DUFLO-SERGANOVA FUNCTOR AND SUPERDIMENSION FORMULA FOR THE PERIPLECTIC LIE SUPERALGEBRA

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Keywords: Superalgebras, Lie superalgebras, superdimension, periplectic Lie superalgebra, Duflo-Serganova functor.
MSC: 17B10, 17B55.

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$$\Pi M \xrightarrow{x} M \xrightarrow{x} \Pi M,$$

where Π denotes the parity-change functor $- \otimes \mathbb{C}^{0|1}$.

In particular, we show that this $\mathfrak{p}(n-1)$-module is multiplicity-free.

We then use this result to give a simple explicit combinatorial formula for the superdimension of a simple integrable finite-dimensional $\mathfrak{p}(n)$-module, based on its highest weight.

In particular, this reproves the Kac-Wakimoto conjecture for $\mathfrak{p}(n)$, which was proved earlier by the authors.

Declarations:

- *Funding*: I.E.-A. was supported by the ISF grant no. 711/18. V.S. was supported by NSF grant 1701532. Part of the work was carried out during the visit of V.S. to Ben Gurion University of the Negev, which was supported by the Faculty of Natural Sciences Distinguished Scientist Visitors Program and by the Center of Advanced Studies in Mathematics in Ben Gurion University.
- *Conflicts of interest/Competing interests*: N/A.
- *Availability of data and material*: N/A.
- *Code availability*: N/A.
- *Authors' contributions*: N/A.

Date: June 13, 2021.

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To Pavel Etingof for his 50th birthday

Abstract. In this paper, we study the representations of the periplectic Lie superalgebra using the Duflo-Serganova functor. Given a simple $\mathfrak{p}(n)$-module L and a certain odd element $x \in \mathfrak{p}(n)$ of rank 1, we give an explicit description of the composition factors of the $\mathfrak{p}(n-1)$-module $DS_x(L)$, which is defined as the homology of the complex

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1. Introduction

1.1. Consider a complex finite-dimensional vector superspace V, and let $\mathbb{C}^{0|1}$ be the odd one-dimensional vector superspace.

The (complex) periplectic Lie superalgebra $\mathfrak{p}(V)$ is the Lie superalgebra of endomorphisms of a complex vector superspace V preserving a non-degenerate symmetric form $\omega : S^2V \to \mathbb{C}^{0|1}$ (this form is also referred to as an "odd form"). Note that ω exists if and only $\dim V_{\bar{0}} = \dim V_{\bar{1}}$, and in this case it is unique up to the action of the group $\mathrm{Aut}(V)$. Assume that $V_n = \mathbb{C}^{n|n}$ and $\omega_n : S^2V_n \to \mathbb{C}^{0|1}$ pairs the even and odd parts of V_n, we denote the corresponding Lie superalgebra by $\mathfrak{p}(n) := \mathfrak{p}(V_n)$.

The periplectic Lie superalgebra $\mathfrak{p}(n)$ has an interesting non-semisimple representation theory; some results on the category of finite-dimensional integrable representations of $\mathfrak{p}(n)$ can be found in [BaDE+16, Che15, Cou16, DeLZ15, Gor01, HoIR19, IRS19, Moo03, Ser02].

We denote by $\mathcal{F}_n$ the category of finite-dimensional $\mathfrak{p}(n)$-modules such that the $\mathfrak{p}(n)_{\bar{0}} \cong \mathfrak{gl}_n$ action can be lifted to an action of $GL(n)$. An important tool in studying representations of Lie superalgebras, particularly the connection between representation theory of Lie superalgebras of same type but different rank, is the Duflo-Serganova functor. Given an odd element $x \in \mathfrak{p}(n)$ satisfying $[x, x] = 0$, and a $\mathfrak{p}(n)$-module M, we denote by DS_xM the homology of the complex

$$\Pi M \xrightarrow{x} M \xrightarrow{x} \Pi M,$$

where we denote by Π the parity-change functor $- \otimes \mathbb{C}^{0|1}$. The resulting homology is a module over the Lie superalgebra $DS_x(\mathfrak{p}(n))$, and DS_x can be seen as a symmetric monoidal functor

$$\mathcal{F}_n \to \mathrm{Rep}(DS_x(\mathfrak{p}(n))).$$

Date: June 13, 2021.

This functor is called the Duflo-Serganova functor.

This functor has been introduced in [DuS05] in a general Lie superalgebra setting. The Duflo-Serganova functor has been studied extensively for different Lie superalgebras, see for example [EnS18, EnS19, GS17, HeW14, HoR18, IRS19, Ser11]. Its precise effect in the periplectic case has been unknown until now, although it was shown that it can be used to compute Grothendieck rings for $\mathfrak{p}(n)$, see [IRS19].

Note that $GL(n)$ acts on $\mathfrak{p}(n)_{\bar{1}}$ via the adjoint action. It is easy to see that this action has a unique orbit of minimal positive dimension consisting of odd elements of rank 1. For any $x \in \mathfrak{p}(n)$ of rank 1, the Lie superalgebra $DS_x(\mathfrak{p}(n))$ is isomorphic to $\mathfrak{p}(n-1)$. Hence in this case, the Duflo-Serganova functor becomes a symmetric monoidal functor $DS_x : \mathcal{F}_n \to \mathcal{F}_{n-1}$.

Although this DS functor is not exact on either side, it turns out to be extremely useful to carry information between the categories.

1.2. We recall that $\mathfrak{p}(n)_{\bar{0}} \cong \mathfrak{gl}_n(\mathbb{C})$ and we will use the set of simple roots

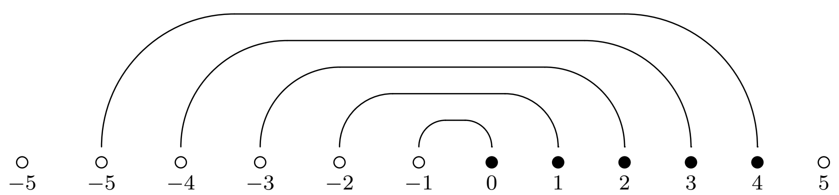
$$\varepsilon_2 - \varepsilon_1, \ldots, \varepsilon_n - \varepsilon_{n-1}, -\varepsilon_{n-1} - \varepsilon_n$$

where the last root is odd and all others are even. Thus the dominant integral weights of $\mathfrak{p}(n)$ are of the form $\lambda = \sum_i \lambda_i \varepsilon_i$, where $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ are integers. The set of dominant integral weights for $\mathfrak{p}(n)$ will be denoted by Λ_n.

Let $L_n(\lambda)$ be a simple module in $\mathcal{F}_n$ with highest weight λ whose highest weight space is purely even. All simple modules in $\mathcal{F}_n$ are of the form $L_n(\lambda)$ or $\Pi L_n(\lambda)$ for some $\lambda \in \Lambda_n$.

For each such weight λ we can construct its *cap diagram* d_λ: namely, we consider the integer line, and draw a black bullet $\bullet$ in each position $\lambda_i + (i-1)$, $i = 1, 2, \ldots, n$; the rest of the positions are empty (we draw the white bullet symbol $\circ$ in all empty positions). We then draw "caps" in this diagram. Each such "cap" is an arc connecting two positions in a diagram; it has a bullet on the right end and an empty position on the left end. The cap diagram is drawn iteratively: at each step, we take the leftmost black bullet which is not yet part of a cap, and draw a cap connecting this bullet with the closest empty position on its left, which is not yet part of any cap.

Here is an example of a cap diagram, corresponding to weight $\lambda = 0$ for $\mathfrak{p}(5)$:



There is a bijection between weight diagrams and cap diagrams. When considering cap diagrams, we will usually not draw bullets since they can be inferred directly from the cap diagram (being the right endpoints of the caps drawn).

We will use the following terminology. If a cap c' is sitting "inside" another cap c, we say that the c' is internal to c (we will also set c to be internal to itself); if $c' \neq c$ and there are no intermediate caps to which c' is internal and which are internal to c (different from c and c'), we say that c' is a successor of c.

A cap c is called *maximal* if it is not internal to any cap other than itself.

Let $x \in \mathfrak{p}(n)_{\bar{1}}$ correspond to the root $2\varepsilon_n$. The first main result of this article, concerning the action of the DS_x functor on simple modules, is as follows:

Theorem 1 (See Theorem 3.1.1, Corollary 3.1.4).

The $\mathfrak{p}(n-1)$-module $DS_x(L_n(\lambda))$ is multiplicity free. Its composition factors can be explicitly described as simple modules $\Pi^{z(\lambda,\mu)}L_{n-1}(\mu)$, where the cap diagram of μ is obtained by removing a single maximal cap from the cap diagram of λ.

The parity $z(\lambda,\mu)\in\mathbb{Z}/2\mathbb{Z}$ is given by $z(\lambda,\mu)\equiv z \mod 2$, where $\lambda_{n-z}+(n-z-1)$ is the rightmost end of the removed cap.

Remark 1.2.1. A similar result for the general linear Lie superalgebra was proved in [HeW14] using a similar technique. However, in contrast with the $\mathfrak{gl}(m|n)$-case, $DS_x(L_n(\lambda))$ may be not semisimple. For example, consider the case $n=2$ and the simple module $V_2\cong\mathbb{C}^{2|2}$ with the tautological action of $\mathfrak{p}(2)$. Then $DS_x(V_2)\cong V_1$ (the $(1|1)$-dimensional tautological representation of $p(1)$), which is indecomposable but not simple. Another example is $n=3$ with $L_n(\lambda)$ being isomorphic to the simple ideal $\mathfrak{sp}(3)$ of matrices with zero supertrace. Then $DS_x(L_n(\lambda))$ is isomorphic to $\mathfrak{sp}(2)$ which is indecomposable but not simple $\mathfrak{p}(2)$-module.

In Section 3.4, we state some corollaries of this theorem, such as a criterion describing when the $\mathfrak{p}(n-1)$-module $DS_x(L_n(\lambda))$ is simple.

1.3. We next proceed to compute the superdimension of any simple finite-dimensional $\mathfrak{p}(n)$-module. This is done by defining a subset of Λ_n consisting of *worthy* weights. For any worthy weight λ, we construct a rooted forest graph F_λ. If λ is not worthy, we show that $\mathrm{sdim}L_n(\lambda)=0$. If λ is worthy, then $\mathrm{sdim}L_n(\lambda)\neq 0$, and we give a simple combinatorial formula for $\mathrm{sdim}L_n(\lambda)$ based on the rooted forest graph F_λ. Below we elaborate on this result.

To state the result on superdimensions, we will need additional terminology.

A cap c in a cap diagram is called *odd* if there is an odd number of caps internal to c, including c itself. A weight $\lambda\in\Lambda_n$ is called *worthy* if each cap c in d_λ has at most one odd successor, and there is at most one maximal odd cap (such a cap will appear for worthy weights only when n is odd).

If λ is worthy, we will construct a rooted forest F_λ corresponding to λ as follows.

We begin by constructing a *reduced* cap diagram d_λ^{red}: this is done by erasing the odd caps in d_λ. The partial order on the caps of d_λ induces a partial order on the caps of d_λ^{red}. The notion of "successor" for caps in d_λ^{red} is defined accordingly.

The reduced cap diagram defines a rooted forest F_λ:

Definition 1.3.1. Let λ be a worthy weight. We construct a rooted forest F_λ as follows.

- The nodes of F_λ are caps c in the reduced cap diagram d_λ^{red}.
- There is an edge from a node c to a node c' in F_λ if c' is a successor of c.

Remark 1.3.2. This is a slightly different (but equivalent) version of Definition 4.1.11.

We can now state our second main theorem. Recall that $\mathrm{sdim}V=\dim V_{\bar 0}-\dim V_{\bar 1}$ for any finite dimensional vector superspace V.

Theorem 1.3.3 (See Theorem 4.2.1).

Let $\lambda\in\Lambda_n$ and let $L_n(\lambda)$ be the corresponding simple module in $\mathcal{F}_n$ (as in Section 1.2).

If the weight λ is not worthy, then

$$\mathrm{sdim}L_n(\lambda)=0.$$

If the weight λ is worthy, let F_λ be the corresponding rooted forest. Then

$$\mathrm{sdim} L_n(\lambda) = \frac{|F_\lambda|!}{F_\lambda!}$$

where $|F_\lambda| = \lfloor \frac{n+1}{2} \rfloor$ is the number of nodes in the forest F_λ, and

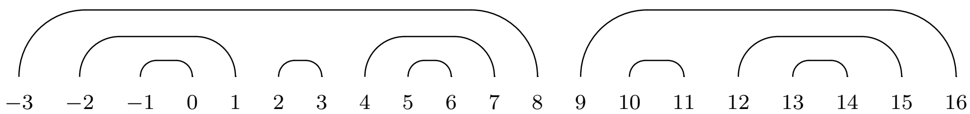
$$F_\lambda! = \prod_{v \text{ a node of } F_\lambda} \sharp \text{ descendants of } v \text{ in } F_\lambda$$

is the forest factorial of F_λ[1].

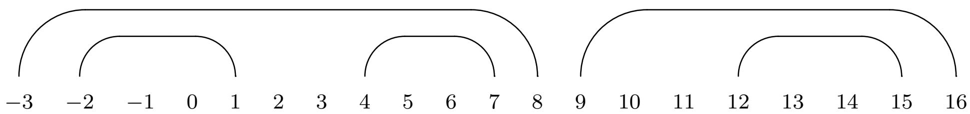
Example 1.3.4. For the weight

$$\lambda = \varepsilon_3 + 3\varepsilon_4 + 3\varepsilon_5 + 3\varepsilon_6 + 5\varepsilon_7 + 7\varepsilon_8 + 7\varepsilon_9 + 7\varepsilon_{10}$$

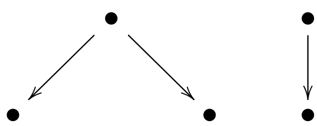
of $\mathfrak{p}(10)$, the cap diagram is



This is a worthy weight, with odd caps $(-1, 0), (2, 3), (5, 6), (10, 11), (13, 14)$; the rest of the caps are even. The reduced cap diagram is



The rooted forest is



Hence $\mathrm{sdim} L_n(\lambda) = \frac{5!}{3 \cdot 1 \cdot 1 \cdot 2 \cdot 1} = 20$.

As a corollary, we recover the result of [EnS19] proving the Kac-Wakimoto conjecture for $\mathfrak{p}(n)$: any module lying in a "non-principal" block of $\mathcal{F}_n$ (in the sense of [EnS19]) has superdimension zero.

1.4. **Acknowledgements.** I.E.-A. was supported by the ISF grant no. 711/18. V.S. was supported by NSF grant 1701532. Part of the work was carried out during the visit of V.S. to Ben Gurion University of the Negev, which was supported by the Faculty of Natural Sciences Distinguished Scientist Visitors Program and by the Center of Advanced Studies in Mathematics in Ben Gurion University.

The authors would like to thank Catharina Stroppel for her explanations on the cap diagrams, and the anonymous referees for carefully reading an earlier version of this manuscript and for their very helpful comments.

[1] Each node is considered its own descendant.

2. Preliminaries

2.1. **General.** Throughout this paper, we will work over the base field $\mathbb{C}$, and all the categories considered will be $\mathbb{C}$-linear.

A *vector superspace* is defined as a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. The *parity* of a homogeneous vector $v \in V$ is denoted by $p(v) \in \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ (whenever the notation $p(v)$ appears in formulas, we always assume that v is homogeneous).

2.2. The periplectic Lie superalgebra.

2.2.1. *Definition of the periplectic Lie superalgebra.* Let $n \in \mathbb{Z}_{>0}$, and let V_n be an $(n|n)$-dimensional vector superspace equipped with a non-degenerate odd symmetric form

(1) $\quad \omega : V_n \otimes V_n \to \mathbb{C}, \quad \omega(v, w) = \omega(w, v), \quad$ and $\quad \omega(v, w) = 0$ if $p(v) = p(w)$.

Then $\mathrm{End}_{\mathbb{C}}(V_n)$ inherits the structure of a vector superspace from V_n. We denote by $\mathfrak{p}(n)$ the Lie superalgebra of all $X \in \mathrm{End}_{\mathbb{C}}(V_n)$ preserving ω, i.e. satisfying

$$\omega(Xv, w) + (-1)^{p(X)p(v)}\omega(v, Xw) = 0.$$

Remark 2.2.1. Choosing dual bases $v_1, v_2, \dots, v_n$ in $V_{\bar{0},n}$ and $v_{1'}, v_{2'}, \dots v_{n'}$ in $V_{\bar{1},n}$, we can write the matrix of $X \in \mathfrak{p}(n)$ as $\left(\begin{smallmatrix} A & B \\ C & -A^t \end{smallmatrix}\right)$ where A, B, C are $n \times n$ matrices such that $B^t = B$, $C^t = -C$.

We will also use the triangular decomposition $\mathfrak{p}(n) \cong \mathfrak{p}(n)_{-1} \oplus \mathfrak{p}(n)_0 \oplus \mathfrak{p}(n)_1$ where

$$\mathfrak{p}(n)_0 \cong \mathfrak{gl}(n), \;\; \mathfrak{p}(n)_{-1} \cong \Pi \wedge^2 (\mathbb{C}^n)^*, \;\; \mathfrak{p}(n)_1 \cong \Pi S^2\mathbb{C}^n.$$

Then the action of $\mathfrak{p}(n)_{\pm 1}$ on any $\mathfrak{p}(n)$-module is $\mathfrak{p}(n)_0$-equivariant.

2.2.2. *Weights for the periplectic superalgebra.* The integral weight lattice for $\mathfrak{p}(n)$ will be $span_{\mathbb{Z}}\{\varepsilon_i\}_{i=1}^n$.

- ⋆ We denote by $\mathfrak{b}_{0,n}^-$ the Borel subalgebra of $\mathfrak{p}(n)_0$ consisting of lower triangular matrices A under the identification $\mathfrak{p}(n)_0 \cong \mathfrak{gl}(n)$ as in Remark 2.2.1.

  We also fix the "lower-triangular" Borel subalgebra $\mathfrak{b}_n^- = \mathfrak{b}_{0,n}^- + \mathfrak{p}(n)_{-1}$ in $\mathfrak{p}(n)$. In terms of the matrix description given in Remark 2.2.1, the elements of $\mathfrak{b}_n^-$ are given by matrices $\begin{pmatrix} A & 0 \\ C & -A^t \end{pmatrix}$ in $\mathfrak{p}(n)$ where A is lower-triangular.
- ⋆ The choice of the Borel subalgebra $\mathfrak{b}_n^-$ gives us the set of simple roots $\varepsilon_2 - \varepsilon_1, \dots, \varepsilon_n - \varepsilon_{n-1}, -\varepsilon_{n-1} - \varepsilon_n$ for $\mathfrak{p}(n)$, where the last root is odd and all others are even. The set of all dominant integral weights for $\mathfrak{p}(n)$ will be denoted by Λ_n.
- ⋆ The dominant integral weights with respect to this choice of the Borel subalgebra are of the form $\lambda = \sum_i \lambda_i \varepsilon_i$, where $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$.
- ⋆ We fix an order on the integral weights of $\mathfrak{p}(n)$: for weights μ, λ, we say that $\mu \ge \lambda$ if $\mu_i \le \lambda_i$ for each i.

  Remark 2.2.2. It was shown in [BaDE+16, Section 3.3] that the order $\le$ corresponds to a highest-weight structure on the category of finite-dimensional representations of $\mathfrak{p}(n)$. Note that in the cited paper we use slightly different set of simple roots $-\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n$. Our choice of a different Borel subalgebra is a matter of convenience since we would like to avoid the shift in the combinatorial algorithm for Duflo-Serganova functor. Indeed, as we use an embedding $\mathfrak{p}(n-1) \subset \mathfrak{p}(n)$ it is natural to require that the Weyl vector $\rho^{(n)}$ defined below is given by the uniform formula for all n. The results of [BaDE+16] are applicable in this case since the only difference is in permutation of indices $1, \dots, n$.

- ⋆ The simple finite-dimensional representation of $\mathfrak{p}(n)$ corresponding to the weight λ whose highest weight vector is *even* will be denoted by $L_n(\lambda)$.

  Example 2.2.3. Let $n \geq 2$. The natural representation V_n of $\mathfrak{p}(n)$ has highest weight $-\varepsilon_1$, with odd highest-weight vector; hence $V_n \cong \Pi L_n(-\varepsilon_1)$. The representation $\bigwedge^2 V_n$ (the second exterior power of the vector superspace V_n) has highest weight $-2\varepsilon_1$, and the representation $S^2 V_n$ (the second symmetric power of the vector superspace V_n) has highest weight $-\varepsilon_1 - \varepsilon_2$; both have even highest weight vectors, so

  $$\wedge^2 V_n \twoheadrightarrow L_n(-2\varepsilon_1), \quad L_n(-\varepsilon_1 - \varepsilon_2) \hookrightarrow S^2 V_n.$$

- ⋆ Set $\rho^{(n)} = \sum_{i=1}^{n} (i-1)\varepsilon_i$, and for any weight λ, denote

  $$\bar{\lambda} = \lambda + \rho^{(n)}.$$

2.2.3. *Representations of* $\mathfrak{p}(n)$*.* We denote by $\mathcal{F}_n$ the category of finite-dimensional representations of $\mathfrak{p}(n)$ whose restriction to $\mathfrak{p}(n)_{\bar{0}} \cong \mathfrak{gl}(n)$ integrates to an action of $GL(n)$.

By definition, the morphisms in $\mathcal{F}_n$ will be *grading-preserving* $\mathfrak{p}(n)$-morphisms, i.e., $\mathrm{Hom}_{\mathcal{F}_n}(X, Y)$ is a vector space and not a vector superspace. This is important in order to ensure that the category $\mathcal{F}_n$ be abelian.

The category $\mathcal{F}_n$ is not semisimple. In fact, this category is a highest-weight category; more about the highest-weight structure can be found in [BaDE$^+$16].

2.2.4. *Weight diagrams and arrows.* The following notation has been introduced in [BaDE$^+$16].

For λ a dominant weight we define the map

$$f_\lambda : \mathbb{Z} \to \{0, 1\} \quad \text{as} \quad f_\lambda(i) \quad = \quad \begin{cases} 1 & \text{if } i \in \{\bar{\lambda}_j,\, j = 1, \dots, n\}, \\ 0 & \text{else.} \end{cases}$$

The corresponding *weight diagram* d_λ is the labeling of the integer line by symbols $\bullet$ ("black bullet") and $\circ$ ("empty") such that i has label $\bullet$ if $f(i) = 1$, and label $\circ$ otherwise.

Definition 2.2.4. For $\lambda \in \Lambda_n$ we define the function $g_\lambda : \mathbb{Z} \to \{-1, 1\}$ by setting

$$g_\lambda(i) = (-1)^{f_\lambda(i)+1}.$$

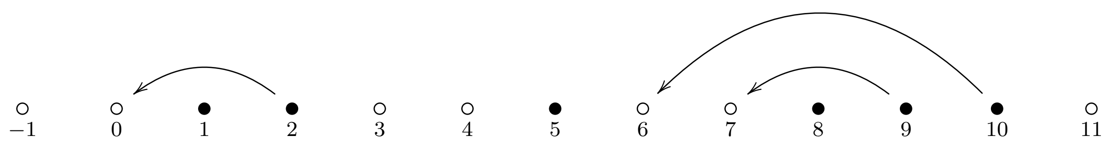
So $g_\lambda(i) = 1$ if d_λ has a black bullet at the i-th position and $g_\lambda(i) = -1$ otherwise.

Notation 2.2.5. For any $i < j$ set $r_\lambda(j, i) \;=\; \sum_{s=i}^{j-1} g_\lambda(s)$.

As in [BaDE$^+$16, Section 6.2], in the diagram d_λ we will draw a solid[2] arrow from position j to position $i < j$ if $f_\lambda(j) = 1 = g_\lambda(j)$, and if

$$r_\lambda(j, i) = 0, \quad \text{and } \forall\, i < s < j,\ r_\lambda(j, s) \geq 0.$$

Example 2.2.6. Let $n = 6$, $\lambda = \varepsilon_1 + \varepsilon_2 + 3\varepsilon_3 + 5\varepsilon_4 + 5\varepsilon_5 + 5\varepsilon_6$. The diagram d_λ with solid arrows is given by



[2]In this paper we do not use any other types of arrows, but in [BaDE$^+$16] "dual" dashed arrows were introduced as well.

and all other positions on the integer line are empty.

Definition 2.2.7. Let $\lambda \in \Lambda_n$. Consider the solid arrows in the diagram d_λ. We will call a solid arrow *maximal* if there is no solid arrow above it; in other words, a solid arrow from j to i is called maximal if there is no solid arrow from k to l where $l \leq i$, $k \geq j$ and $(k, l) \neq (j, i)$.

Example 2.2.8. In Example 2.2.6, the two maximal solid arrows are $(0, 2), (6, 10)$.

Definition 2.2.9. A (black) cluster in a weight diagram d_α is a sequence of consecutive black bullets:

$$d_\alpha = \underset{i-1}{\circ} \quad \underset{i}{\bullet} \quad \underset{i+1}{\bullet} \quad \ldots \quad \underset{j-1}{\bullet} \quad \underset{j}{\bullet} \quad \underset{j+1}{\circ}$$

In other words, it is a segment in of the form $[i, j]$, $i < j$ such that

$$f_\alpha(i-1) = 0,\ f_\alpha(i) = f_\alpha(i+1) = \ldots = f_\alpha(j-1) = f_\alpha(j) = 1,\ f_\alpha(j+1) = 0.$$

Position i is called the beginning of the cluster, and position j is called the end of the cluster.

2.2.5. *Cap diagrams.* Consider the weight diagram d_λ of λ. Instead of drawing arrows, we can draw a cap diagram on the integer line $\mathbb{Z}$. Each "cap" is an arc connecting two positions in our diagram. The cap diagram is drawn iteratively: at each step, we take the leftmost black bullet which is not yet part of a cap, and draw a cap connecting this bullet with the closest empty position on its left, which is not yet part of any cap.

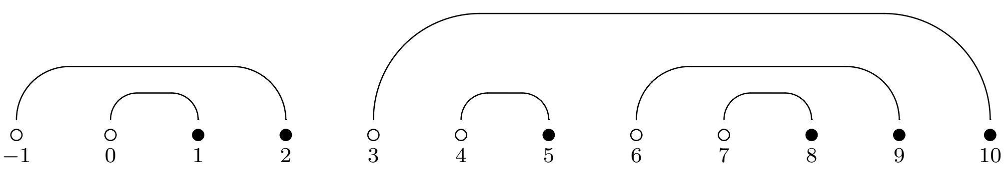
We denote by (i, j) the cap whose left end is in position i and right end is in position j (so $f_\lambda(i) = 0, f_\lambda(j) = 1$).

Clearly, every black bullet in d_λ is the right end of exactly one cap and the obtained caps are non-crossing. The weight diagram d_λ can be uniquely determined from the cap diagram (by abuse of notation, the cap diagram is also denoted d_λ).

Definition 2.2.10.

- A cap (i, j) is called *internal* to a cap (i', j') if $i' \leq i < j \leq j'$. We denote: $(i, j) \trianglelefteq (i', j')$. If these caps to not coincide (that is, if $(i, j) \neq (i', j')$) we denote $(i, j) \lhd (i', j')$.
- A cap (i, j) is called *maximal* if it is not internal to any other cap.
- A cap (i, j) is called *a successor* of a cap (i', j') if $(i, j) \lhd (i', j')$ and there is no cap (i'', j'') such that $(i, j) \lhd (i'', j'') \lhd (i', j')$.

Example 2.2.11. Consider the weight $\lambda = \varepsilon_1 + \varepsilon_2 + 3\varepsilon_3 + 5\varepsilon_4 + 5\varepsilon_5 + 5\varepsilon_6$ for $\mathfrak{p}(6)$, as in Example 2.2.6. Here we draw the cap diagram for λ on top of the weight diagram d_λ:



The partial order on the caps in this diagram is:

$$(0, 1) \lhd (-1, 2), \quad (4, 5) \lhd (3, 10),\ (7, 8) \lhd (6, 9) \lhd (3, 10).$$

The maximal caps here are $(-1, 2)$ and $(3, 10)$. The successors of the cap $(3, 10)$ are $(4, 5)$, $(6, 9)$.

Remark 2.2.12. Every solid arrow goes from the right end of a cap to the left end of one of its successor caps. In particular, the total number of solid arrows equals n minus the number of maximal caps.

Lemma 2.2.13. *Let (i, j) be a cap in the cap diagram d_λ. Then exactly one of the following is true:*

- *We have $i + 1 = j$.*
- *There is a solid arrow from j to $i+1$, and this is the longest solid arrow originating in j.*

Proof. First of all, if $i+1 = j$ then clearly there is no solid arrow from j to $i+1$. Assume $i + 1 \neq j$. By the construction of the cap diagram, we have:

$$\forall i+1 \leq l \leq j-1,\ \ r_\lambda(j,l) = \sum_{s=l}^{j-1} g_\lambda(s) \geq 0,\ r_\lambda(j, i+1) = \sum_{s=i+1}^{j-1} g_\lambda(s) = 0,\ r_\lambda(j,i) < 0$$

Hence the statement follows. ☐

Corollary 2.2.14. *Let (i, j) be a maximal cap in the cap diagram of d_λ. Then either $i+1 = j$ or there is a solid arrow from j to $i+1$, and this solid arrow is maximal.*

2.2.6. *Tensor Casimir and translation functors.* The constructions in this section follow [BaDE+16, Section 4].

Recall that $\mathfrak{p}(n)$ is the set of fixed points of the involutive anti-automorphism σ : $\mathfrak{gl}(n|n) \to \mathfrak{gl}(n|n)$ defined as

$$\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)^\sigma := \begin{pmatrix} -D^t & B^t \\ -C^t & -A^t \end{pmatrix}.$$

Then $\mathfrak{p}(n) \subset \mathfrak{gl}(n|n)$ is given by all elements fixed by σ and we have a $\mathfrak{p}(n)$-equivariant decomposition $\mathfrak{gl}(n|n) \cong \mathfrak{p}(n) \oplus \mathfrak{p}(n)^*$ where

$$\{x \in \mathfrak{gl}(n|n) \mid x^\sigma = -x\} = \mathfrak{p}(n)^*.$$

Both $\mathfrak{p}(n)$ and $\mathfrak{p}(n)^*$ are maximal isotropic subspaces with respect to the invariant symmetric form on $\mathfrak{gl}(n|n)$ given by the supertrace, and hence this form defines a non-degenerate pairing $\mathfrak{p}(n)^* \otimes \mathfrak{p}(n) \to \mathbb{C}$.

Definition 2.2.15 (Tensor Casimir)**.** For any $M \in \mathcal{F}_n$, let Ω_M be twice the composition

$$M \otimes V_n \xrightarrow{\mathrm{Id} \otimes coev \otimes \mathrm{Id}} M \otimes \mathfrak{p}(n) \otimes \mathfrak{p}(n)^* \otimes V_n \xrightarrow{i_* \otimes \mathrm{Id}} M \otimes \mathfrak{p}(n) \otimes \mathfrak{gl}(V_n) \otimes V_n \xrightarrow{act \otimes (\tau \circ act)} M \otimes V_n$$

where $i_* : \mathfrak{p}(n)^* \to \mathfrak{gl}(V_n)$ denotes the $\mathfrak{p}(n)$-equivariant embedding defined above, and $coev : \mathbb{C} \to \mathfrak{p}(n) \otimes \mathfrak{p}(n)^*$ denotes the coevaluation morphism (sending 1 to $\sum_i X_i \otimes X_i^*$ where X_i form a basis in $\mathfrak{p}(n)$ and X_i^* form the dual basis).

Finally, $act : \mathfrak{gl}(V_n) \otimes V_n \to V_n$, $act : \mathfrak{p}(n) \otimes M \to M$ denote the action maps and $\tau : M \otimes \mathfrak{p}(n) \to \mathfrak{p}(n) \otimes M$ the (super) symmetry morphism.

We write $\Omega^{(n)}$ for the corresponding endomorphism of the endofunctor $(-) \otimes V_n$ of $\mathcal{F}_n$.

Definition 2.2.16 (Translation functors)**.** For $k \in \mathbb{C}$, we define a functor ${\Theta'}_k^{(n)} : \mathcal{F}_n \to \mathcal{F}_n$ as the functor $\Theta^{(n)} = (-) \otimes V_n$ followed by the projection onto the generalized k-eigenspace for $\Omega^{(n)}$, i.e.

$$(2) \qquad {\Theta'}_k^{(n)}(M) := \bigcup_{m>0} \mathrm{Ker}(\Omega_M - k\,\mathrm{Id})^m_{|M \otimes V_n}$$

and set $\Theta_k^{(n)} := \Pi^k {\Theta'}_k^{(n)}$ in case $k \in \mathbb{Z}$ (it was proved in [BaDE+16, Proposition 4.1.9] that $\forall k \notin \mathbb{Z},\ \Theta_k^{(n)} \cong 0$).

The functors $\Theta_k^{(n)}$ are exact (since $-\otimes V_n$ is an exact functor) and $\Theta_k^{(n)}$ is left adjoint to $\Theta_{k-1}^{(n)}$ for each $k \in \mathbb{Z}$ (see [BaDE$^+$16, Proposition 4.4.1]). Furthermore, we have the following result, proved in [BaDE$^+$16, Corollary 8.2.1].

Theorem 2.2.17 (See [BaDE$^+$16].)**.** *Let L, L' be non-isomorphic simple modules in $\mathcal{F}_n$. Let $i \in \mathbb{Z}$.*

(1) The module $\Theta_i^{(n)} L$ is multiplicity free.
(2) The modules $\Theta_i^{(n)}(L)$ and $\Theta_i^{(n)}(L')$ have no common simple subquotients (that is, their sets of composition factors are disjoint).

For more details on the structure of $\mathcal{F}_n$, see [BaDE$^+$16].

Lemma 2.2.18. *Let $\lambda \in \Lambda_n$.*

(1) We have: $\Theta_i^{(n)}(L_n(\lambda)) \neq 0$ iff $f_\lambda(i) = 1$, $f_\lambda(i-1) = 0$.
(2) Assume we have: $f_\lambda(i) = 1$, $f_\lambda(i-1) = 0$. Let $\lambda' \in \Lambda_n$ such that $d_{\lambda'}$ be obtained from d_λ by moving $\bullet$ from position i to position $i-1$.
  (a) We have: $[\Theta_i^{(n)}(L_n(\lambda)) : \Pi^{i+1} L_n(\lambda')] = 1$.
  (b) If $[\Theta_i^{(n)}(L_n(\lambda)) : \Pi^z L_n(\mu)] \neq 0$ for some $z \in \{0,1\}$ and $\mu \neq \lambda'$, then $f_\mu(i) \neq 0$ or $f_\mu(i-1) \neq 1$.
  (c) If $[\Theta_i^{(n)}(L_n(\lambda)) : \Pi^z L_n(\mu)] \neq 0$ for some $z \in \{0,1\}$ and $f_\mu(i) = f_\mu(i-1) = 0$, then $f_\mu(s) = f_\lambda(s)$ for any $s \leq i-1$.

Proof. The implication "$\Theta_i^{(n)}(L_n(\lambda)) \neq 0$ implies $f_\lambda(i) = 1$, $f_\lambda(i-1) = 0$" of (1) has been proved in in [BaDE$^+$16, Corollary 8.2.2]. In the other direction, the implication follows from (2a) proved below.

To prove the remaining statements, let $\nabla_n(\lambda)$ denote the thin Kac module for the weight λ. This is the *costandard module* in the highest weight category $\mathcal{F}_n$ with the highest weight structure given by our Borel subalgebra $\mathfrak{b}_n^-$ in $\mathfrak{p}(n)$. The modules $\nabla_n(\lambda)$ were introduced in [BaDE$^+$16, Section 3.1].

To prove (2a), we recall an exact sequence established in [BaDE$^+$16, Proposition 5.2.2]:

$$0 \to \Pi^{i+1} \nabla_n(\lambda') \to \Theta_i^{(n)}(\nabla_n(\lambda))$$

The cokernel of the rightmost map is either 0 or $\nabla_n(\lambda'')$ where $d_{\lambda''}$ is obtained from d_λ by moving $\bullet$ from i to $i+1$ if it is possible. Therefore we have an embedding $\Pi^{i+1} L_n(\lambda') \to \Theta_i^{(n)}(\nabla_n(\lambda))$. On the other hand, by [BaDE$^+$16, Theorem 6.3.3], all composition factors (up to change of parity) $L_n(\nu)$ of $\nabla_n(\lambda)$ satisfy the condition $\nu = \lambda + \sum_{j,k} a_{jk}(\varepsilon_j + \varepsilon_k)$ for some $a_{jk} \in \mathbb{N}$. That ensures that $[L_n(\nu) \otimes V_n : L_n(\lambda')] = 0$ unless $\nu = \lambda$. Hence $[\Theta_i^{(n)}(L_n(\lambda)) : \Pi^{i+1} L_n(\lambda')] = 1$.

To show (2b), assume the opposite, i.e., $f_\mu(i) = 0$ and $f_\mu(i-1) = 1$. Let d_ν be obtained from d_μ by moving black bullet from $i-1$ to i.

Then by (2a), we have $[\Theta_i^{(n)}(L_n(\nu)) : \Pi^{i+1} L_n(\mu)] = 1$. Therefore $L_n(\mu)$ (up to change of parity) appears as a composition factor in both $\Theta_i^{(n)}(L_n(\lambda))$ and $\Theta_i^{(n)}(L_n(\nu))$. This contradicts Theorem 2.2.17 (2).

The statement in (2c) is proved in the same methods as in the proof of [BaDE$^+$16, Corollary 8.2.2]. Assume that $[\Theta_i^{(n)}(L_n(\lambda)) : \Pi^z L_n(\mu)] \neq 0$ for some $z \in \{0,1\}$ and that $f_\mu(i) = f_\mu(i-1) = 0$. Denote by $P_n(\lambda), P_n(\mu)$ the projective covers of $L_n(\lambda), L_n(\mu)$

respectively. Then by the adjointness of $\Theta_{i+1}^{(n)}$ and $\Theta_i^{(n)}$, we have:

$$\dim \operatorname{Hom}_{\mathfrak{p}(n)}(\Theta_{i+1}^{(n)} P_n(\mu), L_n(\lambda)) = \dim \operatorname{Hom}_{\mathfrak{p}(n)}(P_n(\mu), \Theta_i^{(n)}(L_n(\lambda)))$$
$$= [\Theta_i^{(n)}(L_n(\lambda)) : \Pi^z L_n(\mu)] \neq 0.$$

Now, by [BaDE+16, Lemma 7.2.3], the statement of (2c) follows. □

2.2.7. *Blocks.* It was proved in [BaDE+16, Theorem 9.1.2] that there are $2(n+1)$ blocks in the category $\mathcal{F}_n$. These blocks are in bijection with the set $\{-n, -n+2, \dots, n-2, n\} \times \{+, -\}$.

We have a decomposition

$$\mathcal{F}_n = \bigoplus_{k \in \{-n,-n+2,\dots,n-2,n\}} (\mathcal{F}_n)_k^+ \oplus \bigoplus_{k \in \{-n,-n+2,\dots,n-2,n\}} (\mathcal{F}_n)_k^- ,$$

where the functor Π (parity change) induces an equivalence $(\mathcal{F}_n)_k^+ \cong (\mathcal{F}_n)_k^-$. Hence we may define *up-to-parity blocks*

$$\mathcal{F}_n^k := (\mathcal{F}_n)_k^+ \oplus (\mathcal{F}_n)_k^- .$$

The block $\mathcal{F}_n^k$ contains all simple modules $L(\lambda)$ with

$$\sum_i (-1)^{\bar{\lambda}_i} = k$$

By abuse of terminology, we will just call these "blocks" throughout the paper. The following theorem was proved in [BaDE+16, Corollary 9.2.1]:

Theorem 2.2.19 (See [BaDE+16].)**.** *Let $i \in \mathbb{Z}$, $k \in \{-n, -n+2, \dots, n-2, n\}$. Then we have*

$$\Theta_i^{(n)} \mathcal{F}_n^k \subset \begin{cases} \mathcal{F}_n^{k+2} & \text{if } i \text{ is odd} \\ \mathcal{F}_n^{k-2} & \text{if } i \text{ is even} \end{cases}$$

2.3. **The Duflo-Serganova functor.** Let $n \geq 2$, and let $x \in \mathfrak{p}(n)$ be an odd element such that $[x, x] = 0$. Let $s := \operatorname{rk}(x)$. We define the following correspondence of vector superspaces:

Definition 2.3.1 (See [DuS05])**.** Let $M \in \mathcal{F}_n$, and consider the complex

$$\Pi M \xrightarrow{x} M \xrightarrow{x} \Pi M$$

We define $DS_x(M)$ to be the homology of this complex.

The vector superspace $\mathfrak{p}_x := DS_x \mathfrak{p}(n)$ is naturally equipped with a Lie superalgebra structure. One can check by direct computations that $\mathfrak{p}_x$ is isomorphic to $\mathfrak{p}(n-s)$ where s is the rank of x. The above correspondence defines a symmetric monoidal functor $DS_x : \mathcal{F}_n \to \mathcal{F}_{n-s}$, called the *Duflo-Serganova functor*. Such functors were introduced in [DuS05].

An important feature of the Duflo-Serganova functors is that they preserve categorical dimensions ("superdimensions"). That can be proved by direct computation (see [DuS05, Lemma 2.2(6)]. For completeness of presentation, we give a short proof of this classical statement using the fact that DS_x is symmetric monoidal:

Lemma 2.3.2. *For any finite dimensional vector superspace M and linear map $x : M \to \Pi M$ such that $\Pi x \circ x = 0$, we can define $DS_x(M)$ as the homology of the above complex and have:* $\operatorname{sdim} DS_x(M) = \operatorname{sdim} M$.

Proof. The superdimension of M is defined as follows: $\mathrm{sdim}M\,\mathrm{Id}_{\mathbb{C}}$ is defined to be the composition

$$\mathbb{C} \xrightarrow{coev} M \otimes M^* \xrightarrow{\tau} M^* \otimes M \xrightarrow{ev} \mathbb{C}$$

where $coev : \mathbb{C} \to M \otimes M^*$ denotes the coevaluation morphism (sending 1 to $\sum_i e_i \otimes e_i^*$ where e_i form a basis in M and e_i^* form the dual basis in M^*), $\tau : M \otimes M^* \to M^* \otimes M$ denotes the (super) symmetry morphism and $ev : M^* \otimes M \to \mathbb{C}, f \otimes v \to f(v)$ denotes the evaluation morphism. Monoidal functors take coevaluation morphisms to coevaluation morphisms and evaluation morphisms to evaluation morphisms (see [EtGNO15, Exercise 2.10.6]) and the fact that DS_x is symmetric means that it takes symmetry morphisms to symmetry morphisms. Hence

$$\mathrm{sdim}M\,DS_x(\mathrm{Id}_{\mathbb{C}}) = DS_x(\mathrm{sdim}M\,\mathrm{Id}_{\mathbb{C}}) = \mathrm{sdim}DS_x(M)\,\mathrm{Id}_{\mathbb{C}}$$

and thus $\mathrm{sdim}DS_x(M) = \mathrm{sdim}M$. ☐

The following lemmata are used extensively throughout this paper (see also [DuS05][3], a similar result appears in [HeW14, Lemma 2.1]).

Lemma 2.3.3 (Hinich Lemma)**.** *Given a short exact sequence*

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

in $\mathcal{F}_n$, we have an exact sequence

$$0 \to E \to DS_x(M_1) \xrightarrow{DS_x(f)} DS_x(M_2) \xrightarrow{DS_x(g)} DS_x(M_3) \to \Pi E \to 0$$

for some $E \subset DS_x(M_1)$ in $\mathcal{F}_{n-s}$.

Proof. Applying the Zig-Zag Lemma to the following infinite complex (vertically periodic, with period 2):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \xrightarrow{f} & M_2 & \xrightarrow{g} & M_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} & & \\
0 & \longrightarrow & \Pi M_1 & \xrightarrow{\Pi f} & \Pi M_2 & \xrightarrow{\Pi g} & \Pi M_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Pi x} & & \downarrow{\scriptstyle \Pi x} & & \downarrow{\scriptstyle \Pi x} & & \\
0 & \longrightarrow & M_1 & \xrightarrow{f} & M_2 & \xrightarrow{g} & M_3 & \longrightarrow & 0,
\end{array}$$

we obtain an infinite periodic long exact sequence

$$\ldots \to \Pi DS_x(M_3) \xrightarrow{d} DS_x(M_1) \xrightarrow{DS_x(f)} DS_x(M_2) \xrightarrow{DS_x(g)} DS_x(M_3) \xrightarrow{\Pi d} \Pi DS_x(M_1) \to \ldots,$$

for some linear map $d : \Pi DS_x(M_3) \to DS_x(M_1)$. Taking $E := Im(d) = \mathrm{Ker}(DS_x(f))$ we obtain the required result. ☐

In particular, if L is a simple composition factor of $DS_x(M_2)$, then it is a simple composition factor of $DS_x(M_1)$ or of $DS_x(M_3)$.

Lemma 2.3.4 (See [EnS19])**.** *The functor DS_x commutes with translation functors, that is we have a natural isomorphism of functors*

$$DS_x\Theta_k^{(n)} \xrightarrow{\sim} \Theta_k^{(n-s)}DS_x$$

for any $k \in \mathbb{Z}$.

[3]The lemma appears in an unpublished version.

3. The Duflo-Serganova functor: main theorem

Let $x_n \in \mathfrak{p}(n)_1, x_n \neq 0$ be an odd element corresponding to the root $2\varepsilon_n$. Then $[x_n, x_n] = 0$ and we may define a Duflo-Serganova functor

$$DS_{x_n} : \mathcal{F}_n \to \mathcal{F}_{n-1}$$

as in Section 2.3.

Throughout this section, we will write $DS = DS_{x_n}$ for short.

3.1. Statement of the theorem.

Let $\lambda \in \Lambda_n$.

As before, we denote by $L_n(\lambda)$ the simple finite-dimensional integrable $\mathfrak{p}(n)$-module with an even highest weight vector of weight λ. We consider the simple subquotients of $DS(L_n(\lambda))$ in $\mathcal{F}_{n-1}$.

Theorem 3.1.1. *Let $\lambda \in \Lambda_n$ and $\mu \in \Lambda_{n-1}$.*

The following are equivalent:

(1) *$[DS(L_n(\lambda)) : \Pi^z L_{n-1}(\mu)] \neq 0$ for some $z \in \mathbb{Z}$.*
(2) *The diagram d_μ is obtained by removing one black bullet from position i in d_λ, where i satisfies the Initial Segment Condition:*

$$\forall j > i+1, \ r_\lambda(j, i+1) \leq 0.$$

In other words, $f_\lambda(i) = 1, \ f_\lambda(i+1) = 0$ and there is no solid arrow in d_λ ending in $i+1$.

Furthermore, if these conditions hold, then

$$[DS(L_n(\lambda)) : \Pi^z L_{n-1}(\mu)] = 1$$

where $i = \bar{\lambda}_{n-z}$ (that is, $0 \leq z \leq n-1$ and $n-z$ is the sequential number of the removed black bullet (counting from the left)).

Remark 3.1.2. For any position i in d_λ, the following is an equivalent formulation of the Initial Segment Condition: for any $j \geq i+1$, in the segment $[i+1, j]$ in d_λ the number of empty positions is greater or equal to the number of black bullets in that segment.

Proof of Theorem 3.1.1. The proof goes as follows:

(1) Assume $[DS(L_n(\lambda)) : DS(L_{n-1}(\mu))] \neq 0$.
   - First, we prove:

   $$f_\mu(i-1) = 0, \ f_\mu(i) = 1 \implies f_\lambda(i-1) = 0, \ f_\lambda(i) = 1.$$

   In other words, the clusters in d_μ begin in the same positions as in d_λ. This is proved in Lemma 3.2.1.
   - Secondly, we prove:

   $$\forall i, \ f_\lambda(i) \geq f_\mu(i).$$

   In other words, if a position in d_λ was empty, so is the corresponding position in d_μ. This is proved in Proposition 3.2.2.

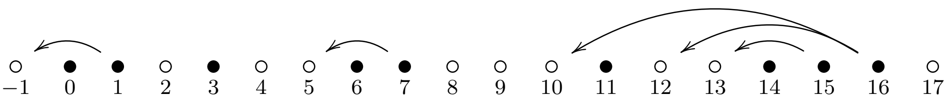
   Hence we conclude: if $[DS(L_n(\lambda)) : L_{n-1}(\mu)] \neq 0$ then d_μ is obtained from d_λ by removing one black bullet from the right end of some cluster.
(2) Next, we prove Proposition 3.2.8, stating that black bullets which do not satisfy the Initial Segment Condition (2) cannot be removed.
(3) We prove Proposition 3.3.2, which completes the proof of the Theorem.

□

Example 3.1.3. For the weight

$$\lambda = \varepsilon_3 + 3\varepsilon_4 + 3\varepsilon_5 + 6\varepsilon_6 + 8\varepsilon_7 + 8\varepsilon_8 + 8\varepsilon_9$$

of $\mathfrak{p}(9)$, the arrow diagram is

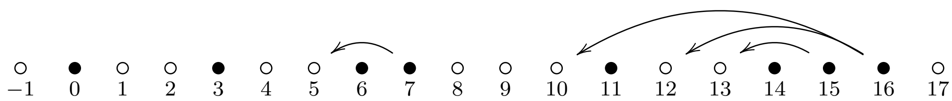


Then the simple factors of $DS_{x_9}(L_9(\lambda))$ are $\Pi L_8(\mu_1), L_8(\mu_2), L_8(\mu_3), L_8(\mu_4)$ where

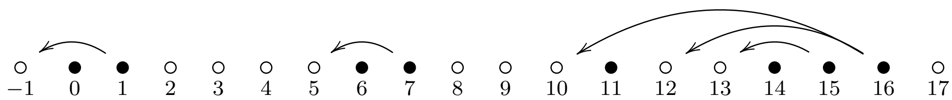
$$\begin{aligned}
\mu_1 &= 2\varepsilon_2 + 4\varepsilon_3 + 4\varepsilon_4 + 7\varepsilon_5 + 9\varepsilon_6 + 9\varepsilon_7 + 9\varepsilon_8,\\
\mu_2 &= 4\varepsilon_3 + 4\varepsilon_4 + 7\varepsilon_5 + 9\varepsilon_6 + 9\varepsilon_7 + 9\varepsilon_8,\\
\mu_3 &= \varepsilon_3 + 3\varepsilon_4 + 7\varepsilon_5 + 9\varepsilon_6 + 9\varepsilon_7 + 9\varepsilon_8,\\
\mu_4 &= \varepsilon_3 + 3\varepsilon_4 + 3\varepsilon_5 + 6\varepsilon_6 + 8\varepsilon_7 + 8\varepsilon_8.
\end{aligned}$$

are weights in Λ_8 with arrow diagrams

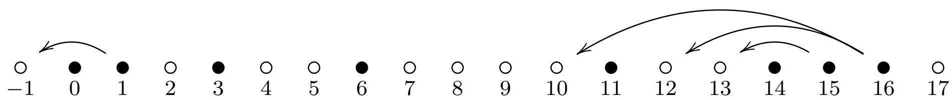
μ_1



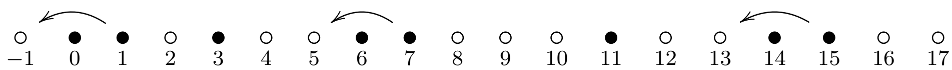
μ_2



μ_3



μ_4 .



We also give a formulation of the theorem using cap diagrams, which will suit our needs better when computing superdimensions.

The following is a rephrasing of the statement of Theorem 3.1.1, using Corollary 2.2.14:

Corollary 3.1.4. *Let $\lambda \in \Lambda_n$, $\mu \in \Lambda_{n-1}$. The following are equivalent:*

(1) $[DS(L_n(\lambda)) : \Pi^z L_{n-1}(\mu)] \neq 0$ for some $z \in \mathbb{Z}$.

(2) The diagram d_μ is obtained from d_λ by removing one maximal cap.

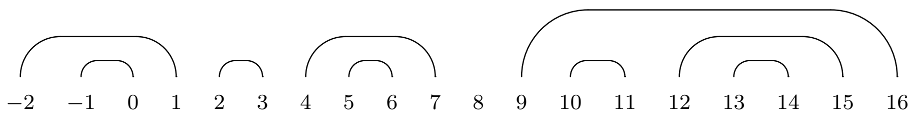
Furthermore, if these conditions hold, then $[DS(L_n(\lambda)) : \Pi^z L_{n-1}(\mu)] = 1$, where position $\bar{\lambda}_{n-z}$ is the rightmost end of the removed cap.

Remark 3.1.5. Equivalently, z is the number of caps whose right end is (strictly) to the right of the removed cap.

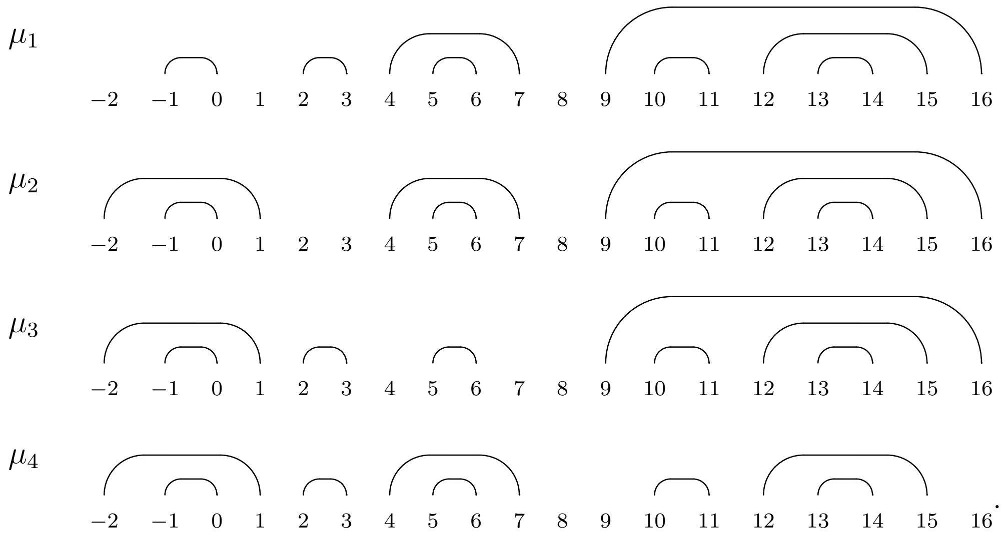
Example 3.1.6. For the weight

$$\lambda = \varepsilon_3 + 3\varepsilon_4 + 3\varepsilon_5 + 6\varepsilon_6 + 8\varepsilon_7 + 8\varepsilon_8 + 8\varepsilon_9$$

of $\mathfrak{p}(9)$ as described in Example 3.1.3, the cap diagram is



Then the simple factors of $DS_{x_9}(L_9(\lambda))$ are $\Pi L_8(\mu_1), L_8(\mu_2), L_8(\mu_3), L_8(\mu_4)$ as in Example 3.1.3, and the corresponding cap diagrams are as follows:



3.2. **Proof of Theorem 3.1.1: auxiliary results, part 1.**

Throughout this subsection, we consider all modules in $\mathcal{F}_n$, $\mathcal{F}_{n-1}$ up to parity switch.

Lemma 3.2.1. *Let $L_n(\lambda)$ as above. If $[DS(L_n(\lambda)) : L_{n-1}(\mu)] \neq 0$ then we have:*
$f_\mu(i-1) = 0, f_\mu(i) = 1$ *implies* $f_\lambda(i-1) = 0, f_\lambda(i) = 1$.

Proof. Assume the contrary. Then there exists a position i which is the beginning of a cluster in d_μ but not in d_λ.

Apply the translation functors $\Theta_i^{(n)}$, $\Theta_i^{(n-1)}$ to modules $L_n(\lambda)$ and $L_{n-1}(\mu)$ respectively. By Lemma 2.2.18(1), the functor $\Theta_i^{(n)} : \mathcal{F}_m \to \mathcal{F}_m$ $(m \geq 1)$ annihilates any simple module $L_m(\tau)$ unless d_τ has a black bullet in position i and an empty position ("white bullet") in position $i-1$. Hence

$$\Theta_i^{(n)}(L_n(\lambda)) = 0, \;\; \Theta_i^{(n-1)}(L_{n-1}(\mu)) \neq 0.$$

But $\Theta_i^{(n-1)}$ is an exact functor, so $\Theta_i^{(n-1)}(L_{n-1}(\mu))$ is a subquotient of $\Theta_i^{(n-1)}(DS(L_n(\lambda))) \cong DS(\Theta_i^{(n)}(L_n(\lambda))) = 0$. This contradicts our observation that $\Theta_i^{(n-1)}(L_{n-1}(\mu)) \neq 0$, and the claim of the Lemma follows.

□

Proposition 3.2.2. *Assume $[DS(L_n(\lambda)) : L_{n-1}(\mu)] \neq 0$.*

Then for any $i \in \mathbb{Z}$, we have: $f_\lambda(i) \geq f_\mu(i)$. That is, if a position in d_λ was empty, so is the corresponding position in d_μ.

Proof. Define $\mathcal{M}$ as the set of all quintuples (λ, μ, i, j, k) satisfying the following conditions

(1) $[DS(L_n(\lambda)) : L_{n-1}(\mu)] \neq 0$ (recall that modules are considered up to parity shift!);
(2) $f_\lambda(j) = 0$, $f_\mu(j) = 1$ and j is minimal with this property (that is, for any $s < j$ we have: $f_\lambda(s) \geq f_\mu(s)$);
(3) $i \leq j$ and $f_\mu(i) = f_\mu(i+1) = \cdots = f_\mu(j-1) = 1$, $f_\mu(i-1) = 0$;
(4) k is the number of $s < j$ such that $f_\mu(s) = 1$.

By Lemma 3.2.1 we have that

$$k \geq 1, \quad i < j, \quad f_\lambda(i) = f_\lambda(i+1) = \cdots = f_\lambda(j-1) = 1. \tag{3}$$

Our goal is to prove that $\mathcal{M} = \emptyset$. Let us assume that $\mathcal{M}$ is not empty and let k be minimal with property $(\lambda, \mu, i, j, k) \in \mathcal{M}$ for some λ, μ, i, j. Let λ' and μ' be obtained from λ and μ respectively by moving $\bullet$ from i to $i-1$. We are going to prove the following

Lemma 3.2.3. *If $(\lambda, \mu, i, j, k) \in \mathcal{M}$, where k is minimal then $(\lambda', \mu', i+1, j, k) \in \mathcal{M}$.*

Proof. By Lemma 2.2.18 (2a) $\Theta_i^{(n-1)}(L_{n-1}(\mu))$ has a composition factor $L_{n-1}(\mu')$. This composition factor appears in $DS(\Theta_i^{(n)}(L_n(\lambda)))$. Therefore it appears in $DS(L_n(\nu))$ for some composition factor $L_n(\nu)$ in $\Theta_i^{(n)}(L_n(\lambda))$. We claim that $\nu = \lambda'$. Indeed, by Lemma 3.2.1 we have $f_\nu(i) = 0$, $f_\nu(i+1) = 1$ since $f_{\mu'}(i) = 0$, $f_{\mu'}(i+1) = 1$.

Assume $\nu \neq \lambda'$. Then Lemma 2.2.18 (2b) implies that $f_\nu(i-1) = 0 < f_{\mu'}(i-1) = 1$.

Let us show that $i-1$ is the minimal position with such property. Indeed, $f_\nu(i-1) = f_\nu(i) = 0$. Hence by Lemma 2.2.18 (2c) we have:

$$\forall s \leq i-1,\ f_\lambda(s) = f_\nu(s)$$

Furthermore, by our assumption $(\lambda, \mu, i, j, k) \in \mathcal{M}$, so

$$\forall s < i-1 < j,\ f_\nu(s) = f_\lambda(s) \geq f_\mu(s) = f_{\mu'}(s).$$

Hence $(\nu, \mu', i', i-1, k') \in \mathcal{M}$ for some $i' < i-1$ and $k' < k$. Since k is chosen minimal this is impossible. Hence $\nu = \lambda'$ and clearly $(\lambda', \mu', i+1, j, k) \in \mathcal{M}$. ☐

The statement of the Proposition follows from this lemma since after applying it several times we get a tuple of the form $(\lambda'', \mu'', j, j, k) \in \mathcal{M}$ which is impossible by (3). ☐

The next statements will rely on the following corollary of Lemma 3.2.1 and Proposition 3.2.2:

Corollary 3.2.4. *If $[DS(L_n(\lambda)) : L_{n-1}(\mu)] \neq 0$ then d_μ is obtained from d_λ by removing one black bullet from the end of some cluster.*

Definition 3.2.5. Let α be a dominant integral weight for $\mathfrak{p}(n)$. Denote by $\alpha^{\clubsuit}$ the weight whose diagram is obtained from d_α by moving each black bullet through the longest solid arrow originating at this position.

Lemma 3.2.6. *Let α be a dominant integral weight for $\mathfrak{p}(n)$. Let α^* be the highest weight of the dual module $L_n(\alpha)^*$. Then d_{α^*} is obtained from $d_{\alpha^\clubsuit}$ by reflecting with respect to position 0.*

Proof. This is a direct consequence of [BaDE+16, Propositions 3.6.1, 8.3.1]. ☐

Remark 3.2.7. In Proposition 3.3.2, we also use the weight $\alpha^\dagger$, defined in [BaDE+16, Section 5.3]. Its weight diagram $d_{\alpha^\dagger}$ is obtained from d_{α^*} by reflecting with respect to the position $(n-1)/2$. Hence $d_{\alpha^\dagger}$ is a shift of $d_{\alpha^\clubsuit}$ to the right by $n-1$ positions.

Proposition 3.2.8. *Assume $[DS(L_n(\lambda)) : L_{n-1}(\mu)] \neq 0$. Then d_μ satisfies the Initial Segment Condition in Theorem 3.1.1*(2).

Proof. By Corollary 3.2.4, d_μ was obtained from d_λ by removing a single black bullet.

Assume that the statement of the proposition is false: that is, $[DS(L_n(\lambda)) : L_{n-1}(\mu)] \neq 0$ and d_μ was obtained from d_λ by removing a black bullet in position i, where i satisfies:

- $f_\lambda(i) = 1$, $f_\lambda(i+1) = 0$.
- There exists $j \geq i+1$ such that $r_\lambda(j+1, i+1) > 0$. That is, the segment $[i+1, j]$ contains more black bullets than it has empty positions.

Consider the minimal $j \geq i+1$ as above. In that case, we must have:

- $f_\lambda(j) = 1$,

- $r_\lambda(j, i+1) = 0$ (that is, the segment $[i+1, j-1]$ contains equal amounts of black bullets and empty positions).
- $\forall i < k < j,\ r_\lambda(k+1, i+1) \leq 0$. That is, the segment $[i+1, k]$ contains no more black bullets than it has empty positions.

From this, we conclude that in the diagram d_λ, there is a solid arrow from j to $i+1$:

$d_\lambda =$ 

Since $f_\lambda(i) = 1$, we may conclude that this is **not** the longest solid arrow originating at j in d_λ.

On the other hand, in d_μ, we have: $f_\mu(i) = 0$, $f_\mu(s) = f_\lambda(s)$ for any $s \neq i$.

Hence in d_μ we also have a solid arrow from j to $i+1$:

$d_\mu =$ 

and it is the longest solid arrow originating at j in d_λ.

We now construct $\lambda^\clubsuit$ and $\mu^\clubsuit$. These are obtained by moving each black bullet through the longest solid arrow originating at this position. Hence we have:

$d_{\lambda^\clubsuit} =$  and $d_{\mu^\clubsuit} =$ 

By the Lemma 3.2.6, we have:

$d_{\lambda^*} =$  and $d_{\mu^*} =$ 

Hence $f_{\lambda^*}(-i-1) = 0$, $f_{\mu^*}(-i-1) = 1$.

Yet the DS functor commutes with the duality functor (up to isomorphism), so

$$[DS(L_n(\lambda^*)) : L_{n-1}(\mu^*)] = [DS(L_n(\lambda)^*) : L_{n-1}(\mu)^*] =$$
$$= [DS(L_n(\lambda))^* : L_{n-1}(\mu)^*] = [DS(L_n(\lambda)) : L_{n-1}(\mu)] \neq 0$$

Hence we may apply Proposition 3.2.2, and conclude that

$$\forall k \in \mathbb{Z},\ f_{\lambda^*}(k) \geq f_{\mu^*}(k).$$

But this contradicts our previous conclusion that $f_{\lambda^*}(-i-1) = 0$, $f_{\mu^*}(-i-1) = 1$.

This completes the proof of the proposition. □

3.3. **Proof of Theorem 3.1.1: auxiliary results, part 2.** In this subsection we distinguish between simple representations varying by a parity switch. We will also use cap diagrams instead of arrow diagrams, since they suit our needs better in this instance.

Lemma 3.3.1. *If d_μ is obtained from d_λ by removing the rightmost black bullet, then*

$$[DS(L_n(\lambda)) : L_{n-1}(\mu)] = 1.$$

Proof. The module $L_n(\lambda)$ is a highest weight module with respect to the Borel subalgebra $\mathfrak{b}_n^- = \mathfrak{b}_0^- \oplus \mathfrak{p}(n)_{-1} \subset \mathfrak{p}(n)$. The roots corresponding to $\mathfrak{p}(n)_{-1}$ are $-\varepsilon_i - \varepsilon_j$ for $\varepsilon_i \neq \varepsilon_j$.

Thus we have the following observation: any weight α in $L_n(\lambda)$ can be written as

$$\alpha = \lambda + \sum_{1 \leq i \neq j \leq n} s_{ij}(\varepsilon_i + \varepsilon_j) + \sum_{1 \leq i < j \leq n} t_{ij}(\varepsilon_i - \varepsilon_j)$$

for some $s_{ij} \in \{0, 1\}$ and $t_{ij} \geq 0$.

Now, we show that $[DS(L_n(\lambda)) : L_{n-1}(\mu)] \leq 1$. Indeed, given a weight α in $L_n(\lambda)$ such that $\alpha_i = \lambda_i$ for all $i < n$, we necessarily have $\alpha = \lambda$ by the observation above. The weight λ appears in $L_n(\lambda)$ with multiplicity 1, hence $[DS(L_n(\lambda)) : L_{n-1}(\mu)] \leq 1$.

Finally, we show that $[DS(L_n(\lambda)) : L_{n-1}(\mu)] \neq 0$: Let $v \neq 0$ be the (even) highest weight vector in $L_n(\lambda)$ with respect to the Borel subalgebra $\mathfrak{b}_n^-$. Then $x.v$ must have weight $\lambda + 2\varepsilon_n$, which by the observation above is not a weight of $L_n(\lambda)$. Hence $x.v = 0$.

Now, assume that $v \in \mathrm{Im}(x)$. Let us write $v = x.w$ for some $w \in L_n(\lambda)$. Then w has weight $\lambda - 2\varepsilon_n$, which by the reasoning above is impossible. Hence $v \notin \mathrm{Im}(x)$. This implies that v has non-zero (even) image $\tilde{v}$ in $DS(L_n(\lambda)) = \mathrm{Ker}(x)/\mathrm{Im}(x)$, and its image has weight μ.

Now, the vector v is a primitive vector with respect to the Borel subalgebra $\mathfrak{b}_n^-$, hence the (even) vector $\tilde{v}$ is a primitive vector with respect to the Borel subalgebra $\mathfrak{b}_{n-1}^-$ of $\mathfrak{p}(n-1)$, as required. This completes the proof of the lemma. □

Proposition 3.3.2. *Let d_μ be obtained from d_λ by removing a black bullet whose cap is maximal. Then there exists a unique $z \in \mathbb{Z}/2\mathbb{Z}$ such that $[DS(L_n(\lambda)) : \Pi^z L_{n-1}(\mu)] = 1$, moreover z equals the parity of number of black bullets to the right of the removed black bullet.*

In order to prepare for the proof of Proposition 3.3.2, we begin by proving the following.

Lemma 3.3.3. *Let $n > 1$. Suppose that d_λ and d_μ have the leftmost black bullet in the same position and $d_{\lambda'}$ $d_{\mu'}$ are obtained from d_λ and d_μ by removing this black bullet. Then we have*

$$[DS(L_n(\lambda)) : \Pi^z L_{n-1}(\mu)] = [DS(L_{n-1}(\lambda')) : \Pi^z L_{n-2}(\mu')]$$

where z as in Proposition 3.3.2.

Proof. Let $h_1, \ldots, h_n$ be the basis in the Cartan subalgebra of $\mathfrak{p}(n)_0 \subset \mathfrak{p}(n)$ dual to $\varepsilon_1, \ldots, \varepsilon_n$. We have a decomposition

$$L_n(\lambda) = \bigoplus_{i \geq \lambda_1} L_n(\lambda)^i$$

where $L_n(\lambda)^i$ is the eigenspace of h_1 with eigenvalue i. Every component $L_n(\lambda)^i$ is a module over the centralizer $\mathfrak{l}$ of h_1. Since $x \in \mathfrak{l}$ we have

$$DS(L_n(\lambda)) = \bigoplus_{i \geq \lambda_1} DS(L_n(\lambda)^i).$$

Note that $\mathfrak{l}$ is the direct sum $\mathbb{C}h_1 \oplus \mathfrak{l}'$ where $\mathfrak{l}'$ is another copy of $\mathfrak{p}(n-1)$ inside $\mathfrak{p}(n)$. Furthermore, $L_n(\lambda)^{\lambda_1}$ is isomorphic $L_{n-1}(\lambda')$ since $L_n(\lambda)$ is a quotient of the parabolically induced module $U(\mathfrak{p}(n)) \otimes_{U(\mathfrak{b}_n^- + \mathfrak{l})} L_n(\lambda)^{\lambda_1}$. Now it is clear that if $\mu_1 = \lambda_1$ then $L_{n-1}(\mu)$ occurs in $DS(L_n(\lambda))$ with the same multiplicity as $L_{n-1}(\mu)^{\lambda_1}$ occurs in $DS(L_n(\lambda)^{\lambda_1})$. The statement follows. □

Consider the "mixed triangular" Borel subalgebra $\mathfrak{b}_n^\dagger = \mathfrak{b}_{0,n}^- + \mathfrak{p}(n)_1$ of $\mathfrak{p}(n)$. In terms of the matrix description given in Remark 2.2.1, the elements of $\mathfrak{b}_n^\dagger$ are given by matrices $\begin{pmatrix} A & B \\ 0 & -A^t \end{pmatrix}$ in $\mathfrak{p}(n)$ where A is lower-triangular. The corresponding simple roots are $2\varepsilon_1, \varepsilon_2 - \varepsilon_1, \ldots, \varepsilon_n - \varepsilon_{n-1}$, and the corresponding Borel subalgebra $\mathfrak{b}_{n-1}^\dagger$ of $\mathfrak{p}(n-1)$ has simple roots $2\varepsilon_1, \varepsilon_2 - \varepsilon_1, \ldots, \varepsilon_{n-1} - \varepsilon_{n-2}$. Let $\lambda^\dagger$ denote the highest weight of $L_n(\lambda)$ with respect to $\mathfrak{b}_n^\dagger$, and similarly for weights of $\mathfrak{p}(n-1)$. We will denote by $L_n^\dagger(\nu)$ the simple $\mathfrak{p}(n)$-module of highest weight ν with respect to $\mathfrak{b}_n^\dagger$ having an even highest weight vector, and similarly for simple $\mathfrak{p}(n-1)$-modules.

As in the proof of Lemma 3.3.1, one readily sees that $\lambda^\dagger = \lambda + \sum_{1 \leq i \neq j \leq n} s_{ij}(\varepsilon_i + \varepsilon_j)$ for some $s_{ij} \in \{0, 1\}$, and

$$L_n(\lambda) \simeq \Pi^{\sum_{i \neq j} s_{ij}} L_n^\dagger(\lambda^\dagger).$$

But $\sum_{i\neq j} s_{ij} = \frac{1}{2}\left(\sum_{i=1}^{n} \lambda_i^\dagger - \lambda_i\right)$ so we obtain:

$$L_{n-1}(\mu) \simeq \Pi^s L_{n-1}^\dagger(\mu^\dagger), \quad L_n(\lambda) \simeq \Pi^t L_n^\dagger(\lambda^\dagger)$$

where

$$s = \frac{1}{2}\left(\sum_{i=1}^{n-1} \mu_i^\dagger - \mu_i\right), \quad t = \frac{1}{2}\left(\sum_{i=1}^{n} \lambda_i^\dagger - \lambda_i\right). \tag{4}$$

Let $y \in \mathfrak{p}(n)$ be a root vector of weight $2\varepsilon_1$. Then by the same argument as in the proof Lemma 3.3.1, we have:

Lemma 3.3.4. *Let d_ν be obtained from $d_{\lambda^\dagger}$ by removing the leftmost black bullet and shifting all other black bullets one position left, then $[DS_y(L_n^\dagger(\lambda^\dagger)) : L_{n-1}^\dagger(\nu)] = 1$.*

Remark 3.3.5. The shift is necessary due to renumeration $2 \mapsto 1, \dots, n \mapsto n-1$.

A combinatorial algorithm of computing $\lambda^\dagger$ in terms of weight diagrams is given in [BaDE+16, Section 5.3]. Enumerate the black bullets from left to right. Let $1 \leq a<b \leq n$. Define the operation $D_{a,b}$ on the set of diagrams as follows: if the positions next right to both a-th and b-th bullets in a diagram d are empty, then $D_{a,b}(d)$ is obtained by moving both bullets one position right. Otherwise $D_{a,b}(d) = d$. Then

$$d_{\lambda^\dagger} = D_{1,2} \dots D_{1,n} D_{2,3} \dots D_{2,n} \dots D_{n-2,n-1} D_{n-2,n} D_{n-1,n}(d_\lambda).$$

Definition 3.3.6. We will say that a cap $c = (i,j), i < j$ *covers* a black bullet in a given weight diagram d_λ if the position k of the black bullet satisfies: $i < k < j$.

We also denote by m_i the number of caps which cover the i-th black bullet in d_λ.

Lemma 3.3.7. *We have $\bar{\lambda}_1^\dagger - \bar{\lambda}_1 = n - m_1 - 1$. In particular, if the cap ending at the first black bullet is maximal then $\bar{\lambda}_1^\dagger - \bar{\lambda}_1 = n - 1$.*

Proof. One proves the statement by induction on n. **Base:** let $n = 1$. Then $m_1 = 0$ and $\bar{\lambda}_1^\dagger - \bar{\lambda}_1 = 0$ as required.

Step: Let $n > 1$ and assume the statement holds for $n - 1$.

Let $\alpha \in \Lambda_n$ be the weight defined by

$$d_\alpha := D_{2,3} \dots D_{2,n} \dots D_{n-1,n}(d_\lambda)$$

and let $\lambda', \alpha' \in \Lambda_{n-1}$ be the weights whose diagrams $d_{\lambda'}, d_{\alpha'}$ are obtained from d_λ, d_α respectively by removing the leftmost black bullet in each diagram. Then $\alpha' = \lambda'^\dagger$, so by the induction assumption, we have:

$$\bar{\alpha}'_1 - \bar{\lambda}'_1 = \bar{\alpha}_2 - \bar{\lambda}_2 = n - 2 - m_2.$$

Now, consider first the case when $m_1 > 0$. Then $\bar{\lambda}_2 - \bar{\lambda}_1 = m_2 - m_1 + 2$. Recall that we have: $\bar{\alpha}_2 - \bar{\lambda}_2 = n - 2 - m_2$ and hence $\bar{\alpha}_2 - \bar{\lambda}_1 = n - m_1$.

Using $d_{\lambda^\dagger} = D_{1,2} \dots D_{1,n}(d_\alpha)$ we get that we can move the first black bullet until it stays next to the second black bullet of d_α, namely exactly $n - 1 - m_1$ times. Hence $\bar{\lambda}_1^\dagger - \bar{\lambda}_1 = n - m_1 - 1$.

Now let $m_1 = 0$. Then $\bar{\lambda}_2 - \bar{\lambda}_1 \geq m_2 + 2$. Recall that we have: $\bar{\alpha}_2 - \bar{\lambda}_2 = n - 2 - m_2$ and so $\bar{\alpha}_2 - \bar{\lambda}_1 \geq n$. Thus we move the first black bullet $n-1$ times and so $\bar{\lambda}_1^\dagger - \bar{\lambda}_1 = n - 1$. ☐

Now we are ready to prove Proposition 3.3.2:

Proof of Proposition 3.3.2. Note that the fact that a black bullet is the end of a maximal cap depends only on positions of the black bullets to its right. Therefore Lemma 3.3.3 implies that it suffices to prove the statement of Proposition 3.3.2 in the case when the removed black bullet is the leftmost black bullet in the diagram d_λ.

Assume d_μ is of this form: namely, d_μ is obtained from d_λ by removing the leftmost black bullet (from position λ_1). Since λ, μ should satisfy the condition of Proposition 3.3.2, the cap ending in position λ_1 is maximal, hence $m_1 = 0$ in the notation of Lemma 3.3.7.

Let d_ν be the diagram obtained from $d_{\lambda^\dagger}$ as in Lemma 3.3.4. Then we have $\nu = \mu^\dagger$ and

$$[DS_y L^\dagger_n(\lambda^\dagger) : L^\dagger_{n-1}(\nu)] = [DS_y L^\dagger_n(\lambda^\dagger) : L^\dagger_{n-1}(\mu^\dagger)] = 1.$$

Note that DS_y and $DS = DS_x$ are isomorphic functors since y and x are conjugate by the adjoint action of $GL(n)$. Let t, s as in (4). We obtain:

$$[DSL_n(\lambda) : L_{n-1}(\mu)] = [\Pi^t DS_y L^\dagger_n(\lambda^\dagger) : \Pi^s L^\dagger_{n-1}(\mu^\dagger)].$$

Finally, we have: $\mu_i = \lambda_{i+1} + 1$, $\mu^\dagger_i = \lambda^\dagger_{i+1}$ for each $1 \leq i \leq n-1$, while $\lambda^\dagger_1 - \lambda_1 = n - 1$ by Lemma 3.3.7. Thus

$$t - s = \frac{1}{2}\left(\sum_{i=1}^{n} \lambda^\dagger_i - \lambda_i + \sum_{i=1}^{n-1} \mu_i - \mu^\dagger_i\right) = n - 1$$

which gives us the required statement.

□

3.4. **Action of the DS functor: corollaries.** Let $x_n \in \mathfrak{p}(n)_1$, and $DS = DS_{x_n}$ as before. The following are direct corollaries of Theorem 3.1.1:

Corollary 3.4.1. *Let $\lambda \in \Lambda_n$. The number of composition factors of $DS(L_n(\lambda))$ is precisely the number of maximal arrows (or maximal caps).*

Corollary 3.4.2. *Let $\lambda \in \Lambda_n$. Then $DS(L_n(\lambda))$ is simple iff there exists exactly one maximal solid arrow (one maximal cap) in d_λ.*

4. Computation of superdimensions

In this section we compute the superdimension of the simple $\mathfrak{p}(n)$-modules in $\mathcal{F}_n$.

4.1. **Forests.** Let $\lambda \in \Lambda_n$ be a dominant integral weight, and let d_λ be its weight diagram with caps. Let $(C(\lambda), \trianglelefteq)$ be the poset of caps in d_λ with partial order $\trianglelefteq$ described in Definition 2.2.10.

We define an augmented poset

$$(\widehat{C}(\lambda), \trianglelefteq), \quad \widehat{C}(\lambda) = C(\lambda) \sqcup \{c_*\}$$

where c_* is a *"virtual cap"* which is defined to be the greatest element in $\widehat{C}(\lambda)$: namely, we have

$$c_* \notin C(\lambda), \quad \text{and} \quad \forall c \in C(\lambda),\, c \lhd c_*.$$

We define the successors of c_* as in Definition 2.2.10. These are precisely the maximal caps in $C(\lambda)$.

Definition 4.1.1.

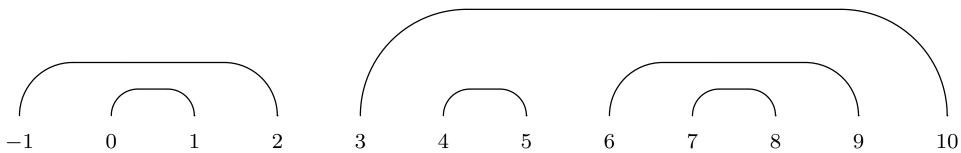
- Given a cap $c \in \widehat{C}(\lambda)$, let
$$int(c) = \sharp\{c' \in \widehat{C}(\lambda) : c \trianglerighteq c'\}$$
be the number of caps internal to c, including c itself.
If $c = (i, j)$ is a non-virtual cap, then $int(c)$ is the number of black bullets in d_λ between positions i and j (including position j), and $int(c_*) = n + 1$.
- A cap $c \in \widehat{C}(\lambda)$ with $int(c) \equiv 0 \mod 2$ is called an *even* cap; otherwise it is called an *odd* cap.
- If every cap $c \in \widehat{C}(\lambda)$ has at most one odd successor, we call such a weight λ *worthy.*

Remark 4.1.2. The virtual cap c_* is even iff $n \equiv 1 \mod 2$.

Definition 4.1.3. Given a worthy weight λ, we consider the subset $\widehat{C}(\lambda)^{even} \subset \widehat{C}(\lambda)$ consisting of even caps only. One can think of it as corresponding to a *reduced* cap diagram d_λ^{red}: this diagram is obtained by erasing the odd caps in d_λ, with an additional maximal virtual cap c_* if n is odd.

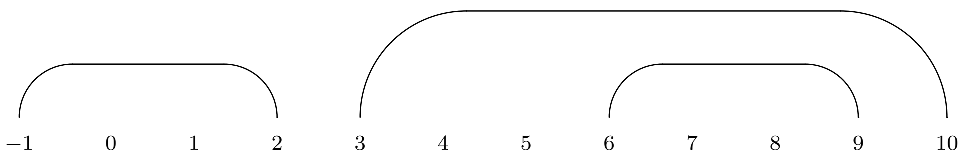
The inclusion $\widehat{C}(\lambda)^{even} \subset \widehat{C}(\lambda)$ induces a partial order on the set $\widehat{C}(\lambda)^{even}$. The notion of "successor" for caps in d_λ^{red} is defined accordingly.

Example 4.1.4. Consider the weight $\lambda = \varepsilon_1 + \varepsilon_2 + 3\varepsilon_3 + 5\varepsilon_4 + 5\varepsilon_5 + 5\varepsilon_6$ for $\mathfrak{p}(6)$ as in Examples 2.2.6, 2.2.11. The cap diagram for λ is:

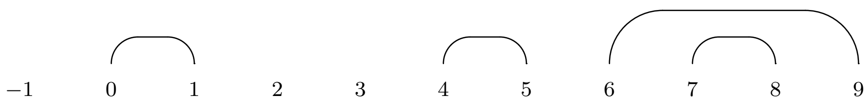


Here c_* has two successors: $(-1, 2), (3, 10)$ (both even caps), and we have:
$$int(c_*) = 7,\ int((0,1)) = int((4,5)) = int((7,8)) = 1,$$
$$int((-1,2)) = int((6,9)) = 2,\ int((3,10)) = 4.$$
The odd caps here are c_* (the virtual cap) as well as $(0, 1), (4, 5), (7, 8)$; the rest of the caps are even. In this case, each cap in $\widehat{C}(\lambda)$ has at most one odd successor, so the weight λ is worthy.

The reduced diagram d_λ^{red} in this case is



Example 4.1.5. Consider the weight $\lambda = \varepsilon_1 + 4\varepsilon_2 + 6\varepsilon_3 + 6\varepsilon_4$ for $\mathfrak{p}(4)$. The cap diagram for λ is:



Here

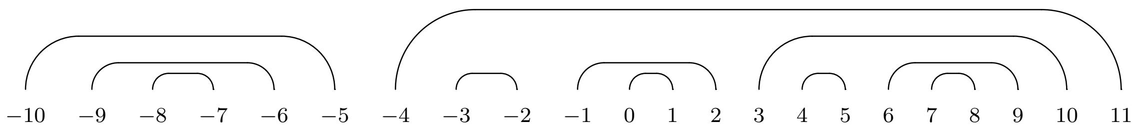
$$int((0,1)) = int((4,5)) = int((7,8)) = 1, \; int((6,9)) = 2.$$

The odd caps here are $(0,1),(4,5),(7,8)$, and the $(6,9)$ is an even cap. The maximal (non-virtual) caps in $C(\lambda)$ are $(0,1),(4,5),(6,9)$. Hence the virtual cap has two odd successors, and the weight λ is not worthy.

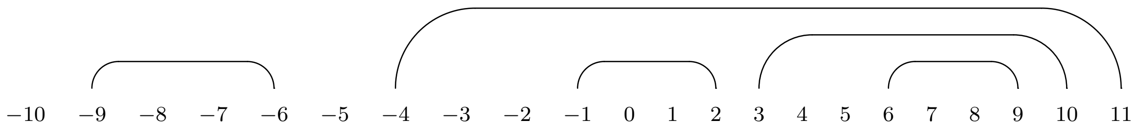
Example 4.1.6. Consider the weight

$$\lambda = -7\varepsilon_1 - 7\varepsilon_2 - 7\varepsilon_3 - 5\varepsilon_4 - 3\varepsilon_5 - 3\varepsilon_6 - \varepsilon_7 + \varepsilon_8 + \varepsilon_9 + \varepsilon_{10} + \varepsilon_{11}$$

for $\mathfrak{p}(11)$. The cap diagram for λ is:



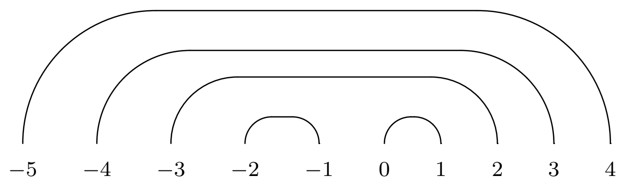
In this case, each cap in $\widehat{C}(\lambda)$ has at most one odd successor, so the weight λ is worthy. The reduced diagram d_λ^{red} in this case is



and we have a virtual cap c_* in this diagram as well (not drawn).

Example 4.1.7. The zero weight $\lambda = 0$ is always worthy (for any $n \geq 1$), since it gives a linear order on the augmented set of its caps $\widehat{C}(\lambda)$.

Example 4.1.8. The weight $\lambda = -\varepsilon_1$ is not worthy for any $n \geq 2$. For example, for $n = 5$, the cap diagram of λ is



The cap $(-3,2)$ has two odd successors, hence λ is not worthy.

The following lemma is straightforward:

Lemma 4.1.9. *Given any weight $\lambda \in \Lambda_n$, any even cap $c \in \widehat{C}(\lambda)$ has an odd number of odd successors, and any odd cap $c \in \widehat{C}(\lambda)$ has an even number of odd successors.*

This immediately leads to the following conclusion:

Corollary 4.1.10. *Given a worthy weight $\lambda \in \Lambda_n$, we have:*

(1) Given any odd cap $c \in \widehat{C}(\lambda)$, all its successors are even caps.
(2) Given any even cap $c \in \widehat{C}(\lambda)$, it has exactly one odd successor.

Definition 4.1.11. Let λ be a worthy weight. We construct a rooted forest F_λ as follows.

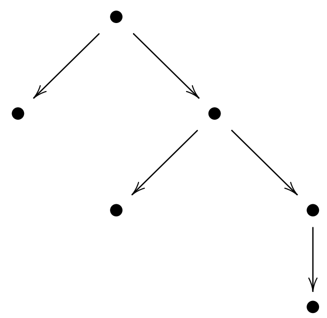
- The nodes of F_λ are caps $c \in \widehat{C}(\lambda)^{even}$.
- There is an edge from a node c to a node c' in F_λ if c' is a successor of c in $\widehat{C}(\lambda)^{even}$.

The forest F_λ is called *the rooted forest corresponding to* λ.

Example 4.1.12.

(1) For $\lambda = 0$, F_λ is a linear rooted tree with $\lfloor \frac{n+1}{2} \rfloor$ nodes.
(2) For λ as in Example 4.1.4, the rooted forest is

(3) For λ as in Example 4.1.6, the rooted forest is



We also recall the following definitions (cf. [HeW14]):

Definition 4.1.13. Let F be a rooted forest.

- We denote by $|F|$ the number of nodes in the forest.
- For any node v in F, we denote by $F^{(v)}$ the rooted subtree of F whose root is v.
- For any root v in F (that is, v has no parent), we denote by $F \setminus \{v\}$ the rooted forest obtained from F by removing v and all the edges originating in it.
- We define the *forest factorial* $F!$ by

$$F! = \prod_v \left|F^{(v)}\right|$$

in particular, for $F = \varnothing$ the empty forest, we define $F! = 1$.

Remark 4.1.14. Given a worthy weight $\lambda \in \Lambda_n$, $|F_\lambda| = \lfloor \frac{n+1}{2} \rfloor$.

Example 4.1.15.

(1) For $\lambda = 0$, we have:

$$F_\lambda! = \lfloor \frac{n+1}{2} \rfloor !$$

(2) For λ as in Example 4.1.4, we have

$$F_\lambda! = 1 \cdot 2 \cdot 1 = 2, \ |F_\lambda| = 3.$$

(3) For λ as in Example 4.1.6, we have

$$F_\lambda! = 6 \cdot 1 \cdot 4 \cdot 1 \cdot 2 \cdot 1 = 48, \ |F_\lambda| = 6.$$

The following statements will be useful for Theorem 4.2.1:

Lemma 4.1.16. *The integer $\frac{|F|!}{F!}$ counts the number of heap-orderings on the rooted forest F. Here a heap-ordering on a rooted forest is a bijection*

$$\alpha : \{ \text{ nodes of } F \} \longrightarrow \{1, 2, 3, \ldots, |F|\}$$

such that $\alpha(v) \leq \alpha(v')$ whenever v is an ancestor of v' (equivalently, on any subtree, the number corresponding to the root is less or equal to the numbers corresponding the rest of the nodes in that subtree).

Proof. We prove the statement by (complete) induction on $|F|$.

Base: if $|F| = 0$ then the statement is clearly true.

Step: let F be a rooted forest with at least 1 node, and assume the statement holds for any rooted forest with fewer nodes.

Let $v_1, \ldots, v_m$ be the roots of F, and let $T_i := F^{(v_i)}$ be the subtree whose root is v_i. Then

$$\frac{|F|!}{F!} = \frac{|F|!}{\prod_{i=1}^{m} |T_i|\,!} \cdot \frac{\prod_{i=1}^{m} |T_i|\,!}{F!} = \binom{|F|!}{|T_1|, |T_2|, \ldots, |T_m|} \cdot \prod_{i=1}^{m} \frac{|T_i|!}{T_i!} =$$

$$= \binom{|F|!}{|T_1|, |T_2|, \ldots, |T_m|} \cdot \prod_{i=1}^{m} \frac{|T_i \setminus \{v_i\}|\,!}{(T_i \setminus \{v_i\})!}$$

The multinomial coefficient $\binom{|F|!}{|T_1|, |T_2|, \ldots, |T_m|}$ counts the number of ways to partition the set $\{1, 2, 3, \ldots, |F|\}$ into an ordered multiset of unordered subsets, whose sizes are $|T_1|, |T_2|, \ldots, |T_m|$. Each such subset will be the set of numbers corresponding under the heap-ordering to the rooted tree T_i, with the smallest number corresponding to the root v_i of T_i.

By the induction assumption, for each i we have: the value $\frac{|T_i \setminus \{v_i\}|!}{(T_i \setminus \{v_i\})!}$ counts the number of heap-orderings on the rooted forest $T_i \setminus \{v_i\}$, which implies the statement of the lemma. □

From Lemma 4.1.16 we immediately obtain:

Corollary 4.1.17. *Given a rooted forest F, we have the following identity:*

$$\frac{|F|!}{F!} = \sum_{v \text{ a root of } F} \frac{|F \setminus \{v\}|!}{(F \setminus \{v\})!}$$

4.2. Computation of superdimensions.

Theorem 4.2.1. *Let $\lambda \in \Lambda_n$ and let $L_n(\lambda)$ be the corresponding simple module in $\mathcal{F}_n$ (with an even highest weight vector, as before).*

Consider the cap diagram d_λ, as described in Section 2.2.5.

If the weight λ is not worthy (see Definition 4.1.1), then

$$\mathrm{sdim} L_n(\lambda) = 0.$$

If the weight λ is worthy, let F_λ be the corresponding rooted forest (as in Definition 4.1.11 above). Then

$$\mathrm{sdim} L_n(\lambda) = \frac{|F_\lambda|!}{F_\lambda!}.$$

Example 4.2.2.

(1) For $\lambda = 0$ and any $n \geq 1$, we have: $\mathrm{sdim} L_n(0) = \frac{|F_\lambda|!}{F_\lambda!} = 1$.

(2) For $\lambda = -\varepsilon_1$ and $n \geq 2$, we have: $\mathrm{sdim} L_n(-\varepsilon_1) = -\mathrm{sdim} V_n = 0$.

(3) For λ as in Example 4.1.4, we have: $\mathrm{sdim} L_6(\lambda) = \frac{|F_\lambda|!}{F_\lambda!} = 3$.

(4) For λ as in Example 4.1.5, we have: $\mathrm{sdim} L_4(\lambda) = 0$.
(5) For λ as in Example 4.1.6, we have: $\mathrm{sdim} L_{11}(\lambda) = \frac{|F_\lambda|!}{F_\lambda!} = 15$.

Proof of Theorem 4.2.1. We prove the required statement by induction on $n \geq 1$, done separately for odd and even n.

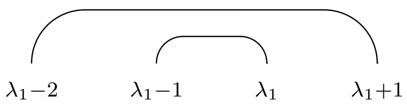
Base: For $n = 1$, any (dominant) integral $\mathfrak{p}(1)$-weight $\lambda \in \Lambda_1$ has a cap diagram with a single cap. So it is worthy, and its rooted forest (tree) F_λ consists of just one node. The simple $\mathfrak{p}(1)$-module $L_1(\lambda)$ has superdimension 1. Hence

$$\frac{|F_\lambda|!}{F_\lambda!} = 1 = \mathrm{sdim} L_1(\lambda)$$

as required.

For $n = 2$, we have two types of (dominant) integral $\mathfrak{p}(2)$-weights $\lambda \in \Lambda_1$:

(1) If $\lambda_1 = \lambda_2$, then the cap diagram has exactly two caps, one internal to the other:



So λ is worthy and $\widehat{C}(\lambda)^{even}$ has just one element (the cap $(\lambda_1 - 2, \lambda_1 + 1)$). Its rooted forest (tree) F_λ consists of one node. The simple $\mathfrak{p}(2)$-module $L_2(\lambda)$ is a tensor power of the determinant representation of $\mathfrak{p}(2)_0 = \mathfrak{gl}_2$, and has superdimension 1. Hence

$$\frac{|F_\lambda|!}{F_\lambda!} = 1 = \mathrm{sdim} L_2(\lambda)$$

as required.

(2) If $\lambda_1 \neq \lambda_2$, then the cap diagram has exactly two disjoint caps:

The virtual cap in this case has two odd successors, hence λ is not worthy. The simple $\mathfrak{p}(2)$-module $L_2(\lambda)$ is typical and has superdimension 0, as required.

Step: Assume the statement of the theorem holds for $n-2, n-1$. We now prove it for n.

Recall that the Duflo-Serganova functor DS_x (for any $x \in \mathfrak{p}(n)_{\bar{1}}$) preserves categorical superdimensions, by Lemma 2.3.2.

For each $k = n-1, n$, let $x_k \in \mathfrak{p}(k)_1, x_k \neq 0$ be the odd element corresponding to the root $2\varepsilon_k$. Let $DS_{x_{n-1}}$, DS_{x_n} be the corresponding Duflo-Serganova functors.

First we consider the case when $n \equiv 1 \mod 2$.

Let $\lambda \in \Lambda_n$. Then

$$\mathrm{sdim} L_n(\lambda) = \mathrm{sdim} DS_{x_n}(L_n(\lambda)) = \sum_{c \in C(\lambda) \text{ maximal}} (-1)^{z(\lambda,c)} \mathrm{sdim} L_{n-1}(\mu_c) \tag{5}$$

Here for each maximal (non-virtual) cap c in $C(\lambda)$, we denote by μ_c the weight in Λ_{n-1} such that d_{μ_c} is obtained from d_λ by removing the cap c (see Corollary 3.1.4), and $z(\lambda, c) = z$ is the parity of the composition factor $L_{n-1}(\mu_c)$ in $DS_{x_n}(L_n(\lambda))$.

Consider a maximal cap $c \in C(\lambda)$ as above, and let $\mu := \mu_c$. Then $\widehat{C}(\mu) = \widehat{C}(\lambda) \setminus \{c\}$ with induced partial order.

We then have the following sublemma:

Sublemma 4.2.3. *Assume $n \equiv 1 \mod 2$. Then we have:*

- *If λ is not worthy, then neither is μ.*
- *If λ is worthy, and c is even, then μ is not worthy.*
- *If λ is worthy, and c is odd, then μ is worthy.*

Proof of Sublemma.

- Assume λ is not worthy.

  Let $c' \in \widehat{C}(\lambda)$ be a cap with at least 2 odd successors. Then we have three cases:

  (1) Case $c' = c$. In this case $c_* \in \widehat{C}(\mu)$ will have at least 2 odd successors.

  (2) Case $c' = c_*$. Recall that since $n \equiv 1 \mod 2$, the virtual cap $c_* \in \widehat{C}(\lambda)$ is even, hence it has an odd number of odd successors, by Lemma 4.1.9. Thus it has at least 3 odd successors in $\widehat{C}(\lambda)$, and $c_* \in \widehat{C}(\mu)$ will have at least 2 odd successors in $\widehat{C}(\mu)$.

  (3) Case $c' \neq c, c_*$. In this case $c' \in \widehat{C}(\mu)$ will have at least 2 odd successors.

  In all these cases μ is not worthy.
- Assume λ is worthy, and c is even.

  Since $n \equiv 1 \mod 2$, the virtual cap $c_* \in \widehat{C}(\lambda)$ is even. So c_* has one odd successor in $\widehat{C}(\lambda)$ which is not c, and will gain one more odd successor (a former successor of c) after c is removed. Thus $c_* \in \widehat{C}(\mu)$ will still have at least 2 odd successors, and μ is not worthy.
- Assume λ is worthy, and c is odd. Then by Corollary 4.1.10 the number of odd successors of any given cap has not grown when passing from d_λ to d_μ, and hence μ is worthy.

The sublemma is proved. □

Thus in case $n \equiv 1 \mod 2$, we have: if λ is not worthy then $\mathrm{sdim} L_n(\lambda) = 0$; if λ is worthy then

$$\mathrm{sdim} DS_{x_n}(L_n(\lambda)) = (-1)^{z(\lambda,c)} \mathrm{sdim} L_{n-1}(\mu)$$

where $\mu \in \Lambda_{n-1}$ is the weight whose cap diagram d_μ is obtained by removing the unique non-virtual *odd* maximal cap c in d_λ (recall that $c_* \in \widehat{C}(\lambda)$ has exactly one odd successor, by Corollary 4.1.10, and it is precisely c).

This implies that the rooted forest F_μ is obtained from the rooted tree F_λ by removing its root, hence

$$\frac{|F_\mu|!}{F_\mu!} = \frac{|F_\lambda|!}{F_\lambda!}.$$

The parity $z(\lambda, c)$ appearing in Corollary 3.1.4 is 0: indeed, since c is the only odd cap in d_λ, there is an even number of caps whose right end is to the right of c, hence $z(\lambda, c) = 0$ by Remark 3.1.5.

Applying the induction assumption to $L_{n-1}(\mu)$, we obtain:

$$\mathrm{sdim} L_n(\lambda) = \mathrm{sdim} DS_{x_n}(L_n(\lambda)) = \mathrm{sdim} L_{n-1}(\mu) = \frac{|F_\mu|!}{F_\mu!} = \frac{|F_\lambda|!}{F_\lambda!}$$

as required. This completes the proof of the theorem in case n is odd.

We now consider the case when n is even.

Again, let $\lambda \in \Lambda_n$.

We consider the functor

$$\overline{DS} : \mathcal{F}_n \to \mathcal{F}_{n-2}, \quad \overline{DS} := DS_{x_{n-1}} \circ DS_{x_n}$$

Then $\overline{DS}$ is a symmetric monoidal functor preserving superdimensions.

Computing the action of $\overline{DS}$ on $L_n(\lambda)$ explicitly, we have:

$$\text{(6)} \qquad \mathrm{sdim} L_n(\lambda) = \mathrm{sdim}\overline{DS}(L_n(\lambda)) = \sum_{\underline{c}=(c_1,c_2),\ c_1,c_2 \in C(\lambda)} (-1)^{\tilde{z}(\lambda,\underline{c})} \mathrm{sdim} L_{n-2}(\mu_{\underline{c}})$$

Here the sum goes over all ordered pairs of caps $\underline{c} = (c_1, c_2)$ where c_1 is a maximal (non-virtual) cap in $C(\lambda)$, while $c_2 \in C(\lambda)\backslash\{c_1\}$ is a successor of either c_* or c_1. The weight $\mu_{\underline{c}} \in \Lambda_{n-2}$ is such that $d_{\mu_{\underline{c}}}$ is obtained from d_λ by removing c_1 and then c_2. The parity $\tilde{z}(\lambda, \underline{c})$ is computed using Corollary 3.1.4:

$$\tilde{z}(\lambda, \underline{c}) = z(\lambda, c_1) + z(\lambda_{c_1}, c_2)$$

where the notation is as in (5).

Let $\underline{c} = (c_1, c_2)$ be a pair of caps as above, and let $\mu := \mu_{\underline{c}}$. Then $\widehat{C}(\mu) = \widehat{C}(\lambda) \setminus \{c_1, c_2\}$ with the induced partial order.

We begin our study of the sum (6) above with the following observation:

Assume c_1, c_2 are both successors of c_*. Then both (c_1, c_2) and (c_2, c_1) are ordered pairs appearing as indices in the sum (6), and $\mu_{(c_1,c_2)} = \mu_{(c_2,c_1)}$. By Remark 3.1.5, we have:

$$\tilde{z}(\lambda, (c_1, c_2)) \equiv \tilde{z}(\lambda, (c_2, c_1)) + 1 \mod 2.$$

Hence the corresponding terms in the sum (6) cancel out, and from now on we will consider the sum (6) so that the sum goes over the ordered pairs (c_1, c_2) where c_2 is a successor of c_1.

Let us consider the case when λ is not worthy.

Let $c' \in \widehat{C}(\lambda)$ be a cap (perhaps virtual) with at least 2 odd successors.

Sublemma 4.2.4. *The weight $\mu = \mu_{\underline{c}} \in \Lambda_{n-2}$ is not worthy as well.*

Proof. Assume the contrary: μ is worthy.

Recall that since $n \equiv 0 \mod 2$, the virtual cap $c_* \in \widehat{C}(\lambda)$ is odd, hence it has an even number of odd successors, by Lemma 4.1.9. After the removal of c_1, c_2 it inherits their odd successors, so we have a disjoint union:

$$\left\{ {\text{odd successors of} \atop c_* \text{ in } \widehat{C}(\mu)} \right\} = \left\{ {\text{odd successors of} \atop c_* \text{ in } \widehat{C}(\lambda)} \right\} \setminus \{c_1\} \sqcup \left\{ {\text{odd successors of} \atop c_1 \text{ in } \widehat{C}(\lambda)} \right\} \setminus \{c_2\} \sqcup \left\{ {\text{odd successors of} \atop c_2 \text{ in } \widehat{C}(\lambda)} \right\}.$$

Since $c_* \in \widehat{C}(\mu)$ has at most one odd successor, the above union contains only one element. Now Lemma 4.1.9 implies that $c_* \in \widehat{C}(\lambda)$ has no odd successors, and thus c_1 is even. Applying Lemma 4.1.9 again we conclude that c_1 must have at least one odd successor, and the same goes for c_2 if it is even. But since the set

$$\left\{ {\text{odd successors of} \atop c_1 \text{ in } \widehat{C}(\lambda)} \right\} \setminus \{c_2\} \sqcup \left\{ {\text{odd successors of} \atop c_2 \text{ in } \widehat{C}(\lambda)} \right\}$$

contains only one element. we conclude that the following must hold in $\widehat{C}(\lambda)$: $c_* \in \widehat{C}(\lambda)$ has no odd successors, c_1 is even and has precisely one odd successor: c_2, which has no odd successors itself.

Hence we must have $c' \neq c_*, c_1, c_2$. In this case $c' \in \widehat{C}(\mu)$ will have at least 2 odd successors, and μ is not worthy, contradicting our assumption. This proves the statement of the sublemma. □

Applying the induction assumption to each $\mu_{\underline{c}}$, we conclude that if λ is not worthy, then

$$\mathrm{sdim} L_n(\lambda) = \mathrm{sdim} DS(L_n(\lambda)) = 0.$$

Now let us consider the case when λ is worthy. Then c_* is odd, and all the maximal (non-virtual) caps in $C(\lambda)$ are even. Hence c_1 is necessarily even.

Assume c_2 is even. Then both c_1 and c_2 have odd successors, and after the removal of these caps both odd successors are "inherited" by $c_* \in \widehat{C}(\mu)$. Hence $c_* \in \widehat{C}(\mu)$ will have at least 2 odd successors in $\widehat{C}(\mu)$, and μ is not worthy.

Applying the induction assumption to μ, we conclude: if λ is worthy, the sum in (6) becomes

$$\text{sdim} L_n(\lambda) = \text{sdim} DS(L_n(\lambda)) = \sum_{\underline{c}=(c_1,c_2),\ c_1,c_2\in C(\lambda)} (-1)^{\tilde{z}(\lambda,\underline{c})} \text{sdim} L_{n-2}(\mu_{\underline{c}}) \tag{7}$$

over ordered pairs $\underline{c} = (c_1, c_2)$ where c_1 is a maximal (non-virtual, even) cap in $C(\lambda)$ and c_2 is its unique odd successor.

In that case, the rooted forest $F_{\mu_{\underline{c}}}$ is obtained from F_λ by removing exactly one node, corresponding to the even cap c_1.

The parity $\tilde{z}(\lambda, \overline{c})$ is then necessarily 0: indeed, there is an even number of caps whose right end is to the right of the cap c_1, and after its removal, the same is true for the cap c_2. By Remark 3.1.5, this implies:

$$\tilde{z}(\lambda, \underline{c}) = 0 + 0 = 0.$$

Applying the induction assumption to all $\mu_{\underline{c}}$ and using Corollary 4.1.17, we obtain:

$$\text{sdim} L_n(\lambda) = \text{sdim} DS(L_n(\lambda)) = \sum_{\substack{\underline{c}=(c_1,c_2)\ c_1,c_2\in C(\lambda),\\ c_2 \text{ unique odd successor of } c_1,\\ c_1 \text{ is maximal}}} \text{sdim} L_{n-2}(\mu_{\underline{c}}) =$$

$$\sum_{\substack{\underline{c}=(c_1,c_2)\ c_1,c_2\in C(\lambda),\\ c_2 \text{ unique odd successor of } c_1,\\ c_1 \text{ is maximal}}} \frac{\left|F_{\mu_{\underline{c}}}\right|!}{F_{\mu_{\underline{c}}}!} = \sum_{v \text{ a root of } F_\lambda} \frac{|F_\lambda \setminus \{v\}|!}{(F_\lambda \setminus \{v\})!} = \frac{|F_\lambda|!}{F_\lambda!}$$

as required. This completes the proof of Theorem 4.2.1. □

As s special case of the statement of Theorem 4.2.1, we have:

Proposition 4.2.5. *Let $L \in \mathcal{F}_n^k$ be a simple module, and $k \neq 0, \pm 1$. Then* $\text{sdim} L = 0$.

Proof. Recall from Theorem 4.2.1 that

$$\text{sdim} L_n(\lambda) \neq 0 \iff \lambda \text{ is worthy.}$$

So let $\lambda \in \Lambda_n$ be a worthy weight. We will show that $L_n(\lambda) \in \mathcal{F}_n^k$ with $k = 0$ if n is even and $k = \pm 1$ otherwise. In other words, we will prove that

$$\sum_{i=1}^{n} (-1)^{\bar{\lambda}_i} = \begin{cases} 0 & \text{if } n \equiv 0 \mod 2 \\ \pm 1 & \text{if } n \equiv 1 \mod 2 \end{cases}. \tag{8}$$

where $\{\bar{\lambda}_i\}_{i=1}^n$ are precisely the right ends of the caps in the cap diagram for λ.

Let us prove this by complete induction on $n \geq 1$.

Base case: For $n = 1$, the category $\mathcal{F}_1$ only has two blocks: $\mathcal{F}_1^{\pm 1}$, so there is nothing to prove. For $n = 2$, the category $\mathcal{F}_2$ has three blocks: $\mathcal{F}_2^0, \mathcal{F}_2^{\pm 2}$. The worthy weights in this case have the form $\lambda \in \Lambda_2$ where $\lambda_1 = \lambda_2$, hence $\sum_{i=1}^2 (-1)^{\bar{\lambda}_i} = 0$ as required.

Step: Let $n \geq 3$, and assume the statement holds up to rank $n - 1$. Let $\lambda \in \Lambda_n$ be a worthy weight.

If n is even, the cap diagram for λ has at least one maximal (non-virtual) even cap c. Let c' be its unique odd successor. Let j, j' be the indices of the right ends of c, c' respectively. Then $j \not\equiv j' \mod 2$, hence $(-1)^j + (-1)^{j'} = 0$. If we remove both caps c, c', we are left with a cap diagram for a worthy weight in Λ_{n-2}. By the induction assumption, the statement of (8) holds for this weight, so

$$\sum_{i:\,\bar{\lambda}_i\neq j,j'}(-1)^{\bar{\lambda}_i}=0 \implies \sum_{i=1}^{n}(-1)^{\bar{\lambda}_i}=0$$

as required.

If n is odd, the cap diagram for λ has precisely one maximal (non-virtual) odd cap c. Let j be the index of its right end. If we remove this cap, we are left with a cap diagram for a worthy weight in Λ_{n-1}. By the induction assumption, the statement of (8) holds for this weight, so

$$\sum_{i:\,\bar{\lambda}_i\neq j}(-1)^{\bar{\lambda}_i}=0 \implies \sum_{i=1}^{n}(-1)^{\bar{\lambda}_i}=\pm 1.$$

This completes the proof of the proposition.

□

Finally, we recover the Kac-Wakimoto conjecture for $\mathfrak{p}(n)$ proved in [EnS19]:

Corollary 4.2.6. *Let $M \in \mathcal{F}_n^k$ where $k \neq 0, \pm 1$. Then* $\mathrm{sdim} M = 0$.

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