Journal of Fourier Analysis and Applications (2022) 28:77
https://doi.org/10.1007/s00041-022-09971-1

Paley's Inequality for Discrete Groups

Chian Yeong Chuah[1] · **Yazhou Han**[2] · **Zhen-Chuan Liu**[3] · **Tao Mei**[3]

Received: 19 October 2021 / Revised: 15 July 2022 / Accepted: 7 September 2022 /
Published online: 26 September 2022

Abstract

This article studies Paley's theory of lacunary Fourier series for von Neumann algebra of discrete groups. The results unify and generalize the work of Rudin (Fourier Analysis on Groups, Reprint of the 1962 original. Wiley Classics Library, A Wiley-Interscience Publication, Wiley, New York, 1990, Section 8) for abelian discrete ordered groups and the work of Lust-Piquard and Pisier (Ark Mat 29(2):241–260, 1991) for operator valued functions, and provide new examples of Paley sets and $\Lambda(p)$ sets on free groups.

Keywords Paley's inequality · Semigroup of operators · Group von Neumann algebra · Noncommutative L^p spaces · Free group

Mathematics Subject Classification Primary: 46L52, 46L54 · Secondary: 42A55

Communicated by Oscar Blasco.

✉ Yazhou Han
hanyazhou@foxmail.com

Chian Yeong Chuah
chuah.21@osu.edu

Zhen-Chuan Liu
Zhen-chuan_Liu1@baylor.edu

Tao Mei
tao_mei@baylor.edu

1 Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, OH 43210-1174, USA

2 College of Mathematics, Taiyuan University of Technology, Taiyuan 030024, China

3 Department of Mathematics, Baylor University, 1301 S University Parks Dr, Waco, TX 76798, USA

Birkhäuser

Introduction

Denote by $\mathbb{T}$ the unit circle. Consider a sequence $(j_k)_{k\in\mathbb{N}}$ of elements of $\mathbb{Z}$ which is *lacunary à la Hadamard*, i.e. there exists $\delta > 0$ such that for all $k \in \mathbb{N}$,

$$\frac{|j_{k+1}|}{|j_k|} > 1 + \delta.$$

A classical Khintchine-Pisier type inequality states that there exists $C_\delta < \infty$ such that

$$\left\| \sum_{k=1}^{\infty} c_k z^{j_k} \right\|_{L^1(\mathbb{T})} \leq \left(\sum_{k=1}^{\infty} |c_k|^2 \right)^{\frac{1}{2}} \leq C_\delta \left\| \sum_{k=1}^{\infty} c_k z^{j_k} \right\|_{L^1(\mathbb{T})}.$$

This shows that $\ell_2(\mathbb{N})$ embeds into $L^1(\mathbb{T})$. However, the map

$$P : f \mapsto \left(\hat{f}(j_k)\right)_{k\in\mathbb{N}}$$

does not extend to a bounded map from the whole space $L^1(\mathbb{T})$ to ℓ_2. Here $\hat{f}$ denotes the Fourier transform of f. This can be easily seen by looking at the so-called Riesz products,

$$f(z) = \prod_{k=1}^{N}(1 + \frac{z^{2^k} + z^{-2^k}}{2}),$$

which have norm $\|f\|_{L^1(\mathbb{T})} = \hat{f}(0) = 1$ while $(\hat{f}(2^k))_{1\leq k\leq N}$ has norm $\frac{\sqrt{N}}{2}$ since $\hat{f}(2^k) = \frac{1}{2}$ for $k = 1, \cdots, N$. Paley's theory [33] is a variant of Khintchine's inequality. Let $H^1(\mathbb{T})$ be the real Hardy space on the unit circle, that consists of integrable functions such that both their analytic and the anti-analytic parts are integrable. Equivalently,

$$H^1(\mathbb{T}) = \left\{ f \in L^1(\mathbb{T}) : \|f\|_{H^1} = \|f\|_{L^1} + \|H(f)\|_{L^1} < \infty \right\},$$

with H the Hilbert transform of f. Paley's theory says that

$$\left(\sum_k |c_k|^2 \right)^{\frac{1}{2}} \simeq_\delta \inf \left\{ \|f\|_{H^1} : f \in H^1(\mathbb{T}), \hat{f}(j_k) = c_k \right\} \tag{0.1}$$

This shows that the map P is bounded from $H^1(\mathbb{T})$ (or the analytic L^1) to ℓ_2. These Khintichine/Paley type inequalities for lacunary series have important applications to Grothendieck's theory (see [40, Section 5], [26, Appendix]).

A subset $E \subset \mathbb{N}$ is called a Paley set (see [37, Section 3]) if the above equivalence (0.1) holds for all choices of $(c_k)_k \in \ell_2$, $j_k \in E$ with constants depending only on E. Rudin [48, Section 8] proved that E is a Paley set only if

$$\sup_{n \in \mathbb{N}} \# E \cap [2^n, 2^{n+1}] < C$$

which is equivalent to say that E is a finite union of lacunary sequences.

By Fefferman-Stein's H^1-BMO duality theory, (0.1) has an equivalent formulation that, for any $(c_k) \in \ell_2$,

$$\left(\sum_k |c_k|^2 \right)^{\frac{1}{2}} \simeq_\delta \left\| \sum_k c_k z^{j_k} \right\|_{BMO(\mathbb{T})}. \tag{0.2}$$

Here, BMO($\mathbb{T}$) denotes the bounded mean oscillation (semi)norm

$$\|f\|_{BMO(\mathbb{T})} = \sup_I \frac{1}{|I|} \int_I |f - f_I| \, ds \tag{0.3}$$

with the supremum taking over all arcs $I \subseteq \mathbb{T}$.

In the first part of this article, we give an interpretation of Paley's theory in the semigroup language which allows an extension to non-abelian discrete groups. For each $t > 0$, let P_t be the Poisson integral operator that sends $e^{ik\theta}$ to $e^{-|k|t} e^{ik\theta}$. Here is an equivalent characterization of the classical BMO and H^1-norms by P_t's. For $f \in L^1(\mathbb{T})$,

$$\|f\|_{BMO(\mathbb{T})} \simeq \sup_{t>0} \left\| P_t \left[|f - P_t(f)|^2 \right] \right\|_{L^\infty(\mathbb{T})}^{\frac{1}{2}} \tag{0.4}$$

$$\|f\|_{H^1(\mathbb{T})} \simeq \left\| \left(\int_0^\infty \left| \frac{\partial}{\partial t} P_t f \right|^2 t dt \right)^{\frac{1}{2}} \right\|_{L^1(\mathbb{T})}. \tag{0.5}$$

Consider a discrete group G and a conditionally negative definite length ψ on G. By that, we mean ψ is a $\mathbb{R}_+$-valued function on G satisfying $\psi(g) = 0$ if and only if $g = e$, $\psi(g) = \psi(g^{-1})$, and

$$\sum_{g,h} \overline{a_g} a_h \psi(g^{-1}h) \leq 0 \tag{0.6}$$

for any finite collection of coefficients $a_g \in \mathbb{C}$ with $\sum_g a_g = 0$. We say a sequence $(h_k)_{k \in \mathbb{N}}$ of elements of G is ψ-lacunary if there exists a constant $\delta > 0$ such that

$$\psi(h_k) \geq (1 + \delta)\psi(h_j) \tag{0.7}$$

$$\psi(h_j^{-1} h_k) \geq \delta \psi(h_k). \tag{0.8}$$

for any $k > j$.

Note that the condition (0.8) follows from (0.7) (with a smaller constant) if we further require ψ to be sub-additive, i.e. $\psi(hg) \leq \psi(h) + \psi(g)$. In fact, assuming $\psi(h_k) = (1+\varepsilon)\psi(h_j) \geq (1+\delta)\psi(h_j)$, then the sub-additivity of ψ implies that $\psi(h_k) = \psi(h_j h_j^{-1} h_k) \leq \psi(h_j) + \psi(h_j^{-1} h_k)$, thus $\psi(h_j^{-1} h_k) \geq \psi(h_k) - \psi(h_j) \geq \varepsilon\psi(h_j) = \frac{\varepsilon}{1+\varepsilon}\psi(h_k) \geq \frac{\delta}{1+\delta}\psi(h_k)$.

Let λ be the left regular representation of G. Given a sequence $c_k \in \ell^2(\mathbb{C})$, we view $f = \sum_k c_k \lambda_{h_k}$ as a lacunary "Fourier series" and will study the related Paley's theory. To state our first main result, let us recall the semigroup type H^1 and BMO-norms introduced in [20, 28], which have been frequently used in recent study of noncommutative analysis (see [8–10, 12, 21, 23, 34] etc.). Let

$$T_t : \lambda_g \mapsto e^{-t\psi(g)}\lambda_g \tag{0.9}$$

be the semigroup of operators on the group von Neumann algebra $\hat{G}$ associated with ψ. For $f = \sum_g c_g \lambda_g \in L^1(\hat{G})$, let

$$\|f\|_{H_c^1(\psi)} = \tau\left[\left(\int_0^\infty \left|\frac{\partial}{\partial s}T_s(f)\right|^2 s\,ds\right)^{\frac{1}{2}}\right]$$

$$\|f\|_{\mathrm{BMO}_c(\psi)} = \sup_{s>0}\left\|T_s\left[|f - T_s(f)|^2\right]\right\|^{\frac{1}{2}},$$

with τ being the canonical trace on the group von Neumann algebra of G. Please see Sect. 1.1 for the introduction of the group von Neumann algebra and $L^1(\hat{G})$.

Theorem 0.1 *Assume (h_k) is a ψ-lacunary sequence. Then, for any sequence $(c_k)_{k=1}^\infty \in \ell^2(\mathbb{C})$, the series $\sum_{k=1}^\infty c_k \lambda_{h_k}$ converges in $BMO_c(\psi)$ and*

$$\left\|\sum_{k=1}^\infty c_k \lambda_{h_k}\right\|_{BMO_c(\psi)}^2 \simeq_\delta \sum_{k=1}^\infty |c_k|^2. \tag{0.10}$$

with a constant of equivalence depending only on the lacunary constant δ. Moreover,

$$\inf\left\{\|f\|_{H_c^1(\psi)} : \hat{f}(h_k) = c_k, \text{ for all } k \in \mathbb{N}\right\} \simeq_\delta \left(\sum_k |c_k|^2\right)^{\frac{1}{2}}. \tag{0.11}$$

By the interpolation result proved in [20] (see Lemma 1.2 below), one obtains that every subset of a ψ-lacunary sequence is a $\Lambda(p)$ set for all $2 < p < \infty$. More precisely, we have that, for any $2 < p < \infty$ and any f of the form $f = \sum_k c_k \lambda_{h_k}$,

$$\|f\|_{L^p(\hat{G})} \simeq_\delta p\left(\sum_k |c_k|^2\right)^{\frac{1}{2}}. \tag{0.12}$$

Passing to the dual, we have for any $1 < p < 2$ and any f of the form $f = \sum_k c_k \lambda_{h_k}$,

$$\|f\|_{L^p(\hat{G})} \simeq_\delta \frac{1}{p-1} \left(\sum_k |c_k|^2 \right)^{\frac{1}{2}}. \tag{0.13}$$

We will prove Theorem 0.1 for operator valued c_k. See Theorem 2.3.

In the second part of the article, we assume that the group G is equipped with a bi-invariant order "$\leq$", that is a total order which is invariant under the left and right multiplications. Let $G_+ = \{g \in G : e \leq g\}$. Following Rudin's terminology [48, Section 8.6], we say a subset $E \subseteq G_+$ is lacunary if there exists a constant K such that

$$N(E) := \sup_{g \in G_+} \#\{h \in E : g \leq h \leq g^2\} \leq K.$$

Theorem 0.2 *For any sequence $(c_k)_{k=1}^\infty \in \mathbb{C}$, and any sequence $(h_k)_{k=1}^\infty$ in a lacunary subset $E \subseteq G_+$, we have*

$$\inf \left\{ \tau(|f|) : \hat{f}(h_k) = c_k, \operatorname{supp}(\hat{f}) \subset G_+ \right\} \simeq \left(\sum_k |c_k|^2 \right)^{\frac{1}{2}}, \tag{0.14}$$

Theorem 0.2 follows from a factorization theorem of noncommutative analytic Hardy spaces and an adaptation of Lust-Piquard and Pisier's argument [26] to Rudin's terminology [48, Section 8.6] of lacunary sets. We also provide interesting examples of Paley sequences and $\Lambda(p)$ sets (see e.g. Corollary 4.3) on free groups.

We prove all these results for operator-valued functions f, that is, for f of the form $f = \sum_k c_k \otimes \lambda_{h_k}$ with c_k being operators. See Theorems 2.4 and 3.2. Sect. 1 introduces some preliminaries. The main results are proved in Sects. 2 and 3. In Sect. 4, we apply our theories to the case of free groups and construct new examples of Paley sets and $\Lambda(p)$ sets.

1 Preliminaries

1.1 Noncommutative L^p-Space

Let $\mathcal{M}$ be a semifinite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a normal semifinite faithful trace τ. For $0 < p < \infty$, denote by $L^p(\mathcal{M})$ the noncommutative L^p space associated with the (quasi)norm $\|f\|_p = \left[\tau(|f|^p)\right]^{\frac{1}{p}}$. As usual, we set $L^\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm. For a (possibly nonabelian) discrete group G, the von Neumann algebra is the closure of the linear span of the elements λ_g given by the left regular representation of G with respect to the weak operator topology. The trace τ is defined by

$$\tau(f) = c_e,$$

for $f = \sum_g c_g \lambda_g$. The associated L^p norm is defined as

$$\|f\|_p = \left[\tau(|f|^p)\right]^{\frac{1}{p}}$$

for $1 \leq p < \infty$.

When G is abelian, e.g. $G = \mathbb{Z}^d$, the non-commutative L^p space obtained is isometrically isomorphic to the classical L^p space on the dual group. We denote by $\hat{G}$ the group von Neumann algebra of G, and by $L^p(\hat{G})$ the associated non-commutative L^p spaces. So, $\hat{G}$ will also be denoted by $L^\infty(\hat{G})$ sometimes. Let $\mathcal{M} = \mathcal{B}(\mathcal{H})$, the algebra of all bounded operators on $\mathcal{H}$ equipped with the usual trace tr, then the associated L^p-space $L^p(\mathcal{M})$ is the Schatten-p class $S^p(\mathcal{H})$. We denote τ defined above as the canonical trace on the group von Neumann algebra $\hat{G}$ and tr as the usual trace on $\mathcal{B}(\mathcal{H})$. We will often consider the tensor product of von Neumann algebra $B(H)\bar{\otimes}\hat{G}$ with the tensor trace $tr \otimes \tau$ and the associated noncommutative L^p space $L^p(B(H)\bar{\otimes}\hat{G})$. For $1 \leq p < \infty$, this is the closure of the collection of all finite sums $f = \sum_g c_g \otimes \lambda_g$ with $c_g \in S^p$ with respect to the L^p-norm

$$\|f\|_{L^p} = \left((tr \otimes \tau)(|f|^p)\right)^{\frac{1}{p}}.$$

The classical Cauchy–Schwartz inequality and Hölder's inequality extend to the non-commutative setting. In particular, we have the Kadison-Cauchy-Schwartz inequality

$$|(id \otimes \tau)(f)|^2 \leq (id \otimes \tau)(|f|^2), \tag{1.1}$$

for all $f \in B(H)\bar{\otimes}\hat{G}$, and Hölder's inequality,

$$\|f\varphi\|_{L^r} \leq \|f\|_{L^p}\|\varphi\|_{L^q}, \tag{1.2}$$

for all $f \in L^p(B(H)\bar{\otimes}\hat{G})$, $\varphi \in L^q(B(H)\bar{\otimes}\hat{G})$, where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Here id denotes the identity operator on $B(H)$. For the brevity of notation, we will omit the "id" in "$id \otimes \tau$", and omit the "$\otimes$" in "$c_k \otimes \lambda_{h_k}$" in the statement of theorems or proofs and simply write τ and $c_k\lambda_{h_k}$ when no confusion arises.

We refer readers to the survey paper [44] for information on non-commutative L^p spaces.

1.2 Column and Row Spaces

Let $0 < p \leq \infty$ and let $(c_n)_{n\geq 0}$ be a finite sequence in $L^p(\mathcal{M})$. Define

$$\|(c_n)\|_{L^p(\mathcal{M},\ell_c^2)} = \left\| \left(\sum_{n\geq 0} |c_n|^2 \right)^{1/2} \right\|_p, \quad \|(c_n)\|_{L^p(\mathcal{M},\ell_r^2)} = \left\| \left(\sum_{n\geq 0} |c_n^*|^2 \right)^{1/2} \right\|_p.$$

For $0 < p < \infty$, we define $L^p(\mathcal{M}, \ell_c^2)$(respectively $L^p(\mathcal{M}, \ell_r^2)$) as the completion of the family of all finite sequences in $L^p(\mathcal{M})$ with respect to $\|\cdot\|_{L^p(\mathcal{M},\ell_c^2)}$ (respectively $\|\cdot\|_{L^p(\mathcal{M},\ell_r^2)}$). For $p = \infty$, we define $L^\infty(\mathcal{M}, \ell_c^2)$ (respectively $L^\infty(\mathcal{M}, \ell_r^2)$) as the Banach space of (possible infinite) sequences in $\mathcal{M}$ such that $\sum_n c_n^* c_n$ (respectively $\sum_n c_n c_n^*$) converges in the weak*-topology.

Let $0 < p \leq \infty$. We define the space $L^p(\mathcal{M}, \ell_{cr}^2)$ as follows:

(1) If $0 < p < 2$,

$$L^p(\mathcal{M}, \ell_{cr}^2) = L^p(\mathcal{M}, \ell_c^2) + L^p(\mathcal{M}, \ell_r^2)$$

equipped with the norm:

$$\|(c_k)_{n\geq 0}\|_{L^p(\mathcal{M},\ell_{cr}^2)} = \inf_{c_k = c_k' + c_k''} \left\{ \|(c_k')\|_{L^p(\mathcal{M},\ell_c^2)} + \|(c_k'')\|_{L^p(\mathcal{M},\ell_r^2)} \right\}$$

where the infimum is taken over all decompositions for which

$$\|(c_k')\|_{L^p(\mathcal{M},\ell_c^2)} < \infty \text{ and } \|(c_k'')\|_{L^p(\mathcal{M},\ell_r^2)} < \infty.$$

(2) If $p \geq 2$,

$$L^p(\mathcal{M}, \ell_{cr}^2) = L^p(\mathcal{M}, \ell_c^2) \cap L^p(\mathcal{M}, \ell_r^2)$$

equipped with the norm:

$$\|(c_k)\|_{L^p(\mathcal{M},\ell_{cr}^2)} = \max \left\{ \|(c_k)\|_{L^p(\mathcal{M},\ell_r^2)}, \|(c_k)\|_{L^p(\mathcal{M},\ell_c^2)} \right\}.$$

The spaces $L^p(\mathcal{M}, \ell_{cr}^2)$ (resp. $L^p(\mathcal{M}, \ell_c^2)$, $L^p(\mathcal{M}, \ell_r^2)$) form an interpolation scale for the complex interpolation method.

Lemma 1.1

$$L^p(\mathcal{M}, \ell_c^2) = [L^\infty(\mathcal{M}, \ell_c^2), L^1(\mathcal{M}, \ell_c^2)]_{\frac{1}{p}} \tag{1.3}$$

$$L^p(\mathcal{M}, \ell_r^2) = [L^\infty(\mathcal{M}, \ell_r^2), L^1(\mathcal{M}, \ell_r^2)]_{\frac{1}{p}} \tag{1.4}$$

$$L^p(\mathcal{M}, \ell_{cr}^2) \simeq [L^\infty(\mathcal{M}, \ell_{cr}^2), L^1(\mathcal{M}, \ell_{cr}^2)]_{\frac{1}{p}} \tag{1.5}$$

for $1 < p < \infty$.

Proof $L^p(\mathcal{M}, \ell_c^2)$ are complemented subspaces of $L^p(\mathcal{M}\bar{\otimes}B(H))$ via the embedding

$$c_k :\mapsto c_k \otimes e_{1,k}.$$

Therefore, they inherit the duality relation and the interpolation relation (1.3) from $L^p(\mathcal{M}\bar{\otimes}B(H))$. The interpolation (1.4) holds because of a similar reason. The interpolation equivalence (1.5) is proved by Pisier in [38, Theorem 8.4.8] for $\mathcal{M} = B(H)$. The argument for the general case is in the same spirit, which we sketch here. Let $\mathbb{F}_\infty$ be the free nonabelian group generated by countable many free generators $\{g_k, k \in \mathbb{N}\}$, and denote by $L^\infty(\hat{\mathbb{F}}_\infty)$ the associated group von Neumann algebra. Let E_p, $1 \leq p < \infty$ $(p = \infty)$ be the norm (weak $*$) closure of $span\{\lambda_{g_k}, k \in \mathbb{N}\}$. Then E_p are complemented subspaces of $L^p(\mathcal{M}\bar{\otimes}L^\infty(\hat{\mathbb{F}}_\infty))$ and form an interpolation scale for complex interpolation. This is [38, Corollary 8.3.3]. By [38, Theorem 8.4.10] and the duality relation, $L^p(\mathcal{M}, \ell_{cr}^2)$ is completely isomorphic to E_p for all $1 \leq p \leq \infty$ via the map

$$c_k :\mapsto c_k \otimes \lambda_{g_k}.$$

Therefore, the interpolation relation (1.5) holds for $L^p(\mathcal{M}, \ell_{cr}^2)$. □

We denote by $S^p(\ell_c^2)$, $S^p(\ell_r^2)$ and $S^p(\ell_{cr}^2)$ the spaces $L^p(\mathcal{M}, \ell_c^2)$, $L^p(\mathcal{M}, \ell_r^2)$ and $L^p(\mathcal{M}, \ell_{cr}^2)$ when $\mathcal{M} = \mathcal{B}(\mathcal{H})$, respectively. Please refer to [19, 38, 44] for details on these spaces.

1.3 Semigroup BMO Spaces

Given a conditionally negative definite length ψ on discrete group G, Schoenberg's theorem [2, p. 74, Theorem2.2] says that the functions $\phi_t(g) = e^{-t\psi(g)}$ are positive definite on G for all $t > 0$. Therefore, the family of operators,

$$T_t : \lambda_g = e^{-t\psi(g)}\lambda_g$$

extends to a symmetric Markov semigroup of completely positive operators on the group von Neumann algebra $\hat{G}$. That is to say, for every $t > 0$, T_t is

(1) Unital, i.e. $T_t(\lambda_e) = \lambda_e$.
(2) Normal, i.e. T_t is weak $*$ continuous.
(3) Symmetric, i.e. $\tau(T_t(f)\varphi) = \tau(f T_t(\varphi)), \forall f, \varphi \in \hat{G}$.
(4) Completely positive, i.e. $id \otimes T_t$ is positive preserving on $\mathcal{K}(\ell^2) \otimes \hat{G}$ with $\mathcal{K}(\ell_2)$ the algebra of compact operators on $\ell^2(\mathbb{N})$.

and the family $(T_t)_{t>0}$ is weak-$*$ continuous at $0+$, i.e. $\tau(T_t(f)\varphi) \to \tau(f\varphi)$ as $t \to 0$ for all $f \in \mathcal{L}(G), \varphi \in L^1(\hat{G})$. It is well known that a unital map on C^*-algebras (in particular on von Neumann algebras) is (completely) positive preserving iff it is (completely) contractive. Because of this fact and the symmetric assuption (2), each

T_t extends to a contraction on $L^1(\hat{G})$ by duality. By interpolation, each T_t extends to a contraction on $L^p(\hat{G})$ for every $1 \leq p \leq \infty$.

Given a semifinite von Neumann algebra $\mathcal{M}$, denote by $\mathcal{N} = \mathcal{M}\bar{\otimes}\hat{G}$. Following [20] and [28], for finite sums $f = \sum_k c_g \otimes \lambda_g$ with $c_g \in \mathcal{M}$ set

$$\|f\|_{\mathrm{BMO}_c(\psi)} = \sup_{0<t<\infty} \left\|(id \otimes T_t)\left[|f - (id \otimes T_t)(f)|^2\right]\right\|_{\mathcal{N}}^{\frac{1}{2}}. \tag{1.6}$$

We will often omit id and simply write T_t instead of $T_t \otimes id$ when no confusion arises in the context. Set

$$\|f\|_{BMO(\psi)} = \max\{\|f\|_{BMO_c(\psi)}, \|f^*\|_{BMO_c(\psi)}\}. \tag{1.7}$$

When G is the integer group $\mathbb{Z}$, $L^p(\hat{\mathbb{Z}})$ is the L^p-space of p-integrable functions with respect to the Haar measure on the torus $\mathbb{T}$. The semigroup T_t is the heat semigroup (respectively Poisson semigroup) if we set $\psi(g) = |g|^2$ (respectively $|g|$) for $g \in \mathbb{Z}$. It is an elementary calculation that the semigroup BMO norm defined above coincides with those defined in (0.3), (0.4). When f is operator-valued, the semigroup BMO norm defined above coincides with the ones studied in [27, Section 1.3]. The semigroup BMO norms may differ from each other for different semigroups, see [12, Section 4] for examples.

As in [20] and [28], we define the space BMO(ψ) as an abstract closure of all the finite sums $f = \sum_k c_g \otimes \lambda_g$. We omit the definition of this closure as it will not be needed in this article. We refer the readers to [28, p. 3377] or [20, Sect. 5.2] for the definition of this closure. The following interpolation result was proved in [20, Theorem 0.2]. We refer the readers to Theorem 5.2 and Remark 5.5 of [20] for related results. E. Ricard proved in [46] that the type of semigroup of operators considered in this article satisfies the Markov dilation assumption in [20, Theorem 0.2].

Lemma 1.2 [20, Theorem 0.2] *Let* $L_0^p(\mathcal{N}) = \{f \in L^p(\mathcal{N}), \tau(f) = 0\}$. *The following interpolation result holds for* $1 < p < \infty$,

$$[BMO(\psi), L_0^1(\mathcal{N})]_{\frac{1}{p}} \simeq L_0^p(\mathcal{N})$$

1.4 Analytic H^p-Space on Ordered Groups

The theory of analytic noncommutative Hardy spaces were developed by Marsalli and West etc. in [3, 5, 29–31] in the general context of Arveson's subdiagonal operator algebras [1]. We recall here the definition of these spaces and the corresponding factorization, duality and interpolation results in the case of group von Neuman algebra associated with ordered groups. Please see [44, Sect. 8] for a survey of these results.

Let $(G, \leq)$ be a countable (possibly non-abelian) discrete group with a total order. Denote by e the unit element. Let $L^p(\hat{G})$ be the noncommutative L^p spaces associated with the canonical trace τ.

For $1 \leq p \leq \infty$, let $\mathcal{A}_p \subset L^p(\hat{G})$ be the collection of all the finite sum $\sum_{g \geq e} c_g \lambda_g$. Let $H^p(\hat{G})$ be the norm (respectively weak operator) closure of $\mathcal{A}_p$ in $L^p(\hat{G})$ for $1 \leq p < \infty$ (respectively $p = \infty$).

Like the classical case, one can define an analytic BMO space as the dual of $H^1(\hat{G})$ using an analogue of the Hilbert transform, that we will introduce at below. Let H be the linear map on $L^2(\hat{G})$ such that

$$H\left(\sum_g c_g \lambda_g\right) = -i\left(\sum_{g \geq e} c_g \lambda_g - \sum_{g \leq e} c_g \lambda_g\right). \tag{1.8}$$

It is clear that H is bounded on $L^2(\hat{G})$. It was proved in [30] that H extends to a bounded map on $L^p(\hat{G})$ for all $1 < p < \infty$.[1] So $H^p(\hat{G})$ is complemented in $L^p(\hat{G})$, and the dual of $H^p(\hat{G})$ is isomorphic to $H^{p'}(\hat{G})$ for $\frac{1}{p} + \frac{1}{p'} = 1, 1 < p < \infty$. For $f = \sum_g c_g \lambda_g \in L^2(\hat{G})$, set

$$\|f\|_{BMO(\hat{G})} = \inf\{\|u\|_{L^\infty(\hat{G})} + \|v\|_{L^\infty(\hat{G})} : f = u + Hv\}$$

where the infimum is taken over all $u, v \in L^\infty(\hat{G})$. Note $Hv \in L^2(\hat{G})$ as $v \in L^\infty(\hat{G}) \subset L^2(\hat{G})$. Let $BMOA(\hat{G})$ be the space of all $f \in H^2(\hat{G})$ with finite $\|\cdot\|_{BMO(\hat{G})}$-norms. It is easy to show that the dual of $H^1(\hat{G})$ is isomorphic to BMOA$(\hat{G})$ following the classical arguments. This was explained in [31, Theorem 5] and [44, p. 1499]. Please note that BMOA was denoted by A, and the Hilbert transform was called the conjugate map and was denoted by "$\sim$" in Theorem 5 of [31].

We will also need the operator-valued version of $H^p(\hat{G})$ and BMOA$(\hat{G})$. Let $(\mathcal{M}, tr)$ be a semifinite von Neumann algebra. Let $\mathcal{N} = \mathcal{M}\overline{\otimes}\hat{G}$ with the trace $tr \otimes \tau$. For $1 \leq p \leq \infty$, let $H^p(\mathcal{N})$ be the norm (respectively weak operator) closure in $L^p(\mathcal{N})$ of the collection of all finite sums $\sum_{g \geq e} c_g \otimes \lambda_g$ with $c_g \in L^p(\mathcal{M})$. In this case, $H^1(\mathcal{N})$ coincides with the projective tensor product $L^1(\mathcal{M})\hat{\otimes}H^1(\hat{G})$, and its dual is isomorphic to BMOA$(\mathcal{N}) = \mathcal{M}\bar{\otimes}BMOA(\hat{G})$ the injective tensor product. The Hilbert transform $id \otimes H$ extends to a bounded map on $L^p(\mathcal{N})$ for all $1 < p < \infty$. So, for $1 < p < \infty$, $H^p(\mathcal{N})$ is a complemented subspace of $L^p(\mathcal{N})$, and we have the following equivalence for $f = \sum_g c_g \otimes \lambda_g \in L^p(\mathcal{N})$,

$$\|f\|_p \simeq \left\|\sum_{g \geq e} c_g \otimes \lambda_g\right\|_p + \left\|\sum_{g < e} c_g \otimes \lambda_g\right\|_p. \tag{1.9}$$

$H^p(\mathcal{N})$ and BMOA$(\mathcal{N})$ were studied in [3, 29–31, 44] etc. in the general context of Arveson's subdiagonal operator algebras. The following factorization theorem was proved in [30, Theorem 4.3] and [3, Theorem 3.2].

[1] One can check directly that H satisfies the so-called Cotlar's identity. So its p-boundedness follows from the classical iteration and interpolation argument. See [44, Lemma 8.5] for the details.

Lemma 1.3 *Given any* $f \in H^1(\mathcal{N})$ *and* $\varepsilon > 0$*, there exist* $y, z \in H^2(\mathcal{N})$ *such that* $f = yz$ *and* $\|y\|_2\|z\|_2 \le \|f\|_1 + \varepsilon$.

It was proved in [31] and [3] that the dual of $H^1(\mathcal{N})$ is isomorphic to BMOA($\mathcal{N}$), and $H^p(\mathcal{N})$ is an interpolation space between them.

Lemma 1.4 [3, Proposition 4.1] *Let* $1 < p < \infty$ *then*

$$H^p(\mathcal{N}) \simeq [BMOA(\mathcal{N}), H^1(\mathcal{N})]_{\frac{1}{p}}$$

with equivalent norms.

Lemma 1.4 has been known to experts after the work of Pisier [36], Marsallie/West [30], and Pisier/Xu [44, p. 1499]. It was formally proved in [3] for the semifinite von Neumann algebras.

2 Proof of Theorem 0.1

Lemma 2.1 *For* $a_s \in \mathbb{R}_+, c_s, b_s \in B(H)$*, we have, for any* $0 < p, q, r < \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$,

$$\left\|\sum_s a_s c_s^* b_s\right\| \le \left\|\sum_s a_s |c_s|^2\right\|^{\frac{1}{2}} \left\|\sum_s a_s |b_s|^2\right\|^{\frac{1}{2}} \tag{2.1}$$

$$\left\|\sum_k a_s c_s^* b_s\right\|_{S^r} \le \left\|\sum_s a_s |c_s|^2\right\|_{S^p}^{\frac{1}{2}} \left\|\sum_s a_s |b_s|^2\right\|_{S^q}^{\frac{1}{2}}. \tag{2.2}$$

Proof Let $e_{s,t}$ be the canonical basis of $B(H)$. Let $f = \sum_s a_s^{\frac{1}{2}} c_s^* \otimes e_{1,s}$ and $\varphi = \sum_s a_s^{\frac{1}{2}} b_s \otimes e_{s,1}$. Then (2.1) follows from the fact that $\|f\varphi\| \le \|f\|\|\varphi\| = \|ff^*\|^{\frac{1}{2}}\|\varphi^*\varphi\|^{\frac{1}{2}}$. The inequality (2.2) follows from Hölder's inequality (1.2), $\|f\varphi\|_{S^r} \le \|f\|_{S^p}\|\varphi\|_{S^q}$. □

Lemma 2.2 *Let* $f = \sum_k c_k \otimes \lambda_{h_k} \in L^2(B(H)\bar{\otimes}\hat{G})$*, we have*

$$\frac{1}{2}\left(\sum_k |c_k|^2\right)^{\frac{1}{2}} \ge \tau\left[\left(\int_0^\infty \left|\frac{\partial}{\partial s}T_s f\right|^2 sds\right)^{\frac{1}{2}}\right] \tag{2.3}$$

This means that subtracting the right hand side from the left hand gives a nonnegative self-adjoint element of $B(H)$*. Moreover, if we assume* (h_k) *is a* ψ*-lacunary sequence, then*

$$\left\|\int_0^\infty \left|\frac{\partial}{\partial s}T_s f\right|^2 sds\right\| \le \left(1 + \frac{2}{\delta}\right)\left\|\sum_k |c_k|^2\right\|. \tag{2.4}$$

Proof An elementary calculation shows that

$$\begin{aligned}\int_0^\infty \left|\frac{\partial}{\partial s}T_s f\right|^2 sds &= \int_0^\infty \sum_k (c_k\lambda_{h_k})^* \psi(h_k)e^{-s\psi(h_k)} \sum_j (c_j\lambda_{h_j})\psi(h_j)e^{-s\psi(h_j)}sds \\ &= \sum_{k,j}(c_k\lambda_{h_k})^* c_j\lambda_{h_j}\psi(h_k)\psi(h_j)\int_0^\infty e^{-s(\psi(h_k)+\psi(h_j))}sds \\ &= \sum_{k,j} a_{k,j}(c_k\lambda_{h_k})^* c_j\lambda_{h_j}, \end{aligned} \tag{2.5}$$

with

$$a_{k,j} = \frac{\psi(h_k)\psi(h_j)}{(\psi(h_k)+\psi(h_j))^2} \geq 0$$

since $\int_0^\infty e^{-\alpha s} s\, ds = \frac{1}{\alpha^2}$. So, by Cauchy-Schwartz inequality (1.1), we get

$$\begin{aligned}\tau\left[\left(\int_0^\infty \left|\frac{\partial}{\partial s}T_s f\right|^2 sds\right)^{\frac{1}{2}}\right] &\leq \left[\tau\left(\int_0^\infty \left|\frac{\partial}{\partial s}T_s f\right|^2 sds\right)\right]^{\frac{1}{2}} \\ &= \left(\sum_k |c_k|^2 a_{k,k}\right)^{\frac{1}{2}} = \frac{1}{2}\left(\sum_k |c_k|^2\right)^{\frac{1}{2}}.\end{aligned}$$

On the other hand, for any given j, we have that

$$\begin{aligned}\sum_k a_{k,j} &= \sum_{k\leq j}\frac{\psi(h_k)\psi(h_j)}{(\psi(h_k)+\psi(h_j))^2} + \sum_{k>j}\frac{\psi(h_k)\psi(h_j)}{(\psi(h_k)+\psi(h_j))^2} \\ &\leq \sum_{k\leq j}\frac{\psi(h_k)}{\psi(h_j)} + \sum_{k>j}\frac{\psi(h_j)}{\psi(h_k)} \\ &\leq \frac{1+\delta}{\delta} + \frac{1}{\delta} = 1 + \frac{2}{\delta}.\end{aligned}$$

In the last step of the estimate above, we have used the ψ-lacunary property (0.7). Taking the supremum over j, we get

$$\sup_j \sum_k a_{k,j} \leq 1 + \frac{2}{\delta}.$$

A similar estimate shows,

$$\sup_k \sum_j a_{k,j} \leq 1 + \frac{2}{\delta}.$$

Applying Lemma 2.1 to (2.5), we have

$$\begin{aligned}\left\|\int_0^\infty \left|\frac{\partial}{\partial s}T_s f\right|^2 sds\right\| &\leq \left\|\left(\sum_{k,j}|c_k|^2 a_{k,j}\right)^{\frac{1}{2}}\right\| \left\|\left(\sum_{k,j}|c_j|^2 a_{k,j}\right)^{\frac{1}{2}}\right\| \\ &= \left\|\left(\sum_{k}|c_k|^2 \sum_j a_{k,j}\right)^{\frac{1}{2}}\right\| \left\|\left(\sum_{j}|c_j|^2 \sum_k a_{k,j}\right)^{\frac{1}{2}}\right\| \\ &\leq \left(1+\frac{2}{\delta}\right)\left\|\sum_k |c_k|^2\right\|.\end{aligned}$$

□

Theorem 2.3 *Suppose that $(h_k)_k \subseteq G$ is a ψ-lacunary sequence. Then, for any $N \in \mathbb{N}$ and $f = \sum_{k=1}^N c_k \lambda_{h_k}$ with $c_k \in B(H)$, we have*

$$\|f\|^2_{BMO_c(\psi)} \simeq_\delta \left\|\sum_{k,h_k\neq e}|c_k|^2\right\|. \tag{2.6}$$

At the other end, we have, for any $(c_k) \in S^1(\ell^2_c)$,

$$\mathrm{tr}\left[\left(\sum_k |c_k|^2\right)^{\frac{1}{2}}\right] \simeq_\delta \inf\left\{(tr\otimes\tau)\left[\left(\int_0^\infty \left|\frac{\partial}{\partial s}T_s f\right|^2 s\, ds\right)^{\frac{1}{2}}\right] : \tau(f\lambda^*_{h_k}) = c_k\right\}. \tag{2.7}$$

where the infimum runs over all $f \in L^1(B(H)\bar{\otimes}\hat{G})$.

Proof We prove the BMO estimate (2.6) first. An easy calculation shows that

$$\begin{aligned}T_t\left[|f-T_t(f)|^2\right] &= T_t\left[\left(\sum_k c_k(1-e^{-t\psi(h_k)}\lambda_{h_k})\right)^*\left(\sum_k c_k(1-e^{-t\psi(h_k)}\lambda_{h_k})\right)\right] \\ &= T_t\left[\left(\sum_{k,j}(1-e^{-t\psi(h_k)})(1-e^{-t\psi(h_j)})(c_k\lambda_{h_k})^* c_j\lambda_{h_j}\right)\right] \\ &= \sum_{k,j} a_{k,j}(c_k\lambda_{h_k})^* c_j \lambda_{h_j},\end{aligned} \tag{2.8}$$

with

$$a_{k,j} = e^{-t\psi(h_k^{-1}h_j)}(1-e^{-t\psi(h_k^{-1})})(1-e^{-t\psi(h_j)}) \geq 0.$$

Note

$$a_{k,j} \leq \min\{e^{-t\psi(h_k^{-1}h_j)}, 1 - e^{-t\psi(h_k^{-1})}\} = \min\{e^{-t\psi(h_k^{-1}h_j)}, 1 - e^{-t\psi(h_k)}\}.$$

We have, for a fixed j,

$$\sum_k a_{k,j} \leq \sum_{t\psi(h_k)\leq 1} 1 - e^{-t\psi(h_k)} + \sum_{t\psi(h_k)>1} e^{-t\psi(h_k^{-1}h_j)}.$$

By the lacunary property (0.8) $\psi(h_k^{-1}h_j) \geq \delta\psi(h_k)$ and the inequality $1 - e^{-s} \leq s$, we get

$$\sum_k a_{k,j} \leq \sum_{t\psi(h_k)\leq 1} t\psi(h_k) + \sum_{t\psi(h_k)>1} e^{-t\delta\psi(h_k)} \tag{2.9}$$

By the lacunary property (0.7) we see that

$$\frac{t\psi(h_k)}{t\psi(h_{k+1})} \leq \frac{1}{1+\delta}, \quad \frac{e^{-t\delta\psi(h_{k+1})}}{e^{-t\delta\psi(h_k)}} = e^{t\delta(\psi(h_k)-\psi(h_{k+1}))} \leq e^{-t\delta^2\psi(h_k)}.$$

So the first term on the right hand side of (2.9) is bounded by a geometric series with ratio $(1+\delta)^{-1}$ and starting with 1. So it is smaller than $\frac{1+\delta}{\delta}$. The second term on the right hand side of (2.9) is bounded by a geometric series with ratio $e^{-\delta^2}$ and starting with $e^{-\delta}$. So it is smaller than $\frac{e^{-\delta}}{1-e^{-\delta^2}}$. We then conclude that

$$\sup_j \sum_k a_{k,j} \leq 1 + \delta^{-1} + \frac{e^{-\delta}}{1 - e^{-\delta^2}} \leq 1 + \delta^{-1} + \delta^{-2} =: c_\delta.$$

A similar estimate shows that

$$\sup_k \sum_j a_{k,j} \leq c_\delta.$$

Applying Lemma 2.1 to (2.8), we have

$$\begin{aligned} \left\| T_t\left[|f - T_t(f)|^2\right]\right\| &\leq \left\|\sum_{k,j} |c_k|^2 a_{k,j}\right\|^{\frac{1}{2}} \left\|\sum_{k,j} |c_j|^2 a_{k,j}\right\|^{\frac{1}{2}} \\ &\leq (1 + \delta^{-1} + \delta^{-2}) \left\|\sum_k |c_k|^2\right\|. \end{aligned}$$

Taking the supremum over t, we get $\|f\|^2_{BMO_c} \leq c_\delta \left\|\sum_k |c_k|^2\right\|$. For the lower estimate, note that for any $f = \sum_k c_k \lambda_{h_k}$ with $\|\sum_k |c_k|^2\| < \infty$, we have

$$\|T_t\left[|f - T_t(f)|^2\right]\| \geq \left\|(id \otimes \tau)\left(T_t\left[|f - T_t(f)|^2\right]\right)\right\|$$
$$= \left\|\sum_{k, h_k \neq e} \left|\left[1 - e^{-t\psi(h_k)}\right] c_k\right|^2\right\|.$$

Since $1-e^{-t\psi(h_k)}$ tends to 1 uniformly in k as t tends to ∞, we get the lower estimate. Taking the adjoint, we prove the estimate for the BMO norms.

We now turn to the H^1-estimate (2.7). By duality, we may choose b_k such that $\|\sum |b_k|^2\| = 1$ and

$$\operatorname{tr}\left[\left(\sum_k |c_k|^2\right)^{\frac{1}{2}}\right] = \operatorname{tr}\left(\sum c_k^* b_k\right) = \sup_{f,\varphi}(tr \otimes \tau)(f^*\varphi),$$

where the superum runs over all finite sum $\varphi = \sum_{k=1}^N b_k \lambda_{h_k}$, $f = \sum_{k=1}^N c_k \lambda_{h_k}$. We then have

$$(tr \otimes \tau)(f^*\varphi) = 4(tr \otimes \tau)\left(\int_0^\infty \left(\frac{\partial}{\partial s} T_s f^*\right)\left(\frac{\partial}{\partial s} T_s \varphi\right) s ds\right)$$
$$\leq 4(tr \otimes \tau)\left[\left(\int_0^\infty \left|\frac{\partial}{\partial s} T_s f\right|^2 s ds\right)^{\frac{1}{2}}\right] \left\|\int_0^\infty \left|\frac{\partial}{\partial s} T_s \varphi\right|^2 s ds\right\|^{\frac{1}{2}}.$$

In the last step of the inequalities above, we used Hölder's inequality (1.2) for $x \in L^1(B(H \otimes L^2(0,\infty)) \otimes \hat{G})$, $y \in B(H \otimes L^2(0,\infty)) \otimes \hat{G}$ with $x = \int_0^\infty \sqrt{s} e_{1,s} \otimes \frac{\partial}{\partial s} T_s f^* ds$ and $y = \int_0^\infty \sqrt{s} e_{s,1} \otimes \frac{\partial}{\partial s} T_s \varphi ds$. Combining the above estimates with Lemma 2.2, we obtain

$$\operatorname{tr}\left[\left(\sum_k |c_k|^2\right)^{\frac{1}{2}}\right] \leq 4\left(1 + \frac{2}{\delta}\right)^{\frac{1}{2}} (\tau \otimes \operatorname{tr})\left[\left(\int_0^\infty \left|\frac{\partial}{\partial s} T_s f\right|^2 s ds\right)^{\frac{1}{2}}\right].$$

This proves one direction of (2.7), the other direction follows by taking tr on the both sides of (2.3). □

Remark 2.4 Lust-Piquard and Pisier's work [26] is the first one studying noncommutative Paley's inequalities. The second part of this paper generalize Lust-Piquard and Pisier's work to the group von Neumann algebras of ordered groups. An interesting point of Theorem 2.3 is that it gives interpretations of the row (and column) version of the noncommutative Paley's inequality separately. Lust-Piquard and Pisier consider the the analytic H^1 norm which is equivalent to the Littlewood-Paley type H^1 norm

(0.5) for scalar valued functions, but not for operator-valued functions [32]. Applying Theorem 2.3 to the Euclidean length on the integer group, we see that these two norms coincide for lacunary Fourier series with operator valued coefficients.

Given a ψ-lacunary sequence $(h_k)_{k=1}^{\infty}$ of elements in G, define the linear map T from $L^{\infty}(B(H), \ell^2_{cr})$ to BMO by

$$T[(c_k)_{k=1}^{\infty}] = \sum_k c_k \lambda_{h_k}.$$

Then T has a norm c'_δ from $L^{\infty}(B(H), \ell^2_{cr})$ to BMO and norm 1 from $L^2(B(H), \ell^2_{cr})$ to $L^2(B(H) \otimes \hat{G})$, where $c'_\delta \leq c_\delta^{\frac{1}{2}}$ from the proof of Theorem 2.3. Applying the interpolation result Lemma 1.2, we get

Corollary 2.5 *Suppose that (h_k) is a ψ-lacunary sequence for some conditionally negative definite ψ. We have that, for any $p > 2$ and any f of the form $f = \sum_k c_k \lambda_{h_k}$,*

$$\|f\|_{L^p} \lesssim_\delta p \max \left\{ \left\| \left(\sum_k |c_k|^2 \right)^{\frac{1}{2}} \right\|_p, \left\| \left(\sum_k |c_k^*|^2 \right)^{\frac{1}{2}} \right\|_p \right\}. \tag{2.10}$$

By duality, we get the following result: For any $1 < p < 2$,

$$\|(c_k)_k\|_{S^p(\ell^2_{cr})} \lesssim \inf\{\|f\|_p : \hat{f}(h_k) = c_k\}. \tag{2.11}$$

We will prove a column version of (2.11) in the next section.

Remark 2.6 Corollary 2.5 can also be obtained by combining the noncommutative H^{∞}-calculus techniques developed in [19, p. 118], and the dilation theory proved in [46]. There is another approach via noncommutative Riesz transforms [22]. The order p in (2.10) is better than what is implied by these two approaches. For ψ being the usual length function on the integer group, (2.10) holds with constants in the order of $\sqrt{p}$ [47] as $p \to \infty$, it is unclear whether this is the same for general ψ-lacunary sequences.

Remark 2.7 If $G = \mathbb{F}_n$ and ψ is the reduced word length, Haagerup proved in [13] that ψ is a conditionally negative function. For this ψ, it is easy to verify that a set consisting of a ψ-lacunary sequence is the so-called $Z(2)$ set [16, Definition 1.11]. So it is a $\Lambda(4)$ set by Harcharras's work [16, Theorem 1.13]. This does not seem clear for $Z(p)$ with $p > 2$.

Remark 2.8 The sequence of free generators $\{g_i : i \in \mathbb{N}\}$ of $\mathbb{F}_\infty$ is a ψ-lacunary sequence for some ψ. Indeed, let π be the group homomorphism from $\mathbb{F}_\infty$ to $\mathbb{F}_\infty$ sending g_i to $g_i^{2^i}$. Then $\psi(h) = |\pi(h)|$ is a conditionally negative definite function. Because, the matrix

$$(|\pi(h_k^{-1} h_j)|)_{1 \leq k,j \leq N} = (|(\pi(h_k))^{-1} \pi(h_j)|)_{1 \leq k,j \leq N}$$

is conditionally negative definite for any $h_k \in \mathbb{F}_\infty$ since the reduced word length is conditionally negative definite [13]. Note $\{g_i : i \in \mathbb{N}\}$ is obviously ψ-lacunary because $\psi(g_i) = 2^i$.

Remark 2.9 One can extend (2.11) to the range $0 < p \le 1$ as a Khintchine-type inequality

$$\|(c_k)_k\|_{S^p(\ell^2_{cr})} \lesssim \left\| \sum_k c_k \lambda_{h_k} \right\|^2_{\frac{p}{2}} \tag{2.12}$$

by applying Pisier–Ricard's theorem [45]. For the case $p = 1$, one may follow Haagerup–Musat's argument in [14] to get a better constant in (2.12).

Remark 2.10 The assumption of ψ being conditionally negative definite is needed merely by the interpolation result Lemma 1.2. The arguments for other results of this section only need the assumptions (0.7), (0.8).

Remark 2.11 Let P_t be the Poisson semigroup for bounded functions on the torus $\mathbb{T}$. As we pointed out before, the semigroup BMO associated with P_t coincides with the classical BMO. Let $S_t = P_t \otimes id_{M_n}$ be its extension to the von Neumann algebra of bounded n by n matrix-valued functions on $\mathbb{T}$. The semigroup BMO_c associated with S_t coincides with the matrix-valued BMO_{so} introduced in the literature (e.g. [32, 35]). Note that for the dyadic BMO_{so}-norm, the example of $f = \sum r_k \otimes e_{k,1}$, with $r_k = sgn\left[\sin\left(2^k \pi\theta\right)\right]$ being the k-th Rademacher function on $\mathbb{T}$ (which is identified by $[0, 1]$), shows that the estimate

$$\|f\|_{BMO^d_{so}} \lesssim \sqrt{n}\|f^*\|_{BMO^d_{so}}$$

is optimal. The upper bound of the estimate is of the order at most $\sqrt{n}$ owing to the inequality

$$\left\| \sum_k |c_k|^2 \right\| \le \mathrm{tr}\left(\sum_k |c_k^*|^2 \right) \le n \left\| \sum_k |c_k^*|^2 \right\|$$

which holds true for any sequence of n by n matrix c_k. There was no easy method to show that $\sqrt{n}$ is also optimal for the usual (non-dyadic) BMO_{so} norm before the writing of this article. Note that the BMO by dyadic BMO trick does not help in producing such a concrete example. Theorem 2.3 provides such an example by taking $f = \sum_{0<k\le n} z^{2^k} \otimes e_{k,1}$.

3 Proof of Theorem 0.2

Throughout this section, we assume $(G, \le)$ is a countable (possibly non-abelian) discrete group with a bi-invariant total order. This is equivalent to say that G contains a normal subsemigroup G_+ such that, for $G_- = (G_+)^{-1}$,

$$G_+ \cup G_- = G, G_+ \cap G_- = \{e\}.$$

In this case, one has $G_+ = \{g \in G : g \geq e\}$ and $x \leq y$ if and only if $x^{-1}y \in G_+$. It is well-known that the free groups have bi-invariant total orders (see Sect. 4). We use the notation $x < y$ if $x \leq y$ and $x \neq y$.

For each $g \in G_+$, let $L_g = \{h : g \leq h \leq g^2\}$. For $E \subset G_+$, let $N(E, g)$ be the number of elements of $E \cap L_g$, i.e. $N(E, g) = \#(L_g \cap E)$. Following Rudin's terminology [48, Section 8.6], we say $E \subset G_+$ is lacunary, if there is a constant K such that

$$N(E) = \sup_{g \in G_+} N(E, g) \leq K.$$

For a general subset $E \subset G$, let $E_+ = E \cap G_+$, $E_- = E - E_+$. We say E is lacunary if $N(E) = N(E_+) + N((E_-)^{-1})$ is finite. Please see Sect. 4 for examples of such E.

Let $(\mathcal{M}, tr)$ be a semifinite von Neumann algebra. Let $\mathcal{N} = \mathcal{M}\overline{\otimes}\hat{G}$ with the trace $tr \otimes \tau$. For $f \in L^1(\mathcal{N})$, $g \in G$, denote by $\hat{f}(g) = \tau[f(1 \otimes \lambda_{g^{-1}})]$. It is clear that, for a finite sum $f = \sum_g c_g \lambda_g$, $\hat{f}(g) = c_g$.

Theorem 3.1 *Assume that E is a lacunary subset of G_+. Then, for any sequence $(c_k)_k \subset L^1(\mathcal{M})$, and any sequence $(g_k)_{k=1}^\infty \subseteq E$, we have*

$$\begin{aligned} &\|(c_k)_{k=1}^\infty\|_{L^1(\mathcal{M}, \ell^2_{cr})} \\ &\simeq \inf\left\{(tr \otimes \tau)(|f|) : f \in L^1(\mathcal{N}), \hat{f}(g_k) = c_k, \hat{f}(g) = 0, \forall g < e\right\}, \end{aligned} \tag{3.1}$$

Proof By the convexity of $|\cdot|^2$ and the complete positivity of τ, we have that, for any finite sequence $g_k \in G$,

$$\tau|\sum_k a_k \lambda_{g_k}| \leq \left(\tau|\sum_k a_k \lambda_{g_k}|^2\right)^{\frac{1}{2}} = \left(\sum_k |a_k|^2\right)^{\frac{1}{2}}.$$

So

$$\left\|\sum_k a_k \lambda_{g_k}\right\|_1 = tr\left(\tau\left|\sum_k a_k \lambda_{g_k}\right|\right) \leq tr\left(\sum_k |a_k|^2\right)^{\frac{1}{2}} = \|(a_k)_{k=1}^\infty\|_{L^1(\mathcal{M}, \ell^2_c)}.$$

Taking the adjoints, we get

$$\left\|\sum_k b_k \lambda_{g_k}\right\|_1 \leq \|(b_k)_{k=1}^\infty\|_{L^1(\mathcal{M}, \ell^2_r)}.$$

Writing $c_k = a_k + b_k$, we get,

$$\left\| \sum_k c_k \lambda_{g_k} \right\|_1 \leq \|(c_k)_{k=1}^{\infty}\|_{L^1(\mathcal{M}, \ell^2_{cr})}. \tag{3.2}$$

This shows that the right hand side of (3.1) is smaller.

We now prove the other direction of the equivalence. By approximation, without loss of generality, we may assume E is a finite set and $\mathcal{N}$ is a finite von Neumann algebra. For $f \in H^1(\mathcal{N})$ and $\varepsilon > 0$, by Lemma 1.3, there exist $y, z \in H^2(\mathcal{N})$ such that $f = yz$ and $\|y\|_2 \|z\|_2 \leq \|f\|_1 + \varepsilon$.

Given an element $g_i \in E$ with $\widehat{f}(g_i) \neq 0$. Recall that $\hat{f}(g) = \tau(f \lambda_g^*)$, we have

$$\widehat{f}(g_i) = \sum_{e \leq h \leq g_i} \widehat{y}(h) \widehat{z}(h^{-1} g_i), \tag{3.3}$$

since $\widehat{y}(g) = \widehat{z}(g) = 0$ for all $g < e$.

Thus, the sum in (3.3) can be split into two parts;

$$\widehat{f}(g_i) = \sum_{e \leq h \leq g_i < h^2} \widehat{y}(h) \widehat{z}(h^{-1} g_i) + \sum_{e \leq h \leq h^2 \leq g_i} \widehat{y}(h) \widehat{z}(h^{-1} g_i). \tag{3.4}$$

Let

$$Z_i = \sum_{e \leq h \leq h^2 < g_i} \widehat{z}(h) \lambda_h = \sum_{e \leq h \leq g_i < h^2} \widehat{z}(h^{-1} g_i) \lambda_{h^{-1} g_i}.$$

Similarly let

$$Y_i = \sum_{e \leq h \leq h^2 \leq g_i} \widehat{y}(h) \lambda_h.$$

It follows that

$$Z_i \lambda_{g_i^{-1}} = \sum_{e \leq h \leq g_i < h^2} \widehat{z}(h^{-1} g_i) \lambda_{h^{-1}}$$

and

$$\lambda_{g_i^{-1}} Y_i = \sum_{e \leq h \leq h^2 \leq g_i} \widehat{y}(h) \lambda_{g_i^{-1} h}.$$

Let

$$A_i := \tau\left(yZ_i\lambda_{g_i^{-1}}\right) \tag{3.5}$$

$$B_i := \tau\left(\lambda_{g_i^{-1}}Y_iz\right). \tag{3.6}$$

Then

$$\hat{f}(g_i) = A_i + B_i, \tag{3.7}$$

because

$$\begin{aligned}
A_i &= \tau\left(yZ_i\lambda_{g_i^{-1}}\right)\\
&= \tau\left[\left(\sum_{g\geq e}\widehat{y}(g)\lambda_g\right)\left(\sum_{e\leq h\leq g_i<h^2}\widehat{z}(h^{-1}g_i)\lambda_{h^{-1}}\right)\right]\\
&= \tau\left[\sum_{g\geq e}\sum_{e\leq h\leq g_i<h^2}\widehat{y}(g)\widehat{z}(h^{-1}g_i)\lambda_{gh^{-1}}\right]\\
&= \sum_{e\leq h\leq g_i<h^2}\widehat{y}(h)\widehat{z}(h^{-1}g_i)
\end{aligned}$$

and

$$\begin{aligned}
B_i &= \tau\left(\lambda_{g_i^{-1}}Y_iz\right)\\
&= \tau\left[\left(\sum_{e\leq h\leq h^2\leq g_i}\widehat{y}(h)\lambda_{g_i^{-1}h}\right)\left(\sum_{g\geq e}\widehat{z}(g)\lambda_g\right)\right]\\
&= \tau\left[\left(\sum_{e\leq h\leq h^2\leq g_i}\widehat{y}(h)\lambda_{g_i^{-1}h}\right)\left(\sum_{f\leq g_i}\widehat{z}(f^{-1}g_i)\lambda_{f^{-1}g_i}\right)\right]\\
&= \tau\left[\sum_{e\leq h\leq h^2\leq g_i}\sum_{f\leq g_i}\widehat{y}(h)\widehat{z}(f^{-1}g_i)\lambda_{g_i^{-1}h}\lambda_{f^{-1}g_i}\right]\\
&= \sum_{e\leq h\leq h^2\leq g_i}\widehat{y}(h)\widehat{z}(h^{-1}g_i).
\end{aligned}$$

Applying the convexity of τ and Jensen's inequality to (3.5), we have

$$\begin{aligned}\left\|(A_i)_{i=1}^n\right\|_{L^1(\mathcal{M},\ell_r^2)} &\le (tr\otimes\tau)\left[\left(\sum_i |(yZ_i\lambda_{g_i^{-1}})^*|^2\right)^{\frac{1}{2}}\right]\\ &= (tr\otimes\tau)\left[\left(y\left(\sum_i Z_i Z_i\right)y^*\right)^{\frac{1}{2}}\right]\\ &\le \left[(tr\otimes\tau)(|y|^2)\right]^{\frac{1}{2}}\left[(tr\otimes\tau)\left(\sum_i Z_i Z_i^*\right)\right]^{\frac{1}{2}}\\ &= \left[(tr\otimes\tau)(|y|^2)\right]^{\frac{1}{2}}\left[\sum_i \sum_{e\le h\le g_i<h^2} \|\widehat{z}(h)\|_{L^2(\mathcal{M})}^2\right]^{\frac{1}{2}}.\end{aligned}$$

On the other hand, we note that $e \le h \le g_i < h^2$ implies that $g_i \in L_h$. Since $N(E, g) \le K$, we get

$$\begin{aligned}\|(A_i)_{i=1}^n\|_{L^1(\mathcal{M},\ell_r^2)} &\le \left[(tr\otimes\tau)(|y|^2)\right]^{\frac{1}{2}}\left[K\sum_h \|\widehat{z}(h)\|_{L^2(\mathcal{M})}^2\right]^{\frac{1}{2}}\\ &= K^{\frac{1}{2}}\|z\|_{L^2(\mathcal{N})}\|y\|_{L^2(\mathcal{N})} \le K^{\frac{1}{2}}(\|f\|_{L^1(\mathcal{N})}+\varepsilon).\end{aligned}$$

Now, we consider $(B_i)_i$.

$$\begin{aligned}\|(B_i)_{i=1}^n\|_{L^1(\mathcal{M},\ell_c^2)} &\le (tr\otimes\tau)\left[\left(z^*\left(\sum_i |\lambda_{g_i^{-1}}Y_i|^2\right)z\right)^{\frac{1}{2}}\right]\\ &\le \left[(tr\otimes\tau)(|z|^2))\right]^{\frac{1}{2}}\left[(tr\otimes\tau)\left(\sum_i |\lambda_{g_i^{-1}}Y_i|^2\right)\right]^{\frac{1}{2}}\\ &= \|z\|_2\left[\sum_i (tr\otimes\tau)(Y_i^*Y_i)\right]^{\frac{1}{2}}\\ &= \|z\|_2\left[\sum_i \sum_{e\le h\le h^2\le g_i} \|\widehat{y}(h)\|_{L^2(\mathcal{M})}^2\right]^{\frac{1}{2}}.\end{aligned}$$

Note that the condition $e \le h \le h^2 \le g_i$ implies

$$h^{-1}g_i \le g_i \le hg_i < g_i h^{-1} g_i$$

and

$$h' \leq g_i \leq h'^2$$

with $h' = h^{-1} g_i \geq e$ because "$\leq$" is bi-invariant. We then get

$$\sum_{e \leq h \leq h^2 \leq g_i} \|\widehat{y}(h)\|^2_{L^2(\mathcal{M})} \leq \sum_{e \leq h' \leq g_i \leq h'^2} \|\widehat{y}(h')\|^2_{L^2(\mathcal{M})}.$$

By the lacunary assumption $N(E) \leq K$, we get

$$\begin{aligned} \|(B_i)_1^n\|_{L^1(\mathcal{M}, \ell_c^2)} &\leq \|z\|_2 \left[\sum_i \sum_{e \leq h' \leq g_i \leq h'^2} \|\widehat{y}(h')\|^2_{L^2(\mathcal{M})} \right]^{\frac{1}{2}} \\ &\leq K^{\frac{1}{2}} \|z\|_{L^2(\mathcal{N})} \|y\|_{L^2(\mathcal{N})} \leq K^{\frac{1}{2}} (\|f\|_{L^1(\mathcal{N})} + \varepsilon). \end{aligned}$$

Therefore,

$$\begin{aligned} \|(\widehat{f}(g_i))_{i=1}^n\|_{L^1(\mathcal{M}, \ell^2_{cr})} &\leq \|(B_i)_{i=1}^n\|_{L^1(\mathcal{M}, \ell_c^2)} + \|(A_i)_{i=1}^n\|_{L^1(\mathcal{M}, \ell_r^2)} \\ &\leq 2K^{\frac{1}{2}} (\|f\|_{L^1(\mathcal{N})} + \varepsilon), \end{aligned}$$

This completes the proof by letting $\varepsilon \to 0$. □

Corollary 3.2 *Assume that E is a lacunary subset of G_+. Then, for any sequence $(c_k)_k \subset L^p(\mathcal{M})$, and any sequence $(g_k)_{k=1}^\infty \subseteq E$, we have*

$$\|(c_k)_{k=1}^\infty\|_{L^\infty(\mathcal{M}, \ell^2_{cr})} \simeq \left\| \sum_{k=1}^\infty c_k \lambda_{g_k} \right\|_{BMO(\mathcal{N})}, \tag{3.8}$$

$$\|(c_k)_{k=1}^\infty\|_{L^1(\mathcal{M}, \ell^2_{cr})} \simeq \left\| \sum_{k=1}^\infty c_k \lambda_{g_k} \right\|_1, \tag{3.9}$$

$$\|(c_k)_{k=1}^\infty\|_{L^p(\mathcal{M}, \ell^2_{cr})} \simeq \left\| \sum_{k=1}^\infty c_k \lambda_{g_k} \right\|_p, \ 1 < p < \infty. \tag{3.10}$$

$$\|(c_k)_{k=1}^\infty\|_{L^p(\mathcal{M}, \ell^2_{cr})} \simeq \inf \left\{ \|f\|_p : f \in L^p(\mathcal{N}), \ \hat{f}(g_k) = c_k \right\}, \ 1 \leq p \leq 2, \tag{3.11}$$

Proof The first equivalence follows from Theorem 3.1 and the duality. The second equivalence follows from Theorem 3.1 and the inequality (3.2). The third equivalence follows from the first two equivalences and the interpolation between $H^1(\mathcal{N})$ and $\mathrm{BMOA}(\mathcal{N})$. For the forth equivalence, the case $p = 2$ is trivial, the case $p = 1$ follows from Theorem 3.1, the case $1 < p < 2$ follows from the interpolation between $H^1(\mathcal{N})$ and $H^2(\mathcal{N})$. □

Corollary 3.3 *For any sequence* $(g_i)_{i=1}^{\infty}$ *in a lacunary subset* $E \in G$

$$\left\| \sum_{i=1}^{\infty} \lambda_{g_i} \otimes c_{g_i} \right\|_p \simeq \|(c_{g_i})_{i=1}^{\infty}\|_{L^p(\mathcal{M},\ell^2_{cr})}, \quad 0 < p < \infty. \tag{3.12}$$

$$\|(c_{g_i})_{i=1}^{\infty}\|_{L^p(\mathcal{M},\ell^2_{cr})} \simeq \inf \left\{ \|f\|_p : f \in L^p(\mathcal{N}),\, \hat{f}(g_i) = c_{g_i} \right\}, 1 < p < \infty. \tag{3.13}$$

Proof Note $g_i \in G_-$ implies $g_i^{-1} \in G_+$. Taking adjoints, we see that all the equivalences in Corollary 3.2 also hold for lacunary subsets $E \subset G_-$ instead of G_+.

We write $E = E_+ \cup E_-$ with $E_+ = \{g \in E, g \geq e\} \subset G_+$ and $E_- = \{g \in E, g < e\} \subset G_-$. By (1.9) and (3.10), we have, for $1 < p < \infty$,

$$\begin{aligned}
\left\| \sum_{g_i \in E} c_{g_i} \otimes \lambda_{g_i} \right\|_p &\simeq \left\| \sum_{g_i \in E_+} c_{g_i} \otimes \lambda_{g_i} \right\|_p + \left\| \sum_{g_i \in E_-} c_{g_i} \otimes \lambda_{g_i} \right\|_p \\
&\simeq \|(c_{g_i})_{g_i \in E_+}\|_{L^p(\mathcal{M},\ell^2_{cr})} + \|(c_{g_i})_{g_i \in E_-}\|_{L^p(\mathcal{M},\ell^2_{cr})} \\
&= \|(c_{g_i})_{g_i \in E}\|_{L^p(\mathcal{M},\ell^2_{cr})}.
\end{aligned}$$

This proves (3.12) for the case $1 < p < \infty$. The case where $0 < p \leq 1$ follows from the case $1 < p < \infty$ and [45, Corollary 2.2] and [7, Theorem 2.6].

(3.12) implies that the right hand side of (3.13) is dominated by its left hand side up to a constant. We now prove the other direction of (3.13). Given $f = \sum_g \hat{f}(g) \otimes \lambda_g \in L^p(\mathcal{N})$, $p \geq 2$, we have

$$\left(\sum_g |\hat{f}(g)|^2 \right)^{\frac{1}{2}} = (\tau |f|^2)^{\frac{1}{2}} \leq (\tau |f|^p)^{\frac{1}{p}}.$$

So, if $\hat{f}(g_i) = c_{g_i}$,

$$\|(c_{g_i})_{g_i \in E}\|^p_{L^p(\mathcal{M},\ell^2_c)} = tr \left(\sum_{g_i} |c_{g_i}|^2 \right)^{\frac{p}{2}} \leq tr \left(\sum_g |\hat{f}(g)|^2 \right)^{\frac{p}{2}} \leq tr \otimes \tau(|f|^p) = \|f\|_p^p.$$

Taking adjoints, we get $\|(c_{g_i})_{g_i \in E}\|^p_{L^p(\mathcal{M},\ell^2_r)} \leq \|f\|_p^p$. Therefore,

$$\|(c_{g_i})_{g_i \in E}\|^p_{L^p(\mathcal{M},\ell^2_{cr})} \leq \|f\|_p^p.$$

Given $f = \sum_g \hat{f}(g) \otimes \lambda_g \in L^p(\mathcal{N})$, $1 < p < 2$ with $\hat{f}_{g_i} = c_{g_i}$, we have, by (3.11) and its adjoint version,

$$
\begin{aligned}
\|(c_{g_i})_{g_i\in E}\|_{L^p(\mathcal{M},\ell^2_{cr})} &= \|(c_{g_i})_{g_i\in E_+}\|_{L^p(\mathcal{M},\ell^2_{cr})} + \|(c_{g_i})_{g_i\in E_-}\|_{L^p(\mathcal{M},\ell^2_{cr})} \\
&\lesssim \left\|\sum_{g\geq e}\hat{f}(g)\otimes\lambda_g\right\|_p + \left\|\sum_{g<e}\hat{f}(g)\otimes\lambda_g\right\|_p \\
&\simeq \|f\|_p
\end{aligned}
$$

In the last step, we have applied (1.9). This completes the proof of (3.13). □

Remark 3.4 It would be interesting to see whether Pisier–Ricard's argument [45] can push Theorem 3.1 to the case where $p<1$.

4 The Case of Free Groups

Let $G=\mathbb{F}_2$ be the nonabelian group with two free generators a,b. Denote by $|g|$ the reduced word length of $g\in\mathbb{F}_2$. Every $g\in\mathbb{F}_2$ can be uniquely expressed as

$$
g=a^{j_1}b^{k_1}\cdots a^{j_N}b^{k_N} \tag{4.1}
$$

with $j_i,k_i\in\mathbb{Z}$ and $j_i\neq 0$ for $1<i\leq N$ and $k_i\neq 0$ for $1\leq i<N$. For $0<q\leq 2$, set the q-length of g to be

$$
\|g\|_q=\sum_i |j_i|^q+\sum_i |k_i|^q. \tag{4.2}
$$

Then, $\psi:\mathbb{F}_2\to\mathbb{R}$ defined by $\psi(g)=\|g\|_q$ is a conditionally negative definite function for all $0<q\leq 2$. When $q=1$, $\|g\|_q$ is the reduced word length function. The property of it being conditional negative definite was studied in [13] for $q=1$ and in [6, Corollary 1] for $0<q\leq 2$. All results contained in Sect. 2, 3 apply to this ψ. In particular, all $\|\cdot\|_q$-lacunary sequences are completely unconditional in $L^p(\hat{\mathbb{F}}_2)$ for all $0<p<\infty$. However, this is not clear for $p=\infty$.

Given a conditionally negative definite length ψ with $\ker(\psi)=\{e\}$, we say a subset $A\subseteq G$ is a (respectively complete) ψ Paley-set, if there exists a constant C_A such that

$$
\left\|\sum_{h_k\in A}c_k\lambda_{h_k}\right\|_{BMO(\psi)} \leq C_A\max\left\{\left\|\sum_{h_k\in A}|c_k|^2\right\|^{\frac{1}{2}}, \left\|\sum_{h_k\in A}|c_k^*|^2\right\|^{\frac{1}{2}}\right\}, \tag{4.3}
$$

for any choice of finitely many $c_k\in\mathbb{C}$ (respectively $K(H)$). We say A is a Paley-set if it is a ψ Paley-set for some conditionally negative definite length ψ with $\ker(\psi)=\{e\}$. This definition coincides with the classical "Paley"-set, when $G=\mathbb{Z}$, and ψ is the word length on $\mathbb{Z}$. In that case, every Paley set is a Sidon set. One may wonder to what extent this is still true. The concept of Sidon sets had been studied in the noncommutative setting for a long time, and was recently re-investigated by Pisier [41, 42] and by Wang [50]. Pisier defines noncommutative completely Sidon sets using the full C^*-algebras of discrete groups and proves the stability of his completely Sidon sets by taking finite unions. An interesting feature of Pisier's definition is that only non-amenable groups can have infinite completely Sidon sets. The authors wish to consider a weaker definition in the hope of covering the case of lacunary sequence studied in this article.

We say a subset $A \in G$ is a (completely) unconditional Sidon set, if $\{\lambda_h : h \in A\}$ is (completely) unconditional in the reduced C^*-algebra of the group, i.e. there exists a constant C_A such that

$$\left\| \sum_{h_k \in A} \varepsilon_k c_k \lambda_{h_k} \right\| \leq C_A \left\| \sum_{h_k \in A} c_k \lambda_{h_k} \right\|,$$

for any choice $\varepsilon_k = \pm 1$, $c_k \in \mathbb{C}$ (respectively $K(H)$). In the case that $G = \mathbb{F}_2$ and ψ being the reduced word length (or q-length defined in (4.2)), every length-lacunary set is a Paley set and a completely $\Lambda(p)$ set for all $2 < p < \infty$ as shown in this article. The question is as follows:

Question Suppose that $(h_k)_{k=1}^{\infty}$ is a length-lacunary sequence of elements in $\mathbb{F}_2$, e.g. $\frac{|h_{k+1}|}{|h_k|} > 2$. Is $\{h_k\}$ a (completely) unconditional Sidon set? In other words, does there exist a constant C such that

$$\left\| \sum_k \varepsilon_k c_k \lambda_{h_k} \right\| \leq C \left\| \sum_k c_k \lambda_{h_k} \right\|,$$

for any choice $\varepsilon_k = \pm 1$ and $c_k \in \mathbb{C}$ (respectively $K(H)$)?

The transference method used in the work [10] is powerful for the study of harmonic analysis on the quantum tori. A similar method applies to the free group case. For $g \in \mathbb{F}_2$ in the form of (4.1), let

$$|g|_z = \left| \sum_{i=1}^{N} j_i \right|^2 + \left| \sum_{i=1}^{N} k_i \right|^2.$$

Then

$$\psi_z : g \mapsto |g|_z \tag{4.4}$$

is another conditionally negative definite function on $\mathbb{F}_2$,[2] and the unbounded linear operator $L_z : \lambda_g \mapsto \psi_z \lambda_g$ generates a symmetric Markov semigroup on the free

[2] One can see the conditional negativity of ψ_z by identifying $\mathbb{F}_2$ as a subgroup of the direct product $\mathbb{F}_2 \times \mathbb{Z}^2$ via the group homomorphism

$$g :\mapsto \left(g, \sum_{i=1}^{N} j_i, \sum_{i=1}^{N} k_i \right).$$

group von Neumann algebra $\hat{\mathbb{F}}_2$. For $(z_1, z_2) \in \mathbb{T}^2$, let π_z be the $*$-homomorphism on $\hat{\mathbb{F}}_2$ such that

$$\pi_z(\lambda_a) = z_1\lambda_a, \quad \pi_z(\lambda_b) = z_2\lambda_b.$$

Given $f \in \hat{\mathbb{F}}_2$, viewing $\pi_z(f)$ as an operator valued function on $\mathbb{T}^2$, one can see that

$$\pi_z^{-1}(\Delta \otimes id)\pi_z(f) = L_z(f), \tag{4.5}$$

with Δ being the Laplacian on $\mathbb{T}^2$. This identity allows one to transfer classical results to free groups with L_z taking the role of the Laplacian, including the corresponding Paley's inequality proved in this article. The disadvantage is that this transference method cannot produce any helpful information on the large subgroup $ker(\psi_z)$. We will show that the second part of this paper implies a Paley's theory on $ker(\psi_z)$.

Let us first recall a bi-invariant order on free groups $\mathbb{F}_2$. For notational convenience, we denote the free generators by a, b. We define the ring $\Lambda = \mathbb{Z}[A, B]$ to be the ring of formal power series in the non-commuting variables A and B. Let μ be the group homomorphism from $\mathbb{F}_2$ to the group generated by $\{1 + A, 1 + B\}$ in Λ such that:

$$\mu(a) = 1 + A,\ \mu(a^{-1}) = 1 - A + A^2 - A^3 + \cdots,$$
$$\mu(b) = 1 + B,\ \mu(b^{-1}) = 1 - B + B^2 - B^3 + \cdots.$$

Then μ is injective. Denote by "$\leq$" the dictionary order on Λ assuming $0 \leq B \leq A$. To be precise, write the element of Λ in a standard form, with lower degree terms preceding higher degree terms, and within a given degree, list the terms in the sequence according to the dictionary ordering assuming $0 \leq B \leq A$. Compare two elements of Λ by writing them both in standard form and order them according to the natural ordering of the coefficients in the first term at which they differ. We then formally define the ordering on the free group $\mathbb{F}_2$ by setting

$$g \leq h \text{ in } \mathbb{F}_2 \text{ if } \mu(g) \leq \mu(h) \text{ in } \Lambda.$$

This biinvariant order was introduced by Vinogradov [11, 49]. But the corresponding normal semigroup is not finitely generated [17].

Let $J_A(g)$ (respectively $J_B(g)$) be the coefficient of the A term (respectively B term) in $\mu(g)$; and $J_{AB}(g)$ (respectively $J_{BA}(g)$) be the coefficient of the AB term (respectively BA term) in $\mu(g)$. More generally, for any word X of A, B, denote by $J_X(g)$ the coefficient of the X term in $\mu(g)$. Note that $J_A(g) = J_A(a^{J_A(g)})$, and $J_B(g) = J_B(b^{J_B(g)})$. For $g \in \mathbb{F}_2$ in the form of (4.1), that is

$$g = a^{j_1}b^{k_1}\cdots a^{j_N}b^{k_N} \tag{4.6}$$

with $j_i, k_i \in \mathbb{Z}$ and $j_i \neq 0$ for $1 < i \leq N$ and $k_i \neq 0$ for $1 \leq i < N$, we get by direct computations,

$$J_A(g) = \sum_{s=1}^{N} j_s, \quad J_B(g) = \sum_{s=1}^{N} k_s, \tag{4.7}$$

$$J_{AB}(g) = \sum_{1\leq s\leq t\leq N} j_s k_t, \quad J_{BA}(g) = \sum_{1\leq t<s\leq N} j_s k_t. \tag{4.8}$$

From (4.7), (4.8), we see that

$$J_{AB}(g) + J_{BA}(g) = J_A(g)J_B(g).$$

Using that μ is a group homomorphism, we have

$$J_A(gh) = J_A(g) + J_A(h), \tag{4.9}$$

$$J_{AB}(gh) = J_A(g)J_B(h) + J_{AB}(g) + J_{AB}(h), \tag{4.10}$$

Let

$$\mathbb{F}_2^0 = \ker(\psi_z) = \{g \in \mathbb{F}_2 : J_A(g) = J_B(g) = 0\},$$
$$\mathbb{F}_2^{00} = \{g \in \mathbb{F}_2^0 : J_{AB}(g) = 0\} = \{g \in \mathbb{F}_2^0 : J_{AB}(g) = J_{BA}(g) = 0\}.$$

Then, $\mathbb{F}_2^0, \mathbb{F}_2^{00}$ are subgroups because of (4.9), (4.10), and $\mathbb{F}_2^0 = \ker(\psi_z)$ with ψ_z defined in (4.4). For $g \in \mathbb{F}_2^0$, $g > e$ if $J_{AB}(g) > 0$ since $J_{AA}(g) = 0$. Recall that we say a sequence of $\ell_n \neq 0 \in \mathbb{Z}$ is lacunary if there exists $\delta > 1$ such that $\inf_n \frac{\ell_{n+1}}{\ell_n} \geq \delta$. We then get the following property by definition.

Proposition 4.1 *Given a sequence $g_n \in \mathbb{F}_2$, then $E = \{g_n : n \in \mathbb{N}\}$ is a lacunary subset of $\mathbb{F}_2$ if any of the following holds:*

- *The sequence $J_A(g_n) \in \mathbb{Z}$ is lacunary.*
- *$J_A(g_n) = 0$ for all n and the sequence $J_B(g_n) \in \mathbb{Z}$ is lacunary.*
- *$J_A(g_n) = J_B(g_n) = 0$ for all n, and $J_{AB}(g_n)$ is lacunary.*

For instance, $\{a^{2^i} b^{k_i} \in \mathbb{F}_2 : i, k_i \in \mathbb{N}_+\}$ and $\{a^{2^k} b^{2^k} a^{-2^k} b^{-2^k} : k \in \mathbb{N}\}$ are lacunary subsets of $\mathbb{F}_2$.

Remark 4.2 Corollary 3.3 implies that the sets E given in Proposition 4.1 are all completely $\Lambda(p)$ sets [16].

Corollary 4.3 *Suppose $(g_k)_k \in \mathbb{F}_2^0$ is a sequence with $(J_{AB}(g_k))_k \in \mathbb{Z}$ lacunary. Then for any $(c_k)_k$ with elements in $S^p(H)$, we have*

$$\|(c_k)\|^p_{S^p(\ell^2_{cr})} \simeq (\mathrm{tr} \otimes \tau)\left(\left|\sum_k c_k \otimes \lambda_{g_k}\right|^p\right) \tag{4.11}$$

for all $0 < p < \infty$. *Moreover, for* $p = 1$, *we have*

$$\|(c_k)\|_{S^1(\ell^2_{cr})} \simeq \inf \left\{ (tr \otimes \tau) \left(\left| \sum_{J_{AB}(g) \geq 0} \hat{f}(g) \otimes \lambda_g \right| + \left| \sum_{J_{AB}(g) < 0} \hat{f}(g) \otimes \lambda_g \right| \right) \right\} \tag{4.12}$$

Here, the infimum runs over all $f \in S^1(H) \otimes L^1(\hat{\mathbb{F}_2})$ *with* $\hat{f}(g_k) = c_k$.

Proof (4.11) follows from Corollary 3.3. For (4.12), we only need to prove the relation "$\lesssim$" as the other direction is trivial. Since $\mathbb{F}_2^0$ and $\mathbb{F}_2^{00}$ are subgroups, the projection P_0 (and P_{00}) onto $L^1(\hat{\mathbb{F}_2^0})$ (and $L^1(\hat{\mathbb{F}_2^{00}})$) is completely contractive. Given $f \in S^1(H) \otimes L^1(\hat{\mathbb{F}_2})$ with $\hat{f}(g_k) = c_k$, let $y = P_0 f - P_{00} f$. Then, we have $\hat{y}(g_k) = c_k$. By Theorem 3.1, we have

$$\begin{aligned}
&\|(c_k)_{k=1}^n\|_{S^1(\ell^2_{cr})} \\
&\lesssim (tr \otimes \tau) \left[\left| \sum_{g \geq e} \hat{y}(g) \otimes \lambda_g \right| \right] + (tr \otimes \tau) \left[\left| \sum_{g < e} \hat{y}(g) \otimes \lambda_g \right| \right] \\
&= (tr \otimes \tau) \left[\left| \sum_{g \in \mathbb{F}_2^0, J_{AB}(g) > 0} \hat{f}(g) \otimes \lambda_g \right| \right] + (tr \otimes \tau) \left[\left| \sum_{g \in \mathbb{F}_2^0, J_{AB}(g) < 0} \hat{f}(g) \otimes \lambda_g \right| \right] \\
&\leq (tr \otimes \tau) \left[\left| \sum_{g \in \mathbb{F}_2^0, J_{AB}(g) \geq 0} \hat{f}(g) \otimes \lambda_g \right| + |P_{00} f| + \left| \sum_{g \in \mathbb{F}_2^0, J_{AB}(g) < 0} \hat{f}(g) \otimes \lambda_g \right| \right] \\
&\leq 2(tr \otimes \tau) \left[\left| \sum_{g \in \mathbb{F}_2^0, J_{AB}(g) \geq 0} \hat{f}(g) \otimes \lambda_g \right| + \left| \sum_{g \in \mathbb{F}_2^0, J_{AB}(g) < 0} \hat{f}(g) \otimes \lambda_g \right| \right] \\
&\leq 2(tr \otimes \tau) \left[\left| \sum_{g \in \mathbb{F}_2, J_{AB}(g) \geq 0} \hat{f}(g) \otimes \lambda_g \right| + \left| \sum_{g \in \mathbb{F}_2, J_{AB}(g) < 0} \hat{f}(g) \otimes \lambda_g \right| \right].
\end{aligned}$$

□

Remark 4.4 The associated positive cone of any total left order (including the one introduced above) on free groups is NOT represented by a regular language [17]. This increases the mystery of the associated noncommutative Hardy spaces (norms). Corollary 4.3 shows that there are more transparent alternatives (e.g. (4.12)) of the noncommutative real H^1-norm that may be used to formulate the corresponding Paley's inequalities.

Remark 4.5 Interested readers are invited to prove a similar theory by computing $J_{AAB}(g)$.

Acknowledgements The authors would like to thank anonymous referees for constructive comments that improved this paper. Chuah, Liu, and Mei are partially supported by NSF grant DMS 1700171. Han is partially supported by NSFC grant 11761067.

References

1. Arveson, W.B.: Analyticity in operator algebras. Am. J. Math. **89**, 578–642 (1967)
2. Berg, C., Christensen, J., Ressel, P.: Harmonic Analysis on Semigroups: Theory of Positive Definite and Related Functions. Graduate Text in Mathematics. Springer (1984)
3. Bekjan, T., Ospanov, K.: Complex interpolation of Noncommutative hardy spaces associated with semifinite von Neumann algebras. Acta Math. Sci. Ser. B **40**(1), 245–260 (2020)
4. Blasco, O., Pełczyński, A.: Theorems of Hardy and Paley for vector-valued analytic functions and related classes of Banach spaces. Trans. Am. Math. Soc. **323**(1), 335–367 (1991)
5. Blecher, D., Labuschagne, L.: Von Neumann Algebraic H^p Theory. Function Spaces, Contemp. Math., vol. 435, pp. 89–114. Amer. Math. Soc., Providence (2007)
6. Bożejko, M.: Positive definite functions on the free group and the noncommutative Riesz product. (Italian summary) Boll. Un. Mat. Ital. A (6) **5**(1), 13–21 (1986)
7. Cadilhac, L.: Noncommutative Khintchine inequalities in interpolation spaces of L^p-spaces. Adv. Math. **352**, 265–296 (2019)
8. Caspers, M.: Harmonic analysis and BMO-spaces of free Araki-Woods factors. Studia Math. **246**(1), 71–107 (2019)
9. Caspers, M., Junge, M., Sukochev, F., Zanin, D.: BMO-estimates for non-commutative vector valued Lipschitz functions. J. Funct. Anal. **278**(3), 108317 (2020)
10. Chen, Z., Xu, Q., Yin, Z.: Harmonic analysis on quantum tori. Commun. Math. Phys. **322**(3), 755–805 (2013)
11. Dicks, W., Warren, Šunić, Z.: Orders on trees and free products of left-ordered groups. Can. Math. Bull. **63**(2), 335–347 (2020)
12. Ferguson, T., Mei, T., Simanek, B.: H^∞-calculus for semigroup generators on BMO. Adv. Math. **347**, 408–441 (2019)
13. Haagerup, U.: An example of a nonnuclear C^*-algebra, which has the metric approximation property. Invent. Math. **50**(3), 279–293 (1979)
14. Haagerup, U., Musat, M.: On the best constants in noncommutative Khintchine-type inequalities. J. Funct. Anal. **250**(2), 588–624 (2007)
15. Haagerup, U., Pisier, G.: Bounded linear operators between C*-algebras. Duke Math. J. **71**(3), 889–925 (1993)
16. Harcharras, A.: Fourier analysis, Schur multipliers on S^p and non-commutative $\Lambda(p)$-sets. Studia Math. **137**(3), 203–260 (1999)
17. Hermiller, S., Šunić, Z.: No positive cone in a free product is regular. Int. J. Algebra Comput. **27**(8), 1113–1120 (2017)
18. Hare, K.E., Yang, R.: Sidon sets are proportionally Sidon with small Sidon constants. Can. Math. Bull. **62**(4), 798–809 (2019)
19. Junge, M., Le Merdy, C., Xu, Q.: H^∞ functional calculus and square functions on noncommutative L^p-spaces. Astèrisque No. 305 (2006)
20. Junge, M., Mei, T.: BMO spaces associated with semigroups of operators. Math. Ann. **352**(3), 691–743 (2012)
21. Junge, M., Mei, T., Parcet, J.: Smooth Fourier multipliers on group von Neumann algebras. Geom. Funct. Anal. **24**(6), 1913–1980 (2014)
22. Junge, M., Mei, T., Parcet, J., Xu, Q.: Work in progress
23. Junge, M., Mei, T., Parcet, J., Xia, R.: Algebraic Calderón-Zygmund theory. Adv. Math. **376** (2021), Paper No. 107443 (2021)
24. Kochneff, E., Sagher, Y., Zhou, K.C.: BMO estimates for lacunary series. Ark. Mat. **28**(2), 301–310 (1990)
25. Leliévre, H.: Espaces BMO, inégalités de Paley et multiplicateurs idempotents. Studia Math. **123**(3), 249–274 (1997)

26. Lust-Piquard, F., Pisier, G.: Noncommutative Khintchine and Paley inequalities. Ark. Mat. **29**(2), 241–260 (1991)
27. Mei, T.: Operator valued Hardy spaces. Mem. Am. Math. Soc. **188**(881) (2007)
28. Mei, T.: Tent spaces associated with semigroups of operators. J. Funct. Anal. **255**, 3356–3406 (2008)
29. Marsalli, M.: Noncommutative H^2 spaces. Proc. Am. Math. Soc. **125**, 779–784 (1997)
30. Marsalli, M., West, G.: Noncommutative H_p spaces. J. Oper. Theory **40**, 339–355 (1998)
31. Marsalli, M., West, G.: The dual of noncommutative H^1. Indiana Univ. Math. J. **47**(2), 489–500 (1998)
32. Nazarov, F., Pisier, G., Treil, S., Volberg, A.: Sharp estimates in vector Carleson imbedding theorem and for vector Paraproducts. J. Reine Angew. Math. **542**, 147–171 (2002)
33. Paley, R.E.A.C.: On the lacunary coefficients of power series. Ann. Math. **2**(34), 615–616 (1933)
34. Parcet, J., Rogers, K.: Twisted Hilbert transforms vs Kakeya sets of directions. J. Reine Angew. Math. **710**, 137–172 (2016)
35. Petermichl, S.: Dyadic shifts and a logarithmic estimate for Hankel operators with matrix symbol. C. R. Acad. Sci. Paris Sér. I Math. **330**(6), 455–460 (2000)
36. Pisier, G.: Interpolation between H^p spaces and noncommutative generalizations. I. Pac. J. Math. **155**(2), 341–368 (1992)
37. Pisier, G.: Multipliers and lacunary sets in non-amenable groups. Am. J. Math. **117**(2), 337–376 (1995)
38. Pisier, G.: Non-commutative Vector Valued L^p-spaces and Completely p-summing Map. Astérisque, **247** (1998)
39. Pisier, G.: Real interpolation between row and column spaces. Bull. Pol. Acad. Sci. Math. **59**(3), 237–259 (2011)
40. Pisier, G.: Grothendieck's theorem, past and present. Bull. Am. Math. Soc. (N.S.) **49**(2), 237–323 (2012)
41. Pisier, G.: Completely Sidon sets in discrete groups. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **20**(3), 891–923 (2020)
42. Pisier, G.: Completely Sidon sets in C^*-algebras. Monatsh. Math. **187**(2), 357–374 (2018)
43. Pisier, G.: Spectral gap properties of the unitary groups: around Rider's results on non-commutative Sidon sets. In: Baranov, A., Kisliakov, S., Nikolski, N. (eds.) 50 Years with Hardy Spaces. Operator Theory: Advances and Applications, vol. 261. Birkhauser, Cham (2018)
44. Pisier, G., Xu, Q.: Noncommutative L_p-Spaces. Handbook of the Geometry of Banach Spaces, vol. 2, pp. 1459–1517, North-Holland, Amsterdam (2003)
45. Pisier, G., Ricard, E.: The non-commutative Khintchine inequalities for $0<p<1$. J. Inst. Math. Jussieu **16**(5), 1103–1123 (2017)
46. Ricard, E.: A Markov dilation for self-adjoint Schur multipliers. Proc. Am. Math. Soc. **136**(12), 4365–4372 (2008)
47. Rudin, W.: Trigonometric series with gaps. J. Math. Mech. **9**, 203–227 (1960)
48. Rudin, W.: Fourier Analysis on Groups, Reprint of the 1962 original. Wiley Classics Library. A Wiley-Interscience Publication. Wiley, New York (1990)
49. Vinogradov, A.A.: On the free product of ordered groups. Mat. Sbornik N.S. (Russian) **25**(67), 163–168 (1949)
50. Wang, S.: Lacunary Fourier series for compact quantum groups. Commun. Math. Phys. **349**, 895–945 (2017)
51. Xu, Q.: On the maximality of subdiagonal algebras. J. Oper. Theory **54**, 137–146 (2005)
52. Zygmund, A.: On Fourier coefficients and transforms of functions of two variables. Studia Math. 189–201 (1974)