



Deformation Theory of Log Pseudo-holomorphic Curves and Logarithmic Ruan–Tian Perturbations

Mohammad Farajzadeh-Tehrani¹

Received: 10 April 2021 / Revised: 16 February 2023 / Accepted: 9 March 2023
© Peking University 2023

Abstract

In a previous paper (Farajzadeh-Tehrani in Geom Topol 26:989–1075, 2022), for any logarithmic symplectic pair (X, D) of a symplectic manifold X and a simple normal crossings symplectic divisor D , we introduced the notion of log pseudo-holomorphic curve and proved a compactness theorem for the moduli spaces of stable log curves. In this paper, we introduce a natural Fredholm setup for studying the deformation theory of log (and relative) curves. As a result, we obtain a logarithmic analog of the space of Ruan–Tian perturbations for these moduli spaces. For a generic compatible pair of an almost complex structure and a log perturbation term, we prove that the subspace of simple maps in each stratum is cut transversely. Such perturbations enable a geometric construction of Gromov–Witten type invariants for certain semi-positive pairs (X, D) in arbitrary genera. In future works, we will use local perturbations and a gluing theorem to construct log Gromov–Witten invariants of arbitrary such pair (X, D) .

Keywords Gromov–Witten invariants · Ruan–Tian perturbations · Normal crossing divisors

Mathematics Subject Classification 53D45 · 14N35

1 Introduction

Studying pairs (X, D) of a smooth complex projective variety X and a normal crossings (or NC) divisor D has a long history in complex algebraic geometry. In the symplectic category, McLean, Zinger, and the author recently introduced topological notations of NC symplectic divisor and configuration in arbitrary dimension; see [12, 14]. We also constructed a class of almost Kähler structures that is suitable for defining and studying J -holomorphic curves relative to an NC symplectic divisor; see

✉ Mohammad Farajzadeh-Tehrani
mohammad-tehrani@uiowa.edu

¹ Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA

Definitions 2.1 and 2.5. The theory of J -holomorphic curves has been a key tool in the study of symplectic manifolds ever since its inception by Gromov in the 1980s. Given a closed symplectic manifold (X, ω) , an ω -compatible or tame almost complex structure J on X , $g, k \in \mathbb{N}$, and $A \in H_2(X, \mathbb{Z})$ in the stable range (either $A \neq 0$ or $2g+k \geq 3$), the objects of interest are the moduli spaces $\mathcal{M}_{g,k}(X, A)$ of equivalence classes of genus g degree A k -marked J -holomorphic maps into X . Similarly, in the presence of a Lagrangian $L \subset X$, the objects of interests are the moduli spaces of marked bordered J -holomorphic curves with boundary on L . In the presence of an NC symplectic divisor D , we study moduli spaces of J -holomorphic curves that intersect D at finitely many points with pre-determined tangency orders. The particular choice of the symplectic structure ω is often not important and might be hidden. In particular, we say $D \subset X$ is an SNC symplectic divisor whenever there is a symplectic form ω on X with respect to which D is an SNC symplectic divisor. Construction of such moduli spaces and the related enumerative/algebraic invariants involves: (i) compactifying moduli spaces of such curves, (ii) putting some sort of oriented smooth structure on the moduli space (transversality and orientation problems, and the gluing analysis), and (iii) calculating/analyzing/using the resulting invariants or algebraic structures.

In a previous paper [7], we introduced a natural and explicit way of compactifying moduli spaces of pseudo-holomorphic curves (called log compactification) relative to a simple normal crossings (or **SNC**) symplectic divisor. Even if D is smooth, our log compactification is somewhat different and smaller than the well-known relative compactification in [21, 23, 24]. As the naming suggest, we expect it to be closely related to the log compactification of Gross–Siebert [18] and Abramovich–Chen [1] in the algebraic setting. In this paper, we introduce a natural setup for studying the deformation theory of log curves based on the logarithmic tangent bundle $TX(-\log D)$ associated with any symplectic **logarithmic pair** (X, D) , and address the transversality problem to some extent.

1.1 Log Pseudo-holomorphic Curves

Let

$$[N] \equiv \{1, \dots, N\}, \quad \forall N \in \mathbb{N}.$$

Let $D \equiv \bigcup_{i \in [N]} D_i \subset X$ be an SNC symplectic divisor and J be an ω -tame almost complex structure on X such that

$$J(TD_i) = TD_i, \quad \forall i \in [N].$$

For every J -holomorphic map $u : \Sigma \rightarrow X$ (with smooth domain) representing the homology class $A \in H_2(X, \mathbb{Z})$, either $\text{Im}(u) \subset D$ or there is a finite set of points $\{z_1, \dots, z_k\} \subset \Sigma$ and a corresponding set $\mathfrak{s}$ of vectors with non-negative integer coefficients

$$\mathfrak{s} = (s_a = (s_{ai})_{i \in [N]})_{a \in [k]} \in (\mathbb{N}^N)^k, \quad \text{satisfying} \quad \sum_{a \in [k]} s_{ai} = A \cdot D_i, \quad \forall i \in [N], \quad (1.1)$$

such that $u^{-1}(D) \subset \{z_1, \dots, z_k\} \subset \Sigma$ and u has a tangency of order s_{ai} with D_i at z_a and nowhere else. Thus, $\mathfrak{s}$ classifies the intersection type of k -marked J -holomorphic maps that are not mapped into D . Let

$$\mathcal{M}_{g,\mathfrak{s}}(X, D, A) \subset \mathcal{M}_{g,k}(X, A)$$

denote the subspace¹ of all such J -holomorphic curves of type $\mathfrak{s}$. Constructing a nice compactification² of $\mathcal{M}_{g,\mathfrak{s}}(X, D, A)$ and the related enumerative/algebraic invariants in the sense of (i)–(iii) above has been a challenging question that people have been working on from various perspectives for at least two decades. Technically speaking, the main goal is:

($\star$) to construct a natural geometric compactification $\overline{\mathcal{M}}_{g,\mathfrak{s}}(X, D, A)$ of $\mathcal{M}_{g,\mathfrak{s}}(X, D, A)$ so that the definition of the tangency order vector $\mathfrak{s}$ naturally extends to every element of $\overline{\mathcal{M}}_{g,\mathfrak{s}}(X, D, A)$, and $\overline{\mathcal{M}}_{g,\mathfrak{s}}(X, D, A)$ is (virtually) smooth enough to admit a natural class of cobordant Kuranishi structures of the expected real dimension

$$2 \left(c_1^{TX}(A) + (n-3)(1-g) + k - A \cdot D \right). \quad (1.2)$$

We refer to [11, 27] for the technical terms in ($\star$). For each $a \in [k]$, let

$$I_a = \{i \in [N] \text{ s.t. } s_{ai} > 0\} \subset [N] \quad (1.3)$$

be the subset of indices where u intersects D_i at the a -th marked point z_a . Let

$$D_I = \bigcap_{i \in I} D_i \quad \text{and} \quad \partial D_I = \bigcap_{J \supsetneq I} D_J, \quad \forall I \subset [N]. \quad (1.4)$$

In ($\star$), we furthermore expect the natural stabilization and evaluation maps

$$\begin{aligned} \text{st} : \overline{\mathcal{M}}_{g,\mathfrak{s}}(X, D, A) &\longrightarrow \overline{\mathcal{M}}_{g,k} \quad \text{and} \\ \text{ev} \equiv (\text{ev}_a)_{a \in [k]} : \overline{\mathcal{M}}_{g,\mathfrak{s}}(X, D, A) &\longrightarrow X^{\mathfrak{s}} \equiv \prod_{a \in [k]} D_{I_a} \end{aligned} \quad (1.5)$$

to be continuous (or smooth). Similar to the classical case, if $\overline{\mathcal{M}}_{g,\mathfrak{s}}(X, D, A)$ has a “nice” orbifold structure of the expected real dimension (1.2), Gromov–Witten (or

¹ A marked point z_a with $s_a = 0 \in \mathbb{Z}^N$ is a classical marked point. If D is smooth, we can arrange the points $z_1, \dots, z_k$ such that

$$\mathfrak{s} = (s_1, \dots, s_{k_1}, 0, \dots, 0) \in \mathbb{N}^k, \quad s_i \neq 0, \quad \forall i \in [k_1].$$

Then, the common practice in the relative theory is to denote $\mathcal{M}_{g,\mathfrak{s}}(X, D, A)$ by

$$\mathcal{M}_{g,k_1,\ell}(X, D, A) \subset \mathcal{M}_{g,k}(X, A), \quad \ell = k - k_1.$$

Separating marked points into different types is notationally cumbersome and not useful.

² I.e., a naturally defined compact space that contains $\mathcal{M}_{g,\mathfrak{s}}(X, D, A)$ as a subset.

GW) invariants of the logarithmic pair (X, D) with primary insertions can be defined by intersecting the image of $\overline{\mathcal{M}}_{g,s}(X, D, A)$ under $\text{st} \times \text{ev}$ with appropriate cycles and counting (with sign and $\mathbb{Q}$ -weights) the number of intersection points. With J carefully chosen from an appropriate space of almost Kähler structures, the resulting rational numbers will be independent of J , and only depend on the genus g , homology class A , tangency data s , cohomology classes of the aforementioned cycles in $\overline{\mathcal{M}}_{g,k}$ and X^s , and the deformation equivalence class $[X, D, \omega]$ of (X, D, ω) .

The case where D is smooth was treated at the turn of the millennium. The so-called “relative” theory of [24] in algebraic geometry, and [21, 23] in the symplectic category³ address $(\star)$ when D is smooth. More recently, Gross–Siebert and Abramovich–Chen [1, 3, 18] used sophisticated techniques from logarithmic complex algebraic geometry to fully address $(\star)$ in the algebraic case. In the algebraic construction, $\overline{\mathcal{M}}_{g,s}^{\log}(X, D, A)$ is required to have a richer structure. Roughly speaking, it should be a coarse moduli space for the functor which assigns to a log scheme B the set of all families of “good” log curves in X with base B . Ultimately, completing all the steps needed for $(\star)$ will allow the formulation of a symplectic analogue of logarithmic GW invariants in [1, 18] as well-defined invariants of the deformation equivalence class of logarithmic symplectic pairs (X, D) . In that regard, there has been a series of papers by B. Parker (see [30, 31] and the references therein) and a paper by Ionel [20] that aim to address $(\star)$ in the symplectic category; see [8] for some comments on these works. Our approach in [7] and here is substantially different from these approaches. We avoid changing the target (as in [20]) or putting extra sheaves on it (as in [30, 31]).

In [7], we introduced a geometric notion of **log J -holomorphic curve** relative to an arbitrary SNC symplectic divisor and proved the following compactness result for the moduli space $\overline{\mathcal{M}}_{g,s}^{\log}(X, D, A)$ of stable log J -holomorphic curves.

Theorem 1.1 ([7, Thm. 1.4]) *Let (X, ω) be a closed symplectic manifold and $D \subset X$ be an SNC symplectic divisor. For “suitable” choice of J , the Gromov sequential convergence topology on $\overline{\mathcal{M}}_{g,k}(X, A)$ lifts to a compact metrizable sequential convergence topology on $\overline{\mathcal{M}}_{g,s}^{\log}(X, D, A)$ such that the forgetful map*

$$\iota: \overline{\mathcal{M}}_{g,s}^{\log}(X, D, A) \longrightarrow \overline{\mathcal{M}}_{g,k}(X, A) \quad (1.6)$$

is a continuous local embedding. If $g=0$, then ι is globally an embedding.

Intuitively, if $g > 0$, the map ι behaves like an immersion. If D is smooth, there exists a surjective map from the relative moduli space $\overline{\mathcal{M}}_{g,s}^{\text{rel}}(X, D, A)$ in [21, 23, 24] to the log moduli space $\overline{\mathcal{M}}_{g,s}^{\log}(X, D, A)$; see [7, Prop. 4.5].

³ The construction is not complete in any of these two papers; see [16].

For each $i \in [N]$, let $\mathcal{N}_X D_i$ denote the normal bundle of D_i in X . An element of $\overline{\mathcal{M}}_{g,s}^{\log}(X, D, A)$ is the equivalence⁴ class of a tuple

$$f_{\log} = (u_v, \Sigma_v, j_v, \bar{z}_v, \zeta_v = (\zeta_{v,i})_{i \in I_v})_{v \in \mathbb{V}} \quad (1.7)$$

such that

- I. for each $v \in \mathbb{V}$, $I_v \subset [N]$ is the maximal subset where the image of u_v is contained in D_{I_v} ;
- II. for each $v \in \mathbb{V}$ and $i \in I_v$, $\zeta_{v,i}$ is a meromorphic section of $u_v^* \mathcal{N}_X D_i$;
- III. forgetting ζ_v , $f = (u_v, \Sigma_v, j_v, \bar{z}_v)_{v \in \mathbb{V}}$ defines a classical nodal map in $\overline{\mathcal{M}}_{g,k}(X, A)$;
- IV. and, the conditions listed below are satisfied.

First, for each $x \in \Sigma_v$, the pair (u_v, ζ_v) gives rise to a well-defined **tangency order** vector in $\mathbb{Z}^N$ with D at x , which we denote by $\text{ord}_x(u_v, \zeta_v)$; see (3.33), [7, (2-15)], or [21, Lem. 3.4]. Then, the conditions in (IV) are:

1. every point in Σ with a non-trivial tangency order vector is either a marked point or a nodal point;
2. the tangency order vector at the marked point z_a is the pre-determined vector $s_a \in \mathbb{Z}^N$ in $\mathfrak{s}$;
3. the tangency order vectors at the nodes are the opposite of each other;
4. there exists a vector-valued function $s : \mathbb{V} \rightarrow \mathbb{R}^N$ such that

$$s_v = s(v) \in \mathbb{R}_+^{I_v} \times \{0\}^{[N]-I_v}, \quad \forall v \in \mathbb{V},$$

and $s_{v'} - s_v$ is a positive multiple of the tangency order vector of any nodal point on Σ_v connected to $\Sigma_{v'}$, for all $v, v' \in \mathbb{V}$;

5. and, certain group element $\text{ob}_\Gamma(f_{\log}) \in \mathcal{G}(\Gamma)$ associated to $f_{\log}$ is equal to 1.

Conditions 1–4 are combinatorial conditions on the dual graph of $f_{\log}$. A tuple $f_{\log}$ satisfying all the conditions except the last one⁵ is called a **pre-log curve**. The last two conditions ensure that each boundary stratum of $\overline{\mathcal{M}}_{g,s}^{\log}(X, D, A)$ has positive expected complex co-dimension, and every nodal log curve is virtually smoothable, respectively. The moduli space $\overline{\mathcal{M}}_{g,s}^{\log}(X, D, A)$ can be described without mentioning the meromorphic sections $\zeta_{v,i}$. Whenever such a section exists, it is unique up to multiplication by a non-zero constant. More specifically, the data of the meromorphic sections can be read from the corresponding polar divisors and we want divisors whose associated line bundles are something specific; see Remark 4.10. Such divisors represent a particular point in the Picard group. In the genus zero case, each connected component of the Picard group is a point. Therefore, every divisor with correct tangency order coefficients is a valid choice. By (1.6), different lifts of a stable map in

⁴ Two tuples $f_{\log} = ((u_v, \Sigma_v, j_v, \zeta_v = (\zeta_{v,i})_{i \in I_v})_{v \in \mathbb{V}}, \bar{z})$ and $f'_{\log} = ((u'_v, \Sigma'_v, j'_v, \zeta'_v = (\zeta'_{v,i})_{i \in I_v})_{v \in \mathbb{V}}, \bar{z}')$ are equivalent if there exists a holomorphic reparametrization $h : \Sigma \rightarrow \Sigma'$ such that $u' \circ h = u$, $h(\bar{z}) = \bar{z}'$, and $h^* \zeta'_{h(v),i} = c_{v,i} \zeta_{v,i}$ for all $v \in \mathbb{V}$ and some $c_{v,i} \in \mathbb{C}^*$. Here $v \rightarrow h(v)$ is the induced map on the vertices of the dual graph.

⁵ Or sometimes, the last two conditions.

$\overline{\mathcal{M}}_{g,k}(X, A)$ to $\overline{\mathcal{M}}_{g,5}^{\log}(X, D, A)$ are characterized by different choices of tangency order vectors at the nodal points, satisfying Condition 3 and (1.1) on each smooth component. See Sect. 3.3 for the details.

1.2 Deformation Theory

Our first goal in this paper is to introduce a natural Fredholm setup for studying the deformation theory of log pseudo-holomorphic curves. This setup can also be used to address the transversality issue in other applications of moduli spaces of curves relative to a smooth or SNC divisor, such as in the construction of the relative Fukaya category [4]. In the latter, it would also be useful for properly addressing the orientation problem. The key to this setup is the observation that the deformation theory of log J -holomorphic curves relies on the linearization of CR equation (1.10) as an operator acting on the set of sections of the **log tangent bundle** $TX(-\log D)$ instead of TX . In the holomorphic case, the log tangent sheaf is the sheaf of holomorphic tangent vector fields in TX whose restriction to each D_i is tangent to D_i . The construction in the symplectic case is similar but depends on some auxiliary data. The deformation equivalence class of the complex vector bundle $TX(-\log D)$ only depends on the deformation equivalence class of (X, D, ω) . Furthermore, we have

$$c_1^{TX(-\log D)} = c_1^{TX} - \sum_{i \in [N]} \text{PD}(D_i) \in H^2(X, \mathbb{Z});$$

see Sect. 2.3 or [13, 15]. Therefore, in analogy with the classical dimension formula

$$d = \exp\text{-dim}_{\mathbb{R}} \overline{\mathcal{M}}_{g,k}(X, A) = 2 \left(c_1^{TX}(A) + (n-3)(1-g) + k \right), \quad (1.8)$$

the expected dimension in (1.2) can be re-written as

$$d^{\log} = \exp\text{-dim}_{\mathbb{R}} \overline{\mathcal{M}}_{g,5}^{\log}(X, D, A) = 2 \left(c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k \right). \quad (1.9)$$

The analytical setup in the classical case follows the following steps. Given a smooth domain (Σ, j) , let $\text{Map}_A(\Sigma, X)$ denote the space of all smooth maps $u : \Sigma \rightarrow X$ that represent the homology class A . Let

$$\mathcal{E}_A \longrightarrow \text{Map}_A(\Sigma, X)$$

be the infinite dimensional bundle whose fiber over u is $\Gamma(\Sigma, \Omega_{\Sigma,j}^{0,1} \otimes_{\mathbb{C}} u^*TX)$. The Cauchy–Riemann (or **CR**) equation

$$\bar{\partial}u \equiv \frac{1}{2} (du + Jdu \circ j) \quad (1.10)$$

can be seen as a section of this infinite dimensional bundle. To be precise, we need to consider a Sobolev completion of these spaces for the Implicit-Function Theorem to apply, but, by elliptic regularity, every solution of $\bar{\partial}u = 0$ will be smooth. The linearization of the $\bar{\partial}$ -section at any J -holomorphic map u is an $\mathbb{R}$ -linear map

$$D_u \bar{\partial} : \Gamma(\Sigma, u^*TX) \longrightarrow \Gamma(\Sigma, \Omega_{\Sigma, j}^{0,1} \otimes_{\mathbb{C}} u^*TX)$$

that is the sum of a $\mathbb{C}$ -linear $\bar{\partial}$ -operator and a compact operator. Therefore, it is a Fredholm operator and Riemann–Roch applies; i.e., it has finite dimensional kernel and co-kernel, and

$$\dim_{\mathbb{R}} \text{Def}(u) - \dim_{\mathbb{R}} \text{Obs}(u) = 2 \left(\deg(u^*TX) + \dim_{\mathbb{C}} X(1-g) \right), \quad (1.11)$$

where $\text{Def}(u) = \ker(D_u \bar{\partial})$ and $\text{Obs}(u) = \text{coker}(D_u \bar{\partial})$. The first space corresponds to infinitesimal deformations of u (over the fixed smooth marked domain) and the second one is the obstruction space for integrating elements of $\text{Def}(u)$ to actual deformations. If $\text{Obs}(u) \equiv 0$, by Implicit-Function Theorem [26, Thm. A.3.3], in a small neighborhood $B_{\epsilon}(u)$ of u in $\text{Map}_A(\Sigma, X)$ the set of J -holomorphic maps $V_u \equiv \bar{\partial}^{-1}(0) \cap B_{\epsilon}(u)$ is a smooth manifold of real dimension (1.11), all the elements of $\text{Def}(u)$ are smooth, and $T_u V_u \cong \text{Def}(u)$; see [26, Thm. 3.1.5]. The manifold V_u carries a natural orientation.

In the logarithmic case, given $(\Sigma, j, \vec{z} = (z_1, \dots, z_k))$, A , and $\mathfrak{s}$, we generalize this construction in a natural way. In Sect. 4.1, we construct a configuration space

$$\text{Map}_{A, \mathfrak{s}}((\Sigma, \vec{z}), (X, D)) \subset \text{Map}_A(\Sigma, X)$$

whose elements are smooth maps that have tangency order type $\mathfrak{s}$ with D at $\vec{z}$ in a suitable sense. Let

$$\mathcal{E}_{A, \mathfrak{s}} \longrightarrow \text{Map}_{A, \mathfrak{s}}((\Sigma, \vec{z}), (X, D))$$

be the infinite dimensional bundle whose fiber over u is

$$\Gamma\left(\Sigma, \Omega_{\Sigma, j}^{0,1} \otimes_{\mathbb{C}} u^*TX(-\log D)\right).$$

The section $\bar{\partial}$ of $\mathcal{E}_A$ restricts to a section $\bar{\partial}^{\log}$ of $\mathcal{E}_{A, \mathfrak{s}}$. Recall that there is a $\mathbb{C}$ -linear homomorphism

$$\iota : TX(-\log D) \longrightarrow TX \quad (1.12)$$

(covering id_X) that is an isomorphism away from D . This homomorphism induces $\mathbb{C}$ -linear maps

$$\begin{aligned} \iota_1 : \Gamma(\Sigma, u^*TX(-\log D)) &\longrightarrow \Gamma(\Sigma, u^*TX), \\ \iota_2 : \Gamma(\Sigma, \Omega_{\Sigma, j}^{0,1} \otimes_{\mathbb{C}} u^*TX(-\log D)) &\longrightarrow \Gamma(\Sigma, \Omega_{\Sigma, j}^{0,1} \otimes_{\mathbb{C}} u^*TX). \end{aligned}$$

The linearization $D_u^{\log \bar{\partial}}$ of $\bar{\partial}^{\log}$ is a Fredholm⁶ linear map

$$D_u^{\log \bar{\partial}}: \Gamma(\Sigma, u^*TX(-\log D)) \longrightarrow \Gamma(\Sigma, \Omega_{\Sigma, j}^{0,1} \otimes_{\mathbb{C}} u^*TX(-\log D))$$

such that $\iota_2 \circ D_u^{\log \bar{\partial}} = D_u \bar{\partial} \circ \iota_1$ (on the subset of smooth sections). Furthermore, by Riemann–Roch and Implicit-Function Theorem, if $\text{coker}(D_u^{\log \bar{\partial}}) = 0$, the set of J -holomorphic maps of tangency order type $\mathfrak{s}$ close to u form an oriented smooth manifold of real dimension

$$2 \left(\deg(u^*TX(-\log D)) + \dim_{\mathbb{C}} X(1-g) \right).$$

Considering the deformations of the marked domain $(\Sigma, j, \vec{z})$, we get the dimension formula (1.9) and the deformation-obstruction long exact sequence

$$\begin{aligned} 0 \longrightarrow \text{aut}(C) &\xrightarrow{\delta} \text{Def}_{\log}(u) \longrightarrow \text{Def}_{\log}(f) \longrightarrow \text{Def}(C) \\ &\xrightarrow{\delta} \text{Obs}_{\log}(u) \longrightarrow \text{Obs}_{\log}(f) \longrightarrow 0, \end{aligned}$$

where $f = (u, C = (\Sigma, j, \vec{z}))$,

$$\begin{aligned} \text{aut}(C) &= H_{\bar{\partial}}^0(T\Sigma(-\log \vec{z})), \quad \text{Def}(C) \cong H_{\bar{\partial}}^1(T\Sigma(-\log \vec{z})), \\ \text{Def}_{\log}(u) &= \ker(D_u^{\log \bar{\partial}}), \quad \text{Obs}_{\log}(u) = \text{coker}(D_u^{\log \bar{\partial}}). \end{aligned}$$

If $\text{Obs}_{\log}(f) = 0$, then a small neighborhood $B(f)$ of f in $\mathcal{M}_{g, \mathfrak{s}}(X, D, A)$ is a smooth orbifold of the expected dimension (1.9). In Sects. 4.2 and 4.3, we will extend this setup to log maps with smooth domain and image in a stratum D_I , and to general nodal log maps, respectively.

1.3 Transversality

For the general construction of Gromov–Witten type invariants of every arbitrary pair (X, D) , we need a gluing theorem that generalizes the known gluing theorem in the classical case. We also need to generalize the theory of Kuranishi structures to allow toric singularities. In the classical case, by restricting to the subset of simple maps, transversality can be achieved by global perturbations of the $\bar{\partial}$ -equation and J . Then, in the case of *semi-positive* symplectic manifolds, as worked out by McDuff–Salamon [25] in genus 0 and Ruan–Tian [32] in arbitrary genera, the classical analogue of the map $\text{st} \times \text{ev}$ in (1.5) over each stratum of non-simple maps factors through a positive complex co-dimension space of simple maps, and thus can be ignored. Therefore, in semi-positive situations, gluing and virtual techniques are not needed, and GW invariants can be defined by a direct count of perturbed pseudo-holomorphic curves in the following sense.

⁶ After taking Sobolev completions of these spaces.

Definition 1.2 A closed real $2n$ -dimensional symplectic manifold (X, ω) is called **semi-positive** if

$$c_1^{TX}(A) \geq 3 - n \implies c_1^{TX}(A) \geq 0$$

for all $A \in \pi_2(X)$ such that $\omega(A) > 0$. It is called **positive** if

$$c_1^{TX}(A) \geq 3 - n \implies c_1^{TX}(A) > 0$$

for all $A \in \pi_2(X)$ such that $\omega(A) > 0$. We say $[X, \omega]$ is semi-positive/positive if (X, ω') for some ω' deformation equivalent to ω is semi-positive/positive.

In particular, every symplectic manifold of real dimension 6 or less is semi-positive and every symplectic 4-manifold is positive. Let

$$\mathcal{M}_{g,k}^*(X, A) \subset \mathcal{M}_{g,k}(X, A)$$

denote the subspace of **simple** (not multiple-cover) maps. By [26, Thm. 3.1.5], for generic J , $\mathcal{M}_{g,k}^*(X, A)$ is a naturally oriented smooth manifold of the expected real dimension (1.8). If $g=0$ and (X, ω) is positive, for generic J , a deliberate dimension counting argument shows that the image in $\overline{\mathcal{M}}_{0,k} \times X^k$ of the complement

$$\overline{\mathcal{M}}_{0,k}(X, A) \setminus \mathcal{M}_{0,k}^*(X, A)$$

under

$$\text{st} \times \text{ev} : \overline{\mathcal{M}}_{0,k}(X, A, v) \longrightarrow \overline{\mathcal{M}}_{0,k} \times X^k$$

is a set of real codimension at least 2 and can be ignored. In other words, the inclusion $\mathcal{M}_{0,k}^*(X, A) \subset \overline{\mathcal{M}}_{0,k}(X, A)$ gives rise to a **pseudocycle** whose homology class

$$\text{GW}_{0,k,A}^X \subset H_d(\overline{\mathcal{M}}_{0,k} \times X^k, \mathbb{Z})$$

is independent of the choice of J and the particular choice of ω in its deformation equivalence class; see [26, Thm. 6.6.1] for more details.

In the higher genus case, the same argument does not work and one needs to perturb the Cauchy–Riemann equation to take care of constant and multiple-cover maps. Given a smooth Riemann surface Σ with complex structure j and a (sufficiently) smooth map $u : \Sigma \rightarrow X$, the space of **perturbation terms** for the pair (u, Σ) is the infinite dimensional vector space

$$\Gamma(\Sigma, \Omega_{\Sigma,j}^{0,1} \otimes_{\mathbb{C}} u^*TX)$$

of smooth u^*TX -valued $(0, 1)$ -forms with respect to j on Σ and J on TX . Given a perturbation term v , we say u is (J, v) -holomorphic if it satisfies the perturbed CR

equation $\bar{\partial}u = v$. If $2g+k \geq 3$, Ruan–Tian defined a class of global perturbation terms v , where each v is a section in

$$\Gamma\left(\bar{\mathfrak{U}}_{g,k} \times X, \pi_1^* \Omega_{g,k}^{0,1} \otimes_{\mathbb{C}} \pi_2^* TX\right). \quad (1.13)$$

Here, $\bar{\mathfrak{U}}_{g,k}$ is a universal family over a “regular” covering $\bar{\mathfrak{M}}_{g,k}$ of $\bar{\mathcal{M}}_{g,k}$. For each v , the moduli space of interest is the set

$$\bar{\mathcal{M}}_{g,k}(X, A, J, v)$$

of the equivalence classes of stable degree A genus g k -marked nodal (J, v) -holomorphic maps (ϕ, u, C) . Here $\phi: C \rightarrow \bar{\mathfrak{U}}_{g,k}$ is a holomorphic map⁷ from a genus g k -marked nodal curve C onto a fiber of the universal family $\bar{\mathfrak{U}}_{g,k}$ and $\bar{\partial}u = (\phi, u)^*v$; see Sect. 3. By [32, Cor. 3.9], the space $\bar{\mathcal{M}}_{g,k}(X, A, J, v)$ is Hausdorff and compact with respect to a similarly defined Gromov convergence topology. By [32, Thm. 3.16], for each ω -tame J and generic perturbation term v , the main stratum $\mathcal{M}_{g,k}(X, A, J, v)$ consisting of maps with smooth domain⁸ is cut out transversely by the $\{\bar{\partial} - v\}$ -section and thus it is a smooth manifold of the expected dimension (1.8). By [32, Thm. 3.11], it has a canonical orientation. Furthermore, if (X, ω) is semi-positive, by [32, Prop. 3.21], for generic (J, v) , the image of the complement of $\mathcal{M}_{g,k}(X, A, J, v)$ in $\bar{\mathcal{M}}_{g,k}(X, A, J, v)$ under $\text{st} \times \text{ev}$ is contained in images of maps from smooth even-dimensional manifolds of at least 2 real dimension less than the main stratum. Thus, similar to the positive case and after dividing by the degree of the regular covering used to define v , the inclusion

$$\mathcal{M}_{g,k}(X, A, J, v) \subset \bar{\mathcal{M}}_{g,k}(X, A, J, v)$$

gives rise to a GW homology class

$$\text{GW}_{g,k,A}^X \subset H_d(\bar{\mathcal{M}}_{g,k} \times X^k, \mathbb{Q})$$

independent of the choice of the admissible almost complex structure J , the perturbation v , or the particular choice of ω in its deformation equivalence class. We conclude that, in the semi-positive situations, the resulting GW invariants are *enumerative* in the sense that they can be interpreted as a finite $\mathbb{Q}$ -weighted count of (J, v) -holomorphic maps of fixed degree and genus meeting some prescribed cycles at the marked points.

Our second goal is to introduce a logarithmic analogue of Ruan–Tian perturbations of the Cauchy–Riemann section for log moduli spaces, and use them to achieve transversality over the subspace of simple maps.

⁷ It inductively contracts bubble components with unstable domain.

⁸ Since $2g+k \geq 3$, every such map is automatically simple according to Definition 3.4.

The definition of the obstruction bundle $\mathcal{E}_{A,s}$ above and (1.13) suggest that a logarithmic perturbation term should be an element of

$$\Gamma\left(\overline{\mathcal{U}}_{g,k} \times X, \pi_1^* \Omega_{g,k}^{0,1} \otimes_{\mathbb{C}} \pi_2^* TX(-\log D)\right).$$

For each $v_{\log}$ in this space, with ι as in (1.12), $v = \iota(v_{\log})$ is the associated classical perturbation term in (1.13). The construction of $\log(J, v)$ -holomorphic moduli spaces works for arbitrary $v = \iota(v_{\log})$. However, for simplicity, we restrict to a subclass of such sections $\mathcal{H}_{g,k}(X, D)$, described below, whose elements have a standard form near each stratum D_I of D . With the perturbation term $v_{\log}$ as above and a similarly defined log linearization map $D_u^{\log}\{\bar{\partial} - v\}$ in place of $D_u\{\bar{\partial} - v\}$, the proof of the main results follows the same general steps as in [32]. However, we face some difficulties in dealing with non-simple nodal maps.

We consider a particular set of almost Kähler auxiliary data $\text{AK}(X, D, \omega)$ defined in [12] and compatible logarithmic perturbations in the following sense. Since [7, Thm. 1.4] also includes integrable J , similar results for perturbations compatible with integrable almost complex structures can be obtained. An element $(\mathcal{R}, J)$ in $\text{AK}(X, D, \omega)$ consists of

- a regularization $\mathcal{R}$, that is a compatible set of symplectic identifications of neighborhoods of $\{D_I\}_{I \subset [N]}$ in their normal bundles with their neighborhoods in X in the sense of Definition 2.3;
- and an ω -tame almost complex structure J compatible with $\mathcal{R}$ in the sense of Definition 2.5.

The space $\text{AK}(X, D, \omega)$ might be empty for some choices of ω . Let⁹ $\text{Symp}(X, D)$ be the space of all symplectic structures ω on X with respect to which D is an SNC symplectic divisor. Let $\text{AK}(X, D)$ be the space of tuples $(\omega, \mathcal{R}, J)$ where $\omega \in \text{Symp}(X, D)$ and $(\mathcal{R}, J) \in \text{AK}(X, D, \omega)$. As a consequence of [12, Thm. 2.13], the projection map

$$\text{AK}(X, D) \longrightarrow \text{Symp}(X, D), \quad (\omega, \mathcal{R}, J) \longmapsto \omega,$$

is a weak homotopy equivalence. In particular, starting from any ω , we can deform it without changing its cohomology class to another ω' such that $\text{AK}(X, D, \omega') \neq \emptyset$. Therefore, the subspace $\text{AK}(X, D)_{[\omega]}$ of tuples $(\omega', \mathcal{R}, J)$ such that ω' is deformation equivalent to ω in $\text{Symp}(X, D)$ is path connected.

For any fixed tuple $(\omega, \mathcal{R}, J)$ and $g, k \in \mathbb{N}$, with $2g + k \geq 3$, we define a class of “ $(\mathcal{R}, J)$ -compatible” perturbation terms v in Definition 3.7. The compatibility condition requires v to be of a standard form with respect to the regularization $\mathcal{R}$ in the following sense. For every $I \subset [N]$, the normal bundle $\mathcal{N}_X D_I$ of D_I in X admits a decomposition into a direct sum of complex line bundles

$$\mathcal{N}_X D_I \cong \bigoplus_{i \in I} \mathcal{N}_X D_i|_{D_I}. \quad (1.14)$$

⁹ This space is denoted by $\text{Symp}^+(X, D)$ in [12, 14].

The regularization map Ψ_I in $\mathcal{R}$ gives a stratified identification of a neighborhood D_I in $\mathcal{N}_X D_I$ with a neighborhood of that in X so that

- $\Psi_I^* \omega$ is of some standard form,
- on the horizontal subspace $T^{\text{hor}} \mathcal{N}_X D_I \cong \pi_I^* T D_I$ defined via the connections in $\mathcal{R}$, $\Psi_I^* J$ is the pull back of $J|_{T D_I}$ via the projection map $\pi_I: \mathcal{N}_X D_I \rightarrow D_I$,
- and in the vertical subspace $T^{\text{ver}} \mathcal{N}_X D_I \cong \pi_I^* \mathcal{N}_X D_I$, $\Psi_I^* J$ is the direct sum complex structure on the right-hand side of (1.14).

Then, similar to the definition of J , the horizontal component of ν is required to be the pull back from D_I of some ν_I , and the vertical component of ν is required to be $(\mathbb{C}^*)^I$ -equivariant with respect to component-wise action of $\mathbb{C}^*$ on the right-hand side of (1.14). Let $\mathcal{H}_{g,k}(X, D)$ be the set of such tuples $(\omega, \mathcal{R}, J, \nu)$. As a consequence of [12, Thm. 2.13], the projection map

$$\mathcal{H}_{g,k}(X, D) \longrightarrow \text{Symp}(X, D), \quad (\omega, \mathcal{R}, J, \nu) \longmapsto \omega,$$

is again a weak homotopy equivalence. This implies that any invariant of the deformation equivalence classes in $\mathcal{H}_{g,k}(X, D)$ is an invariant of the symplectic deformation equivalence class of (X, D, ω) . The subspace $\mathcal{H}_{g,k}(X, D)_{[\omega]}$ of tuples $(\omega', \mathcal{R}, J, \nu)$ such that ω' is deformation equivalent to ω in $\text{Symp}(X, D)$ is path connected. Given $(\omega, \mathcal{R}, J, \nu) \in \mathcal{H}_{g,k}(X, D)$, in Sect. 3, we construct the moduli space

$$\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\log}(X, D, A, \nu)$$

of equivalence classes of stable k -marked genus g degree A log (J, ν) -holomorphic curves of tangency order type $\mathfrak{s}$, similar to [7, Sect. 3.2].

The following result is the straightforward generalization of [7, Thm. 1.4] to the (J, ν) -holomorphic case. This time, the rescaling map normal to D_i in the proof of [7, Thm. 3.8] yields a meromorphic section with respect to the $\bar{\partial}$ -operator $D_u^{N_i} \{\bar{\partial} - \nu\}$ in (3.15) instead of $u^* \bar{\partial}_{\mathcal{N}_X D_i} = D_u^{N_i} \bar{\partial}$ in [7, Lem. 2.1].

Theorem 1.3 *Suppose X is closed, $D = \bigcup_{i \in [N]} D_i \subset X$ is an SNC symplectic divisor, $A \in H_2(X, \mathbb{Z})$, $g, k \in \mathbb{N}$, and $\mathfrak{s} \in (\mathbb{Z}^N)^k$. For every $(\omega, \mathcal{R}, J, \nu) \in \mathcal{H}_{g,k}(X, D)$, the Gro-mov sequential convergence topology on $\overline{\mathcal{M}}_{g,k}(X, A, \nu)$ lifts to a compact metrizable sequential convergence topology on $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\log}(X, D, A, \nu)$ such that the forgetful map*

$$\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\log}(X, D, A, \nu) \longrightarrow \overline{\mathcal{M}}_{g,k}(X, A, \nu) \quad (1.15)$$

is a continuous local embedding. If $g=0$, then (1.15) is a global embedding.

Furthermore, the forgetful and evaluation maps st and ev in (1.5) are continuous. Each moduli space $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\log}(X, D, A, \nu)$ is coarsely stratified by the subspaces

$$\mathcal{M}_{g,\mathfrak{s}}(X, D, A, \nu)_{\Gamma}$$

consisting of log (J, ν) -holomorphic curves with the decorated dual graph Γ . The set of vertices $\mathbb{V}$ corresponds to the smooth components of Σ , the set of edges $\mathbb{E}$ corresponds to the nodes of Σ , and the set of roots $\mathbb{L}$ corresponds to the marked points. Each $v \in \mathbb{V}$ is decorated by the degree $A_v \in \mathbb{H}_2(X, \mathbb{Z})$, genus $g_v \in \mathbb{N}$, and the index set $I_v \subset [N]$. With the boundary divisor $\partial D_I \subset D_I$ defined as in (1.4), an edge $e \in \mathbb{E}$ is decorated by the index set $I_e \subset [N]$ if $u(q_e) \in D_{I_e} \setminus \partial D_{I_e}$. Each edge can be oriented in two ways $\bar{e}$ and $\underline{e}$ such that the node q_e is obtained by identifying two nodal points $q_{\bar{e}}$ and $q_{\underline{e}}$. Each oriented edge $\bar{e}$ is decorated by the tangency order vector $s_{\bar{e}}$ satisfying Condition 3 and (1.1) on each smooth component. The set of such Γ is finite. Similarly to [7, Sect. 3], associated to every admissible decorated dual graph Γ we get a $\mathbb{Z}$ -linear map

$$\varrho = \varrho_\Gamma: \mathbb{D} = \mathbb{Z}^{\mathbb{E}} \oplus \bigoplus_{v \in \mathbb{V}} \mathbb{Z}^{I_v} \longrightarrow \mathbb{T} = \bigoplus_{e \in \mathbb{E}} \mathbb{Z}^{I_e};$$

see (3.39). Condition 4 means that either Γ is the trivial one-vertex graph ($\mathbb{D} = \mathbb{T} = 0$, corresponding to the virtually main stratum $\mathcal{M}_{g,s}(X, D, A, \nu)$) or $\ker(\varrho)$ has an element in the positive quadrant. The complex torus in Condition 5 has Lie algebra $\text{coker}(\varrho) \otimes \mathbb{C}$.

Definition 1.4 A (J, ν) -holomorphic log map is called **simple** if the underlying (J, ν) -holomorphic map is simple.

For each decorated dual graph Γ , let

$$\mathcal{M}_{g,s}^*(X, D, A, \nu)_\Gamma \subset \mathcal{M}_{g,s}(X, D, A, \nu)_\Gamma$$

denote the open subspace consisting of simple maps.

Theorem 1.5 Suppose X is a closed symplectic manifold, $D = \bigcup_{i \in [N]} D_i$ is an SNC symplectic divisor, $A \in H_2(X, \mathbb{Z})$, $g, k \in \mathbb{N}$, and $s \in (\mathbb{Z}^N)^k$. For each admissible decorated dual graph Γ , the following statements hold.

- (1) If $2g + k \geq 3$, for any given choice of universal family in (3.5), there exists a Baire set of second category $\mathcal{H}_{g,k}^\Gamma(X, D) \subset \mathcal{H}_{g,k}(X, D)$ such that for each $(\omega, \mathcal{R}, J, \nu) \in \mathcal{H}_{g,k}^\Gamma(X, D)$, the subspace of simple maps

$$\mathcal{M}_{g,s}^*(X, D, A, \nu)_\Gamma \subset \mathcal{M}_{g,s}(X, D, A, \nu)_\Gamma$$

is a naturally oriented smooth manifold of the real dimension

$$2 \left(c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k - \dim \ker(\varrho) \right); \quad (1.16)$$

the restriction of $\text{st} \times \text{ev}$ in (1.5) to $\mathcal{M}_{g,s}^*(X, D, A, \nu)_\Gamma$ is smooth.

- (2) If $g = 0$, $\nu \equiv 0$, and $A \neq 0$ or $k \geq 3$, the same statement holds for J in a Baire set of second category $\text{AK}^\Gamma(X, D) \subset \text{AK}(X, D)$.

In a future work, by considering local perturbations, in the sense of Kuranishi structures, and construing a gluing map (outlined in [7, Sect. 3.4]), we will use the same techniques used in the proof of Theorem 1.5 to construct Kuranishi charts for $\overline{\mathcal{M}}_{g,s}(X, D, A)$.

In the classical case (no D), ϱ is the trivial map $\mathbb{Z}^{\mathbb{E}} \rightarrow 0$. If $N = 1$ (D smooth), $\dim \mathbb{D} - \dim \mathbb{T} \geq 0$ and $\dim \ker(\varrho)$ is often very large. If $N > 1$, there are nodal configurations Γ with arbitrary many nodes, $\dim \mathbb{D} - \dim \mathbb{T} < 0$, and $\dim_{\mathbb{R}} \mathbb{K}_{\mathbb{R}}(\Gamma) = 1$; see [7, Example 2.13]. As a consequence, if D is not empty or smooth, we need a sharp dimension counting argument for dealing with non-simple nodal maps.

1.4 Semi-Positive Pairs

Our final goal is to introduce a class of semi-positive pairs (X, D) for which one can use the perturbed moduli spaces above to construct log Gromov–Witten invariants in arbitrary genera without constructing a virtual fundamental class.

For $(X, D = \bigcup_{i \in [N]} D_i, \omega)$ as before, we say D is **Nef**¹⁰ if $A \cdot D_i \geq 0$ for all $A \in \pi_2(X)$ such that $\omega(A) > 0$. Let

$$\ell_{I,A} = \min \left(A \cdot \bigcup_{j \in [N]-I} D_j, 2 \right), \quad \forall A \in H_2(X, \mathbb{Z}), \quad I \subset [N].$$

Definition 1.6 Let (X, ω) be a closed $2n$ -dimensional symplectic manifold and D be a Nef SNC symplectic divisor in (X, ω) . We say (X, D, ω) is **semi-positive** if

$$c_1^{TX(-\log D)}(A) \geq 3 - n + |I| - \ell_{I,A} \implies c_1^{TX(-\log D)}(A) \geq 0 \quad (1.17)$$

for all $A \in \pi_2(D_I)$ such that $\omega(A) > 0$. We say (X, D, ω) is **positive** if

$$c_1^{TX(-\log D)}(A) \geq 3 - n + |I| - \ell_{I,A} \implies c_1^{TX(-\log D)}(A) > 0 \quad (1.18)$$

for all $A \in \pi_2(D_I)$ such that $\omega(A) > 0$.

We say $[X, D, \omega]$ is semi-positive/positive if (X, D, ω') for some ω' deformation equivalent to ω is semi-positive/positive. One may remove the Nef condition at the expense of altering the left-hand side.

In Sect. 4.4, we show that under the semi-positivity condition (1.17) (resp. positivity condition (1.18)), multiple-cover log J -holomorphic spheres happen in dimensions less than or equal (resp. less) than somewhere injective maps. As we explain below, unlike the classical case, this is not sufficient for finding a suitable upper bound for the dimension of the image of non-simple maps in $\overline{\mathcal{M}}_{g,k} \times X^5$ when D is not smooth.

For every decorated dual graph $\Gamma = \Gamma(\mathbb{V}, \mathbb{E}, \mathbb{L})$ with $|\mathbb{V}| \geq 2$, let

$$\mathcal{M}_{g,s}^{\text{ns}}(X, D, A, \nu)_{\Gamma} = \mathcal{M}_{g,s}(X, D, A, \nu)_{\Gamma} - \mathcal{M}_{g,s}^{\star}(X, D, A, \nu)_{\Gamma}$$

¹⁰ The terminology is inspired by the corresponding notion in algebraic geometry but is different.

be the subset of **non-simple** (or **multiply covered**) maps. An arbitrary stable log (J, ν) -map fails to be simple if either it contains a non-trivial bubble component that is a multiple-cover, or if it contains two non-trivial bubbles with the same image.

Proposition 1.7 *If D is smooth,*

- (1) *under the semi-positivity condition of Definition 1.6, for $(\omega', \mathcal{R}, J, \nu)$ in a Baire set of second category $\mathcal{H}_{g,k}^{\Gamma, \text{ns}}(X, D)_{[\omega]} \subset \mathcal{H}_{g,k}(X, D)_{[\omega]}$, the image of $\mathcal{M}_{g,\mathfrak{s}}^{\text{ns}}(X, A, J, \nu)_{\Gamma}$ under $\text{st} \times \text{ev}$ lies in the image of smooth maps from finitely many smooth even-dimensional manifolds of at least 2 real dimension less than the dimension of the main stratum;*
- (2) *similarly, under the same condition, if $g = 0$, $k \leq 2$, and $\nu \equiv 0$, for $(\omega', \mathcal{R}, J)$ in a Baire set of second category $\text{AK}^{\Gamma, \text{ns}}(X, D)_{[\omega]} \subset \text{AK}(X, D)_{[\omega]}$, the image of $\mathcal{M}_{0,\mathfrak{s}}^{\text{ns}}(X, D, A)_{\Gamma}$ under $\text{st} \times \text{ev}$ lies in the image of smooth maps from finitely many smooth even-dimensional manifolds of at least 2 real dimension less than the expected dimension.*

For D smooth, the semi-positivity/positivity conditions of Definition 1.6 are essentially the same as [16, Def. 4.7]. There is some confusion in [21, 22] regarding the proper semi-positivity requirements in the relative case; see [16, Remark 4.9]. Their work does not include a detailed proof of the relative analogue of Proposition 1.7. The claim is that the result follows from the classical result of Ruan–Tian by looking at the image of the moduli space in $\overline{\mathcal{M}}_{g,k}(X, A, \nu)$.

The first statement below follows from Theorem 1.5 (1) and Proposition 1.7 (1). The second follows from a standard family version of these results. The third, follows by lifting two perturbations ν_1 and ν_2 obtained from two regular families $\mathfrak{U}_{g,k}^{(1)} \rightarrow \mathfrak{M}_{g,k}^{(1)}$ and $\mathfrak{U}_{g,k}^{(2)} \rightarrow \mathfrak{M}_{g,k}^{(2)}$ to their fiber product regular family and connecting them by a regular path, as in [35, pp. 34–35].

Corollary 1.8 *Suppose (X, ω) is a closed symplectic manifold, D is a smooth symplectic divisor, $A \in H_2(X, \mathbb{Z})$, $g, k \in \mathbb{N}$ with $2g + k \geq 3$, and $\mathfrak{s} \in \mathbb{N}^k$. If $[X, D, \omega]$ is semi-positive, for any choice of a regular universal family as in (3.5), there exists a Baire subset*

$$\mathcal{H}_{g,k}^{\text{reg}}(X, D)_{[\omega]} \in \mathcal{H}_{g,k}(X, D)_{[\omega]}$$

of the second category such that for every $(\omega', \mathcal{R}, J, \nu)$ in this set

- (1) *the map*

$$\text{st} \times \text{ev}: \mathcal{M}_{g,\mathfrak{s}}(X, D, A, \nu) \longrightarrow \overline{\mathcal{M}}_{g,k} \times X^{\mathfrak{s}}$$

defines a pseudo-cycle of real dimension

$$d^{\log} = 2 \left(c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k \right);$$

- (2) the integral homology class $\widetilde{\text{GW}}_{g,\mathfrak{s},A}^{X,D}$ in $\overline{\mathcal{M}}_{g,k} \times X^{\mathfrak{s}}$ determined by this pseudo-cycle is independent of the choice of $(\omega', \mathcal{R}, J, \nu)$;
- (3) furthermore, the rational class

$$\text{GW}_{g,\mathfrak{s},A}^{X,D} \equiv \frac{1}{\deg p} \widetilde{\text{GW}}_{g,\mathfrak{s},A}^{X,D} \in H_{2d^{\log}}(\overline{\mathcal{M}}_{g,k} \times X^{\mathfrak{s}}, \mathbb{Q}),$$

where $\deg p$ is the degree of the regular covering used to define ν , is an invariant of the deformation equivalence class of $\omega \in \text{Symp}(X, D)$.

Similarly, the first statement below follows from Theorem 1.5 (2), Proposition 1.7 (2), and Lemma 4.16. The second one follows from a family version of these results.

Corollary 1.9 Suppose (X, ω) is a closed symplectic manifold, D is a smooth symplectic divisor, $A \in H_2(X, \mathbb{Z})$, $k \leq 2$, and $\mathfrak{s} \in \mathbb{N}^k$. If $[X, D, \omega]$ is positive, there exists a Baire subset

$$AK^{\text{reg}}(X, D)_{[\omega]} \in AK(X, D)_{[\omega]}$$

of the second category such that for every $(\omega', \mathcal{R}, J)$ in this set

- (1) the map

$$\text{st} \times \text{ev}: \mathcal{M}_{0,\mathfrak{s}}^{\star}(X, D, A) \longrightarrow \overline{\mathcal{M}}_{0,k} \times X^{\mathfrak{s}}$$

defines a pseudo-cycle of real dimension

$$2 \left(c_1^{TX(-\log D)}(A) + n - 3 + k \right);$$

- (2) and, the integral homology class $\text{GW}_{0,\mathfrak{s},A}^{X,D}$ in $\overline{\mathcal{M}}_{0,k} \times X^{\mathfrak{s}}$ determined by this pseudo-cycle is independent of the choice of $(\omega', \mathcal{R}, J)$.

For the reason stated at the last paragraph of Sect. 1.3, Definition 1.6 is not strong enough for proving Proposition 1.7 (and consequently Corollaries 1.8 and 1.9) for arbitrary SNC case ($N > 1$). More precisely, in the proof of the classical version of Proposition 1.7, we get a stratification

$$\mathcal{M}_{g,k}^{\text{ns}}(X, A, \nu)_{\Gamma} = \bigcup_{\gamma} \mathcal{M}_{g,k}^{\gamma}(X, A, \nu)_{\Gamma}$$

where $\gamma = (\Gamma, \Gamma'')$ and Γ'' characterizes the topological type of the underlying simple curves. For each γ , we get a fibration

$$\pi_{\gamma}: \mathcal{M}_{g,k}^{\gamma}(X, A, \nu)_{\Gamma} \longrightarrow \mathcal{M}_{g,k}(X, A, \nu)_{\Gamma''}$$

where the image is the space of underlying simple curves, and $\text{st} \times \text{ev}$ factors through π_γ . In the logarithmic/relative case, we first consider a pre-log space $\mathcal{M}_{g,s}^{\text{plog}}(X, D, A, \nu)_\Gamma$ for which similar fibrations

$$\pi_\gamma: \mathcal{M}_{g,s}^{\text{plog},\gamma}(X, D, A, \nu)_\Gamma \longrightarrow \mathcal{M}_{g,s}(X, D, A, \nu)_{\Gamma''}$$

can be constructed. Then

$$\mathcal{M}_{g,s}^\gamma(X, D, A, \nu)_\Gamma = \text{ob}_\Gamma^{-1}(1),$$

where

$$\text{ob}_\Gamma: \mathcal{M}_{g,s}^{\text{plog},\gamma}(X, D, A, \nu)_\Gamma \longrightarrow \mathcal{G}(\Gamma)$$

is the obstruction map in Condition 5. Intuitively, the obstruction map ob_Γ arises from gluing considerations in the following way. In order to smooth out the nodes and realize a log curve $f_{\log} \in \mathcal{M}_{g,s}^{\text{plog},\gamma}(X, D, A, \nu)_\Gamma$ (see (1.7)) as a limit of curves in the main stratum, we need to glue the connected components at nodes q_e by some local gluing equations $z_e z_{\bar{e}} = \varepsilon_e$ and, at the same time, push-out each map u_v out of D_{I_v} in the direction of the meromorphic sections ζ_v by some multiple factors $t_v \zeta_v$. The condition $\text{ob}_\Gamma(f_{\log}) = 1 \in \mathcal{G}(\Gamma)$ is equivalent to the existence of a compatible set of gluing and push-out parameters $(\{\varepsilon_e\}_{e \in \mathbb{E}}, \{t_v\}_{v \in \mathbb{V}})$. It is evident from this explanation that ob_Γ is very sensitive to the topology of the configuration and does not factor through π_γ ; see after (3.48) for a more detailed discussion.

If D is smooth, we can ignore ob_Γ because $\dim \mathbb{D} - \dim \mathbb{T} \geq 0$. Otherwise, as Example 4.24 shows, $\mathcal{M}_{g,s}^\gamma(X, D, A, \nu)_\Gamma$ can be larger than the main stratum. For this reason, we need a stronger semi-positivity condition that yields a bound on the dimension of $\mathcal{M}_{g,s}^{\text{plog},\gamma}(X, D, A, \nu)_\Gamma$.

Given $[X, D, \omega]$, for each $I \subset [N]$ and $A \in \pi_2(X)$, let $\delta_{I,A}$ denote the minimum number of (geometric) intersection points of a degree A J -holomorphic sphere¹¹ in D_I with ∂D_I , for all $(\omega', J) \in \text{AK}(X, D)_{[\omega]}$.

Definition 1.10 Let (X, ω) be a closed $2n$ -dimensional symplectic manifold and D be a Nef SNC symplectic divisor in (X, ω) . We say (X, D, ω) is **strongly-semi-positive** if

$$c_1^{TX(-\log D)}(A) \geq 3 - n + |I| - \ell_{I,A} \implies c_1^{TX(-\log D)}(A) \geq \max\{0, 2 - \delta_{I,A}\} \quad (1.19)$$

for all $A \in \pi_2(D_I)$ such that $\omega(A) > 0$ (except possibly $(I, \delta_{I,A}) \neq (\emptyset, 0)$). We say (X, D, ω) is **strongly-positive** if it is strongly-semi-positive and positive.

In many examples, Condition (1.19) and (1.17) are equivalent.

¹¹ It is possible to write down a weaker definition of $\delta_{I,A}$ without mentioning J -holomorphic spheres. Often, this number is independent of the choice of almost Kähler structure. Also, we define this number to be zero if such a J -holomorphic sphere does not exist.

Example 1.11 A transverse union of d hyperplanes in $\mathbb{P}^n$ (with Fubini–Study symplectic form) is semi-positive (= strongly-semi-positive) whenever $d \notin [n+2, 2n+1]$ and it is positive (= strongly-positive) whenever $d \notin [n+1, 2n+1]$. An interesting case is when $X = \mathbb{P}^n$ and D is the degree $n+1$ toric boundary divisor $\partial\mathbb{P}^n$. In this case $TX(-\log D)$ is the trivial complex vector bundle generated by (the pre-image in $TX(-\log D)$ of $n+1$ vector fields

$$x_0 \partial x_0, \dots, x_n \partial x_n, \quad \text{satisfying } \sum_{i=0}^n x_i \partial x_i = 0.$$

In other words, $(\mathbb{P}^n, \partial\mathbb{P}^n)$ is a log Calabi–Yau pair. If furthermore $n=3$, then

$$\exp\text{-dim}_{\mathbb{C}} \overline{\mathcal{M}}_{g,\mathfrak{s}}^{\log}(X, D, A) = k$$

is independent of g and A . Note that k is at least 2. Since the quintic Calabi–Yau 3-fold can be degenerated to an SNC configuration whose components are blowups of $(\mathbb{P}^3, \partial\mathbb{P}^3)$, the GW numbers arising from $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\log}(\mathbb{P}^3, \partial\mathbb{P}^3, A)$ should be related to the GW invariants of the quintic CY 3-fold. An interesting example of log Calabi–Yau four-folds is worked out in [9].

The claim is that under the stronger conditions of Definition 1.10, Proposition 1.7 (and consequently Corollaries 1.8 and 1.9) holds for all N .

Conjecture 1.12 Suppose X is closed, $D = \bigcup_{i \in [N]} D_i \subset X$ is an SNC symplectic divisor, $A \in H_2(X, \mathbb{Z})$, $g, k \in \mathbb{N}$ with $2g+k \geq 3$, and $\mathfrak{s} \in (\mathbb{N}^N)^k$. If $[X, D, \omega]$ is strongly-semi-positive (resp. strongly-positive), then Proposition 1.7, and consequently Corollaries 1.8 and 1.9, hold.

We lay the foundation and explain the difficulties for proving Conjecture 1.12 in Sect. 4.5. We provide several examples that illustrate the issues. We also explain the consequence of (1.19). We plan to address this conjecture in a future work. There are, however, some special but interesting cases, such as when X is toric and D is its boundary divisor (see Remark 4.20), or when D_i are $(0, A)$ -hollow (Donaldson divisors of sufficiently high degree) in the sense of [17, Def. 1], where Conjecture 1.12 can be confirmed with easier arguments.

2 SNC Divisors and the Associated Structures: Review

In Sect. 2.1, we recall the notions of simple normal crossings (SNC) symplectic divisor and symplectic regularizations for such objects introduced in [12]. Regularizations allow us to define a suitable space of almost Kähler structures in Sect. 2.2 and perturbations in Sect. 3.2. In Sect. 2.3, we review the notion of logarithmic tangent bundle associated to SNC symplectic divisors introduced in [13]. Readers familiar with the definitions, notations, and results of [12, 13] may skip this section. We refer to [13] for a relatively short review of these concepts.

2.1 SNC Divisors and Regularizations

Let X be a (smooth) manifold. For any submanifold $D \subset X$, let

$$\mathcal{N}_X D \equiv \frac{TX|_D}{TD} \longrightarrow D$$

denote the normal bundle of D in X .

Definition 2.1 ([12, Definition 2.1]) An **SNC symplectic divisor** in a symplectic manifold (X, ω) is a finite transverse union $D \equiv \bigcup_{i \in [N]} D_i$ of smooth symplectic divisors $\{D_i\}_{i \in [N]}$ such that for every $I \subset [N]$ the submanifold

$$D_I \equiv \bigcap_{i \in I} D_i \subset X$$

is symplectic and its symplectic and intersection orientations are the same.

Let $\text{Symp}(X, D)$ (this is denoted by $\text{Symp}^+(X, D)$ in [12]) denote the space of symplectic structures ω on X such that D is an SNC divisor in X with respect to ω . The particular choice of ω in its deformation equivalence class is not important in the construction of (relative) GW invariants.

By the transversality assumption, the homomorphisms

$$\mathcal{N}_X D_I \longrightarrow \bigoplus_{i \in I} \mathcal{N}_X D_i|_{D_I}, \quad \forall I \subset [N], \quad (2.1)$$

induced by the inclusions $TD_I \subset TD_i|_{D_I}$ are isomorphisms. These vector bundle isomorphisms are not symplectic unless $\{D_i\}_{i \in [N]}$ intersect orthogonally. For $I' \subset I \subset [N]$, define

$$\mathcal{N}_{I;I'} = \bigoplus_{i \in I-I'} \mathcal{N}_X D_i|_{D_I};$$

under the decomposition (2.1), $\mathcal{N}_{I;I'}$ is isomorphic to the normal bundle of D_I in $D_{I'}$. We denote by

$$\pi_{I;I'} : \mathcal{N}_{I;I'} \longrightarrow D_I, \quad \pi_I : \mathcal{N}_X D_I \longrightarrow D_I,$$

the natural projection maps.

A **system of regularizations for** $\{D_i\}_{i \in [N]}$ in X is a collection of smooth embeddings

$$\Psi_I : \mathcal{N}'_X D_I \longrightarrow X, \quad \forall I \subset [N],$$

from open neighborhoods $\mathcal{N}'_X D_I \subset \mathcal{N}_X D_I$ so that $\Psi_I|_{D_I} = \text{id}_{D_I}$, $d\Psi_I$ induces the identity map on $\mathcal{N}_X D_I$, and

$$\Psi_I(\mathcal{N}_{I;I'} \cap \text{Dom}(\Psi_I)) = D_{I'} \cap \text{Im}(\Psi_I), \quad \forall I' \subset I \subset [N].$$

This implies that $d\Psi_I$ induces an isomorphism

$$\mathfrak{D}\Psi_{I;I'}: \pi_{I;I'}^* \mathcal{N}_{I;I-I'}|_{\mathcal{N}_{I;I'} \cap \text{Dom}(\Psi_I)} \longrightarrow \mathcal{N}_X D_{I'}|_{D_{I'} \cap \text{Im}(\Psi_I)}; \quad (2.2)$$

see [12, Sect. 2.2]. In the $I = I'$ case, this derivative is the identity map.

Definition 2.2 A **regularization** for D in X is a system of regularizations for $\{D_i\}_{i \in [N]}$ in X as above such that

$$\text{Dom}(\Psi_I) = \mathfrak{D}\Psi_{I;I'}^{-1}(\text{Dom}(\Psi_{I'})), \quad \Psi_I = \Psi_{I'} \circ \mathfrak{D}\Psi_{I;I'}|_{\text{Dom}(\Psi_I)}, \quad \forall I' \subset I \subset [N].$$

For each $i \in [N]$, an **ω -compatible Hermitian structure** on $\mathcal{N}_X D_i$ is a triple $(i_i, \rho_i, \nabla^{(i)})$, where i_i is an ω -compatible (fiber-wise) complex structure on $\mathcal{N}_X D_i$, ρ_i is a Hermitian metric with real part

$$\rho_{\mathbb{R}}(\cdot, \cdot) = \omega|_{\mathcal{N}_X D_i}(\cdot, i_i \cdot),$$

and $\nabla^{(i)}$ is a Hermitian connection compatible with (i_i, ρ_i) . For each $i \in [N]$, the space of ω -compatible Hermitian structures on $\mathcal{N}_X D_i$ is non-empty and contractible. Each triple $(i_i, \rho_i, \nabla^{(i)})$ as above determines a 1-form α_i on $\mathcal{N}_X D_i - D_i$ whose restriction to each fiber $\mathcal{N}_X D_i|_x - \{x\} \cong \mathbb{C}^*$ is the 1-form $d\theta$ with respect to the polar coordinates (r, θ) on $\mathbb{C}$. For each $i \in I \subset [N]$, we denote the tuple induced by $(i_i, \rho_i, \nabla^{(i)}, \alpha_i)$ on $\mathcal{N}_X D_i|_{D_I}$ by $(i_{I;i}, \rho_{I;i}, \nabla^{(I;i)}, \alpha_{I;i})$. We also denote by ρ_i and $\rho_{I;i}$ the square-of-the-norm functions on $\mathcal{N}_X D_i$ and $\mathcal{N}_X D_i|_{D_I}$, respectively.

Definition 2.3 ([12, Definition 2.9]) If $D \equiv \bigcup_{i \in [N]} D_i$ is an SNC symplectic divisor in (X, ω) , an **ω -regularization** for D in X consists of a choice of Hermitian structure $(i_i, \rho_i, \nabla^{(i)})$ on $\mathcal{N}_X D_i$ for all $i \in [N]$ together with a regularization for D in X as in Definition 2.2 so that

$$\Psi_I^* \omega = \pi_I^*(\omega|_{T D_I}) + \frac{1}{2} \sum_{i \in I} d(\rho_{I;i} \alpha_{I;i}), \quad \forall I \subset [N], \quad (2.3)$$

and (2.2) is an isomorphism of split Hermitian vector bundles for all $I' \subset I \subset [N]$.

If $N = 1$, i.e., D is a smooth divisor, an ω -regularization is a single map

$$\Psi: \mathcal{N}'_X D \longrightarrow X \quad (2.4)$$

as in [26, Lem. 3.14] without any further compatibility condition. We define the space of auxiliary data $\text{Aux}(X, D)$ to be the space of pairs $(\omega, \mathcal{R})$, where $\omega \in \text{Symp}(X, D)$ and $\mathcal{R}$ is an ω -regularization of D in X . Let

$$\Pi: \text{Symp}(X, D) \longrightarrow H^2(M; \mathbb{R})$$

be the map sending ω to its de Rham equivalence class $[\omega]$. The following is a weaker version of the main result of [12] for SNC symplectic divisors.

Theorem 2.4 ([12, Theorem 2.13]) *For (X, D) as above, the projection maps*

$$\begin{aligned} \pi : \text{Aux}(X, D) &\longrightarrow \text{Symp}(X, D), \\ \pi|_{\Pi^{-1}(\alpha)} : \{\Pi \circ \pi\}^{-1}(\alpha) &\longrightarrow \Pi^{-1}(\alpha), \quad \alpha \in H_{\text{dR}}^2(M), \end{aligned}$$

are weak homotopy equivalences.

Given ω -regularizations $\mathcal{R}$ and $\mathcal{R}'$, we say $\mathcal{R}'$ is a **shrinking** of $\mathcal{R}$ if the Hermitian data in $\mathcal{R}$ and $\mathcal{R}'$ are the same and

$$\text{Dom}(\Psi'_I) \subset \text{Dom}(\Psi_I) \quad \text{and} \quad \Psi'_I = \Psi_I|_{\text{Dom}(\Psi'_I)}, \quad \forall I \subset [N].$$

Two ω -regularizations $\mathcal{R}$ and $\mathcal{R}'$ are said to be **equivalent** if they have a common shrinking. The latter defines an equivalence relation among ω -regularizations.

If D is an SNC symplectic divisor in (X, ω) , then, for each $I \subset [N]$,

$$\partial D_I = \bigcup_{I \subsetneq J} D_J \quad (2.5)$$

is an SNC symplectic divisor in $(D_I, \omega|_{T D_I})$ and an ω -regularization $\mathcal{R}$ for D in X restricts to an $\omega|_{T D_I}$ -regularization $\mathcal{R}_I$ for ∂D_I in D_I . For $I = \emptyset$, the convention is $D_\emptyset = X$ and $\partial X = D$.

2.2 Almost Kähler Auxiliary Data

In order to define relative GW invariants of (X, D, ω) , we need to consider an almost complex structure J on X that is both ω -tame and D -compatible. The biggest set of such almost complex structures that one may consider is the set of ω -tame (or compatible) J such that $J(T D_i) = T D_i$ for all $i \in [N]$ and

$$N_J(u, v) \in T D_i, \quad \forall i \in [N], \quad x \in D_i, \quad u, v \in T_x X, \quad (2.6)$$

where N_J is Nijenhuis tensor of J . Condition (2.6) is needed to ensure that certain operators are complex linear (see [7, (4-6)]), or equally, certain sequence of almost complex structures on $\mathcal{N}_X D_i$ converges to a standard one (see [7, Lem. 3.5]). In this paper, however, similar to [23] and [6], we restrict to a special class of almost complex structures arising from regularizations in the following sense.

Definition 2.5 Suppose $D = \bigcup_{i \in [N]} D_i$ is an SNC symplectic divisor in (X, ω) , $\mathcal{R}$ is an ω -regularization for D in X as in Definition 2.3, and J is an ω -tame almost complex structure on X such that $J(T D_i) = T D_i$ for all $i \in [N]$. We say J is $\mathcal{R}$ -compatible, if

$$\Psi_I^* J = \pi_I^*(J_I) \oplus \pi_I^* \bigoplus_{i \in I} i_{I; i}, \quad \forall I \subset [N], \quad (2.7)$$

where $J_I \equiv J|_{TD_I}$, $i_{I,i}$ is the complex structure on $\mathcal{N}_X D_i|_{D_I}$ pre-determined in $\mathcal{R}$, and the righthand side of (2.7) is the direct sum complex structure corresponding to the decomposition

$$T\mathcal{N}_X D_I \cong \pi_I^* T D_I \oplus \pi_I^* \mathcal{N}_X D_I, \quad (2.8)$$

given by the connections in $\mathcal{R}$.

Given any $(\omega, \mathcal{R}) \in \text{Aux}(X, D)$, we denote the space of ω -compatible almost complex structures compatible with a shrinking of $\mathcal{R}$ by $\text{AK}(X, D, \omega)_{\mathcal{R}}$. This space is non-empty and contractible. We denote the space of compatible tuples $(\omega, \mathcal{R}, J)$ by $\text{AK}(X, D)$. It follows from Theorem 2.4 that the projection map

$$\text{AK}(X, D) \longrightarrow \text{Symp}(X, D) \quad (2.9)$$

is a weak-homotopy equivalence. This implies that any invariant of the deformation equivalence classes in $\text{AK}(X, D)$ is an invariant of the symplectic deformation equivalence classes in $\text{Symp}(X, D)$.

2.3 Logarithmic Tangent Bundle

In this section, we review the notion of logarithmic tangent bundle in complex geometry and the analogous notion for SNC symplectic divisors introduced in [13]. A detailed description of this construction is provided in [15]. We will show in Sect. 4 that the linearization of Cauchy–Riemann operator for log maps is a lift of the classical linearization map to the logarithmic tangent bundle.

Let X be a smooth complex manifold and $D \subset X$ be a normal crossings divisor; i.e., locally around every point $p \in X$ there are holomorphic coordinates $(x_1, \dots, x_n)$, with $n = \dim_{\mathbb{C}} X$, such that

$$D \equiv (x_1 \cdots x_k = 0) \subset X \quad \text{for some } k \leq n.$$

In such coordinates, the sheaf $\mathcal{T}X$ of holomorphic sections of the complex tangent bundle TX is generated by

$$\partial_{x_1}, \dots, \partial_{x_n}$$

and the **log tangent sheaf** $\mathcal{T}X(-\log D)$ is the sub-sheaf generated by

$$x_1 \partial_{x_1}, \dots, x_k \partial_{x_k}, \quad \partial_{x_{k+1}}, \dots, \partial_{x_n}.$$

It is dual to the sheaf $\Omega_X^1(\log D)$ of meromorphic 1-forms with at most simple poles along D_i . Since $\mathcal{T}X(-\log D)$ is locally free, it is the sheaf of holomorphic sections of a holomorphic vector bundle $TX(-\log D)$. The inclusion $\mathcal{T}X(-\log D) \subset \mathcal{T}X$ gives rise to a holomorphic homomorphism

$$\iota: TX(-\log D) \longrightarrow TX$$

which is an isomorphism away from D .

Given an SNC symplectic divisor $D = \bigcup_{i \in [N]} D_i$ in (X, ω) , an ω -regularization $\mathcal{R}$ for D in X as in Definition 2.3 gives rise to a real rank- $2n$ vector bundle

$$TX(-\log D)_{\mathcal{R}} \longrightarrow X$$

satisfying

$$TX(-\log D)_{\mathcal{R}}|_{D_I} = TD_I(-\log \partial D_I) \oplus D_I \times \mathbb{C}^I, \quad \forall I \subset [N],$$

where $\partial D_I \subset D_I$ is the SNC divisor in (2.5). As in the holomorphic case, there exists a canonical homomorphism

$$\iota: TX(-\log D)_{\mathcal{R}} \longrightarrow TX \quad (2.10)$$

which is an isomorphism away from D . Furthermore, any $\mathcal{R}$ -compatible almost complex structure J on TX gives rise to a similarly denoted complex structure on $TX(-\log D)_{\mathcal{R}}$ such that the homomorphism (2.10) is complex linear. The deformation equivalence class $TX(-\log D)$ of the complex vector bundle $(TX(-\log D)_{\mathcal{R}}, J)$, which we call the **log tangent bundle of (X, D)** , only depends on the deformation equivalence class of (X, D, ω) . Furthermore,

$$c(TX(-\log D)) = c(TX) / \prod_{i \in [N]} (1 + \text{PD}(D_i));$$

see [13, 15]. In particular,

$$c_1^{TX(-\log D)} = c_1^{TX} - \sum_{i \in [N]} \text{PD}(D_i).$$

For a smooth divisor $D \subset X$, with notation as in (2.4), we have

$$\begin{aligned} TX(-\log D)_{\mathcal{R}} &= (\Psi^{-1} * (\pi^* TD \oplus \mathcal{N}'_X D \times \mathbb{C}) \sqcup TX|_{X-D}) / \sim, \\ \Psi^{-1} * (\pi^* TD \oplus \mathcal{N}'_X D \times \mathbb{C})|_{\Psi(\zeta)} &\ni \xi \oplus c \sim d_{\zeta} \Psi(\xi + c\zeta) \in T_{\Psi(\zeta)}(X-D), \end{aligned}$$

where in the last equation, via the isomorphism

$$T_{\zeta} \mathcal{N}_X D \cong (\pi^* TD \oplus \pi^* \mathcal{N}_X D)|_{\zeta}$$

given by the connections in $\mathcal{R}$, we think of $\xi + c\zeta$ as a tangent vector in $T_{\zeta} \mathcal{N}_X D$, for all $\zeta \in \mathcal{N}_X D$. In the general SNC case, with notation as in Sect. 2.1, we have

$$TX(-\log D)_{\mathcal{R}} = \left(\bigsqcup_{I \subset [N]} \Psi_I^{-1} * (\pi_I^* TD_I|_{\mathcal{N}_X^{\circ} D_I} \oplus \mathcal{N}_X^{\circ} D_I \times \mathbb{C}^I) \right) / \sim \quad (2.11)$$

where

$$\mathcal{N}_X^\circ D_I = \pi_I^{-1}(D_I - \partial D_I) \cap \text{Dom}(\Psi_I)$$

and the identification maps that give rise to the equivalence relations $\sim$ on the overlaps are given by

$$\begin{aligned} & \left(\pi_{I'}^* T D_I|_{\mathcal{N}_X^\circ D_I} \oplus \mathcal{N}_X^\circ D_I \times \mathbb{C}^I \right) \Big|_{\zeta} \ni (\xi \oplus (c_i)_{i \in I}) \\ & \longrightarrow (\tilde{\xi} \oplus (c_i)_{i \in I'}) \in \left(\pi_{I'}^* T D_{I'}|_{\mathcal{N}_X^\circ D_{I'}} \oplus \mathcal{N}_X^\circ D_{I'} \times \mathbb{C}^{I'} \right) \Big|_{\tilde{\zeta}}, \quad \forall I' \subset I \subset [N], \end{aligned}$$

with

$$\begin{aligned} p & \in D_I - \partial D_I, \quad \zeta = (\zeta_i)_{i \in I} \in \mathcal{N}_X^\circ D_I|_p, \quad q = \Psi_{I,I'}(p, (\zeta_i)_{i \in I-I'}) \in D_{I'}, \\ \tilde{\zeta} & = \mathfrak{D}\Psi_{I,I'}((p, (\zeta_i)_{i \in I-I'}); (\zeta_j)_{j \in I'}) \in \mathcal{N}_X^\circ D_{I'}|_q, \\ \tilde{\xi} & = d\Psi_{I,I'}|_{(\zeta_i)_{i \in I-I'}} \left(\xi + \sum_{i \in I-I'} c_i \zeta_i \right) \in T_q D_{I'} \end{aligned}$$

on the overlap; see [12, Sect. 2.2] or [15, Sect. 3.3]. For each $\mathcal{R}$ -compatible J , on the local chart

$$\pi_I^* T D_I|_{\mathcal{N}_X^\circ D_I} \oplus \mathcal{N}_X^\circ D_I \times \mathbb{C}^I \longrightarrow \mathcal{N}_X^\circ D_I, \quad (2.12)$$

the complex structure is given by the pull back of $J|_{T D_I}$ on the first summand and the trivial complex structure on the second summand. Furthermore, the $\mathbb{C}$ -linear homomorphism (2.10) is given by

$$\begin{aligned} & \left(\pi_I^* T D_I|_{\mathcal{N}_X^\circ D_I} \oplus \mathcal{N}_X^\circ D_I \times \mathbb{C}^I \right) \Big|_{\zeta} \ni (\xi \oplus (c_i)_{i \in I}) \\ & \longrightarrow d_\zeta \Psi_I \left(\xi \oplus \sum_{i \in I} c_i \zeta_i \right) \in T_{\Psi_I(\zeta)} X, \end{aligned} \quad (2.13)$$

where, again, via the identifications (2.8) and (2.1), for each

$$\zeta \cong (\zeta_i)_{i \in I} \in \mathcal{N}_X D_I \cong \bigoplus_{i \in I} \mathcal{N}_X D_i$$

we think of $\xi \oplus \sum c_i \zeta_i$ as a tangent vector in $T_\zeta \mathcal{N}_X D_I$.

Let h be a Hermitian metric on $TX(-\log D)_{\mathcal{R}}$. For example, at the cost of shrinking $\mathcal{R}$, one can construct h so that on the chart (2.12) it is the direct sum of the standard Hermitian metric on $\mathcal{N}_X^\circ D_I \times \mathbb{C}$ and the pull back of some Hermitian metric from D_I on the first summand. For every $I \subset [N]$, via the inclusion

$$T D_I(-\log \partial D_I) \subset TX(-\log D)|_{D_I}$$

and the identification

$$TD_I(-\log \partial D_I)|_{D_I - \partial D_I} = T(D_I - \partial D_I),$$

h induces a complete Hermitian metric on $D_I - \partial D_I$. Let

$$\exp_I : T(D_I - \partial D_I) \longrightarrow D_I - \partial D_I$$

be the exponential map of h_I . Define

$$\begin{aligned} \exp^{\log} : TX(-\log D)_{\mathcal{R}} &\longrightarrow X, \quad (\xi \oplus c)|_p \longmapsto \exp_I(p, \xi) \in D_I - \partial D_I, \\ \forall I \subset [N], \quad p \in D_I - \partial D_I, \quad \xi \oplus c \in T_p X(-\log D)_{\mathcal{R}} &\cong T_p D_I \oplus \mathbb{C}^I. \end{aligned} \quad (2.14)$$

This map is smooth and it is a logarithmic version of the exponential map in the classical sense. In fact, for any

$$p \in D_I - \partial D_I \quad \text{and} \quad \zeta = (\zeta_i)_{i \in I} \in \mathcal{N}_X^\circ D_I|_p,$$

via the identification (2.13), the map $\exp^{\log}$ is approximately (to the first order in $|\zeta|$) equal to

$$(\zeta, \xi \oplus c) \longrightarrow \left(\exp_I(\xi), \text{Pal}((e^{c_i} \zeta_i)_{i \in I}) \right) \in \mathcal{N}_X^\circ D_I, \quad (2.15)$$

where $\text{Pal}((e^{c_i} \zeta_i)_{i \in I})$ is the parallel translation of the vector $(e^{c_i} \zeta_i)_{i \in I} \in \mathcal{N}_X^\circ D_I$ along the path $\exp_I(t\xi)|_{t \in [0,1]}$ in D_I . Putting $\zeta = 0$, (2.15) becomes $\exp_I$ at p . This logarithmic exponential map will be used in constructing a Banach neighborhood of a (J, ν) -holomorphic map in the space of all smooth maps of the same tangency order type.

3 Moduli Spaces of Log (J, ν) -Curves

In this section, following the description of [32, Sect. 2] and [35, Sect. 2.1], we define a suitable space of perturbation terms ν over a “regular covering” of the Deligne–Mumford space $\overline{\mathcal{M}}_{g,k}$ for any symplectic logarithmic pair (X, D) . Then, following and generalizing the definition of log J -holomorphic curves in [7], we introduce the notion of log (J, ν) -holomorphic curve.

3.1 Regular Coverings

For $g, k \in \mathbb{N}$ with $2g + k \geq 3$, let

$$\pi : \overline{\mathcal{U}}_{g,k} \equiv \overline{\mathcal{M}}_{g,k+1} \longrightarrow \overline{\mathcal{M}}_{g,k} \quad (3.1)$$

be the universal curve, where π is defined by forgetting the last marked point. For a marked curve $C = [\Sigma, j, z_1, \dots, z_k]$ in $\overline{\mathcal{M}}_{g,k}$, if the automorphism group $\text{Aut}(C)$ is non-trivial, then $\pi^{-1}(C) = \Sigma/\text{Aut}(C)$ instead of Σ . Therefore, unless $g = 0$, (3.1) is not a universal family and we can not directly define the perturbation term ν over $\overline{\mathcal{U}}_{g,k}$. One can resolve this problem by taking appropriate finite covers of (3.1).

Denote by $\mathcal{T}_{g,k}$ the Teichmüller space of genus g Riemann surfaces with k marked points (punctures) and by $\mathcal{G}_{g,k}$ the corresponding mapping class group. We have

$$\mathcal{M}_{g,k} = \mathcal{T}_{g,k}/\mathcal{G}_{g,k}.$$

Assume $g = g_1 + g_2$ and $k = k_1 + k_2$ with $2g_i + k_i \geq 3$ for $i = 1, 2$. For any decomposition $S_1 \cup S_2$ of $[k]$ with $|S_i| = k_i$, there exists a canonical immersion

$$\iota = \iota_{S_1, S_2}: \overline{\mathcal{M}}_{g_1, k_1+1} \times \overline{\mathcal{M}}_{g_2, k_2+1} \longrightarrow \partial \overline{\mathcal{M}}_{g,k} \quad (3.2)$$

which assigns to a pair of marked curves

$$(C_i = [\Sigma_i, j_i, z_{i,1}, \dots, z_{i, k_i+1}])_{i=1,2},$$

the marked curve

$$C = [\Sigma, j, z_1, \dots, z_k], \quad \Sigma = \Sigma_1 \sqcup \Sigma_2 / z_{1, k_1+1} \sim z_{2, k_2+1}, \\ \{z_1, \dots, z_k\} = \{z_{1,1}, \dots, z_{1, k_1}\} \cup \{z_{2,1}, \dots, z_{2, k_2}\},$$

so that the remaining marked points are renumbered by $\{1, \dots, k\}$ according to the decomposition $S_1 \cup S_2$. There is also another natural immersion

$$\delta: \overline{\mathcal{M}}_{g-1, k+2} \longrightarrow \partial \overline{\mathcal{M}}_{g,k} \quad (3.3)$$

which is obtained by gluing together the last two marked points.

Definition 3.1 ([35, Def. 2.1]) Let $g, k \in \mathbb{N}$ with $2g + k \geq 3$, and

$$p: \overline{\mathfrak{M}}_{g,k} \longrightarrow \overline{\mathcal{M}}_{g,k} \quad (3.4)$$

be a finite branched cover in the orbifold category. A **universal family over** $\overline{\mathfrak{M}}_{g,k}$ is a tuple

$$\left(\pi: \overline{\mathfrak{U}}_{g,k} \longrightarrow \overline{\mathfrak{M}}_{g,k}, \mathfrak{z}_1, \dots, \mathfrak{z}_k \right) \quad (3.5)$$

where $\overline{\mathfrak{U}}_{g,k}$ is a complex projective variety and π is a projective morphism with disjoint sections $\mathfrak{z}_1, \dots, \mathfrak{z}_k$ such that for each $c \in \overline{\mathfrak{M}}_{g,k}$ the tuple

$$C = ((\Sigma, j) = \pi^{-1}(c), \vec{\mathfrak{z}}(c) = (\mathfrak{z}_1(c), \dots, \mathfrak{z}_k(c)))$$

is a stable k -marked genus g curve whose equivalence class is $[C] = p(c)$.

Definition 3.2 Let $g, k \in \mathbb{N}$ with $2g+k \geq 3$. A cover (3.4) is **regular** if

- (1) it admits a universal family,
- (2) each topological component of $p^{-1}(\mathcal{M}_{g,k})$ is the quotient of $\mathcal{T}_{g,k}$ by a subgroup of $\mathcal{G}_{g,k}$,
- (3) for each boundary divisor (3.2) we have

$$(\overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1}) \times_{(\iota,p)} \overline{\mathfrak{M}}_{g,k} \approx \overline{\mathfrak{M}}_{g_1,k_1+1} \times \overline{\mathfrak{M}}_{g_2,k_2+1},$$

for some regular covers $\overline{\mathfrak{M}}_{g_i,k_i+1}$ of $\overline{\mathcal{M}}_{g_i,k_i+1}$, and

- (4) for the boundary divisor (3.3) we have

$$\overline{\mathcal{M}}_{g-1,k+2} \times_{(\delta,p)} \overline{\mathfrak{M}}_{g,k} \approx \overline{\mathfrak{M}}_{g-1,k+2},$$

for some regular cover $\overline{\mathfrak{M}}_{g-1,k+2}$ of $\overline{\mathcal{M}}_{g-1,k+2}$.

The last two conditions are inductively well-defined. This definition is a modified version of [35, Def. 2.2]. In [35, Def. 2.2], the last condition is missing; furthermore, $\overline{\mathfrak{M}}_{g_i,k_i+1}$ and $\overline{\mathfrak{M}}_{g-1,k+2}$ are only required to be “some” cover of $\overline{\mathcal{M}}_{g_i,k_i+1}$ and $\overline{\mathcal{M}}_{g-1,k+2}$, respectively. The existence of such regular covers is a consequence of [2, Prop. 2.2, Thm. 2.3, Thm. 3.9]; see also moduli space of curves with *level n structures* in [29, p. 285]. In the genus 0 case, for each $k \geq 3$, the moduli space $\overline{\mathcal{M}}_{0,k}$ itself is smooth and the universal curve (3.1) is already a universal family. The regular covers are only branched over the boundaries of the moduli space. Furthermore, the total space of a universal family as in (3.5) over a regular cover only has singularities of the form

$$\{(x, y, t) \in \mathbb{C}^3 : xy = t^m\} \longrightarrow \mathbb{C}, \quad (x, y, t) \longmapsto t$$

at the nodal points of the fibers of π . In the approach of [32], for dealing with such singularities they consider embeddings of a universal family into $\mathbb{P}^N$ for sufficiently large N . In this article, following [23, 35], we consider perturbations supported away from the nodes.

3.2 Logarithmic Ruan–Tian Perturbations

Let $g, k \in \mathbb{N}$ with $2g+k \geq 3$ and fix a regular covering (3.4) and a universal family (3.5). Denote by

$$\overline{\mathfrak{U}}_{g,k}^* \subset \overline{\mathfrak{U}}_{g,k}$$

the complement of the nodes of the fibers of the projection map π in (3.5). Denote by

$$T_{g,k} = \text{Ker } d(\pi|_{\overline{\mathfrak{U}}_{g,k}^*}) \longrightarrow \overline{\mathfrak{U}}_{g,k}^*$$

the vertical tangent bundle. The latter is a complex line bundle; we denote the complex structure by $j_{\mathcal{U}}$. Then

$$\Omega_{g,k}^{0,1} := (T_{g,k}, -j_{\mathcal{U}})^* \longrightarrow \overline{\mathcal{U}}_{g,k}^*$$

is the complex line bundle of vertical $(0, 1)$ -forms. It is possible to extend this construction to the nodal points by allowing simple poles and dual residues, or by embedding $\overline{\mathcal{U}}_{g,k}$ into some $\mathbb{P}^M$ as in [32].

Let (X, ω) be a symplectic manifold and J be an ω -tame almost complex structure on X . The classical space of perturbations considered in [32] (following the modification in [35]) is the infinite dimensional linear space

$$\begin{aligned} \mathcal{H}_{g,k}(X, J) = \left\{ v \in \Gamma \left(\overline{\mathcal{U}}_{g,k}^* \times X, \pi_1^* \Omega_{g,k}^{0,1} \otimes_{\mathbb{C}} \pi_2^* T X \right) \right. \\ \left. \text{s.t. } \text{supp}(v) \subset \left(\overline{\mathcal{U}}_{g,k}^* - \bigcup_{a \in [k]} \text{Im}(\mathfrak{J}_a) \right) \times X \right\}, \end{aligned} \quad (3.6)$$

where π_1, π_2 are projection maps from $\overline{\mathcal{U}}_{g,k}^* \times X$ onto the first and second components, respectively, and $\text{supp}(v)$ is the closure of the complement of the vanishing locus of v in the compact space $\overline{\mathcal{U}}_{g,k} \times X$. Let $\mathcal{H}_{g,k}(X, \omega)$ denote the space of tuples (J, v) where J is ω -tame and $v \in \mathcal{H}_{g,k}(X, J)$. Note that given v and a boundary component as in Definition 3.2 (3) (resp. Definition 3.2 (4)), the restriction of v to $\overline{\mathcal{M}}_{g_1, k_1+1}$ gives a perturbation term in $\mathcal{H}_{g_1, k_1}(X, J)$ (resp. $\mathcal{H}_{g-1, k+2}(X, J)$).

Definition 3.3 Suppose $g, k \in \mathbb{N}$ with $2g+k \geq 3$, $\overline{\mathcal{U}}_{g,k}$ is a universal family as in (3.5), (X, ω) is a symplectic manifold, $A \in H_2(X, \mathbb{Z})$, and $(J, v) \in \mathcal{H}_{g,k}(X, \omega)$. A **degree A genus g k -marked (J, v) -map** is a tuple

$$f = \left(\phi, u, (\Sigma, j, (z_a)_{a \in [k]}) \right) \quad (3.7)$$

where $(\Sigma, j, (z_a)_{a \in [k]})$ is a nodal genus g k -marked complex curve, $\phi : \Sigma \rightarrow \overline{\mathcal{U}}_{g,k}$ is a holomorphic map onto a fiber of $\overline{\mathcal{U}}_{g,k}$ preserving the marked points, and $u : \Sigma \rightarrow X$ represents the homology class A and satisfies

$$\bar{\partial}u = (\phi, u)^* v.$$

Two k -marked (J, v) -holomorphic maps

$$\left(\phi_1, u_1, (\Sigma_1, j_1, (z_{1,a})_{a \in [k]}) \right) \quad \text{and} \quad \left(\phi_2, u_2, (\Sigma_2, j_2, (z_{2,a})_{a \in [k]}) \right)$$

are **equivalent** if there exists a holomorphic identification

$$h : \Sigma_1 \longrightarrow \Sigma_2, \quad h(z_{1,a}) = z_{2,a}, \quad \forall a \in [k],$$

such that $(\phi_1, u_1) = (\phi_2, u_2) \circ h$. A (J, ν) -holomorphic map is **stable** if it has a finite automorphism group. For any fixed J , denote by

$$\overline{\mathcal{M}}_{g,k}(X, A, \nu)$$

the space of equivalence classes of k -marked genus g degree A stable (J, ν) -holomorphic maps. The subspace of maps with smooth domain is denoted by $\mathcal{M}_{g,k}(X, A, \nu)$. A **contracted** component of Σ in (3.7) is a smooth component whose image under the map ϕ is just a point. A map (3.7) is stable if and only if the degree of the restriction of u to every contracted component of Σ containing only one or two special (nodal or marked) points is not zero. If (3.7) is stable, every connected cluster of contracted components is a tree of spheres, with a total of at most 2 special¹² points, at least one of which is a nodal point. For generic ν , the only components of Σ contributing non-trivially to the automorphism group of (3.7) are the contracted components.

Definition 3.4 A genus 0 J -holomorphic map f (with a nodal domain) is called simple if the restriction u_v of u to each irreducible component $\Sigma_v \cong \mathbb{P}^1$ is not multiply covered, whenever u_v is not constant, and the images of two such components in X are distinct. A k -marked (J, ν) -holomorphic map f as in (3.7) is called **simple** if the restriction u_v of u to each irreducible component Σ_v of Σ contracted by ϕ is not multiply covered (or equally it is somewhere injective) whenever¹³ u_v is not constant, and the images of two such components in X are distinct.

If J is an ω -tame almost complex structure on X , let ∇ be the Levi-Civita connection of the metric $\langle u, v \rangle = \frac{1}{2}(\omega(u, Jv) + \omega(v, Ju))$ and

$$\tilde{\nabla}_v \zeta = \nabla_v \zeta - \frac{1}{2} J(\nabla_v J) \zeta = \frac{1}{2} (\nabla_v \zeta - J \nabla_v (J \zeta)), \quad \forall v \in TX, \zeta \in \Gamma(X, TX) \quad (3.8)$$

be the associated Hermitian connection. The torsion T of the modified $\mathbb{C}$ -linear connection

$$\hat{\nabla}_v \zeta = \tilde{\nabla}_v \zeta - A(\zeta)v, \quad A(\zeta) = \frac{1}{4}(\nabla_{J\zeta} J + J \nabla_{\zeta} J), \quad \forall v \in TX, \zeta \in \Gamma(X, TX) \quad (3.9)$$

is related to the Nijenhuis tensor normalized as in [26, p. 18] by

$$T_{\hat{\nabla}}(v, w) = -\frac{1}{4} N_J(v, w), \quad \forall v, w \in TX.$$

¹² Either a marked point or a nodal point connecting the cluster to an irreducible component of Σ outside the cluster.

¹³ This is automatically satisfied if f is stable.

If J is ω -compatible, $\tilde{\nabla}$ coincides with $\widehat{\nabla}$. See [26, Chapter 3.1 and Appendix C] for details. By [26, (3.1.6)], for a (J, ν) -holomorphic map (ϕ, u) , the linearization of $\bar{\partial} - \nu$ has the form

$$\begin{aligned} D_u\{\bar{\partial} - \nu\}: \Gamma(\Sigma, u^*TX) &\longrightarrow \Gamma(\Sigma, \Omega_{\Sigma, j}^{0,1} \otimes_{\mathbb{C}} u^*TX), \\ D_u\{\bar{\partial} - \nu\}(\zeta) &= (\widehat{\nabla}\zeta)^{(0,1)} + \frac{1}{4}N_J(\zeta, du) - \tilde{\nabla}_{\zeta}\nu + B(\nu)\zeta, \end{aligned} \quad (3.10)$$

where

$$B(\nu) = \frac{1}{4}(J\nabla_{\nu}J + \nabla_{J\nu}J)$$

and

$$(\widehat{\nabla}\zeta)^{(0,1)} = \frac{1}{2}(\widehat{\nabla}\zeta + J\widehat{\nabla}\zeta \circ j)$$

is the $(0, 1)$ -part of $\widehat{\nabla}\zeta$, see [26, Chapter 3.1]. The last term in (3.10) is zero if J is ω -compatible.

In the relative Gromov–Witten theory (i.e., when D is smooth), the most general almost complex structure that one may consider is an ω -tame or compatible almost complex structures J on X that preserves TD and satisfies the Nijenhuis condition (2.6) along D . For such an ω -tame almost complex structure J , the most general perturbation term ν in $\mathcal{H}_{g,k}(X, J)$ that one may consider is one satisfying

$$\begin{aligned} \nu|_{\overline{\mathcal{M}}_{g,k} \times D} &\in \mathcal{H}_{g,k}(D, J|_{TD}) \quad \text{and} \\ \frac{1}{2}(J\nabla_{\nu}J + \nabla_{J\nu}J)w - (\tilde{\nabla}_w\nu + J\tilde{\nabla}_{Jw}\nu) &\in \Omega_{g,k}^{0,1} \otimes_{\mathbb{C}} T_x D, \quad \forall x \in D, w \in T_x X; \end{aligned} \quad (3.11)$$

see [16, 17, 21]. The first parenthesis in the second line of (3.11) is zero if J is ω -compatible. If the image of u lies in D , by the first condition in (3.11), $D_u\{\bar{\partial} - \nu\}$ induces an operator

$$D_u^{\mathcal{N}}\{\bar{\partial} - \nu\}: \Gamma(\Sigma, u^*\mathcal{N}_X D) \longrightarrow \Gamma(\Sigma, \Omega_{\Sigma, j}^{0,1} \otimes_{\mathbb{C}} u^*\mathcal{N}_X D). \quad (3.12)$$

The second condition in (3.11) together with (2.6) imply that (3.12) is $\mathbb{C}$ -linear (i.e., it is a $\bar{\partial}$ -operator); see [21] and [17, (2.14)]. Furthermore,

$$D_u^{\mathcal{N}}\{\bar{\partial} - \nu\}([\zeta]) = \left[(\widehat{\nabla}\zeta)^{(0,1)} - \frac{1}{2}(\tilde{\nabla}_{\zeta}\nu - J\tilde{\nabla}_{J\zeta}\nu) \right]^{\mathcal{N}}, \quad \forall [\zeta]^{\mathcal{N}} \in \Gamma(\Sigma, u^*\mathcal{N}_X D), \quad (3.13)$$

where $[\zeta]^\mathcal{N}$ denotes the image in $\Gamma(\Sigma, u^*\mathcal{N}_X D)$ of a section $\zeta \in \Gamma(\Sigma, u^*TX)$. If $v \equiv 0$, $D_u^{\mathcal{N}} \bar{\partial}$ is simply $u^* \bar{\partial}_{\mathcal{N}_X D}$ where

$$\bar{\partial}_{\mathcal{N}_X D}: \Gamma(D, \mathcal{N}_X D) \longrightarrow \Gamma(D, \Omega_{D,J}^{0,1} \otimes_{\mathbb{C}} \mathcal{N}_X D)$$

is the $\bar{\partial}$ -operator associated to J in [7, Lem. 2.1].

Generalizing (3.11) to the SNC case, the most general pairs (J, v) that one may consider are those satisfying (2.6) and (3.11) along each D_i . In other words, we need v to satisfy

$$\begin{aligned} v|_{\bar{\Omega}_{g,k} \times D_I} &\in \mathcal{H}_{g,k}(D_I, J_I), \quad \forall I \subset [N] \quad \text{and} \\ \frac{1}{2}(J\nabla_v J + \nabla_{Jv} J)w - (\tilde{\nabla}_w v + J\tilde{\nabla}_{Jw} v) &\in \Omega_{g,k}^{0,1} \otimes_{\mathbb{C}} T_x D_I, \quad \forall x \in D_I, \quad w \in T_x X. \end{aligned} \quad (3.14)$$

For (J, v) satisfying the first condition of (3.14), if (ϕ, u) is a (J, v) -holomorphic map as in (3.7) with smooth domain, for every $i \in [N]$, either $\text{Im}(u) \subset D_i$ or u intersects D_i at finitely many points with positive tangency orders; see [17, p. 10]. Therefore, there exists a maximal subset $I \subset [N]$ (called **depth** of u) such that $\text{Im}(u) \subset D_I$ and u intersects every D_i , with $i \notin I$, non-negatively. If the image of u lies in D_I , $D_u\{\bar{\partial} - v\}$ induces $\mathbb{C}$ -linear $\bar{\partial}$ -operators

$$D_u^{\mathcal{N}_i}\{\bar{\partial} - v\}: \Gamma(\Sigma, u^*\mathcal{N}_X D_i) \longrightarrow \Gamma(\Sigma, \Omega_{\Sigma,j}^{0,1} \otimes_{\mathbb{C}} u^*\mathcal{N}_X D_i), \quad \forall i \in I. \quad (3.15)$$

Meromorphic sections defined with respect to these $\bar{\partial}$ -operators will be used to define $\log(J, v)$ -holomorphic curves.

Working with the conditions in (3.14) is hard. *In the following, instead of imposing the conditions (3.14) on v , we define a class of logarithmic perturbation terms $v_{\log}$, associated to each of which we get a classical perturbation term v satisfying (3.14).*

Let D be an SNC symplectic divisor in (X, ω) , $\mathcal{R}$ be an ω -regularization, and J be an ω -tame and $\mathcal{R}$ -compatible almost complex structure J on X . In Sect. 4.1, we will define a class of smooth maps $u: \Sigma \rightarrow X$ containing representatives of $\mathcal{M}_{g,s}(X, D, A)$ for which $\bar{\partial}u$ lifts to a log CR section

$$\bar{\partial}^{\log} u \in \Gamma(\Sigma, \Omega_{\Sigma,j}^{0,1} \otimes_{\mathbb{C}} u^*TX(-\log D)). \quad (3.16)$$

In comparison with (3.6), (3.16) indicates that the right set of perturbation terms for the log moduli spaces is (a subspace of)

$$\begin{aligned} \left\{ v_{\log} \in \Gamma(\bar{\Omega}_{g,k}^* \times X, \pi_1^* \Omega_{g,k}^{0,1} \otimes_{\mathbb{C}} \pi_2^* TX(-\log D)_{\mathcal{R}}) \right. \\ \left. \text{s.t. } \text{supp}(v_{\log}) \subset \left(\bar{\Omega}_{g,k}^* - \bigcup_{a \in [k]} \text{Im}(\mathfrak{z}_a) \right) \times X \right\}. \end{aligned} \quad (3.17)$$

Associated to each $v_{\log}$ we get a classical perturbation term

$$\nu = \iota(v_{\log}) \in \mathcal{H}_{g,k}(X, J), \quad (3.18)$$

where by abuse of notation ι denotes the $\mathbb{C}$ -linear homomorphism

$$\pi_1^*(\Omega_{g,k}^{0,1}) \otimes_{\mathbb{C}} \pi_2^*TX(-\log D)_{\mathcal{R}} \longrightarrow \pi_1^*(\Omega_{g,k}^{0,1}) \otimes_{\mathbb{C}} \pi_2^*TX$$

induced by (2.10). Conversely, we may think of log perturbations as those ν that lift to a section of

$$\pi_1^*\Omega_{g,k}^{0,1} \otimes_{\mathbb{C}} \pi_2^*TX(-\log D)_{\mathcal{R}}.$$

Lemma 3.5 *For every $v_{\log}$ in (3.17), the associated classical perturbation term $\nu = \iota(v_{\log})$ satisfies (3.14).*

Proof Given a logarithmic perturbation term $v_{\log}$ as in (3.17), for each $I \subset [N]$, restricted to the neighborhood $\Psi_I(\mathcal{N}'_X D_I) \subset X$ of D_I and with respect to the decomposition

$$\Psi_I^*TX(-\log D)_{\mathcal{R}} \cong \pi_I^*TD_I(-\log \partial D_I) \oplus \mathcal{N}'_X D_I \times \mathbb{C}^I$$

we get a decomposition

$$v_{\log, I} \equiv \Psi_I^*v_{\log} = \tilde{v}_{I, \log} \oplus \tilde{\theta}_I \quad (3.19)$$

where

$$\tilde{\theta}_I = (\tilde{\theta}_{I,i})_{i \in I} \in \Gamma(\overline{\Omega}_{g,k}^{\star} \times \mathcal{N}'_X D_I, \pi_1^*\Omega_{g,k}^{0,1} \otimes_{\mathbb{C}} \mathbb{C}^I)$$

is a tuple of $(0, 1)$ -forms. From (3.18), (2.13), and the decomposition (2.8) we get

$$\nu_I \equiv \Psi_I^*\nu = \tilde{\nu}_I \oplus n_I, \quad (3.20)$$

where

$$\begin{aligned} n_I|_{v_I}(w) &= (\tilde{\theta}_I|_{v_I}(w)) \cdot v_I \equiv (\tilde{\theta}_{I,i}|_{v_I}(w)v_{I,i})_{i \in I} \in \mathcal{N}_X D_I|_x, \\ \forall x \in D_I, \quad v_I &= (v_{I,i})_{i \in I} \in \mathcal{N}'_X D_I|_x, \quad w \in T_{g,k}. \end{aligned} \quad (3.21)$$

Note that $n_I|_{D_I} = 0$; thus, ν satisfies the first condition in (3.14). Furthermore, by (2.3), (2.7), (3.21), and the first condition in (3.14)

$$J\nabla_{\nu}Jw, \nabla_{J\nu}Jw \in T_x D_I, \quad \forall x \in D_I, \quad w \in T_x X,$$

and, with $\theta_I = \tilde{\theta}_I|_{D_I}$,

$$\begin{aligned} [\tilde{\nabla}_w v]^{\mathcal{N}} &= \theta_I|_x \cdot w_I \in \Omega_{g,k}^{0,1} \otimes_{\mathbb{C}} \mathcal{N}_X D_I|_x, \quad \forall x \in D_I, \\ w &= w^{\text{hor}} \oplus w_I \in T_x X \cong T_x D_I \oplus \mathcal{N}_X D_I|_x. \end{aligned}$$

Therefore, v also satisfies the second condition in (3.14) and, by (3.13), (3.15) has the form

$$\begin{aligned} D_u^{\mathcal{N}_i} \{\bar{\partial} - v\}(\zeta_{I,i}) &= (\widehat{\nabla} \zeta_{I,i})^{(0,1)} - (\phi, u)^* \theta_{I,i} \cdot \zeta_{I,i}, \quad \forall i \in I, \\ \forall \zeta_I &= (\zeta_{I,i})_{i \in I} \in \Gamma(\Sigma, u^* \mathcal{N}_X D) \cong \bigoplus_{i \in I} \Gamma(\Sigma, u^* \mathcal{N}_X D_i). \end{aligned} \quad (3.22)$$

□

Remark 3.6 Similar to [7], $\log(J, v)$ -holomorphic curves can be defined for arbitrary $v_{\log}$ in (3.17). However, as in [7, Remark 1.5], certain steps in the proof of the compactness and the rest of analytical work are complicated for arbitrary such $v_{\log}$ (or arbitrary J satisfying (2.6)). To avoid these complications, almost complex structures and perturbations considered in this paper are rather special. In the definition below, in a neighborhood of each stratum D_I , via the identification map Ψ_I , $\tilde{v}_I$ and $\tilde{\theta}_I$ are required to be pull back via π_I of similar terms along D_I .

Definition 3.7 Given an SNC symplectic divisor $D \equiv \bigcup_{i \in [N]} D_i$ in (X, ω) , an ω -regularization $\mathcal{R}$, an ω -tame and $\mathcal{R}$ -compatible almost complex structure J on X (see Definition 2.5), and a perturbation term $v_{\log}$ in (3.17), we say $v_{\log}$ is $\mathcal{R}$ -compatible if

$$\tilde{v}_{I, \log} = \pi_I^*(v_{I, \log}) \quad \text{and} \quad \tilde{\theta}_I = \pi_I^*(\theta_I), \quad \forall I \subset [N], \quad (3.23)$$

for some

$$v_{I, \log} \in \Gamma(\bar{\mu}_{g,k}^* \times D_I, \pi_1^* \Omega_{g,k}^{0,1} \otimes_{\mathbb{C}} \pi_2^* T D_I (-\log \partial D_I)_{\mathcal{R}_I})$$

and

$$\theta_I = (\theta_{I,i})_{i \in I} \in \Gamma(\bar{\mu}_{g,k}^* \times D_I, \pi_1^* \Omega_{g,k}^{0,1} \otimes_{\mathbb{C}} \mathbb{C}^I). \quad (3.24)$$

The condition (3.23) implies that

$$\begin{aligned} n_I|_{\alpha v}(w) &= \alpha n_I|_v(w) \in \mathcal{N}_X D_I|_x, \\ \forall \alpha &\in (\mathbb{C})^I, \quad x \in D_I, \quad v, \alpha v \in \mathcal{N}'_X D_I|_x, \quad w \in T_{g,k}. \end{aligned} \quad (3.25)$$

In other words, in a neighborhood of each stratum D_I in X , identified with a neighborhood of D_I in $\mathcal{N}_X D_I$ via the regularization map Ψ_I in $\mathcal{R}$, the horizontal component of v is the pull back from D_I of some perturbation term v_I on D_I , and the vertical component of v is $(\mathbb{C}^*)^I$ -equivariant with respect to the component-wise

multiplicative action of $(\mathbb{C}^*)^I$ on $\mathcal{N}_X D_I$. As explained in [17, p. 11], under these assumptions, $\Psi_I^* \nu$ extends to a $(\mathbb{C}^*)^I$ -equivariant perturbation term over the fiber product of $\mathbb{P}^1$ -bundles

$$\prod_{i \in I} \mathbb{P}(\mathcal{N}_X D_i|_{D_I} \oplus \mathbb{C}). \quad (3.26)$$

Define $\mathcal{H}_{g,k}(X, D, \omega)$ (resp. $\mathcal{H}_{g,k}(X, D)$) to be set of such tuples $(\mathcal{R}, J, \nu)$ (resp. $(\omega, \mathcal{R}, J)$) where ν is the perturbation term associated¹⁴ to $\nu_{\log}$ as in Definition 3.7. For any fixed $(\omega, \mathcal{R}, J)$ (resp. $(\omega, \mathcal{R})$), define $\mathcal{H}_{g,k}(X, D)_{\mathcal{R}, J}$ (resp. $\mathcal{H}_{g,k}(X, D)_{\mathcal{R}}$) to be the set of perturbation terms $\nu \in \mathcal{H}_{g,k}(X, J)$ (resp. $(J, \nu) \in \mathcal{H}_{g,k}(X, \omega)$) such that $\nu_{\log}$ is compatible with a shrinking $\mathcal{R}'$ of $\mathcal{R}$. Similar to (2.9), the natural projection map

$$\mathcal{H}_{g,k}(X, D) \longrightarrow \text{Symp}(X, D)$$

is a weak homotopy equivalence.

The first component $(\widehat{\nabla} \zeta_{I,i})^{(0,1)}$ in (3.22) is a $\bar{\partial}$ -operator on $u^* \mathcal{N}_X D_i$ by itself and is independent of ν . In what follows, we will denote it by

$$\bar{\partial}_{u^* \mathcal{N}_X D_i} \zeta_{I,i}.$$

The $\bar{\partial}$ -operators $\bar{\partial}_{u^* \mathcal{N}_X D_i}$ and $D_u^{\mathcal{N}_i} \{\bar{\partial} - \nu\}$ define (usually different) holomorphic structures on the pull-back complex line bundle $u^* \mathcal{N}_X D_i$. The latter is a deformation of the former via the $(0, 1)$ -form $(\phi, u)^* \theta_{I,i}$. The one defined by $\bar{\partial}_{u^* \mathcal{N}_X D_i}$ will be used as a reference. The perturbation caused by $\theta_{I,i}$ will allow us to achieve transversality.

By (2.7), (3.23), and (3.21), the X -component $u : \Sigma \rightarrow X$ of any (J, ν) -holomorphic map (ϕ, u) , with image in a sufficiently small neighborhood of D_I (more precisely, in $\text{Im}(\Psi_I)$) is determined by its projection

$$u_I : \Sigma \longrightarrow D_I \quad (3.27)$$

and a set of sections

$$\zeta_I \equiv (\zeta_{I,i})_{i \in I} \in \bigoplus_{i \in I} \Gamma(u_I^* \mathcal{N}_X D_i) \quad (3.28)$$

such that

$$\bar{\partial} u_I = \nu_I \quad \text{and} \quad \bar{\partial} \zeta_I \equiv \bigoplus_{i \in I} \bar{\partial}_{u_I^* \mathcal{N}_X D_i} \zeta_{I,i} = n_I(u_I, \zeta_I) = ((\phi, u_I)^* \theta_I) \cdot \zeta_I. \quad (3.29)$$

¹⁴ Since the map $\nu_{\log} \mapsto \nu$ is one-to-one, it is safe to use ν in place of $\nu_{\log}$ to keep the notation simple.

The second equation in (3.29) can also be written in the compact form

$$D_{u_I}^{\mathcal{N}_I} \{\bar{\partial} - \nu\}(\zeta_{I,i}) = 0, \quad \forall i \in I. \quad (3.30)$$

By (3.29) or equally (3.30), for any $c_I \equiv (c_i)_{i \in I} \in \mathbb{C}^I$, if (u_I, ζ_I) satisfies (3.29) then

$$(u_I, c_I \cdot \zeta_I = (c_i \zeta_{I,i})_{i \in I})$$

satisfies (3.29) as well. Therefore, in a neighborhood of each stratum D_I in X , identified with a neighborhood of D_I in $\mathcal{N}_X D_I$ via the regularization map Ψ_I in $\mathcal{R}$, the set of (J, ν) -holomorphic maps is invariant under the component-wise multiplicative action of $\mathbb{C}^I$ on $\mathcal{N}_X D_I$. This explains the motivation behind the extra assumption (3.23). In general, as a family $\{u_t\}_{t \rightarrow 0}$ of (J, ν) -holomorphic maps sinks into D_I , the corresponding sections $\zeta_{t,I}$ in (3.28) converge to zero. Then the idea is that, by rescaling $\zeta_{t,I}$ we get a (J_I, ν_I) -holomorphic map u_I with image in D_I and a similarly denoted holomorphic (or meromorphic) section

$$\zeta_I = \lim_{t \rightarrow 0} \frac{\zeta_{t,I}}{|\zeta_{t,I}|} \in u_I^* \mathcal{N}_X D_I$$

that remembers the direction at which the maps have approached D_I . Conversely, in the gluing construction, given (u_I, ζ_I) , gluing is done by pushing u_I out in the direction of ζ_I . These local observations explain the motivation behind Definition 3.9 in the next section.

3.3 Construction of Perturbed Moduli Spaces

Let (X, D) and $(\omega, \mathcal{R}, J, \nu)$ be as in the previous section, with ν coming from an $\mathcal{R}$ -compatible logarithmic perturbation term $\nu_{\log}$ as in Definition 3.7. Suppose

$$u_I: \Sigma \longrightarrow D_I \quad (3.31)$$

is a (J_I, ν_I) -holomorphic map with smooth domain not mapped into ∂D_I (so its depth is I) and

$$\zeta_I \equiv (\zeta_{I,i})_{i \in I} \in \Omega_{\text{mero}} \left(\bigoplus_{i \in I} u_I^* \mathcal{N}_X D_i \right) \quad (3.32)$$

is a tuple of non-zero meromorphic sections with respect to the holomorphic structure defined by

$$(D_u^{\mathcal{N}_I} \{\bar{\partial} - \nu\})_{i \in I} = (\bar{\partial}_{u^* \mathcal{N}_X D_i} - (\phi, u)^* \theta_{I,i})_{i \in I}.$$

In other words, $u \cong (u_I, \zeta_I)$ is as in (3.27) and (3.28), but $\zeta_{I,i}$ are allowed to have poles. For each $x \in \Sigma$, the **tangency order vector**

$$\text{ord}_x(u_I, \zeta_I) = (\text{ord}_x^i(u_I, \zeta_I))_{i \in [N]} \in \mathbb{Z}^N \quad (3.33)$$

is defined by

$$\begin{aligned} \text{ord}_x^i(u_I, \zeta_I) &= \text{ord}_x(u, D_i) \geq 0, \quad \forall i \in [N] - I \\ \text{and } \text{ord}_x^i(u_I, \zeta_I) &= \text{ord}_x \zeta_{I,i}, \quad \forall i \in I. \end{aligned} \quad (3.34)$$

The first item in (3.34) is the order of tangency of u to D_i at x ; this is zero if $u(x) \notin D_i$ and is positive otherwise. The second item in (3.34) is the order of zero/pole of $\zeta_{I,i}$ at x . In particular, it only depends on the $\mathbb{C}^*$ -equivalence class $[\zeta_{I,i}]$ of the section ζ_i ; changing each $\zeta_{I,i}$ with a non-zero constant multiple of that does not change $\text{ord}_x^i(u_I, \zeta_I)$.

Remark 3.8 For each $i \in I$, in a local holomorphic trivialization of $u_I^* \mathcal{N}_X D_i$ around any point $x \in \Sigma$, with respect to the holomorphic structure defined by $\bar{\partial}_{u^* \mathcal{N}_X D_i}$, the second equation in (3.29) has the form

$$\bar{\partial} f_i = \theta_i f_i \quad (3.35)$$

where f_i is a complex valued function and θ_i is a $(0, 1)$ -form. Let g_i be any $\mathbb{C}$ -valued function such that $\bar{\partial} g_i = \theta_i$. Then every solution of (3.35) is of the form

$$f_i = e^{g_i} h_i$$

where $\bar{\partial} h_i \equiv 0$. Thus, $\text{ord}_x f_i$ in (3.34) can also be defined to be the order of meromorphic function h_i at x which is independent of the choice of g_i and hence v . This shows that tangency order remains the same when we deform v .

In order to define nodal (J, v) -holomorphic maps, we use decorated dual graphs of the following sort associated to every k -marked nodal domain $(\Sigma, z_1, \dots, z_k)$ as in [7, Sect. 3.1]. Let $\Gamma = \Gamma(\mathbb{V}, \mathbb{E}, \mathbb{L})$ be a graph with the set of vertices $\mathbb{V}$, edges $\mathbb{E}$, and legs $\mathbb{L}$; the latter, also called flags or roots, are half edges that have a vertex at one end and are open at the other end. Let $\mathbb{E}$ be the set of edges with an orientation. Given $\underline{e} \in \mathbb{E}$, let $\bar{e}$ denote the same edge with the opposite orientation. For each $\underline{e} \in \mathbb{E}$, let $v_1(\underline{e})$ and $v_2(\underline{e})$ in $\mathbb{V}$ denote the starting and ending points of the oriented edge, respectively. For $v, v' \in \mathbb{V}$, let $\mathbb{E}_{v,v'}$ denote the subset of edges between the two vertices and $\mathbb{E}_{v,v'}$ denote the subset of oriented edges from v to v' . For every $v \in \mathbb{V}$, let $\mathbb{E}_v$ denote the subset of oriented edges starting from v . Such decorated graphs Γ characterize different topological types of nodal marked surfaces $(\Sigma, \vec{z} = (z_1, \dots, z_k))$ in the following way. Each vertex $v \in \mathbb{V}$ corresponds to a smooth component Σ_v of Σ with genus g_v . Each edge $e \in \mathbb{E}$ corresponds to a node q_e obtained by connecting Σ_v and $\Sigma_{v'}$ at the points $q_{\bar{e}} \in \Sigma_v$ and $q_e \in \Sigma_{v'}$, where $e \in \mathbb{E}_{v,v'}$ and $\bar{e}$ is an orientation on e with $v_1(\bar{e}) = v$. The

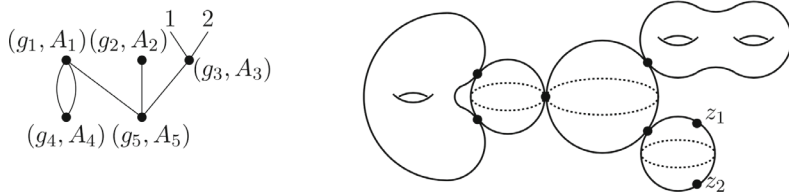


Fig. 1 A nodal curve in $\overline{\mathcal{M}}_{4,2}$ and its dual graph

last condition uniquely specifies e unless e is a loop connecting v to itself. Finally, each leg $l \in \mathbb{L}$ connected to the vertex $v(l)$ corresponds to a marked point $z_{a(l)} \in \Sigma_{v(l)}$ disjoint from the connecting nodes. Thus we have

$$(\Sigma, \vec{z}) = \coprod_{v \in \mathbb{V}} (\Sigma_v, \vec{z}_v \cup q_v) / \sim, \quad q_e \sim q_e, \quad \forall e \in \mathbb{E}, \quad (3.36)$$

where

$$\vec{z}_v = \vec{z} \cap \Sigma_v \quad \text{and} \quad q_v = \{q_e : e \in \mathbb{E}_v\}, \quad \forall v \in \mathbb{V}.$$

We treat q_v as an un-ordered set of marked points on Σ_v . We say Γ is the **decorated dual graph** of $(\Sigma, \vec{z})$. Initially, each vertex $v \in \mathbb{V}$ is decorated by the genus $g_v \in \mathbb{N}$ of the corresponding irreducible component. Further decorations will be introduced below. A complex structure j on Σ is a set of complex structures $(j_v)_{v \in \mathbb{V}}$ on its components. By a (complex) marked nodal curve, we mean a marked nodal real surface together with a complex structure $(\Sigma, j, \vec{z})$. Given a map $u : \Sigma \rightarrow X$, each vertex $v \in \mathbb{V}$ receives an additional decoration that is the homology class $A_v \in H_2(X, \mathbb{Z})$ represented by $u_v = u|_{\Sigma_v}$. Figure 1-Left illustrates a decorated graph with 2 flags and Fig. 1-Right is the associated marked nodal domain with $(g_1, \dots, g_5) = (0, 0, 2, 1, 0)$.

Assume $D = \bigcup_{i \in S} D_i \subset X$ is an SNC symplectic divisor, $(\omega, \mathcal{R}, J, \nu) \in \mathcal{H}_{g,k}(X, D)$, and

$$u = (u_v)_{v \in \mathbb{V}} : (\Sigma, j) \longrightarrow (X, J)$$

is the X -component of a possibly nodal (J, ν) -holomorphic map (ϕ, u) . In this situation, the dual graph of (u, Σ) carries additional labelings

$$I : \mathbb{V}, \mathbb{E} \longrightarrow \text{subsets of } [N], \quad v \longmapsto I_v, \quad \forall v \in \mathbb{V}, \quad e \longmapsto I_e, \quad \forall e \in \mathbb{E}, \quad (3.37)$$

recording the minimal stratum D_I that contains the image of u_v and $u(q_e)$, respectively.

Definition 3.9 Suppose $D = \bigcup_{i \in [N]} D_i \subset X$ is an SNC symplectic divisor, $(\omega, \mathcal{R}, J, \nu) \in \mathcal{H}_{g,k}(X, D)$, and

$$C \equiv (\Sigma, j, \vec{z}) = \left(\coprod_{v \in \mathbb{V}} C_v \equiv (\Sigma_v, j_v, \vec{z}_v \cup q_v) \right) / \sim, \quad q_{\underline{e}} \sim q_{\underline{e}'}, \quad \forall \underline{e} \in \mathbb{E},$$

is a connected nodal k -marked curve with smooth components C_v and dual graph $\Gamma = \Gamma(\mathbb{V}, \mathbb{E}, \mathbb{L})$ as in (3.36). A pre-log (J, ν) -holomorphic map of tangency order type $\mathfrak{s} = (s_a)_{a \in [k]} \in (\mathbb{Z}^N)^k$ from C to X is a collection

$$f \equiv \left(\phi, (f_v \equiv (u_v, \zeta_v, C_v))_{v \in \mathbb{V}} \right), \quad (3.38)$$

such that

- (1) the tuple $(\phi, (u_v, C_v)_{v \in \mathbb{V}})$ is a k -marked genus g degree A (J, ν) -holomorphic map as in (3.7);
- (2) for each $v \in \mathbb{V}$, $(u_v, \zeta_v \equiv (\zeta_{v,i})_{i \in I_v})$ satisfies (3.31)–(3.32) (with $I = I_v$, $u_I = u_v$, $\zeta_I = \zeta_v$);
- (3) for each $v \in \mathbb{V}$, $\text{ord}_{u_v, \zeta_v}$ is **supported** at the special points $z_v \cup q_v$ in the sense that

$$\text{ord}_X(u_v, \zeta_v) \neq 0 \implies x \in z_v \cup q_v;$$

- (4) $s_{\underline{e}} \equiv \text{ord}_{q_{\underline{e}}}(u_v, \zeta_v) = -\text{ord}_{q_{\underline{e}}}(u_{v'}, \zeta_{v'}) \equiv -s_{\underline{e}}$ for all $v, v' \in \mathbb{V}$ and $\underline{e} \in \mathbb{E}_{v,v'}$;
- (5) and $\text{ord}_{z_a}(u_v, \zeta_v) = s_a \in \mathbb{Z}^N$ for all $v \in \mathbb{V}$ and $z_a \in \vec{z}_v$.

In other words, a pre-log map is simply a nodal (J, ν) -map with a bunch of meromorphic sections on each smooth component (with zeros and poles only at the special points), dual tangency orders at the nodes, and prescribed tangency orders at the marked points. Two k -marked pre-log (J, ν) -holomorphic maps

$$f = (\phi, (u_v, \Sigma_v, j_v, \zeta_v = (\zeta_{v,i})_{i \in I_v})_{v \in \mathbb{V}}, \vec{z})$$

$$\text{and } f' = (\phi', (u'_v, \Sigma'_v, j'_v, \zeta'_v = (\zeta'_{v,i})_{i \in I_v})_{v \in \mathbb{V}}, \vec{z}')$$

are equivalent if there exists a holomorphic reparametrization $h: \Sigma \rightarrow \Sigma'$ such that

$$u' \circ h = u, \quad \phi = \phi' \circ h, \quad h(\vec{z}) = \vec{z}', \quad \text{and } h_v^* \zeta'_{h(v),i} = c_{v,i} \zeta_{v,i}$$

for all $v \in \mathbb{V}$ and some $c_{v,i} \in \mathbb{C}^*$. Here $v \rightarrow h(v)$ is the induced map on the vertices of the dual graph. In particular, rescaling any of the meromorphic sections by a non-zero complex number does not change the equivalence class. The space

$$\mathcal{M}_{g,\mathfrak{s}}^{\text{plog}}(X, D, A, \nu)_{\Gamma}$$

of the equivalence classes of stable pre-log maps of the fixed combinatorial type $(g, A, \mathfrak{s}, \Gamma)$ is too big; see [7, Example 2.13]. Similar to [7], in the following, we

will put some restrictions on Γ and take out a subspace that would give us a nice compactification with the correct expected dimension.

In [7, Sect. 5], corresponding to a decorated dual graph $\Gamma = \Gamma(\mathbb{V}, \mathbb{E}, \mathbb{L})$ as in Definition 3.9 and an arbitrary orientation $O = \{e\}_{e \in \mathbb{E}} \subset \mathbb{E}$ on the edges, we constructed a homomorphism of $\mathbb{Z}$ -modules

$$\mathbb{D} = \mathbb{D}(\Gamma) \equiv \mathbb{Z}^{\mathbb{E}} \oplus \bigoplus_{v \in \mathbb{V}} \mathbb{Z}^{I_v} \xrightarrow{\varrho = \varrho_O} \mathbb{T} = \mathbb{T}(\Gamma) \equiv \bigoplus_{e \in \mathbb{E}} \mathbb{Z}^{I_e} \quad (3.39)$$

whose kernel $\mathbb{K}$ and cokernel $\mathbb{CK}$ are independent of the choice of the orientation O on $\mathbb{E}$ and are invariants of the decorated graph Γ . For each $v \in \mathbb{V}$, $e \in \mathbb{E}$, and $s_v \in \mathbb{Z}^{I_v} \in \mathbb{D}$, the e -th component of $\varrho(s_v)$ is equal to $s_v \in \mathbb{Z}^{I_e}$, if v is the starting point of $\vec{e}$ and e is not a loop; $\varrho(s_v)$ is equal to $-s_v \in \mathbb{Z}^{I_e}$, if v is the ending point of $\vec{e}$ and e is not a loop; and, is zero otherwise. In this definition and (3.40), via the identity $I_e = I_v \cup I_{v'}$ (see [7, (2-21)]) for all $e \in \mathbb{E}_{v,v'}$, and the inclusion

$$\mathbb{Z}^{I_v} \cong \mathbb{Z}^{I_v} \times \{0\}^{I_e - I_v} \subset \mathbb{Z}^{I_e},$$

we can think of s_v as a vector also in $\mathbb{Z}^{I_e}$. For each $e \in \mathbb{E}$, and $\lambda_e \in \mathbb{Z}_e \in \mathbb{Z}^{\mathbb{E}} \in \mathbb{D}$, the e -th component of $\varrho(\lambda_e)$ is equal to $\lambda_e s_e$, and the rest are zero. In particular,

$$\mathbb{K} = \left\{ ((\lambda_e)_{e \in \mathbb{E}}, (s_v)_{v \in \mathbb{V}}) \in \mathbb{Z}^{\mathbb{E}} \oplus \bigoplus_{v \in \mathbb{V}} \mathbb{Z}^{I_v} : s_{v'} - s_v = \lambda_e s_e, \quad \forall v, v' \in \mathbb{V}, e \in \mathbb{E}_{v,v'} \right\}. \quad (3.40)$$

In [7, (2-38)], to every (equivalence class of) pre-log J -holomorphic map f we associate a group element

$$\text{obr}(f) \in \mathcal{G}(\Gamma), \quad (3.41)$$

where

$$\mathcal{G}(\Gamma) = \prod_{e \in \mathbb{E}} (\mathbb{C}^*)^{I_e} / \text{Im}(\exp(\varrho_{\mathbb{C}})) \quad (3.42)$$

is the complex torus with Lie algebra $\mathbb{CK}_{\mathbb{C}} = \text{coker}(\varrho_{\mathbb{C}})$, and $\varrho_{\mathbb{C}}$ is the natural extension of ϱ over $\mathbb{C}$. The reasoning leading to [7, (2-38)] also applies to the (J, ν) -holomorphic maps and yields a group element (3.41) for every pre-log (J, ν) -holomorphic map. More explicitly, given

1. local holomorphic coordinates w_e around¹⁵ each nodal point $q_e \in \Sigma_v$, for all $v \in \mathbb{V}$ and $e \in \mathbb{E}_v$,
2. and representatives ζ_v in the $\mathbb{C}^*$ -equivalence class $[\zeta_v]$, for all $v \in \mathbb{V}$,

¹⁵ I.e., $w_e(q_e) = 0$.

for each $\underline{e} \in \mathbb{E}_v$ and $i \in I_e - I_v$, locally around $q_{\underline{e}} \in \Sigma_v$, by (3.27) and (3.28) we have

$$u_v(w_{\underline{e}}) \cong (u_{v,i}(w_{\underline{e}}), \tilde{\eta}_{\underline{e},i}(w_{\underline{e}})w_{\underline{e}}^{s_{\underline{e},i}}) \in \mathcal{N}_X D_i, \quad (3.43)$$

such that

$$0 \neq \eta_{\underline{e},i} \equiv \tilde{\eta}_{\underline{e},i}(0) \in \mathcal{N}_X D_i|_{u_v(q_{\underline{e}})}. \quad (3.44)$$

In other words, $\eta_{\underline{e},i}$ is the $s_{\underline{e},i}$ -th order derivative of u_v with respect to $w_{\underline{e}}$ at $q_{\underline{e}}$ in the normal direction to D_i . Also, for each $\underline{e} \in \mathbb{E}_v$ and $i \in I_v$, locally around $q_{\underline{e}} \in \Sigma_v$, we have

$$\zeta_{v,i}(w_{\underline{e}}) = \tilde{\eta}_{\underline{e},i}(w_{\underline{e}})w_{\underline{e}}^{s_{\underline{e},i}} \in u_v^* \mathcal{N}_X D_i, \quad (3.45)$$

where

$$0 \neq \eta_{\underline{e},i} \equiv \tilde{\eta}_{\underline{e},i}(0) \in \mathcal{N}_X D_i|_{u_v(q_{\underline{e}})}. \quad (3.46)$$

Then, by [7, (2-38)], the class $\text{ob}_{\Gamma}(f)$ of

$$\prod_{e \in \mathbb{E}} \prod_{i \in I_e} \frac{\eta_{\underline{e},i}}{\eta_{\underline{e},i}} \in \prod_{e \in \mathbb{E}} (\mathbb{C}^*)^{I_e} \quad (3.47)$$

in $\mathcal{G}$ is independent of the choice such coordinates in 1 and representatives in 2. In other words, $\text{ob}_{\Gamma}(f) = 1$ if and only if there exists such coordinates in 1 and representatives in 2 such that

$$\eta_{\underline{e},i} = \eta_{\underline{e},i}, \quad \forall e \in \mathbb{E}, i \in I_e. \quad (3.48)$$

This condition will play a major role in the construction of gluing map in [10] in the following way. Forgetting the transversality problem, a gluing theorem would identify a neighborhood of $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\log}(X, D, A, \nu)_{\Gamma}$ in $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\log}(X, D, A, \nu)$ with a neighborhood of that in a product

$$\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\log}(X, D, A, \nu) \times \text{Gl}_{\Gamma},$$

where Gl_{Γ} is an explicit space of gluing parameters. For the classical moduli space of stable maps $\overline{\mathcal{M}}_{g,k}(X, A)$, the space of gluing parameters along the stratum $\overline{\mathcal{M}}_{g,k}(X, A)_{\Gamma}$ is simply $\mathbb{C}^{\mathbb{E}}$, which corresponds to smoothings the nodes of Σ . For a

log map f as in (3.38), the space of gluing parameters is

$$\mathrm{Gl}_\Gamma = \left\{ ((\varepsilon_e)_{e \in \mathbb{E}}, (t_{v,i})_{v \in \mathbb{V}, i \in I_v}) \in \mathbb{C}^{\mathbb{E}} \times \prod_{v \in \mathbb{V}} \mathbb{C}^{I_v} : \right. \\ \left. \varepsilon_e \stackrel{s_{e,i}}{t_{v,i}} t_{v',i} = t_{v',i}, \quad \forall \vec{e} \in \mathbb{E}_{v,v'}, i \in I_e \text{ s.t. } s_{e,i} \geq 0 \right\}. \quad (3.49)$$

Here, ε_e is the gluing parameter at the node q_e and $t_{v,i}$ are the parameters for pushing the map u_v out of D_i along the normal vector field $\zeta_{v,i}$. In [7, Prop. 5.7], we show that Gl_Γ is an affine toric variety corresponding to some explicit toric fan defined by Γ . At each node q_e connecting Σ_v and $\Sigma_{v'}$ with $\mathrm{ord}_{q_e}(u, D_i) = s_{e,i} > 0$, the data on $\Sigma_{v'}$ includes a section $\zeta_{v',i}$ of $u_v^* \mathcal{N}_X D_i$ with a pole of order $s_{e,i}$ at the nodal point $q_e \in \Sigma_{v'}$. Near q_e , and with respect to a local trivialization of $\mathcal{N}_X D_i$, u_v has the product form

$$u_v(z_e) = (\eta_{e,i} \stackrel{s_{e,i}}{z_e}, \bar{u}_v) \in \mathbb{C} \times D_i.$$

On the other hand, $\zeta_{v',i}$ has a local expansion $\zeta_{v',i}(z_e) = \eta_{e,i} \stackrel{s_{e,i}}{z_e} + \dots$. Condition (3.48) is equivalent to the existence of a choice of coordinates z_e and representatives $\zeta_{v,i}$ in their equivalence classes such that

$$\eta_{e,i} \stackrel{s_{e,i}}{z_e} = \eta_{e,i} \stackrel{s_{e,i}}{z_e} \quad (3.50)$$

defines a function from the neck region $z_e z_{\bar{e}} = \varepsilon_e$ of the glued domain into $\mathcal{N}_X D_i$. The so-called regularization maps Ψ_i used in the definition of J identify a neighborhood of the zero section in $\mathcal{N}_X D_i$ and a neighborhood of D_i in X . Thus, we get a J -holomorphic map from the neck region associated to the node q_e into X . We then construct the approximate-gluing log map $\tilde{f}$ in the following way. On each neck region, unlike in the classical gluing construction where the approximate-gluing map is often defined to be constant, we (need to) define the approximate-gluing map to be (3.50) in the i -th direction. Away from the nodes, $\tilde{f}$ is defined to be the push out of u_v via the section $\sum_{i \in I_v} t_{v,i} \zeta_{v,i}$ on the v -th component. In between the two regions, $\tilde{f}$ interpolates between the two maps. Then an argument similar to the classical gluing argument, with the logarithmic linearization of the Cauchy–Riemann operator in place of the classical operator $D\bar{\partial}$ in [26, Chapter 10], allows us to find a log (J, v) -holomorphic map close to $\tilde{f}$.

Remark 3.10 To avoid the compatibility requirement (3.50) and thus avoid the complications of working with the (possibly) singular gluing parameter space (3.49), in [5, Sect. 7], Daemi–Fukaya consider “inconsistent solutions”. These are zero points of certain Kuranishi map that do not correspond to actual maps into any space. The same idea can be followed in the normal crossings case but the space of inconsistent solutions is very hard to describe and control.

Definition 3.11 Suppose $D = \bigcup_{i \in [N]} D_i \subset (X, \omega)$ is an SNC symplectic divisor and $(\omega, \mathcal{R}, J, \nu) \in \mathcal{H}_{g,k}(X, D)$. A **log** (J, ν) -**holomorphic** map is a pre-log (J, ν) -holomorphic map f with the decorated dual graph Γ such that

(1) there exist functions

$$s: \mathbb{V} \longrightarrow \mathbb{R}^N, \quad v \longmapsto s_v, \quad \text{and} \quad \lambda: \mathbb{E} \longrightarrow \mathbb{R}_+, \quad e \longmapsto \lambda_e,$$

such that

- (a) $s_v \in \mathbb{R}_+^{I_v} \times \{0\}^{[N]-I_v}$ for all $v \in \mathbb{V}$,
- (b) $s_{v_2(e)} - s_{v_1(e)} = \lambda_e s_e$ for every $e \in \mathbb{E}$;

(2) and $\text{ob}_\Gamma(f) = 1 \in \mathcal{G}(\Gamma)$.

The first condition is a combinatorial condition on the decorated dual graph Γ and is equal to the condition that the subspace

$$\mathbb{K}_{\mathbb{R}} \subset \mathbb{R}^{\mathbb{E}} \oplus \bigoplus_{v \in \mathbb{V}} \mathbb{R}^{I_v}$$

defined as in (3.40) over $\mathbb{R}$ has a non-empty intersection with the positive quadrant. This implies that

$$\sigma = \mathbb{K}_{\mathbb{R}} \cap \left(\mathbb{R}_{\geq 0}^{\mathbb{E}} \oplus \bigoplus_{v \in \mathbb{V}} \mathbb{R}_{\geq 0}^{I_v} \right) \subset \mathbb{K}_{\mathbb{R}}$$

is a maximal strictly convex rational polyhedral cone; see [7, Lem 5.3]. The affine toric variety associated to σ is the space of gluing parameters (up to some multiplicity) that will be used in [10]; see [7, Sect. 5].

A marked log map is **stable** if it has a finite automorphism group. It is easy to see that a marked log map is stable if and only if the underlying (J, ν) -map is stable. The equivalence class of a stable marked log map is called a **stable marked log curve**. We denote the space of stable k -marked degree A genus g log (J, ν) -holomorphic curves of tangency order type $\mathfrak{s}$ by

$$\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\log}(X, D, A, \nu).$$

Similar to the J -holomorphic case, given $\mathfrak{s} \in (\mathbb{Z}^N)^k$, for every k -marked stable nodal map f representing an element of $\overline{\mathcal{M}}_{g,k}(X, A, \nu)$ with dual graph Γ and a choice of decorations $\{s_e\}_{e \in \mathbb{E}}$ satisfying the necessary combinatorial conditions

$$s_{\bar{e}} = -s_e, \quad \forall e \in \mathbb{E}, \quad \sum_{e \in \mathbb{E}_v} s_e + \sum_{l \in \mathbb{L}_v} s_l = (A_v \cdot D_i)_{i \in [N]}, \quad \forall v \in \mathbb{V}, \quad (3.51)$$

and Definition 3.11 (1), there exists at most one element $f_{\log} \in \overline{\mathcal{M}}_{g, \mathfrak{s}}^{\log}(X, D, A, \nu)$ lifting f with this decorated dual graph. This is because every section $\zeta_{v,i}$ is uniquely determined up to the action of $\mathbb{C}^*$ by the location and order of its zeros/poles. While $f_{\log}$ is stable if and only if f is stable, the automorphism groups are sometimes different; see [7, Examples 2.17–2.18].

Remark 3.12 If $g = 0$, a pre-log lift exists iff there are vectors $\{s_{\mathfrak{e}}\}_{\mathfrak{e} \in \mathbb{E}}$ satisfying the combinatorial condition (3.51) and the lift is unique. In other words,

$$\overline{\mathcal{M}}_{0, \mathfrak{s}}^{\log}(X, D, A, \nu) \subset \overline{\mathcal{M}}_{0, k}(X, A, \nu)$$

is the subset of (J, ν) -curves $(\phi, (u_v, C_v)_{v \in \mathbb{V}})$ for which there exist vectors $\{s_{\mathfrak{e}}\}_{\mathfrak{e} \in \mathbb{E}}$ satisfying (3.51), Definition 3.11 (1), and (2). The first two conditions are combinatorial. The latter is a condition on the derivatives of u at the nodes which depends on u , ν , and the configuration of special points on each component. In theory, it can be stated without mentioning the meromorphic sections.

Example 3.13 Let $g = 0$, $k \geq 3$, $A = 0 \in H_2(X, \mathbb{Z})$, and $\mathfrak{s} = \vec{0} \equiv (0^N)^k \in (\mathbb{N}^N)^k$ (no perturbation). We show that

$$\overline{\mathcal{M}}_{0, \vec{0}}(X, D, 0) \cong \overline{\mathcal{M}}_{0, k} \times X.$$

Every element of $\overline{\mathcal{M}}_{0, \vec{0}}(X, D, 0)$ is (the equivalence class of) a k -marked nodal domain with the constant map into a point $p \in D_I$, for some maximal $I \subset X$, and $|I|$ meromorphic functions on each component. The claim is that all k -marked nodal domains are allowed and (up to equivalence) there is only one possibility for the meromorphic functions. The dual graph is a tree. Starting from a vertex $v \in \mathbb{V}$ with only one edge $\mathfrak{e} \in \mathbb{E}_v$, since all the s_a are trivial, $s_{\mathfrak{e}}$ should be trivial as well. Removing v and continuing inductively we conclude that all the vector decorations $s_{\mathfrak{e}}$ should be the trivial vector. Definition 3.11 (1) holds with $s_v \equiv s$ for all $v \in \mathbb{V}$ and any fixed $s \in \mathbb{Z}_+^I$. The meromorphic functions $\zeta_{v,i} : \Sigma_v \rightarrow \mathbb{C}$ can be taken to be constant 1. Therefore, $\text{ob}_{\Gamma}(f) = 1$ (no matter what $\mathcal{G}$ is). Furthermore, $I_v = I_e = I$ for all $v \in \mathbb{V}$ and $e \in \mathbb{E}$, the map

$$\varrho : \mathbb{Z}^{\mathbb{E}} \oplus \bigoplus_{v \in \mathbb{V}} \mathbb{Z}^I \longrightarrow \bigoplus_{e \in \mathbb{E}} \mathbb{Z}^I$$

in (3.39) is surjective, and

$$\ker(\varrho) = \mathbb{Z}^{\mathbb{E}} \oplus \Delta,$$

where $\Delta = \{(a)_{v \in \mathbb{V}} \in \bigoplus_{v \in \mathbb{V}} \mathbb{Z}^I : a \in \mathbb{Z}^I\} \cong \mathbb{Z}^I$ is the diagonal (note that $|\mathbb{V}| - |\mathbb{E}| = 1$). We conclude that $\mathcal{G}$ is trivial. In this example, transversality holds and perturbation is not needed. $\square$

Remark 3.14 In the previous example, one may replace $\mathfrak{s} = \vec{0}$ with any $\mathfrak{s} = (s_1, \dots, s_k)$ satisfying $\sum_{a \in [k]} s_a = 0^N$. Then,

$$\overline{\mathcal{M}}_{0, \vec{0}}(X, D, 0) \cong \overline{\mathcal{M}}_{0, k} \times D_{I_0},$$

where $[N] - I_0$ is the maximal subset such that $s_{a,i} = 0$ for all $a \in [k]$ and $i \in [N] - I_0$. For every k -marked genus 0 nodal domain $(\Sigma, \vec{z})$, there is a unique set of decorations $\{s_{\underline{e}}\}_{\underline{e} \in \mathbb{E}}$ such that Σ can be equipped with meromorphic functions that have zeros/poles of order s_a at z_a . The group $\mathcal{G}$ is still trivial, but $\ker(\varrho)$ will be different.

Example 3.15 Extending Example 3.13 to the higher genus case, let $2g + k \geq 3$, $A = 0 \in H_2(X, \mathbb{Z})$, and $\mathfrak{s} = \vec{0}$ (and no perturbation yet). We show that still

$$\overline{\mathcal{M}}_{g, \vec{0}}(X, D, 0) \cong \overline{\mathcal{M}}_{g, k} \times X,$$

but transversality does not hold if $g > 0$. The obstruction bundle is the rank ng orbi-bundle

$$\pi_1^* \mathcal{E}_g^* \otimes \pi_2^* TX(-\log D) \longrightarrow \overline{\mathcal{M}}_{g, k} \times X,$$

where π_1, π_2 are the projection maps onto the first and second components, respectively, and $\mathcal{E}_g \rightarrow \overline{\mathcal{M}}_{g, k}$ is the **Hodge bundle**. Every element of $\overline{\mathcal{M}}_{g, \vec{0}}(X, D, 0)$ is (the equivalence class of) a k -marked nodal domain with the constant map into a point $p \in D_I$, for some maximal $I \subset X$, together with $|I|$ meromorphic functions on each component. If the dual graph is not a tree, a priori, there are infinitely many possibilities for the vector decorations $\{s_{\underline{e}}\}_{\underline{e} \in \mathbb{E}}$. The claim is that all k -marked nodal domains are allowed, but only the trivial decoration satisfies the conditions of Definition 3.11. By Definition 3.11 (1), there should exist vectors $s_v \in \mathbb{Z}_+^I$ such that

$$s_{v_2(\underline{e})} - s_{v_1(\underline{e})} = \lambda_{\underline{e}} s_{\underline{e}} \quad \text{for some } \lambda_{\underline{e}} > 0, \quad \forall \underline{e} \in \mathbb{E}.$$

For each $i \in I$, choose $v \in \mathbb{V}$ such that $s_{v,i} \in \mathbb{Z}_+$ is maximal. By the previous identity, $s_{\underline{e},i} \leq 0$ for all $\underline{e} \in \mathbb{E}_v$. Since

$$\sum_{\underline{e} \in \mathbb{E}_v} s_{\underline{e},i} = A_v \cdot D_i = 0,$$

we conclude that $s_{\underline{e},i} = 0$ for all $\underline{e} \in \mathbb{E}_v$. From this we conclude that all s_v should be the same; therefore, $s_{\underline{e}} = 0$ for all $\underline{e} \in \mathbb{E}$. The meromorphic functions $\zeta_{v,i} : \Sigma_v \rightarrow \mathbb{C}$ can be taken to be constant 1. Therefore, $\text{ob}_{\Gamma}(f) = 1$ (no matter what $\mathcal{G}$ is). In this case, the map

$$\varrho : \mathbb{Z}^{\mathbb{E}} \oplus \bigoplus_{v \in \mathbb{V}} \mathbb{Z}^I \longrightarrow \bigoplus_{\underline{e} \in \mathbb{E}} \mathbb{Z}^I$$

in (3.39) is not necessarily surjective. In this example, transversality does not hold for two reasons. The logarithmic linearization of Cauchy–Riemann operator in (4.7) is not surjective and 1 is not a regular value of ob_Γ . Passing to a regular cover

$$p: \overline{\mathcal{M}}_{g,k} \longrightarrow \overline{\mathcal{M}}_{g,k}$$

as in (3.4) and taking a generic perturbation term¹⁶ $v_{\log}$ in

$$\Gamma(\overline{\mathcal{M}}_{g,k} \times X, \pi_1^* \mathcal{E}_g^* \otimes \pi_2^* TX(-\log D)),$$

we get

$$\overline{\mathcal{M}}_{g,\tilde{0}}(X, D, 0, v) \cong v_{\log}^{-1}(0).$$

Therefore, the Virtual Fundamental Class of $\overline{\mathcal{M}}_{g,\tilde{0}}(X, D, 0)$ is the Euler class of the orbi-bundle $\pi_1^* \mathcal{E}_g^* \otimes \pi_2^* TX(-\log D)$ in $\overline{\mathcal{M}}_{g,k} \times X$ (generalizing the classical and relative examples in [17, Sect. 4.1]). $\square$

Each moduli space $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\log}(X, D, A, v)$ is coarsely stratified by the subspaces

$$\mathcal{M}_{g,\mathfrak{s}}(X, D, A, v)_\Gamma = \text{ob}_\Gamma^{-1}(1) \subset \mathcal{M}_{g,\mathfrak{s}}^{\log}(X, D, A, v)_\Gamma$$

consisting of $\log(J, v)$ -holomorphic curves with the decorated dual graph Γ . Here a decoration consists of genus and degree decorations on vertices, ordering of the marked points and $\mathfrak{s}$, labelings by subsets of $[N]$ in (3.37), and vectors $\{s_\ell\}_{\ell \in \mathbb{E}}$ satisfying the combinatorial condition (3.51). The vectors $\{s_\ell\}_{\ell \in \mathbb{E}}$ are also required to satisfy Definition 3.11 (1) but the vectors $\{s_v\}_{v \in \mathbb{V}}$ are not part of the decoration. By Theorem 1.3, for each $(g, \mathfrak{s}, A)$, the set of such decorated dual graphs Γ is finite.

Remark 3.16 Suppose f as in (3.38) is a (J, v) -log curve in $\mathcal{M}_{g,\mathfrak{s}}(X, D, A, v)_\Gamma$ and $\Gamma' = (\mathbb{V}', \mathbb{E}', \mathbb{L}')$ is a connected subgraph of Γ . The new set of legs $\mathbb{L}'$ consists of those legs $l \in \mathbb{L}$ such that $v(l) \in \mathbb{V}'$, as well as those oriented edges $e \in \mathbb{E}$ such that $v_1(e) \in \mathbb{V}'$ but $v_2(e) \notin \mathbb{V}'$. Let f' be the tuple as in (3.38) obtained by restricting to the connected sub-nodal curve

$$\Sigma' = \bigcup_{v \in \mathbb{V}'} \Sigma_v.$$

Then f' is also a $\log(J, v|_{\Sigma'})$ -curve with

$$s' = \left((s_l)_{l \in \mathbb{L}: v(l) \in \mathbb{V}'}, (s_e)_{e \in \mathbb{E}: v_1(e) \in \mathbb{V}', v_2(e) \notin \mathbb{V}'} \right) \quad \text{and} \quad A' = \sum_{v \in \mathbb{V}'} A_v.$$

¹⁶ This argument needs some justification, as we should explain the relation between such a perturbation with perturbations in (3.17).

The combinatorial conditions of Definition 3.9 and Definition 3.11 (1) are clearly satisfied. By definition, $\text{ob}_{\Gamma'}(f') = 1 \in \mathcal{G}(\Gamma')$ iff there are local holomorphic coordinates around each nodal point of Σ' and representatives ζ_v in the $\mathbb{C}^*$ -equivalence class $[\zeta_v]$, for all $v \in \mathbb{V}'$, such that (3.48) holds. The restriction to f' of such local holomorphic coordinates and representatives for f has the required property.

4 Deformation Theory and Transversality

4.1 Main Stratum

In this section, similar to the classical case, we realize $\mathcal{M}_{g,\mathfrak{s}}(X, D, A, \nu)$ as the zero set of a section $\bar{\partial}^{\log} - \nu_{\log}$ of some infinite dimensional bundle over an appropriately defined configuration space. The linearization of this section is a logarithmic lift $D_u^{\log}\{\bar{\partial} - \nu\}$ of the classical linearization map $D_u\{\bar{\partial} - \nu\}$. Then, it follows from Sard–Smale theorem that $\mathcal{M}_{g,\mathfrak{s}}(X, D, A, \nu)$ is cut transversely for generic ν . If $\nu \equiv 0$, the same statement holds for generic J if we restrict to the subspace of simple maps $\mathcal{M}_{g,\mathfrak{s}}^*(X, D, A)$.

We prove the following transversality statement.

Proposition 4.1 *Suppose (X, ω) is a closed symplectic manifold, $D = \bigcup_{i \in [N]} D_i$ is an SNC symplectic divisor, $A \in H_2(X, \mathbb{Z})$, $g, k \in \mathbb{N}$, and $\mathfrak{s} \in (\mathbb{N}^N)^k$.*

- (1) *If $2g + k \geq 3$, for any given choice of universal family in (3.5), there exists a Baire set of second category $\mathcal{H}_{g,k}^{\emptyset}(X, D)_{\mathcal{R},J} \subset \mathcal{H}_{g,k}(X, D)_{\mathcal{R},J}$ such that for each $\nu \in \mathcal{H}_{g,k}^{\emptyset}(X, D)_{\mathcal{R},J}$, $\mathcal{M}_{g,\mathfrak{s}}(X, D, A, \nu)$ is a naturally oriented smooth manifold of the real dimension*

$$2(c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k).$$

The restriction of $\text{st} \times \text{ev}$ in (1.5) to $\mathcal{M}_{g,\mathfrak{s}}(X, D, A, \nu)$ is smooth.

- (2) *If¹⁷ $g=0$, $\nu \equiv 0$, and $A \neq 0$ or $k \geq 3$, the same statement holds for J in a Baire set of second category $\text{AK}^{\emptyset}(X, D)_{\mathcal{R}} \subset \text{AK}(X, D)_{\mathcal{R}}$ if we restrict to the subspace of simple maps $\mathcal{M}_{0,\mathfrak{s}}^*(X, D, A)$.*

We start by setting up a suitable analytical frame work for studying the deformation theory of $\log(J, \nu)$ -holomorphic maps. This setup is in some sense the main step of the proof.

Definition 4.2 Fix a smooth k -marked genus g curve $(\Sigma, j, \vec{z})$, local holomorphic coordinates¹⁸ around the marked points, $A \in H_2(X, \mathbb{Z})$, a regularization $\mathcal{R}$ for D in X , and $\mathfrak{s}$ as in (1.1). With $I_a \subset [N]$ as in (1.3), for each $a \in [k]$, we say a smooth map $u : \Sigma \rightarrow X$ has **tangency order type $\mathfrak{s}$ with D at $\vec{z}$** if

¹⁷ Restriction to $g=0$ is not necessary here but this is the case that we will need later. Furthermore, the generalization of this statement in part (2) of Proposition 4.8 requires the $g=0$ assumption or allowing deformations of $\mathcal{R}$.

¹⁸ This is not needed here, but it will be needed in constructing a Banach completion of $\text{Map}_{A,\mathfrak{s}}((\Sigma, \vec{z}), (X, D))$; see the end of Remark 4.5.

- $u^{-1}(D) = \mathfrak{d} \subset \{z_1, \dots, z_k\}$,
- $u(z_a) = p_a \in D_{I_a} - \partial D_{I_a}$, for all $a \in [k]$,
- and

$$\Psi_{I_a}^{-1} \circ u(w_a) = \left(u_a(w_a), \bigoplus_{i \in I_a} w_a^{s_{ai}} \eta_{a,i}(w_a) \right) \in \mathcal{N}_X D_{I_a}, \quad \forall a \in [k], \quad (4.1)$$

where U_a is a sufficiently small neighborhood of the marked point z_a in Σ , w_a is the local holomorphic coordinate on U_a with $w_a(z_a) = 0$, $u_a: U_a \rightarrow D_{I_a}$ is the projection of $\Psi_{I_a}^{-1} \circ u$ to D_{I_a} , and $\eta_a = (\eta_{a,i})_{i \in I_a}$ is a smooth section of

$$u_a^* \mathcal{N}_X D_{I_a} \cong \bigoplus_{i \in I_a} u_a^* \mathcal{N}_X D_i$$

satisfying

$$\eta_{a,i}(0) \neq 0, \quad \forall i \in I_a.$$

Let $\text{Map}_{A,s}((\Sigma, \vec{z}), (X, D))$ denote the set of smooth degree A maps of tangency order type s with D at $\vec{z}$. This space is an infinite dimensional Fréchet manifold whose tangent space at any u is the infinite dimensional vector space

$$\Gamma(\Sigma, u^* T X(-\log D)_{\mathcal{R}}).$$

More explicitly, if $\{u_t\}_{t \in [0, \epsilon]}$ is a 1-parameter family of maps in $\text{Map}_{A,s}((\Sigma, \vec{z}), (X, D))$, restricted to $\Sigma - \mathfrak{d}$, by the first bullet in Definition 4.2, we get

$$\xi_{\emptyset} = \frac{d}{dt} u_t|_{t=0} \in \Gamma(\Sigma - \mathfrak{d}, \{u_0|_{\Sigma - \mathfrak{d}}\}^* T X(-\log D)) \cong \Gamma(\Sigma - \mathfrak{d}, \{u_0|_{\Sigma - \mathfrak{d}}\}^* T X).$$

On the other hand, for each $a \in [k]$ with $z_a \in \mathfrak{d}$, restricted to the chart U_a in the third bullet above, by (4.1), we have

$$\Psi_{I_a}^{-1} \circ u_t = \left(u_{t,a}, \bigoplus_{i \in I_a} w_a^{s_{ai}} \eta_{t,a,i} \right).$$

Therefore,

$$\frac{d}{dt} (\Psi_{I_a}^{-1} \circ u_t)|_{t=0} = \pi_{I_a}^* \xi_{I_a} \oplus \bigoplus_{i \in I_a} w_a^{s_{ai}} \eta_{0,a,i} c_{a,i}, \quad (4.2)$$

where $\pi^* \xi_{I_a}$ is the horizontal lift of

$$\xi_{I_a} = \frac{d}{dt} u_{t,a}|_{t=0} \in \Gamma(U_a, u_{0,a}^* T D_{I_a})$$

to the horizontal subspace $T^{\text{hor}}\mathcal{N}_X D_{I_a} \cong \pi_{I_a}^* T D_{I_a}$, and

$$c_{a,i} = \frac{\frac{d}{dt} \eta_{t,a,i}(w_a)|_{t=0}}{\eta_{0,a,i}(w_a)} : U_a \longrightarrow \mathbb{C}, \quad \forall i \in I_a.$$

By (2.11)–(2.13) and (4.2), $\xi_{\emptyset}$ and $\{\xi_{I_a} \oplus (c_{a,i})_{i \in I_a}\}_{z_a \in \mathfrak{d}}$ define a global section $\xi_{\log}$ of $u^* T X(-\log D)$ that maps to

$$\xi = \frac{d}{dt} u_t|_{t=0} \in \Gamma(\Sigma, u_0^* T X)$$

under the homomorphism ι in (2.10). Conversely, given a section $\xi_{\log} \in \Gamma(\Sigma, u_0^* T X \cdot (-\log D))$, logarithmic exponentiation (see (2.14)) of $\xi_{\log}$, corresponding to a Hermitian metric on $T X(-\log D)$, produces a 1-parameter family of maps in $\text{Map}_{A,\mathfrak{s}}((\Sigma, \vec{z}), (X, D))$ with tangent vector $\xi = \iota(\xi_{\log})$ at $t=0$.

Lemma 4.3 *For every $u \in \text{Map}_{A,\mathfrak{s}}((\Sigma, \vec{z}), (X, D))$, there exists a logarithmic Cauchy–Riemann section*

$$\bar{\partial}^{\log} u \in \Gamma(\Sigma, \Omega_{\Sigma,j}^{0,1} \otimes_{\mathbb{C}} u^* T X(-\log D)) \quad (4.3)$$

such that the following diagram commutes:

$$\begin{array}{ccc} & & u^* T X(-\log D) \\ & \nearrow \bar{\partial}^{\log} u & \downarrow \iota \\ T^{0,1} \Sigma & \xrightarrow{\bar{\partial} u} & u^* T X. \end{array} \quad (4.4)$$

Proof Away from $\mathfrak{d}$, by the first bullet in Definition 4.2 and the identification

$$T X(-\log D)|_{X-D} \cong T X|_{X-D},$$

we define $\bar{\partial}^{\log} u = \bar{\partial} u$. For each $a \in [k]$ with $z_a \in \mathfrak{d}$, restricted to the chart U_a in Definition 4.2, by (4.1), we have

$$\Psi_{I_a}^{-1} \circ \bar{\partial} u = \pi^* \bar{\partial} u_a \oplus \bigoplus_{i \in I_a} \bar{\partial}_{u^* \mathcal{N}_X D_i} (w_a^{s_{ai}} \eta_{a,i}) = \pi_{I_a}^* \bar{\partial} u_a \oplus \bigoplus_{i \in I_a} w_a^{s_{ai}} \bar{\partial}_{u^* \mathcal{N}_X D_i} (\eta_{a,i}).$$

Restricted to U_a , we define $\bar{\partial}^{\log} u$ to be

$$\Psi_{I_a}^{-1} \circ \bar{\partial}^{\log} u = \pi_{I_a}^* \bar{\partial} u_a \oplus \bigoplus_{i \in I_a} \frac{\bar{\partial}_{u^* \mathcal{N}_X D_i} \eta_{a,i}}{\eta_{a,i}}.$$

By (2.11)–(2.13), these local sections define a global section (4.3) that maps to $\bar{\partial} u$ under the homomorphism ι in (2.10). $\square$

Let $\text{Map}_A(\Sigma, X)$ denote the space of all smooth degree A maps from Σ into X ,

$$\mathcal{E}_A(\Sigma, X) \longrightarrow \text{Map}_A(\Sigma, X) \quad (4.5)$$

denote the infinite dimensional vector space whose fiber over every map u is

$$\Gamma(\Sigma, \Omega_{\Sigma, j}^{0,1} \otimes_{\mathbb{C}} u^*TX),$$

and

$$\mathcal{E}_A(\Sigma, X) \longrightarrow \text{Map}_A(\Sigma, X) \quad (4.6)$$

denote the infinite dimensional vector space whose fiber over every map u is

$$\Gamma(\Sigma, \Omega_{\Sigma, j}^{0,1} \otimes_{\mathbb{C}} u^*TX(-\log D)).$$

The classical CR operator (1.10) can be seen as a section of (4.5). By Lemma 4.3, the restriction of this section to $\text{Map}_{A, s}((\Sigma, \vec{z}), (X, D))$ defines a section of (4.6). Similarly, for every $v \in \mathcal{H}_{g, k}(X, D)_{\mathcal{R}, J}$ associated to $v_{\log}$ as in (3.18) and an identification ϕ of $(\Sigma, j, \vec{z})$ with a fiber of the universal family $\pi : \overline{\mathcal{U}}_{g, k} \rightarrow \overline{\mathcal{M}}_{g, k}$ (used to define v), the restriction of $\bar{\partial} - v$ to $\text{Map}_{A, s}((\Sigma, \vec{z}), (X, D))$ lifts to the section $\bar{\partial}^{\log} - v_{\log}$ of (4.6) so that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{E}_{A, s}((\Sigma, \vec{z}), (X, D)) & \longrightarrow & \mathcal{E}_A(\Sigma, X) \\ \bar{\partial}^{\log} - v_{\log} \uparrow & & \bar{\partial} - v \uparrow \\ \text{Map}_{A, s}((\Sigma, \vec{z}), (X, D)) & \longrightarrow & \text{Map}_A(\Sigma, X). \end{array}$$

The linearization of $\bar{\partial}^{\log} - v_{\log}$ along the zero set is then the restriction/lift

$$D_u^{\log}\{\bar{\partial} - v\} \equiv D_u\{\bar{\partial}^{\log} - v_{\log}\}$$

of the classical linearization map $D_u\{\bar{\partial} - v\}$ to $\Gamma(\Sigma, u^*TX(-\log D))$ so that the following diagram commutes:

$$\begin{array}{ccc} \Gamma(\Sigma, u^*TX(-\log D)) & \xrightarrow{D_u^{\log}\{\bar{\partial} - v\}} & \Gamma(\Sigma, \Omega_{\Sigma, j}^{0,1} \otimes_{\mathbb{C}} u^*TX(-\log D)) \\ \downarrow \iota_1 & & \downarrow \iota_2 \\ \Gamma(\Sigma, u^*TX) & \xrightarrow{D_u\{\bar{\partial} - v\}} & \Gamma(\Sigma, \Omega_{\Sigma, j}^{0,1} \otimes_{\mathbb{C}} u^*TX). \end{array} \quad (4.7)$$

Fix a j -Hermitian metric on $T\Sigma$ and a J -Hermitian metric on $TX(-\log D)$, an integer $\ell \geq 1$, and a real number $p > 2$. Via the logarithmic exponentiation map, we

can then construct a completion $W_{A,s}^{\ell,p}((\Sigma, \bar{z}), (X, D))$ of $\text{Map}_{A,s}((\Sigma, \bar{z}), (X, D))$ which is a smooth separable Banach manifold with tangent space

$$T_u W_{A,s}^{\ell,p}((\Sigma, \bar{z}), (X, D)) = W^{k,p}(\Sigma, u^*TX(-\log D)). \quad (4.8)$$

The completion $\mathcal{E}_{A,s}^{\ell-1,p}((\Sigma, \bar{z}), (X, D))$ of $\mathcal{E}_{A,s}((\Sigma, \bar{z}), (X, D))$ is a Banach complex vector bundle over $W_{A,s}^{\ell,p}((\Sigma, \bar{z}), (X, D))$. For every $v \in \mathcal{H}_{g,k}(X, D)_{\mathcal{R},J}$ and any identification ϕ of $(\Sigma, \bar{j}, \bar{z})$ with a smooth fiber of the universal family $\bar{\mathcal{U}}_{g,k}$ used to define v , $\bar{\partial}^{\log} - v_{\log}$ defines a smooth section of Banach bundle

$$\mathcal{E}_{A,s}^{\ell-1,p}((\Sigma, \bar{z}), (X, D)) \longrightarrow W_{A,s}^{\ell,p}((\Sigma, \bar{z}), (X, D)). \quad (4.9)$$

As in the classical case (3.10), $D_u^{\log}\{\bar{\partial} - v\}$ can be written as the sum of a complex linear map (a CR operator on $u^*TX(-\log D)$) and a compact operator (It is the restriction of the corresponding operators). Thus, by Riemann–Roch, it is a Fredholm operator with index

$$\dim_{\mathbb{R}} \text{Def}_{\log}(u) - \dim_{\mathbb{R}} \text{Obs}_{\log}(u) = 2(\deg(u^*TX(-\log D)) + \dim_{\mathbb{C}} X(1-g)),$$

where

$$\text{Def}_{\log}(u) \equiv \ker(D_u^{\log}\{\bar{\partial} - v\}) \quad \text{and} \quad \text{Obs}_{\log}(u) \equiv \text{coker}(D_u^{\log}\{\bar{\partial} - v\}).$$

From Implicit-Function Theorem [26, Thm. A.3.3], we deduce the following corollary.

Corollary 4.4 *If $u \in \text{Map}_{A,s}((\Sigma, \bar{z}), (X, D))$ is (J, v) -holomorphic and $\text{Obs}_{\log}(u) \equiv 0$, in a small neighborhood $B(u)$ of u in $W_{A,s}^{\ell,p}((\Sigma, \bar{z}), (X, D))$ the set of (J, v) -holomorphic maps*

$$V_u \equiv \{\bar{\partial}^{\log} - v_{\log}\}^{-1}(0) \cap B(u)$$

is a smooth manifold of real dimension (1.11).

Furthermore, by elliptic regularity, all the elements of V_u and $\text{Def}_{\log}(u) \cong T_u V_u$ are smooth (see [26, Thm. 3.1.5]). The manifold V_u carries a *natural orientation*. Starting with the complex linear part of $D_u^{\log}\{\bar{\partial} - v\}$, both the kernel and cokernel of that are complex linear and thus naturally oriented. By deforming $D_u^{\log}\{\bar{\partial} - v\}$ into its complex linear part via a 1-parameter family of compact operators, [26, Prop. A.2.4] gives us a natural orientation on $\text{Def}_{\log}(u)$.

Next, we consider deformations of the marked domain $C = (\Sigma, \bar{j}, \bar{z})$. Given a regular covering $p : \bar{\mathcal{M}}_{g,k} \rightarrow \bar{\mathcal{M}}_{g,k}$ and a universal family $\pi : \bar{\mathcal{U}}_{g,k} \rightarrow \bar{\mathcal{M}}_{g,k}$ as in (3.4)–(3.5), let

$$\mathfrak{M}_{g,k} = p^{-1}(\mathcal{M}_{g,k}), \quad \mathcal{U}_{g,k} = \pi^{-1}(\mathfrak{M}_{g,k}), \quad \pi : \mathcal{U}_{g,k} \longrightarrow \mathfrak{M}_{g,k},$$

be the restrictions to the subspace of smooth curves. Choose a projective embedding $\overline{\mathfrak{U}}_{g,k} \rightarrow \mathbb{P}^M$, for some sufficiently large M , and define

$$W_{A,\mathfrak{s}}^{\ell,p}((\mathfrak{U}_{g,k}, \vec{\mathfrak{z}}), (X, D)) \equiv \left\{ (c, u) : c \in \mathfrak{M}_{g,k}, u \in W_{A,\mathfrak{s}}^{\ell,p}((\pi^{-1}(c), \vec{\mathfrak{z}}(c)), (X, D)) \right\} \quad (4.10)$$

where the metric considered on

$$((\Sigma, j_c), \vec{z}) \equiv (\pi^{-1}(c), \vec{\mathfrak{z}}(c)),$$

for each $c \in \mathfrak{M}_{g,k}$, is the restriction of Fubini–Study metric on $\mathbb{P}^M$ to the image of Σ .

Remark 4.5 By definition, each fiber of the projection map

$$W_{A,\mathfrak{s}}^{\ell,p}((\mathfrak{U}_{g,k}, \vec{\mathfrak{z}}), (X, D)) \longrightarrow \mathfrak{M}_{g,k} \quad (4.11)$$

has a Banach manifold structure but the total space does not a priori come with a natural smooth structure. In order to define a Banach manifold structure on (4.10), we need to fix a smooth trivialization of $\mathfrak{U}_{g,k} \rightarrow \mathfrak{M}_{g,k}$. Such a trivialization gives us a trivialization of (4.10) and thus a product Banach manifold structure on that; see [11, Sect. 6.1]. The genus- g surface-bundle $\mathfrak{U}_{g,k} \rightarrow \mathfrak{M}_{g,k}$ does not necessarily admit a global smooth trivialization. However, for $\mathcal{B} \subset \mathfrak{M}_{g,k}$ sufficiently small around any $b \in \mathfrak{M}_{g,k}$,

$$\mathcal{C} = \pi^{-1}(\mathcal{B}) \longrightarrow \mathcal{B} \quad (4.12)$$

is smoothly trivial. In other words, locally around every $((\Sigma, j_b), \vec{z}) \equiv (\pi^{-1}(b), \vec{\mathfrak{z}}(b))$ there exists an $\text{Aut}((\Sigma, j_b), \vec{z})$ -equivariant diffeomorphism

$$\varphi : \mathcal{C} \longrightarrow \Sigma \times \mathcal{B} \quad (4.13)$$

such that $\pi \circ \varphi^{-1}$ is the projection onto the second factor, each section $\varphi \circ \mathfrak{z}_a$ is constant, and $\varphi|_{\pi^{-1}(b)} = \text{id}_\Sigma$. The smooth trivialization φ gives rise to a Banach manifold structure on the restriction

$$W_{A,\mathfrak{s}}^{\ell,p}((\mathcal{C}, \vec{\mathfrak{z}}|_{\mathcal{B}}), (X, D)) \equiv \left\{ (c, u) : c \in \mathcal{B}, u \in W_{A,\mathfrak{s}}^{\ell,p}((\pi^{-1}(c), \vec{\mathfrak{z}}(c)), (X, D)) \right\} \quad (4.14)$$

which we denote by $W_{A,\mathfrak{s}}^{\ell,p}((\mathcal{C}, \vec{\mathfrak{z}}|_{\mathcal{B}}), (X, D))_\varphi$. If φ_1 and φ_2 are two such smooth trivialization maps, the map

$$W_{A,\mathfrak{s}}^{\ell,p}((\mathcal{C}, \vec{\mathfrak{z}}|_{\mathcal{B}}), (X, D))_{\varphi_1} \longrightarrow W_{A,\mathfrak{s}}^{\ell,p}((\mathcal{C}, \vec{\mathfrak{z}}|_{\mathcal{B}}), (X, D))_{\varphi_2}$$

induced by the change of trivialization map $\varphi_2 \circ \varphi_1^{-1}$ is not smooth (unless $\varphi_2 \circ \varphi_1^{-1}$ is constant in c); see [28, Sect. 3.1]. Therefore, there is no natural way of putting

a Banach manifold structure on (4.11). On the other hand, restricted to the moduli space which is the zero set of $\bar{\partial}^{\log} - \nu_{\log}$, (by elliptic regularity) the transition maps are smooth. Therefore, in the proof of Proposition 4.1 below, we cover $\mathfrak{M}_{g,k}$ with countably many such charts, find a Baire set of regular perturbation terms for each one, and then take intersection which yields a Baire set again. It is for the same reason that we fix local coordinates around the marked points in Definition 4.2. If w_a and w'_a are two local coordinates around the marked point z_a , they are related by a $\mathbb{C}^*$ -valued reparametrization map φ ; i.e. $w'_a = \varphi(w_a)w_a$. The Banach smooth structure $W_{A,\mathfrak{s}}^{\ell,p}((\Sigma, \vec{z}), (X, D))$ with respect to w_a and w'_a will not be the same unless φ is constant. In the family version (4.12), we will fix local defining equations $w_a : \mathcal{C} \rightarrow \mathbb{C}$ for Cartier divisors $\mathfrak{z}_a(\mathcal{B}) \subset \mathcal{C}$. The restriction of w_a to each fiber is the local coordinate needed in Definition 4.2; see [11, (5.20)–(5.21)] for details.

Let

$$\mathcal{E}_{A,\mathfrak{s}}^{\ell-1,p}((\mathfrak{U}_{g,k}, \vec{\mathfrak{z}}), (X, D)) \longrightarrow W_{A,\mathfrak{s}}^{\ell,p}((\mathfrak{U}_{g,k}, \vec{\mathfrak{z}}), (X, D)) \quad (4.15)$$

denote the natural extension of (4.9) over $\mathfrak{M}_{g,k}$. Similar to the previous paragraph, locally over any sufficiently small neighborhood $\mathcal{B}$ of $b \in \mathfrak{M}_{g,k}$, a smooth trivialization φ as in (4.13) gives rise to a Banach vector bundle structure on the restriction

$$\mathcal{E}_{A,\mathfrak{s}}^{\ell-1,p}((\mathcal{C}, \vec{\mathfrak{z}}|_{\mathcal{B}}), (X, D)) \longrightarrow W_{A,\mathfrak{s}}^{\ell,p}((\mathcal{C}, \vec{\mathfrak{z}}|_{\mathcal{B}}), (X, D)) \quad (4.16)$$

such that $\bar{\partial}^{\log} - \nu_{\log}$ is a smooth section of that.

Remark 4.6 Let $f = (u, C) \in \mathcal{M}_{g,\mathfrak{s}}(X, D, A)$ (no perturbation here). Similar to the classical case (see [19, Sect. 24.1] and [11, Remark 6.2.1]), deformation theory of f , i.e., if we allow deformations of both u and C , is described by the long exact sequence

$$\begin{aligned} 0 \longrightarrow \text{aut}(C) &\xrightarrow{\delta} \text{Def}_{\log}(u) \longrightarrow \text{Def}_{\log}(f) \longrightarrow \text{Def}(C) \\ &\xrightarrow{\delta} \text{Obs}_{\log}(u) \longrightarrow \text{Obs}_{\log}(f) \longrightarrow 0, \end{aligned} \quad (4.17)$$

where

$$\text{aut}(C) = H_{\bar{\partial}}^0(T\Sigma(-\log \vec{z})) \quad \text{and} \quad \text{Def}(C) \cong T_b\mathcal{B} \cong H_{\bar{\partial}}^1(T\Sigma(-\log \vec{z})).$$

If $\text{Obs}_{\log}(f) = 0$, then a small neighborhood $B(f)$ of f in $\mathcal{M}_{g,\mathfrak{s}}(X, D, A)$ is a smooth orbifold of the expected real dimension (1.9). The long exact sequence (4.17) is the hyper-cohomology of a short exact sequence of complexes of fine sheaves constructed in the following way. In order to simplify the notation, for a complex vector bundle $E \rightarrow (\Sigma, \mathfrak{j})$ let $\Omega^0(E)$ and $\Omega^{0,1}(E)$ denote the associated fine sheaves of smooth sections of E and of smooth E -valued $(0, 1)$ -forms, respectively. The map $du : T\Sigma \rightarrow TX$ gives rise to a *logarithmic derivative* map

$$d^{\log}u : T\Sigma(-\log \vec{z}) \longrightarrow u^*TX(-\log D)$$

such that the following diagram commutes:

$$\begin{array}{ccc} T\Sigma(-\log \vec{z}) & \xrightarrow{d^{\log u}} & u^*TX(-\log D) \\ \downarrow \iota_{\Sigma, \vec{z}} & & \downarrow \iota_{X, D} \\ T\Sigma & \xrightarrow{du} & u^*TX. \end{array} \quad (4.18)$$

Away from the D -intersection points $\mathfrak{d}$, by the first bullet in Definition 4.2 and the identification

$$TX(-\log D)|_{X-D} \cong^{\iota_{X,D}} TX|_{X-D},$$

we have $d^{\log u} = \iota_{X,D}^{-1} \circ du \circ \iota_{\Sigma, \vec{z}}$. For each intersection point $z_a \in \vec{z}$, restricted to the chart U_a in Definition 4.2, by (4.1), $d^{\log u}$ is given by

$$\Psi_{I_a}^{-1} \circ d^{\log u} = \pi_{I_a}^* du_a \oplus \bigoplus_{i \in I_a} \left(s_{ai} \frac{dw_a}{w_a} + \frac{\nabla^{(I_a; i)} \eta_{a,i}}{\eta_{a,i}} \right),$$

which maps the local generating section $w_a \partial w_a$ to

$$\pi_{I_a}^* \partial u_a (w_a \partial w_a) \oplus \bigoplus_{i \in I_a} \left(s_{ai} + w_a \frac{\nabla_{\partial w_a}^{(I_a; i)} \eta_{a,i}}{\eta_{a,i}} \right). \quad (4.19)$$

The following commutative diagram has exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^0(T\Sigma(-\vec{z})) & \longrightarrow & \Omega^0(T\Sigma(-\vec{z})) & & \\ \downarrow & & \downarrow d^{\log u \oplus \bar{\partial}} & & \downarrow \bar{\partial} & & \\ \Omega^0(u^*TX(-\log D)) & \longrightarrow & \Omega^0(u^*TX(-\log D)) \oplus \Omega^{0,1}(T\Sigma(-\vec{z})) & \longrightarrow & \Omega^{0,1}(T\Sigma(-\vec{z})) & & \\ \downarrow D_u^{\log \bar{\partial}} & & \downarrow D_u^{\log \bar{\partial} - d^{\log u}} & & \downarrow & & \\ \Omega^{0,1}(u^*TX(-\log D)) & \longrightarrow & \Omega^{0,1}(u^*TX(-\log D)) & \longrightarrow & 0; & & \end{array}$$

i.e., it is an exact sequence of chain complexes given by the columns. Then, the deformation/obstruction long exact sequence (4.17) is the hyper-cohomology of this diagram. By (4.19) and similar to the classical case [33, pp. 284–285], if u is an immersion away from $\vec{z}$ and

$$u^{-1}(D) = \mathfrak{d} = \{z_1, \dots, z_k\}$$

(i.e., $s_a \neq 0$ for all $a \in [k]$), then $d^{\log u}$ is an embedding, the quotient

$$\mathcal{N}_X \Sigma(-\log D) \equiv u^*TX(-\log D) / (d_u^{\log} T\Sigma(-\vec{z}))$$

is a complex vector bundle, and $D_u^{\log \bar{\partial}}$ descends to a Fredholm operator $D_{\mathcal{N}}^{\log \bar{\partial}}$ on smooth sections of $\mathcal{N}_X \Sigma(-\log D)$ such that

$$\text{Def}_{\log}(f) = \ker(D_{\mathcal{N}}^{\log \bar{\partial}}) \quad \text{and} \quad \text{Obs}_{\log}(f) = \text{coker}(D_{\mathcal{N}}^{\log \bar{\partial}}). \quad (4.20)$$

If $d^{\log} u$ is not an embedding, we still obtain a short exact sequence of sheaves of $\mathcal{O}_{\Sigma}$ -modules

$$0 \longrightarrow \mathcal{O}(T\Sigma(-\log \vec{z})) \xrightarrow{d^{\log} u} \mathcal{O}(u^*TX(-\log D)) \longrightarrow \mathcal{N} \longrightarrow 0$$

such that

$$\mathcal{N} = \mathcal{O}(\mathcal{N}_X \Sigma(-\log D)) \oplus \mathcal{N}^{\text{tor}}$$

is the direct sum of sheaf of holomorphic sections of an $(n-1)$ -dimensional holomorphic vector bundle $\mathcal{N}_X \Sigma(-\log D)$ and a skyscraper sheaf $\mathcal{N}^{\text{tor}}$. Furthermore, $D_u^{\log \bar{\partial}}$ descends to a Fredholm operator $D_{\mathcal{N}}^{\log \bar{\partial}}$ on smooth sections of $\mathcal{N}_X \Sigma(-\log D)$ such that

$$\text{Def}_{\log}(f) = \ker(D_{\mathcal{N}}^{\log \bar{\partial}}) \oplus H^0(\mathcal{N}^{\text{tor}}) \quad \text{and} \quad \text{Obs}_{\log}(f) = \text{coker}(D_{\mathcal{N}}^{\log \bar{\partial}}). \quad (4.21)$$

In particular, $\text{Obs}_{\log}(f) = 0$ whenever $\dim_{\mathbb{C}} X = 1$.

Proof of Proposition 4.1 Since every map u in $W_{A,\mathfrak{s}}^{\ell,p}((\mathfrak{U}_{g,k}, \vec{\mathfrak{z}}), (X, D))$ meets D only at finitely many points, by substituting $D_u\{\bar{\partial} - \nu\}$ with $D_u^{\log}\{\bar{\partial} - \nu\}$, Proposition 4.1 essentially follows from restricting the arguments of the proof of [32, Thm. 3.1] and [26, Thm. 3.1.5] to maps in $W_{A,\mathfrak{s}}^{\ell,p}((\mathfrak{U}_{g,k}, \vec{\mathfrak{z}}), (X, D))$.

More precisely, for $m > \ell$, let $\mathcal{H}_{g,k}^m(X, D)_{\mathcal{R},J}$ denote the completion of the vector space $\mathcal{H}_{g,k}(X, D)$ in C^m -topology. The universal moduli space

$$\begin{aligned} \mathfrak{M}_{g,\mathfrak{s}}(X, D, A) = & \left\{ ((c, u), \nu) \in W_{A,\mathfrak{s}}^{\ell,p}((\mathfrak{U}_{g,k}, \vec{\mathfrak{z}}), (X, D)) \times \mathcal{H}_{g,k}^m(X, D)_{\mathcal{R},J} : \right. \\ & \left. \bar{\partial}^{\log} u(x) = \nu_{\log}(x, u(x)), \quad \forall x \in \pi^{-1}(c) \right\} \end{aligned} \quad (4.22)$$

is the zero set of the section¹⁹

$$\bar{\partial}^{\log} - \nu_{\log} : W_{A,\mathfrak{s}}^{\ell,p}((\mathfrak{U}_{g,k}, \vec{\mathfrak{z}}), (X, D)) \times \mathcal{H}_{g,k}^m(X, D)_{\mathcal{R},J} \rightarrow E_{A,\mathfrak{s}}^{\ell,p}((\mathfrak{U}_{g,k}, \vec{\mathfrak{z}}), (X, D)) \quad (4.23)$$

¹⁹ To be precise, the right-hand side should be $\pi_1^* E_{A,\mathfrak{s}}^{\ell,p}((\mathfrak{U}_{g,k}, \vec{\mathfrak{z}}), (X, D))$, where π_1 is projection map to the first component

$$W_{A,\mathfrak{s}}^{\ell,p}((\mathfrak{U}_{g,k}, \vec{\mathfrak{z}}), (X, D)) \times \mathcal{H}_{g,k}^m(X, D)_{\mathcal{R},J} \longrightarrow W_{A,\mathfrak{s}}^{\ell,p}((\mathfrak{U}_{g,k}, \vec{\mathfrak{z}}), (X, D)).$$

We avoid these details to keep the notation short.

and is independent of (ℓ, p) by the elliptic regularity. Restricting to each sufficiently small sub-universal family $\mathcal{C} \rightarrow \mathcal{B}$ and fixing a smooth trivialization φ as above, the restricted section

$$\bar{\partial}^{\log} - v_{\log}: W_{A, \mathfrak{s}}^{\ell, p}((\mathcal{C}, \vec{\mathfrak{z}}|_{\mathcal{B}}), (X, D)) \times \mathcal{H}_{g, k}^m(X, D)_{\mathcal{R}, J} \rightarrow E_{A, \mathfrak{s}}^{\ell, p}((\mathcal{C}, \vec{\mathfrak{z}}|_{\mathcal{B}}), (X, D)) \quad (4.24)$$

is $C^{m-\ell}$ -smooth. With the same reasoning as in the argument leading to the surjectivity of [32, (3.12)], for $c \in \mathcal{B}$ and $((c, u), v)$ in the universal moduli space (4.22), the linearization map

$$\begin{aligned} D_{((c, u), v)}^{\log} \{\bar{\partial} - v\}: T_{(c, u)} W_{A, \mathfrak{s}}^{\ell, p}((\mathcal{C}, \vec{\mathfrak{z}}|_{\mathcal{B}}), (X, D)) \oplus T_v \mathcal{H}_{g, k}^m(X, D)_{\mathcal{R}, J} \\ \longrightarrow \Gamma(\Sigma, \Omega_{\pi^{-1}(c)}^{0, 1} \otimes_{\mathbb{C}} u^* T X(-\log D)) \end{aligned} \quad (4.25)$$

of the section (4.23) is surjective. This is due to the fact that $\text{coker}(D_u^{\log} \{\bar{\partial} - v\})$ can be represented by sections supported away from the intersection points where everything has a classical form. Therefore, the universal moduli space

$$\mathfrak{M}_{g, \mathfrak{s}}(X, D, A)|_{\mathcal{B}}$$

is a separable $C^{m-\ell}$ -smooth Banach manifold. Here the restriction to $\mathcal{B}$ means we are only considering (J, v) -maps with domain in $\mathcal{B}$ corresponding to (4.23). Then by the Sard–Smale Theorem, the set of regular values $\mathcal{H}_{g, k}^{\text{reg}(\mathcal{B})}(X, D)_{\mathcal{R}, J}$ of the projection map

$$\pi_2: \mathfrak{M}_{g, \mathfrak{s}}(X, D, A)|_{\mathcal{B}} \longrightarrow \mathcal{H}_{g, k}^m(X, D)_{\mathcal{R}, J}$$

is Baire set of second category. For every $v \in \mathcal{H}_{g, k}^{\text{reg}(\mathcal{B})}(X, D)_{\mathcal{R}, J}$,

$$\mathcal{M}_{g, \mathfrak{s}}(X, D, A, v)|_{\mathcal{B}} = \pi_2^{-1}(v)$$

is a smooth manifold of the expected dimension. Cover $\mathfrak{M}_{g, k}$ with countably many charts $\{\mathcal{B}_i\}_{i=1}^{\infty}$ and let

$$\mathcal{H}_{g, k}^{\text{reg}}(X, D)_{\mathcal{R}, J} = \bigcap_{i=1}^{\infty} \mathcal{H}_{g, k}^{\text{reg}(\mathcal{B}_i)}(X, D)_{\mathcal{R}, J}.$$

This is still a Baire set of second category so that for each $v \in \mathcal{H}_{g, k}^{\text{reg}}(X, D)_{\mathcal{R}, J}$,

$$\mathcal{M}_{g, \mathfrak{s}}(X, D, A, v)$$

is a smooth manifold of the expected dimension. With an argument similar to the Taubes' trick in the proof of [26, Thm. 3.1.6 (ii)], we conclude that the subset of

smooth perturbations

$$\mathcal{H}_{g,k}^{\emptyset}(X, D)_{\mathcal{R},J} = \mathcal{H}_{g,k}^{\text{reg}}(X, D)_{\mathcal{R},J} \cap \mathcal{H}_{g,k}(X, D)_{\mathcal{R},J}$$

satisfies the first statement in Proposition 4.1.

With the modifications above, proof of the second statement is similar to the proof of [26, Thm. 3.1.5]. More precisely, for $g = 0$ and no perturbation, if $k \geq 3$, let

$$\mathfrak{U}_{0,k} = \mathcal{M}_{0,k+1} \longrightarrow \mathfrak{M}_{0,k} = \mathcal{M}_{0,k}$$

denote the universal curve, and if $k < 3$, let $\mathfrak{U}_{0,k} = \mathbb{P}^1$, $\mathfrak{M}_{0,k}$ be a point, and $\text{aut}_k = \text{Aut}(\mathbb{P}^1, \vec{z})$. Let

$$\begin{aligned} & \tilde{\mathfrak{M}}_{0,s}(X, D, A) \\ &= \left\{ ((c, u), J) \in W_{A,s}^{\ell,p}((\mathfrak{U}_{0,k}, \vec{z}), (X, D)) \times \text{AK}^m(X, D)_{\mathcal{R}} : \bar{\partial}^{\log} u(x) = 0 \right\} \end{aligned} \quad (4.26)$$

be the zero set of the section

$$\bar{\partial}^{\log}: W_{A,s}^{\ell,p}((\mathfrak{U}_{0,k}, \vec{z}), (X, D)) \times \text{AK}^m(X, D)_{\mathcal{R}} \longrightarrow E_{A,s}^{\ell,p}((\mathfrak{U}_{0,k}, \vec{z}), (X, D)). \quad (4.27)$$

The universal moduli space $\mathfrak{M}_{0,s}(X, D, A)$ is the quotient of $\tilde{\mathfrak{M}}_{0,s}(X, D, A)$ with respect to aut_k . Since $\text{coker}(D_u \bar{\partial}^{\log})$ can be represented by sections supported away from the intersection points, the same reasoning as in the proof of [26, Thm. 3.15] shows that the linearization map

$$\begin{aligned} D_{((c,u),J)} \bar{\partial}^{\log}: T_{(c,u)} W_{A,s}^{\ell,p}((C, \vec{z}|_B), (X, D)) \oplus T_J \text{AK}^m(X, D)_{\mathcal{R}} \\ \longrightarrow \Gamma(\Sigma, \Omega_{\pi^{-1}(c)}^{0,1} \otimes_{\mathbb{C}} u^* T X(-\log D)) \end{aligned} \quad (4.28)$$

is surjective, whenever u is simple. Therefore, the subset of simple maps $\mathfrak{M}_{0,s}^*(X, D, A)$ in the universal moduli space is a separable $C^{m-\ell}$ -smooth Banach manifold. Then by the Sard–Smale Theorem, the set of regular values $\text{AK}^{\text{reg}}(X, D)_{\mathcal{R}}$ of the projection map

$$\pi_2: \mathfrak{M}_{0,s}^*(X, D, A) \longrightarrow \text{AK}^{\text{reg}}(X, D)_{\mathcal{R}}$$

is Baire set of second category. With an argument similar to the Taubes' trick in the proof of [26, Thm. 3.1.6 (ii)], we conclude that the subset of smooth perturbations

$$\text{AK}^{\emptyset}(X, D)_{\mathcal{R}} = \text{AK}^{\text{reg}}(X, D)_{\mathcal{R}} \cap \text{AK}(X, D)_{\mathcal{R}}$$

satisfies the second statement in Proposition 4.1. $\square$

Moving to the simple nodal case in Sect. 4.3, we will need to show that certain evaluation maps on the universal moduli spaces are transverse. In the analytical setup of [26], the proof of transversality of evaluation maps in [26, Prop. 6.2.8] uses [26, Prop. 3.4.2] and induction on the number of edges. Proposition 3.4.2 in [26], itself, is a consequence of [26, Lem. 3.4.3]. We will need the following natural generalization of [26, Lem. 3.4.3] to show that the evaluation maps at the nodes and ob_Γ are transverse. With $\bar{\partial}^{\log}$ and $D_u^{\log} \bar{\partial}$ in place of $\bar{\partial}$ and $D_u \bar{\partial}$, respectively, its proof is similar to the (long and explicit) proof of [26, Lem. 3.4.3].

Lemma 4.7 *Let $A \neq 0$ and $(\Sigma, \vec{z})$ be a smooth k -marked curve.²⁰ With notation as above, let*

$$(u, J) \in \tilde{\mathfrak{M}}_{g,s}^*(X, D, A) \subset W_{A,s}^{\ell,p}((\Sigma, \vec{z}), (X, D)) \times \text{AK}^m(X, D)_{\mathcal{R}}.$$

For each $a \in [k]$, let ξ_a be a log tangent vector in $T_{u(z_a)} X(-\log D)_{\mathcal{R}}$. For each open set $U \subset \Sigma - \vec{z}$, there exists

$$\xi \in W^{\ell,p}(\Sigma, u^*TX(-\log D)_{\mathcal{R}}) \quad \text{and} \quad Y \in T_J \text{AK}^m(X, D)_{\mathcal{R}}$$

such that

$$\xi(z_a) = \xi_a, \quad \forall a \in [k], \quad \text{Supp}(Y|_{u(\Sigma)}) \subset U, \quad \text{and} \quad D_u \bar{\partial}^{\log} \xi + \frac{1}{2} Y \circ du \circ j = 0.$$

4.2 Depth- I Maps

For each $v \in \mathcal{H}_{g,k}(X, D)_{\mathcal{R},J}$, consider a log map

$$f = [u, (\zeta_i)_{i \in I}, (\Sigma, j, z_1, \dots, z_k)] \in \overline{\mathcal{M}}_{g,s}^{\log}(X, D, A, v)$$

where Σ is smooth, i.e., $u(\Sigma) \subset D_I$ for a non-trivial maximal subset $I \subset S$, $\text{ord}_{z_a}(u, D_i) = s_{ai} \geq 0$ for all $i \in [N] - I$, and $\text{ord}_{z_a}(\zeta_i) = s_{ai}$ for all $i \in I$. We allow s_{ai} to be negative for $i \in I$. Let

$$\mathcal{M}_{g,s}(X, D, A, v)_I \subset \overline{\mathcal{M}}_{g,s}(X, D, A, v)$$

be the stratum of such maps. The stratum $\mathcal{M}_{g,s}(X, D, A, v)_I$ is a generalization of the main stratum

$$\mathcal{M}_{g,s}(X, D, A, v)_{\emptyset} = \mathcal{M}_{g,s}(X, D, A, v)$$

where the domain is still smooth, but the image could lie in a non-trivial stratum of the divisor.

²⁰ We just need the sphere case.

Forgetting the meromorphic sections ζ_i , for the same reason²¹ as in [7, Remark 2.3], we get a topological embedding

$$\begin{aligned} \mathcal{M}_{g,\mathfrak{s}}(X, D, A, \nu)_I &\hookrightarrow \mathcal{M}_{g,\bar{\mathfrak{s}}}(D_I, \partial D_I, A, \nu_I), \\ [\phi, u, (\zeta_i)_{i \in I}, \Sigma, \bar{z}] &\longmapsto [\phi, u, \Sigma, \bar{z}], \end{aligned} \quad (4.29)$$

where $\partial D_I \subset D_I$ is the boundary divisor as in (2.5) and

$$\bar{\mathfrak{s}} = (s_a = (s_{ai})_{i \in [N]-I})_{a \in [k]} \in (\mathbb{N}^{[N]-I})^k.$$

In this section we prove the following transversality argument.

Proposition 4.8 *Suppose (X, ω) is a closed symplectic manifold, $D = \bigcup_{i \in [N]} D_i$ is an SNC symplectic divisor, $A \in H_2(X, \mathbb{Z})$, $g, k \in \mathbb{N}$, and $\mathfrak{s} \in (\mathbb{Z}^N)^k$, with $\bar{\mathfrak{s}} \in (\mathbb{N}^{[N]-I})^k$.*

- (1) *If $2g + k \geq 3$, for any given choice of universal family in (3.5), there exists a Baire set of second category $\mathcal{H}_{g,k}^I(X, D)_{\mathcal{R},J} \subset \mathcal{H}_{g,k}(X, D)_{\mathcal{R},J}$ such that for each $\nu \in \mathcal{H}_{g,k}^I(X, D)_{\mathcal{R},J}$, $\mathcal{M}_{g,\mathfrak{s}}(X, D, A, \nu)_I$ is a naturally oriented smooth manifold of the real dimension*

$$2(c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k - |I|).$$

- (2) *If $g=0$, $\nu \equiv 0$, and $A \neq 0$ or $k \geq 3$, the same statement holds for J in a Baire set of second category $\text{AK}^I(X, D)_{\mathcal{R}} \subset \text{AK}(X, D)_{\mathcal{R}}$, if we restrict to the subspace of simple maps $\mathcal{M}_{0,\mathfrak{s}}^*(X, D, A)_I$.*

Proposition 4.1 is a special case of Proposition 4.8, where $I = \emptyset$. By (3.20) and (3.23), in a neighborhood of D_I , every $\nu \in \mathcal{H}_{g,k}(X, D)_{\mathcal{R},J}$ can be decomposed as $\pi_I^* \nu_I \oplus n_I$, where

$$\nu_I \in \mathcal{H}_{g,k}(D_I, \partial D_I)_{\mathcal{R}_I, J_I},$$

and n_I is determined by a family of $\mathbb{C}^I$ -valued $(0, 1)$ -forms

$$\begin{aligned} \theta_I \in \Theta_{g,k}(D_I) &\equiv \left\{ (\theta_{I,i})_{i \in I} \in \Gamma(\overline{\mathfrak{L}}_{g,k}^* \times D_I, \pi_1^* \Omega_{g,k}^{0,1} \otimes_{\mathbb{C}} \mathbb{C}^I) : \right. \\ &\quad \left. \text{supp}(\theta_{I,i}) \subset \left(\overline{\mathfrak{L}}_{g,k}^* - \bigcup_{a \in [k]} \text{Im}(\mathfrak{z}_a) \right) \times D_I \right\}. \end{aligned}$$

The map

$$\text{Res} : \mathcal{H}_{g,k}(X, D)_{\mathcal{R},J} \longrightarrow \mathcal{H}_{g,k}(D_I, \partial D_I)_{\mathcal{R}_I, J_I} \times \Theta_{g,k}(D_I), \quad \nu \longmapsto (\nu_I, \theta_I), \quad (4.30)$$

²¹ Fixing a set of marked points, up to multiplication by a constant, there is at most one meromorphic section of any holomorphic line bundle with prescribed zeros/poles at the marked points and nowhere else.

is surjective and continuous. Therefore, in order to prove the first statement of Proposition 4.8, it is enough to find a Baire set of second category

$$\mathcal{H}\Theta_{g,k}^{\text{reg}}(D_I, \partial D_I)_{\mathcal{R}_I, J_I} \subset \mathcal{H}_{g,k}(D_I, \partial D_I)_{\mathcal{R}_I, J_I} \times \Theta_{g,k}(D_I) \quad (4.31)$$

such that for each v with $\text{Res}(v) := (v_I, \theta_I)$ in this set, the statement of Proposition 4.8 holds. Similarly, in case (2), the map

$$\text{AK}(X, D)_{\mathcal{R}} \longrightarrow \text{AK}(D_I, \partial D_I)_{\mathcal{R}_I}$$

is surjective and continuous. Below, we show that we can take $\text{AK}^I(X, D)_{\mathcal{R}}$ to be the preimage of $\text{AK}^{\emptyset}(D_I, \partial D_I)_{\mathcal{R}_I}$ given by the second part of Proposition 4.1. Therefore, the main goal of this section is to describe the normal bundle of the embedding (4.29) and $\mathcal{H}\Theta_{g,k}^{\text{reg}}(D_I, \partial D_I)_{\mathcal{R}_I, J_I}$. Proposition 4.8 can also be obtained from (proof of) Proposition 4.1 by a looking $(u, (\zeta_i)_{i \in I})$ as a log map into the fiber product (3.26) of the projectivizations of $\mathcal{N}_X D_i$. We will explain this argument in Remark 4.14.

For each complex curve (Σ, j) , let $\text{Pic}^0(\Sigma, j)$ be the group of degree 0 holomorphic line bundles on (Σ, j) , $\mathcal{O} = \mathcal{O}_{\Sigma, j} \in \text{Pic}^0(\Sigma, j)$ be the trivial line bundle, and $\mathcal{O}^I \equiv \bigoplus_{i \in I} \mathcal{O}$. Let

$$\text{Pic}^0(\mathfrak{U}_{g,k}) \longrightarrow \mathfrak{M}_{g,k}$$

be the fiber bundle whose fiber over every $c \in \mathfrak{M}_{g,k}$ is $\text{Pic}^0(\pi^{-1}(c))$. In the following, by $\mathcal{O}$ we mean the section

$$\mathcal{O}: \mathfrak{M}_{g,k} \longrightarrow \text{Pic}^0(\mathfrak{U}_{g,k})$$

that takes c to the trivial line bundle $\mathcal{O}_{\pi^{-1}(c)}$. Image of $\mathcal{O}$ has complex codimension g . By abuse of notation, we also let $\text{Pic}^0(\mathfrak{U}_{g,k})$ to denote the pull back of $\text{Pic}^0(\mathfrak{U}_{g,k})$ to $\mathcal{M}_{g,s}(D_I, \partial D_I, A, v_I)$ (or any other configuration space).

The next Lemma describes the (virtual) normal bundle of the embedding (4.29).

Lemma 4.9 *For each $(v_I, \theta_I) = \text{Res}(v)$ as in (4.30), there exists a natural map*

$$P_{\theta_I} = (P_{\theta_{I,i}})_{i \in I}: \mathcal{M}_{g,s_I}(D_I, \partial D_I, A, v_I) \longrightarrow \text{Pic}^0(\mathfrak{U}_{g,k})^I \quad (4.32)$$

such that

$$\mathcal{M}_{g,s}(X, D, A, v)_I = P_{\theta_I}^{-1}(\mathcal{O}^I).$$

In particular,

$$\mathcal{M}_{0,s}(X, D, A, v)_I = \mathcal{M}_{0,s_I}(D_I, \partial D_I, A, v_I). \quad (4.33)$$

Proof For each $i \in I$ and $(\phi, u, (\Sigma, j, \vec{z})) \in \mathcal{M}_{g, \mathfrak{s}_I}(D_I, \partial D_I, A, v_I)$, define

$$P_{\theta_{I,i}}(\phi, u, (\Sigma, j, \vec{z})) = u^* \mathcal{N}_X D_i \otimes \mathcal{O}_\Sigma \left(- \sum_{a \in [k]} s_{ai} z_a \right) \in \text{Pic}^0(\Sigma, j),$$

where the holomorphic structure on the line bundle $u^* \mathcal{N}_X D_i$ is given by the $\bar{\partial}$ -operator

$$D_u^{\mathcal{N}_i} \{ \bar{\partial} - v \} = \bar{\partial}_{u^* \mathcal{N}_X D_i} - (\phi, u)^* \theta_{I,i}$$

and the second term is the line bundle corresponding to the divisor $\sum_{a=1}^k s_{ai} z_a$. By definition, $P_{\theta_{I,i}}(\phi, u, \Sigma, \vec{z}) = \mathcal{O}$ if and only if there exists a non-trivial meromorphic section ζ_i of $u^* \mathcal{N}_X D_i$ with zeros/poles of order s_{ai} and z_a , for all $a = 1, \dots, k$ (and nowhere else). $\square$

Remark 4.10 In light of Lemma 4.9, the moduli space $\overline{\mathcal{M}}_{g, \mathfrak{s}}^{\log}(X, D, A, v)$ can be described without mentioning the meromorphic sections $\zeta_{v,i}$ in the following way. This explains the absence of these sections in the proof of Proposition 4.8 part (1) and other proofs. An element of $\overline{\mathcal{M}}_{g, \mathfrak{s}}^{\log}(X, D, A, v)$ is the equivalence class of a stable (J, v) -holomorphic map

$$(u_v, \Sigma_v, j_v, \vec{z}_v \cup q_v)_{v \in \mathbb{V}},$$

together with a choice of decorations $\{s_e\}_{e \in \mathbb{E}}$ on the nodal points such that

- (1) $s_e = -s_{\bar{e}}, \quad \forall e \in \mathbb{E}, \quad \sum_{e \in \mathbb{E}_v} s_e + \sum_{l \in \mathbb{L}_v} s_l = (A_v \cdot D_i)_{i \in [N]}, \quad \forall v \in \mathbb{V};$
- (2) for each $v \in \mathbb{V}$, $e \in \mathbb{E}_v$, and $i \notin I_v$, u_v has a tangency of order $s_{e,i}$ with D_i at q_e ;
- (3) for each $v \in \mathbb{V}$, $l \in \mathbb{L}_v$, and $i \notin I_v$, u_v has a tangency of order $s_{al,i}$ with D_i at z_{al} ;
- (4) there exists a vector-valued function $s : \mathbb{V} \rightarrow \mathbb{R}^N$ such that $s_v = s(v) \in \mathbb{R}_+^{I_v} \times \{0\}^{[N]-I_v}$ for all $v \in \mathbb{V}$, and

$$s_{v_2(e)} - s_{v_1(e)} = \lambda_e s_e \quad \text{for some } \lambda_e > 0, \quad \forall e \in \mathbb{E};$$

- (5) $u_v^* \mathcal{N}_X D_i \cong \mathcal{L}_{v,i} \equiv \mathcal{O}_{\Sigma_v} \left(\sum_{l \in \mathbb{L}_v} s_{al,i} z_{al} + \sum_{e \in \mathbb{E}_v} s_{e,i} q_e \right)$, for all $v \in \mathbb{V}$ and $i \in I_v$;
- (6) and, $\text{ob}_\Gamma(f) = 1$.

The last condition can (in theory) be expressed in terms of u_v , the canonical sections of $\mathcal{L}_{v,i}$, and the isomorphisms of the holomorphic line bundles in (5).

The following statements are immediate corollaries of the first and second statements of Lemma 4.9, respectively.

Corollary 4.11 Replacing (X, D) with $(D_I, \partial D_I)$ in Proposition 4.1, let $\mathcal{H}_{g,k}^\emptyset(D_I, \partial D_I)_{\mathcal{R}_I, J_I}$ and $\text{AK}^\emptyset(D_I, \partial D_I)_{\mathcal{R}_I}$ be the resulting sets of regular perturbations and almost complex structures on D_I , respectively.

(1) If $v_I \in \mathcal{H}_{g,k}^\emptyset(D_I, \partial D_I)_{\mathcal{R}_I, J_I}$ and the image of P_{θ_I} is transverse to $\mathcal{O}^I$ then $\mathcal{M}_{g,s}(X, D, A, v)_I$ is a naturally oriented manifold of the expected real dimension

$$\begin{aligned} & 2(c_1^{T D_I(-\log \bar{D})}(A) + (n - |I| - 3)(1 - g) + k - |I| \dim_{\mathbb{C}} \text{Pic}^0(\Sigma)) \\ & = 2(c_1^{T X(-\log D)}(A) + (n - 3)(1 - g) + k - |I|). \end{aligned} \quad (4.34)$$

(2) Similarly, if $J_I = J|_{T D_I} \in \text{AK}^\emptyset(D_I, \partial D_I)_{\mathcal{R}_I}$ is a naturally oriented manifold of the expected real dimension

$$2(c_1^{T D_I(-\log \bar{D})}(A) + (n - |I| - 3) + k) = 2(c_1^{T X(-\log D)}(A) + (n - 3) + k - |I|).$$

The second statement establishes Proposition 4.8 (2). It also shows that in the genus 0 case, Condition (5) in Remark 4.10 is automatically satisfied. By the first statement, in order to prove Proposition 4.8 (1), we need to show that for generic θ_I and v_I , the image of P_{θ_I} is transverse to $\mathcal{O}^I$.

Proof of Proposition 4.8 (1) Let $\Theta_{g,k}^m(D_I)$ be the completion of $\Theta_{g,k}^m(D_I)$ in the C^m -norm. Fix a sub-universal family $\tilde{\mathcal{C}} \rightarrow \mathcal{B}$ of $\mathfrak{U}_{g,k}$ around $C = (\Sigma, j, z_1, \dots, z_k)$ and a smooth trivialization φ of $\mathcal{C}$ as in (4.12) and (4.13), respectively. Consider the configuration space

$$W_I = W_{\bar{s}, A}^{\ell, p}((C, \tilde{j}|_{\mathcal{B}}), (D_I, \partial D_I))_\varphi \times \mathcal{H}_{g,k}^m(D_I, \partial D_I)_{\mathcal{R}_I, J_I} \times \Theta_{g,k}^m(D_I). \quad (4.35)$$

The map P in Lemma 4.9, extends to a map

$$P: W_I \longrightarrow \text{Pic}^0(\mathfrak{U}_{g,k})^I, \quad ((c, u), v_I, \theta_I) \longmapsto P_{\theta_I}(\text{id}_{\pi^{-1}(c)}, u, (\pi^{-1}(c), \tilde{j}(c))). \quad (4.36)$$

The universal moduli space

$$\begin{aligned} \mathfrak{M}_{g,s}(X, D, A)_I = & \left\{ ((c, u), v_I, \theta_I) \in W_I : \bar{\partial}^{\log} u(x) = v_{\log, I}(x, u(x)), \right. \\ & \left. \forall x \in \pi^{-1}(c), P((c, u), v_I, \theta_I) = \mathcal{O}^I \right\} \end{aligned} \quad (4.37)$$

is the $(0 \oplus \mathcal{O}^I)$ -level set of

$$(\bar{\partial}^{\log} - v_{\log, I}) \times P: W_I \longrightarrow E_{\bar{s}, A}^{\ell, p}((C, \tilde{j}|_{\mathcal{B}}), (D_I, \partial D_I)) \times (\text{Pic}^0(\mathfrak{U}_{g,k}))^I. \quad (4.38)$$

We show that $0 \times \mathcal{O}^I$ is a regular value.

Since $\Theta_{g,k}^m(D_I)$ is a linear space we have

$$T_{\theta_I} \Theta_{g,k}^m(D_I) \cong \Theta_{g,k}^m(D_I), \quad \forall \theta_I \in \Theta_{g,k}^m(D_I).$$

For every $((c, u), v_I, \theta_I) \in \mathfrak{M}_{g,\mathfrak{s}}(X, D, A)_I$, the normal component of the linearization of (4.38) at $((c, u), v_I, \theta_I)$ has the form

$$\begin{aligned} D_{((c,u),v_I,\theta_I)}(\bar{\partial}^{\log} - v_{\log,I}) \times P : \\ T_{(c,u)} W_{\bar{s},A}^{\ell,p}((C, \bar{\mathfrak{z}}|_B), (D_I, \partial D_I)) \oplus T_{v_I} \mathcal{H}_{g,k}^m(D_I, \partial D_I)_{\mathcal{R},J} \oplus T_{\theta_I} \Theta_{g,k}^m(D_I) \\ \longrightarrow W^{\ell-1,p}(\Sigma, \Omega_{\pi^{-1}(c)}^{0,1} \otimes_{\mathbb{C}} u^* T D_I(-\log \partial D_I)) \oplus H^{0,1}(\Sigma, \mathfrak{j})^I. \end{aligned} \quad (4.39)$$

It is given by (4.25) on the first two components and sends $0 \oplus 0 \oplus \tilde{\theta}_I$ to the cohomology class

$$0 \oplus [(\text{id}_{\pi^{-1}(c)}, u)^* \tilde{\theta}_I] \in H^{0,1}(\Sigma, \mathfrak{j})^I.$$

For the same reason as in the proof of Proposition 4.1, every element of

$$W^{\ell-1,p}(\Sigma, \Omega_{\pi^{-1}(c)}^{0,1} \otimes_{\mathbb{C}} u^* T D_I(-\log \partial D_I))$$

is in the image of the restriction of (4.39) to the first two summands of the domain, followed by the projection to the first component of the target. Since a representative of every cohomology class in $H^{0,1}(\Sigma, \mathfrak{j})^I$ can be extended to a global $(0, 1)$ -form on $\mathcal{U}_{g,k}$, the restriction of (4.39) to the third summand of the domain is a map onto the second summand of the target. Therefore, (4.39) is surjective. Consequently, by Implicit-Function Theorem, the universal moduli space $\mathfrak{M}_{g,\mathfrak{s}}(X, D, A)_I$ is a separable $C^{m-\ell}$ -smooth Banach manifold. Then by the Sard–Smale Theorem, the set of regular values $\mathcal{H}\Theta_{g,k}^{\text{reg},m}(D_I, \partial D_I)_{\mathcal{R},J_I}$ of the projection map

$$\text{proj}: \mathfrak{M}_{g,\mathfrak{s}}(X, D, A)_I \longrightarrow \mathcal{H}\Theta_{g,k}^m(D_I, \partial D_I)_{\mathcal{R},J_I} \times \Theta_{g,k}^m(D_I)$$

is Baire set of second category. By construction, for every v with

$$\text{Res}(v) = (v_I, \theta_I) \in \mathcal{H}\Theta_{g,k}^{\text{reg},m}(D_I, \partial D_I)_{\mathcal{R},J_I}$$

the stratum

$$\mathfrak{M}_{g,\mathfrak{s}}(X, D, A, v)_I = \text{proj}^{-1}(v_I, \theta_I)$$

is a naturally oriented smooth manifold of the expected dimension (4.34). With an argument similar to the Taubes' trick in the proof of [26, Thm. 3.1.6 (ii)], we conclude

that the subset of smooth perturbations

$$\begin{aligned} \mathcal{H}\Theta_{g,k}^{\text{reg}}(D_I, \partial D_I)_{\mathcal{R}_I, J_I} \\ = \mathcal{H}\Theta_{g,k}^{\text{reg}, \text{m}}(D_I, \partial D_I)_{\mathcal{R}_I, J_I} \cap (\mathcal{H}_{g,k}(D_I, \partial D_I)_{\mathcal{R}_I, J_I} \times \Theta_{g,k}(D_I)) \end{aligned}$$

satisfies the requirement of (4.31). The set

$$\mathcal{H}_{g,k}^I(X, D)_{\mathcal{R}, J} = \text{Res}^{-1}(\mathcal{H}\Theta_{g,k}^{\text{reg}}(D_I, \partial D_I)_{\mathcal{R}_I, J_I})$$

satisfies Proposition 4.8 (1). $\square$

Remark 4.12 Suppose

$$f = [\phi, u, \zeta = (\zeta_i)_{i \in I}, \Sigma, \vec{z}] \in \mathcal{M}_{g, \mathfrak{s}}(X, D, A, v)_I.$$

Restricted to D_I , by (2.11), we have

$$TX(-\log D)|_{D_I} = TD_I(-\log \partial D_I) \oplus D_I \times \mathbb{C}^I. \quad (4.40)$$

Replacing (X, D, v) in (4.7) with $(D_I, \partial D_I, v_I)$, we get the linearized CR operator

$$\begin{aligned} D_u^{\log} \{\bar{\partial} - v_I\} : W^{\ell, p}(\Sigma, u^*TD_I(-\log \partial D_I)) \\ \longrightarrow W^{\ell-1, p}(\Sigma, \Omega_{\Sigma}^{0,1} \otimes_{\mathbb{C}} u^*TD_I(-\log \partial D_I)). \end{aligned} \quad (4.41)$$

Let

$$\bar{\partial}_{\text{std}} : W^{\ell, p}(\Sigma, \mathbb{C}^I) \longrightarrow W^{\ell-1, p}(\Sigma, \Omega_{\Sigma, j}^{0,1} \otimes_{\mathbb{C}} \mathbb{C}^I) \quad (4.42)$$

denote the standard $\bar{\partial}$ -operator on the trivial bundle $\Sigma \times \mathbb{C}^I$. In order to extend (4.7) to the case of maps with smooth domain but image in a stratum D_I , using the decomposition (4.40), define

$$\begin{aligned} D_u^{\log} \{\bar{\partial} - v\} : W^{\ell, p}(\Sigma, u^*TX(-\log D)) \rightarrow W^{\ell-1, p}(\Sigma, \Omega_{\Sigma}^{0,1} \otimes_{\mathbb{C}} u^*TX(-\log D)), \\ D_u^{\log} \{\bar{\partial} - v\}(\xi \oplus \eta = (\eta_i)_{i \in I}) = D_u^{\log} \{\bar{\partial} - v_I\}(\xi) \oplus \left(\bar{\partial}_{\text{std}}(\eta_i) - D_{\xi} D_u^{\mathcal{N}_i} \{\bar{\partial} - v\} \right)_{i \in I}, \end{aligned} \quad (4.43)$$

where

$$D_{\xi} D_u^{\mathcal{N}_i} \{\bar{\partial} - v\} \in \Gamma(\Sigma, \Omega_{\Sigma, j}^{0,1})$$

is the derivative of the $\bar{\partial}$ -operator $D_u^{\mathcal{N}_i} \{\bar{\partial} - v\}$ in (3.15) in the direction of ξ . Note that the derivative of a 1-parameter family of $\bar{\partial}$ -operators on a complex line bundle is a

$(0, 1)$ -form. In other words, with respect to the decomposition (4.40), $D_{(u, \zeta)}^{\log} \{\bar{\partial} - v\}$ is the lower diagonal operator

$$\begin{bmatrix} D_u^{\log} \{\bar{\partial} - v_I\} & 0 \\ * & \bar{\partial}_{\text{std}} \end{bmatrix}$$

with diagonal entries (4.41) and (4.42). Cohomology class of

$$(D_{\xi} D_u^{\mathcal{N}_i} \{\bar{\partial} - v\})_{i \in I} \in \Gamma(\Sigma, \Omega_{\Sigma, i}^{0,1})^I$$

is

$$D_u P(\xi) \in H^{0,1}(\Sigma, \mathfrak{j})^I.$$

In light of the proof of Proposition 4.8, Corollary 4.11 can be rephrased in the following way.

Corollary 4.13 *If $D_u^{\log} \{\bar{\partial} - v\}$ is surjective then $\mathcal{M}_{g, \mathfrak{s}}(X, D, A, v)_I$ is cut transversely in a neighborhood of f . Furthermore, if $\mathfrak{s} \in (\mathbb{N}^{[N]})^k$, then the whole moduli space $\overline{\mathcal{M}}_{g, \mathfrak{s}}(X, D, A, v)$ is cut transversely in a neighborhood of f .*

The elements of the form $0 \oplus (c_i)_{i \in I}$ in the kernel of $D_u^{\log} \{\bar{\partial} - v\}$, where c_i is a constant section of the trivial line bundle $\Sigma \times \mathbb{C}$, correspond to those deformations of u that push the image of u out of D_i in the direction of ζ_i by $c_i \zeta_i$.

Proposition 4.8 can essentially be obtained from Proposition 4.1 by looking at $(u, (\zeta_i)_{i \in I})$ as a log map into the fiber product (3.26) of the projectivizations of $\mathcal{N}_X D_i$, in the following way.

Remark 4.14 Given $(X, D, \omega, \mathcal{R})$, let

$$\tilde{X} \equiv \prod_{i \in I} \mathbb{P}(\mathcal{N}_X D_i|_{D_I} \oplus \mathbb{C}) \longrightarrow D_I \quad (4.44)$$

be the associated $(\mathbb{P}^1)^I$ -fiber bundle over D_I . For $i \in [N] - I$, let $\tilde{D}_i = \pi^{-1} D_{i \cup I}$. For $i \in I$, let

$$\tilde{D}_i = (D_{i,0} \cup D_{i,\infty}) \times \prod_{j \in I-i} \mathbb{P}(\mathcal{N}_X D_j|_{D_I} \oplus \mathbb{C}),$$

where $D_{i,0}$ and $D_{i,\infty}$ are the zero and infinity divisors of $\mathbb{P}(\mathcal{N}_X D_i|_{D_I} \oplus \mathbb{C})$, respectively. The inclusion

$$\tilde{D} = \bigcup_{i \in [N]} \tilde{D}_i \subset \tilde{X}$$

is an SNC symplectic divisor with respect to the symplectic structure obtained from the standard symplectic structure on $\mathcal{N}_X D_I$. As explained in [17, p. 11], from any $\mathcal{R}$ -compatible almost complex structure J on (X, D) and any $v \in \mathcal{H}_{g,k}(X, D)$ we obtain a compatible $(\mathbb{C}^*)^I$ -equivariant almost complex structure $\tilde{J}$ on $(\tilde{X}, \tilde{D})$ and a $(\mathbb{C}^*)^I$ -equivariant perturbation term $\tilde{v} \in \mathcal{H}_{g,k}(\tilde{X}, \tilde{D})$. The latter only depends on v_I and θ_I . Every tuple $(\phi, u, \zeta = (\zeta_i)_{i \in I}, \Sigma, \vec{z})$ representing an element of $\mathcal{M}_{g,s}(X, D, A, v)_I$ can be seen as a log $(\tilde{J}, \tilde{v})$ -map $\tilde{u} : \Sigma \rightarrow \tilde{X}$ representing an element of

$$\mathcal{M}_{g,s}(\tilde{X}, \tilde{D}, \tilde{A}, \tilde{v})$$

where, for each $i \in I$, $s_{ai} > 0$ denotes a tangency of order s_{ai} with $D_{i,0}$ and $s_{ai} < 0$ denotes a tangency of order $|s_{ai}|$ with $D_{i,\infty}$. We have

$$\mathcal{M}_{g,s}(X, D, A, v)_I \cong \mathcal{M}_{g,s}(\tilde{X}, \tilde{D}, \tilde{A}, \tilde{v})/(\mathbb{C}^*)^I. \quad (4.45)$$

Following the proof of Proposition 4.1, we can show that for generic $(\tilde{J}, \tilde{v})$, $\mathcal{M}_{g,s}(\tilde{X}, \tilde{D}, \tilde{A}, \tilde{v})$ is a smooth oriented $(\mathbb{C}^*)^I$ -equivariant manifold of real dimension

$$2(c_1^{T\tilde{X}(-\log \tilde{D})}(\tilde{A}) + (n-3)(1-g) + k) = 2(c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k).$$

Then Proposition 4.1 follows from (4.45), and $D_u^{\log}\{\bar{\partial} - v\}$ in (4.43) is equivalent to $D_u^{\log}\{\bar{\partial} - \tilde{v}\}$ defined in (4.7).

4.3 Simple Nodal Maps

Moving to the nodal case, let

$$\mathcal{M}_{g,s}(X, D, A, v)_\Gamma \subset \overline{\mathcal{M}}_{g,s}(X, D, A, v)$$

be the stratum of stable nodal log (J, v) -holomorphic curves with the decorated dual graph Γ (and $|\mathbb{V}| \geq 2$). With a setup similar to [11, Sect. 6.3], the deformation/obstruction theory of $\mathcal{M}_{g,s}(X, D, A)_\Gamma$ around any $f = (u, [\zeta], \Sigma, z_1, \dots, z_k)$ is given by (1) the sum of $D_{u_v}^{\log}\{\bar{\partial} - v\}$ over the irreducible components Σ_v , and (2) the obstruction map in (3.41), i.e.,

$$\text{ob}_\Gamma : \times_{v \in \mathbb{V}} \mathcal{M}_{g_v, s_v}(X, D, A_v, v)_{I_v} \longrightarrow \mathcal{G}(\Gamma), \quad (4.46)$$

where $\times_{v \in \mathbb{V}}$ denotes the fiber product over the evaluation maps at the nodes.

We write $\mathbb{V} = \mathbb{V}_p \cup \mathbb{V}_b$, where $\mathbb{V}_p$ corresponds to set of non-contracted or **principal** components and $\mathbb{V}_b$ corresponds to set of contracted or **bubble** components. For each $v \in \mathbb{V}$, let $\mathcal{M}_{g_v, s_v}^*(X, D, A_v, v)_{I_v}$ be the space of simple maps corresponding to the v -th component Σ_v of Σ in (3.38). An element of $\mathcal{M}_{g_v, s_v}^*(X, D, A_v, v)_{I_v}$ is the equivalence class of a tuple

$$f_v = (\phi_v = \phi|_{\Sigma_v}, u_v, \zeta_v = (\zeta_{v,i})_{i \in I_v}, C_v = (\Sigma_v, j_v, \vec{z}_v \cup q_v))$$

where $\vec{z}_v$ are the marked points on Σ_v , $q_v = \{q_e\}_{e \in \mathbb{E}_v}$ is the set of nodal points on Σ_v , $\mathfrak{s}_v$ is the set of tangency orders at $\vec{z}_v \cup q_v$, u_v is a map into D_{I_v} satisfying

$$\bar{\partial}u_v(x) = v(\phi_v(x), u_v(x)), \quad \forall x \in \Sigma_v,$$

and $\zeta_v = (\zeta_{v,i})_{i \in I_v}$ is a meromorphic section of $\mathcal{N}_X D_{I_v}$ with zeros/poles of orders determined by $\mathfrak{s}_v$ at $\vec{z}_v \cup q_v$. Locally around any f , we will fix a random ordering $\vec{q}_v$ of q_v that we will forget at the end. If Σ_v is not a bubble component, then

$$\mathcal{M}_{g_v, \mathfrak{s}_v}^*(X, D, A_v, v)_{I_v} = \mathcal{M}_{g_v, \mathfrak{s}_v}(X, D, A_v, v)_{I_v};$$

otherwise, Σ_v is a sphere, ϕ_v is a constant map (that we will drop from the notation), $\bar{\partial}u_v = 0$, and by (4.33)

$$\mathcal{M}_{g_v, \mathfrak{s}_v}^*(X, D, A_v, v)_{I_v} = \mathcal{M}_{0, \bar{\mathfrak{s}}_v}^*(D_{I_v}, \partial D_{I_v}, A_v).$$

For each $e \in \mathbb{E}_v$, let

$$\text{ev}_e: \mathcal{M}_{g_v, \mathfrak{s}_v}(X, D, A_v, v)_{I_v} \longrightarrow D_{I_e}, \quad [u_v, \zeta_v, C_v] \longmapsto u_v(q_e)$$

denote the evaluation map at the nodal point q_e . Recall that for $I = \emptyset$ the convention is $D_I = X$. Let

$$\text{ev}_{\mathbb{E}} \equiv \prod_{e \in \mathbb{E}} (\text{ev}_e \times \text{ev}_e): \prod_{v \in \mathbb{V}} \mathcal{M}_{g_v, \mathfrak{s}_v}(X, D, A_v, v)_{I_v} \longrightarrow \prod_{e \in \mathbb{E}} (D_{I_e})^2$$

denote the overall evaluation maps at the nodal points. Then the fiber product space in (4.46) is

$$\mathcal{M}_{g, \mathfrak{s}}^{\text{plog}}(X, D, A, v)_{\Gamma} = \times_{v \in \mathbb{V}} \mathcal{M}_{g_v, \mathfrak{s}_v}(X, D, A_v, v)_{I_v} = \text{ev}_{\mathbb{E}}^{-1} \left(\prod_{e \in \mathbb{E}} \Delta_e \right) \quad (4.47)$$

where

$$\Delta_e \subset D_{I_e} \times D_{I_e}, \quad \forall e \in \mathbb{E},$$

is the diagonal subspace. The obstruction map in (3.41) is the map (4.46) from this fiber product into the obstruction group $\mathcal{G}$ and

$$\mathcal{M}_{g, \mathfrak{s}}(X, D, A, v)_{\Gamma} = \text{ob}_{\Gamma}^{-1}(1).$$

Let

$$\prod_{v \in \mathbb{V}}^* \mathcal{M}_{g_v, \mathfrak{s}_v}^*(X, D, A_v, v)_{I_v} \subset \prod_{v \in \mathbb{V}} \mathcal{M}_{g_v, \mathfrak{s}_v}^*(X, D, A_v, v)_{I_v} \quad (4.48)$$

be the subset of tuples where the images of every two non-constant bubble components in X are distinct. Also, let

$$\mathcal{M}_{g,s}^{\text{plog},\star}(X, D, A, \nu)_{\Gamma} = \mathcal{M}_{g,s}^{\text{plog}}(X, D, A, \nu)_{\Gamma} \cap \prod_{v \in \mathbb{V}}^{\star} \mathcal{M}_{g_v, s_v}^{\star}(X, D, A_v, \nu)_{I_v}. \quad (4.49)$$

By (3.6) and Definition 3.1, the map ϕ has image in a product of universal families

$$(\pi_v : \overline{\mathfrak{U}}_{g_v, k_v + \ell_v} \longrightarrow \overline{\mathfrak{M}}_{g_v, k_v + \ell_v, \mathfrak{z}_v \cup \mathfrak{q}_v})_{v \in \mathbb{V}_p}. \quad (4.50)$$

The restriction of a perturbation term ν (or $\nu_{\log}$) in $\mathcal{H}_{g,k}(X, D)$ to $\overline{\mathfrak{U}}_{g_v, |z_v| + |q_v|}$ defines an element of $\mathcal{H}_{g_v, |z_v| + |q_v|}(X, D)$. Furthermore, recall from (4.30) that if $\nu \in \mathcal{H}_{g,k}(X, D)_{\mathcal{R}, J}$, then the restriction of that to $\mathfrak{U}_{g_v, |z_v| + |q_v|}$ and D_{I_v} is made of components

$$\nu_v \in \mathcal{H}_{g_v, |z_v| + |q_v|}(D_{I_v}, \partial D_{I_v})_{\mathcal{R}_{I_v}, J_{I_v}} \quad (4.51)$$

and

$$\theta_v = (\theta_{v,i})_{i \in I_v} \in \Theta_{g_v, |z_v| + |q_v|}(D_{I_v}) \quad (4.52)$$

such that

$$\bar{\partial} u_v = (\phi_v, u_v)^* \nu_v \quad \text{and} \quad \bar{\partial}_{u_v^* \mathcal{N}_X D_i} \zeta_{I_v, i} = \theta_{v,i} \zeta_{I_v, i}, \quad \forall i \in I_v. \quad (4.53)$$

Lemma 4.15 *With notation as above, if*

- (1) $J \in \text{AK}^{I_v}(X, D)_{\mathcal{R}}$ in the sense of Proposition 4.8 (2) for all $v \in \mathbb{V}_b$,
- (2) $\nu \in \mathcal{H}_{g_v, |z_v| + |q_v|}^{I_v}(X, D)_{\mathcal{R}, J}$ in the sense of Proposition 4.8 (1) for all $v \in \mathbb{V}_p$,
- (3) the map $\text{ev}_{\mathbb{E}}$ restricted to $\prod_{v \in \mathbb{V}}^{\star} \mathcal{M}_{g_v, s_v}^{\star}(X, D, A_v, \nu)_{I_v}$ is transverse, and
- (4) the map ob_{Γ} restricted to $\mathcal{M}_{g,s}^{\text{plog},\star}(X, D, A, \nu)_{\Gamma}$ is transverse,

then $\mathcal{M}_{g,s}^{\star}(X, D, A, \nu)_{\Gamma}$ is a naturally oriented smooth manifold of the real dimension (1.16).

Proof By Proposition 4.8, under the first two conditions, each $\mathcal{M}_{g_v, s_v}^{\star}(X, D, A_v, \nu)_{I_v}$ is a naturally oriented smooth manifold of the real dimension

$$2(c_1^{TX(-\log D)}(A_v) + (n-3)(1-g_v) + k_v + \ell_v - |I_v|), \quad \text{where } k_v = |\vec{z}_v|, \ell_v = |q_v|.$$

By (3), since each D_I is oriented, the fiber product space $\mathcal{M}_{g,s}^{\text{plog},\star}(X, D, A, \nu)_\Gamma$ is a naturally oriented smooth manifold of the real dimension

$$\begin{aligned} & 2 \left(\sum_{v \in \mathbb{V}} (c_1^{TX(-\log D)}(A_v) + (n-3)(1-g_v) + k_v + \ell_v - |I_v|) - \sum_{e \in \mathbb{E}} (n - |I_e|) \right) \\ &= 2 \left(c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k - |\mathbb{E}| - \sum_{v \in \mathbb{V}} |I_v| + \sum_{e \in \mathbb{E}} |I_e| \right). \end{aligned}$$

By (3.39),

$$\dim_{\mathbb{R}} \mathbb{K}_{\mathbb{R}}(\Gamma) - \dim_{\mathbb{C}}(\mathcal{G}) = |\mathbb{E}| + \sum_{v \in \mathbb{V}} |I_v| - \sum_{e \in \mathbb{E}} |I_e|.$$

Therefore, by (4), and since $\mathcal{G}$ is a complex manifold, $\mathcal{M}_{g,s}^{\star}(X, D, A, \nu)_\Gamma$ is a naturally oriented smooth orbifold of the real dimension

$$\begin{aligned} & 2 \left(c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k - |\mathbb{E}| - \sum_{v \in \mathbb{V}} |I_v| + \sum_{e \in \mathbb{E}} |I_e| - \dim_{\mathbb{C}}(\mathcal{G}) \right) \\ &= 2 \left(c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k - \dim_{\mathbb{R}} \mathbb{K}_{\mathbb{R}}(\Gamma) \right). \quad \square \end{aligned}$$

Recall that, by the first condition in Definition 3.11,

$$\dim_{\mathbb{R}} \mathbb{K}_{\mathbb{R}}(\Gamma) > 0$$

unless Γ is the trivial one-vertex graph ($\mathbb{V} = \{v\}$, $\mathbb{E} = \emptyset$ with $I_v = \emptyset$) which corresponds to the main stratum. In the classical case (no D), the map ϱ in (3.39) is the trivial map $\mathbb{Z}^{\mathbb{E}} \rightarrow 0$. Therefore, $\dim_{\mathbb{R}} \mathbb{K}_{\mathbb{R}}(\Gamma) = |\mathbb{E}|$ is the number of the nodes. In the logarithmic case, there are configurations with arbitrary large number of nodes and $\dim_{\mathbb{R}} \mathbb{K}_{\mathbb{R}}(\Gamma) = 1$; see [7, Example 2.13].

Proof of Theorem 1.5 With modifications as in Sects. 4.1 and 4.2, proof of this proposition is similar to the proof of [32, Prop. 3.16], [26, Thm. 6.2.6], and [35, Prop. 4.3]. The main difference is the extra evaluation-type map ob_Γ that needs to be transversal as in Lemma 4.15 (4).

For each $v \in \mathbb{V}_p$, let $\overline{\mathcal{U}}_{g_v, k_v + \ell_v}$, ν_v , and θ_v be as in (4.50), (4.51), and (4.52), respectively. For each $v \in \mathbb{V}_b$, since $\Sigma_v = \mathbb{P}^1$, if C_v is stable, let $\overline{\mathcal{U}}_{0, k_v + \ell_v}$ denote the universal curve, and if C_v is not stable, i.e., if it is a $\mathbb{P}^1$ with less than 3 points, then let $\overline{\mathcal{U}}_{0, k_v + \ell_v} = C_v$ and $\overline{\mathcal{M}}_{0, k_v + \ell_v}$ be just a point. For $v \in \mathbb{V}_b$, $\nu_v = \phi^* \nu|_{\Sigma_v}$ is zero and we are dealing with log J -holomorphic curves.

For each $v \in \mathbb{V}$, fix a local family $\pi_v|_{C_v} : C_v \rightarrow \mathcal{B}_v$ around C_v as in (4.12) and a smooth trivialization φ_v of that as in (4.13).

For $v \in \mathbb{V}_p$, let

$$W_v \equiv W_{\bar{s}_v, A_v}^{\ell, p}((\mathcal{C}_v, \{\mathfrak{z}_v \cup \mathfrak{q}_v\}|_{\mathcal{B}_v}), (D_{I_v}, \partial D_{I_v}))_{\varphi_v} \\ \times \mathcal{H}_{g_v, k_v + \ell_v}^m(D_{I_v}, \partial D_{I_v})_{[\omega]} \times_J \Theta_{g_v, k_v + \ell_v}^m(D_{I_v}),$$

where

$$\bar{s}_v = (s_v; a = (s_v; a_i)_{i \in [N] - I_v})_{a \in [k_v + \ell_v]} \in (\mathbb{N}^{[N] - I_v})^{k_v + \ell_v}$$

is as in (4.29), be the corresponding configuration space as in (4.35). This time, $(\mathcal{R}, J)$ is not fixed and can change as well. Thus, the notation $\times_J$ means fiber product over $\text{AK}(X, D)_{[\omega]}$. For each $v \in \mathbb{V}_b$, the configuration space is

$$W_v \equiv W_{\bar{s}_v, A_v}^{\ell, p}((\mathcal{C}_v, \{\mathfrak{z}_v \cup \mathfrak{q}_v\}|_{\mathcal{B}_v}), (D_{I_v}, \partial D_{I_v}))_{\varphi_v} \times \text{AK}^m(X, D)_{[\omega]}$$

as in (4.26) with $(D_{I_v}, \partial D_{I_v})$ in place of (X, D) . The evaluation map $\text{ev}_{\mathbb{E}}$ at the nodal points extends to the product $\prod_{v \in \mathbb{V}} W_v$. For $v \in \mathbb{V}$, let

$$\mathcal{E}_v \equiv \mathcal{E}_{\bar{s}_v, A_v}^{\ell, p}((\mathcal{C}_v, \{\mathfrak{z}_v \cup \mathfrak{q}_v\}|_{\mathcal{B}_v}), (D_{I_v}, \partial D_{I_v})) \longrightarrow W_v$$

be the Banach bundles in (4.16). Let

$$W_{\Gamma} \equiv \times_{v \in \mathbb{V}} W_v = \text{ev}_{\mathbb{E}}^{-1} \left(\prod_{e \in \mathbb{E}} \Delta_e \right),$$

$\pi_v: W_{\Gamma} \rightarrow W_v$ denote the projection map into the v -th component, and

$$E_{\Gamma} \equiv \bigoplus_{v \in \mathbb{V}} \pi_v^* \mathcal{E}_v \longrightarrow W_{\Gamma}$$

denote the obstruction Banach bundle. The tangent space of W_{Γ} has the form

$$T_f W_{\Gamma} \cong T_f^{\text{ver}} W_{\Gamma} \oplus \bigoplus_{v \in \mathbb{V}} T_{C_v} \mathcal{B}_v \oplus T_{(\mathcal{R}, J)} \text{AK}^m(X, D)_{[\omega]} \\ \oplus \bigoplus_{v \in \mathbb{V}_p} (T_{v_v} \mathcal{H}_{g_v, k_v + \ell_v}^m(D_{I_v}, \partial D_{I_v})_{\mathcal{R}_{I_v}, J_{I_v}} \oplus T_{\theta_v} \Theta_{g_v, k_v + \ell_v}^m),$$

where $T_f^{\text{ver}} W_{\Gamma}$ has the following description. By (4.8),

$$T_{u_v} W_{\bar{s}_v, A_v}^{\ell, p}((\Sigma_v, \vec{z}_v \cup q_v), (D_{I_v}, \partial D_{I_v})) = W^{\ell, p}(\Sigma_v, u_v^* T D_{I_v}(-\log D_{I_v})).$$

For every $\xi \in W^{k,p}(\Sigma_v, u_v^* T D_{I_v}(-\log D_{I_v}))$, the section $\iota(\xi) \in C^0(\Sigma_v, u_v^* T D_{I_v})$ defined via (2.10) satisfies

$$\iota(\xi)(q_e) \in T D_{I_e}, \quad \forall e \in \mathbb{E}_v.$$

Then

$$T_f^{\text{ver}} W_\Gamma = \left\{ (\xi_v)_{v \in \mathbb{V}} \in \bigoplus T_{u_v} W_{\bar{s}_v, A_v}^{\ell,p}((\Sigma_v, \bar{z}_v \cup q_v), (D_{I_v}, \partial D_{I_v})) : \right. \\ \left. \iota(\xi_v)(q_e) = \iota(\xi_{v'})(q_e), \quad \forall v, v' \in \mathbb{V}, e \in \mathbb{E}_{v,v'} \right\}. \quad (4.54)$$

Summing the maps (4.36) over all principal components we get

$$P = \prod_{v \in \mathbb{V}_p} P_v : W_\Gamma \longrightarrow \text{Pic}_\Gamma^0 \equiv \prod_{v \in \mathbb{V}_p} \text{Pic}^0(\mathcal{U}_{g_v, k_v + \ell_v})^{I_v}. \quad (4.55)$$

Let $\tilde{\mathfrak{M}}_{g,s}^{\text{plog}}(X, D, A)_\Gamma$ denote the $(0 \times \mathcal{O}_\Gamma)$ -level set of

$$(\bar{\partial}^{\log} - \nu_{\log}) \times P : W_\Gamma \longrightarrow E_\Gamma \times \text{Pic}_\Gamma^0, \quad (4.56)$$

where $\mathcal{O}_\Gamma \equiv \prod_{v \in \mathbb{V}_p} \mathcal{O}^{I_v}$. The universal pre-log moduli space $\mathfrak{M}_{g,s}^{\text{plog}}(X, D, A)_\Gamma$ is the quotient of $\tilde{\mathfrak{M}}_{g,s}^{\text{plog}}(X, D, A)_\Gamma$ by the automorphism group. The latter is the product of automorphism groups of the bubble components $\Sigma_v \cong \mathbb{P}^1$ with $k_v + \ell_v < 3$.

Claim 1 *Restricted to the subset of simple maps $\tilde{\mathfrak{M}}_{g,s}^{\text{plog},*}(X, D, A)_\Gamma$, $0 \times \mathcal{O}_\Gamma$ is a regular value of (4.56).*

Proof For every $(f, J, \nu) \in \tilde{\mathfrak{M}}_{g,s}^{\text{plog}}(X, D, A)_\Gamma$, the linearization $D_{f,J,\nu}^{\log}(\{\bar{\partial} - \nu\} \times P)$ of (4.56) is the direct sum linearization map

$$D_{f,J,\nu}^{\log}(\{\bar{\partial} - \nu\} \times P) \equiv \bigoplus_{v \in \mathbb{V}_b} D_{u_v, J}^{\log} \bar{\partial} \oplus \bigoplus_{v \in \mathbb{V}_p} D_{u_v, J, \nu_v}^{\log}(\{\bar{\partial} - \nu_v\} \times P_v) \quad (4.57)$$

with summands as in (4.28) and (4.39). By the proof of Proposition 4.8 (1), fixing J , for each $v \in \mathbb{V}_p$, $D_{u_v, \nu_v}^{\log}(\{\bar{\partial} - \nu_v\} \times P_v)$ is surjective onto

$$\mathcal{E}_v \oplus T_{\mathcal{O}^{I_v}} \text{Pic}^0(\mathcal{U}_{g_v, k_v + \ell_v})^{I_v}.$$

Furthermore, transversality can be achieved by sections supported away from the nodes. Therefore, for any fixed J , the direct sum

$$\bigoplus_{v \in \mathbb{V}_p} D_{u_v, \nu_v}^{\log}(\{\bar{\partial} - \nu_v\} \times P_v)$$

is surjective onto

$$\bigoplus_{v \in \mathbb{V}_p} \mathcal{E}_v \oplus T_{\mathcal{O}_\Gamma \text{Pic}_\Gamma^0}.$$

By the proof of Proposition 4.8 (2), for each $v \in \mathbb{V}_b$, $D_{u_v, J}^{\log} \bar{\partial}$ is surjective onto $\mathcal{E}_v$. Furthermore, if the bubble components are all simple with mutually different images, as in the proof of [26, Prop. 6.2.7], by Proposition 4.8 (2) and Lemma 4.7, transversality can be achieved simultaneously by deformations of J that are supported along different images of these components. We conclude that (4.57) is surjective along the subset of simple maps.

Claim 2 *The map $\text{ob}_\Gamma: \tilde{\mathcal{M}}_{g, s}^{\text{plog}, \star}(X, D, A)_\Gamma \rightarrow \mathcal{G}(\Gamma)$ is transverse.*

Proof Fix a tuple

$$f = \left((c, u_v, \zeta_v = (\zeta_{v,i})_{i \in I_v}, C_v = (\Sigma_v, j_v, \bar{z}_v \cup q_v)), J, v \right) \in \text{ob}_\Gamma^{-1}(1).$$

By definition, we can choose local holomorphic coordinates w_e around each nodal point $q_e \in \Sigma_v$ and representatives ζ_v such that

$$\eta_{e,i}/\eta_{e,i} = 1, \quad \forall e \in \mathbb{E}, i \in I_e; \quad (4.58)$$

see (3.48). Let q_e be a nodal point on Σ_v connecting that to $\Sigma_{v'}$. By Lemma A.1, we may assume that either $v \in \mathbb{V}_p$ or $v \in \mathbb{V}_b$ and $A_v \neq 0$. Equally, one may use the method of proof of [26, Thm. 6.3.1] in [26, p. 155] to address the nodes connecting two ghost bubbles. Choose any $i \in I_e$.

(i) If $v \in \mathbb{V}_p$ and $i \in I_v$, we have a meromorphic section $\zeta_{v,i}$ included in f satisfying

$$\bar{\partial}_{u_v^* \mathcal{N}_X D_i} \zeta_{v,i} = \theta_{v,i} \zeta_{v,i};$$

see (4.53) above. Recall from (3.45) that

$$\zeta_{v,i}(w_e) = \tilde{\eta}_{e,i}(w_e) w_e^{s_{e,i}} \in u_v^* \mathcal{N}_X D_i,$$

such that

$$0 \neq \eta_{e,i} \equiv \tilde{\eta}_{e,i}(0) \in \mathcal{N}_X D_i|_{u_v(q_e)}.$$

Let

$$\beta: \Sigma_v \longrightarrow \mathbb{C}$$

be a smooth function that is supported in a neighborhood of q_e , is constant in a smaller neighborhood of q_e , and satisfies

$$\frac{d}{dt}(e^{t\beta(q_e)} \eta_{e,i}/\eta_{e,i})|_{t=0} = 1. \quad (4.59)$$

Let

$$(f_t)_{t \in [0,1]} \in \tilde{\mathfrak{M}}_{g,s}^{\text{plog},\star}(X, D, A, v)_\Gamma$$

be the path obtained by deforming $\zeta_{v,i}$ to $e^{t\beta}\zeta_{v,i}$ and $\theta_{v,i}$ to $\theta_{v,i} + t\bar{\partial}\beta$. Then, by (4.59),

$$\frac{d}{dt} \text{ob}_\Gamma(f_t) = [1_{e,i}] \in T_1 \mathcal{G}_\Gamma = \text{coker}(\varrho_\mathbb{C}), \quad (4.60)$$

where $[1_{e,i}]$ denotes the image of $1_{e,i} \in \bigoplus_{e \in \mathbb{E}} \mathbb{C}^{I_e}$ in $\text{coker}(\varrho_\mathbb{C})$.

(ii) If $v \in \mathbb{V}_p$ and $i \in I_e - I_v$, near q_e , u_v has the local form

$$u_v(w_e) \cong (u_{v,i}(w_e), \tilde{\eta}_{e,i}(w_e) w_e^{s_{e,i}}) \in \mathcal{N}_X D_i,$$

such that

$$0 \neq \eta_{e,i} \equiv \tilde{\eta}_{e,i}(0) \in \mathcal{N}_X D_i|_{u_v(q_e)};$$

see (3.43). Therefore, instead, we can deform u_v by deforming $\tilde{\eta}_{e,i}(w_e)$ to $e^{t\beta}\tilde{\eta}_{e,i}(w_e)$ as above and get the same conclusion as in (4.60).

(iii) If $v \in \mathbb{V}_b$ and $i \in I_e - I_v$, we need to consider a deformation of J . By Lemma 4.7, for any open set $U \subset \Sigma - \{\bar{z}_v \cup q_v\}$, there exist

$$\xi \in W^{\ell,p}(\Sigma_v, u_v^* T D_{I_v}(-\log \partial D_{I_v})) \quad \text{and} \quad Y \in T_{J_{I_v}} \text{AK}^m(D_{I_v}, \partial D_{I_v}) \mathcal{R}_{I_v}$$

such that

$$\xi(q_e) = 0 \oplus 1_{e,i} \in T_{u_v(q_e)} D_{I_v}(-\log \partial D_{I_v}) \cong T_{u_v(q_e)} D_{I_e}(-\log \partial D_{I_e}) \oplus \mathbb{C}^{I_e - I_v},$$

$$\xi(z_a) = 0, \quad \forall z_a \in \bar{z}_v, \quad \xi(q_{e'}) = 0, \quad \forall e' \in \mathbb{E}_v - e,$$

$$\text{Supp}(Y|_{\Sigma_v}) \subset U, \quad D_{u_v} \bar{\partial}^{\log} \xi + \frac{1}{2} Y \circ du_v \circ j = 0.$$

By trivial extension of ξ to other components and horizontal extension of Y to a deformation of J on the entire X , such a pair (ξ, Y) defines a tangent vector in

$$T_f \tilde{\mathfrak{M}}_{g,s}^{\text{plog},\star}(X, D, A, v)_\Gamma$$

such that

$$D_{f\text{ob}\Gamma}(\xi, Y) = [1_{e,i}] \in T_1\mathcal{G}_\Gamma.$$

(iv) If $v \in \mathbb{V}_b$ and $i \in I_v$, we need to consider a deformation of J in the normal direction to D_i . That is, we need a deformation of $\mathcal{R}$ by deforming the connection $\nabla^{(i)}$ in Definition 2.3. This is the only step where we need to deform $\mathcal{R}$. Deformation of $\nabla^{(i)}$ by a 1-form results in a deformation of the corresponding $\bar{\partial}$ -operator $\bar{\partial}_{\mathcal{N}_X D_i}$ on $\mathcal{N}_X D_i$ in [7, Lem. 2.1]. Deformation of $\bar{\partial}_{\mathcal{N}_X D_i}$, then, yields a deformation of the associated almost complex structure on the total space of $\mathcal{N}_X D_i$; see [34, Lem. 2.2] or [7, Sect. 2.1]. In other words, the isomorphism

$$T\mathcal{N}_X D_i \cong \pi_i^* T D_i \oplus \pi^* \mathcal{N}_X D_i,$$

and thus the construction of J on $T\mathcal{N}_X D_i$ via $J|_{TD_i}$ and i_i on $\mathcal{N}_X D_i$, depends on $\nabla^{(i)}$. Deforming the latter results in a deformation of the former. A deformation

$$u^* \bar{\partial}_{\mathcal{N}_X D_i} \longrightarrow u^* \bar{\partial}_{\mathcal{N}_X D_i} + t\beta$$

(supported on an open set of Σ_v whose image in X is disjoint from the image of the rest of non-trivial bubble components) as in (i) such that (4.59) holds yields a path $(f_t)_{t \in [0,1]}$ as in (i) such that (4.60) holds. $\square$

Finally, by Claims 1 and 2, and Implicit-Function Theorem, the universal moduli space

$$\mathfrak{M}_{g,s}^*(X, D, A)_\Gamma$$

is a separable $C^{m-\ell}$ -smooth Banach manifold. Then by the Sard–Smale Theorem, the set of regular values $\mathcal{H}\Theta_{g,k}^{\text{reg},m}(X, D)_{[\omega]}$ of the projection map

$$\text{proj}: \mathfrak{M}_{g,s}^*(X, D, A)_I \longrightarrow \prod_{v \in \mathbb{V}_p} \mathcal{H}_{g_v, k_v + \ell_v}^m(D_{I_v}, \partial D_{I_v})_{[\omega]} \times_J \Theta_{g_v, k_v + \ell_v}^m(D_{I_v})$$

is Baire set of second category. By definition and Lemma 4.15, for every $(\omega', \mathcal{R}, J, v) \in \mathcal{H}_{g,k}(X, D)_{[\omega]}$ such that

$$\text{Res}(\omega', \mathcal{R}, J, v) = \left(\omega', \mathcal{R}, J, ((v_v, \theta_v) = v|_{\Sigma_v})_{v \in \mathbb{V}_p} \right) \in \mathcal{H}\Theta_{g,k}^{\text{reg},m}(X, D)_{[\omega]},$$

the stratum

$$\mathfrak{M}_{g,s}(X, D, A, v)_I = \text{proj}^{-1} \left(\omega', \mathcal{R}, J, ((v_v, \theta_v) = v|_{\Sigma_v})_{v \in \mathbb{V}_p} \right)$$

is a naturally oriented smooth manifold of the expected dimension (1.16). With an argument similar to the Taubes' trick in the proof of [26, Thm. 3.1.6 (ii)], we conclude

that the subset of smooth perturbations

$$\mathcal{H}_{g,k}^{\Gamma}(X, D)_{[\omega]} = \text{Res}^{-1}(\mathcal{H}\Theta_{g,k}^{\text{reg,m}}(X, D)_{[\omega]}) \cap \mathcal{H}_{g,k}^{\Gamma}(X, D)_{[\omega]}$$

satisfies Theorem 1.5 (1). If we restrict this proof to genus 0 J -holomorphic log curves and the resulting set of regular values in $\text{AK}(X, D)_{[\omega]}$, we get Theorem 1.5 (2). $\square$

4.4 Genus Zero Multiple-Cover Maps

A main step in proving Proposition 1.7 (for arbitrary D) is to address the transversality issue at multiple-cover log J -holomorphic spheres. In this section we show that under the positivity/semi-positivity conditions in Definition 1.6, multiple-cover log spheres do not happen in families of larger than the expected dimension.

Suppose

$$[u, \mathbb{P}^1, z_1, \dots, z_{\ell}] \in \mathcal{M}_{0,t}^{\star}(X, D, B)$$

with $t = (t_1, \dots, t_{\ell})$ and $t_a \neq 0$ for all $a \in [\ell]$. All the marked points have non-trivial tangency orders; therefore, none of them can be ignored (as in the classical case) to decrease the expected dimension. The other cases can be reduced to this case by ignoring those marked points that have trivial tangency order with D . Every such point has a finite number of pre-images in any multiple-cover of u . Let d and $k_1, \dots, k_{\ell} \leq d$ be positive integers and set $k = k_1 + \dots + k_{\ell}$. For each $a \in [\ell]$ let

$$\alpha_{1,a} + \dots + \alpha_{a,k_a} = d$$

be an ordered partition of d into a sum of k_a positive numbers. We are interested in those tuples

$$\alpha \equiv (\alpha_{a,b})_{a \in [\ell], b \in [k_a]} \quad (4.61)$$

such that there exists a degree d covering map

$$h: \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \quad (4.62)$$

satisfying

$$h^{-1}(z_a) = \{z_{ab}\}_{b \in [k_a]}, \quad \text{ord}_{z_{ab}} h = \alpha_{a,b}.$$

If h is such a covering map, the k -marked degree d map

$$f_h = (h, \mathbb{P}^1, (z_{ab})_{a \in [\ell], b \in [k_a]})$$

(with the lexicographic order on the marked points z_{ab}) defines a point of the log moduli space

$$\mathcal{M}_{0,\alpha}(\mathbb{P}^1, D_z, [d])$$

where $D_z = \{z_1, \dots, z_\ell\}$ and we are treating $\alpha \in \mathbb{N}^k$ as the tangency order data with D_z at those k points. By (4.21), if $\mathcal{M}_{0,\alpha}(\mathbb{P}^1, D_z, [d])$ is non-empty, it is a smooth manifold of the expected dimension

$$d_{\text{fiber}} \equiv \dim_{\mathbb{C}} \mathcal{M}_{0,\alpha}(\mathbb{P}^1, D_z, [d]) = (d-1)(2-\ell) + k - \ell. \quad (4.63)$$

Also, the k -marked degree $A = dB$ composition map

$$(u \circ h, \mathbb{P}^1, (z_{ab})_{a \in [\ell], b \in [k_a]}) \quad (4.64)$$

defines a point of the log moduli space $\mathcal{M}_{0,\mathfrak{s}}(X, D, A)$, with

$$\mathfrak{s} = (s_{ab})_{a \in [\ell], b \in [k_a]}, \quad \mathbb{N}^N \ni s_{ab} = \alpha_{a,b} s_a.$$

Let

$$\mathcal{M}_{0,\mathfrak{s}}^\alpha(X, D, A) \subset \mathcal{M}_{0,\mathfrak{s}}(X, D, A)$$

denote the subspace of multiple-cover maps of type α . There is a projection map

$$\mathcal{M}_{0,\mathfrak{s}}^\alpha(X, D, A) \longrightarrow \mathcal{M}_{0,\mathfrak{t}}^*(X, D, B) \quad (4.65)$$

whose fiber over $[u, \mathbb{P}^1, z_1, \dots, z_\ell]$ is $\mathcal{M}_{0,\alpha}(\mathbb{P}^1, D_z, [d])$. We have

$$\begin{aligned} d_{\text{down}} &\equiv \exp\text{-dim}_{\mathbb{C}} \mathcal{M}_{0,\mathfrak{t}}^*(X, D, B) = c_1^{TX(-\log D)}(B) + n - 3 + \ell, \\ d_{\text{up}} &\equiv \exp\text{-dim}_{\mathbb{C}} \mathcal{M}_{0,\mathfrak{s}}^*(X, D, A) = c_1^{TX(-\log D)}(A) + n - 3 + k. \end{aligned}$$

In order for the image of $\mathcal{M}_{0,\mathfrak{s}}^\alpha(X, D, A)$ under ev to have a smaller (resp. smaller or equal) dimension than the image of the main stratum $\mathcal{M}_{0,\mathfrak{s}}^*(X, D, A)$, whenever $d_{\text{fiber}}, d_{\text{down}} \geq 0$, we need $d_{\text{down}} \leq d_{\text{up}}$, i.e.,

$$\begin{aligned} c_1^{TX(-\log D)}(B) + n - 3 + \ell &\geq 0 \\ \implies (d-1)c_1^{TX(-\log D)}(B) + k - \ell &> 0 \quad (\text{resp. } \geq 0). \end{aligned} \quad (4.66)$$

In other words, we want to **avoid** a situation where $d_{\text{fiber}} \geq 0$, $B \in \pi_2(X)$ with $\omega(B) > 0$, and

$$3 - n - \ell \leq c_1^{TX(-\log D)}(B) \leq \frac{\ell - k}{d-1} \quad \left(\text{resp. } < \frac{\ell - k}{d-1} \right).$$

For $\ell = 0, 1, 2$, the condition $d_{\text{fiber}} \geq 0$ automatically holds. For $\ell > 2$, the condition $d_{\text{fiber}} \geq 0$ implies

$$\frac{\ell - k}{d - 1} \leq 2 - \ell.$$

Therefore, (4.66) can be replaced with the stronger requirement

$$\begin{cases} c_1^{TX(-\log D)}(B) \notin [3 - n - \ell, 0] \quad (\text{resp. } [3 - n - \ell, 0)) & \text{if } \ell = 0, 1, 2, \\ c_1^{TX(-\log D)}(B) \notin [3 - n - \ell, 2 - \ell] \quad (\text{resp. } [3 - n - \ell, 2 - \ell)) & \text{if } \ell > 2. \end{cases} \quad (4.67)$$

More generally, we will consider maps with image in D_I . Then n should be replaced with $n - |I|$. For $N > 1$ and $(I, \ell) \neq (\emptyset, 0)$, we will further need $d_{\text{fiber}} + d_{\text{down}} \leq d_{\text{up}}$. The latter is equivalent to $c_1^{TX(-\log D)}(B) \geq 2 - \ell$. This explains Definition 1.10. The following lemma summarizes the outcome of these calculations.

Lemma 4.16 *With D_I and ∂D_I in place of (X, D) in (4.65), if $[X, D, \omega]$ is semi-positive (resp. positive) in the sense of Definition 1.6 and $\ell \leq 2$, then*

$$\exp\text{-dim } \mathcal{M}_{0,t}^*(D_I, \partial D_I, B) \geq 0 \quad (4.68)$$

implies

$$\exp\text{-dim } \mathcal{M}_{0,s}^\alpha(D_I, \partial D_I, dB) \geq \exp\text{-dim } \mathcal{M}_{0,t}^*(D_I, \partial D_I, B) \quad (\text{resp. } >),$$

for all $d \geq 1$ and (α, s, t) as above. Furthermore, if $[X, D, \omega]$ is strongly-semi-positive in the sense of Definition 1.10, then (4.68) implies

$$\begin{aligned} \exp\text{-dim } \mathcal{M}_{0,s}^\alpha(D_I, \partial D_I, A) \\ \geq \exp\text{-dim } \mathcal{M}_{0,t}^*(D_I, \partial D_I, B) + \exp\text{-dim } \mathcal{M}_{0,\alpha}(\mathbb{P}^1, D_z, [d]). \end{aligned}$$

Regarding (4.66), the example below illustrates a non-positive situation where (4.66) does not hold, $\mathcal{M}_{0,s}^*(X, D, A)$ is empty, and $\mathcal{M}_{0,s}^\alpha(X, D, A)$ is always positive dimensional.

Example 4.17 Let $X = \mathbb{P}^2$, D be a smooth quartic hypersurface, $B = [1] \in H_2(\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z}$, $\ell = 2$, $k = 3$, $t = (2, 2)$, and $s = (2a, 2b, 2d)$ with $a, b > 0$ and $a + b = d$. For generic J ,

$$\mathcal{M}_{0,t}^*(X, D, B) = \overline{\mathcal{M}}_{0,t}^{\log}(X, D, B)$$

is the (zero-dimensional) moduli space of lines with 2 intersections of order 2 with D and has 160 points. Also, $d_{\text{fiber}} = 1$; if $(z_1, z_2) = (0, \infty)$, the map h in (4.62) is of the form

$$h(z) = \lambda \frac{(z - z_{11})^a (z - z_{12})^b}{(z - z_{21})^d} \quad \text{for some } \lambda \in \mathbb{C}^*.$$

On the other hand, $d_{\text{up}} = 2 - d$; therefore, if $d > 2$, $\mathcal{M}_{0,s}^*(X, D, A)$ is empty for generic J while $\mathcal{M}_{0,s}^\alpha(X, D, A)$ will always be positive dimensional. $\square$

For (J, v) -holomorphic curves, every connected cluster Σ' of contracted components is a tree of spheres, with a total of at most 2 special points at least one of which is a nodal point. Here, by a special point we mean either a marked point or a nodal point connecting the cluster to an irreducible component of Σ outside the cluster.²² Because of this restriction on the number of special points, it is natural to expect that the condition $\ell \leq 2$ in Lemma 4.16 to be always satisfied. The following lemma shows that this is indeed the case under the Nef condition of Sect. 1.4.

For each bubble component $\mathbb{P}^1 \cong \Sigma_v \subset \Sigma'$, the restriction of f' to Σ_v is a tuple

$$f_v = (u_v, \zeta_v, C_v = (\Sigma_v, j_v, z_v \cup q_v)),$$

where z_v is empty or is the only 1 marked point allowed on Σ' , $q_v = \{q_{\underline{e}}\}_{\underline{e} \in \mathbb{E}_v}$ is the set of nodal points on Σ_v , $\mathfrak{s}_v$ is the tuple of tangency orders at $z_v \cup q_v$, u_v is a J -holomorphic map into D_{I_v} , and $\zeta_v = (\zeta_{v,i})_{i \in I_v}$ is a meromorphic section of $u_v^* \mathcal{N}_X D_{I_v}$ with zeros/poles of orders determined by $\mathfrak{s}_v$ at $z_v \cup q_v$. Note that while f' has at most 2 special points, q_v can be arbitrary large. The case we are interested in is when u_v is a multiple-cover map of the form $u_v = \bar{u}_v \circ h$ as in (4.64), where $\bar{u}_v$ represents the homology class B_v and has tangency order type $\mathfrak{t}_v$. We say a point $p \in \Sigma_v$ is an **actual intersection point** with D , if $I_v \neq [N]$ and

$$p \in u_v^{-1} \left(\bigcup_{i \in [N] - I_v} D_i \right).$$

Let δ_v (resp. $\bar{\delta}_v$) denote the number of actual intersection points of u_v (resp. $\bar{u}_v$) with D . We say $p \in \Sigma_v$ is a **positive** point if there exists $i \in [N]$ such that

$$\text{ord}_{u_v, \zeta_v}^i(p) > 0.$$

Let δ_v^+ denote the number of positive points on Σ_v . Each positive point is either a nodal point or a marked point, and $\delta_v^+ \geq \delta_v$, as every actual intersection point is a positive point.

Lemma 4.18 *If D is Nef, then $\delta_v^+ \leq 2$ for all $v \in \mathbb{V}_b$.*

Proof Assume more than two points in $z_v \cup q_v$ are positive. Since Σ' has at most 2 special points, removing Σ_v from Σ' we get some sub-clusters, at least one of which, say Σ'' has the following properties:

- it is not connected to the non-contracted (principal) part of Σ ,
- it does not carry any of the marked points,

²² By Remark 3.16, such a cluster defines a log J -holomorphic curve $f' \in \mathcal{M}_{0,s'}(X, D, A')_{\Gamma'}$ where s' records the tangency order data at those (one or two) special points (which now act as marked points for f'), Γ' is the decorated sub-graph of the cluster, and A' is the total homology class of the cluster.

- it is connected to Σ_v at a node $q_e \in \Sigma_v$ which is a positive point.

Let Σ_{v_1} be the component of Σ'' connected to Σ_v at $\Sigma_{v_1} \ni q_e \sim q_e \in \Sigma_v$. Since, by assumption, $\text{ord}_{u_v, \zeta_v}^i(q_e) > 0$, for some $i \in [N]$, the image of u_{v_1} must lie in D_i and the meromorphic section $\zeta_{v_1, i}$ on Σ_{v_1} corresponding to that should have a non-trivial pole (of the same order as $s_{e, i}$) at q_e . By the Nef assumption, the line bundle

$$u_{v_1}^* \mathcal{N}_X D_i \longrightarrow \Sigma_{v_1} \cong \mathbb{P}^1$$

has a non-negative degree. Therefore, $\zeta_{v_1, i}$ should have a non-trivial zero at another nodal point $q_{e_1} \in \Sigma_{v_1}$ and there is another component Σ_{v_2} of Σ'' connected to Σ_{v_1} at $\Sigma_{v_2} \ni q_{e_1} \sim q_{e_1} \in \Sigma_{v_1}$. Continuing inductively we will see an infinite chain of irreducible components in Σ'' , which is a contradiction. $\square$

As a conclusion of this lemma, for every bubble component f_v in f , since those points in $z_v \cup q_v$ which are not actual intersection points (with ∂D_{I_v}) are not relevant to the argument leading to (4.66), by ignoring these points, replacing n with $n - |I_v|$, and assuming $\ell = \bar{\delta}_v \leq k = \delta_v \leq 2$ in (4.67), the semi-positivity condition (1.17) in Definition 1.6 guaranties that

$$c_1^{TX(-\log D)}(B_v) + (n - |I_v|) - 3 + \ell \geq 0 \implies c_1^{TX(-\log D)}(B_v) \geq 0. \quad (4.69)$$

The following statement summarizes the main result of this section.

Corollary 4.19 *Suppose (X, D, ω) is semi-positive in the sense of Definition 1.6, and f is a log (J, v) -curve in $\mathcal{M}_{g, s}(X, D, A, v)$. If J belongs to the Baire sets $\text{AK}^{I_v}(X, D)_{\mathcal{R}}$ associated with $\mathcal{M}_{0, t_v}^*(X, D, B_v)_{I_v}$ in Proposition 4.8, for all $v \in \mathbb{V}_b$, then $c_1^{TX(-\log D)}(B_v) \geq 0$ for all $v \in \mathbb{V}_b$.*

Proof By the assumption on J , each moduli space $\mathcal{M}_{0, t_v}^*(X, D, B_v)_{I_v}$ is a non-empty smooth manifold of the expected complex dimension

$$c_1^{TX(-\log D)}(B_v) + (n - |I_v|) - 3 + \bar{\delta}_v \geq 0.$$

The conclusion follows from (4.69). $\square$

Lemma 4.18 puts a major restriction on a contracted cluster Σ' . In general, each cluster will be of one of the following types.

- A cluster with one node $q_{e_0} \in \Sigma_{v_0}$ that connects Σ' to the principal part and no marked points. In this case Σ' is a rooted tree with the root v_0 . Distance from v_0 defines a partial order $<$ on the vertices of Σ' with v_0 being the minimal vertex. By the same inductive reasoning as in the proof of Lemma 4.18, if $v < v'$ and $q_e \in \mathbb{E}_{v, v'}$ then q_e can not be a positive point. Therefore each $\Sigma_{v'}$ in the cluster has at most one positive point, that will be the unique nodal point q_e connecting $\Sigma_{v'}$ to the unique component

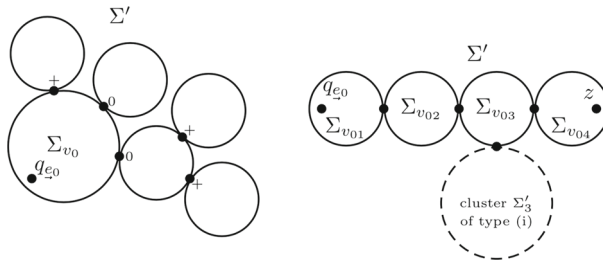


Fig. 2 (Left): A cluster of type (i). (Right): A cluster of type (ii)

Σ_v with $v < v'$. Furthermore, $I_v \supseteq I_{v'}$ whenever $v < v'$, and $I_v \not\supseteq I_{v'}$ only if q_e is an actual intersection point. Figure 2 (Left) illustrates the situation with the “+” sign indicating a positive point. At a positive nodal point q_e , we have $0 \neq s_e \in \mathbb{N}^N$. At nodal points without a “+” sign on any side, s_e must be zero. We put a 0 to indicate those points. If each D_i is positive, in the sense that $A \cdot D_i > 0$ for all $A \in \pi_2(X)$ such that $\omega(A) > 0$, then each bubble component will have exactly one positive point.

(ii) A cluster with one node $q_{e_0} \in \Sigma_{v_{01}}$ that connects Σ' to the principal part and one marked point z on $\Sigma_{v_{0k}}$ (possibly $v_{01} = v_{0k}$). In this case, Σ' includes a chain of components $\Sigma_{v_{01}}, \dots, \Sigma_{v_{0k}}$. For each $\Sigma_{v_{0i}}$, Σ' might include a cluster Σ_i (or more) of type (i) that is attached to $\Sigma_{v_{0i}}$ at a unique nodal point $q_{e_{i0}} \in \Sigma_i$; see Fig. 2 (Right).

(iii) A cluster with two nodes $q_{e_0} \in \Sigma_{v_{01}}$ and $q_{e_k} \in \Sigma_{v_{0k}}$ that connect Σ' to the principal part. With q_{e_k} in place of z , this case is like case (ii) above.

Remark 4.20 If $\delta_v \geq 2$ for some geometric reason, then the only possibility for Σ' is a chain of bubble components as in (ii) or (iii) between the two special points. This is for example the case if X is a toric variety and D is the boundary divisor.

4.5 Non-simple Maps

In [32], in order to prove the classical analogue of Proposition 1.7 (i.e., [32, Thm. 3.11]), in a process which we will call it **RT-process** here, they replace a non-simple map f with an underlying simple map f' with **multi-nodes**. A multi-node m is a point at which more than two components of the domain are connected to each other. In a nodal domain, a node q_e is obtained by attaching two irreducible components Σ_{v_1} and Σ_{v_2} at nodal points $q_{e_1} \in \Sigma_{v_1}$ and $q_{e_2} \in \Sigma_{v_1}$. A multi-node q_m is obtained by attaching more than two components $\Sigma_{v_1}, \dots, \Sigma_{v_\ell}$ at nodal points $q_{e_i} \in \Sigma_{v_i}$, with $i = 1, \dots, \ell$. To keep the notation inline with the rest of the paper, we let $\mathbb{E}$ to denote the set of nodes and multi-nodes, and $\mathbb{E}$ to denote the set of nodal points on different components. When there is no multi-node we have $|\mathbb{E}| = 2|\mathbb{E}|$; otherwise, $|\mathbb{E}| > 2|\mathbb{E}|$. In the

presence of an SNC divisor $D = \bigcup_{i \in [N]} D_i$, the sets $\mathbb{E}$ and $\mathbb{E}$ admit decompositions

$$\mathbb{E} = \bigcup_{I \subset [N]} \mathbb{E}_I \quad \text{and} \quad \mathbb{E} = \bigcup_{I \subset [N]} \mathbb{E}_I$$

by the type of these points. Similar to Sect. 4.3, we write $\mathbb{V} = \mathbb{V}_p \cup \mathbb{V}_b$, where $\mathbb{V}_p$ corresponds to set of non-contracted or **principal** components and $\mathbb{V}_b$ corresponds to set of contracted or **bubble** components. Furthermore, we write $\mathbb{V}_b = \mathbb{V}_{b,\bullet} \cup \mathbb{V}_{b,\circ}$, where $A_v \neq 0$ for $v \in \mathbb{V}_{b,\bullet}$ and $\mathbb{V}_{b,\circ}$ is the set of ghost bubbles.

Ruan–Tian proved that the dimension of the space of such f' is at least 2 real dimension less than the dimension of the main stratum. The same argument does not directly work for log maps. If we apply the same process to the underlying (J, ν) -map $\bar{f}$ of a log map f , the resulting simple map $\bar{f}'$ may not lift to a log map f' for at least two reasons: (1) since we replace a multiple-cover map with its underlying simple map, the matching condition in Definition 3.9 (4) may no longer hold, (2) since we identify different components with the same image, the vectors s_v satisfying Definition 3.11 (1) may no longer exist.

Suppose

$$f = \left(\phi, (u_v, \zeta_v = (\zeta_{v,i})_{i \in I_v}, C_v = (\Sigma_v, j_v, \vec{z}_v \cup \vec{q}_v))_{v \in \mathbb{V}}, J, \nu \right) \\ \in \mathcal{M}_{g,5}^{\text{ns}}(X, D, A, \nu)_{\Gamma}.$$

The RT-process, described after [32, Def. 3.10], changes the underlying (J, ν) -map

$$\bar{f} = \left(\phi, (u_v, C_v = (\Sigma_v, j_v, \vec{z}_v \cup \vec{q}_v))_{v \in \mathbb{V}}, J, \nu \right) \in \mathcal{M}_{g,k}^{\text{ns}}(X, A, \nu)_{\Gamma}$$

to another (J, ν) -map $\bar{f}''$ given as

$$\left(\phi, (u_v, C_v = (\Sigma_v, j_v, \vec{z}_v \cup \vec{q}_v))_{v \in \mathbb{V}_p}, (u_{v''}, C_{v''} = (\Sigma_{v''}, j_{v''}, \vec{z}_{v''} \cup \vec{q}_{v''}))_{v'' \in \mathbb{V}_b'}, J, \nu \right)$$

with multi-nodes in the following way.

- (i) It collapses the ghost bubbles (and any marked point on it will be thrown away). As a result we get some multi-nodes.
- (ii) It replaces each multiple-cover bubble component by its image. Since some of the special (marked or nodal) points may have the same image, this step may produce further multi-nodes.
- (iii) It collapses each sub-tree of the bubbles whose components have the same image.

None of these three steps changes the genus though. Let Γ' be the resulting combinatorial type of the domain with components indexed by $\mathbb{V}'$, nodes and multi-nodes indexed by $\mathbb{E}'$, and nodal points indexed by $\mathbb{E}'$. With notation as in Sect. 4.3, $\mathbb{V}'$ decomposes as $\mathbb{V}'_p \cup \mathbb{V}'_b$. The first component is identical to $\mathbb{V}_p$. There is a collapsing map

$$\text{red}: \mathbb{V}_{b,\bullet} \longrightarrow \mathbb{V}'_b \quad (4.70)$$

and a multiplicity map

$$d: \mathbb{V}'_{\mathbf{b}} \longrightarrow \mathbb{Z}_+ \quad (4.71)$$

such that

$$\sum_{v \in \text{red}^{-1}(v')} A_v = d_{v'} A_{v'}, \quad \forall v' \in \mathbb{V}'_{\mathbf{b}}.$$

Here $A_{v'}$ is the homology class of the resulting simple curve. After these three steps, there might still be bubble components (not adjacent to each other) which have the same image or their nodal points have the same image.

(iv) We identify components with the same image and nodal points with the same image.

After step (iv), we get a domain with possibly further multi-nodes and higher genus. Let Γ'' be the resulting combinatorial type of this domain with components indexed by $\mathbb{V}''$, nodes and multi-nodes indexed by $\mathbb{E}''$, and nodal points indexed by $\mathbb{I}''$. The maps (4.70) and (4.71) descend to

$$\mathbb{V}_{\mathbf{b}, \bullet} \xrightarrow{\text{red}} \mathbb{V}'_{\mathbf{b}} \xrightarrow{\text{red}} \mathbb{V}''_{\mathbf{b}}, \quad d: \mathbb{V}''_{\mathbf{b}} \longrightarrow \mathbb{Z}_+,$$

with the same properties as above. For the same reason as in [32, Cor. 3.17], we have

$$|\mathbb{E}''_{\mathbf{I}}| - |\mathbb{E}'_{\mathbf{I}}| = |\mathbb{E}'_{\mathbf{I}}| - |\mathbb{E}_{\mathbf{I}}|. \quad (4.72)$$

Each nodal point of f'' is still decorated by a well-defined subset $I \subset [N]$ such that all of its pre-image nodal points in f have the same decoration I . Also, if a bubble component $\Sigma_{v''}$ of f'' has image in $D_{I_{v''}}$, then all of its pre-image bubble components have image in the same stratum, i.e.,

$$I_v = I_{v''}, \quad \forall v \in \text{red}^{-1}(v'').$$

The combinatorial type γ of this process is encoded in the triple $(\Gamma, \Gamma', \Gamma'')$, and the associated maps red and d . By Theorem 1.3, the set of such γ is finite. Let $\mathcal{M}_{g,k}(X, A, v)_{\Gamma''}$ denote the classical moduli space of such (J, v) curves $\bar{f}''$. By [32, Prop. 3.21], if (X, ω) is semi-positive, for generic (J, v) , $\mathcal{M}_{g,k}(X, A, v)_{\Gamma''}$ is a smooth moduli space of the $\mathbb{C}$ -dimension at most

$$c_1^{TX}(A) + (n-3)(1-g) + k - (|\mathbb{E}''| - |\mathbb{E}|). \quad (4.73)$$

Comparing (4.73) with [32, Prop. 3.21], note that

$$|\mathbb{E}''| - |\mathbb{E}| = |\mathbb{E}'| - |\mathbb{E}| = n_{\phi} + |\mathbb{V}'_{\mathbf{b}}|,$$

where n_{ϕ} is the number of nodes of the stable domain $\phi(\Sigma) \in \bar{\mathcal{U}}_{g,k}$.

Away from the principal components, the map $\overline{f}''$ does not lift to a log curve. However, some of the information still passes to $\overline{f}''$. First, let us consider the pre-log space

$$\mathcal{M}_{g,s}^{\text{plog}}(X, D, A, v)_{\Gamma},$$

that is we forget about Condition (2) in Definition 3.11. Instead of $\mathcal{M}_{g,k}(X, A, v)_{\Gamma''}$, we consider the set $\mathcal{M}_{g,s}^{\text{plog}}(X, D, A, v)_{\Gamma''}$ of tuples

$$f'' = \left(\phi, (u_v, \zeta_v, C_v = (\Sigma_v, j_v, \vec{z}_v \cup \vec{q}_v))_{v \in \mathbb{V}_p}, \right. \\ \left. (u_{v''}, C_{v''} = (\Sigma_{v''}, j_{v''}, \vec{z}_{v''} \cup \vec{q}_{v''}))_{v'' \in \mathbb{V}_b}, J, v \right)$$

obtained from the elements of $\mathcal{M}_{g,s}^{\text{plog}}(X, D, A, v)_{\Gamma}$, where the principal components still carry the information of the meromorphic sections ζ_v . Let $\mathcal{M}_{g,s}^{\text{plog},\gamma}(X, D, A, v)_{\Gamma}$ denote its pre-image in $\mathcal{M}_{g,s}^{\text{plog}}(X, D, A, v)_{\Gamma}$, i.e., those pre-log maps for which the RT-process is of type γ . The projection

$$\pi_{\gamma}: \mathcal{M}_{g,s}^{\text{plog},\gamma}(X, D, A, v)_{\Gamma} \longrightarrow \mathcal{M}_{g,s}^{\text{plog}}(X, D, A, v)_{\Gamma''} \quad (4.74)$$

is a surjective fiber bundle. The key point is that by (4.33), if Σ_v is genus 0, as long as the second combinatorial condition in (3.51) is satisfied, for each $i \in I_v$, there are meromorphic sections $\zeta_{v,i}$ of $u_v^* \mathcal{N}_X D_i$ with zeros/poles of orders $s_{\underline{e},i}$ and s_a at $q_{\underline{e}}$ and z_a , respectively, for all $\underline{e} \in \mathbb{E}_v$ and $z_a \in z_v$. The fiber $\mathcal{M}_{f''}$ of (4.74) over any f'' is a product of the manifolds of the form described below and

$$\text{st} \times \text{ev}: \mathcal{M}_{g,s}^{\text{plog},\gamma}(X, D, A, v)_{\Gamma} \longrightarrow \overline{\mathcal{M}}_{g,k} \times X^s$$

factors through π_{γ} .

(1) For each $v \in \mathbb{V}_{b,o}$, we have the configuration space $\mathcal{M}_{0,k_v+\ell_v}$ of the special points on the ghost bubble Σ_v in $\pi^{-1}(f'')$.

(2) For all $v'' \in \mathbb{V}''$ and $v \in \tau \epsilon \mathfrak{d}^{-1}(v'')$, we have $u_v = u_{v''} \circ h_v$ for some degree d_v covering map $h_v: \Sigma_v \rightarrow \Sigma_{v''}$ as in (4.62). Note that

$$d_{v''} = \sum_{v \in \tau \epsilon \mathfrak{d}^{-1}(v'')} d_v.$$

The combinatorial type α_v of h_v is determined by the image $z_{v''} \cup q_{v''}$ of $z_v \cup q_v$ in $\Sigma_{v''}$ and the branching order of h_v at $z_v \cup q_v$ (i.e., the partition of d_v as in (4.61) at the actual intersection points

$$u_{v''}^{-1}(\partial D_{I''}) \subset z_{v''} \cup q_{v''}$$

and the branching orders at the rest of the points). Therefore, as in (4.65), in the fiber over f'' we get the relative moduli space

$$\mathcal{M}_{0,\alpha_v}(\Sigma_{v''} = \mathbb{P}^1, z_{v''} \cup q_{v''}, [d_v]) \quad (4.75)$$

of tuples

$$(h_v: \Sigma_v = \mathbb{P}^1 \longrightarrow \Sigma_{v''} = \mathbb{P}^1, \vec{z}_v \cup \vec{q}_v)$$

relative to the divisor $z_{v''} \cup q_{v''} \subset \Sigma_{v''}$, with the ramification/tangency order data α_v . The diffeomorphism type of (4.75) is independent of the location of $z_{v''} \cup q_{v''}$ (and thus f'').

We conclude that

$$\mathcal{M}_{f''} = \prod_{v \in \mathbb{V}_{b,o}} \mathcal{M}_{0,k_v+\ell_v} \times \prod_{v'' \in \mathbb{V}''} \prod_{v \in \text{red}^{-1}(v'')} \mathcal{M}_{0,\alpha_v}(\mathbb{P}^1, z_{v''} \cup q_{v''}, [d_v]). \quad (4.76)$$

Similar to [32, Thm. 3.16], and with a similar proof as in Sect. 4.3, for $(\omega', \mathcal{R}, J, v)$ in a subset of second category $\mathcal{H}_{g,k}^{\gamma}(X, D)_{[\omega]} \subset \mathcal{H}_{g,k}(X, D)_{[\omega]}, \mathcal{M}_{g,s}^{\text{plog}}(X, D, A, v)_{\Gamma''}$ is a smooth manifold of $\mathbb{C}$ -dimension

$$\begin{aligned} k'' + |\mathbb{E}''| + \sum_{v \in \mathbb{V}_p} (c_1^{TX(-\log D)}(A_v) + (n-3)(1-g_v) - |I_v|) \\ + \sum_{v \in \mathbb{V}_b''} (c_1^{TX(-\log D)}(A_{v''}) + (n-3) - |I_{v''}|) \\ - \sum_{I \subset [N]} (n - |I|)(|\mathbb{E}_I''| - |\mathbb{E}_I''|). \end{aligned}$$

Note the number $h_{\overline{D}}$ in [32, pp. 485–486] is $|\mathbb{E}''|$ in our notation and the number $t_{\overline{D}}$ there is $|\mathbb{E}''|$. Also k'' denotes the number of surviving²³ marked points. For generic $(\omega', \mathcal{R}, J)$,

$$c_1^{TX(-\log D)}(A_{v''}) + (n-3) - |I_{v''}| + \delta_{v''} \geq 0, \quad (4.77)$$

where $\delta_{v''}$ is the number of intersection points of $u_{v''}$ with $\partial D_{I_{v''}}$ as in the proof of Corollary 4.19. If some bubble in Γ'' happens to be the image of two or more bubbles

²³ Which will be k or $k-1$ since the contracted part carries at most one of the marked points.

in Γ' , by adding (4.77) to the dimension, and by (4.72), we get

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{M}_{g, \mathfrak{s}}^{\text{plog}}(X, D, A, v)_{\Gamma''} &\leq k' + |\mathbb{E}'| + \sum_{v \in \mathbb{V}_p} (c_1^{TX(-\log D)}(A_v) + (n-3)(1-g_v) - |I_v|) \\ &\quad + \sum_{v' \in \mathbb{V}'_b} (c_1^{TX(-\log D)}(A_{v'}) + (n-3) - |I_{v'}|) \\ &\quad - (n-1)(|\mathbb{E}'| - |\mathbb{E}'|) + \sum_{I \subset [N]} (|I| - 1)(|\mathbb{E}'_I| - |\mathbb{E}'_I|). \end{aligned}$$

Since

$$1 - g = \sum_{v \in \mathbb{V}'} (1 - g_v) - (|\mathbb{E}'| - |\mathbb{E}'|),$$

we get

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{M}_{g, \mathfrak{s}}^{\text{plog}}(X, D, A, v)_{\Gamma''} &\leq k' + (n-3)(1-g) + \sum_{v' \in \mathbb{V}'} (c_1^{TX(-\log D)}(A_{v'}) - |I_{v'}|) \\ &\quad + (2|\mathbb{E}'| - |\mathbb{E}'|) + \sum_{I \subset [N]} (|I| - 1)(|\mathbb{E}'_I| - |\mathbb{E}'_I|). \end{aligned} \quad (4.78)$$

By the semi-positivity condition, we have $c_1^{TX(-\log D)}(A_{v'}) \geq 0$ for all $v' \in \mathbb{V}'_b$, see Corollary 4.19. Also $|\mathbb{E}'| \geq 2|\mathbb{E}'|$. Therefore, the last equation is less than or equal to

$$\begin{aligned} c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k &- \sum_{v' \in \mathbb{V}'} |I_{v'}| + \sum_{I \subset [N]} (|I| - 1)(|\mathbb{E}'_I| - |\mathbb{E}'_I|). \end{aligned} \quad (4.79)$$

Also, note that if the equality happens, then

$$\sum_{v'' \in \mathbb{V}''} |I_{v''}| = \sum_{v' \in \mathbb{V}'} |I_{v'}| \quad \text{and} \quad |\mathbb{E}'| = 2|\mathbb{E}'|.$$

The first equality implies that step (iv) in RT-process is trivial. The second one implies that step (i) is trivial and no multi-node is created in steps (ii) and (iii) of the process.

Then it is easy to see that

$$\begin{aligned} & \sum_{v'' \in \mathbb{V}''} |I_{v''}| + \sum_{I \subset [N]} (|I| - 1)(|\mathbb{E}_I''| - |\mathbb{E}_I''|) \\ &= -|\mathbb{E}| - \sum_{v \in \mathbb{V}} |I_v| + \sum_{e \in \mathbb{E}} |I_e| = \dim_{\mathbb{C}}(\mathcal{G}) - \dim_{\mathbb{R}} \mathbb{K}_{\mathbb{R}}(\Gamma). \end{aligned}$$

Proof of Proposition 1.7 If D is smooth, i.e. $N = 1$, then the second line (4.79) is negative. We conclude that

$$\dim_{\mathbb{C}} \mathcal{M}_{g,s}^{\text{plog}}(X, D, A, v)_{\Gamma''} < \dim_{\mathbb{C}} \mathcal{M}_{g,s}(X, D, A, v).$$

In this sense, for D smooth, Proposition 1.7 essentially follows from the classical result of Ruan–Tian by looking at the image of non-simple maps in $\bar{\mathcal{M}}_{g,k}(X, A, v)$. $\square$

For an arbitrary SNC divisor D , in order to take care of the extra term

$$(2|\mathbb{E}'| - |\mathbb{E}'|) - \sum_{v' \in \mathbb{V}'} |I_{v'}| + \sum_{I \subset [N]} (|I| - 1)(|\mathbb{E}_I'| - |\mathbb{E}_I'|) \quad (4.80)$$

in (4.78), we need to use Condition (2) in Definition 3.11 to reduce the dimension. We encounter the following two problems.

1. The map ob_{Γ} into $\mathcal{G}(\Gamma)$ is defined on $\mathcal{M}_{g,s}^{\text{plog},\gamma}(X, D, A, v)_{\Gamma}$. As the examples below show, unlike $\text{st} \times \text{ev}$, ob_{Γ} does not necessarily factor through π_{γ} . Therefore, we need to work with the larger space $\mathcal{M}_{g,s}^{\text{plog},\gamma}(X, D, A, v)_{\Gamma}$.
2. Since the elements of $\mathcal{M}_{g,s}^{\text{plog},\gamma}(X, D, A, v)$ are not simple, transversality of ob_{Γ} in the sense of Claim 2 of proof of Theorem 1.5 might not be achievable. We need to replace $\mathcal{G}(\Gamma)$ with a smaller group $\mathcal{G}(\gamma)$ that admits a surjective homomorphism $h: \mathcal{G}(\Gamma) \rightarrow \mathcal{G}(\gamma)$, and such that

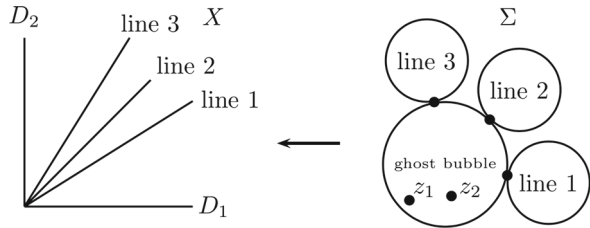
$$\text{ob}_{\gamma} = h \circ \text{ob}_{\Gamma}: \mathcal{M}_{g,s}^{\text{plog},\gamma}(X, D, A, v) \rightarrow \mathcal{G}(\gamma)$$

can be transversal (i.e. $1 \in \mathcal{G}(\gamma)$ is a regular value of that).

The two problems above are not specific to the particular compactification considered in this work and should appear, either explicitly or implicitly, in any other analytical approach.

The first example below illustrates a simple situation where ob_{Γ} depends on the location of special points on a ghost bubble. In Appendix A, we study this dependence in details. The second example below illustrates a situation where ob_{Γ} depends on the choice of the covering map h_v for some multiple-cover bubble Σ_v . The third example below illustrates a situation where different bubbles have the same image, and ob_{Γ} can not be transversal. Finally, the fourth example below illustrates the necessity of the extra condition (1.19), whenever $N > 1$.

Fig. 3 A nodal configuration in the boundary of $\overline{\mathcal{M}}_{0,((2,1)(1,2))}^{\log}(\mathbb{P}^2, D, [3])$



Example 4.21 Let $X = \mathbb{P}^2$, $D = D_1 \cup D_2$ be the transverse union of two coordinate lines,

$$g=0, \quad k=2, \quad A=[3], \quad \text{and} \quad \mathfrak{s}=(s_1, s_2)=((2, 1), (1, 2)).$$

Let Γ be the configuration in Fig. 3; it is the nodal configuration obtained by three lines (not contained in D) passing through the point D_{12} , together with a ghost bubble mapped to D_{12} that connects the domains of these three lines and carries the marked points z_1, z_2 .

Let $\mathbb{V} = \{v_0, v_1, v_2, v_3\}$ where v_1, v_2, v_3 correspond to the lines 1, 2, and 3, respectively, and v_0 corresponds to the ghost bubble. Let $\mathbb{E} = \{e_1, e_2, e_3\}$ where e_i corresponds to the node connecting the domain of the i -th line to the ghost bubble. We can choose the orientation e_i to be the one ending at v_0 . We have

$$I_{v_0} = \{1, 2\}, \quad I_{v_i} = \emptyset, \quad I_{e_i} = \{1, 2\}, \quad s_{e_i} = (1, 1), \quad \forall i = 1, 2, 3.$$

The map

$$\varrho: \mathbb{Z}^{\mathbb{E}} \oplus \mathbb{Z}^{I_{v_0}} \longrightarrow \bigoplus_{i=1}^3 \mathbb{Z}^{I_{e_i}}$$

has a 1-dimensional kernel and a 2-dimensional cokernel. The kernel is generated by

$$(\lambda_{e_1}, \lambda_{e_2}, \lambda_{e_3}, s_{v_0}) = (1, 1, 1, (1, 1)).$$

Therefore, Condition (1) of Definition 3.11 is satisfied. The obstruction group $\mathcal{G}(\Gamma)$ is 2-dimensional. The homomorphism

$$\begin{aligned} (\mathbb{C}^*)^{I_{e_1}} \times (\mathbb{C}^*)^{I_{e_2}} \times (\mathbb{C}^*)^{I_{e_3}} &\longrightarrow (\mathbb{C}^*)^2, \\ ((x_1, y_1), (x_2, y_2), (x_3, y_3)) &\longmapsto \left(\frac{x_1}{y_1} \frac{y_2}{x_2}, \frac{x_2}{y_2} \frac{y_3}{x_3} \right) \end{aligned}$$

descends to an isomorphism $\mathcal{G}(\Gamma) \rightarrow (\mathbb{C}^*)^2$. We take the marked point z_1 to be 0, z_2 to be ∞ , and the nodal points q_{e_1} , q_{e_2} , and q_{e_3} to be 1, α_2 , and α_3 , respectively. Thus, α_2 and α_3 are parametrizing the two-dimensional configuration space $\mathcal{M}_{0,5}$ of the five

special points on Σ_{v_0} . The meromorphic sections (functions) $\zeta_{v_0,1}$ and $\zeta_{v_0,2}$ are given by

$$\zeta_{v_0,i}(z) = \frac{\beta_i z^{s_{1,i}}}{(z-1)^{s_{e_1,i}} (z-\alpha_2)^{s_{e_2,i}} (z-\alpha_3)^{s_{e_3,i}}} = \begin{cases} \frac{\beta_1 z^2}{(z-1)(z-\alpha_2)(z-\alpha_3)} & i=1, \\ \frac{\beta_2 z}{(z-1)(z-\alpha_2)(z-\alpha_3)} & i=2, \end{cases}$$

where $\beta_i \in \mathbb{C}^*$. We conclude that

$$\text{ob}_\Gamma(f) = \left(\frac{m_1}{m_2 \alpha_2}, \frac{m_2 \alpha_2}{m_3 \alpha_3} \right) \in (\mathbb{C}^*)^2,$$

where m_i is the slope of the i -th line. The RT-process removes the ghost bubble Σ_{v_0} and creates a map f' with one multi-node. In general, as we will show in Appendix A, the element $\text{ob}_\Gamma(f)$ can be expressed as a product of two terms

$$\text{ob}_\Gamma(f) = \text{ob}_\Gamma(f') \cdot o_{v_0}(\mathbb{P}^1, z_1, z_2, q_{e_1}, q_{e_2}, q_{e_3}).$$

Here, the decomposition is

$$\left(\frac{m_1}{m_2 \alpha_2}, \frac{m_2 \alpha_2}{m_3 \alpha_3} \right) = \left(\frac{m_1}{m_2}, \frac{m_2}{m_3} \right) \cdot \left(\alpha_2^{-1}, \frac{\alpha_2}{\alpha_3} \right). \quad (4.81)$$

However, as (4.81) shows, the second term on the right-hand side can be a non-trivial function from $\mathcal{M}_{0,k_{v_0}+\ell_{v_0}}$ into $\mathcal{G}(\Gamma)$. $\square$

Next, we consider a similar configuration in one dimension higher with a multiple-cover map in place of the ghost bubble above.

Example 4.22 Let $X = \mathbb{P}^3$, $D = D_1 \cup D_2$ be the transverse union of two hyperplanes,

$$g=0, \quad k=2, \quad A=[5], \quad \text{and} \quad \mathfrak{s}=(s_1)=((5,0), (0,5)).$$

Let Γ be the configuration in Fig. 4; it is the nodal configuration obtained by three lines (not contained in D) passing through $D_{12} \cong \mathbb{P}^1$, together with a degree 2 multiple-cover map

$$h_{v_0}: \Sigma_{v_0} = \mathbb{P}^1 \longrightarrow D_{12}$$

that connects the domains of these three lines and carries the marked points z_1, z_2 .

The dual graph Γ and the isomorphism $\mathcal{G} \rightarrow (\mathbb{C}^*)^2$ is the same as in Example 4.21. After a reparametrization, we may assume that the double points are at $a, b \in \mathbb{D}_{12} = \mathbb{P}^1$ and $0, \infty \in \Sigma_{v_0} = \mathbb{P}^1$. Therefore,

$$h_{v_0}(z) = \frac{a - bz^2}{1 - z^2}.$$

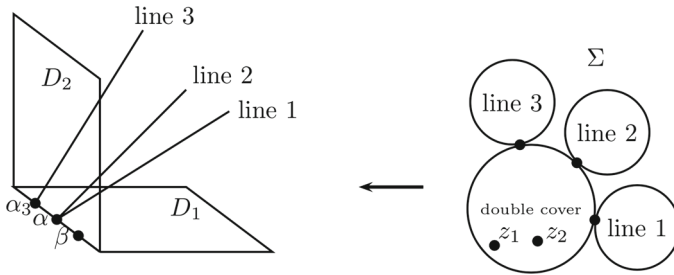


Fig. 4 A nodal configuration in the boundary of $\overline{\mathcal{M}}_{0,((5,0),(0,5))}^{\log}(\mathbb{P}^3, D, [5])$

The location of each special point on Σ_{v_0} is determined up to a sign by its image in D_{12} , i.e.

$$h_{v_0}(q_{e_i}) = \alpha_i \leftrightarrow q_{e_i} \in \pm \sqrt{\frac{a - \alpha_i}{b - \alpha_i}}, \quad h_{v_0}(z_i) = \beta_i \leftrightarrow z_i \in \pm \sqrt{\frac{a - \beta_i}{b - \beta_i}}.$$

The RT-process replaces u_{v_0} with the identification $u_{v'_0}: \Sigma_{v'_0} = \mathbb{P}^1 \rightarrow D_{12}$ together with the marked points $\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2\} \in \Sigma_{v'_0} = \mathbb{D}_{12}$ (if two of these are the same, they will be identified). In the fibers $\mathcal{M}_{f'}$ of (4.74) over f' , the parameters $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$ will be fixed, but a, b are allowed to change and they parametrize the moduli space of multiple covers that yield the same underlying simple marked curve f' . The type of Γ' depends on the location of $\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2\}$. For example, we may assume that

$$\alpha_1 = \alpha_2 = \alpha \quad \text{and} \quad \beta_1 = \beta_2 = \beta$$

as in Fig. 4. Then Γ' has a multi-node at α , just one marked point at β , and a regular node at α_3 .

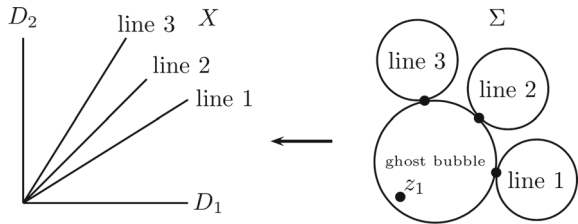
In any case, the meromorphic sections $\zeta_{v_0,1}$ and $\zeta_{v_0,2}$ of $u_{v_0}^* \mathcal{N}_X D_i = \mathcal{O}(2)$ are given by

$$\begin{aligned} \zeta_{v_0,i}(z) &= \frac{c_i (z - z_1)^{s_{1,i}} (z - z_2)^{s_{2,i}}}{z^{s_{\epsilon_1,i}} (z - 1)^{s_{\epsilon_2,i}} (z - \alpha)^{s_{\epsilon_3,i}}} \\ &= \begin{cases} c_1 (z - z_1)^5 ((z - q_{e_1})(z - q_{e_2})(z - q_{e_3}))^{-1} & \text{if } i = 1, \\ c_2 (z - z_2)^5 ((z - q_{e_1})(z - q_{e_2})(z - q_{e_3}))^{-1} & \text{if } i = 2, \end{cases} \end{aligned}$$

where $c_i \in \mathbb{C}^*$. We conclude that

$$\text{obr}_{\Gamma}(f) = \left(\frac{m_1}{m_2} \left(\frac{(q_{e_1} - z_1)(q_{e_2} - z_2)}{(q_{e_1} - z_2)(q_{e_2} - z_1)} \right)^5, \frac{m_2}{m_3} \left(\frac{(q_{e_2} - z_1)(q_{e_3} - z_2)}{(q_{e_2} - z_2)(q_{e_3} - z_1)} \right)^5 \right) \in (\mathbb{C}^*)^2,$$

Fig. 5 A nodal configuration in the boundary of $\overline{\mathcal{M}}_{0,((3,3))}^{\log}(\mathbb{P}^2, D, [3])$



where m_i is the slope of the i -th line with respect to D_1 and D_2 . The fractions

$$\frac{(q_{e_1} - z_1)(q_{e_2} - z_2)}{(q_{e_1} - z_2)(q_{e_2} - z_1)} \quad \text{and} \quad \frac{(q_{e_2} - z_1)(q_{e_3} - z_2)}{(q_{e_2} - z_2)(q_{e_3} - z_1)}$$

are the cross-ratios of $(q_{e_1}, q_{e_2}, z_1, z_2)$ and $(q_{e_2}, q_{e_3}, z_1, z_2)$, respectively. Their dependence on a and b , and thus on h_{v_0} , is non-trivial. $\square$

Example 4.23 Let $X = \mathbb{P}^2$, $D = D_1 \cup D_2$ be the transverse union of two coordinate lines,

$$g=0, \quad k=1, \quad A=[3], \quad \text{and} \quad \mathfrak{s}=(s_1)=((3, 3)).$$

Let Γ be the configuration in Fig. 5; it is the nodal configuration obtained by three lines (not contained in D) passing through the point D_{12} , together with a ghost bubble mapped to D_{12} that connects the domains of these three lines and carries the marked point z_1 .

The dual graph Γ is as in Example 4.21 and thus the isomorphism $\mathcal{G} \rightarrow (\mathbb{C}^*)^2$ is induced by

$$(\mathbb{C}^*)^{l_{e_1}} \times (\mathbb{C}^*)^{l_{e_2}} \times (\mathbb{C}^*)^{l_{e_3}} \longrightarrow (\mathbb{C}^*)^2, \\ ((x_1, y_1), (x_2, y_2), (x_3, y_3)) \longmapsto \left(\frac{x_1 y_2}{y_1 x_2}, \frac{x_2 y_3}{y_2 x_3} \right).$$

We take the marked point z_1 to be ∞ and the nodal points q_{e_1} , q_{e_2} , and q_{e_3} to be 0, 1, and α , respectively. Thus, α is parametrizing the configuration space $\mathcal{M}_{0,4}$ of the four special points on Σ_{v_0} . The meromorphic sections (functions) $\zeta_{v_0,1}$ and $\zeta_{v_0,2}$ are given by

$$\zeta_{v_0,i}(z) = \frac{\beta_i}{z^{s_{e_1,i}}(z-1)^{s_{e_2,i}}(z-\alpha)^{s_{e_3,i}}} = \frac{\beta_i}{z(z-1)(z-\alpha)}, \quad i=1, 2,$$

where $\beta_i \in \mathbb{C}^*$. We conclude that

$$\text{ob}_{\Gamma}(f) = \left(\frac{m_1}{m_2}, \frac{m_2}{m_3} \right) \in (\mathbb{C}^*)^2,$$

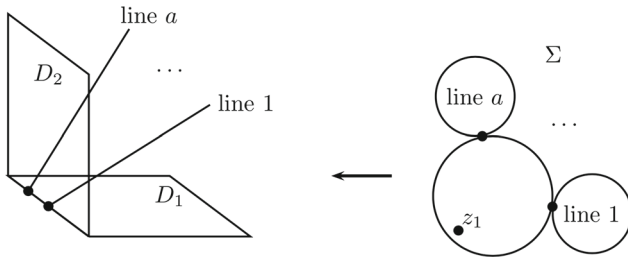


Fig. 6 A nodal configuration in the boundary of $\overline{\mathcal{M}}_{0,((3(d+a),d+a))}^{\log}(\mathbb{P}^4, D, [d+a])$

where m_i is the slope of the i -th line. In particular, it does not depend on α (which is good). However, if we assume $m_1 = m_2 = m_3$, i.e., the 3-lines have the same image, we obtain a configuration for which the step (iii) of the RT-process will be non-trivial. Step (i) will collapse the ghost bubble and ob_Γ does not depend on the location of 4 special points on that. Step (ii) is trivial, and step (iii) yields just one line with the slope $m = m_1 = m_2 = m_3 \in \mathbb{C}^*$. Then

$$\text{ob}_\Gamma: \mathcal{M}_{g,s}^{\text{plog},\gamma}(X, D, A)_\Gamma \cong \mathbb{C}^* \times \mathcal{M}_{0,4} \longrightarrow (\mathbb{C}^*)^2$$

is the constant map 1 and 1 is not a regular value of ob_Γ . $\square$

Example 4.24 Let $X = \mathbb{P}^4$, and $D = D_1 \cup D_2$ where D_1 is a degree 3 hypersurface and D_2 is a hyperplane. The intersection is a cubic surface with 27 lines in it. Let

$$g=0, \quad k=1, \quad A=[d+a], \quad \text{and} \quad s=(s_1)=((3(d+a), d+a)).$$

Let Γ be the configuration in Fig. 6; it is the nodal configuration obtained by a lines (not contained in D) each of which intersects D_1 and D_2 at a single point along D_{12} with tangency order $(3, 1)$, together with a degree d rational curve in D_{12} that connects these a lines and carries the marked point z_1 . If we assume that the latter is a d -fold multiple-cover of a line in D_{12} , we obtain a non-simple configuration γ .

The associated map

$$\varrho: \mathbb{D} = \mathbb{Z}^a \oplus \mathbb{Z}^2 \longrightarrow \mathbb{Z}^{2a}$$

has a 1-dimensional kernel generated by $((1, \dots, 1), (4, 1))$. Therefore, $\mathcal{G}(\Gamma)$ is $(a-1)$ -dimensional. A simple calculation shows that the expected dimension of $\mathcal{M}_{0,s}^\gamma(X, D, A)_\Gamma$ is $2d+a$. On the other hand, the expected dimension of the main stratum is $d+a+2$. For $d \geq 2$, the former is not smaller than the latter. In the classical case, D_{12} is a positive manifold and multiple-covers cause no issue. Here, though, the map ob_Γ depends on the covering map and can not be ignored. Note that Condition (1.19) is not satisfied for the line class in D_{12} . $\square$

Regarding Problem 1, the stronger condition of Definition 1.10 allows us to get the upper²⁴ bound

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{M}_{g,s}^{\text{plg},\gamma}(X, D, A, \nu)_{\Gamma} &\leq \mathcal{Q}(\Gamma) \\ &= c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k + \dim_{\mathbb{C}}(\mathcal{G}) - \dim_{\mathbb{R}} \mathbb{K}_{\mathbb{R}}(\Gamma). \end{aligned} \quad (4.82)$$

We get (4.82), by following steps (i)–(iv) of the RT-process and tracking the change of the quantity

$$\begin{aligned} \mathcal{Q}(\Gamma) &= \sum_{v \in \mathbb{V}} c_1^{TX(-\log D)}(A_v) + k + (2|\mathbb{E}| - |\mathbb{E}_I|) \\ &\quad - \sum_{v \in \mathbb{V}} |I_v| + \sum_{I \subset [N]} (|I| - 1)(|\mathbb{E}_I| - |\mathbb{E}_I|). \end{aligned}$$

- Collapsing a ghost bubble $C_v = (\Sigma_v = \mathbb{P}^1, z_v \cup q_v)$, with $k_v = |z_v|$, $\ell_v = |q_v|$, creates a dual graph $\bar{\Gamma}$ with

$$\mathcal{Q}(\Gamma) - \mathcal{Q}(\bar{\Gamma}) = k_v + \ell_v - 3 = \dim_{\mathbb{C}} \mathcal{M}_{0,k_v+\ell_v}. \quad (4.83)$$

Let $\Gamma_{(i)}$ be the result of collapsing all the ghost bubbles in the step (i) of the RT-process. Applying (4.83) inductively, we conclude that

$$\mathcal{Q}(\Gamma) = \mathcal{Q}(\Gamma_{(i)}) + \dim \prod_{v \in \mathbb{V}_{b,o}} \mathcal{M}_{0,k_v+\ell_v}.$$

- Replacing a multiple-cover bubble $(u_v, C_v = (\Sigma_v = \mathbb{P}^1, z_v \cup q_v))$, with the underlying simple map $(u_{\bar{v}}, C_{\bar{v}} = (\Sigma_{\bar{v}} = \mathbb{P}^1, z_{\bar{v}} \cup q_{\bar{v}}))$, with $A_v = d_v A_{\bar{v}}$, creates a dual graph $\bar{\Gamma}$ with the same set of vertices and

$$\mathcal{Q}(\Gamma) - \mathcal{Q}(\bar{\Gamma}) = (d_v - 1)c_1^{TX(-\log D)}(A_{\bar{v}}) + (k_v + \ell_v) - (k_{\bar{v}} + \ell_{\bar{v}}). \quad (4.84)$$

By (1.19) and (4.63), the righthand side is smaller than or equal to

$$\dim_{\mathbb{C}} \mathcal{M}_{0,\alpha_v}(\mathbb{P}^1, z_{\bar{v}} \cup q_{\bar{v}}, [d_v]) = (d_v - 1)(2 - \delta_{\bar{v}}) + \delta_v - \delta_{\bar{v}},$$

where δ_v (resp. $\delta_{\bar{v}}$) is the number of intersection points of u_v (resp. $u_{\bar{v}}$) with D . Let $\Gamma_{(ii)}$ be the result of reducing all the multiple-cover bubbles of $\Gamma_{(i)}$ in the step (ii) of the RT-process. Starting with $\Gamma_{(i)}$ and applying (4.84) inductively, we conclude that

$$\mathcal{Q}(\Gamma) \geq \mathcal{Q}(\Gamma_{(ii)}) + \dim_{\mathbb{C}} \mathcal{M}_{f''},$$

²⁴ Better upper bounds can be achieved.

where $\mathcal{M}_{f''}$ is the fiber of (4.74) in (4.76).

Step (iii) will further decrease this quantity; therefore,

$$Q(\Gamma) \geq Q(\Gamma_{\text{(iii)}}) + \dim_{\mathbb{C}} \mathcal{M}_{f''}.$$

The inequality (4.82) follows.

Regarding Problem 2, we need to replace $\mathcal{G}(\Gamma)$ with a smaller group $\mathcal{G}(\gamma)$ for which the obstruction map $\text{ob}_{\gamma}: \mathcal{M}_{g,s}^{\text{plog},\gamma}(X, D, A, v)_{\Gamma} \rightarrow \mathcal{G}(\gamma)$ can be transversed and

$$\dim_{\mathbb{C}} \mathcal{M}_{g,s}^{\text{plog},\gamma}(X, D, A, v)_{\Gamma} - \dim_{\mathbb{C}} \mathcal{G}(\gamma) < c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k.$$

This should be done by selecting a subset $\mathbb{E}^*$ of $\mathbb{E}$ where the Eq. (3.48) can be transversed at the nodes corresponding to $\mathbb{E}^*$. We plan to address these in a future work.

Acknowledgements I would like to thank A. Zinger, D. Pomerleano, and the referee of [7] for their helpful comments.

Data Availability Not applicable.

Appendix A: Dependence of ob_{Γ} on Ghost Bubbles

In this section, we study the dependence of the obstruction map ob_{Γ} on the configuration of special points on a ghost bubble.

First, we discuss the effect of collapsing a tree of ghost bubbles into one ghost bubble, and further into a multi-node. Consider a decorated dual graph $\tilde{\Gamma} = (\tilde{\mathbb{V}}, \tilde{\mathbb{E}})$ associated to a nodal pre-log map

$$\tilde{f} = (\phi_v = \phi|_{\Sigma_v}, u_v, \zeta_v = (\zeta_{v,i})_{i \in I_v}, C_v = (\Sigma_v, j_v, \vec{z}_v \cup \vec{q}_v))_{v \in \tilde{\mathbb{V}}}$$

as in Definition 3.9. Let $\Gamma' = (\mathbb{V}', \mathbb{E}')$ be a sub-tree of contracted components in $\tilde{\mathbb{V}}_{\mathbf{b}}$ with $A_v = 0$ for all $v \in \mathbb{V}'$. We can replace Γ' with a single vertex v_0 (keeping the decorations unchanged at the rest of the edges) to obtain a new decorated dual graph $\Gamma = (\mathbb{V}, \mathbb{E})$ satisfying the combinatorial conditions (3.51) (but not necessarily Definition 3.11 (1)). Conversely, starting from Γ we obtain various $\tilde{\Gamma}$ by replacing a ghost bubble v_0 with a tree of ghost bubbles Γ' . Note that

$$\tilde{\mathbb{V}} = (\mathbb{V} - \{v_0\}) \cup \mathbb{V}' \quad \text{and} \quad \tilde{\mathbb{E}} = \mathbb{E} \cup \mathbb{E}'. \quad (\text{A.1})$$

Also,

$$I_e = I_v = I_{v'}, \quad \forall v, v' \in \mathbb{V}', \quad e \in \mathbb{E}'.$$

This common value is what I_{v_0} is in Γ . For each $v \in \mathbb{V}'$, u_v is the constant map into some $p \in D_{I_{v_0}} - \partial D_{I_{v_0}}$. The map u_{v_0} is defined to be the constant map into p as well.

Since $v_0 \in \mathbb{V}_b$ and the combinatorial condition (3.51) is satisfied, for any arrangement of special point $z_{v_0} \cup q_{v_0}$ on Σ_{v_0} , there exists a set of meromorphic functions

$$\zeta_{v_0} = (\zeta_{v_0, i})_{i \in I_{v_0}}$$

such that

$$\begin{aligned} f &= (\phi_v, u_v, \zeta_v, C_v)_{v \in \mathbb{V} - \{v_0\}} \cup (u_{v_0}, \zeta_{v_0}, C_{v_0} = (\Sigma_{v_0}, j_{v_0}, \vec{z}_{v_0} \cup \vec{q}_{v_0})) \\ &\in \mathcal{M}_{g, s}^{\text{plog}}(X, D, A, v)_{\Gamma}. \end{aligned}$$

The reduced graph Γ may not satisfy Condition (1) of Definition 3.11, however, since $\mathcal{G}(\Gamma')$ is trivial (see Example 3.13 and remark 3.14), Lemma A.1 below shows that the natural homomorphism from the obstruction group $\mathcal{G}(\Gamma)$ associated with Γ to $\mathcal{G}(\tilde{\Gamma})$ is onto. Expanding v_0 will increase the kernel of ϱ in (3.39) and reduces its cokernel.

Let

$$\iota_{\mathbb{D}}: \mathbb{D} = \mathbb{Z}^{\mathbb{E}} \oplus \bigoplus_{v \in \mathbb{V}} \mathbb{Z}^{I_v} \longrightarrow \tilde{\mathbb{D}} = \mathbb{Z}^{\tilde{\mathbb{E}}} \oplus \bigoplus_{v \in \tilde{\mathbb{V}}} \mathbb{Z}^{I_v}$$

be the embedding which maps $\mathbb{Z}^{I_{v_0}}$ diagonally into $\bigoplus_{v \in \mathbb{V}'} \mathbb{Z}^{I_v}$, and is the identity map on the rest of the terms with respect to the identifications (A.1). Let

$$\iota_{\mathbb{T}}: \mathbb{T} = \bigoplus_{e \in \mathbb{E}} \mathbb{Z}^{I_e} \longrightarrow \tilde{\mathbb{T}} = \bigoplus_{e \in \tilde{\mathbb{E}}} \mathbb{Z}^{I_e}$$

denote the natural inclusion map corresponding to the second identification in (A.1).

Lemma A.1 *With notation as above, for compatible choices of orientations on $\mathbb{E}$ and $\tilde{\mathbb{E}}$, the commutative diagram*

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\iota_{\mathbb{D}}} & \tilde{\mathbb{D}} \\ \downarrow \varrho & & \downarrow \tilde{\varrho} \\ \mathbb{T} & \xrightarrow{\iota_{\mathbb{T}}} & \tilde{\mathbb{T}} \end{array}$$

induces a surjective homomorphism $\varphi_{\Gamma, \tilde{\Gamma}}: \mathcal{G}(\Gamma) \longrightarrow \mathcal{G}(\tilde{\Gamma})$.

Proof Since $I_v = I_{v_0}$ for all $v \in \mathbb{V}'$, the map $\varrho': \mathbb{D}' \rightarrow \mathbb{T}'$ corresponding to Γ' descends to the similarly denoted map

$$\varrho': \mathbb{D}' = \mathbb{Z}^{\mathbb{E}'} \oplus \frac{\bigoplus_{v \in \mathbb{V}'} \mathbb{Z}^{I_v}}{\mathbb{Z}^{I_{v_0}}} \longrightarrow \mathbb{T}' = \bigoplus_{e \in \mathbb{E}'} \mathbb{Z}^{I_e},$$

where $\mathbb{Z}^{I_{v_0}} \hookrightarrow \bigoplus_{v \in \mathbb{V}'} \mathbb{Z}^{I_v}$ is the diagonal embedding. Let

$$\pi_{\mathbb{D}} : \tilde{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}' \quad \text{and} \quad \pi_{\mathbb{T}} : \tilde{\mathbb{T}} \longrightarrow \mathbb{T}'$$

denote the natural projection maps. For each $e \in \tilde{\mathbb{E}}$, let ϱ denote the chosen orientation on e for defining ϱ . By restriction, this gives us the choice of orientations on $\mathbb{E}$ and $\mathbb{E}'$ used to define ϱ and ϱ' . The commutative diagram

$$\begin{array}{ccccc} \mathbb{D} & \xrightarrow{\iota_{\mathbb{D}}} & \tilde{\mathbb{D}} & \xrightarrow{\pi_{\mathbb{D}}} & \overline{\mathbb{D}}' \\ \downarrow \varrho & & \downarrow \tilde{\varrho} & & \downarrow \varrho' \\ \mathbb{T} & \xrightarrow{\iota_{\mathbb{T}}} & \tilde{\mathbb{T}} & \xrightarrow{\pi_{\mathbb{T}}} & \mathbb{T}' \end{array}$$

has exact rows. Therefore, since $\text{coker}(\varrho') = 0$, we get the long exact sequence,

$$0 \longrightarrow \ker(\varrho) \longrightarrow \ker(\tilde{\varrho}) \longrightarrow \ker(\varrho') \longrightarrow \text{coker}(\varrho) \longrightarrow \text{coker}(\tilde{\varrho}) \longrightarrow 0.$$

The last map gives us the surjective homomorphism $\varphi_{\Gamma, \tilde{\Gamma}} : \mathcal{G}(\Gamma) \rightarrow \mathcal{G}(\tilde{\Gamma})$. $\square$

Suppose $\{f_t\}_{t=1}^{\infty}$ is a sequence of pre-log maps f_t are obtained by the deformations

$$(\Sigma_{v_0}, \vec{z}_{v_0, t} \cup \vec{q}_{v_0, t})_{t=1}^{\infty} \in \mathcal{M}_{0, k_{v_0} + \ell_{v_0}} \quad (\text{A.2})$$

of $(\Sigma_{v_0}, \vec{z}_{v_0} \cup \vec{q}_{v_0})$ and keeping the other components fixed. Suppose that, as $t \rightarrow \infty$, f_t converges to $\tilde{f} \in \mathcal{M}_{g, s}^{\text{plog}}(X, D, A, v)_{\tilde{\Gamma}}$ that has the nodal configuration C' in place of C_{v_0} . For each $i \in I_{v_0}$, let $\zeta_{v_0, t, i}$ be a meromorphic function on Σ_{v_0} with zeros/poles of the given order at $\vec{z}_{v_0, t} \cup \vec{q}_{v_0, t}$. By [7, Cor. 3.12], for each $v \in \mathbb{V}'$, as $t \rightarrow \infty$, restricted to Σ_v and up to scaling, $\zeta_{v_0, t, i}$ converges to $\zeta_{v, i}$. We conclude that

$$\lim_{t \rightarrow \infty} \varphi_{\Gamma, \tilde{\Gamma}} \circ \text{ob}_{\Gamma}(f_t) = \text{ob}_{\tilde{\Gamma}}(\tilde{f}) \in \mathcal{G}(\tilde{\Gamma}). \quad (\text{A.3})$$

Next, we study the dependence of ob_{Γ} on the location of the special points $\vec{z}_{v_0} \cup q_{v_0}$ on a single ghost bubble Σ_{v_0} . For any pre-log map

$$\begin{aligned} f &= (\phi_v, u_v, \zeta_v, C_v)_{v \in \mathbb{V} - \{v_0\}} \cup (u_{v_0}, \zeta_{v_0}, C_{v_0} = (\Sigma_{v_0}, \vec{z}_{v_0} \cup \vec{q}_{v_0})) \\ &\in \mathcal{M}_{g, s}^{\text{plog}}(X, D, A, v)_{\Gamma} \end{aligned}$$

with domain $\Sigma = \bigcup_{v \in \mathbb{V}} \Sigma_v$, let

$$\bar{f} = (\phi_v, u_v, \zeta_v, C_v)_{v \in \overline{\mathbb{V}} = \mathbb{V} - \{v_0\}} \in \mathcal{M}_{g, s}^{\text{plog}}(X, D, A, v)_{\bar{\Gamma}} \quad (\text{A.4})$$

denote the tuple obtained by forgetting the v_0 -th component, defined on the domain $\overline{\Sigma}$ obtained by removing Σ_{v_0} from Σ . This is no longer a nodal domain; $\overline{\Sigma}$ is a domain

with a multi-node m in place of v_0 . Let $\bar{\Gamma} = (\bar{\mathbb{V}}, \bar{\mathbb{E}})$ denote the combinatorial type of $\bar{\Sigma}$. We have

$$\bar{\mathbb{V}} = \mathbb{V} - \{v_0\}, \quad \bar{\mathbb{E}} = (\mathbb{E} - \{e_1, \dots, e_{\ell_0}\}) \cup \{m\}, \quad \bar{\mathbb{E}} = \bar{\mathbb{E}} - \{e_1, \dots, e_{\ell_0}\}.$$

Note that $I_{v_0} = I_{e_1} = \dots = I_{e_{\ell_0}}$. We let I_m to also denote this common value. Associated to $\bar{\Gamma}$ (or in general, any domain with such multi-nodes) we consider the linear map

$$\bar{\varrho}: \bar{\mathbb{D}} = \mathbb{Z}^{\mathbb{E}} \oplus \bigoplus_{v \in \bar{\mathbb{V}}} \mathbb{Z}^{I_v} \longrightarrow \bar{\mathbb{T}} = \bigoplus_{e \in \mathbb{E} - \{e_1, \dots, e_{\ell_0}\}} \mathbb{Z}^{I_e} \oplus \frac{\bigoplus_{e \in \{e_1, \dots, e_{\ell_0}\}} \mathbb{Z}^{I_e}}{\mathbb{Z}^{I_m}}, \quad (\text{A.5})$$

where

$$\mathbb{Z}^{I_m} \longrightarrow \bigoplus_{e \in \{e_1, \dots, e_{\ell_0}\}} \mathbb{Z}^{I_e}$$

is the diagonal embedding, and $\bar{\varrho}$ is the composition of the natural inclusion²⁵ inclusion $\bar{\mathbb{D}} \longrightarrow \mathbb{D}$, ϱ , and the natural projection $\pi_{\mathbb{T}}: \mathbb{T} \longrightarrow \bar{\mathbb{T}}$. Similar to (3.42), the obstruction group $\mathcal{G}(\bar{\Gamma})$ associated to $\bar{\Gamma}$ is the quotient

$$\mathcal{G}(\bar{\Gamma}) = \left(\prod_{e \in \mathbb{E} - \{e_1, \dots, e_{\ell_0}\}} (\mathbb{C}^*)^{I_e} \times \frac{\prod_{e \in \{e_1, \dots, e_{\ell_0}\}} (\mathbb{C}^*)^{I_e}}{(\mathbb{C}^*)^{I_m}} \right) / \text{Image}(\exp(\bar{\varrho}_{\mathbb{C}})).$$

Similar to (3.47), after fixing a choice of local holomorphic coordinates w_e around each nodal point $q_e \in \Sigma_v$, for all $v \in \bar{\mathbb{V}}$ and $e \in \bar{\mathbb{E}}_v$, we define

$$\text{ob}_{\bar{\Gamma}}(\bar{f}) \in \mathcal{G}(\bar{\Gamma})$$

to be class of

$$\begin{aligned} \bar{\eta} &= \prod_{e \in \mathbb{E} - \{e_1, \dots, e_{\ell_0}\}} \prod_{i \in I_e} \frac{\eta_{e,i}}{\eta_{e,i}} \times \prod_{i \in I_m} [\eta_{e_1,i}, \dots, \eta_{e_{\ell_0},i}] \\ &\in \prod_{e \in \mathbb{E} - \{e_1, \dots, e_{\ell_0}\}} (\mathbb{C}^*)^{I_e} \times ((\mathbb{C}^*)^{\ell_0} / \mathbb{C}^*)^{I_m} \end{aligned}$$

in $\mathcal{G}(\bar{\Gamma})$. Here, we use the fact that, for each $i \in I_m$, the class

$$[\eta_{e_1,i}, \dots, \eta_{e_{\ell_0},i}] \in (\mathbb{C}^*)^{\ell_0} / \mathbb{C}^*$$

²⁵ Since $\bar{\mathbb{D}}$ is just missing a summand in $\mathbb{D}$, we get both natural inclusion and projection maps $\bar{\mathbb{D}} \hookrightarrow \mathbb{D}$ and $\pi_{\mathbb{D}}: \mathbb{D} \rightarrow \bar{\mathbb{D}}$, respectively.

is independent of the identification $\mathcal{N}_X D_i|_p = \mathbb{C}$. For the same reason as in [7, pp. 1013–1015], the class $\text{ob}_{\bar{\Gamma}}(\bar{f}) = [\bar{\eta}]$ of $\bar{\eta}$ in $\mathcal{G}(\bar{\Gamma})$ is independent of the choice of local holomorphic coordinates w_e and $\{\zeta_v\}_{v \in \bar{\mathbb{V}}}$ up to rescaling.

Lemma A.2 *The obstruction groups $\mathcal{G}(\Gamma)$ and $\mathcal{G}(\bar{\Gamma})$ are naturally isomorphic. There exists a holomorphic map $o_{v_0}: \mathcal{M}_{0,k_{v_0}+\ell_{v_0}} \rightarrow \mathcal{G}(\bar{\Gamma})$ such that, under the natural isomorphism $\mathcal{G}(\Gamma) \cong \mathcal{G}(\bar{\Gamma})$, we have*

$$\text{ob}_{\bar{\Gamma}}(\bar{f}) \circ_{v_0}(C_{v_0})^{-1} = \text{ob}_{\Gamma}(f).$$

In other words, $\text{ob}_{\Gamma}(f) = 1$ if and only if

$$\text{ob}_{\bar{\Gamma}}(\bar{f}) = o_{v_0}(C_{v_0}).$$

Proof The commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}^{I_{v_0}} & \xrightarrow{\iota_{\mathbb{D}}} & \mathbb{D} & \xrightarrow{\pi_{\mathbb{D}}} & \bar{\mathbb{D}} \\ \downarrow \cong & & \downarrow \varrho & & \downarrow \bar{\varrho} \\ \mathbb{Z}^{I_m} & \xrightarrow{\iota_{\mathbb{T}}} & \mathbb{T} & \xrightarrow{\pi_{\mathbb{T}}} & \bar{\mathbb{T}} \end{array}$$

has exact rows. We get the long exact sequence,

$$0 \longrightarrow \ker(\varrho) \longrightarrow \ker(\bar{\varrho}) \longrightarrow 0 \longrightarrow \text{coker}(\varrho) \longrightarrow \text{coker}(\bar{\varrho}) \longrightarrow 0.$$

The last map gives us the isomorphism $\varphi_{\Gamma, \bar{\Gamma}}: \mathcal{G}(\Gamma) \rightarrow \mathcal{G}(\bar{\Gamma})$.

By definition, $\varphi_{\Gamma, \bar{\Gamma}} \circ \text{ob}_{\Gamma}(f)$ is the class $[\eta]$ of

$$\begin{aligned} \eta &= \prod_{e \in \mathbb{E} - \{e_1, \dots, e_{\ell_0}\}} \prod_{i \in I_e} \frac{\eta_{e,i}}{\eta_{e,i}} \times \prod_{i \in I_m} \left[\frac{\eta_{e_1,i}}{\eta_{e_1,i}}, \dots, \frac{\eta_{e_{\ell_0},i}}{\eta_{e_{\ell_0},i}} \right] \\ &\in \prod_{e \in \mathbb{E} - \{e_1, \dots, e_{\ell_0}\}} (\mathbb{C}^*)^{I_e} \times ((\mathbb{C}^*)^{\ell_0} / \mathbb{C}^*)^{I_m} \end{aligned}$$

in $\mathcal{G}(\bar{\Gamma})$. We have

$$\eta = \bar{\eta} \cdot g^{-1},$$

where

$$\begin{aligned} g &= \prod_{e \in \mathbb{E} - \{e_1, \dots, e_{\ell_0}\}} \prod_{i \in I_e} 1 \times \prod_{i \in I_m} \left[\eta_{e_1,i}, \dots, \eta_{e_{\ell_0},i} \right] \\ &\in \prod_{e \in \mathbb{E} - \{e_1, \dots, e_{\ell_0}\}} (\mathbb{C}^*)^{I_e} \times ((\mathbb{C}^*)^{\ell_0} / \mathbb{C}^*)^{I_m}. \end{aligned}$$

Up to rescaling, the section ζ_{v_0} only depends on the location of the special points $z_{v_0} \cup q_{v_0}$ (and the pre-determined multiplicities at those points). The map

$$\zeta_{v_0} \longrightarrow \prod_{i \in I_m} [\eta_{e_1, i}, \dots, \eta_{e_{\ell_0}, i}] \in ((\mathbb{C}^*)^{\ell_0} / \mathbb{C}^*)^{I_m}$$

only depends on the $(\mathbb{C}^*)^{I_m}$ -equivalence class of ζ_{v_0} and, via the inclusion

$$((\mathbb{C}^*)^{\ell_0} / \mathbb{C}^*)^{I_m} \hookrightarrow \prod_{e \in \mathbb{E} - \{e_1, \dots, e_{\ell_0}\}} (\mathbb{C}^*)^{I_e} \times ((\mathbb{C}^*)^{\ell_0} / \mathbb{C}^*)^{I_m}$$

descends to a well-defined map

$$o_{v_0} : \mathcal{M}_{0, k_{v_0} + \ell_{v_0}} \longrightarrow \mathcal{G}(\bar{\Gamma}).$$

□

The projection map

$$\pi_{\Gamma, \bar{\Gamma}} : \mathcal{M}_{g, \mathfrak{s}}^{\text{plog}}(X, D, A, \nu)_{\Gamma} \longrightarrow \mathcal{M}_{g, \mathfrak{s}}^{\text{plog}}(X, D, A, \nu)_{\bar{\Gamma}}, \quad f \longmapsto \bar{f}, \quad (\text{A.6})$$

is a fiber bundle with fibers $\mathcal{M}_{0, g_{v_0} + \ell_{v_0}}$. By Lemma A.2, there exists a map

$$\text{ob}_{\bar{\Gamma}} : \mathcal{M}_{g, \mathfrak{s}}^{\text{plog}}(X, D, A, \nu)_{\bar{\Gamma}} \longrightarrow \mathcal{G}_{\bar{\Gamma}}$$

such that

$$\pi_{\Gamma, \bar{\Gamma}}(\mathcal{M}_{g, \mathfrak{s}}(X, D, A, \nu)_{\Gamma}) = \text{ob}_{\bar{\Gamma}}^{-1}(\text{Im}(o_{v_0})).$$

In general, the map o_{v_0} can be non-trivial; see Example 4.21 or A.4 below. There are, however, situations where the map is constant and ob_{Γ} factors through the fibration (A.6); see Example 4.23. There are also examples, such as Example A.3 below, where the map o_{v_0} is constant but the constant value is not 1.

Example A.3 Let $X = \mathbb{P}^3$, $D = D_1 \cup D_2 \cup D_3$ be the transverse union of three coordinate hyperplanes,

$$g = 0, \quad k = 3, \quad A = [3], \quad \mathfrak{s} = (s_1, s_2, s_3) = ((3, 0, 0), (0, 3, 0), (0, 0, 3)).$$

Let Γ be the dual graph in Fig. 7 with the set of vertices $\mathbb{V} = \{v_1, v_2, v_3, v_0\}$ and the set of edges $\mathbb{E} = \{e_1, e_2, e_3\}$ such that e_i connects v_0 and v_i , for all $i = 1, 2, 3$. Choose the orientations $\underline{e}_i$ to end at v_0 , for all $i = 1, 2, 3$. We have

$$\begin{aligned} I_{v_0} &= [N] = \{1, 2, 3\}, \quad s_{\underline{e}_1} = (-2, 1, 1) \in \mathbb{Z}^3, \quad s_{\underline{e}_2} = (1, -2, 1) \in \mathbb{Z}^3, \\ s_{\underline{e}_3} &= (1, 1, -2) \in \mathbb{Z}^3, \quad I_{v_i} = \{i\}, \quad A_{v_i} = [1] \in H_2(X_i, \mathbb{Z}), \quad \forall i = 1, 2, 3. \end{aligned}$$

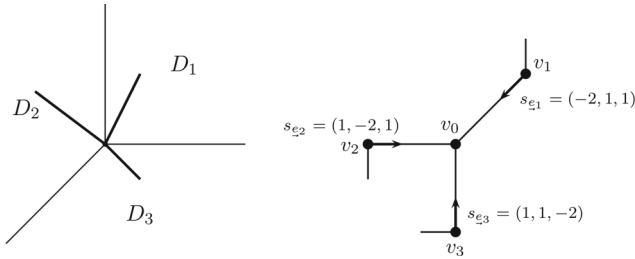


Fig. 7 Dual graph Γ and the image of a map belonging to $\mathcal{M}_{0,3}(\mathbb{P}^3, D, [3])_\Gamma$ in $\mathbb{P}^3$

A pre-log curve with this dual graph is made of a line $\ell_i = \text{Im}(u_{v_i})$ in $D_i \cong \mathbb{P}^2$ passing through the point D_{123} , for each $i \in [3]$, and a log tuple

$$(u_{v_0}, [\zeta_{v_0,i}]_{i \in [3]}, (\Sigma_{v_0}, j_{v_0})) \cong \mathbb{P}^1, q_{v_0} = \{q_{e_i}\}_{i \in [3]},$$

where u_{v_0} is the constant map onto D_{123} , and $\zeta_{v_0,i}$, for each $i \in [3]$, is a meromorphic function with poles/zeros of the prescribed order at q_{v_0} . Each ℓ_i also carries a meromorphic section ζ_{v_i} of $\mathcal{O}(1)|_{\ell_i}$ with a pole of order 2 at q_{e_i} , and a zero of order 3 at the marked point z_i . Both kernel and cokernel of

$$\varrho: \mathbb{D} = \mathbb{Z}^{\mathbb{E}} \oplus \bigoplus_{i=0}^3 \mathbb{Z}^{I_{v_i}} \longrightarrow \mathbb{T} = \bigoplus_{i=1}^3 \mathbb{Z}^{I_{e_i}}$$

are 1-dimensional. The homomorphism

$$(\mathbb{C}^*)^3 \times (\mathbb{C}^*)^3 \times (\mathbb{C}^*)^3 \longrightarrow \mathbb{C}^*, \quad \prod_{i=1}^3 (x_{i1}, x_{i2}, x_{i3}) \longmapsto \frac{x_{12}x_{23}x_{31}}{x_{13}x_{32}x_{21}} \quad (\text{A.7})$$

descends to an isomorphism $\mathcal{G} \rightarrow \mathbb{C}^*$.

In the pre-log space $\mathcal{M}_{0,5}^{\text{plog}}(\mathbb{P}^3, D, [3])_\Gamma$, the three lines ℓ_1, ℓ_2, ℓ_3 are allowed to be any line passing through the point D_{123} with non-trivial slopes in $\mathbb{C}^*$. For each $i \in [3]$, the line ℓ_i is the completion of the image of a map of the form

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{C}^3, \quad w \longmapsto (w_{ij})_{j=1,2,3} \subset \mathbb{C}^3, \\ w_{ii} &= 0, \quad w_{ij} = a_{ij}w_i, \quad a_{ij} \in \mathbb{C}^*, \quad \forall j \in [3] - i. \end{aligned}$$

Here D_i corresponds to the subspace $(x_i = 0) \subset \mathbb{C}^3$. For $i = 1, 2, 3$, we have $\zeta_{v_i}(w_i) = \alpha_{ii}w_i^{-2}$. Putting $(q_{e_1}, q_{e_2}, q_{e_3}) = (0, 1, \infty)$, we get

$$\zeta_{v_0,1} = \alpha_1 \frac{z^2}{z-1}, \quad \zeta_{v_0,2} = \alpha_2 \frac{(z-1)^2}{z}, \quad \zeta_{v_0,3} = \alpha_3 \frac{1}{z(z-1)}.$$

We conclude that

$$\text{ob}_\Gamma(f) = -\frac{a_{12}a_{23}a_{31}}{a_{13}a_{32}a_{21}};$$

i.e., f is a log map if and only if the product of the slopes of three lines (in the cyclic order) is -1 . On the other hand,

$$\text{ob}_{\bar{\Gamma}}(\bar{f}) = \frac{a_{12}a_{23}a_{31}}{a_{13}a_{32}a_{21}}.$$

We conclude that the image of o_{v_0} in $\mathcal{G}(\bar{\Gamma}) \cong \mathbb{C}^*$ is -1 . $\square$

For each expansion $\tilde{\Gamma}$ of Γ obtained by expending v_0 into a tree of ghost bubbles Γ' (or in other words, expanding Σ_{v_0} to C'), the pre-log space

$$\mathcal{M}_{g,s}^{\text{plog}}(X, D, A, \nu)_{\tilde{\Gamma}}$$

is obtained by taking the closure of fibers of (A.6) and letting C_{v_0} to converge to a nodal domain

$$C' = \bigcup_{v \in \mathbb{V}'} C_v \in \partial \overline{\mathcal{M}}_{0,g_{v_0}+\ell_{v_0}}. \quad (\text{A.8})$$

Let

$$\mathcal{M}_{\Gamma'} \subset \partial \overline{\mathcal{M}}_{0,g_{v_0}+\ell_{v_0}}$$

denote the stratum of nodal configurations (A.8). Taking union over all such $\tilde{\Gamma}$, we get the fiber bundle

$$\bigcup_{\tilde{\Gamma}} \mathcal{M}_{g,s}^{\text{plog}}(X, D, A, \nu)_{\tilde{\Gamma}} \longrightarrow \mathcal{M}_{g,s}^{\text{plog}}(X, D, A, \nu)_{\bar{\Gamma}}$$

with compact fibers $\overline{\mathcal{M}}_{0,g_{v_0}+\ell_{v_0}}$. Similar to Lemma A.2, for each $\tilde{\Gamma}$, we get a map

$$o_{\Gamma'}: \mathcal{M}_{\Gamma'} \longrightarrow \mathcal{G}(\tilde{\Gamma})$$

such that

$$\text{ob}_{\tilde{\Gamma}}(\tilde{f}) = \varphi_{\bar{\Gamma},\tilde{\Gamma}} \circ \text{ob}_{\bar{\Gamma}}(\bar{f}) \cdot o_{\Gamma'}(C')^{-1}, \quad \varphi_{\bar{\Gamma},\tilde{\Gamma}} = \varphi_{\Gamma,\tilde{\Gamma}} \circ \varphi_{\Gamma,\bar{\Gamma}}^{-1}: \mathcal{G}(\bar{\Gamma}) \longrightarrow \mathcal{G}(\tilde{\Gamma}).$$

In other words, for the fiber bundle

$$\pi_{\tilde{\Gamma},\bar{\Gamma}}: \mathcal{M}_{g,s}^{\text{plog}}(X, D, A, \nu)_{\tilde{\Gamma}} \longrightarrow \mathcal{M}_{g,s}^{\text{plog}}(X, D, A, \nu)_{\bar{\Gamma}}, \quad \tilde{f} \longmapsto \bar{f},$$

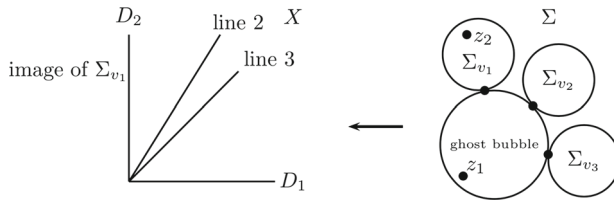


Fig. 8 A nodal configuration in the boundary of $\overline{\mathcal{M}}_{0,((3,3))}^{\log}(\mathbb{P}^2, D, [3])$

we have

$$\pi_{\tilde{\Gamma}, \tilde{\Gamma}}(\mathcal{M}_{g,s}(X, D, A, v)_{\tilde{\Gamma}}) = (\varphi_{\tilde{\Gamma}, \tilde{\Gamma}} \circ \text{ob}_{\tilde{\Gamma}})^{-1}(\text{Im}(o_{\Gamma'})).$$

Note that as Γ' gets bigger, both $\mathcal{M}_{\Gamma'}$ and $\mathcal{G}(\tilde{\Gamma})$ get smaller. The Deligne–Mumford convergence in (A.2) is compatible with the projection $\mathcal{G}(\Gamma) \rightarrow \mathcal{G}(\tilde{\Gamma})$ in the sense that

$$\begin{array}{ccc} \mathcal{M}_{0, k_{v_0} + \ell_{v_0}} & \xrightarrow{o_{v_0}} & \mathcal{G}(\Gamma) \cong \mathcal{G}(\bar{\Gamma}) \\ \downarrow \text{DM-convergence} & & \downarrow \varphi_{\Gamma, \tilde{\Gamma}} \cong \varphi_{\tilde{\Gamma}, \tilde{\Gamma}} \\ \mathcal{M}_{\Gamma'} & \xrightarrow{o_{\Gamma'}} & \mathcal{G}(\tilde{\Gamma}); \end{array}$$

i.e., if a family of smooth marked curves $\{C_t\}_{t=1}^{\infty}$ converges to $C' \in \mathcal{M}_{\Gamma'}$, then $\pi_{\Gamma, \tilde{\Gamma}}(o_{v_0}(C_t))$ converges to $o_{\Gamma'}(C')$. In Example A.4, as α converges to 0, 1, ∞ , we get an expansion $\tilde{\Gamma}$ of Γ that replaces Σ_{v_0} by a nodal sphere with 2 components. Then $\mathcal{M}_{\Gamma'}$ is just a point and $\mathcal{G}(\tilde{\Gamma})$ is the trivial group.

Example A.4 Let $X = \mathbb{P}^2$, $D = D_1 \cup D_2$ be the transverse union of two coordinate lines,

$$g=0, \quad k=2, \quad A=[4], \quad \text{and} \quad \mathfrak{s}=(s_1, s_2)=((5, 4), (0, 1)).$$

Let Γ be the configuration in Fig. 8. It is the nodal configuration obtained by 2 lines passing through the point D_{12} , a double cover $u_{v_1} : \Sigma_{v_1} \rightarrow D_2$ of the line D_2 ramified at D_{12} containing the second marked point z_2 . The three of them are connected by a ghost bubble Σ_{v_0} mapped to D_{12} carrying the marked point z_1 .

Let $\mathbb{V} = \{v_0, v_1, v_2, v_3\}$ where v_1 corresponds to the double cover of D_2 , v_2, v_3 correspond to the lines 2 and 3, respectively, and v_0 corresponds to the ghost bubble. Let $\mathbb{E} = \{e_1, e_2, e_3\}$ where e_i corresponds to the node connecting the domain of the i -th component to the ghost bubble. We can choose the orientation e_i to be the one ending at v_0 . We have

$$\begin{aligned} I_{v_0} &= \{1, 2\}, \quad I_{v_1} = \{2\}, \quad I_{v_i} = \emptyset, \quad \forall i = 2, 3, \quad I_{e_i} = \{1, 2\}, \quad \forall i = 1, 2, 3, \\ s_{e_1} &= (2, 1), \quad s_{e_2} = (1, 1), \quad s_{e_3} = (1, 1). \end{aligned}$$

The map

$$\varrho: \mathbb{Z}^{\mathbb{E}} \oplus \mathbb{Z}^{I_{v_0}} \oplus \mathbb{Z}^{I_{v_1}} \longrightarrow \bigoplus_{i=1}^3 \mathbb{Z}^{I_{e_i}}$$

has 1-dimensional kernel cokernel. The group homomorphism

$$(\mathbb{C}^*)^{I_{e_1}} \times (\mathbb{C}^*)^{I_{e_2}} \times (\mathbb{C}^*)^{I_{e_2}} \longrightarrow \mathbb{C}^*, \quad ((x_1, y_1), (x_2, y_2), (x_3, y_3)) \longmapsto \frac{x_2 y_3}{y_2 x_3}$$

descends to an isomorphism $\mathcal{G} \cong \mathbb{C}^*$. We take the marked point z_1 to be ∞ and the nodal points q_{e_1} , q_{e_2} , and q_{e_3} to be α , 0, and 1, respectively. Thus, α parametrizes the configuration space $\mathcal{M}_{0,4}$ of the four special points on Σ_{v_0} . The meromorphic sections (functions) $\zeta_{v_0,1}$ and $\zeta_{v_0,2}$ are given by

$$\zeta_{v_0,i}(z) = \frac{\beta_i}{z^{\frac{s_{e_1,i}}{s_{e_1,i}}}(z-1)^{\frac{s_{e_2,i}}{s_{e_2,i}}}(z-\alpha)^{\frac{s_{e_3,i}}{s_{e_3,i}}}} = \begin{cases} \beta_1 (z(z-1)(z-\alpha)^2)^{-1} & \text{if } i=1, \\ \beta_2 (z(z-1)(z-\alpha))^{-1} & \text{if } i=2, \end{cases}$$

where $\beta_i \in \mathbb{C}^*$. Then

$$\text{ob}_\Gamma(f) = \frac{m_2}{m_3} \frac{\alpha - 1}{\alpha} \in \mathbb{C}^*,$$

where m_2 and m_3 are slopes of the lines 2 and 3, respectively. Therefore, we get

$$o_{v_0}: \mathcal{M}_{0,4} \longrightarrow \mathbb{C}^*, \quad \alpha \longmapsto \frac{\alpha}{\alpha - 1}, \quad \text{and} \quad \text{ob}_{\overline{\Gamma}}(\overline{f}) = \frac{m_2}{m_3}.$$

For any $\overline{f}$ with $m_2/m_3 \neq 1$, there exists a unique α such that $f \in \mathcal{M}_{0,5}(X, D, A)_\Gamma$. The three special values $m_2/m_3 = 0, 1, \infty$ correspond to the limiting situations where α converges to 0, ∞ , 1, respectively, and we get a tree Γ' of two ghost bubbles instead of Σ_{v_0} . $\square$

References

1. Abramovich, D., Chen, Q.: Stable logarithmic maps to Deligne–Faltings pairs II. *Asian J. Math.* **18**(3), 465–488 (2014)
2. Boggi, M., Pikaart, M.: Galois covers of moduli of curves. *Compos. Math.* **120**(2), 171–191 (2000)
3. Chen, Q.: Stable logarithmic maps to Deligne–Faltings pairs I. *Ann. Math.* **180**(2), 455–521 (2014)
4. Daemi, A., Fukaya, K.: Monotone Lagrangian Floer theory in smooth divisor complements: I. [arXiv:1808.08915](https://arxiv.org/abs/1808.08915) (2018)
5. Daemi, A., Fukaya, K.: Monotone Lagrangian Floer theory in smooth divisor complements: II. [arXiv:1809.03409](https://arxiv.org/abs/1809.03409) (2018)
6. Eliashberg, Y., Givental, A., Hofer, H.: Introduction to symplectic field theory. In: *GAFA 2000* (Tel Aviv, 1999). *Geom. Funct. Anal., Special Volume, Part II*, pp. 560–673. Birkhäuser, Basel (2000)
7. Farajzadeh-Tehrani, M.: Pseudoholomorphic curves relative to a normal crossings symplectic divisor: compactification. *Geom. Topol.* **26**(3), 989–1075 (2022)

8. Farajzadeh-Tehrani, M.: Limits of stable maps in a semi-stable degeneration. *Geom. Dedicata* **216**(6), 66 (2022)
9. Farajzadeh-Tehrani, M.: BPS invariants of symplectic log Calabi–Yau fourfolds. [arXiv:2206.13589](https://arxiv.org/abs/2206.13589) (2022)
10. Farajzadeh-Tehrani, M.: Pseudoholomorphic curves relative to a normal crossings symplectic divisor: gluing (**in progress**)
11. Farajzadeh-Tehrani, M., Fukaya, K.: Gromov–Witten theory via Kuranishi structures. In: *Virtual Fundamental Cycles in Symplectic Topology. Mathematical Surveys and Monographs*, vol. 237, pp. 111–252. Amer. Math. Soc., Providence, RI (2019)
12. Farajzadeh-Tehrani, M., McLean, M., Zinger, A.: Normal crossings singularities for symplectic topology. *Adv. Math.* **339**, 672–748 (2018)
13. Farajzadeh-Tehrani, M., McLean, M., Zinger, A.: Singularities and semistable degenerations for symplectic topology. *C. R. Math. Acad. Sci. Paris* **356**(4), 420–432 (2018)
14. Farajzadeh-Tehrani, M., McLean, M., Zinger, A.: Normal crossings singularities for symplectic topology, II. [arXiv:1908.09390](https://arxiv.org/abs/1908.09390) (2019)
15. Farajzadeh-Tehrani, M., McLean, M., Zinger, A.: Normal crossings singularities for symplectic topology; structures. [arXiv:2112.13125](https://arxiv.org/abs/2112.13125) (2021)
16. Farajzadeh-Tehrani, M., Zinger, A.: On symplectic sum formulas in Gromov–Witten theory. [arXiv:1404.1898](https://arxiv.org/abs/1404.1898) (2014)
17. Farajzadeh-Tehrani, M., Zinger, A.: Absolute vs. relative Gromov–Witten invariants. *J. Symplect. Geom.* **14**(4), 1189–1250 (2016)
18. Gross, M., Siebert, B.: Logarithmic Gromov–Witten invariants. *J. Am. Math. Soc.* **26**(2), 451–510 (2013)
19. Hori, K., Katz, Sh., Klemm, A., Pandharipande, R., Thomas, R., Vafa, C., Vakil, R., Zaslow, E.: *Mirror Symmetry*, Clay Mathematics Monographs, vol. 1. Amer. Math. Soc., Providence, RI (2003)
20. Ionel, E.-N.: GW invariants relative to normal crossing divisors. *Adv. Math.* **281**, 40–141 (2015)
21. Ionel, E.-N., Parker, T.: Relative Gromov–Witten invariants. *Ann. Math.* **157**(1), 45–96 (2003)
22. Ionel, E., Parker, T.: The symplectic sum formula for Gromov–Witten invariants. *Ann. Math.* **159**(3), 935–1025 (2004)
23. Li, A.-M., Ruan, Y.: Symplectic surgery and Gromov–Witten invariants of Calabi–Yau 3-folds. *Invent. Math.* **145**(1), 151–218 (2001)
24. Li, J.: Stable morphisms to singular schemes and relative stable morphisms. *J. Differ. Geom.* **57**(3), 509–578 (2001)
25. McDuff, D., Salamon, D.: *J-Holomorphic Curves and Quantum Cohomology*, University Lecture Series, vol. 6. Amer. Math. Soc., Providence, RI (1994)
26. McDuff, D., Salamon, D.: *J-Holomorphic Curves and Symplectic Topology*. American Mathematical Society Colloquium Publications, vol. 52. Amer. Math. Soc., Providence, RI (2004)
27. McDuff, D., Wehrheim, K.: Smooth Kuranishi atlases with isotropy. *Geom. Topol.* **21**(5), 2725–2809 (2017)
28. McDuff, D., Wehrheim, K.: The fundamental class of smooth Kuranishi atlases with trivial isotropy. *J. Topol. Anal.* **10**(1), 71–243 (2018)
29. Mumford, D.: Towards an enumerative geometry of the moduli space of curves. In: *Arithmetic and Geometry*, vol. II, *Progr. Math.*, vol. 36, pp. 271–328. Birkhäuser, Basel (1983)
30. Parker, B.: Gromov–Witten invariants of exploded manifolds. [arXiv:1102.0158](https://arxiv.org/abs/1102.0158) (2011)
31. Parker, B.: Exploded manifolds. *Adv. Math.* **229**(6), 3256–3319 (2012)
32. Ruan, Y., Tian, G.: Higher genus symplectic invariants and sigma models coupled with gravity. *Invent. Math.* **130**(3), 455–516 (1997)
33. Siebert, B., Tian, G.: Lectures on pseudo-Holomorphic curves and the symplectic isotopy problem. In: *Symplectic 4-Manifolds and Algebraic Surfaces*, *Lecture Notes in Mathematics*, vol. 1938, pp. 269–341. Springer, Berlin (2008)
34. Zinger, A.: Basic Riemannian geometry and Sobolev estimates used in symplectic topology. [arXiv:1012.3980](https://arxiv.org/abs/1012.3980) (2010)
35. Zinger, A.: Real Ruan–Tian perturbations. [arXiv:7010.1420](https://arxiv.org/abs/7010.1420) (2017)

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.