SIMPLE WEIGHT MODULES WITH FINITE WEIGHT MULTIPLICITIES OVER THE LIE ALGEBRA OF POLYNOMIAL VECTOR FIELDS

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ABSTRACT. Let $\mathcal{W}_n$ be the Lie algebra of polynomial vector fields. We classify simple weight $\mathcal{W}_n$-modules M with finite weight multiplicities. We prove that every such nontrivial module M is either a tensor module or the unique simple submodule in a tensor module associated with the de Rham complex on $\mathbb{C}^n$.

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1. INTRODUCTION

Lie algebras of vector fields have been studied since the fundamental works of S. Lie and E. Cartan in the late 19th century and the early 20th century. A classical example of such Lie algebra is the Lie algebra $\mathcal{W}_n$ consisting of the derivations of the polynomial algebra $\mathbb{C}[x_1, ..., x_n]$, or, equivalently, the Lie algebra of polynomial vector fields on $\mathbb{C}^n$. The first classification results concerning representations of $\mathcal{W}_n$ and other Cartan type Lie algebras were obtained by A. Rudakov in 1974-1975, [16], [17]. These results address the classification of a class of irreducible $\mathcal{W}_n$-representations that satisfy some natural topological conditions. The modules of Rudakov are a particular class of the so-called *tensor modules*.

General tensor modules $T(P, V)$ are introduced by Shen and Larson, [18], [11], and are defined for a $\mathcal{D}_n$-module P and $\mathfrak{gl}(n)$-module V, where $\mathcal{D}_n$ is the algebra of polynomial differential operators on $\mathbb{C}^n$ (see §2.8 for details). The modules $T(P, V)$ have nice geometric interpretations. If V is finite dimensional, then we have a natural map from $\mathcal{W}_n$ to the algebra of differential operators in the section of a trivial vector bundle on $\mathbb{C}^n$ with fiber V. This map is a specialization of a Lie algebra homomorphism $\mathcal{W}_n \to \mathcal{D}_n \otimes U(\mathfrak{gl}(n))$. The tensor module $T(P, V)$ is nothing but the pull back of the $\mathcal{D}_n \otimes U(\mathfrak{gl}(n))$-module $P \otimes V$.

Tensor $\mathcal{W}_1$-modules and their extensions were studied extensively in the 1970's and in the 1980's by B. Feigin, D. Fuks, I. Gelfand, and others, see for example, [4], [6].

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Important results on general tensor modules $T(P, V)$ have been recently established by G. Liu, R. Lu, Y. Xue, K. Zhao, and others, see [19] and the references therein.

In this paper we focus on the category of weight representations of $\mathcal{W}_n$, namely those that decompose as direct sums of weight spaces relative to the subalgebra $\mathfrak{h}$ of $\mathcal{W}_n$ spanned by the derivations $x_1\partial_1,...,x_n\partial_n$. The study of weight representations of Lie algebras of vector fields is a subject of interest by both mathematicians and theoretical physicists in the last 30 years. Two particular cases in this study have attracted special attention - the cases of $\mathcal{W}_n$ and of the Witt algebra Witt_n. Recall that Witt_n is the Lie algebra of the derivations of the Laurent polynomial algebra $\mathbb{C}[x_1^{\pm}, ..., x_n^{\pm 1}]$, or, equivalently, the Lie algebra of polynomial vector fields on the n-dimensional complex torus. In particular, Witt_1 is the centerless Virasoro algebra. The classification of all simple weight representations with finite weight multiplicities of $\mathcal{W}_1$ and Witt_1 (and hence of the Virasoro algebra) was obtained by O. Mathieu in 1992, [13]. Following a sequence of works of S. Berman, Y. Billig, C. Conley, X. Guo, C. Martin, O. Mathieu, V. Mazorchuk, V. Kac, G. Liu, R. Lu, A. Piard, S. Eswara Rao, Y. Su, K. Zhao, recently, Y. Billig and V. Futorny managed to extend Mathieu's classification result to Witt_n for arbitrary $n \geq 1$ (see [1] and the references therein).[1]

The classification of simple bounded (i.e. with a bounded set of weight multiplicities) modules of $\mathcal{W}_n$ was completed in [19]. The result in [19] states that every simple bounded module is a tensor module $T(P, V)$ or a submodule of a tensor module. In order $T(P, V)$ to be bounded, P must be a weight $\mathcal{D}_n$-module and V must be a finite-dimensional module.

In this paper we classify all simple weight $\mathcal{W}_n$-modules M with finite weight multiplicities. The main result is surprisingly easy to formulate - every such nontrivial module M is either a tensor module $T(P, V)$ or the unique simple submodule of $T(P, \bigwedge^k \mathbb{C}^n)$ for $k = 1, ..., n$. The necessary and sufficient condition for P and V so that $T(P, V)$ has finite weight multiplicities is given in Theorem 3.5. This condition is expressed in terms of the subsets of roots $\mathcal{W}_n$ and $\mathfrak{gl}(n)$ that act locally finitely or injectively on P and V, respectively. For our classification result, we first use a theorem of [15] stating that M is parabolically induced from a bounded simple module N over a subalgebra $\mathfrak{g} = \mathcal{W}_m \ltimes (\mathfrak{k} \otimes \mathcal{O}_m)$ of $\mathcal{W}_n$. This subalgebra $\mathfrak{g}$ plays the role of a Levi subalgebra of a parabolic subalgebra of $\mathcal{W}_n$. The classification of simple bounded $\mathfrak{g}$-modules is one of the most difficult parts of the proof. By introducing the so called $(\mathfrak{g}, \mathcal{O}_m)$-modules, we prove that N is either the unique submodule of a tensor module, or it is a special generalized tensor module $\mathcal{F}(T(P, V), S)$, see Theorem 5.17. The essential tool for proving this theorem is the twisted localization functor intrduced in [14]. For the main theorem we show that the parabolic induction functor maps $\mathcal{F}(T(P, V), S)$ to a tensor module.

[1] Note that the Witt algebra Witt_n is denoted by $\mathcal{W}_n$ in [1].

The content of the paper is as follows. In Section 2 we collect some important definitions and preliminary results on weight modules, twisted localization, parabolic induction, and tensor modules. In Section 3 we prove the necessary and sufficient condition for the tensor module $T(P,V)$ to be a weight module with finite weight multiplicities. We also show that $T(P,V)$ has a unique simple submodule and explain how the restricted duality functor acts on the tensor modules. The main theorem of this paper is also stated in Section 3. Section 4 is devoted to a few results concerning the parabolic induction theorem. The study of bounded $\mathfrak{g}$-modules and the classification of all possible $\mathfrak{g}$-modules N that appear in the parabolic induction theorem are included in Section 5. In Section 6 we complete the proof of the main theorem by showing that the application of the parabolic induction functor on all possible N described in the previous section leads to modules M that are either tensor modules or the unique simple submodules of $T(P, \bigwedge^k \mathbb{C}^n)$ for $k = 1, ..., n$.

2. Preliminaries

2.1. **Notation and convention.** Throughout the paper the ground field is $\mathbb{C}$. All vector spaces, algebras, and tensor products are assumed to be over $\mathbb{C}$ unless otherwise stated.

2.2. **Weight modules in general setting.** Let $\mathcal{U}$ be an associative unital algebra and $\mathcal{H} \subset \mathcal{U}$ be a commutative subalgebra. We assume in addition that $\mathcal{H}$ is a polynomial algebra identified with the symmetric algebra of a vector space $\mathfrak{h}$, and that we have a decomposition

$$\mathcal{U} = \bigoplus_{\mu \in \mathfrak{h}^*} \mathcal{U}^\mu,$$

where

$$\mathcal{U}^\mu = \{x \in \mathcal{U} | [h,x] = \mu(h)x, \forall h \in \mathfrak{h}\}.$$

Let $Q_\mathcal{U} = \mathbb{Z}\Delta_\mathcal{U} = \Delta_\mathcal{U} \cup (-\Delta_\mathcal{U})$ be the $\mathbb{Z}$-lattice in $\mathfrak{h}^*$ generated by $\Delta_\mathcal{U} = \{\mu \in \mathfrak{h}^* \mid \mathcal{U}^\mu \neq 0\}$. We obviously have $\mathcal{U}^\mu \mathcal{U}^\nu \subset \mathcal{U}^{\mu+\nu}$.

We call a $\mathcal{U}$-module M *a weight module*, or a $(\mathcal{U},\mathcal{H})$*-module*, if $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda$, where

$$M^\lambda = \{m \in M \mid hm = \lambda(h)m \text{ for all } h \in \mathfrak{h}\}.$$

We call M^λ the weight space of M, $\dim M^\lambda$ the λ-*weight multiplicity* of M, and $\operatorname{supp} M = \{\lambda \in \mathfrak{h}^* \mid M^\lambda \neq 0\}$ the *support* of M. Note that

$$\mathcal{U}^\mu M^\lambda \subset M^{\mu+\lambda}.$$

for every weight module M.

We will call a weight $\mathcal{U}$-module *bounded* if its set of weight multiplicities is a bounded set. For a bounded $\mathcal{U}$-module M, the degree $d(M)$ is the maximal weight multiplicity of M. A weight $\mathcal{U}$-module M with finite weight multiplicities is *cuspidal* if all nonzero elements of $\mathcal{U}^\mu$ act injectively on M. If $\Delta_\mathcal{U} = -\Delta_\mathcal{U}$, then every cuspidal

$\mathcal{U}$-module is bounded. We use this notion in the case when $\mathcal{U}$ is the Weyl algebra or the universal enveloping algebra of a reductive Lie algebra where the latter property holds.

In the particular case when $\mathcal{U} = U(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ and $\mathcal{H} = S(\mathfrak{h})$ for a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we have that a weight $\mathcal{U}$-module is a weight $\mathfrak{g}$-module.

2.3. **Twisted localization in general setting.** In this subsection we collect some facts about the twisted localization functor. This functor was originally introduced in [14] and for proofs and more details we refer the reader to §7 of [10].

We retain the notation of the previous subsection. Let a be an ad-nilpotent element of $\mathcal{U}$. Then the set $\langle a \rangle = \{a^n \mid n \geq 0\}$ is an Ore subset of $\mathcal{U}$ which allows us to define the $\langle a \rangle$-localization $D_{\langle a \rangle}\mathcal{U}$ of $\mathcal{U}$. For a $\mathcal{U}$-module M by $D_{\langle a \rangle}M = D_{\langle a \rangle}\mathcal{U} \otimes_{\mathcal{U}} M$ we denote the $\langle a \rangle$-localization of M. Note that if a is injective on M, then M is isomorphic to a submodule of $D_{\langle a \rangle}M$. In the latter case we will identify M with that submodule.

We next recall the definition of the generalized conjugation of $D_{\langle a \rangle}\mathcal{U}$ relative to $x \in \mathbb{C}$. This is the automorphism $\phi_x : D_{\langle a \rangle}\mathcal{U} \to D_{\langle a \rangle}\mathcal{U}$ defined by the formula

$$\phi_x(u) = \sum_{i \geq 0} \binom{x}{i} \operatorname{ad}(a)^i(u) a^{-i}.$$

If $x \in \mathbb{Z}$, then $\phi_x(u) = a^x u a^{-x}$. With the aid of ϕ_x we define the twisted module $\Phi_x(M) = M^{\phi_x}$ of any $D_{\langle a \rangle}\mathcal{U}$-module M. Finally, we set $D^x_{\langle a \rangle}M = \Phi_x D_{\langle a \rangle}M$ for any $\mathcal{U}$-module M and call it the *twisted localization* of M relative to a and x. We will use the notation $a^x \cdot m$ (or simply $a^x m$) for the element in $D^x_{\langle a \rangle}M$ corresponding to $m \in D_{\langle a \rangle}M$. In particular, the following formula holds in $D^x_{\langle a \rangle}M$:

$$u(a^x m) = a^x \left(\sum_{i \geq 0} \binom{-x}{i} \operatorname{ad}(a)^i(u) a^{-i} m \right)$$

for $u \in \mathcal{U}$, $m \in D_{\langle a \rangle}M$.

If $a_1, ..., a_k$ are commuting ad-nilpotent elements in $\mathcal{U}$ and $\mathbf{c} = (c_1, ..., c_k)$ is in $\mathbb{C}^k$, then we set $D_{\langle a_1,...,a_k \rangle}M = \prod_{i=1}^k D_{\langle a_i \rangle}M$ and $D^{\mathbf{c}}_{\langle a_1,...,a_k \rangle}M = \prod_{i=1}^k D^{c_i}_{\langle a_i \rangle}M$. Note that the products $\prod_{i=1}^k D_{\langle a_i \rangle}$ and $\prod_{i=1}^k D^{c_i}_{\langle a_i \rangle}$ are well defined because the functors involved pairwise commute.

If $a \in \mathcal{U}$ is an ad-nilpotent weight element and M is a weight module then $D^x_{\langle a \rangle}M$ is again a weight module.

Lemma 2.1. *Let $a \in \mathcal{U}$ be an* ad*-nilpotent weight element in $\mathcal{U}$, M be a simple a-injective weight $\mathcal{U}$-module, and $z \in \mathbb{C}$. If N is a simple nontrivial $\mathcal{U}$-submodule of $D^z_{\langle a \rangle}M$, then $D_{\langle a \rangle}M \simeq D^{-z}_{\langle a \rangle}N$. In particular, if a acts bijectively on M, $M \simeq D^{-z}_{\langle a \rangle}N$.*

Proof. We use that since M is a simple a-injective weight $\mathcal{U}$-module, then $D_{\langle a \rangle}M$ and $D^z_{\langle a \rangle}M$ are simple $D_{\langle a \rangle}\mathcal{U}$-modules. Indeed, $D_{\langle a \rangle}M$ is a simple $D_{\langle a \rangle}\mathcal{U}$-module because

$\theta(M)$ generates the $D_{\langle a\rangle}\mathcal{U}$-module $D_{\langle a\rangle}M$, where $\theta : M \to D_{\langle a\rangle}M$ is the localization map. The simplicity of $D^z_{\langle a\rangle}M$ follows straightforward from the simplicity of $D_{\langle a\rangle}M$.

Since N is submodule of $D^z_{\langle a\rangle}M$, $D_{\langle a\rangle}N$ is a submodule of $D^z_{\langle a\rangle}M$. The simplicity of N implies $D_{\langle a\rangle}N \simeq D^z_{\langle a\rangle}M$ and $D^{-z}_{\langle a\rangle}N \simeq D_{\langle a\rangle}M$. If a acts bijectively, then $M \simeq D_{\langle a\rangle}M$. □

We will also consider the following particular case of the twisted localization functor for $\mathcal{U} = U(\mathfrak{g})$, where $\mathfrak{g} = \mathcal{W}_m + \mathfrak{k}\otimes\mathcal{O}_m$ where $\mathfrak{k}$ is a reductive Lie algebra isomorphic to $\mathfrak{gl}(p)\oplus\mathfrak{gl}(q)$. Let $a_i \in \mathfrak{k}^{\alpha_i}$, $i = 1, ..., \ell$, and $\Gamma = \{\alpha_1, ..., \alpha_\ell\}$ is a set of commuting roots of $\mathfrak{k}$ that is linearly independent in $\mathbb{Z}\Delta_{\mathfrak{k}}$. Let also $\lambda \in \mathfrak{h}^*$ be such that $\lambda = \sum_{i=1}^{\ell} z_i\alpha_i$. We set $D^\lambda_\Gamma = D^{z_1}_{\langle a_1\rangle}...D^{z_\ell}_{\langle a_\ell\rangle}$. If $M \simeq D^\lambda_\Gamma \bar{M}$ we will say that M is *obtained by a twisted localization from* $\bar{M}$. If $\bar{M}$ is bounded, then M is bounded, [14] Lemma 4.4.

2.4. **The algebras $\mathcal{O}_n$, $\mathcal{D}_n$, and $\mathcal{W}_n$.** In what follows, $\mathcal{O}_n = \mathbb{C}[x_1, ..., x_n]$ and $\mathcal{D}_n$ will stand for the associative algebra of differential operators in $\mathcal{O}_n$. In other words, $\mathcal{D}_n$ is the n-th Weyl algebra. We will often use the fact that $\mathcal{D}_n \simeq \mathcal{D}_1 \otimes ... \otimes \mathcal{D}_1$ (n copies). Also, $\mathcal{W}_n$ will stand for the Lie algebra of vector fields on $\mathbb{C}^n$, i.e. $\mathcal{W}_n = \mathrm{Der}(\mathcal{O}_n)$.

Henceforth, we fix $\mathfrak{h} = \mathrm{Span}\{x_1\partial_1, \dots, x_n\partial_n\}$. Note that $\mathfrak{h}$ is a Cartan subalgebra of $\mathcal{W}_n$ and $\mathcal{H} = \mathbb{C}[x_1\partial_1, \dots, x_n\partial_n]$ is a maximal commutative subalgebra of $\mathcal{D}_n$. We will use the setting of §2.2 and §2.3 both for $\mathcal{U} = U(\mathcal{W}_n)$ and $\mathcal{U} = \mathcal{D}_n$, and in both cases $\mathfrak{h}$ is the one that we fixed above. The set of roots of $\mathcal{W}_n$ is:

$$\Delta = \left\{ \sum_{j=1}^{n} m_j\varepsilon_j, -\varepsilon_i + \sum_{j\neq i} m_j\varepsilon_j \mid m_j \in \mathbb{Z}_{\geq 0}, i = 1, ..., n \right\},$$

where $\varepsilon_i(x_j\partial_j) = \delta_{ij}$. Let $\Delta' := \Delta \cap -\Delta$. One can see that Δ' is a root system of type A_n.

A $\mathcal{W}_n$-module M is a $(\mathcal{W}_n, \mathcal{O}_n)$*-module* if M is an $\mathcal{O}_n$-module satisfying

$$X(fv) = fX(v) + X(f)v, \quad \forall v \in M, f \in \mathcal{O}_n, X \in \mathcal{W}_n.$$

If M is a weight $\mathcal{W}_n$-module with finite weight multiplicities, then the *restricted dual* M_* of M is by definition the maximal semisimple $\mathfrak{h}$-submodule of M^*. The following properties of the restricted dual functor are straightforward.

Lemma 2.2. *Let M be a weight $\mathcal{W}_n$-module with finite weight multiplicities. Then*

(1) $\mathrm{supp}\, M_* = -\,\mathrm{supp}\, M$;
(2) $\dim M_*^\mu = \dim M^{-\mu}$;
(3) *M is simple if and only if M_* is simple.*

Consider the embedding $\mathbb{C}^n \to \mathbb{C}P^n$. The Lie algebra of vector fields on $\mathbb{C}P^n$ is isomorphic to $\mathfrak{sl}(n+1)$ and is a Lie subalgebra of $\mathcal{W}_n$. In other words we have a canonical embedding $\mathfrak{sl}(n+1) \subset \mathcal{W}_n$ of Lie algebras.

Lemma 2.3. *Let M be a bounded weight $\mathcal{W}_n$-module such that* $\mathrm{supp}\, M \subset \lambda + \mathbb{Z}\Delta_{\mathcal{W}_n}$ *for some weight λ. Then M has finite length.*

Proof. The result holds for $\mathfrak{sl}(n+1)$-modules, see Lemma 3.3 in [14], and hence it holds for $\mathcal{W}_n$ by using the natural embedding of $\mathfrak{sl}(n+1)$ in $\mathcal{W}_n$. □

2.5. **Simple weight $\mathcal{D}_n$-modules.** According to §2.2, a $\mathcal{D}_n$-module M is a weight module if

$$M = \bigoplus_{\lambda \in \mathbb{C}^n} M^\lambda,$$

where $M^\lambda = \{m \in M \mid x_i \partial_i m = \lambda_i m, \text{ for } i = 1, ..., n\}$. Below we recall the classification of the simple weight $\mathcal{D}_n$-modules.

We will use the automorphism $\sigma_F : \mathcal{D}_n \to \mathcal{D}_n$ defined by $\sigma_F(x_i) = \partial_i, \sigma_F(\partial_i) = -x_i$ for all i. We call σ_F the (full) *Fourier transform* of $\mathcal{D}_n$. If M is a $\mathcal{D}_n$-module, by M^F we denote the module M twisted by σ_F.

The following gives the classification of all simple weight $\mathcal{D}_n$-modules, see for example Corollary 2.9 in [8]

Proposition 2.4. (i) *Every simple weight module of $\mathcal{D}_1$ is isomorphic to one of the following: $\mathcal{O}_1 = \mathbb{C}[x]$, $\mathcal{O}_1^F$, $x^\lambda \mathbb{C}[x^{\pm 1}]$, $\lambda \in \mathbb{C} \setminus \mathbb{Z}$.*

(ii) *Every simple weight module of $\mathcal{D}_n$ is isomorphic to $P_1 \otimes ... \otimes P_n$ where P_i is a simple weight $\mathcal{D}_1$-module.*

We note also that every simple nontrivial weight $\mathcal{D}_n$-module M has degree 1, i.e. all its weight multiplicities equal 1. Moreover, for every i, x_i (respectively, ∂_i) acts either injectively, or locally nilpotently on M. Let $I^+(M)$ denote the subset of indices in $\{1, \dots, n\}$ such that ∂_i acts locally nilpotently on M and x_i acts injectively on M, $I^-(M)$ the subset of indices such that x_i acts locally nilpotently on M and ∂_i acts injectively on M, and $I^0(M)$ the subset of indices such that both x_i and ∂_i act injectively on M. Note that $\{1, \dots, n\} = I^-(M) \sqcup I^0(M) \sqcup I^+(M)$. Furthermore, there exists $\lambda \in \operatorname{supp} M$ such that

$$\operatorname{supp} M = \lambda + \sum_{i \in I^+(M)} \mathbb{Z}_{\geq 0} \varepsilon_i + \sum_{j \in I^0(M)} \mathbb{Z} \varepsilon_j + \sum_{k \in I^-(M)} \mathbb{Z}_{\leq 0} \varepsilon_k.$$

2.6. **Parabolic induction in general.** Let $\mathfrak{g}$ be any Lie algebra with Cartan subalgebra $\mathfrak{h}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$. Let $\gamma \in \mathfrak{h}^*$. Then the subalgebra

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{Re\langle \gamma, \alpha \rangle \geq 0} \mathfrak{g}_\alpha$$

is called the *parabolic subalgebra* of $\mathfrak{g}$ corresponding to γ. The Levi subalgebra of $\mathfrak{p}$ is

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{Re\langle \gamma, \alpha \rangle = 0} \mathfrak{g}_\alpha,$$

and the nilradical of $\mathfrak{p}$ is

$$\mathfrak{n} = \bigoplus_{Re\langle \gamma, \alpha \rangle > 0} \mathfrak{g}_\alpha.$$

For a $\mathfrak{g}$-module M we set

$$M^{\mathfrak{n}} = \{m \in M \mid \mathfrak{n}m = 0\}.$$

It is easy to see that $M^{\mathfrak{n}}$ is $\mathfrak{p}$-submodule of M.

We are going to use extensively the following standard result.

Proposition 2.5. *(a) Let N be a simple $\mathfrak{l}$-module, considered also as simple $\mathfrak{p}$-modules by letting $\mathfrak{n}$ act trivially on N. Then the $\mathfrak{g}$-module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} N$ has a unique simple quotient.*

(b) If L is a simple $\mathfrak{g}$-module such that $L^{\mathfrak{n}} \neq 0$, then $L^{\mathfrak{n}}$ is a simple $\mathfrak{l}$-module.

(c) If L and M are simple $\mathfrak{g}$-modules such that $M^{\mathfrak{n}}$ and $L^{\mathfrak{n}}$ are isomorphic simple $\mathfrak{l}$-modules, then $M \simeq L$ as $\mathfrak{g}$-modules.

Remark 2.6. If M is a simple weight $\mathfrak{g}$-module then $M^{\mathfrak{n}} = \bigoplus_{\lambda \in S} M^{\lambda}$ where S is the subset of $\operatorname{supp} M$ such that $\lambda + \alpha \notin \operatorname{supp} M$ for any $\alpha \in \Delta(\mathfrak{n})$. For an arbitrary weight module M we call $\oplus_{\lambda \in S} M^{\lambda}$ the $\mathfrak{p}$-*top* of M and denote it by M^{top}.

2.7. **Parabolic induction for $\mathcal{W}_n$.** In this subsection we recall one of the main results in [15]. Recall the definitions of Δ and Δ' from §2.4. Let $\gamma = a_1\varepsilon_1 + \cdots + a_n\varepsilon_n$ for some $a_i \in \mathbb{R}$. Set

$$\Delta_0 = \{\alpha \in \Delta \mid (\gamma, \alpha) = 0\}, \quad \Delta_{\pm} = \{\alpha \in \Delta \mid (\gamma, \alpha) > (< 0)\},$$
$$\Delta_0' = \Delta_0 \cap \Delta', \quad \Delta_{\pm}' = \Delta_{\pm} \cap \Delta'.$$

Let

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_0 \cup \Delta_+} (\mathcal{W}_n)_{\alpha}, \quad \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_0} (\mathcal{W}_n)_{\alpha}.$$

Theorem 2.7. *Let M be a simple weight $\mathcal{W}_n$-module.*

(a) There exists a weight $\lambda \in \operatorname{supp} M$ and γ such that

$$\operatorname{supp} M \subset \lambda + \mathbb{Z}_{\geq 0}(\Delta_-' \cup \Delta_0').$$

(b) One can choose γ in such a way that $\mathbb{Z}\Delta_0' = \mathbb{Z}\Delta_0$ and

$$\lambda + \mathbb{Z}\Delta_0 \subset \operatorname{supp} M.$$

(c) M is a unique simple quotient of the parabolically induced module $U(\mathcal{W}_n) \otimes_{U(\mathfrak{p})} M_0$ for some simple weight $\mathfrak{g}$-module M_0 that is extended in the natural way to a simple $\mathfrak{p}$-module.

2.8. **Tensor modules over $\mathcal{W}_n$.** Let V be a $\mathfrak{gl}(n)$-module and $\tilde{V} := \mathcal{O}_n \otimes V$. One can look at $\tilde{V}$ as the space of sections of the $\mathfrak{gl}(n)$-bundle on $\mathbb{C}^n$ with fiber V. Thus, $\tilde{V}$ has the natural structure of a $(\mathcal{W}_n, \mathcal{O}_n)$-module.

For a $\mathcal{D}_n$-module P and a $\mathfrak{gl}(n)$-module V, we define the tensor $(\mathcal{W}_n, \mathcal{O}_n)$-module by

$$T(P, V) := P \otimes_{\mathcal{O}_n} \tilde{V}$$

and call it *the tensor $\mathcal{W}_n$-module relative to P and V*. If P is a weight $\mathcal{D}_n$-module and V is a weight $\mathfrak{gl}(n)$-module then $P \otimes_{\mathcal{O}_n} \tilde{V}$ is a weight module and

$$\operatorname{supp}(P \otimes_{\mathcal{O}_n} \tilde{V}) = \operatorname{supp} P + \operatorname{supp} V.$$

Alternatively, we can define $T(P,V)$ as follows. Consider $T(P,V)$ as the vector space $T(P,V) = P \otimes_{\mathbb{C}} V$ and define $\mathcal{W}_n$-action and $\mathcal{O}_n$-action by the formulas

$$\begin{aligned} x^\alpha \partial_j \cdot (f \otimes v) &= x^\alpha \partial_j f \otimes v + \sum_{i=1}^n \partial_i (x^\alpha) f \otimes E_{ij} v, \\ x^\alpha \cdot (f \otimes v) &= x^\alpha f \otimes v, \end{aligned}$$

for $f \in P$, $v \in V$.

In what follows, the k-th exterior power $\bigwedge^k \mathbb{C}^n$ of the natural representation of $\mathfrak{gl}(\mathfrak{n})$ will be called the k-th fundamental representation. We have the following result from [12] (Theorem 3.1 and Lemma 3.7):

Proposition 2.8. (i) *Let P be a simple $\mathcal{D}_n$-module and V be a simple $\mathfrak{gl}(n)$-module that is not isomorphic to a fundamental representation. Then $T(P,V)$ is a simple $\mathcal{W}_n$-module.*

(ii) *Let P_1 and P_2 be simple $\mathcal{D}_n$-modules and let V_1 and V_2 be simple $\mathfrak{gl}(n)$-modules such that neither of them is isomorphic to a fundamental representation. Then $T(P_1,V_1) \simeq T(P_2,V_2)$ if and only if $P_1 \simeq P_2$ and $V_1 \simeq V_2$.*

We next consider tensor modules $T(P,V)$ for which V is a fundamental representation. For any $\mathcal{D}_n$-module P, the *differential map*

$$d : T(P, \bigwedge \mathbb{C}^n) \to T(P, \bigwedge \mathbb{C}^n),$$

is defined by $d(f \otimes v) = \sum_{i=1}^n (\partial_i f) \otimes (e_i \wedge v)$, where $(e_1, ..., e_n)$ is the standard basis of $\mathbb{C}^n$ associated to the coordinates $x_1, ..., x_n$ of $\mathbb{C}^n$. The map d is a homomorphsim of $\mathcal{W}_n$-modules but not $\mathcal{O}_n$-modules. One readily sees that $d^2 = 0$. As a result we have the following generalized de Rham complex:

$$0 \xrightarrow{d} T(P, \bigwedge^0 \mathbb{C}^n) \xrightarrow{d} T(P, \bigwedge^1 \mathbb{C}^n) \xrightarrow{d} \cdots \xrightarrow{d} T(P, \bigwedge^n \mathbb{C}^n) \xrightarrow{d} 0.$$

By Theorem 3.5 in [12] we have the following.

Proposition 2.9. *Let P be a simple $\mathcal{D}_n$-module.*

(i) *If $k = 0, ..., n-1$, then the module $T(P, \bigwedge^k \mathbb{C}^n)$ has a simple quotient isomorphic to $dT(P, \bigwedge^k \mathbb{C}^n)$.*

(ii) *The module $T(P, \bigwedge^0 \mathbb{C}^n)$ is simple if and only if P is not isomorphic to $\mathcal{O}_n$. If $P \simeq \mathcal{O}_n$ then $T(P, \bigwedge^0 \mathbb{C}^n)$ contains a trivial $\mathcal{W}_n$-submodule $\mathbb{C}$ and $dT(P, \bigwedge^0 \mathbb{C}^n) \simeq T(P, \bigwedge^0 \mathbb{C}^n)/\mathbb{C}$.*

(iii) *The module $T(P, \bigwedge^n \mathbb{C}^n)$ is simple if and only if $\sum_i \partial_i P = P$.*

We finish this subsection by stating the main result from [19] concerning the classification of the simple bounded $\mathcal{W}_n$-modules.

Theorem 2.10. *Let M be a nontrivial simple bounded $\mathcal{W}_n$-module. Then M is isomorphic to one of the following:*

(a) *the module $T(P, V)$, where P is a simple weight $\mathcal{D}_n$ module and V is a simple finite-dimensional $\mathfrak{gl}(n)$-module that is not isomorphic to a fundamental representation;*

(b) *a simple submodule of $T(P, \bigwedge^k \mathbb{C}^n)$, where $k \in \{1, 2, ..., n\}$, and P is a simple weight $\mathcal{D}_n$ module.*

Remark 2.11. Proposition 3.7 implies that $T(P, \bigwedge^k \mathbb{C}^n)$ has a unique simple submodule.

3. Tensor modules with finite weight multiplicities

3.1. **Tensor product of weight $\mathfrak{gl}(n)$-modules.** Let M be a simple weight $\mathfrak{gl}(n)$-module with finite weight multiplicities. Recall from [3] that M induces the following the following *shadow decomposition* of the root system:

$$\Delta(\mathfrak{gl}(n)) = \Delta_M^F \sqcup \Delta_M^I \sqcup \Delta_M^+ \sqcup \Delta_M^-,$$

such that the α-root vectors X_α act locally nilpotently on M for all roots $\alpha \in \Delta_M^+ \sqcup \Delta_M^F$ and injectively for all roots $\alpha \in \Delta_M^- \sqcup \Delta_M^I$. Moreover, $\Delta_M^I \sqcup \Delta_M^F$ and Δ_M^+ are the roots of the Levi subalgebra $\mathfrak{g}^I + \mathfrak{g}^F$ and the nilradical $\mathfrak{g}^+$, respectively, of a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{gl}(n)$, and M is a quotient a parabolically induced module $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{gl}(n)} \left(M^F \otimes M^I\right)$, for some cuspidal simple $\mathfrak{g}^I$-module M^I and some finite-dimensional simple $\mathfrak{g}^F$-module M^F.

Lemma 3.1. *Let $\mathfrak{l}$ be the Levi subalgebra of some parabolic $\mathfrak{p}$ in $\mathfrak{gl}(n)$. Assume that M' and N' are weight $\mathfrak{l}$-modules and that $M' \otimes N'$ has finite weight multiplicities. Then $(\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{gl}(n)} M') \otimes (\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{gl}(n)} N')$ has finite weight multiplicities.*

Proof. Let $\mathfrak{m}$ denote the nilradical of the opposite to $\mathfrak{p}$ parabolic subalgebra $\mathfrak{p}^-$, and let $U = U(\mathfrak{m})$. Then U has a $\mathbb{Z}_{\geq 0}$-grading $U = \bigoplus_{p \geq 0} U_p$ such that $U_0 = \mathbb{C}$ and each U_p is a finite-dimensional $\mathfrak{l}$-module. This grading induces $\mathbb{Z}_{\geq 0}$-gradings on both $M = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{gl}(n)} M'$ and $N = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{gl}(n)} N'$ so that $M_p = M' \otimes U_p$ and $N_p = N' \otimes U_p$. Then $M \otimes N$ is also graded and its mth graded component is

$$(M \otimes N)_m = \bigoplus_{p+q=m} M' \otimes N' \otimes U_p \otimes U_q.$$

Hence, $M \otimes N$ has finite weight multiplicities. $\square$

Lemma 3.2. *Let M and N be simple weight $\mathfrak{gl}(n)$-modules. Then $M \otimes N$ has finite weight multiplicities if and only if $(\Delta_M^I \sqcup \Delta_M^-) \subset (\Delta_N^F \sqcup \Delta_N^-)$ or, equivalently, $(\Delta_M^I \sqcup \Delta_M^-) \cap (\Delta_N^I \sqcup \Delta_N^+) = \emptyset$.*

Proof. First assume that the condition is not true. There exists a root $\alpha \in \Delta_M^I \sqcup \Delta_M^-$ such that $-\alpha \in \Delta_N^I \sqcup \Delta_N^-$. If $\mu \in \operatorname{supp} M$ and $\nu \in \operatorname{supp} N$ then $\mu + \mathbb{Z}_{\geq 0}\alpha \subset \operatorname{supp} M$ and $\nu - \mathbb{Z}_{\geq 0}\alpha \in \operatorname{supp} N$. Hence $\mu + \nu$ has infinite multiplicity in $M \otimes N$.

Next assume that the condition holds. Then $\Delta_M^I \subset \Delta_N^F$, $\Delta_N^I \subset \Delta_M^F$ and $\Delta_M^- \subset (\Delta_N^F \sqcup \Delta_N^-)$. Choose $\gamma_M \in \mathbb{Q}\Delta$ such that $(\gamma_M, \alpha) = 0$ for all $\alpha \in \Delta_M^I \sqcup \Delta_M^F$ and $(\gamma_M, \alpha) < 0$ for all $\alpha \in \Delta_M^-$. Similarly choose γ_N, and let $\gamma = \gamma_M + \gamma_N$. Then $(\gamma, \Delta_M^I) = (\gamma, \Delta_N^I) = 0$ and $(\gamma, \alpha) < 0$ for any $\alpha \in \Delta_M^- \cup \Delta_N^-$. Let $\mathfrak{p}$ be the parabolic defined by γ. Then both M and N are quotients of the parabolically induced modules $\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{gl}(n)} M'$ and $\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{gl}(n)} N'$, respectively. The Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ is isomorphic to $\mathfrak{g}_M^I \oplus \mathfrak{g}_N^I \oplus (\mathfrak{g}_M^F \cap \mathfrak{g}_N^F)$. Furthermore, $M' = M^i \otimes M^f$ where M^i is a simple cuspidal $\mathfrak{g}_M^I$-module and M^f is some finite-dimensional $\mathfrak{g}_N^I \oplus (\mathfrak{g}_M^F \cap \mathfrak{g}_N^F)$-module. Similarly, $N' = N^i \otimes N^f$ where N^i is a simple cuspidal $\mathfrak{g}_N^I$-module and N^f is some finite-dimensional $\mathfrak{g}_M^I \oplus (\mathfrak{g}_M^F \cap \mathfrak{g}_N^F)$-module. Therefore $M' \otimes N'$ has finite weight multiplicities and the statement follows from Lemma 3.1. □

3.2. **Weight tensor modules.**

Lemma 3.3. *Let P be a simple weight $\mathcal{D}_n$-module. Then $P = \bigoplus_\kappa P_\kappa$, where P_κ is the eigenspace of $\sum_{i=1}^n x_i \partial_i$ with eigenvalue κ. Furthermore, every nonzero P_κ is a simple $\mathfrak{gl}(n)$-module and all nonzero P_κ have the same shadow.*

Proof. The first assertion is obvious. Since the adjoint action of $\mathfrak{gl}(n)$ on $\mathcal{D}_n$ is locally finite, every root vector $X_\alpha \in \mathfrak{gl}(n)$ either acts locally nilpotently or injectively on all nonzero vectors of P. Therefore all P_κ have the same shadow. By the classification of simple weight $\mathcal{D}_n$-modules, every P_κ is multiplicity free and $\operatorname{supp} P_\kappa \subset \lambda + \mathbb{Z}\Delta(\mathfrak{gl}(n))$ for any weight $\lambda \in \operatorname{supp} P_\kappa$. Using these and the fact that $U(\mathfrak{gl}(n))P_\kappa^\lambda = P_\kappa$, we obtain that P_κ is simple. □

Remark 3.4. Lemma 3.3 implies that every simple $\mathcal{D}_n$-module P has a well-defined $\mathfrak{gl}(n)$-shadow. Below we give an explicit description of this shadow in terms of the subsets $I^\pm(P)$, $I^0(P)$ of $\{1, \dots, n\}$ defined in §2.5:

$$\Delta_P^I = \{\varepsilon_i - \varepsilon_j \mid i, j \in I^0(P)\}, \quad \Delta_P^F = \{\varepsilon_i - \varepsilon_j \mid i, j \in I^+(P) \,\text{or}\, i, j \in I^-(P)\},$$

$$\Delta_P^- = \{\varepsilon_i - \varepsilon_j \mid i \in I^+(P), j \notin I^+(P) \text{ or } i \notin I^-(P), j \in I^-(P)\}, \quad \Delta_P^+ = -\Delta_P^-.$$

Theorem 3.5. *Let P be a simple weight $\mathcal{D}_n$-module and V be a simple weight $\mathfrak{gl}(n)$-module. Then the $\mathcal{W}_n$-module $T(P,V)$ has finite weight multiplicities if and only if $(\Delta_P^I \sqcup \Delta_P^-) \subset (\Delta_V^F \sqcup \Delta_V^-)$.*

Proof. For every semisimple $\mathfrak{h}$-module X we denote by X_κ the eigenspace of $\sum x_i \partial_i$ with eigenvalue κ. By Lemma 3.3, $P = \bigoplus_{\tau \in \tau_0 + \mathbb{Z}} P_\tau$ for some $\tau_0 \in \mathbb{C}$. Then

$$T(P,V) = \bigoplus_{\tau \in \tau_0 + \mathbb{Z}} P_\tau \otimes V,$$

and the statement follows from Lemma 3.2. □

Example 3.6. Consider a simple highest weight module $\mathfrak{gl}(4)$-module V such that

$$\Delta_V^+ = \{\varepsilon_i - \varepsilon_j \mid i = 1,2; j = 3,4\}\,,\ \Delta_V^F = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_3 - \varepsilon_4)\}\,.$$

Let P be a simple weight $\mathcal{D}_4$-module P on which $x_1, x_3, \partial_2, \partial_4$ act injectively and $\partial_1, \partial_3, x_2, x_4$ act locally nilpotently. Then by Remark 3.4 and Theorem 3.5, $T(P,V)$ has infinite weight multiplicities as $\varepsilon_1 - \varepsilon_4 \in \Delta_P^- \cap \Delta_V^+$. On the other hand, if P' is a simple $\mathcal{D}_4$-module on which $x_1, \partial_2, \partial_3, \partial_4$ act locally nilpotently and $\partial_1, x_2, x_3, x_4$ act injectively, then $T(P', V)$ has finite weight multiplicities.

Proposition 3.7. *For any simple weight $\mathcal{D}_n$-module P and any simple weight $\mathfrak{gl}(n)$-module V, the $\mathcal{W}_n$-module $T(P,V)$ has a unique simple submodule.*

Proof. If V is not a fundamental representation the statement follows from Proposition 2.8(i). Now let $V = \bigwedge^k \mathbb{C}^n$. It is shown in [9] that if P is cuspidal, i.e., $I^+(P) = I^-(P) = \emptyset$, then $T(P, \bigwedge^k \mathbb{C}^n)$ is simple for $k = 0, n$ and an indecomposable $\mathfrak{sl}(n+1)$-module of length two for $k = 1, \dots, n-1$. This implies the statement for a cuspidal module P. For a general module P, consider

$$\gamma = s \sum_{i \in I^-(P)} \varepsilon_i - \sum_{j \in I^+(P)} \varepsilon_j$$

for some irrational $s > 1$. Let $\mathfrak{p}$ be the corresponding parabolic subalgebra of $\mathcal{W}_n$ and $\mathfrak{n}$ be the nilradical of $\mathfrak{p}$. The Levi subalgebra $\mathfrak{g}$ is isomorphic to $\mathfrak{gl}(p) \oplus \mathfrak{gl}(q) \oplus \mathcal{W}_m$ where $p = |I^-(P)|$, $q = |I^+(P)|$, and $m = |I^0(P)|$. Note that

$$P \simeq \mathcal{O}_p^F \otimes \mathcal{O}_q \otimes P_m$$

for some cuspidal $\mathcal{D}_m$-module P_m. Since V is finite dimensional and simple, $V^{\mathfrak{n} \cap \mathfrak{gl}(n)} \simeq V_p \otimes V_q \otimes V_m$ is a simple module over $\mathfrak{gl}(p) \oplus \mathfrak{gl}(q) \oplus \mathfrak{gl}(m)$. It is easy to compute that

$$T(P,V)^{\mathfrak{n}} \simeq V_p \otimes V_q \otimes T(P_m, V_m).$$

Since P_m is cuspidal, $T(P_m, V_m)$ has a unique simple $\mathcal{W}_m$-submodule and hence $T(P,V)^{\mathfrak{n}}$ has a unique simple $\mathfrak{g}$-submodule N. If M is a simple $\mathcal{W}_n$ submodule of $T(P,V)$ then $M^{\mathfrak{n}} \neq 0$ and hence $N \subset M$. That implies the uniqueness of M. $\square$

3.3. **Duality for tensor modules.** Recall that P^F denotes the Fourier transform of a $\mathcal{D}_n$-module P.

Lemma 3.8. *Let P be a simple weight $\mathcal{D}_n$-module. Consider P as a $\mathcal{W}_n$-module via the natural homomorphism $\mathcal{W}_n \to \mathcal{D}_n$. Then $P_* \simeq T(P^F, \Lambda^n \mathbb{C}^n)$.*

Proof. Recall the definition of $I^{\pm}(P), I^0(P)$. As a vector space

$$P = \prod_{i \in I^0(P)} x_i^{\lambda_i} \otimes \mathbb{C}[x_j]_{j \in I^+(P)} \otimes \mathbb{C}[\partial_k]_{k \in I^-(P)} \otimes \mathbb{C}[x_\ell^{\pm 1}]_{\ell \in I^+(P)},$$

where λ_i are nonintegral for all $i \in I^0(P)$. We denote the monomial basis of P by $e(\mu)$ where $\mu_i \in \lambda_i + \mathbb{Z}$ for $i \in I^0(P)$, $\mu_i \in \mathbb{Z}_{\geq 0}$ for $i \in I^+(P) \sqcup I^-(P)$. We have

$$x_i e(\mu) = \begin{cases} e(\mu+\varepsilon_i) \text{ if } i \in I^0(P) \cup I^+(P) \\ -\mu_i e(\mu+\varepsilon_i) \text{ if } i \in I^-(P) \end{cases},$$

$$\partial_i e(\mu) = \begin{cases} \mu_i e(\mu-\varepsilon_i) \text{ if } i \in I^0(P) \cup I^+(P) \\ e(\mu+\varepsilon_i) \text{ if } i \in I^-(P) \end{cases}.$$

Denote the corresponding basis of P^F by $f(\mu)$ where μ runs over the same set as $e(\mu)$. Using identification of P and P^F as vector spaces, we have that if $X(e(\mu)) = ce(\nu)$ then $\sigma_F(X)f(\mu) = cf(\nu)$. This observation allows us to write the action of generators in the basis $f(\mu)$:

$$\partial_i f(\mu) = \begin{cases} -f(\mu+\varepsilon_i) \text{ if } i \in I^0(P) \cup I^+(P), \\ \mu_i f(\mu-\varepsilon_i) \text{ if } i \in I^-(P), \end{cases}$$

$$x_i f(\mu) = \begin{cases} \mu_i f(\mu-\varepsilon_i) \text{ if } i \in I^0(P) \cup I^+(P), \\ f(\mu+\varepsilon_i) \text{ if } i \in I^-(P). \end{cases}$$

Let $\varphi(\mu)$ be a function satisfying

$$\varphi(\mu+\varepsilon_i) = \begin{cases} (\mu_i+1)\varphi(\mu) \text{if } i \in I^0(P) \cup I^+(P), \\ -(\mu_i+1)\varphi(\mu) \text{ if } i \in I^-(P). \end{cases}$$

Define a pairing $P \times P^F \to \mathbb{C}$ by setting $\langle e(\mu), f(\nu)\rangle = \varphi(\mu)\delta_{\mu,\nu}$. Then we have

$$\langle \partial_i e(\mu), f(\nu)\rangle = -\langle e(\mu), \partial_i f(\nu)\rangle, \quad \langle x_i e(\mu), f(\nu)\rangle = \langle e(\mu), x_i f(\nu)\rangle.$$

Hence

$$\langle g(x)\partial_i e(\mu), f(\nu)\rangle = -\langle e(\mu), \partial_i g(x) f(\nu)\rangle.$$

Using that $\partial_i g(x) = g(x)\partial_i + \partial_i(g(x))$ and choosing nonzero $\omega \in \bigwedge^n \mathbb{C}^n$, we obtain

$$g(x)\partial_i(f(\nu)\otimes\omega) = (\partial_i g(x) f(\nu))\otimes\omega.$$

This leads to a nondegenerate $\mathcal{W}_n$-invariant pairing $P \times T(P^F, \bigwedge^n \mathbb{C}^n) \to \mathbb{C}$. □

Lemma 3.9. *Let V and P be such that $T(P,V)$ has finite weight multiplicities, and let V_* be the restricted dual of V. Then $T(P^F, V_* \otimes \bigwedge^n \mathbb{C}^n)$ and $T(P,V)$ are restricted dual to each other in the category of weight $\mathcal{W}_n$-modules.*

Proof. We define a pairing

$$T(P^F, V_* \otimes \bigwedge^n \mathbb{C}^n) \times T(P,V) \to \mathbb{C}$$

by the formula

$$\langle f\otimes v, g\otimes w\rangle = \langle f,g\rangle\langle v,w\rangle, \quad v \in V, w \in V_*, f \in P, g \in T(P^F, \bigwedge^n \mathbb{C}^n).$$

Then we have

$$\langle x^\alpha\partial_j(f)\otimes v+\sum_i \partial_i(x^\alpha)f\otimes E_{ij}v, g\otimes w\rangle + \langle f\otimes v, x^\alpha\partial_j(g)\otimes w+\sum_i \partial_i(x^\alpha)\otimes E_{ij}w\rangle =$$

$$\langle x^\alpha\partial_j(f), g\rangle\langle v,w\rangle + \langle f, x^\alpha\partial_j(g)\rangle\langle v,w\rangle +$$
$$\sum_j \langle \partial_j(x^\alpha)f, g\rangle\langle E_{ij}v, w\rangle + \langle f, \partial_j(x^\alpha)g\rangle\langle v, E_{ij}w\rangle = 0,$$

because of

$$\langle x^\alpha\partial_j(f), g\rangle + \langle f, x^\alpha\partial_j(f)\rangle = 0,$$
$$\langle \partial_j(x^\alpha)f, g\rangle = \langle f, \partial_j(x^\alpha)g\rangle$$

and

$$\langle E_{ij}v, w\rangle + \langle v, E_{ij}w\rangle = 0.$$

□

3.4. **Statement of Main Result.** In this subsection we state and prove the main result in the paper. Some of the results used in the proof will be established in the next three sections.

Theorem 3.10. *Let M be a simple weight $\mathcal{W}_n$-module with finite weight multiplicities. Then M is the unique submodule of some tensor module $T(P,V)$ with finite weight multiplicities. More precisely, exactly one of the following holds:*

- (i) *M is isomorphic to $T(P,V)$ for a simple weight $\mathcal{D}_n$-module P and a simple weight $\mathfrak{gl}(n)$-module V with finite weight multiplicities, such that $(\Delta_P^I \sqcup \Delta_P^-) \subset (\Delta_V^F \sqcup \Delta_V^-)$ and such that V is not isomorphic to a fundamental representation.*
- (ii) *M is isomorphic to $dT(P, \bigwedge^k \mathbb{C}^n)$ for some $k = 0, 1, ..., n-1$, and a simple weight $\mathcal{D}_n$-module P.*
- (iii) *$M \simeq \mathbb{C}$, which is the unique simple submodule of $T(\mathcal{O}_n, \bigwedge^0 \mathbb{C}^n)$.*

Proof. Let M be a simple weight $\mathcal{W}_n$-module with finite weight multiplicities. By Theorem 2.7 and Proposition 4.1, M is a quotient of the parabolically induced module $\mathrm{Ind}_{\mathfrak{p}}^{\mathcal{W}_n} N$ where N is a simple bounded $\mathfrak{g}$-module over the Levi subalgebra $\mathfrak{g}$ of $\mathfrak{p}$. Moreover, by Corollary 4.4, N satisfies the additional conditions (5.1) and (5.2) of Section 5. Theorem 5.17 provides a classification of such N. Finally, Lemma 6.2 and Lemma 6.3 ensure that M is one of the modules listed in the statement. □

4. Applications of the parabolic induction

Recall that Δ stands for the set of roots of $\mathcal{W}_n$. We use the setting of §2.7.

In what follows we always assume that M is a simple weight $\mathcal{W}_n$-module that has finite weight multiplicities. We will use that M is the unique simple quotient of a parabolically induced module $U(\mathcal{W}_n)\otimes_{U(\mathfrak{p})} M_0$, as stated in Theorem 2.7. Let $\mathfrak{p}$ be the parabolic subalgebra associated with $\gamma = \sum_{i=1}^{n} a_i\varepsilon_i$. We assume without loss of generality that

$$a_1 \geq \cdots \geq a_p > 0 = a_{p+1} = \cdots = a_{p+m} > a_{p+m+1} \geq \cdots \geq a_n.$$

Henceforth we fix $\mathfrak{p}$ and denote by $\mathfrak{g}$ the Levi subalgebra of $\mathfrak{p}$. Then $\mathfrak{g} \simeq \mathcal{W}_m \ltimes (\mathfrak{k} \otimes \mathcal{O}_m)$ where $\mathfrak{k}$ is a Levi subalgebra in $\mathfrak{gl}(p) \oplus \mathfrak{gl}(n-m-p)$. Under this assumptions we have the following

Proposition 4.1. *The simple $\mathfrak{g}$-module M_0 is bounded.*

Proof. First we prove three preliminary results.

Lemma 4.2. *Let $\alpha = -\varepsilon_i$ or $\alpha \in \Delta(\mathfrak{k})$. Then*

(1) $\dim \mathfrak{g}_\alpha = 1$ *and any nonzero $X_\alpha \in \mathfrak{g}_\alpha$ can be included in the $\mathfrak{sl}_2$ triple;*
(2) *Either $\mathfrak{g}_\alpha$ acts locally nilpotently on M_0 or $\mathfrak{g}_\alpha : M_0 \to M_0$ is injective.*

Proof. The first assertion is obvious. The second follows from the fact that $\operatorname{ad} \mathfrak{g}_\alpha$ is locally nilpotent in $\mathfrak{g}$. □

Lemma 4.3. *Let $\alpha = -\varepsilon_i$ for $p < i \leq p+m$ or $\alpha \in \Delta(\mathfrak{k})$. Then $\mathfrak{g}_\alpha$ acts injectively on M_0.*

Proof. Suppose that $\mathfrak{g}_\alpha$ is locally nilpotent on M_0. Let h be the Cartan element in the $\mathfrak{sl}_2$-triple containing $X_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$. In particular, $\alpha(h) = 2$. Let $\mu \in \operatorname{supp} M$. Then $\mu + \mathbb{Z}\alpha \subset \operatorname{supp} M$. Furthermore, for any $n > 0$ there exist $k \geq n$ and $v \in M^{\mu+k\alpha}$ such that $\mathfrak{g}_\alpha v = 0$. Let M_k denote the $\mathfrak{sl}(2)$-submodule of M generated by v. For all sufficiently large k we have $\mu \in \operatorname{supp} M_k$. Therefore $\dim M^\mu = \infty$. A contradiction. □

Corollary 4.4. *Let $\alpha = -\varepsilon_i$ or $\alpha \in \Delta(\mathfrak{k})$. For any $\lambda \in \operatorname{supp} M$ and $X \in \mathfrak{g}_\alpha \setminus 0$ the map $X : M^\lambda \to M^{\lambda+\alpha}$ is an isomorphism.*

Proof. From the previous lemma we know that $X : M^\lambda \to M^{\lambda+\alpha}$ is injective. Applying the same lemma to M_* we obtain $X : M_*^{-\lambda-\alpha} \to M_*^{-\lambda}$ is injective. Hence $X : M^\lambda \to M^{\lambda+\alpha}$ is surjective. □

We are now ready to complete the proof of Proposition 4.1. Corollary 4.4 implies $\dim M^\mu = \dim M^{\mu+\gamma}$ for any $\gamma \in \mathbb{Z}\Delta(\mathcal{W}_m) + \mathbb{Z}\Delta(\mathfrak{k})$ and $\mu \in \operatorname{supp} M$. The statement follows. □

5. Bounded simple $\mathfrak{g}$-modules

5.1. **Generalization of tensor modules for the Levi subalgebra $\mathfrak{g}$ of $\mathfrak{p}$.** We retain the notation of the previous section. In this section we assume that $m > 0$. Recall that $\mathfrak{g} = \mathcal{W}_m \ltimes (\mathfrak{k} \otimes \mathcal{O}_m)$. Without loss of generality we may assume $\mathcal{O}_m = \mathbb{C}[x_1, \ldots, x_m]$. In this section we will classify simple bounded $\mathfrak{g}$-modules N satisfying the additional properties:

$$\operatorname{supp} N = \lambda + \mathbb{Z}\Delta(\mathfrak{g}) \quad \text{for any } \lambda \in \operatorname{supp} N. \tag{5.1}$$

(5.2) All weight spaces of N have the same dimension d.

First, we generalize the notion of a $(\mathcal{W}_m, \mathcal{O}_m)$-module to that of a $(\mathfrak{g}, \mathcal{O}_m)$-module.

Definition 5.1. A $\mathfrak{g}$-module N is a $(\mathfrak{g}, \mathcal{O}_m)$-module if N is a $\mathcal{O}_m$-module satisfying

$$X(fv) = fX(v) + X(f)v \ \forall v \in N, f \in \mathcal{O}_m, X \in \mathcal{W}_m, \tag{5.3}$$

$$(h \otimes Y)(fv) = (hf)Yv \ \forall v \in N, \ f, h \in \mathcal{O}_m, Y \in \mathfrak{k}. \tag{5.4}$$

From now all $(\mathcal{W}_m, \mathcal{O}_m)$-modules and all $(\mathfrak{g}, \mathcal{O}_m)$-modules we consider are weight modules.

Lemma 5.2. *Let N be a $\mathfrak{g}$-module satisfying (5.1) and (5.2).*

(a) If $\alpha = -\varepsilon_i$ or $\alpha \in \Delta(\mathfrak{k})$ then $\mathfrak{g}_\alpha$ acts injectively on N.

(b) If in addition N is equipped with $\mathcal{O}_m$-module structure satisfying (5.3) then x_i acts injectively on N for all $i = 1, \dots, m$.

Proof. The proof of (a) is exactly the same as the proof of Lemma 4.3. To prove (b) we consider the subalgebra $\mathcal{W}_1$ generated by $x_i^n \partial_i$ for all n. Then condition (5.3) implies that N is a $(\mathcal{W}_1, \mathcal{O}_1)$-module and the statement follows from [13]. □

Remark 5.3. Consider the associative algebra $\mathcal{A}(m)$ generated by $\mathcal{W}_m \otimes 1$ and $1 \otimes \mathcal{O}_m$ with relations

$$(x \otimes 1)(y \otimes 1) - (y \otimes 1)(x \otimes 1) = [x, y] \otimes 1,$$
$$(1 \otimes f)(1 \otimes g) = 1 \otimes fg,$$
$$(x \otimes 1)(1 \otimes f) - (1 \otimes f)(x \otimes 1) = 1 \otimes x(f)$$

for $x, y \in \mathcal{W}_m$ and $f, g \in \mathcal{O}_m$. Any $(\mathcal{W}_m, \mathcal{O}_m)$ is an $\mathcal{A}(m)$-module, and conversely, any $\mathcal{A}(m)$-module is a $(\mathcal{W}_m, \mathcal{O}_m)$-module. Furthermore, $\mathcal{A}(m)$ is isomorphic to $U(\mathcal{W}_m) \otimes \mathcal{O}_m$ as a vector space by the correspondence $(X \otimes 1)(1 \otimes f) \mapsto X \otimes f$ for all $X \in U(\mathcal{W}_m)$ and $f \in \mathcal{O}_m$. Let $\mathcal{B} := \mathcal{A}(m) \otimes U(\mathfrak{k})$. Then any $(\mathfrak{g}, \mathcal{O}_m)$-module is a $\mathcal{B}$-module.

Example 5.4. Let S be a $\mathfrak{k}$-module. We define a $(\mathfrak{g}, \mathcal{O}_m)$-module structure on the vector space $\mathcal{O}_m \otimes S$ by setting

$$f(h \otimes s) = fh \otimes s, \quad (f \otimes Y)(h \otimes s) = fh \otimes Ys, \quad X(h \otimes s) = X(h) \otimes s$$

for all $f, h \in \mathcal{O}_m$, $Y \in \mathfrak{k}$, $X \in \mathcal{W}_m$ and $s \in S$. One can easily verify that $\tilde{S} := \mathcal{O}_m \otimes S$ is a $(\mathfrak{g}, \mathcal{O}_m)$-module. Moreover, if R is a $(\mathcal{W}_m, \mathcal{O}_m)$-module then $\mathcal{F}(R, S) := R \otimes_{\mathcal{O}_m} \tilde{S}$ is a $(\mathfrak{g}, \mathcal{O}_m)$-module.

Remark 5.5. A simple weight $(\mathcal{W}_m, \mathcal{O}_m)$-module R with finite weight multiplicities is a tensor module $T(P, V)$ for some simple weight $\mathcal{D}_m$-module P and some simple weight $\mathfrak{gl}(m)$-module V, see Theorem 3.7 in [19].

Lemma 5.6. *If R is a simple $(\mathcal{W}_m, \mathcal{O}_m)$-module and S is a simple weight $\mathfrak{k}$-module then $\mathcal{F}(R, S)$ is a simple $(\mathfrak{g}, \mathcal{O}_m)$-module, in the sense that it does not contain proper nontrivial $(\mathfrak{g}, \mathcal{O}_m)$-submodules.*

Proof. Observe that $\mathcal{F}(R,S)$ is isomorphic to $R\otimes S$ as a $\mathcal{B}$-module. Hence it is a simple $\mathcal{B}$-module. This implies the statement. □

Lemma 5.7. *If $N=\mathcal{F}(R,S)$ satisfies conditions (5.1) and (5.2), then S is a simple cuspidal $\mathfrak{k}$-module, and $R=T(P,V)$ for some simple cuspidal $\mathcal{D}_m$-module P a simple finite-dimensional $\mathfrak{gl}(m)$-module V. For any $\lambda\in\operatorname{supp}N$ we have that $\dim N^\lambda=(\dim V)d(S)$, where $d(S)$ is the degree of the cuspidal module S.*

Proof. The lemma follows from the isomorphism of $\mathfrak{h}$-modules $\mathcal{F}(R,S)\simeq R\otimes S$. □

Lemma 5.8. *Let S be a simple nontrivial weight $\mathfrak{k}$-module, P be a simple weight $\mathcal{D}_m$-module, and V be a simple finite-dimensional $\mathfrak{gl}(m)$-module. Then $\mathcal{F}(T(P,V),S)$ is a simple $\mathfrak{g}$-module.*

Proof. Choose a regular $u\in\mathfrak{k}\cap\mathfrak{h}$ that acts nontrivially on S, and denote by $\mathcal{F}(T(P,V),S)^a$ the eigenspace of u with eigenvalue a. Let M be a proper nonzero submodule of $\mathcal{F}(T(P,V),S)$. Then $M^a=M\cap\mathcal{F}(T(P,V),S)^a$ is $\mathcal{O}_m$-invariant for any $a\neq 0$. Using the action of the root elements of $\mathfrak{k}$, we obtain that M^a is $\mathcal{O}_m$-invariant for $a=0$ as well. Hence M is a $(\mathfrak{g},\mathcal{O}_m)$-submodule of $\mathcal{F}(T(P,V),S)$ and we reach a contradiction. □

Lemma 5.9. *Let N be a simple $(\mathfrak{g},\mathcal{O}_m)$-module satisfying (5.1) and (5.2). Then N is isomorphic to $\mathcal{F}(R,S)$ for some $(\mathcal{W}_m,\mathcal{O}_m)$-module R and some simple cuspidal $\mathfrak{k}$-module S.*

Proof. Recall the definition of $\mathcal{B}$ from Remark 5.3. Consider N as a $\mathcal{B}$-module. By definition, for any vector $v\in N$ we have $\mathcal{B}v=U(\mathfrak{g})v$ (this follows from the relation $(f\otimes Y)v=f(Yv)$). Hence, N is a simple $\mathcal{B}$-module. For a simple $\mathfrak{k}$-module S', the subspace $\operatorname{Hom}_{\mathfrak{k}}(S',N)\otimes S'$ of N is $\mathcal{B}$-stable. Hence, there is a unique up to isomorphism S', such that $\operatorname{Hom}_{\mathfrak{k}}(S',N)\neq 0$. The existence of such S' follows from condition on the support of N. Let S be such module. We have that $R=\operatorname{Hom}_{\mathfrak{k}}(S,N)$ is a simple $\mathcal{A}(m)$-module. Therefore, $N\simeq R\otimes S$ as a $\mathcal{B}$-module. The condition (5.4) ensures that $N\simeq\mathcal{F}(R,S)$. □

Recall that $\tilde{V}$ stands for $(\mathcal{W}_m,\mathcal{O}_m)$-module $\mathcal{O}_m\otimes V$.

Lemma 5.10. *Let N be a simple weight $\mathfrak{g}$-module, such that ∂_i acts locally nilpotently for all $i=1,\dots,m$. Then N is isomorphic to a simple submodule of $\mathcal{F}(\tilde{V},S)$ for some simple $\mathfrak{k}$-module S and a simple $\mathfrak{gl}(m)$-module V.*

Proof. Let N_0 be the space of invariants of $\partial_1,\dots,\partial_m$. If $\mathfrak{q}$ is the parabolic subalgebra associated to $\gamma=-(\varepsilon_1+\cdots+\varepsilon_m)$, then N is the unique simple quotient of $U(\mathfrak{g})\otimes_{U(\mathfrak{q})}N_0$. Thus, N_0 is a simple $\mathfrak{gl}(m)\oplus\mathfrak{k}$-module, so $N_0=V\otimes S$ for some simple modules V and S. Then we have a natural homomorphism $\varphi:N_0\to\mathcal{F}(\tilde{V},S)$ of $\mathfrak{gl}(m)\oplus\mathfrak{k}$-modules, hence, also of $\mathfrak{q}$-modules. The homomorphism φ induces a homomorphism $\Phi:U(\mathfrak{g})\otimes_{U(\mathfrak{q})}N_0\to\mathcal{F}(\tilde{V},S)$ of $\mathfrak{g}$-modules. The image of Φ is isomorphic to N. □

Lemma 5.11. *Let N be a simple $\mathfrak{g}$-module satisfying (5.1) and (5.2). Assume that one can define an $\mathcal{O}_m$-module structure on N in such a way that it satisfies (5.3) and*

$$f(g\otimes Y)v=(g\otimes Y)fv,\ \forall v\in N, f,g\in\mathcal{O}_m, Y\in\mathfrak{k}.$$

Then N is a $(\mathfrak{g},\mathcal{O}_m)$-module.

Proof. We need to verify (5.4). First, we claim that verifying (5.4) is equivalent to checking

$$x_1(1\otimes Y)=(x_1\otimes Y),\ \forall Y\in\mathfrak{k}. \tag{5.5}$$

Indeed, let $f\in\mathcal{O}_m$ and $X=f\partial_1$. Then

$$[X,x_1](1\otimes Y)=f(1\otimes Y)=[X,x_1\otimes Y]=f\otimes Y.$$

For $f,g\in\mathcal{O}_m$ we have

$$f(g\otimes Y)=fg(1\otimes Y).$$

By Lemma 5.2, ∂_i and x_i act injectively on N. Therefore we can localize N with respect to x_i and define a $(\mathcal{W}_m,\tilde{\mathcal{O}}_m)$-module structure on N, where

$$\tilde{\mathcal{O}}_m=\mathbb{C}[x_1^{\pm1},x_2^{\pm1},\dots,x_m^{\pm1}].$$

Consider the twisted localization $D^{\mathbf{c}}_{\langle x_1,\dots,x_n\rangle}N$ of N with $\mathcal{U}=\mathcal{B}$, and some $\mathbf{c}\in\mathbb{C}^m$.

Next we observe that that there is $\mathbf{c}$ such that $D^{\mathbf{c}}_{\langle x_1,\dots,x_n\rangle}N$ has a nonzero weight vector annihilated by all ∂_i. Indeed, it is enough to find such $\mathbf{c}$ in the case $n=1$. In this case we first fix an eigenvalue α of the endomorphism $(x_1)(\partial_1)|_{N^\lambda}$ where λ is any weight of N.[2] Then if v is nonzero vector such that $(x_1)(\partial_1)v=\alpha v$, $(\partial_1)(x_1^{-\alpha})v=0$.

Among all vectors in $D^{\mathbf{c}}_{\langle x_1,\dots,x_n\rangle}N$ annihilated by all ∂_i choose a vector u of weight μ with maximal possible $|\mu|_r:=\operatorname{Re}\sum_{i=1}^m\mu_i$. Let N' be a $\mathfrak{g}$-submodule of $D^{\mathbf{c}}_{\langle x_1,\dots,x_n\rangle}N$ generated by u. Note that for any $\nu\in\operatorname{supp}N'$ we have $|\mu|_r\geq|\nu|_r$. Let $v\in N'$ have weight ν with $|\nu|_r=|\mu|_r$. Then $\partial_iv=0$ and for any $Y\in\mathfrak{k}$

$$\partial_i(x_1(1\otimes Y)-x_1\otimes Y)v=0.$$

Hence $u'=(x_1(1\otimes Y)-x_1\otimes Y)v$ is annihilated by all ∂_i. On the other hand, the weight η of u' satisfies $|\eta|_r=|\mu|_r+1$ hence $u'=0$. Let $w\in N'$ be a weight vector of weight λ with minimal $|\lambda|_r$ such that for some $Y\in\mathfrak{k}$

$$(x_1(1\otimes Y)-x_1\otimes Y)w\neq0.$$

We have

$$\partial_i(x_1(1\otimes Y)-x_1\otimes Y)w=(x_1(1\otimes Y)-x_1\otimes Y)\partial_iw=0,$$

which leads to a contradiction. Next we note that $D^{\mathbf{c}}_{\langle x_1,\dots,x_n\rangle}N=\tilde{\mathcal{O}}_m\cdot N'$. Since $x_1(1\otimes Y)-x_1\otimes Y$ commutes with $\tilde{\mathcal{O}}_m$ we have $(x_1(1\otimes Y)-x_1\otimes Y)N^{(\mathbf{c})}=0$. Then $(x_1(1\otimes Y)-x_1\otimes Y)N=0$. This completes the proof. □

[2] Warning: here $x_1\in\mathcal{O}_m$, $\partial_1\in\mathcal{W}_m$ and $(x_1)(\partial_1)$ should not be confused with $x_1\partial_1\in\mathcal{W}_m$.

Lemma 5.12. *Assume that N is a simple $\mathfrak{g}$-module satisfying (5.1) and (5.2). Suppose that there exists a central element $z \in \mathfrak{k}$ which does not act trivially on N. Then N is a $(\mathfrak{g}, \mathcal{O}_m)$-module and hence is isomorphic to a module $\mathcal{F}(R, S)$.*

Proof. Without loss of generality we may assume that z acts as identity on N. Define an $\mathcal{O}_m$-module structure on N by setting $x_i v := (x_i \otimes z)v$. Then N satisfies the assumptions of Lemma 5.11. The statement follows. □

Lemma 5.13. *Let $\mathfrak{k}$ be abelian and N be a simple $\mathfrak{g}$-module satisfying (5.1) and (5.2). If $\mathfrak{k}N = 0$, then $(\mathcal{O}_m \otimes \mathfrak{k})N = 0$.*

Proof. We will show that $(f \otimes h)N = 0$ for any $f \in \mathcal{O}_m,\ h \in \mathfrak{k}$. Note that

$$[\partial_1, x_1 \otimes h]N = hN = 0.$$

Therefore we have the following identites on N:

$$[\partial_1(x_1 \otimes h), \partial_1^2(x_1^2 \otimes h)] = 2\partial_1^2(x_1 \otimes h)^2 = 2(\partial_1(x_1 \otimes h))^2,$$

$$[(\partial_1(x_1 \otimes h))^k, \partial_1^2(x_1^2 \otimes h)] = 2k(\partial_1(x_1 \otimes h))^{k+1}.$$

This implis that, on each weight space N^λ of N, $\mathrm{tr}_{N^\lambda}(\partial_1(x_1 \otimes h))^k = 0$ for all $k > 2$. Since N is bounded this implies nilpotency of $\partial_1(x_1 \otimes h)$ on N. Since ∂_1 is invertible on N we obtain that $x_1 \otimes h$ is nilpotent on N. Let p be the nilpotency degree of $x_1 \otimes h$. There exists $v \in N$ such that $w := (x_1 \otimes h)^{p-1}v \neq 0$. Then for $f \in \mathcal{O}_m$ we have

$$0 = f\partial_1(x_1 \otimes h)^p v = p(f \otimes h)(x_1 \otimes h)^{p-1}v.$$

In other words, w is annihilated by $\mathcal{O}_m \otimes h$. The subspace N' of all vectors annihilated by $\mathcal{O}_m \otimes h$ is $\mathfrak{g}$-invariant, but we just proved that $N' \neq 0$. By the irreducibility of N, we have $N = N'$. Thus $(\mathcal{O}_m \otimes h)N = 0$. □

Proposition 5.14. *Let N be a $\mathfrak{g}$-module satisfying (5.1) and (5.2). Then for any Borel subalgebra $\mathfrak{b} \subset \mathfrak{k}$ there exists a simple bounded $\mathfrak{g}$-module $\bar{N}$ satisfying the following two conditions:*

(i) *There exists a weight $\lambda \in \mathfrak{h}^*$ such that* $\operatorname{supp} \bar{N} \subset \lambda + \sum_{i=1}^m \mathbb{Z}\varepsilon_i - \mathbb{Z}_{\geq 0}\Delta(\mathfrak{b})$ *and $\lambda(\mathfrak{h} \cap \mathfrak{k}) \neq 0$.*

(ii) *The module N is obtained from $\bar{N}$ by a twisted localization with respect to some set of commuting roots $\Gamma \subset -\Delta(\mathfrak{b})$.*

Proof. Let $\hat{\mathfrak{k}} = \mathfrak{k} + \mathfrak{h}$, $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. Then N is a bounded weight $\hat{\mathfrak{k}}$-module, hence, any cyclic $\hat{\mathfrak{k}}$-submodule of N has finite length (see Lemma 3.3 in [14]). Let N_0 be a simple $\hat{\mathfrak{k}}$-submodule of N. Note that N_0 is a cuspidal $\hat{\mathfrak{k}}$-module. By Proposition 4.8 in [14], there exist $\mu \in (\mathfrak{h} \cap \mathfrak{k})^*$ and $\Gamma \subset -\Delta(\mathfrak{b})$ such that $N_0 \simeq D_\Gamma^\mu M_0$ for some simple bounded $\mathfrak{b}$-highest weight $\hat{\mathfrak{k}}$-module M_0. Since D_Γ^μ is well defined for $\mathfrak{g}$-modules and commutes with the restriction functor $\mathrm{Res}^{\mathfrak{g}}_{\hat{\mathfrak{k}}}$, $M := D_\Gamma^{-\mu} N$ contains an $\mathfrak{n}$-primitive weight vector $v \in M_0$, while ∂_i act injectively on M for all $i = 1, \dots, m$. Since

$U(\hat{\mathfrak{k}})v$ is bounded it has finite $\hat{\mathfrak{k}}$-length and hence there is $\lambda' \in \operatorname{supp} U(\hat{\mathfrak{k}})v$ such that $\lambda' + \alpha \notin \operatorname{supp} U(\hat{\mathfrak{k}})v$ for all $\alpha \in \Delta(\mathfrak{b})$. The injectivity of the action of ∂_i implies that

$$(\lambda' + \alpha + \sum_{i=1}^{m} \mathbb{Z}_{\geq 0}\varepsilon_i) \cap \operatorname{supp} U(\mathfrak{g})v = \emptyset.$$

If w is a nonzero vector of weight λ' then

$$\operatorname{supp} U(\mathfrak{g})w \subset \lambda' + \sum_{i=1}^{m} \mathbb{Z}\varepsilon_i - \mathbb{Z}_{\geq 0}\Delta(\mathfrak{b}).$$

This implies $\dim(U(\mathcal{O}_m \otimes \mathfrak{n}))u < \infty$ for any $u \in U(\mathfrak{g})w$. By Lemma 2.3, the boundedness of $U(\mathfrak{g})w$ implies that $U(\mathfrak{g})w$ has finite length. Let $\bar{N}$ be a simple submodule of $U(\mathfrak{g})w$. Then there is a nonzero weight vector $u \in \bar{N}$ annihilated by $\mathcal{O}_m \otimes \mathfrak{n}$. Then $\bar{N}$ satisfies (i) with λ being the weight of u, while (ii) follows from the simplicity of N.

It remains to show that $\lambda(\mathfrak{h} \cap \mathfrak{k}) \neq 0$. For the sake of contradiction, assume that the opposite holds. Take a simple root $\alpha \in \Delta(\mathfrak{b})$. Then a simple computation shows that $\mathfrak{g}_{-\alpha}u$ is annihilated by $\mathcal{O}_m \otimes \mathfrak{n}$. The simplicity of $\bar{N}$ hence implies that $\mathfrak{g}_{-\alpha}u = 0$ for all simple roots α and thus M contains a trivial $\mathfrak{k}$-submodule. But the roots of Γ act injectively on N and hence on M. This leads to a contradiction. □

For a weight $\mu \in (\mathfrak{h} \cap \mathfrak{k})^*$ and a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{k}$, by $L_{\mathfrak{b}}(\mu)$ (or simply by $L(\mu)$) we denote the simple $\mathfrak{b}$-highest weight $\mathfrak{k}$-module of highest weight μ.

Lemma 5.15. *The module $\bar{N}$ constructed in Proposition 5.14 is isomorphic to $\mathcal{F}(T(P,V), L(\bar{\lambda}))$ for a cuspidal simple $\mathcal{D}_m$-module P, a simple finite-dimensional $\mathfrak{gl}(m)$-module V, and a simple highest weight $\mathfrak{k}$-module $L(\bar{\lambda})$, where $\bar{\lambda}$ is the restriction of λ to $\mathfrak{h} \cap \mathfrak{k}$.*

Proof. Consider $\gamma \in (\mathfrak{k} \cap \mathfrak{h})^*$ which determines the Borel subalgebra $\mathfrak{b}$. Then γ determines also a parabolic subalgebra $\mathfrak{q}$ in $\mathfrak{g}$. The $\mathfrak{q}$-top of the module $\bar{N}$ is a simple $(\mathcal{W}_m \oplus \mathfrak{h})$-module. Since $\bar{\lambda} \neq 0$, this module is isomorphic to $T(P,V) \otimes \mathbb{C}_{\bar{\lambda}}$ by Lemma 5.12. A simple computation shows that it is also isomorphic to the top of $\mathcal{F}(T(P,V), L(\bar{\lambda}))$. Hence the statement follows from Proposition 2.5(c). □

Corollary 5.16. *If $\mathfrak{k}$ is not abelian then N is isomorphic to $\mathcal{F}(T(P,V), S)$ for some cuspidal simple $\mathcal{D}_m$-module P, a simple finite-dimensional $\mathfrak{gl}(m)$-module V, and a simple cuspidal $\mathfrak{k}$-module S.*

Proof. The result follows immediately from Proposition 5.14, Lemma 5.15, and the isomorphism of $\mathfrak{g}$-modules

$$D_{\Gamma}^{-\mu}\mathcal{F}(T(P,V), L(\bar{\lambda})) \simeq \mathcal{F}(T(P,V), D_{\Gamma}^{-\mu}L(\bar{\lambda})).$$

□

Theorem 5.17. *Let N be a simple bounded $\mathfrak{g}$-module satisfying (5.1) and (5.2). Then we have one of the following two mutually exclusive statements.*

(a) $\mathcal{O}_m \otimes \mathfrak{k}$ acts trivially on N and N is a unique simple submodule of $T(P, V)$ for some simple cuspidal $\mathcal{D}_m$-module P and a simple finite-dimensional $\mathfrak{gl}(m)$-module V. In this case $\mathfrak{k}$ must be abelian.

(b) N is isomorphic to $\mathcal{F}(T(P, V), S)$ for some cuspidal simple $\mathcal{D}_m$-module P, a simple finite-dimensional $\mathfrak{gl}(m)$-module V and a simple nontrivial cuspidal $\mathfrak{k}$-module S.

Proof. If $\mathfrak{k}$ is not abelian the statement follows from Corollary 5.16. If $\mathfrak{k}$ is abelian and $\mathfrak{k}$ acts nontrivially on N, the statement follows from Lemma 5.12. If $\mathfrak{k}$ acts trivially on N, then by Lemma 5.13, $(\mathcal{O}_m \otimes \mathfrak{k})N = 0$. Then N is a simple bounded $\mathcal{W}_m$-module and the statement is a consequence of Theorem 1.1 in [19]. □

6. Back to tensor modules via parabolic induction

6.1. **The case of infinite-dimensional $\mathfrak{g}$.** We retain the notation of Section 4 and assume again that M is a simple weight $\mathcal{W}_n$-module that is also the unique simple quotient of the parabolically induced module $U(\mathcal{W}_n) \otimes_{U(\mathfrak{p})} N$, where N is a simple bounded $\mathfrak{g}$-module satisfying (5.1) and (5.2). We will use the properties of N listed in Theorem 5.17.

Recall that p and m are fixed and defined in §4. Let $\mathfrak{p}' = \mathfrak{p} \cap \mathfrak{gl}(n)$. The Levi subalgebra of $\mathfrak{p}'$ is isomorphic to $\mathfrak{k} \oplus \mathfrak{gl}(m)$. Consider a $\mathfrak{g}$-module $\mathcal{F}(T(P, V), S)$ where V is a finite-dimensional $\mathfrak{gl}(m)$-module, P is a simple cuspidal $\mathcal{D}_m$-module and S be a simple cuspidal $\mathfrak{k}$-module. Note that S might be a trivial $\mathfrak{k}$-module in the case when $\mathfrak{k}$ is abelian. Let U be the one-dimensional $\mathfrak{k}$-module of weight $\sum_{i=1}^{p} \varepsilon_i$ and $S^U = S \otimes U$. Finally, let $\hat{S}$ be the unique simple quotient of $U(\mathfrak{gl}(n)) \otimes_{U(\mathfrak{p}')} (S^U \otimes V)$. Using the isomorphism

$$\mathcal{D}_n \simeq \mathcal{D}_p \otimes \mathcal{D}_m \otimes \mathcal{D}_{n-p-m},$$

define a $\mathcal{D}_n$-module $\tilde{P}$ by

$$\tilde{P} = \mathbb{C}[x_1, \dots, x_p]^F \otimes P \otimes \mathbb{C}[x_{p+m+1}, \dots, x_n],$$

(recall that X^F is the full Fourier transform of X).

Lemma 6.1. *The $\mathfrak{p}$-top of $T(\tilde{P}, \hat{S})$ is isomorphic to $\mathcal{F}(T(P, V), S)$.*

Proof. The statement follows by comparing the supports of the two modules. Let $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{n}$.

$$\operatorname{supp} \tilde{P} = \sum_{i=1}^{p} \mathbb{Z}_{<0}\varepsilon_i + \operatorname{supp} P + \sum_{i=p+m+1}^{n} \mathbb{Z}_{\geq 0}\varepsilon_i,$$

$$\operatorname{supp} \hat{S} \subset \operatorname{supp} S + \sum_{i=1}^{p} \varepsilon_i + \operatorname{supp} V - \operatorname{supp} U(\mathfrak{n}'),$$

where $\mathfrak{n}' = \mathfrak{n} \cap \mathfrak{gl}(n)$. Then we have that

$$\operatorname{supp} \mathcal{F}(T(P,V),S) \subset \operatorname{supp} T(\tilde{P},\hat{S}) \subset \operatorname{supp} \mathcal{F}(T(P,V),S) + \operatorname{supp} U(\mathfrak{n}^-),$$

where $\mathfrak{n}^-$ is the nilradical of the opposite parabolic. Moreover, the multiplicity of any $\mu \in \operatorname{supp} \mathcal{F}(T(P,V),S)$ is the same as its multiplicity in $T(\tilde{P},\hat{S})$. □

Lemma 6.2. *Let $N = \mathcal{F}(T(P,V),S)$ be a simple $\mathfrak{g}$-module. Then the unique simple quotient M of $U(\mathcal{W}_n) \otimes_{U(\mathfrak{p})} N$ is isomorphic to unique simple submodule of $T(\tilde{P},\hat{S})$.*

Proof. The isomorphism of $\mathfrak{p}$-modules $N \to T(\tilde{P},\hat{V})^{\mathrm{top}}$ induces a nonzero homomorphism of $\mathcal{W}_m$-modules $U(\mathcal{W}_n) \otimes_{U(\mathfrak{p})} N \to T(\tilde{P},\hat{V})$. The image of this homomorphism is simple since $T(\tilde{P},\hat{V})$ has a unique simple submodule. Thus, this submodule is isomorphic to M. □

Now assume that $\mathcal{F}(T(P,V),S)$ is not simple. This is only possible if $\mathfrak{k}$ is abelian, S is trivial, and $V = \bigwedge^k \mathbb{C}^m$.

Lemma 6.3. *Assume that N is the simple submodule $T(P, \bigwedge^k \mathbb{C}^m)$ for some $k = 0, \dots, m-1$. Then the unique simple quotient M of $U(\mathcal{W}_n) \otimes_{U(\mathfrak{p})} N$ is isomorphic to the unique simple submodule of $T(\tilde{P}, \bigwedge^{p+k} \mathbb{C}^n)$.*

Proof. We consider the monomorphism of $\mathfrak{p}$-modules $N \to T(\tilde{P}, \bigwedge^{p+k} \mathbb{C}^n)^{\mathrm{top}}$, and the induced map

$$U(\mathcal{W}_n) \otimes_{U(\mathfrak{p})} N \to T(\tilde{P}, \bigwedge^{p+k} \mathbb{C}^n).$$

To complete the proof, we use the same reasoning as the one in the proof of the previous lemma. □

6.2. **The case of finite-dimensional $\mathfrak{g}$.** In this case we have $m = 0$ and $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(n)$. Using arguments similar to the ones used in the previous subsection, one can show that the unique simple quotient of $U(\mathcal{W}_n) \otimes_{U(\mathfrak{p})} S$ is isomorphic to the unique simple submodule of $T(\tilde{P},\hat{S})$.

6.3. **The case $\mathfrak{g} = \mathcal{W}_m$.** This case follows from Theorem 2.10.

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