}^{\infty} m_j b_j(x)\,,$$

where $\{b_j\}_{j\geq 1}$ are $(p', \alpha)$-blocks. We can assume that $\operatorname{supp}(b_j) \subset B_j$. Then, applying Hölder's inequality yields

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} f(x)g(x)\,dx \right| &= \left| \int_{\mathbb{R}^n} f(x) \sum_{j=1}^{\infty} m_j b_j(x)\,dx \right| = \left| \sum_{j=1}^{\infty} m_j \int_{B_j} f(x) b_j(x)\,dx \right| \\
&\leq \sum_{j=1}^{\infty} |m_j|\, \|f\|_{L^p(B_j)} \|b_j\|_{L^{p'}(B_j)} \\
&= \sum_{j=1}^{\infty} |m_j|\, r^{-\alpha/p} \|f\|_{L^p(B_j)}\, r^{\alpha/p} \|b_j\|_{L^{p'}(B_j)} \leq \left( \sum_{j=1}^{\infty} |m_j| \right) \|f\|_{L^{p,\alpha}}\,.
\end{aligned}$$

Hence, the conclusion follows from the definition of $\mathcal{B}^{p',\alpha}(\mathbb{R}^n)$. □

## 2.2 The Space Atomic $\mathbf{H}^1(\mathbb{R}^n)$

**Definition 2.6** We say that a real-valued function $a$ is an atom (or 1-atom) if it is supported in $B(x, r) \subset \mathbb{R}^n$, and

$$\int_{\mathbb{R}^n} a(x)\,dx = 0, \quad \text{and } \|a\|_{L^\infty} \leq r^{-n}\,.$$

We now denote the Hardy space atomic $\mathbf{H}^1(\mathbb{R}^n)$ by

$$\mathbf{H}^1(\mathbb{R}^n) = \left\{ \sum_{k\geq 1} \lambda_k a_k : a_k \text{ atoms },\ \lambda_k \in \mathbb{R},\ \sum_{k\geq 1} |\lambda_k| < \infty \right\}.$$

And we define a norm on $\mathbf{H}^1(\mathbb{R}^n)$ by

$$\|f\|_{\mathbf{H}^1(\mathbb{R}^n)} = \inf \left\{ \sum_{k\geq 1} |\lambda_k| : f = \sum_{k\geq 1} \lambda_k a_k \right\}.$$

Next, we recall the following result, obtained from the elementary properties of $\mathbf{H}^1(\mathbb{R}^n)$ (see, e.g., [21, Lemma 2.1], [20, Lemma 4.3], [22] for proofs).

**Lemma 2.7** *Let $x_0, y_0 \in \mathbb{R}^n$ be such that $|x_0 - y_0| = Mr$, for some $r > 0$, and $M > 100$. If $\int_{\mathbb{R}^n} F(x)\,dx = 0$, and*

$$|F(x)| \le r^{-n}\left(\mathbf{1}_{B(x_0,r)}(x) + \mathbf{1}_{B(y_0,r)}(x)\right), \quad \forall x \in \mathbb{R}^n\,,$$

*then there is a positive constant $C = C(n)$, such that*

$$\|F\|_{\mathbf{H}^1(\mathbb{R}^n)} \le C \log M\,.$$

## 3 Weak Hardy Factorization in Terms of Commutator $[b, T]_l$ on Morrey Spaces

This part is devoted to the proof of Theorems 1.8 and 1.9.

### 3.1 Proof of Theorem 1.8

If $f \in \mathbf{H}^1(\mathbb{R}^n)$, then we will utilize the $\mathbf{H}^1(\mathbb{R}^n)$ decomposition of $f$ in order to construct an approximation to $f$ in terms of $\Pi_l(\ldots)$.

**Lemma 3.1** *If $f \in \mathbf{H}^1(\mathbb{R}^n)$ can be written as*

$$f = \sum_{k\ge1} \lambda_k a_k$$

*then, there exist $\{g^k\}_{k\ge1}, \{h_1^k\}_{k\ge1}, \ldots, \{h_m^k\}_{k\ge1} \subset L_c^\infty(\mathbb{R}^n)$ such that*

$$\left\| a_k - \Pi_l(g^k, h_1^k, \ldots, h_m^k)\right\|_{\mathbf{H}^1} \le C\frac{\log M}{M^\eta}\,, \tag{3.1}$$

*and*

$$\sum_{k\ge1} |\lambda_k|\, \|g^k\|_{\mathcal{B}^{p',\alpha}} \|h_1^k\|_{L^{p_1,\alpha_1}} \ldots \|h_m^k\|_{L^{p_m,\alpha_m}} \le CM^{mn}\|f\|_{\mathbf{H}^1}\,, \tag{3.2}$$

*where $M > 0$ is sufficiently large. Furthermore, we have*

$$\left\| f - \sum_{k\ge1} \lambda_k \Pi_l\left(g^k, h_1^k, \ldots, h_m^k\right)\right\|_{\mathbf{H}^1} \le \frac{1}{2}\|f\|_{\mathbf{H}^1}\,. \tag{3.3}$$

*Proof of Lemma 3.1* Let $a$ be an atom, supported in $B(x_0, r)$, for some $x_0 \in \mathbb{R}^n$, and for $r > 0$, such that

$$\|a\|_{L^\infty} \le r^{-n}, \quad \text{and} \int_{\mathbb{R}^n} a(x)\,dx = 0\,.$$

To apply the homogeneity of $T$, we recall a construction of $(m + 1)$-pairwise disjoint balls $B(x_0, r), B(y_1, r), \ldots, B(y_m, r)$ as in [21] satisfying

$$|x_0 - y_l| = |y_j - y_l| = Mr, \quad j = 1, \ldots, m,\ j \ne l\,.$$

Now, let us set

$$\begin{cases} g(x) = \mathbf{1}_{B(y_l,r)}(x)\,, \\ h_j(x) = \mathbf{1}_{B(y_j,r)}(x)\,, \quad j \ne l\,, \\ h_l(x) = \frac{a(x)}{T_l^*(h_1,\ldots,h_{l-1},g,h_{l+1},\ldots,h_m)(x_0)}\,, \end{cases}$$

where $T_l^*$ is the $l$-th transpose of $T$ as defined in Eq. 1.3. It is obvious that these functions are in $L_c^\infty(\mathbb{R}^n)$. Moreover, we observe that, since T is $mn$-homogeneous, and so is $T_l^*$, for the specific choice of the functions $h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m$ as above, we have that there exists a positive constant $C$ such that

$$|T_l^*(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x_0)| \geq CM^{-mn}.$$

From the definitions of the functions $g$ and $h_j$, we obtain

$$\begin{cases} \|g\|_{\mathcal{B}^{p',\alpha}} \leq Cr^{n+\frac{\alpha-n}{p}}, \\ \|h_j\|_{L^{p_j,\alpha_j}} \approx r^{\frac{n-\alpha_j}{p_j}}, \text{ for } j = 1, \dots, m, \text{ and } j \neq l, \\ \|h_l\|_{L^{p_l,\alpha_l}} = \frac{\|a\|_{L^{p_l,\alpha}}}{|T_l^*(h_1,\dots,h_{l-1},g,h_{l+1},\dots,h_m)(x_0)|} \leq CM^{mn} r^{-n} r^{\frac{n-\alpha_l}{p_l}}. \end{cases} \tag{3.4}$$

Therefore, we get by Lemma 2.4

$$\|g\|_{\mathcal{B}^{p',\alpha}} \|h_1\|_{L^{p_1,\alpha_1}} \dots \|h_m\|_{L^{p_m,\alpha_m}} \leq CM^{mn} r^{n+\frac{\alpha-n}{p}+\frac{n-\alpha_1}{p_1}+\dots+\frac{n-\alpha_m}{p_m}-n} = CM^{mn}. \tag{3.5}$$

Next, we claim that

$$\|a - \Pi_l(g, h_1, \dots, h_m)\|_{\mathbf{H}^1} \leq C\frac{\log M}{M^\eta}. \tag{3.6}$$

Indeed, we have

$$\begin{aligned} a(x) - \Pi_l(g, h_1, \dots, h_m)(x) &= a(x) - a(x)\frac{T_l^*(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x)}{T_l^*(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x_0)} \\ &\quad + g(x)T(h_1, \dots, h_m)(x) \\ &= a(x)\left(1 - \frac{T_l^*(\dots)(x)}{T_l^*(\dots)(x_0)}\right) + g(x)T(h_1, \dots, h_m)(x) \\ &:= \mathbf{J}_1 + \mathbf{J}_2, \end{aligned}$$

where we denote $T_l^*(\dots) = T_l^*(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)$ for short.

For $\mathbf{J}_1$, we use the smoothness of $K$ in Eq. 1.1 in order to obtain

$$\begin{aligned} |\mathbf{J}_1| &= |a(x)| \frac{\left|T_l^*(\dots)(x_0) - T_l^*(\dots)(x)\right|}{\left|T_l^*(\dots)(x_0)\right|} \\ &\leq CM^{mn}\|a\|_{L^\infty} \int_{\Pi_{j=1}^m B(y_j,r)} |K(z_l, z_1, \dots, z_{l-1}, x_0, z_{l+1}, \dots, z_m) \\ &\quad - K(z_l, z_1, \dots, z_{l-1}, x, z_{l+1}, \dots, z_m)| \, dz_1 \dots dz_m \\ &\leq CM^{mn} r^{-n} \int_{\Pi_{j=1}^m B(y_j,r)} \frac{|x_0 - x|^\eta}{\left(\sum_{i=1, i\neq l}^m |z_l - z_i| + |z_l - x_0|\right)^{mn+\eta}} dz_1 \dots dz_m \\ &\leq CM^{mn} r^{-n} r^{mn} \frac{r^\eta}{(Mr)^{mn+\eta}} = CM^{-\eta} r^{-n}. \end{aligned}$$

With this inequality noted, and since $a(x)$ is compactly supported in $B(x_0, r)$, then we deduce

$$|\mathbf{J}_1| \leq CM^{-\eta} r^{-n} \mathbf{1}_{B(x_0,r)}. \tag{3.7}$$

Concerning $\mathbf{J}_2$, since $\int_{\mathbb{R}^n} a(x)\,dx = 0$, and $\operatorname{supp}(a) \subset B(x_0, r)$, then we observe that

$$\begin{aligned}
|\mathbf{J}_2| &= \mathbf{1}_{B(y_l,r)} \left| \int_{\mathbb{R}^{mn}} K(x, z_1, \dots, z_m)\, \Pi_{j=1}^m h_j(z_j)\, dz_1 \dots dz_m \right| \\
&= \mathbf{1}_{B(y_l,r)} \left| \int_{\Pi_{j=1, j\neq l}^m B(y_j,r)\times B(x_0,r)} K(x, z_1, \dots, z_m) \frac{a(z_l)}{T_l^*(\dots)(x_0)}\, dz_1 \dots dz_m \right| \\
&= \frac{\mathbf{1}_{B(y_l,r)}}{|T_l^*(\dots)(x_0)|} \left| \int_{\Pi_{j=1, j\neq l}^m B(y_j,r)\times B(x_0,r)} [K(z_1, \dots, x, \dots, z_m) - K(z_1, \dots, x_0, \dots, z_m)]\, a(z_l)\, dz_1 \dots dz_m \right| \\
&\leq C M^{mn} \mathbf{1}_{B(y_l,r)} \int_{\Pi_{j=1, j\neq l}^m B(y_j,r)\times B(x_0,r)} \|a\|_{L^\infty} \frac{|x - x_0|^\eta}{\left(\sum_{j=1}^m |x_0 - z_j|\right)^{mn+\eta}}\, dz_1 \dots dz_m \\
&\leq C \mathbf{1}_{B(y_l,r)} M^{mn} r^{-n} \frac{r^\eta r^{mn}}{(Mr)^{mn+\eta}} = C \mathbf{1}_{B(y_l,r)} M^{-\eta} r^{-n}\,.
\end{aligned}$$

Combining the last inequality and Eq. 3.7 yields

$$|a(x) - \Pi_l(g, h_1, \dots, h_m)(x)| \leq C M^{-\eta} r^{-n} \left(\mathbf{1}_{B(x_0,r)} + \mathbf{1}_{B(y_l,r)}\right)\,.$$

Now, applying Lemma 2.7 to the function $F(x) = a(x) - \Pi_l(g, h_1, \dots, h_m)(x)$, we obtain

$$\|a - \Pi_l(g, h_1, \dots, h_m)\|_{\mathbf{H}^1} \leq C \frac{\log M}{M^\eta}\,. \tag{3.8}$$

Therefore, we obtain (3.1).

Next, it follows from Eq. 3.5 that

$$\|g^k\|_{\mathcal{B}^{p',\alpha}} \|h_1^k\|_{L^{p_1,\alpha_1}} \dots \|h_m^k\|_{L^{p_m,\alpha_m}} \leq C M^{mn}, \quad \text{for } k \geq 1\,.$$

Thus,

$$\sum_{k\geq 1} |\lambda_k|\, \|g^k\|_{\mathcal{B}^{p',\alpha}} \|h_1^k\|_{L^{p_1,\alpha_1}} \dots \|h_m^k\|_{L^{p_m,\alpha_m}} \leq C M^{mn} \|f\|_{\mathbf{H}^1}\,.$$

Hence, we obtain (3.2).

It remains to prove (3.3). By applying (3.8) to $a = a_k$, $k \geq 1$, we obtain that there exist $\{g^k\}_{k\geq 1}$, $\{h_1^k\}_{k\geq 1}, \dots, \{h_m^k\}_{k\geq 1} \subset L_c^\infty(\mathbb{R}^n)$, such that

$$\left\| a_k - \Pi_l(g^k, h_1^k, \dots, h_m^k) \right\|_{\mathbf{H}^1} \leq C \frac{\log M}{M^\eta}\,.$$

This implies that

$$\begin{aligned}
\left\| f - \sum_{k\geq 1} \lambda_k \Pi_l \left(g^k, h_1^k, \dots, h_m^k\right) \right\|_{\mathbf{H}^1} &\leq \sum_{k\geq 1} |\lambda_k| \left\| a_k - \Pi_l \left(g^k, h_1^k, \dots, h_m^k\right) \right\|_{\mathbf{H}^1} \\
&\leq C \frac{\log M}{M^\eta} \sum_{k\geq 1} |\lambda_{k,1}| \\
&\leq \frac{1}{2} \|f\|_{\mathbf{H}^1}
\end{aligned} \tag{3.9}$$

provided that $M$ is large enough. This ends the proof of Lemma 3.1. □

Now, suppose that $f$ can be written as

$$f = \sum_{k\geq 1} \lambda_k a_k\,.$$

Thanks to Lemma 3.1, there exist $\{g^k\}_{k\geq1}, \{h_1^k\}_{k\geq1}, \ldots, \{h_m^k\}_{k\geq1} \subset L_c^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} \sum_{k\geq1} |\lambda_k| \, \|g^k\|_{\mathcal{B}^{p',\alpha}} \|h_1^k\|_{L^{p_1,\alpha_1}} \ldots \|h_m^k\|_{L^{p_m,\alpha_m}} \leq CM^{mn} \|f\|_{\mathbf{H}^1} , \\ \left\| f - \sum_{k\geq1} \lambda_k \Pi_l \left(g^k, h_1^k, \ldots, h_m^k\right) \right\|_{\mathbf{H}^1} \leq \frac{1}{2} \|f\|_{\mathbf{H}^1} . \end{cases}$$

Let us set

$$f_1 = f - \sum_{k\geq1} \lambda_k \Pi_l \left(g^k, h_1^k, \ldots, h_m^k\right) .$$

Since $f_1 \in \mathbf{H}^1(\mathbb{R}^n)$, then we can decompose $f_1$ as follows:

$$f_1 = \sum_{k\geq1} \lambda_{k,1} a_{k,1} ,$$

where $\{\lambda_{k,1}\}_{k\geq1} \in l^1$, and $\{a_{k,1}\}_{k\geq1}$ are atoms.

By applying Lemma 3.1 to $f_1$, there exist $\{g_1^k\}_{k\geq1}, \{h_{1,1}^k\}_{k\geq1}, \ldots, \{h_{1,m}^k\}_{k\geq1} \subset L_c^\infty(\mathbb{R}^n)$, such that

$$\begin{cases} \left\| f_1 - \sum_{k\geq1} \lambda_{k,1} \Pi_l \left(g_1^k, h_{1,1}^k, \ldots, h_{1,m}^k\right) \right\|_{\mathbf{H}^1} \leq \frac{1}{2} \|f_1\|_{\mathbf{H}^1} \leq \frac{1}{2^2} \|f\|_{\mathbf{H}^1} , \\ \sum_{k\geq1} |\lambda_{k,1}| \, \|g_{k,1}\|_{\mathcal{B}^{p',\alpha}} \|h_{1,1}^k\|_{L^{p_1,\alpha_1}} \ldots \|h_{1,m}^k\|_{L^{p_m,\alpha_m}} \leq CM^{mn} \|f_1\|_{\mathbf{H}^1} \leq CM^{mn} \frac{1}{2} \|f\|_{\mathbf{H}^1} . \end{cases}$$

Similarly, we can repeat the above argument to

$$\begin{aligned} f_2 &= f_1 - \sum_{k\geq1} \lambda_{k,1} \Pi_l \left(g_1^k, h_{1,1}^k, \ldots, h_{1,m}^k\right) \\ &= f - \sum_{k\geq1} \lambda_k \Pi_l \left(g^k, h_1^k, \ldots, h_m^k\right) - \sum_{k\geq1} \lambda_{k,1} \Pi_l \left(g_1^k, h_{1,1}^k, \ldots, h_{1,m}^k\right) . \end{aligned}$$

In summary, we can construct a sequence $\{\lambda_{k,j}\} \in l^1$, $\{g_j^k\}$, $\{h_{j,1}^k\}$,..., $\{h_{j,m}^k\} \subset L_c^\infty(\mathbb{R}^n)$, such that

$$\begin{cases} f = \sum_{j=0}^{N} \sum_{k\geq1} \lambda_{k,j} \Pi_l \left(g_j^k, h_{j,1}^k, \ldots, h_{j,m}^k\right) + f_N , \\ \sum_{j=0}^{N} \sum_{k\geq1} |\lambda_{k,j}| \, \|g_j^k\|_{\mathcal{B}^{p',\alpha}} \|h_{j,1}^k\|_{L^{p_1,\alpha_1}} \ldots \|h_{j,m}^k\|_{L^{p_m,\alpha_m}} \leq CM^{mn} \sum_{j=0}^{N} \frac{1}{2^j} \|f\|_{\mathbf{H}^1} , \\ \|f_N\|_{\mathbf{H}^1} \leq \frac{1}{2^N} \|f\|_{\mathbf{H}^1} , \end{cases} \tag{3.10}$$

where we adopt the notations $\lambda_{k,0} = \lambda_k$, $g_{k,0} = g_k$, $h_{0,1}^k = h_1^k, \ldots, h_{0,m}^k = h_m^k$. Thus, the desired result follows as $N \to \infty$. This puts an end to the proof of Theorem 1.8.

## 3.2 Proof of Theorem1.9

To obtain the upper bound of $[b, T]_l$, we recall the following result (see e.g. [2, 23]).

**Lemma 3.2** *Let $b \in BMO(\mathbb{R}^n)$. Then, for any $1 < q < p$ there exists a positive constant $C$ such that*

$$\mathbf{M}^\sharp\left([b,T]_l(\vec{f})\right)(x) \le C\|b\|_{BMO}\left(\prod_{j=1}^m \mathbf{M}_{q_j}(f_j)(x) + \mathbf{M}_q\left(T(\vec{f})\right)(x)\right). \tag{3.11}$$

*with* $q_j = \frac{qp_j}{p}$.

Since $\|g\|_{L^{p,\alpha}} \lesssim \|\mathbf{M}^\sharp(g)\|_{L^{p,\alpha}}$; and $\|\mathbf{M}_q(g)\|_{L^{p,\alpha}} \lesssim \|g\|_{L^{p,\alpha}}$ for $g \in L^{p,\alpha}(\mathbb{R}^n)$ (see e.g. [13]), then applying Hölder's inequality and Eq. 3.11 yields

$$\begin{aligned}\left\|[b,T]_l(\vec{f})\right\|_{L^{p,\alpha}} &\lesssim \left\|\mathbf{M}^\sharp\left([b,T]_l(\vec{f})\right)\right\|_{L^{p,\alpha}} \lesssim \|b\|_{\mathrm{BMO}}\left\|\left(\prod_{j=1}^m \mathbf{M}_{q_j}(f_j) + \mathbf{M}_q\left(T(\vec{f})\right)\right)\right\|_{L^{p,\alpha}} \\ &\lesssim \|b\|_{\mathrm{BMO}}\left(\left\|\prod_{j=1}^m \mathbf{M}_{q_j}(f_j)\right\|_{L^{p,\alpha}} + \left\|\mathbf{M}_q\left(T(\vec{f})\right)\right\|_{L^{p,\alpha}}\right) \\ &\lesssim \|b\|_{\mathrm{BMO}}\left(\prod_{j=1}^m \|\mathbf{M}_{q_j}(f_j)\|_{L^{p_j,\alpha_j}} + \left\|T(\vec{f})\right\|_{L^{p,\alpha}}\right) \\ &\lesssim \|b\|_{\mathrm{BMO}}\prod_{j=1}^m \|f_j\|_{L^{p_j,\alpha_j}}. \end{aligned} \tag{3.12}$$

Hence, we get the desired result.

It remains to prove the lower bound of $[b,T]_l$. The proof can be obtained via the Hardy decomposition in terms of the multilinear operators $\Pi_l$, and the duality between $\mathrm{BMO}(\mathbb{R}^n)$ and $\mathbf{H}^1(\mathbb{R}^n)$.

Indeed, as a matter of fact, $\mathbf{H}^1(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$ is dense in $\mathbf{H}^1(\mathbb{R}^n)$. Next, for every $L > 0$, let us put

$$b_L(x) = b(x)\mathbf{1}_{B(x_0,L)}(x).$$

For every $f \in \mathbf{H}^1(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$, thanks to Theorem 1.8, there exist sequences $\{\lambda^k_j\} \in l^1$ and functions $g^k_j, h^k_{j,1}, \dots, h^k_{j,m} \in L^\infty_c(\mathbb{R}^n)$, such that

$$f = \sum_{k=1}^\infty \sum_{j=1}^\infty \lambda^k_j \,\Pi_l\left(g^k_j, h^k_{j,1}, \dots, h^k_{j,m}\right).$$

Furthermore, we have

$$\|f\|_{\mathbf{H}^1} \approx \sum_{k=1}^\infty \sum_{j=1}^\infty |\lambda^k_j|\, \|g^k_j\|_{\mathcal{B}^{p',\alpha}}\, \|h^k_{j,1}\|_{L^{p_1,\alpha_1}} \cdots \|h^k_{j,m}\|_{L^{p_m,\alpha_m}}.$$

Now, since $b_L \to b$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $L \to \infty$, and $f \in \mathbf{H}^1(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$, then we have

$$\lim_{L\to\infty}\langle b_L, f\rangle = \langle b, f\rangle.$$

Thus,

$$\begin{aligned}\langle b, f\rangle = \lim_{L\to\infty}\langle b_L, f\rangle &= \lim_{L\to\infty}\left\langle b_L, \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\lambda_j^k\,\Pi_l(g_j^k, h_{j,1}^k, \ldots, h_{j,m}^k)\right\rangle\\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\lambda_j^k \lim_{L\to\infty}\left\langle b_L, \Pi_l\left(g_j^k, h_{j,1}^k, \ldots, h_{j,m}^k\right)\right\rangle\\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\lambda_j^k\left\langle b, \Pi_l\left(g_j^k, h_{j,1}^k, \ldots, h_{j,m}^k\right)\right\rangle.\end{aligned}$$

Note that the last equality follows from the fact that $\Pi_l\left(g_j^k, h_{j,1}^k, \ldots, h_{j,m}^k\right) \in L_c^\infty(\mathbb{R}^n)$ for $k, j \geq 1$. Furthermore, we observe that

$$\begin{aligned}&\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\lambda_j^k\left\langle b, \Pi_l\left(g_j^k, h_{j,1}^k, \ldots, h_{j,m}^k\right)\right\rangle\\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\lambda_j^k\int b(x)\left[h_{j,l}^k(x)T_l^*\left(h_{j,1}^k, \ldots, h_{j,l-1}^k, g_j^k, h_{j,l+1}^k, ...h_{j,m}^k\right)(x)\right.\\ &\quad\left. -g_j^k(x)T\left(h_{j,1}^k, \ldots, h_{j,m}^k\right)(x)\right]dx\\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\lambda_j^k\int[b, T]_l\left(h_{j,1}^k, \ldots, h_{j,m}^k\right)(x)\,g_j^k(x)\,dx.\end{aligned}$$

Therefore, we get

$$\langle b, f\rangle = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\lambda_j^k\int[b, T]_l\left(h_{j,1}^k, \ldots, h_{j,m}^k\right)(x)\,g_j^k(x)\,dx.$$

Since $[b, T]_l$ maps $L^{p_1,\alpha_1}(\mathbb{R}^n) \times \cdots \times L^{p_m,\alpha_m}(\mathbb{R}^n) \to L^{p,\alpha}(\mathbb{R}^n)$, and by Lemma 2.5, then we obtain

$$\begin{aligned}|\langle b, f\rangle| &\leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}|\lambda_j^k|\left\|[b, T]_l\left(h_{j,1}^k, \ldots, h_{j,m}^k\right)\right\|_{L^{p,\alpha}}\|g_j^k\|_{\mathcal{B}^{p',\alpha}}\\ &\lesssim \|[b, T]_l\|_{L^{p_1,\alpha_1}\times\cdots\times L^{p_m,\alpha_m}\to L^{p,\alpha}}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}|\lambda_j^k|\left\|h_{j,1}^k\right\|_{L^{p_1,\alpha_1}}\cdots\|h_{j,m}^k\|_{L^{p_m,\alpha_m}}\|g_j^k\|_{\mathcal{B}^{p',\alpha}}\\ &\lesssim \|[b, T]_l\|_{L^{p_1,\alpha_1}\times\cdots\times L^{p_m,\alpha_m}\to L^{p,\alpha}}\|f\|_{\mathbf{H}^1}.\end{aligned}$$

With this inequality noted, it follows from the duality between BMO($\mathbb{R}^n$) and $\mathbf{H}^1(\mathbb{R}^n)$ and the density argument that

$$\|b\|_{\mathrm{BMO}} \lesssim \|[b, T]_l\|_{L^{p_1,\alpha_1}\times\cdots\times L^{p_m,\alpha_m}\to L^{p,\alpha}}.$$

Hence, we complete the proof of Theorem 1.9.

## 4 Compactness Characterization of Functions in Terms of Multilinear Calderón–Zygmund Operators

Here we show how to extend the boundedness results to related compactness results.

*Proof* **a) Necessity:** Assume that $b \in \mathrm{CMO}(\mathbb{R}^n)$. Let $E$ be a bounded set in $L^{p_1,\alpha_1}(\mathbb{R}^n) \times \cdots \times L^{p_m,\alpha_m}(\mathbb{R}^n)$. It is enough to show that $[b, T]_l(E)$ is relatively compact in $L^{p,\alpha}(\mathbb{R}^n)$.

Since $b \in \mathrm{CMO}(\mathbb{R}^n)$ then, for every $\varepsilon > 0$ there exists a function $b_\varepsilon \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that

$$\|b - b_\varepsilon\|_{\mathrm{BMO}} < \varepsilon . \tag{4.1}$$

By the triangle inequality and Theorem 1.9, we have

$$\begin{aligned}
&\|[b, T]_l(f_1, \dots, f_m)\|_{L^{p,\alpha}} \\
\le\ &\|[b - b_\varepsilon, T]_l(f_1, \dots, f_m)\|_{L^{p,\alpha}} + \|[b_\varepsilon, T]_l(f_1, \dots, f_m)\|_{L^{p,\alpha}} \\
\lesssim\ &\|b - b_\varepsilon\|_{\mathrm{BMO}}\|f_1\|_{L^{p_1,\alpha_1}} \dots \|f_m\|_{L^{p_m,\alpha_m}} + \|[b_\varepsilon, T]_l(f_1, \dots, f_m)\|_{L^{p,\alpha}} \\
\le\ &C\varepsilon + \|[b_\varepsilon, T]_l(f_1, \dots, f_m)\|_{L^{p,\alpha}} ,
\end{aligned}$$

for all $(f_1, \dots, f_m) \in E$. With this inequality noted, it suffices to demonstrate that $[b_\varepsilon, T]_l(E)$ is relatively compact in $L^{p,\alpha}(\mathbb{R}^n)$. To obtain the desired result, we recall a compactness criterion in Morrey space (see, e.g., [3]).

**Lemma 4.1** *Let $0 < \alpha < n$, and $1 \le p < \infty$. Suppose the subset $G$ in $L^{p,\alpha}(\mathbb{R}^n)$ satisfies the following conditions:*

(i) $\sup_{f \in G} \|f\|_{L^{p,\alpha}} < \infty$,

(ii) $\lim_{R\to\infty} \|f \chi_{B_R^c}\|_{L^{p,\alpha}} = 0$, *uniformly in $f \in G$, with $B_R^c = \mathbb{R}^n \backslash B_R$,*

(iii) $\lim_{|h|\to 0} \|f(. + h) - f(.)\|_{L^{p,\alpha}} = 0$, *uniformly in $f \in G$. Then $G$ is a strongly pre-compact set in $L^{p,\alpha}(\mathbb{R}^n)$.*

Since $E$ is a bounded set in $L^{p_1,\alpha_1}(\mathbb{R}^n) \times \cdots \times L^{p_m,\alpha_m}(\mathbb{R}^n)$, then $[b_\varepsilon, T]_l(E)$ satisfies $(i)$ by the upper bound of $[b_\varepsilon, T]_l(E)$, obtained in Eq. 3.12.

Next, we show that $[b_\varepsilon, T]_l(E)$ also satisfies $(ii)$. Indeed, suppose that $\mathrm{supp}(b_\varepsilon) \subset B_{R_\varepsilon}$, for some $R_\varepsilon > 1$. Then, for any $(f_1, \dots, f_m) \in E$, and for $x \in B_R^c$, with $R > 10R_\varepsilon$, we observe that

$$\begin{aligned}
&|[b_\varepsilon, T]_l(f_1, \dots, f_m)(x)| \\
=\ &|T(f_1, \dots, bf_l, \dots, f_m)(x)| \\
\le\ &C_0\|b_\varepsilon\|_{L^\infty} \int_{\mathbb{R}^{mn-1}} \int_{\{|y_l|<R_\varepsilon\}} \frac{\prod_{j=1}^m |f_j(y_j)|}{\left(\sum_{j=1}^m |x - y_j|\right)^{mn}}\, dy_l d\overline{y}_l \\
\le\ &C_0\|b_\varepsilon\|_{L^\infty} \int_{\mathbb{R}^{mn-1}} \int_{\{|y_l|<R_\varepsilon\}} \frac{\prod_{j=1}^m |f_j(y_j)|}{|x - y_l|^{\frac{n}{p_0}} \left(\sum_{j=1}^m |x - y_j|\right)^{(m-1)n+\frac{n}{p_0'}}}\, dy_l d\overline{y}_l ,
\end{aligned}$$

where $d\overline{y}_l = \prod_{j=1, j\neq l}^m dy_j$ and for some $1 < p_0 < \min_{j=1,\dots,m}\{p_j\}$.

Since $|x| > R$, and $R > 10R_\varepsilon > 10$, then for any $|y_l| < R_\varepsilon$ we have $\sum_{j=1}^m |x-y_j| > 1+|x-y_k|$, $\forall k = 1, \dots, m, k \neq l$. By combining the above inequalities and by the change of variables, we obtain

$$
\begin{aligned}
&|[b_\varepsilon, T]_l(f_1, \dots, f_m)(x)| \\
&\le C_0\|b_\varepsilon\|_{L^\infty}\left(\int_{\{|y_l|<R_\varepsilon\}} \frac{|f_l(y_l)|}{|x-y_l|^{\frac{n}{p_0}}}\,dy_l\right)\prod_{j=1, j\neq l}^m \int_{\mathbb{R}^n} \frac{|f_j(y_j)|}{\left(1+|x-y_j|\right)^{n+\frac{n}{(m-1)p_0'}}}\,dy_j \\
&= C_0\|b_\varepsilon\|_{L^\infty}\left(\int_{\{|x-y_l|<R_\varepsilon\}} \frac{|f_l(x-y_l)|}{|y_l|^{\frac{n}{p_0}}}\,dy_l\right)\prod_{j=1, j\neq l}^m \int_{\mathbb{R}^n} \frac{|f_j(x-y_j)|}{\left(1+|y_j|\right)^{n+\frac{n}{(m-1)p_0'}}}\,dy_j \\
&\lesssim \left(\int_{\{|x-y_l|<R_\varepsilon\}} \frac{|f_l(x-y_l)|^{p_l}}{|y_l|^{\frac{np_l}{p_0}}}\,dy_l\right)^{1/p_l}\prod_{j=1, j\neq l}^m \int_{\mathbb{R}^n} \frac{|f_j(x-y_j)|}{\left(1+|y_j|\right)^{n+\frac{n}{(m-1)p_0'}}}\,dy_j\,.
\end{aligned}
$$

With this inequality noted, applying Hölder's inequality yields

$$
\begin{aligned}
&\left\|[b_\varepsilon, T]_l(f_1, \dots, f_m)\mathbf{1}_{B_R^c}\right\|_{L^p(B_r)} \\
&\lesssim \left(\int_{B_r}\left(\int_{\{|x-y_l|<R_\varepsilon\}} \frac{|f_l(x-y_l)|^{p_l}\mathbf{1}_{B_R^c}(x)}{|y_l|^{\frac{np_l}{p_0}}}\,dy_l\right)^{p/p_l}\prod_{j=1, j\neq l}^m\left(\int_{\mathbb{R}^n} \frac{|f_j(x-y_j)|\mathbf{1}_{B_R^c}(x)}{(1+|y_j|)^{n+\frac{n}{(m-1)p_0'}}}\,dy_j\right)^p dx\right)^{1/p} \\
&\lesssim \left(\int_{B_r}\int_{\{|x-y_l|<R_\varepsilon\}} \frac{|f_l(x-y_l)|^{p_l}\mathbf{1}_{B_R^c}(x)}{|y_l|^{\frac{np_l}{p_0}}}\,dy_l dx\right)^{1/p_l}\prod_{j=1, j\neq l}^m\left(\int_{B_r}\left(\int_{\mathbb{R}^n} \frac{|f_j(x-y_j)|\mathbf{1}_{B_R^c}(x)}{(1+|y_j|)^{n+\frac{n}{(m-1)p_0'}}}\,dy_j\right)^{p_j} dx\right)^{1/p_j}
\end{aligned}
\tag{4.2}
$$

for any ball $B_r = B(x_0, r)$ in $\mathbb{R}^n$. Concerning the first term in Eq. 4.2, we have

$$
\begin{aligned}
&\left(\int_{B_r}\int_{\{|x-y_l|<R_\varepsilon\}} \frac{|f_l(x-y_l)|^{p_l}\mathbf{1}_{B_R^c}(x)}{|y_l|^{\frac{np_l}{p_0}}}\,dy_l dx\right)^{1/p_l} \\
&\le r^{\frac{\alpha_l}{p_l}}\|f\|_{L^{p_l,\alpha_l}}\left(\int_{\{|y_l|\ge R-R_\varepsilon\}} |y_l|^{-\frac{np_l}{p_0}}\,dy_l\right)^{1/p_l} \\
&\lesssim r^{\frac{\alpha_l}{p_l}}\|f\|_{L^{p_l,\alpha_l}}(R-R_\varepsilon)^{n(\frac{1}{p_l}-\frac{1}{p_0})}\,.
\end{aligned}
\tag{4.3}
$$

For the second term, it follows from Minkowski's inequality that

$$
\begin{aligned}
&\left(\int_{B_r}\left(\int_{\mathbb{R}^n} \frac{|f_j(x-y_j)|\mathbf{1}_{B_R^c}(x)}{(1+|y_j|)^{n+\frac{n}{(m-1)p_0'}}}\,dy_j\right)^{p_j} dx\right)^{1/p_j} \\
&\le \int_{\mathbb{R}^n}\left(\int_{B_r}|f_j(x-y_j)|^{p_j}\mathbf{1}_{B_R^c}(x)\,dx\right)^{1/p_j}\left(1+|y_j|\right)^{-n-\frac{n}{(m-1)p_0'}}\,dy_j \lesssim r^{\frac{\alpha_j}{p_j}}\|f_j\|_{L^{p_j,\alpha_j}}\,.
\end{aligned}
\tag{4.4}
$$

By a combination of Eqs. 4.2, 4.3, and 4.4, we obtain

$$
\left\|[b_\varepsilon, T]_l(f_1, \dots, f_m)\mathbf{1}_{B_R^c}\right\|_{L^p(B_r)} \lesssim r^{\frac{\alpha}{p}}(R-R_\varepsilon)^{n(\frac{1}{p_l}-\frac{1}{p_0})}\prod_{j=1}^m\|f_j\|_{L^{p_j,\alpha_j}}\,.
$$

Thus,

$$\left\|[b_\varepsilon, T]_l(f_1, \dots, f_m)\mathbf{1}_{B_R^c}\right\|_{L^{p,\alpha}} \lesssim (R - R_\varepsilon)^{n(\frac{1}{p_l} - \frac{1}{p_0})} \prod_{j=1}^m \|f_j\|_{L^{p_j,\alpha_j}} .$$

This implies that $\left\|[b_\varepsilon, T]_l(f_1, \dots, f_m)\mathbf{1}_{B_R^c}\right\|_{L^{p,\alpha}} \to 0$, as $R \to \infty$.

Finally, we prove the equicontinuity of $[b_\varepsilon, T]_l$. To do that, we prove that for every $\delta > 0$, if $|z|$ is sufficiently small (merely depending on $\delta$) then, for every $(f_1, \dots, f_m) \in E$,

$$\|[b_\varepsilon, T]_l(f_1, \dots, f_m)(. + z) - [b_\varepsilon, T]_l(f_1, \dots, f_m)(.)\|_{L^{p,\alpha}} \le C\delta^\eta , \tag{4.5}$$

where the constant $C > 0$ is independent of $(f_1, \dots, f_m)$, $\delta$, $|z|$. Indeed, for any $x \in \mathbb{R}^n$ we express

$$\begin{aligned}
&[b_\varepsilon, T]_l(f_1, \dots, f_m)(x + z) - [b_\varepsilon, T]_l(f_1, \dots, f_m)(x) \\
= &\int_{\mathbb{R}^{mn}} (b_\varepsilon(y_l) - b_\varepsilon(x + z))\, K(x + z, y_1, \dots, y_m) \prod_{j=1}^m f_j(y_j)\, dy_1 \cdots dy_m \\
&- \int_{\mathbb{R}^{mn}} (b_\varepsilon(y_l) - b_\varepsilon(x))\, K(x, y_1, \dots, y_m) \prod_{j=1}^m f_j(y_j)\, dy_1 \cdots dy_m \\
= &\int_{\sum_{j=1}^m |x-y_j|>\delta^{-1}|z|} (b_\varepsilon(x) - b_\varepsilon(x + z))\, K(x, y_1, \dots, y_m) \prod_{j=1}^m f_j(y_j)\, dy_1 \cdots dy_m \\
&+ \int_{\sum_{j=1}^m |x-y_j|>\delta^{-1}|z|} (b_\varepsilon(y_l) - b_\varepsilon(x + z))\, [K(x + z, y_1, \dots, y_m) \\
&- K(x, y_1, \dots, y_m)] \prod_{j=1}^m f_j(y_j)\, dy_1 \cdots dy_m \\
&+ \int_{\sum_{j=1}^m |x-y_j|\le\delta^{-1}|z|} (b_\varepsilon(x) - b_\varepsilon(y_l))\, K(x, y_1, \dots, y_m) \prod_{j=1}^m f_j(y_j)\, dy_1 \cdots dy_m \\
&+ \int_{\sum_{j=1}^m |x-y_j|\le\delta^{-1}|z|} (b_\varepsilon(y_l) - b_\varepsilon(x + z))\, K(x + z, y_1, \dots, y_m) \prod_{j=1}^m f_j(y_j)\, dy_1 \cdots dy_m \\
:= &\ \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \mathbf{I}_4.
\end{aligned}$$

We first consider $\mathbf{I}_1$.

$$\begin{aligned}
|\mathbf{I}_1| &\le |b_\varepsilon(x + z) - b_\varepsilon(x)| \left| \int_{\sum_{j=1}^m |x-y_j|>\delta^{-1}|z|} K(x, y_1, \dots, y_m) \prod_{j=1}^m f_j(y_j)\, dy_1 \cdots dy_m \right| \\
&\le \|\nabla b_\varepsilon\|_{L^\infty} |z|\, \mathcal{T}(f_1, \dots, f_m)(x) .
\end{aligned}$$

Therefore, we obtain

$$\|\mathbf{I}_1\|_{L^{p,\alpha}} \le \|\nabla b_\varepsilon\|_{L^\infty}|z|\,\|\mathcal{T}(f_1,\ldots,f_m)\|_{L^{p,\alpha}}\ , \tag{4.6}$$

for $(f_1,\ldots,f_m)\in E$.

Thanks to Cotlar's inequaltiy in Eq. 1.9, the Hölder inequality in Morrey spaces, and Lemma 1.3, we obtain

$$\begin{aligned}
\|\mathcal{T}(f_1,\ldots,f_m)\|_{L^{p,\alpha}} &\lesssim \|\mathbf{M}_r\,(T(f_1,\ldots,f_m))\|_{L^{p,\alpha}} + \left\|\prod_{j=1}^m \mathbf{M}f_j\right\|_{L^{p,\alpha}}\\
&\lesssim \|T(f_1,\ldots,f_m)\|_{L^{p,\alpha}} + \prod_{j=1}^m \left\|\mathbf{M}f_j\right\|_{L^{p_j,\alpha_j}}\\
&\lesssim \left\|\mathbf{M}^\sharp T(f_1,\ldots,f_m)\right\|_{L^{p,\alpha}} + \prod_{j=1}^m \left\|f_j\right\|_{L^{p_j,\alpha_j}}\\
&\lesssim \left\|\prod_{j=1}^m \mathbf{M}_{q_j}(f_j)\right\|_{L^{p,\alpha}} + \prod_{j=1}^m \left\|f_j\right\|_{L^{p_j,\alpha_j}}\\
&\lesssim \prod_{j=1}^m \left\|\mathbf{M}_{q_j}(f_j)\right\|_{L^{p_j,\alpha_j}} + \prod_{j=1}^m \left\|f_j\right\|_{L^{p_j,\alpha_j}} \lesssim \prod_{j=1}^m \left\|f_j\right\|_{L^{p_j,\alpha_j}}\ .
\end{aligned} \tag{4.7}$$

A combination of Eqs. 4.6 and 4.7 implies that

$$\|\mathbf{I}_1\|_{L^{p,\alpha}} \lesssim \|\nabla b_\varepsilon\|_{L^\infty}|z|\prod_{j=1}^m \left\|f_j\right\|_{L^{p_j,\alpha_j}} \lesssim |z|\,, \tag{4.8}$$

for $(f_1,\ldots,f_m)\in E$.

For $\mathbf{I}_2$, thanks to the smoothness of the kernel $K$ (see Eq. 1.1) and the change of variables, we obtain

$$\begin{aligned}
|\mathbf{I}_2| &\lesssim \|b_\varepsilon\|_{L^\infty}|z|^\eta \int_{\sum_{j=1}^m |x-y_j|>\delta^{-1}|z|} \frac{\prod_{j=1}^m |f_j(y_j)|}{(\sum_{j=1}^m |x-y_j|)^{mn+\eta}}\,dy_1\cdots dy_m\\
&= \|b_\varepsilon\|_{L^\infty}|z|^\eta \int_{\sum_{j=1}^m |y_j|>\delta^{-1}|z|} \frac{\prod_{j=1}^m |f_j(x-y_j)|}{(\sum_{j=1}^m |y_j|)^{mn+\eta}}\,dy_1\cdots dy_m\,.
\end{aligned}$$

Applying Minkowski's inequality and Hölder's inequality yields

$$
\begin{aligned}
&\left(r^{-\alpha}\int_{B_r}|\mathbf{I}_2|^p\,dx\right)^{1/p}\\
&\lesssim |z|^{\eta}r^{-\alpha/p}\left(\int_{B_r}\left(\int_{\sum_{j=1}^m|y_j|>\delta^{-1}|z|}\frac{\prod_{j=1}^m|f_j(x-y_j)|}{(\sum_{j=1}^m|y_j|)^{mn+\eta}}\,dy_1\cdots dy_m\right)^p dx\right)^{1/p}\\
&\le |z|^{\eta}r^{-\alpha/p}\int_{\sum_{j=1}^m|y_j|>\delta^{-1}|z|}\left(\int_{B_r}\prod_{j=1}^m|f_j(x-y_j)|^p\,dx\right)^{1/p}\left(\sum_{j=1}^m|y_j|\right)^{-mn-\eta} dy_1\cdots dy_m\\
&\le |z|^{\eta}\int_{\sum_{j=1}^m|y_j|>\delta^{-1}|z|} r^{-\sum_{j=1}^m\frac{\alpha_j}{p_j}}\prod_{j=1}^m\|f_j\|_{L^{p_j}((B(x_0-y_j,r))}\left(\sum_{j=1}^m|y_j|\right)^{-mn-\eta} dy_1\cdots dy_m\\
&\le |z|^{\eta}\prod_{j=1}^m\|f_j\|_{L^{p_j,\alpha_j}}\int_{\sum_{j=1}^m|y_j|>\delta^{-1}|z|}\left(\sum_{j=1}^m|y_j|\right)^{-mn-\eta} dy_1\cdots dy_m\\
&\lesssim |z|^{\eta}\left(\delta^{-1}|z|\right)^{-\eta}\prod_{j=1}^m\|f_j\|_{L^{p_j,\alpha_j}}\lesssim\delta^{\eta}\,.
\end{aligned}
$$

This implies that

$$\|\mathbf{I}_2\|_{L^{p,\alpha}}\lesssim\delta^{\eta}\,. \tag{4.9}$$

Next, we consider $\mathbf{I}_3$. Thanks to Hölder's inequality, we get

$$
\begin{aligned}
|\mathbf{I}_3| &\lesssim \|\nabla b_\varepsilon\|_{L^\infty}\int_{\sum_{j=1}^m|x-y_j|<\delta^{-1}|z|}|x-y_l|\frac{\prod_{j=1}^m|f_j(y_j)|}{(\sum_{j=1}^m|x-y_j|)^{mn}}\,dy_1\cdots dy_m\\
&\lesssim \int_{\sum_{j=1}^m|y_j|<\delta^{-1}|z|}\frac{\prod_{j=1}^m|f_j(x-y_j)|}{(\sum_{j=1}^m|y_j|)^{mn-1}}\,dy_1\cdots dy_m\,.
\end{aligned}
$$

Arguing as in the proof of $\mathbf{I}_2$, we also obtain

$$
\begin{aligned}
&r^{-\alpha/p}\|\mathbf{I}_3\|_{L^p(B_r)}\\
&\lesssim r^{-\alpha/p}\int_{\sum_{j=1}^m|y_j|<\delta^{-1}|z|}\left(\int_{B_r}\prod_{j=1}^m|f_j(x-y_j)|^p\,dx\right)^{1/p}\left(\sum_{j=1}^m|y_j|\right)^{-mn+1} dy_1\cdots dy_m\\
&\le r^{-\sum_{j=1}^m\alpha_j/p_j}\prod_{j=1}^m\|f_j\|_{L^{p_j}(B(x_0-y_j,r))}\int_{\sum_{j=1}^m|y_j|<\delta^{-1}|z|}\left(\sum_{j=1}^m|y_j|\right)^{-mn+1} dy_1\cdots dy_m\\
&\lesssim \delta^{-1}|z|\prod_{j=1}^m\|f_j\|_{L^{p_j,\alpha_j}}\lesssim\delta^{-1}|z|\,.
\end{aligned}
$$

This implies that

$$\|\mathbf{I}_3\|_{L^{p,\alpha}}\lesssim\delta^{-1}|z|\,. \tag{4.10}$$

Similarly, we also obtain

$$|\mathbf{I}_4| \lesssim \|\nabla b_\varepsilon\|_{L^\infty} \int_{\sum_{j=1}^m |x-y_j|<\delta^{-1}|z|} |x+z-y_l| \frac{\prod_{j=1}^m |f_j(y_j)|}{(\sum_{j=1}^m |x+z-y_j|)^{mn}}\, dy_1 \cdots dy_m$$
$$\lesssim \int_{\sum_{j=1}^m |y_j|<\delta^{-1}|z|+|z|} \frac{\prod_{j=1}^m |f_j(x+z-y_j)|}{(\sum_{j=1}^m |y_j|)^{mn-1}}\, dy_1 \cdots dy_m \,.$$

Therefore,

$$r^{-\alpha/p}\|\mathbf{I}_4\|_{L^p(B_r)}$$
$$\lesssim r^{-\alpha/p} \int_{\sum_{j=1}^m |y_j|<\delta^{-1}|z|+|z|} \left(\int_{B_r} \prod_{j=1}^m |f_j(x+z-y_j)|^p\, dx\right)^{1/p} \left(\sum_{j=1}^m |y_j|\right)^{-mn+1} dy_1 \cdots dy_m$$
$$\leq r^{-\sum_{j=1}^m \alpha_j/p_j} \prod_{j=1}^m \|f_j\|_{L^{p_j}(B(x_0+z-y_j,r))} \int_{\sum_{j=1}^m |y_j|<\delta^{-1}|z|+|z|} \left(\sum_{j=1}^m |y_j|\right)^{-mn+1} dy_1 \cdots dy_m$$
$$\lesssim (\delta^{-1}|z|+|z|) \prod_{j=1}^m \|f_j\|_{L^{p_j,\alpha_j}} \lesssim \delta^{-1}|z|+|z| \,.$$

Thus,

$$\|\mathbf{I}_4\|_{L^{p,\alpha}} \lesssim \delta^{-1}|z|+|z| \,. \tag{4.11}$$

A combination of Eqs. 4.8, 4.9, 4.10, and 4.11 provides us

$$\|[b_\varepsilon, T]_l(f_1,\ldots,f_m)(x+z) - [b_\varepsilon, T]_l(f_1,\ldots,f_m)(x)\|_{L^{p,\alpha}} \lesssim \delta^\eta + \delta^{-1}|z| + 2|z| \,.$$

Therefore,

$$\|[b_\varepsilon, T]_l(f_1,\ldots,f_m)(x+z) - [b_\varepsilon, T]_l(f_1,\ldots,f_m)(x)\|_{L^{p,\alpha}} \lesssim \delta^\eta \,,$$

if provided that $|z| < \delta^2$. This yields (4.5). Then, $[b, T]$ is a compact operator on $L^{p_1,\alpha_1}(\mathbb{R}^n) \times \cdots \times L^{p_m,\alpha_m}(\mathbb{R}^n)$.

**b) Sufficiency:** Suppose that $b \in L^1_{\text{loc}}(\mathbb{R})$, and $[b, T]$ is a compact operator on $L^{p_1,\alpha_1}(\mathbb{R}^n) \times \cdots \times L^{p_m,\alpha_m}(\mathbb{R}^n)$. By Theorem 1.9, we have that $b \in \text{BMO}(\mathbb{R}^n)$. We now show that $b \in \text{CMO}(\mathbb{R}^n)$.

To obtain the result, we need a characterization of a function in $\text{CMO}(\mathbb{R}^n)$ (see, e.g., [30]).

**Lemma 4.2** *A function $b \in CMO(\mathbb{R}^n)$ if and only if $b$ satisfies the following three conditions.*

(i) $$\lim_{\delta\to 0} \sup_{B_r, r<\delta} ⨍_{B_r} |b(z) - b_{B_r}|\, dz = 0 \,,$$

(ii) $$\lim_{R\to\infty} \sup_{B_r, r>R} ⨍_{B_r} |b(z) - b_{B_r}|\, dz = 0 \,,$$

(iii) $$\lim_{R\to\infty} \sup_{\{B_r: B_r\cap B(0,R)=\emptyset\}} ⨍_{B_r} |b(z) - b_{B_r}|\, dz = 0 \,.$$

We also need the following result for technical reasons.

**Lemma 4.3** *There exists a positive constant* $M \geq 100$ *such that for any ball* $B_0 := B(x_0, r) \subset \mathbb{R}^n$, *there exist balls* $B_j := B(x_j, r)$, $j = 1, \ldots, m$, *such that*

$$|x_0 - x_j| \approx Mr\,,$$

*and for any* $x \in B_0$, $T(\mathbf{1}_{B_1}, \ldots, \mathbf{1}_{B_m})(x)$ *does not change sign and*

$$\frac{1}{M^{mn}} \lesssim \left|T(\mathbf{1}_{B_1}, \ldots, \mathbf{1}_{B_m})(x)\right|\,. \tag{4.12}$$

*Proof* For any $x \in B_0$, by the smoothness of $K$, we have

$$\begin{aligned}
&\left|T(\mathbf{1}_{B_1}, \ldots, \mathbf{1}_{B_m})(x) - T(\mathbf{1}_{B_1}, \ldots, \mathbf{1}_{B_m})(x_0)\right| \\
&\leq \int_{B_1} \cdots \int_{B_m} |K(x, y_1, \ldots, y_m) - K(x_0, y_1, \ldots, y_m)|\, dy_1 \cdots dy_m \\
&\leq C_0 \int_{B_1} \cdots \int_{B_m} \frac{|x - x_0|^\eta}{\left(\sum_{k,l=0}^m |x_k - x_l|\right)^{mn+\eta}}\, dy_1 \cdots dy_m \\
&\leq C_0' \int_{B_1} \cdots \int_{B_m} \frac{r^\eta}{(Mr)^{mn+\eta}}\, dy_1 \cdots dy_m \leq \frac{C_0''}{M^{mn+\eta}} \leq \frac{1}{2M^{mn}}\,,
\end{aligned} \tag{4.13}$$

if provided that $M$ large enough, and where $C_0''$ merely depends on $n$. If $T(\mathbf{1}_{B_1}, \ldots, \mathbf{1}_{B_m})(x_0) > 0$, then since $T$ is $mn$-homogeneous then, it follows from the triangle inequality and Eq. 4.13 that

$$\begin{aligned}
T(\mathbf{1}_{B_1}, \ldots, \mathbf{1}_{B_m})(x) &\geq T(\mathbf{1}_{B_1}, \ldots, \mathbf{1}_{B_m})(x_0) - \left|T(\mathbf{1}_{B_1}, \ldots, \mathbf{1}_{B_m})(x) - T(\mathbf{1}_{B_1}, \ldots, \mathbf{1}_{B_m})(x_0)\right| \\
&\geq \frac{1}{M^{mn}} - \frac{1}{2M^{mn}} = \frac{1}{2M^{mn}}\,.
\end{aligned}$$

Similarly, we also obtain the conclusion if $T(\mathbf{1}_{B_1}, \ldots, \mathbf{1}_{B_m})(x_0) < 0$. Hence, we complete the proof of Lemma 4.3. □

Now, we are ready to demonstrate that $b \in \mathrm{CMO}(\mathbb{R}^n)$. Seeking a contradiction, we assume that $b \notin \mathrm{CMO}(\mathbb{R}^n)$. Therefore, $b$ violates $(i)$, $(ii)$, and $(iii)$ in Lemma 4.2. We consider these cases in the order.

**Case 1.** If $(i)$ does not hold true for $b$ then, there exists a sequence of balls $\{B_k = B(x_k, \delta_k)\}_{k\geq 1}$ such that $\delta_k \to 0$ as $k \to \infty$, and

$$⨍_{B_k} |b(x) - b_{B_k}|\, dx \geq c_0 > 0, \quad \text{for every } k \geq 1. \tag{4.14}$$

Since $\delta_k \to 0$, we can choose a subsequence of $\{\delta_k\}_{k\geq 1}$ (still denoted by $\{\delta_k\}_{k\geq 1}$) such that

$$\delta_{k+1} \leq \frac{1}{C}\delta_k, \quad \forall k \geq 1,$$

for some $C > 1$. We emphasize that our approach is different to the one, by the authors in [24, 28]. Here, we introduce $m_b(\Omega)$, the median value of function $b$ on a bounded set $\Omega \subset \mathbb{R}^n$ (possibly non-unique) such that

$$\begin{cases} |\{x \in \Omega : b(x) > m_b(\Omega)\}| \leq \frac{1}{2}|\Omega|\,, \\ |\{x \in \Omega : b(x) < m_b(\Omega)\}| \leq \frac{1}{2}|\Omega|\,. \end{cases} \tag{4.15}$$

Next, for any $k \geq 1$, let $y_k^l \in \mathbb{R}^n$ be such that $|z_k - y_k^l| = M\delta_k$, $M > 10$, and put

$$B_k^l = B(y_k^l, \delta_k), \quad B_{k,1}^l = \left\{y_l \in B_k^l : b(y_l) \leq m_b(B_k^l)\right\}, \quad B_{k,2}^l = \left\{y_l \in B_k^l : b(y_l) \geq m_b(B_k^l)\right\};$$

and

$$B_{k,1} = \left\{x \in B_k : b(x) \geq m_b(B_k^l)\right\}, \quad B_{k,2} = \left\{x \in B_k : b(x) < m_b(B_k^l)\right\};$$

also

$$F_{k,1} = B_{k,1}^l \setminus \bigcup_{j=k+1}^{\infty} B_j^l, \quad F_{k,2} = B_{k,2}^l \setminus \bigcup_{j=k+1}^{\infty} B_j^l .$$

Note that $F_{k,1} \cap F_{j,1} = \emptyset$ whenever $j \neq k$, and

$$|F_{k,1}| \geq |B_{k,1}^l| - \sum_{j=k+1}^{\infty} |B_j^l| \gtrsim \delta_k^n - \sum_{l=k+1}^{\infty} \delta_l^n \gtrsim \left(1 - \frac{1}{C-1}\right) \delta_k^n \approx |B_k^l| . \tag{4.16}$$

By the same analogue above, we also obtain

$$|F_{k,2}| \approx |B_k^l| . \tag{4.17}$$

From the construction, we have

$$\left|b(x) - m_b(B_k^l)\right| \leq |b(x) - b(y_l)|, \quad \forall (x, y_l) \in B_{k,j} \times B_{k,j}^l, \; j = 1, 2 . \tag{4.18}$$

Next, it follows from the triangle inequality and Eq. 4.14 that

$$\begin{aligned} c_0 &\leq \fint_{B_k} |b(x) - b_{B_k}| \, dx \\ &\leq 2 \fint_{B_k} |b(x) - m_b(B_k^l)| \, dx \\ &= \frac{2}{|B_k|} \left( \int_{B_{k,1}} |b(x) - m_b(B_k^l)| \, dx + \int_{B_{k,2}} |b(x) - m_b(B_k^l)| \, dx \right) . \end{aligned} \tag{4.19}$$

With this inequality noted, we deduce that there exists a subsequence with respect to $k$ such that either

$$\frac{1}{|B_k|} \int_{B_{k,1}} |b(x) - m_b(B_k^l)| \, dx \geq \frac{c_0}{2} , \tag{4.20}$$

or

$$\frac{1}{|B_k|} \int_{B_{k,2}} |b(x) - m_b(B_k^l)| \, dx \geq \frac{c_0}{2} , \tag{4.21}$$

for any $k \geq 1$. Thus, one can assume without loss of generality that Eq. 4.20 holds.

For any $k \geq 1$, let us denote $B_k^j = B(y_k^j, \delta_k)$, with $|x_k - y_k^j| = M\delta_k$, $j = 1, \ldots, m$. Applying Lemma 4.3 to $B_0 = B_k$, $B_l = F_{k,1}$, and $B_j = B_k^j$, $j = 1, \ldots, m$, $j \neq l$ for any $k \geq 1$ yields

$$M^{-mn} \lesssim \left| T\left(\mathbf{1}_{B_k^1}, \ldots, \mathbf{1}_{F_{k,1}}, \ldots, \mathbf{1}_{B_k^m}\right)(x)\right|, \quad \forall x \in B_k .$$

Furthermore, $T\left(\mathbf{1}_{B_k^1},\dots,\mathbf{1}_{F_{k,1}},\dots,\mathbf{1}_{B_k^m}\right)(x)$ is a constant sign in $B_k$. Then, it follows from Eqs. 4.18 and 4.20 that

$$\begin{aligned}
&\frac{c_0}{2}M^{-mn}\\
&\le \frac{M^{-mn}}{|B_k|}\int_{B_{k,1}}\left|b(x)-m_b(B_k^l)\right|dx\\
&\lesssim \frac{1}{|B_k|}\int_{B_{k,1}}\left|b(x)-m_b(B_k^l)\right|\left|T\left(\mathbf{1}_{B_k^1},\dots,\mathbf{1}_{F_{k,1}},\dots,\mathbf{1}_{B_k^m}\right)(x)\right|dx\\
&= \frac{1}{|B_k|}\int_{B_{k,1}}\left|\int_{\mathbb{R}^{mn}}\left(b(x)-m_b(B_k^l)\right)K(x,y_1,\dots,y_m)\mathbf{1}_{F_{k,1}}\,dy_l\prod_{j=1,j\neq l}^{m}\mathbf{1}_{B_k^j}\,d\overline{y}_l\right|dx\\
&\le \frac{1}{|B_k|}\int_{B_{k,1}}\left|\int_{\mathbb{R}^{mn}}(b(x)-b(y_l))K(x,y_1,\dots,y_m)\mathbf{1}_{F_{k,1}}\,dy_l\prod_{j=1,j\neq l}^{m}\mathbf{1}_{B_k^j}\,d\overline{y}_l\right|dx\\
&= \frac{1}{|B_k|}\int_{B_{k,1}}\left|[b,T]_l(\mathbf{1}_{B_k^1},\dots,\mathbf{1}_{F_{k,1}},\dots,\mathbf{1}_{B_k^m})(x)\right|dx\,.
\end{aligned}\tag{4.22}$$

Next, we put

$$\begin{cases}\phi_k^j(z)=\delta_k^{\frac{\alpha_j-n}{p_j}}\mathbf{1}_{B_k^j}(z)\,, & j=1,\dots,m,\ j\neq l,\\ \phi_k^l(z)=\delta_k^{\frac{\alpha_l-n}{p_l}}\mathbf{1}_{F_{k,1}}(z)\,,\end{cases}$$

for $k\ge 1$. It is clear that

$$\left\|\phi_k^j\right\|_{L^{p_j,\alpha_j}}\approx 1,\quad \text{for every } k\ge 1. \tag{4.23}$$

Thanks to the compactness of $[b,T]_l$, we have that there exists a subsequence of $\left\{[bT]_l(\phi_k^1,\dots,\phi_k^m)\right\}_{k\ge1}$ (still denoted as $\left\{[bT]_l(\phi_k^1,\dots,\phi_k^m)\right\}_{k\ge1}$) such that

$$[bT]_l(\phi_k^1,\dots,\phi_k^m)\to\Psi\quad\text{in } L^{p,\alpha}(\mathbb{R}^n)\,, \tag{4.24}$$

as $k\to\infty$. By Eq. 4.22, we obtain

$$\|\Psi\|_{L^{p,\alpha}}\approx 1\,. \tag{4.25}$$

On the other hand, for $1<q<p$, let $\gamma=\frac{q}{p}$, and $q_j=\gamma p_j$. Since $[b,T]_l$ maps $L^{q_1}(\mathbb{R}^n)\times\cdots\times L^{q_m}(\mathbb{R}^n)\to L^q(\mathbb{R}^n)$, then we obtain

$$\begin{aligned}
\left\|[b,T]_l(\phi_k^1,\dots,\phi_k^m)\right\|_{L^q} &\lesssim \|b\|_{\mathrm{BMO}}\prod_{j=1}^{m}\|\phi_k^j\|_{L^{q_j}}\\
&= \|b\|_{\mathrm{BMO}}\left\|\delta_k^{\frac{\alpha_l-n}{p_l}}\mathbf{1}_{F_{k,1}}\right\|_{L^{q_l}}\prod_{j=1,j\neq l}^{m}\left\|\delta_k^{\frac{\alpha_j-n}{p_j}}\mathbf{1}_{B_k^j}\right\|_{L^{q_j}}\\
&\lesssim \|b\|_{\mathrm{BMO}}\,\delta_k^{\frac{\alpha}{p}+n(\frac{1}{q}-\frac{1}{p})}\,.
\end{aligned}$$

Thus, $[b,T]_l(\phi_k^1,\dots,\phi_k^m)\to 0$ in $L^q(\mathbb{R}^n)$, as $k\to\infty$. This contradicts Eq. 4.25. In other words, $b$ must satisfy $(i)$. Similarly, we also obtain the desired result if Eq. 4.21 holds true. In conclusion, $b$ cannot violate $(i)$.

**Case 2.** Assume that $b$ violates $(ii)$.

The proof of this case is most like that of **Case 1** by considering $R_k$ in place of $\delta_k$, with $R_k \to \infty$. By repeating the above proof for $R_k$ in place of $\delta_k$, we also obtain (4.24) and (4.25). For any $q > \frac{np}{n-\alpha}$, let $\gamma = \frac{q}{p}$, and $q_j = \gamma p_j$. Then, we get

$$\begin{aligned}\left\| [b, T]_l(\phi_k^1, \ldots, \phi_k^m) \right\|_{L^q} &\lesssim \|b\|_{\mathrm{BMO}} \prod_{j=1}^m \|\phi_k^j\|_{L^{q_j}} \\ &= \|b\|_{\mathrm{BMO}} \left\| R_k^{\frac{\alpha_l - n}{p_l}} \mathbf{1}_{F_{k,1}} \right\|_{L^{q_l}} \prod_{j=1, j\neq l}^m \left\| R_k^{\frac{\alpha_j - n}{p_j}} \mathbf{1}_{B_k^j} \right\|_{L^{q_j}} \\ &\lesssim \|b\|_{\mathrm{BMO}}\, R_k^{\frac{\alpha}{p} + n(\frac{1}{q} - \frac{1}{p})}.\end{aligned}$$

Note that $\frac{\alpha}{p} + n(\frac{1}{q} - \frac{1}{p}) < 0$. Thus, $[b, T]_l(\phi_k^1, \ldots, \phi_k^m) \to 0$ in $L^q(\mathbb{R}^n)$, when $k \to \infty$. As a result, we obtain $\Psi \equiv 0$, which contradicts (4.25). In conclusion, $b$ satisfies $ii$).

**Case 3.** The proof of this case is similar to the one of **Case 2**. Thus, we leave it to the reader. From the above cases, we conclude that $b \in \mathrm{CMO}(\mathbb{R}^n)$. □

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