



Limits of stable maps in a semi-stable degeneration

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Abstract

Given a semistable degeneration with a simple normal crossings central fiber, Abramovich et al. (Compositio Mathematica 156(10):2020–2075, 2020) proved a degeneration (or decomposition) formula that relates the moduli spaces of stable maps in smooth fibers to certain moduli spaces of log-smooth maps in the central fiber. In this paper, we study the same problem from an analytic point of view. We prove that the limiting stable maps in the central fiber satisfy specific combinatorial and analytical conditions. Furthermore, we derive a (conjectural) degeneration formula similar to ACGS's formula, and work out an explicit example. The results are expected to hold in the symplectic category as outlined in an earlier version (Farajzadeh-Tehrani in Towards a degeneration formula for the Gromov–Witten invariants of symplectic manifolds, [arXiv:1710.00599v1](https://arxiv.org/abs/1710.00599v1) [math]) of this paper.

Keywords Gromov–Witten invariants · Degeneration formula · Relative maps

Mathematics Subject Classification 53D45 · 14N35

1 Introduction

In this paper, by a semistable degeneration we mean a one-parameter family

$$\pi : \mathcal{Z} \longrightarrow \Delta, \quad (1.1)$$

where Δ is a disk around the origin in $\mathbb{C}$, $\mathcal{Z}$ is a smooth Kähler manifold, π is a proper map, the central fiber

$$\mathcal{Z}_0 := \pi^{-1}(0) = X_\emptyset := \bigcup_{i \in \mathcal{I}} X_i$$

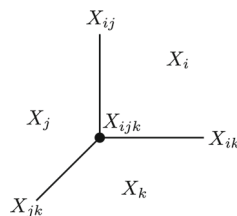
is simple normal crossings (or SNC), and the fibers over $\Delta^* := \Delta - \{0\}$ are smooth. An SNC variety with 3 irreducible components is illustrated in Fig. 1.

For each $i \in \mathcal{I}$, let

$$\mathcal{N}_i \equiv \mathcal{N}_{\mathcal{Z}} X_i$$

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Fig. 1 A threefold SC variety

denote the normal line bundle of X_i in $\mathcal{Z}$. The line bundle

$$\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0) = \otimes_{i \in \mathcal{I}} \mathcal{O}_{\mathcal{Z}}(X_i)$$

is trivial. Let

$$X_I \equiv \bigcap_{i \in I} X_i \quad \forall \emptyset \neq I \subset \mathcal{I}.$$

Any trivialization $\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0) \cong \mathcal{O}_{\mathcal{Z}}$ restricts to a set of compatible trivializations

$$\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)|_{X_I} = \bigotimes_{i \in I} \mathcal{O}_{\mathcal{Z}}(X_i)|_{X_I} = \bigotimes_{i \in I} \mathcal{N}_i|_{X_I} \otimes \bigotimes_{i \in \mathcal{I}-I} \mathcal{O}_{X_I}(X_{I+i}) \cong \mathcal{O}_{X_I}, \quad (1.2)$$

where $\mathcal{O}_{X_I}(X_{I+i})$ is the line bundle corresponding to the hypersurface

$$X_{I+i} := X_{I \cup \{i\}} \subset X_I, \quad \forall i \in \mathcal{I} - I.$$

For every $\lambda \in \Delta^*$, given $g, k \in \mathbb{N}$ and $A \in H_2(\mathcal{Z}_\lambda, \mathbb{Z})$, a k -marked genus g degree A per-stable map into $\mathcal{Z}_\lambda$ is a tuple

$$(u: \Sigma \longrightarrow \mathcal{Z}_\lambda, \vec{z} = (z_1, \dots, z_k))$$

where $C \equiv (\Sigma, \vec{z})$ is a connected genus g nodal curve with k distinct ordered marked points (away from the nodes) and u is a holomorphic map representing the homology class A . Two marked pre-stable maps

$$(u: \Sigma \longrightarrow \mathcal{Z}_\lambda, \vec{z}) \quad \text{and} \quad (\tilde{u}: \tilde{\Sigma} \longrightarrow \mathcal{Z}_\lambda, \vec{\tilde{z}})$$

are equivalent if there exists a bi-holomorphic isomorphism $h: \Sigma \longrightarrow \tilde{\Sigma}$ such that $h(z_a) = \tilde{z}_a$, for all $a = 1, \dots, k$, and $u = \tilde{u} \circ h$. A marked pre-stable map is stable iff the group of self-automorphisms is finite. For $\lambda \in \Delta^*$, let $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}_\lambda, A)$ denote the moduli space (set) of equivalence classes of k -marked genus g degree A stable maps into $\mathcal{Z}_\lambda$.

If $\dim_{\mathbb{C}} \mathcal{Z}_\lambda = n$, the expected $\mathbb{C}$ -dimension of $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}_\lambda, A)$ is

$$c_1^{T\mathcal{Z}_\lambda}(A) + (n-3)(1-g) + k. \quad (1.3)$$

Gromov–Witten (or GW) invariants are obtained by the integration of certain cohomology classes against the virtual fundamental class (or VFC) of $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}_\lambda, A)$. Since there might be different homology classes in $\mathcal{Z}_\lambda$ that are the same as homology classes in $\mathcal{Z}$, for each $A \in H_2(\mathcal{Z}, \mathbb{Z})$, we let $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}_\lambda, A)$ to be the union over all the representatives of A in $H_2(\mathcal{Z}_\lambda, \mathbb{Z})$; see [15] for a careful discussion of this issue.

For any choice of (g, k, A) , the fibration (1.1) gives rise to a 1-parameter family

$$\overline{\mathcal{M}}_{g,k}(\mathcal{Z}^*, A) := \bigcup_{\lambda \in \Delta^*} \overline{\mathcal{M}}_{g,k}(\mathcal{Z}_\lambda, A) \longrightarrow \Delta^* \quad (1.4)$$

with fibers of equal virtual dimension (and cobordant VFC). For $\lambda = 0$, let $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}_0, A)$ denote the space of all stable maps in $\mathcal{Z}$ whose image lies inside $\mathcal{Z}_0$; $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}_0, A)$ is not a moduli space of the correct expected dimension that extends the virtual cobordism (1.4) over $0 \in \Delta$. Therefore, from an analytical perspective, the important questions are:

1. which stable maps in $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}_0, A)$ can arise as the Gromov-limit of a sequence of stable maps in (1.4)?
2. how to complete (1.4) with a moduli space $\overline{\mathcal{M}}_{g,k}^{\text{good}}(\mathcal{Z}_0, A)$, ideally still a subset of $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}_0, A)$, admitting a VFC that is cobordant to VFC of smooth fibers?
3. can $\overline{\mathcal{M}}_{g,k}^{\text{good}}(\mathcal{Z}_0, A)$ and its VFC be expressed in terms of certain moduli spaces in X_i relative to the SNC divisor $\partial X_i \equiv \bigcup_{j \in \mathcal{I}-i} X_{ij} \subset X_i$ and their VFCs?

In the algebraic category and for a semistable degeneration into two pieces $\mathcal{Z}_0 = X_1 \cup_{X_{12}} X_2$ along a smooth divisor, these questions were first answered by Li [24, 25]. For a smooth divisor $D \subset X$, he introduced the notion of a *stable relative map* whose image lives in a natural SNC “expanded degeneration” associated to (X, D) . Similarly, for a semistable degeneration into two pieces $\mathcal{Z}_0 = X_1 \cup_{X_{12}} X_2$, he constructed a compactification $\overline{\mathcal{M}}_{g,k}^{\text{rel}}(\mathcal{Z}_0, A)$ whose (virtually) main components are fiber products of the relative moduli spaces¹ $\overline{\mathcal{M}}_{\chi_1, s}(X_1/X_{12}, A_1)$ and $\overline{\mathcal{M}}_{\chi_2, s}(X_2/X_{12}, A_2)$. In [25], he proved a decomposition formula which expresses the GW invariants of the smooth fibers in terms of the products of relative GW invariants of (X_1, X_{12}) and (X_2, X_{12}) . For a symplectic version of these results see [14, 21, 26]. More recently, Gross-Siebert [19] and Abramovich-Chen [1, 5] introduced moduli spaces of (fine, saturated) stable log maps and proved a degeneration/decomposition formula [3] to answer the first two questions above for an arbitrary semistable degeneration. Also, in [33], Brett Parker uses moduli spaces of curves in exploded manifolds [30–32] to address the first two questions. These constructions work for even a larger class of “log smooth” and “exploded” varieties (see [29]), respectively. The degeneration formula² in [3] can be read as: *virtually, a stable map f in $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}_0, A)$ can arise as a limit of a sequence of stable maps in smooth fibers if and only if f can be enhanced to a log smooth map in the (algebraic) log moduli space $\overline{\mathcal{M}}_{g,k}^{\text{al, log}}(\mathcal{Z}_0, A)$* . A log smooth map is a stable map plus a lift of that to a map between certain sheaves of monoids satisfying some conditions. Parker’s definition in the category of exploded manifolds involves sheaf theory in a similar manner. For a geometric approach using expanded degenerations we refer to [35]. For the comparison of Jun Li’s formula and ACGS’s formula, in the case of a basic semistable degeneration, we refer to [23].

Theorem 5.3.3 in [3], gives a criterion for lift-ability and a formula for the number of lifts for transverse maps. However, in practice, given a stable map f , it is rather hard to check whether f lifts to an element of $\overline{\mathcal{M}}_{g,k}^{\text{log}}(\mathcal{Z}_0, A)$. In other words, the image of the forgetful map

$$\overline{\mathcal{M}}_{g,k}^{\text{log}}(\mathcal{Z}_0, A) \longrightarrow \overline{\mathcal{M}}_{g,k}(\mathcal{Z}_0, A),$$

where $\overline{\mathcal{M}}_{g,k}^{\text{log}}(\mathcal{Z}_0, A)$ is the log moduli space in [3], is hard to describe. For example, in [3, Sec 5], the authors lift to a blowup of the SNC central fiber to find those “star-shaped” maps that can be lifted to a log map.

In this paper, first, we prove that the Gromov-limits of stable maps in the central fiber satisfy two specific combinatorial (called C1) and analytical (called C2) conditions. The combinatorial condition (C1) is equivalent to the basicness condition in [19, Dfn 1.20] but it

¹ Over possibly disconnected domains with Euler characteristic χ_1 and χ_2 .

² Also called an “invariance property” or a “decomposition formula”.

is stated differently. The analytical condition (C2) more or less corresponds to the existence of a morphism to a log point; see Remark 2.12. The two conditions are linked by a linear map associated to the dual graph of the stable map in the question. The positive cone in the kernel of this map gives a toric description of the space of gluing parameters. Second, we provide evidence that the moduli space of log maps satisfying conditions (C1) and (C2) should similarly address the first two questions above. In particular, we formulate an explicit degeneration formula that, in the case of basic degenerations, coincides with Jun Li's formula. The degeneration formula [3, Thm 1.2] and the one that we propose here are both a sum over the same set of combinatorial data, but with different coefficients; see Remark 5.5. In conclusion, this paper partially answers the questions (1)–(3) above (analytically) by providing a natural compactification and some evidence (in the sense of calculating the expected dimension and suggesting a GW degeneration formula) that these moduli spaces have the expected properties.

While the algebraic log GW theory and tropical geometry can be used to simplify the presentation at several places, the proposed approach has been packaged in a way so that it extends to the symplectic category without using sheaf theory. For example, in the symplectic category, the complex line bundles $\mathcal{O}_{X_i}(X_{ij})$ are defined but they don't admit a holomorphic structure or even a canonical $\bar{\partial}$ -operator that can be used to simplify certain definitions. The only obstacle in the way of generalizing these result to the symplectic category is to find a suitable class of almost complex structures compatible with a symplectic semistable degeneration (in the sense of [12]); see [9].

Remark 1.1 Since $\mathcal{Z}_0 \subset \mathcal{Z}$ is an SNC divisor, Theorem 1.3 in [7] with trivial tangency data at the marked points gives us a relatively³ compact log moduli space $\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}, \mathcal{Z}_0, A)$ that contains $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}^*, A)$ as an open subset. However, if $g > 0$, even the expected dimension of the subset of log curves in $\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}, \mathcal{Z}_0, A)$ that live in $\mathcal{Z}_0$ is different from (1.3). The conditions (C1) and (C2) are a refinement of the similar conditions in [7, Dfn 2.8]. Therefore, the compactness result [7, Thm 1.3] does not directly apply; it needs some enhancements.

For a finite set $\mathcal{I}$ and a ring R , let

$$R_{\bullet}^{\mathcal{I}} = \{r = (r_j)_{j \in \mathcal{I}} \in R^{\mathcal{I}} : \sum_{j \in \mathcal{I}} r_j = 0\} \subset R^{\mathcal{I}}.$$

For each $i \in \mathcal{I}$, let ξ_i denote a non-zero holomorphic section of $\mathcal{O}_{\mathcal{Z}}(X_i)$ vanishing (to the order 1) along X_i . Since $\mathcal{Z}_0$ is compact, the restriction of each section ξ_i to $\mathcal{Z}_0$ is unique up to multiplication by a constant. We will choose these sections so that the composition

$$\mathcal{Z} \longrightarrow \mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0) \cong \mathcal{Z} \times \mathbb{C} \longrightarrow \mathbb{C}, \quad (1.5)$$

where the first map is $x \rightarrow \prod_{i \in \mathcal{I}} \xi_i(x)$ and the last map is projection to the second factor, is equal to (1.1). For the trivial holomorphic line bundle $\mathcal{O}$ (on any base), let $\lambda_{\mathcal{O}}$ denote the constant section corresponding to $\lambda \in \mathbb{C}$.

Fix a trivialization of $\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)$. With notation as above, we define an analytical marked nodal log map into $\mathcal{Z}_0$ with the marked nodal domain $(\Sigma, \vec{z}) = \bigcup_{v \in \mathbb{V}} (\Sigma_v, \vec{z}_v)$ to be a collection of tuples

$$f \equiv (u_v : \Sigma_v \longrightarrow X_{I_v}, \vec{z}_v, (\zeta_{v,i})_{i \in I_v})_{v \in \mathbb{V}} \quad (1.6)$$

over the irreducible components Σ_v of Σ such that

³ i.e. the projection map π is proper.

1. $(u \equiv (u_v)_{v \in \mathbb{V}} : \Sigma \longrightarrow \mathcal{Z}_0, \vec{z})$ is a k -marked nodal map in the classical sense,
2. for each $v \in \mathbb{V}$, $\emptyset \neq I_v \subset \mathcal{I}$ is the maximal subset such that $\text{Im}(u_v) \subset X_{I_v} \subset \mathcal{Z}_0$,
3. for each $v \in \mathbb{V}$ and every $i \in I_v$, $\zeta_{v,i}$ is a non-trivial meromorphic section of the holomorphic line bundle $u_v^* \mathcal{N}_i$,
4. for each $v \in \mathbb{V}$, with respect to the isomorphism (1.2), we have

$$\bigotimes_{i \in I_v} \zeta_{v,i} \otimes \bigotimes_{j \in \mathcal{I} - I_v} u_v^* \xi_j = u_v^* (\lambda_{\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)})$$

for some fixed $\lambda = \lambda(f) \in \mathbb{C}^*$ (independent of $v \in \mathbb{V}$),

5. the “contact order vectors” in $\mathbb{Z}_{\bullet}^{\mathcal{I}}$, defined in (2.7) and (2.8), are the opposite of each other at the nodal points of Σ ,
6. every point in Σ with a non-trivial contact vector is a nodal point,
7. (C1): there exists a vector-valued function $s : \mathbb{V} \longrightarrow \mathbb{R}^{\mathcal{I}}$ such that $s_v = s(v) \in \mathbb{R}_{+}^{I_v} \times \{0\}^{\mathcal{I} - I_v}$ for all $v \in \mathbb{V}$, and $s_v - s_{v'}$ is a positive multiple of the contact order vector of any nodal point on Σ_v connected to $\Sigma_{v'}$, for all $v, v' \in \mathbb{V}$,
8. (C2): certain Lie group (a complex torus) element $\text{ob}(f)$ associated to f , defined in (2.21), is equal to 1, or equivalently, there is a choice of local holomorphic coordinates at the nodes such that certain first order terms calculated with respect to these coordinates match (see Remark 2.9);

see Definitions 2.2, 2.4, and 2.10 for the details. In simple words, a log map is a stable map together with a set of meromorphic sections that satisfies certain combinatorial (i.e. (5), (6), and (C1)) and analytical (i.e. (4) and (C2)) conditions.

Two marked log maps

$$\begin{aligned} f &\equiv (u_v : \Sigma_v \longrightarrow X_{I_v}, \vec{z}_v, (\zeta_{v,i})_{i \in I_v})_{v \in \mathbb{V}} \quad \text{and} \\ \tilde{f} &\equiv (\tilde{u}_v : \tilde{\Sigma}_v \longrightarrow X_{I_v}, \vec{\tilde{z}}_v, (\tilde{\zeta}_{v,i})_{i \in I_v})_{v \in \mathbb{V}} \end{aligned}$$

are equivalent if there exists a bi-holomorphic isomorphism

$$(h : \Sigma \longrightarrow \tilde{\Sigma}) \equiv (h_v : \Sigma_v \longrightarrow \tilde{\Sigma}_{h(v)})_{v \in \mathbb{V}}$$

such that

$$h(z_a) = \tilde{z}_a \quad \forall a = 1, \dots, k, \quad \tilde{u} \circ h = u, \quad h_v^* \tilde{\zeta}_{h(v),i} = c_{v,i} \zeta_{v,i} \quad \forall v \in \mathbb{V}, i \in I_v.$$

In particular, given a marked log map f as in (1.6), replacing each meromorphic section $\zeta_{v,i}$ with a non-zero multiple $c_{v,i} \zeta_{v,i}$ of that, satisfying $\prod_{i \in I_v} c_{v,i} = c'$ for all $v \in \mathbb{V}$, produces another marked log map which is equivalent to f . A marked log map is stable if it has a finite automorphism group. For $g, k \in \mathbb{N}$ and $A \in H_2(\mathcal{Z}_0, \mathbb{Z})$, we denote the space of equivalence classes of stable k -marked degree A genus g log maps by

$$\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}_0, A).$$

This moduli space is independent of the choice of the sections ξ_i used in the construction because rescalings of ξ_i can be compensated by rescalings of $\zeta_{v,i}$. The equivalence class of an analytic log map is called an analytic log *curve*. We will often drop the adjective “analytic” and simply say log map or log curve.

There is a natural forgetful map

$$\begin{aligned} \overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}_0, A) &\longrightarrow \overline{\mathcal{M}}_{g,k}(\mathcal{Z}_0, A), \\ (u_v : \Sigma_v \longrightarrow X_{I_v}, \vec{z}_v, (\zeta_{v,i})_{i \in I_v})_{v \in \mathbb{V}} &\longrightarrow (u_v : \Sigma_v \longrightarrow \mathcal{Z}_0, \vec{z}_v)_{v \in \mathbb{V}}. \end{aligned}$$

It turns out that for every k -marked stable nodal curve f in $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}_0, A)$, there exists at most finitely many log curves $f_{\log} \in \overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}_0, A)$ (with distinct decorations on the dual graph) lifting f ; see Remark 2.3. Furthermore, $f_{\log}$ is stable if and only if f is stable, and the automorphism groups are often the same.

Remark 1.2 In Sect. 2, we will construct the analytical log moduli spaces for any arbitrary d -semistable (see [17]) SNC variety $\mathcal{Z}_0$ without using the smoothing $\mathcal{Z}$ that contains it. Here, we used $\mathcal{Z}$ to slightly simplify the notation. Furthermore, it is possible to define the log map without mentioning the meromorphic sections $\zeta_{v,i}$; see Remark 2.3.

By Smirnov's theorem, every paracompact, Hausdorff, and locally metrizable space is metrizable. Therefore, if $f: M \rightarrow N$ is a local embedding from a compact Hausdorff space M to a compact metrizable space N then M is metrizable.

Theorem 1.3 *For every $A \in H_2(\mathcal{Z}_0, \mathbb{Z})$ and $g, k \in \mathbb{N}$, the Gromov sequential convergence topology on $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}, A)$ lifts to a Hausdorff sequential convergence topology on*

$$\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}, A) := \overline{\mathcal{M}}_{g,k}(\mathcal{Z}^*, A) \cup \overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}_0, A)$$

such that

$$\pi: \overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}, A) \rightarrow \Delta$$

is proper (i.e., $\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}_0, A)$ is compact) and the natural forgetful maps

$$\iota: \overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}, A) \rightarrow \overline{\mathcal{M}}_{g,k}(\mathcal{Z}, A) \quad \text{and} \quad \iota: \overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}_0, A) \rightarrow \overline{\mathcal{M}}_{g,k}(\mathcal{Z}_0, A) \quad (1.7)$$

are local (topological) embeddings. In particular, $\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}, A)$ and $\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}_0, A)$ are metrizable. If $g=0$, then the forgetful maps in (1.7) are global embeddings.

If X_0 is just an abstract d -semistable SNC variety, we just get the restriction of the theorem above to $\overline{\mathcal{M}}_{g,k}^{\log}(X_0, A)$. If $\mathcal{Z}_0$ is basic ($\mathcal{Z}_0 = X_1 \cup_{X_{12}} X_2$), it follows from [7, Prp 4.5] that there is a surjective projection map

$$\overline{\mathcal{M}}_{g,k}^{\text{rel}}(\mathcal{Z}_0, A) \rightarrow \overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}_0, A),$$

where the former is Jun Li's relative moduli space. The degeneration formula that we will derive will be the same as Jun Li's formula in this case.

Theorem 1.3 provides necessary conditions for a stable map in $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}_0, A)$ to be the Gromov(-type) limit of a sequence of stable maps in (1.4). We expect these conditions to be sufficient, virtually. We describe the deformation-obstruction long exact sequence in Sect. 4 and show that the moduli space $\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}_0, A)$ is of the expected dimension equal to (1.3). The expectation is that $\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}_0, A)$ admits a VFC that is cobordant to VFC of smooth fibers in (1.4). Assuming that, we formulate an explicit formula for the contributions of the virtually main components of $\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}_0, A)$ to its VFC; see Formula (5.3). In Sect. 6, we work out the details for the same non-trivial example considered in [3, Sec 5] to highlight the similarities and differences. Constructing VFC and proving the degeneration formula (5.3) needs a gluing theorem (with the space of gluing parameters described in (5.4)) that will appear in a future work. This also needs introducing a generalized version of Kuranishi structures/spaces that allow "toroidal singularities" or working with a generalization of *inconsistent solution spaces* as in [6].

2 Analytical log moduli spaces

In this section, associated to any d -semistable SNC Kähler variety $X_\emptyset = \bigcup_{i \in \mathcal{I}} X_i$, $g, k \in \mathbb{N}$, and $A \in H_2(X_\emptyset, \mathbb{Z})$, we construct the (analytic) moduli space $\overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A)$ of k -marked genus g degree A log holomorphic curves (as a set).

Given an SNC variety $X_\emptyset = \bigcup_{i \in \mathcal{I}} X_i$, for $i \neq j \in \mathcal{I}$, let X_{ij} denote the intersection of X_i and X_j (which is a complex hypersurface in both of them), and

$$X_\partial = \bigcup_{\substack{i,j \in \mathcal{I} \\ i \neq j}} X_{ij}$$

denote the singular locus of $X_\emptyset$. In [17], associated to any SNC variety $X_\emptyset$, Friedman constructs a holomorphic line bundle

$$\mathcal{O}_{X_\partial}(X_\emptyset) \longrightarrow X_\partial$$

such that

$$\mathcal{O}_{X_\partial}(X_\emptyset)|_{X_I} = \bigotimes_{i \in I} \mathcal{N}_{X_{I-i}} X_I \otimes \bigotimes_{i \in \mathcal{I}-I} \mathcal{O}_{X_I}(X_{I+i}) \quad \forall I \subset \mathcal{I}, \quad |I| \geq 2, \quad (2.1)$$

where $\mathcal{O}_{X_I}(X_{I+i})$ is the line bundle associated to the smooth divisor $X_{I+i} := X_{I \cup \{i\}} \subset X_I$.

If $X_\emptyset = \mathcal{Z}_0$ is the central fiber of a smoothing $\mathcal{Z}$ as in (1.1), $\mathcal{O}_{X_\partial}(X_\emptyset)$ is the restriction to X_∂ of $\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)$ and (1.2) coincides with (2.1). An SNC variety is smoothable⁴ only if $\mathcal{O}_{X_\partial}(X_\emptyset)$ is trivial, but the converse is not true; see [34, Sec 3] for examples. An SNC variety $X_\emptyset$ is called d -semistable if $\mathcal{O}_{X_\partial}(X_\emptyset)$ is isomorphic to the trivial line bundle; see [17, Dfn 1.13]. Regarding the connection between the d -semistability condition and log geometry, the result is that (see [2, Thm 5.9]):

if $X_\emptyset$ is a normal crossings variety over the spectrum of an algebraically closed field, then $X_\emptyset$ can be equipped with a log structure over the standard log point, such that the structure morphism is log smooth if and only if X is d -semistable.

We will use a reinterpretation of this statement in Sect. 4.

For $I = \{i\}$, we define

$$\mathcal{N}_{X_\emptyset} X_i \equiv \left(\bigotimes_{j \in \mathcal{I}-\{i\}} \mathcal{O}_{X_i}(X_{ij}) \right)^{-1}. \quad (2.2)$$

With this convention, the line bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$ extends to $X_\emptyset$ and the trivialization of (2.1) provided by the d -semistability assumption compatibly extends to the case where $|I| = 1$. If a smoothing $\mathcal{Z}$ of $X_\emptyset$ as in (1.1) is given, then $\mathcal{N}_{X_\emptyset} X_i$ coincides with $\mathcal{N}_i$ in (1.2) and $\mathcal{N}_i|_{X_I} = \mathcal{N}_{X_{I-i}} X_I$ for all $i \in I$ and $|I| > 1$. Therefore, for simplicity, in the following we will write $\mathcal{N}_i$ instead of $\mathcal{N}_{X_\emptyset} X_i$.

In the following construction, for each $\emptyset \neq I \subset \mathcal{I}$ and every $i \in \mathcal{I} - I$, we need to fix a holomorphic section $\xi_{I,i}$ of $\mathcal{O}_{X_I}(X_{I+i})$ vanishing (to order 1) along X_{I+i} ; the section $\xi_{I,i}$ is unique up to multiplication by a constant. Because of the natural isomorphism

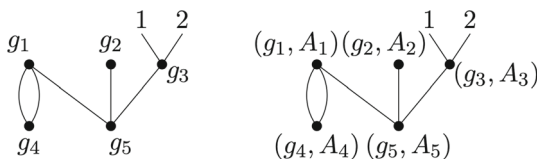
$$\mathcal{O}_{X_I}(X_{I+i})|_{X_J} = \mathcal{O}_{X_J}(X_{J+i}) \quad \forall \emptyset \neq I \subset J, \quad i \in \mathcal{I} - J,$$

we can choose the set $\{\xi_{I,i}\}$ so that

$$\xi_{I,i}|_{X_J} = \xi_{J,i} \quad \forall \emptyset \neq I \subset J, \quad i \in \mathcal{I} - J. \quad (2.3)$$

⁴ Here, we require the total space of the degeneration, and not just the general fiber, to be smooth.

Fig. 2 On left, a labeled graph Γ representing elements of $\mathcal{M}_{g,2}$. On right, a labeled graph Γ representing elements of $\mathcal{M}_{g,2}(X, A)$



Remark 2.1 In [11] and [13], with McLean and Zinger, we introduced topological notions of normal crossings symplectic divisor and variety and established that they are equivalent, in a suitable sense, to the desired geometric notions. In [12], we showed that the direct analogue of d-semistability condition is the only obstruction to smoothability in the symplectic topology category. The process of constructing a 1-parameter family of smoothings $\mathcal{Z}$ in [12] is a multifold analogue of the now classical (twofold) symplectic sum construction. Conversely, we introduced a multifold symplectic cut construction in [16] that, given certain configuration of Hamiltonian torus actions, degenerates a smooth target into an SNC symplectic variety. Subject to the existence of an appropriate⁵ class of almost complex structures on $\mathcal{Z}$, the results of this paper and the rest of the claims will extend to the symplectic category; see [9] for an outline.

Let $\Gamma = \Gamma(\mathbb{V}, \mathbb{E}, \mathbb{L})$ be a graph with the set of vertices $\mathbb{V}$, edges $\mathbb{E}$, and legs $\mathbb{L}$; the latter, also called flags or roots, are half edges that have a vertex at one end and are open at the other end. Let $\vec{\mathbb{E}}$ be the set of edges with an orientation. Given an oriented edge $\vec{e} \in \vec{\mathbb{E}}$, let $\bar{e}$ denote the same edge e with the opposite orientation. For each $\vec{e} \in \vec{\mathbb{E}}$, let $v_1(\vec{e})$ and $v_2(\vec{e})$ in $\mathbb{V}$ denote the starting and ending points of the arrow, respectively. For $v, v' \in \mathbb{V}$, let $\mathbb{E}_{v,v'}$ denote the subset of edges between the two vertices and $\vec{\mathbb{E}}_{v,v'}$ denote the subset of oriented edges from v to v' . For every $v \in \mathbb{V}$, let $\vec{\mathbb{E}}_v$ denote the subset of oriented edges starting from v .

A genus labeling of Γ is a function $g: \mathbb{V} \rightarrow \mathbb{N}$. An ordering of the legs of Γ is a bijection $a: \mathbb{L} \rightarrow \{1, \dots, |\mathbb{L}|\}$. If a decorated graph Γ is connected, the arithmetic genus of Γ is

$$g = g_\Gamma = \sum_{v \in \mathbb{V}} g_v + \text{rank } H_1(\Gamma, \mathbb{Z}),$$

where $H_1(\Gamma, \mathbb{Z})$ is the first homology group of the underlying topological space of Γ . Figure 2-left illustrates a labeled graph with 2 legs.

Such decorated graphs Γ characterize different topological types of nodal marked curves

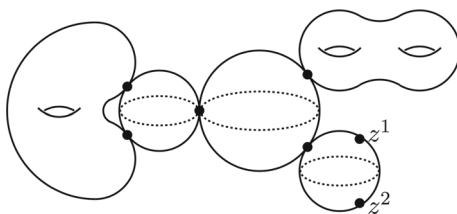
$$(\Sigma, \vec{z} = (z_1, \dots, z_k))$$

in the following way. Each vertex $v \in \mathbb{V}$ corresponds to a smooth⁶ component Σ_v of Σ with genus g_v . Each edge $e \in \mathbb{E}$ corresponds to a node q_e obtained by connecting Σ_v and $\Sigma_{v'}$ at the points $q_{\vec{e}} \in \Sigma_v$ and $q_{\bar{\vec{e}}} \in \Sigma_{v'}$, where $e \in \mathbb{E}_{v,v'}$ and $\vec{e}$ is an orientation on e with $v_1(\vec{e}) = v$. The last condition uniquely specifies $\vec{e}$ unless e is a loop connecting v to itself. Finally, each leg $l \in \mathbb{L}$ connected to the vertex v_l corresponds to a marked point $z_{a_l} \in \Sigma_{v_l}$ disjoint from the connecting nodes. If Σ is connected, then g_Γ is the arithmetic genus of Σ . Thus we have

$$(\Sigma, \vec{z}) = \coprod_{v \in \mathbb{V}} (\Sigma_v, \vec{z}_v, q_v) / \sim, \quad q_{\vec{e}} \sim q_{\bar{\vec{e}}} \quad \forall e \in \mathbb{E}, \quad (2.4)$$

⁵ We need an almost complex structure J on $\mathcal{Z}$ such that the projection map π in (1.1) is (i, J) -holomorphic and the Nijenhuis tensor of J vanishes to the first order (at least) along $\mathcal{Z}_0 \subset \mathcal{Z}$; see [7, (1.3)].

⁶ We mean a smooth closed oriented surface.

Fig. 3 A nodal curve in $\overline{\mathcal{M}}_{4,2}$


where

$$\vec{z}_v = \vec{z} \cap \Sigma_v \quad \text{and} \quad q_v = \{q_e : e \in \mathbb{E}_v\} \quad \forall v \in \mathbb{V}$$

are the set of marked and nodal points on Σ_v , respectively. In this situation, we say Γ is the dual graph of $(\Sigma, \vec{z})$. We treat q_v as an un-ordered set of marked points on Σ_v . If we fix an ordering on the set q_v , we denote the ordered set by $\vec{q}_v$. Figure 3 illustrates a nodal curve with $(g_1, g_2, g_3, g_4, g_5) = (0, 2, 0, 1, 0)$ corresponding to Fig. 2-left.

Similarly, for nodal marked curves mapping into a topological space X , we consider similar decorated graphs where the vertices carry an additional degree labeling

$$A: \mathbb{V} \longrightarrow H_2(X, \mathbb{Z}), \quad v \longrightarrow A_v,$$

recording the homology class of the image of the corresponding component. Figure 2-right illustrates a dual graph associated to a marked nodal map over the graph on the left.

Assume $X_\emptyset = \bigcup_{i \in \mathcal{I}} X_i$ is an SNC variety and Σ is an irreducible smooth curve. Then every holomorphic map $u: \Sigma \longrightarrow X_\emptyset$ has a well-defined depth $\emptyset \neq I \subset \mathcal{I}$, which is the maximal subset of $\mathcal{I}$ such that $\text{Image}(u) \subset X_I$. Similarly, we say a point $x \in \Sigma$ has depth I , if X_I is the minimal stratum containing $u(x)$. Let $\mathcal{P}^*(\mathcal{I})$ be the set of non-empty subsets of $\mathcal{I}$. The dual graph of (u, Σ) carries additional labelings

$$I: \mathbb{V}, \mathbb{E} \longrightarrow \mathcal{P}^*(\mathcal{I}), \quad v \longrightarrow I_v \quad \forall v \in \mathbb{V}, \quad e \longrightarrow I_e \quad \forall e \in \mathbb{E},$$

recording the depths of smooth components and nodes of Σ .

Given a finite set $\mathcal{I}$ and a ring R , let

$$R_\bullet^\mathcal{I} = \left\{ r = (r_j)_{j \in \mathcal{I}} \in R^\mathcal{I} : \sum_{j \in \mathcal{I}} r_j = 0 \right\} \subset R^\mathcal{I}. \quad (2.5)$$

For every $i \in \mathcal{I}$, the projection

$$\pi_i: R_\bullet^\mathcal{I} \longrightarrow R^{\mathcal{I} - \{i\}}, \quad (r_j)_{j \in \mathcal{I}} \longrightarrow (r_j)_{j \in \mathcal{I} - \{i\}} \quad (2.6)$$

is an isomorphism. For every subset $\mathcal{I}' \subset \mathcal{I}$, the natural injective homomorphism $R^{\mathcal{I}'} \hookrightarrow R^\mathcal{I}$ restricts to a homomorphism $R_\bullet^{\mathcal{I}'} \hookrightarrow R_\bullet^\mathcal{I}$. Therefore, via this inclusion, $R_\bullet^{\mathcal{I}'}$ can be thought of as a subspace of $R_\bullet^\mathcal{I}$. We will use the projection maps in (2.6) to identify each component of a pre-log moduli space in $X_\emptyset$ with a fiber product of relative spaces in $\{X_i\}_{i \in \mathcal{I}}$.

Assume $X_\emptyset = \bigcup_{i \in \mathcal{I}} X_i$ is a d-semistable SNC Kähler variety and fix a trivialization of $\mathcal{O}_{X_\emptyset}(X_\emptyset)$. Let $u: \Sigma \longrightarrow X_\emptyset$ be a holomorphic map of depth $I \subset \mathcal{I}$ with smooth domain. Then, for every $i \in \mathcal{I} - I$, the function

$$\text{ord}_u^i: \Sigma \longrightarrow \mathbb{N}, \quad \text{ord}_u^i(x) = \text{ord}_x(u, X_i), \quad (2.7)$$

recording the contact order of u with the smooth divisor $X_{I+i} \subset X_I$ at x is well-defined. It coincides with the vanishing order $\text{ord}_x(u^* \xi_{I,i}) \in \mathbb{Z}_{\geq 0}$ of $\xi_{I,i}$ at x where $\xi_{I,i}$ are the holomorphic sections in (2.3).

For every $\emptyset \neq I \subset \mathcal{I}$, each $i \in I$, and a given meromorphic section ζ of $u^*\mathcal{N}_i$, we will also need the function

$$\text{ord}_\zeta : \Sigma \longrightarrow \mathbb{Z}, \quad \text{ord}_\zeta(x) = \text{ord}_x(\zeta), \quad (2.8)$$

recording the vanishing order of ζ at x (which, by definition, is negative if ζ has a pole at x). Given a holomorphic line bundle $\mathcal{L} \longrightarrow \Sigma$, let $\Gamma_{\text{mero}}(\Sigma, \mathcal{L})$ denote the space of non-trivial meromorphic sections of $\mathcal{L}$.

Definition 2.2 For each $\emptyset \neq I \subset \mathcal{I}$, a log holomorphic tuple (u, ζ, Σ, q) of depth I consists of a smooth irreducible curve Σ , a finite set of distinct points $q = \{q_1, \dots, q_\ell\}$ on Σ , a depth I holomorphic map $u : \Sigma \longrightarrow X_I$, and a set of meromorphic sections

$$\zeta \equiv (\zeta_i)_{i \in I} \in \prod_{i \in I} \Gamma_{\text{mero}}(\Sigma, u^*\mathcal{N}_{X_I-i}X_I)$$

such that:

(a) via the identification (2.1) and the given trivialization of $\mathcal{O}_{X_\emptyset}(X_\emptyset)$, we have

$$\bigotimes_{i \in I} \zeta_i \otimes \bigotimes_{j \in \mathcal{I}-I} u^*\xi_{I,j} = u^*(\lambda_{\mathcal{O}_{X_I}})$$

for some $\lambda \in \mathbb{C}^*$, where $\lambda_{\mathcal{O}_{X_I}}$ is the constant section corresponding to λ ;

(b) for all $x \in \Sigma$

$$\text{if } \text{ord}_{u,\zeta}(x) \neq 0 \Rightarrow x \in q, \quad (2.9)$$

where the vector-valued order function

$$\begin{aligned} \text{ord}_{u,\zeta}(x) &= ((\text{ord}_u^j(x))_{j \in \mathcal{I}-I}, (\text{ord}_{\zeta_i}(x))_{i \in I}) \\ &= ((\text{ord}_{u^*\xi_{I,j}}(x))_{j \in \mathcal{I}-I}, (\text{ord}_{\zeta_i}(x))_{i \in I}) \in \mathbb{Z}_{\bullet}^{\mathcal{I}} \subset \mathbb{Z}^{\mathcal{I}} \end{aligned}$$

is defined via (2.7) and (2.8).

Remark 2.3 Let us elaborate on Definition 2.2.

- In (b) above, it follows from Condition (a) that $\text{ord}_{u,\zeta}(x) \in \mathbb{Z}_{\bullet}^{\mathcal{I}}$.
- If $X_\emptyset = \mathcal{Z}_0$ is the central fiber of a semistable degeneration and u is of degree $A \in H_2(\mathcal{Z}_0, \mathbb{Z})$, then (2.9) implies that

$$(A \cdot X_i)_{i \in \mathcal{I}} = \sum_{q_a \in q} \text{ord}_{u,\zeta}(q_a) \in \mathbb{Z}_{\bullet}^{\mathcal{I}}.$$

Note that $0 = A \cdot \mathcal{Z}_\lambda = A \cdot \mathcal{Z}_0 = \sum_{i \in \mathcal{I}} A \cdot X_i$.

- Changing any of the meromorphic sections ζ_i with a non-zero constant multiple of that has no effect on (a) and (b).
- If $I = \{i\}$, by Condition (a) and (2.2), the section ζ_i is uniquely determined by u and λ . For I with $|I| > 1$, specifying $|I| - 1$ of sections $(\zeta_i)_{i \in I}$ and λ will uniquely determine the remaining one, because their multiplication is the constant section. So there is a redundancy in Definition 2.2. The reason for this redundancy is to avoid a non-symmetric definition that depends on the choice of $i \in I$.

- The set of vectors

$$\mathfrak{s} = \{s_i = (s_{ij})_{j \in \mathcal{I}}\}_{1 \leq i \leq \ell} \subset (\mathbb{Z}_{\bullet}^{\mathcal{I}})^{\ell}$$

describe the “contact type” of the log holomorphic tuple (u, ζ, Σ, q) with the singular locus X_{∂} and will play an important role in defining log moduli spaces. They also appear in the relative compactification of [7] at the marked points. Here, they only appear at the nodal points. Nevertheless, a point with $s_i = \vec{0}$ is simply a marked point in the classical sense, because $u(z_i)$ will be disjoint from any of the boundary divisors.

Definition 2.4 Let $X_{\emptyset} = \bigcup_{i \in \mathcal{I}} X_i$ be a d-semistable SNC Kähler variety. Suppose

$$C \equiv (\Sigma, \vec{z}) = \left(\coprod_{v \in \mathbb{V}} C_v \equiv (\Sigma_v, \vec{z}_v, q_v) \right) / \sim, \quad q_{\vec{e}} \sim q_{\vec{e}} \quad \forall \vec{e} \in \vec{\mathbb{E}},$$

is a k -marked connected nodal curve with smooth components C_v and dual graph $\Gamma = \Gamma(\mathbb{V}, \mathbb{E}, \mathbb{L})$ as in (2.4). A pre-log map from C to $X_{\emptyset}$ is a collection

$$f \equiv (f_v \equiv (u_v, \zeta_v, C_v))_{v \in \mathbb{V}} \quad (2.10)$$

such that

1. for each $v \in \mathbb{V}$, $(u_v, \zeta_v = (\zeta_{v,i})_{i \in I_v}, \Sigma_v, q_v)$ is a depth I_v log holomorphic tuple as in Definition 2.2 for some fixed⁷ $\lambda = \lambda(f)$,
2. $u_v(q_{\vec{e}}) = u_{v'}(q_{\vec{e}})$ for all $\vec{e} \in \vec{\mathbb{E}}_{v,v'}$;
3. $s_{\vec{e}} \equiv \text{ord}_{u_v, \zeta_v}(q_{\vec{e}}) = -\text{ord}_{u_{v'}, \zeta_{v'}}(q_{\vec{e}}) \equiv -s_{\vec{e}}$ for all $v, v' \in \mathbb{V}$ and $\vec{e} \in \vec{\mathbb{E}}_{v,v'}$;

In other words, a pre-log map is a (pre-stable) nodal holomorphic map into $X_{\emptyset}$ with a bunch of meromorphic sections (satisfying Definition 2.2.(a) on each smooth component), opposite contact orders at the nodes, and trivial contact orders at every other point (including the marked points).

Lemma 2.5 With notation as above, we have $I_{\vec{e}} = I_v \cup I_{v'}$ for all $\vec{e} \in \vec{\mathbb{E}}_{v,v'}$ and $s_{\vec{e}} \in \mathbb{Z}_{\bullet}^{I_{\vec{e}}} \subset \mathbb{Z}_{\bullet}^{\mathcal{I}}$.

Proof For every $v \in \mathbb{V}$ and $\vec{e} \in \vec{\mathbb{E}}_v$, let

$$s_{\vec{e}} = (s_{\vec{e},i})_{i \in \mathcal{I}} = ((\text{ord}_{u_v}^i(q_{\vec{e}}))_{i \in \mathcal{I} - I_v}, (\text{ord}_{\zeta_{v,i}}(q_{\vec{e}}))_{i \in I_v}) \in \mathbb{Z}_{\bullet}^{\mathcal{I}} \quad (2.11)$$

be the contact order vector at the nodal point $q_{\vec{e}} \in \Sigma_v$. For $\vec{e} \in \vec{\mathbb{E}}_{v,v'}$, since u_v and $u_{v'}$ have image in X_{I_v} and $X_{I_{v'}}$, respectively, by Condition 2 in Definition 2.4, we have

$$u(q_{\vec{e}}) = u_v(q_{\vec{e}}) = u_{v'}(q_{\vec{e}}) \in X_{I_v} \cap X_{I_{v'}} = X_{I_v \cup I_{v'}};$$

i.e. $I_{\vec{e}} \supset I_v \cup I_{v'}$. If $i \notin I_v \cup I_{v'}$, by (2.7), we have

$$s_{\vec{e},i}, s_{\vec{e},i} \geq 0.$$

Therefore, by Condition 3, they are both zero, i.e.

$$I_{\vec{e}} = I_v \cup I_{v'} \quad \text{and} \quad s_{\vec{e}} \in \mathbb{Z}_{\bullet}^{I_{\vec{e}}} \subset \mathbb{Z}_{\bullet}^{\mathcal{I}} \quad \forall \vec{e} \in \vec{\mathbb{E}}_{v,v'}.$$

□

⁷ i.e., independent of the choice of $v \in \mathbb{V}$.

Definition 2.6 Two pre-log maps $(u, \zeta, C) \equiv (u_v, \zeta_v, C_v)_{v \in \mathbb{V}}$ and $(\tilde{u}, \tilde{\zeta}, \tilde{C}) \equiv (\tilde{u}_v, \tilde{\zeta}_v, \tilde{C}_v)_{v \in \mathbb{V}}$ with isomorphic decorated dual graphs Γ are equivalent if there exists a biholomorphic identification

$$(h: \tilde{C} \longrightarrow C) \equiv (h_v: \tilde{\Sigma}_v \longrightarrow \Sigma_{h(v)})_{v \in \mathbb{V}} \quad (2.12)$$

such that

$$\begin{aligned} h(\tilde{z}_a) &= z_a \quad \forall a = 1, \dots, k, \quad u \circ h = \tilde{u}, \\ h_v^* \zeta_{h(v), i} &= c_{v, i} \tilde{\zeta}_{v, i} \quad \forall v \in \mathbb{V}, \quad i \in I_v, \quad \text{some } c_{v, i} \in \mathbb{C}^*. \end{aligned}$$

A pre-log map f is stable if the group of self-equivalences $\mathfrak{Aut}(f)$ is finite.

Lemma 2.7 *A pre-log map is stable if and only if the underlying marked nodal map is stable. Furthermore, the automorphism group of a pre-log map can be a proper subgroup of the automorphism group of the underlying marked nodal map.*

Proof A per-log map differs from the underlying marked nodal map by the extra collection of meromorphic sections on each irreducible component Σ_v of the domain. Given a set of mutually distinct points $q = \{q_1, \dots, q_\ell\} \subset \Sigma_v$, a holomorphic line bundle $\mathcal{L} \longrightarrow \Sigma_v$, and a set of integers $s_1, \dots, s_\ell \in \mathbb{Z}$, up to multiplication by a non-zero scalar, there is at most one meromorphic section ζ of $\mathcal{L}$ such that

$$\text{ord}_\zeta(q_a) = s_a \quad \forall a = 1, \dots, \ell \quad \text{and} \quad \text{ord}_\zeta(x) = 0 \quad \forall x \in \Sigma - q$$

Therefore, the meromorphic sections of a per-log map are uniquely identified (up to $\mathbb{C}^*$ -action) by the intersection data at the nodal points. This fact and [7, Lmm 2.15] imply the first statement of the Lemma. Furthermore, [7, Exa 2.18] describes an example where the automorphism group of a pre-log map is a proper subgroup of the automorphism group of the underlying marked nodal map. $\square$

Definition 2.8 The equivalence class of a pre-log map is called a pre-log curve. For every choice of a decorated dual graph Γ , we denote the space of stable k -marked degree A pre-log holomorphic curves of type Γ by

$$\mathcal{M}_{g, k}^{\text{plog}}(X_\emptyset, A)_\Gamma.$$

A basic dimension counting reveals that the expected dimension of $\mathcal{M}_{g, k}^{\text{plog}}(X_\emptyset, A)_\Gamma$ could be much larger than (1.3). In Definition 2.10 below, we take out a subspace that would give us a nice compactification with the correct expected dimension.

Let

$$\left(\bigoplus_{v \in \mathbb{V}} \mathbb{Z}^{I_v} \right)^* = \left\{ (s_v)_{v \in \mathbb{V}} \in \bigoplus_{v \in \mathbb{V}} \mathbb{Z}^{I_v} : \sum_{i \in I_v} s_{v, i} = \sum_{i \in I_{v'}} s_{v', i} \quad \forall v, v' \in \mathbb{V} \right\},$$

and

$$\pi : \left(\bigoplus_{v \in \mathbb{V}} \mathbb{Z}^{I_v} \right)^* \longrightarrow \mathbb{Z}, \quad (s_v)_{v \in \mathbb{V}} \longrightarrow \sum_{i \in I_v} s_{v, i}. \quad (2.13)$$

Associated to the decorated dual graph $\Gamma = \Gamma(\mathbb{V}, \mathbb{E}, \mathbb{L})$ of a pre-log map and an arbitrary orientation $O \equiv \{\zeta_e\}_{e \in \mathbb{E}} \subset \mathbb{E}$ on the edges, we define a $\mathbb{Z}$ -linear map

$$\varrho : \mathbb{D} := \mathbb{Z}^{\mathbb{E}} \oplus \left(\bigoplus_{v \in \mathbb{V}} \mathbb{Z}^{I_v} \right)^* \longrightarrow \mathbb{T} := \bigoplus_{e \in \mathbb{E}} \mathbb{Z}^{I_e} \quad (2.14)$$

in the following way:

- ϱ maps the generator 1_e of the e -th component in the summand $\mathbb{Z}^{\mathbb{E}} \subset \mathbb{D}$ to the contact vector $s_{\vec{e}} \in \mathbb{Z}_{\bullet}^{I_e} \subset \mathbb{T}$, where $\vec{e}$ is the chosen orientation on e in O ;
- if $\vec{e}$ is the chosen orientation on e in O , the e -th component of $\varrho(\oplus_{v \in \mathbb{V}} s_v)$ is

$$s_{v_1(\vec{e})} - s_{v_2(\vec{e})} \in \mathbb{Z}_{\bullet}^{I_e}.$$

In other words,

$$\varrho((\lambda_e)_{e \in \mathbb{E}}, (s_v)_{v \in \mathbb{V}}) = \bigoplus_{\vec{e} \in O} (\lambda_e s_{\vec{e}} + s_{v_1(\vec{e})} - s_{v_2(\vec{e})})$$

Let

$$\Lambda = \text{image}(\varrho), \quad K = \text{Ker}(\varrho) \quad \text{and} \quad CK = \mathbb{T}/\Lambda = \text{Coker}(\varrho).$$

The $\mathbb{Z}$ -modules Λ , K , and CK are independent of the choice of the orientation O on $\mathbb{E}$ and are invariants of the decorated graph Γ . In particular,

$$K = \left\{ ((\lambda_e)_{e \in \mathbb{E}}, (s_v)_{v \in \mathbb{V}}) \in \mathbb{Z}^{\mathbb{E}} \oplus \left(\bigoplus_{v \in \mathbb{V}} \mathbb{Z}^{I_v} \right)^* : \right. \\ \left. s_{v'} - s_v = \lambda_e s_{\vec{e}} \quad \forall v, v' \in \mathbb{V}, \vec{e} \in \vec{\mathbb{E}}_{v, v'} \right\}. \quad (2.15)$$

Replacing $\mathbb{Z}$ with another ring R in the equations above, we denote the corresponding terms in (2.14) by $\varrho_R: \mathbb{D}_R \rightarrow \mathbb{T}_R$ and image/kernel/cokernel by Λ_R , K_R , and CK_R , respectively. Via the exponentiation map, let

$$\exp(\Lambda_{\mathbb{C}}) \subset \prod_{e \in \mathbb{E}} (\mathbb{C}^*)_{\bullet}^{I_e}, \quad \text{with} \quad (\mathbb{C}^*)_{\bullet}^I = \left\{ (t_i)_{i \in I} \in (\mathbb{C}^*)^I : \prod_{i \in I} t_i = 1 \right\},$$

be the subgroup corresponding to the sub-Lie algebra $\Lambda_{\mathbb{C}} \subset \mathbb{T}_{\mathbb{C}}$, and denote the quotient group by

$$\mathcal{G} = \mathcal{G}(\Gamma) = \exp(CK_{\mathbb{C}}).$$

In other words,

$$\mathcal{G} = \frac{\prod_{e \in \mathbb{E}} (\mathbb{C}^*)_{\bullet}^{I_e}}{\exp(\varrho) \left((\mathbb{C}^*)^{\mathbb{E}} \times \left(\prod_{v \in \mathbb{V}} (\mathbb{C}^*)^{I_v} \right)^* \right)},$$

where

$$\left(\prod_{v \in \mathbb{V}} (\mathbb{C}^*)^{I_v} \right)^* := \left\{ (t_v)_{v \in \mathbb{V}} \in \prod_{v \in \mathbb{V}} (\mathbb{C}^*)^{I_v} : \prod_{i \in I_v} t_{v,i} = \prod_{i \in I_{v'}} t_{v',i} \quad \forall v, v' \in \mathbb{V} \right\}.$$

Below, similarly to [7], to every pre-log f as in Definition 2.4 we associate a group element

$$\text{ob}_{\Gamma}(f) \in \mathcal{G}(\Gamma)$$

that only depends on the equivalence class of f .

Given a pre-log map $f \equiv (f_v \equiv (u_v, \zeta_v, C_v))_{v \in \mathbb{V}}$ as in (2.10), for each $v \in \mathbb{V}$ and $\vec{e} \in \vec{\mathbb{E}}_v$, let $z_{\vec{e}}$ be an arbitrary holomorphic coordinate in a sufficiently small disk $\Delta_{\vec{e}}$ around the nodal point $(z_{\vec{e}} = 0) = q_{\vec{e}} \in \Sigma_v$.

1. For every $v \in \mathbb{V}$, $e \in \mathbb{E}_v$, and $i \in I_v$, consider a local holomorphic trivialization

$$u_v^* \mathcal{N}_{X_{I_v - \{i\}}} X_{I_v} |_{\Delta_e} \approx \mathcal{N}_{X_{I_v - i}} X_{I_v} |_{u(q_e)} \times \Delta_e,$$

such that the identification between the fibers $\mathcal{N}_{X_{I_v - i}} X_{I_v} |_{u(q_e)}$ on the left and $\mathcal{N}_{X_{I_v - i}} X_{I_v} |_{u(q_e)} \times \{0\}$ on the right is the identity map. By (2.8), we have

$$\zeta_{v,i}(z_e) = z_e^{s_{e,i}} \tilde{\zeta}_{v,i}(z_e) \quad (2.16)$$

such that

$$0 \neq \eta_{e,i} := \tilde{\zeta}_{v,i}(0) \in \mathcal{N}_{X_{I_v - i}} X_{I_v} |_{u(q_e)} = \mathcal{N}_i |_{u(q_e)}.$$

The vector $\tilde{\zeta}_{v,i}(0)$ is independent of the choice of the trivialization because a different choice of trivialization as above correspond to multiplication by a function $h: \Delta_e \rightarrow \mathbb{C}^*$ satisfying $h(0) = 1$. Therefore, it does not change $\tilde{\zeta}_{v,i}(0)$.

2. Similarly, for every $v \in \mathbb{V}$, $e \in \mathbb{E}_v$, and $i \in I_e - I_v$, with $\xi_{I_v,i}$ as in Definition 2.2.(a), we have

$$u_v^* \xi_{I_v,i}(z_e) = z_e^{s_{e,i}} \tilde{\xi}_{v,i}(z_e) \quad (2.17)$$

such that

$$0 \neq \eta_{e,i} := \tilde{\xi}_{v,i}(0) \in \mathcal{N}_{X_{I_v}} X_{I_v + \{i\}} |_{u(q_e)} = \mathcal{N}_i |_{u(q_e)}.$$

Note that the map u_v has a well-defined $s_{e,i}$ -th derivative

$$\eta'_{e,i} \in \mathcal{N}_{X_{I_v}} X_{I_v + i} |_{u(q_e)} \quad (2.18)$$

(with respect to the coordinate z_e) in the normal direction to $X_{I_v + i} \subset X_{I_v}$ at the nodal marked point q_e . The vector $\eta'_{e,i}$ is a multiple of $\eta_{e,i}$ by a factor that only depends on the choice of $\xi_{I_v,i}$.

3. Finally, for every $v \in \mathbb{V}$, $e \in \mathbb{E}_v$, and $i \in \mathcal{I} - I_e$, let

$$\eta_{e,i} \in \mathcal{N}_{X_{I_v}} X_{I_v + \{i\}} |_{u(q_e)} = \mathcal{N}_i |_{u(q_e)}$$

be the non-zero vector $(u_v^* \xi_{I_v,i})(q_e)$.

For each $e \in \mathbb{E}$, if e is the choice of orientation on e in O , let

$$\eta_e := (\eta_{e,i}/\eta_{e,i})_{i \in I_e} \in (\mathbb{C}^*)^{I_e}. \quad (2.19)$$

By Definition 2.4.(1) and because $s_e \in \mathbb{Z}_{\bullet}^{I_e} \subset \mathbb{Z}_{\bullet}^{\mathcal{I}}$ we have

$$\otimes_{i \in \mathcal{I}} \eta_{e,i} = \lambda_{O_{X_{\emptyset}}(X_{\emptyset})} |_{u(q_e)}.$$

By (2.3), we have

$$\eta_{e,i} = \eta_{e,i} \quad \forall i \in \mathcal{I} - I_e$$

Therefore,

$$\eta_e \in (\mathbb{C}^*)_{\bullet}^{I_e} \quad \forall e \in \mathbb{E}.$$

The tuples η_e give rise to an element

$$\eta := (\eta_e)_{e \in \mathbb{E}} \in \prod_{e \in \mathbb{E}} (\mathbb{C}^*)_{\bullet}^{I_e}. \quad (2.20)$$

The action of the subgroup $\exp(\varrho) \left((\mathbb{C}^*)^{\mathbb{E}} \times \left(\prod_{v \in \mathbb{V}} (\mathbb{C}^*)^{I_v} \right)^* \right)$ on η corresponds to admissible rescalings of the sections $\zeta_{v,i}$ and change of coordinates in z_e ; i.e the class

$$\text{ob}_{\Gamma}(f) = [\eta] \quad (2.21)$$

of η in $\mathcal{G}$ is independent of the choices involved. If f and f' are equivalent with respect to a reparametrization $h: \Sigma' \rightarrow \Sigma$ as in (2.12), the associated group elements η and η' , respectively, would be the same with respect to any h -symmetric choice of holomorphic coordinates $\{z_e\}_{e \in \mathbb{E}}$. Therefore, (2.21) only depends on the equivalence class $[f]$ of f and thus yields a well-defined function

$$\text{ob}_{\Gamma}: \mathcal{M}_{g,k}^{\text{plog}}(X_{\emptyset}, A) \rightarrow \mathcal{G}(\Gamma). \quad (2.22)$$

Lemma 2.9 *With notation as above, we have $\text{ob}_{\Gamma}([f]) = 1$ if and only if there exist a representative $f \equiv (f_v \equiv (u_v, \zeta_v, C_v))_{v \in \mathbb{V}}$ of the equivalence class $[f]$ as in (2.10) and choices of local coordinates z_e around the nodal point $(z_e = 0) = q_e \in \Sigma_v$ such that*

$$\eta_{e,i} = \eta_{e,i} \quad \forall e \in \mathbb{E}, i \in I_e.$$

Proof By definition, $\text{ob}_{\Gamma}([f]) = 1 \in \mathcal{G}$ if and only if

$$\eta \in \exp(\varrho) \left((\mathbb{C}^*)^{\mathbb{E}} \times \left(\prod_{v \in \mathbb{V}} (\mathbb{C}^*)^{I_v} \right)^* \right),$$

with η defined as (2.20) using any representative f of $[f]$. By the statement preceding (2.21), multiplying by a tuple of numbers in $(\mathbb{C}^*)^{\mathbb{E}} \times \left(\prod_{v \in \mathbb{V}} (\mathbb{C}^*)^{I_v} \right)^*$ corresponds to rescaling the coordinates z_e and the sections $\zeta_{v,i}$ used in the definition of η_e in (2.19). Therefore, $[\eta] = 1$ if and only if there exist choices of $\zeta_{v,i}$ (i.e. a representative $f \equiv (f_v \equiv (u_v, \zeta_v, C_v))_{v \in \mathbb{V}}$ of the equivalence class $[f]$ as in (2.10)) and choices of local coordinates z_e around the nodal point $(z_e = 0) = q_e \in \Sigma_v$ such that

$$(\eta_e)_{e \in \mathbb{E}} \equiv 1 := (1)_{e \in \mathbb{E}} \Leftrightarrow \eta_{e,i} = \eta_{e,i} \quad \forall e \in \mathbb{E}, i \in I_e.$$

Note the choice of z_e is independent of the choice of $i \in I_e$; otherwise, the condition is vacuous. $\square$

Definition 2.10 Let $X_{\emptyset} = \bigcup_{i \in \mathcal{I}} X_i$ be a d -semistable SNC variety. A log map is a stable pre-log map f with the decorated dual graph Γ such that

- (C1) there exist functions

$$s: \mathbb{V} \rightarrow \mathbb{R}^{\mathcal{I}}, \quad v \mapsto s_v, \quad \text{and} \quad \lambda: \mathbb{E} \rightarrow \mathbb{R}_+, \quad e \mapsto \lambda_e,$$

such that

1. $s_v \in \mathbb{R}_+^{I_v} \times \{0\}^{\mathcal{I}-I_v}$ for all $v \in \mathbb{V}$,
2. $s_{v_2(e)} - s_{v_1(e)} = \lambda_e s_e$ for every $e \in \mathbb{E}$;

- (C2) and $\text{ob}_{\Gamma}(f) = 1 \in \mathcal{G}(\Gamma)$; see Remark 2.9.

Remark 2.11 The description of Condition (C2) in Remark 2.9 is more intuitive and that's the description of the choice for constructing a gluing of f . We need the condition $\text{ob}_{\Gamma}(f) = 1$ to be able to consistently glue the components of f at the nodes; see [7, (5.16)]. The description in terms of a map into $\mathcal{G}(\Gamma)$ is used in the dimension count of Lemma 4.2 and will be needed for proving transversality statements as in [8].

Since Γ is connected, Condition (C1)2 and $s_e \in \mathbb{Z}_{\bullet}^{\mathcal{I}}$, for all $e \in \mathbb{E}$, imply that

$$\sum_{i \in \mathcal{I}} s_{v,i} = r \quad (2.23)$$

is a fixed positive constant $r \in \mathbb{Z}_+$ independent of $v \in \mathbb{V}$ (at the expense of replacing $\mathbb{Z}$ with $\mathbb{R}$, which has no effect on the problem, we may uniformly rescale $s_{v,i}$ so that $r = 1$). The combinatorial condition (C1) is essentially equivalent to the basicness condition in [19, Dfn 1.20] (More precisely, it is equivalent to $\text{GS}(\overline{\mathcal{M}}, \mathbf{u}) \neq \emptyset$). The analytical condition (C2) more or less corresponds to the existence of a morphism to a log point; see Remark 2.12 below. The set of vectors $((s_v)_{v \in \mathbb{V}}, (\lambda_e)_{e \in \mathbb{E}})$ satisfying (C1) is the intersection of the kernel $K_{\mathbb{R}}$ of $\varrho_{\mathbb{R}}$ with the positive quadrant in $\mathbb{D}_{\mathbb{R}}$. Since this intersection is non-empty by assumption, its closure is a convex maximal rational polyhedral cone $\sigma(\Gamma)$ in $K_{\mathbb{R}}$. The space of gluing parameters is a multiple of the affine toric variety Y_{σ} with the toric fan $\sigma(\Gamma) \subset K_{\mathbb{R}}$; see Sect. 5. The projection map π in (2.13) restricts to $\sigma(\Gamma) \rightarrow \mathbb{R}_{\geq 0}$; it gives a projection map between toric varieties $Y_{\sigma} \rightarrow \mathbb{C}$ ($\mathbb{C}$ is the toric variety associated to the toric fan $\mathbb{R}_{\geq 0} \subset \mathbb{R}$) that, via gluing, corresponds to the fibration (1.1).

Remark 2.12 The following details, by an anonymous referee, explain the relation between the condition (C2) and log geometry (which is probably well-known among the experts but it is not clearly written down anywhere in the log GW literature).

A log structure on a (smooth or SNC) variety X is given by: a constructible sheaf of monoids $\overline{M}_X$ (the *ghost* or *characteristic* sheaf), and an assignment of a holomorphic line bundle/section pair $(\mathcal{O}_X(\alpha), s_{\alpha})$ to every section $\alpha \in \overline{M}_X$. The latter must be compatible with the restriction maps of open sets and the monoid structure. A log enhancement of a morphism $f: \Sigma \rightarrow X$ is given by: a pullback map $f^b: f^{-1}\overline{M}_X \rightarrow \overline{M}_{\Sigma}$, and a compatible choice of isomorphisms

$$f^*(\mathcal{O}_X(\alpha), s_{\alpha}) \cong (\mathcal{O}_{\Sigma}(f^b(\alpha)), s_{f^b(\alpha)}) \quad \forall \alpha \in \overline{M}_X. \quad (2.24)$$

When X is an SNC variety $\bigcup_{i \in \mathcal{I}} X_i$, there are $|\mathcal{I}|$ independent sections of the canonically assigned characteristic sheaf $\overline{M}_X$. The line bundle/section pair associated to the i -th section is given by $(\mathcal{O}_{X_j}(X_{ij}), \zeta_{j,i})$ when restricted to X_j with $j \neq i$, and by $(\mathcal{N}_{\mathbb{Z}X_i}, 0)$ when restricted to X_i . The condition (C1) asserts that the pullback map f^b exists (i.e. there is a log structure on Σ such that the pull-back map is well-defined). Then, in order to obtain a log map, it remains to construct the isomorphism (2.24). As explained in [36, Prp 2.4.1], for each smooth component Σ_v , the bundle $\mathcal{O}_{\Sigma}(f^b(\alpha))|_{\Sigma_v}$ is given by an explicit weighted sum of the marked points and nodes recording the contact order at special points. Therefore, the identification

$$\mathcal{O}_{\Sigma}(f^b(\alpha))|_{\Sigma_v} \cong f^*\mathcal{O}_X(\alpha)|_{\Sigma_v}$$

is equivalent to the data of a meromorphic section of $f^*\mathcal{O}_X(\alpha)|_{\Sigma_v}$ with specified orders at the special points. When α is the section of $\overline{M}_X$ corresponding to X_i , this is precisely the section $\zeta_{v,i}$ in our notation. When Σ_v is mapped to X_i , the sections are zero, hence there is a $\mathbb{C}^*$ worth of choices, corresponding to the choice of meromorphic sections $\xi_{v,i}$ in our notation. This explains the identification (2.24) on each component. The condition (C2) is then needed to extend these local identifications globally.

We will study the sufficiency of this condition to compare the moduli spaces in another work.

We will denote the subset of log curve in $\mathcal{M}_{g,k}^{\text{log}}(X_{\emptyset}, A)_{\Gamma}$ by $\mathcal{M}_{g,k}^{\text{log}}(X_{\emptyset}, A)_{\Gamma}$. In other words, we require Γ to satisfy the combinatorial condition (C1) and

$$\mathcal{M}_{g,k}^{\text{log}}(X_{\emptyset}, A)_{\Gamma} = \text{ob}_{\Gamma}^{-1}(1) \subset \mathcal{M}_{g,k}^{\text{log}}(X_{\emptyset}, A)_{\Gamma}.$$

In Sect. 4, we show that the expected $\mathbb{C}$ -dimension of $\mathcal{M}_{g,k}^{\log}(X_{\emptyset}, A)_{\Gamma}$ is equal to

$$c_1^{T^{\log} X_{\emptyset}}(A) + (\dim_{\mathbb{C}} X_{\emptyset} - 3)(1 - g) + k - (\dim_{\mathbb{R}} K_{\mathbb{R}} - 1),$$

where $T^{\log} X_{\emptyset}$ is the log tangent bundle of $X_{\emptyset}$. Therefore, $\mathcal{M}_{g,k}^{\log}(X_{\emptyset}, A)_{\Gamma}$ is virtually a main stratum, i.e. a stratum whose expected dimension is the maximum value $c_1^{T^{\log} X_{\emptyset}}(A) + (\dim_{\mathbb{C}} X_{\emptyset} - 3)(1 - g) + k$, if and only if $K \cong \mathbb{Z}$.

If $\mathcal{Z}$ is a semistable smoothing of $\mathcal{Z}_0 = X_{\emptyset}$ as in (1.1), $c_1^{T^{\log} X_{\emptyset}}(A)$ coincides with $c_1^{T\mathcal{Z}_{\lambda}}(A)$, for all $\lambda \in \Delta^*$. Therefore, the expected dimension of the analytical log moduli space

$$\overline{\mathcal{M}}_{g,k}^{\log}(X_{\emptyset}, A) = \bigcup_{\Gamma} \mathcal{M}_{g,k}^{\log}(X_{\emptyset}, A)_{\Gamma}$$

coincides with the expected dimension of $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}_{\lambda}, A)$. Different components in this union, however, will contribute with different weights to the VFC of $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}_{\lambda}, A)$. The degeneration formula (5.3) describes these weights.

Remark 2.13 There is a slightly different map associated to Γ that will be useful in Sect. 5. Let

$$\varrho_{\bullet} : \mathbb{D}_{\bullet} = \mathbb{Z}^{\mathbb{E}} \oplus \bigoplus_{v \in \mathbb{V}} \mathbb{Z}_{\bullet}^{I_v} \longrightarrow \mathbb{T} \equiv \bigoplus_{e \in \mathbb{E}} \mathbb{Z}_{\bullet}^{I_e}. \quad (2.25)$$

denote the restriction of ϱ to $\mathbb{D}_{\bullet} \subset \mathbb{D}$. Recall from (2.5) that the subscript $\bullet$ denotes the subspace where the sum of the coefficients is zero. Then, the following diagram commutes

$$\begin{array}{ccc} \mathbb{D}_{\bullet} & \xrightarrow{\varrho_{\bullet}} & \mathbb{T} \\ \downarrow & & \downarrow \cong \\ \mathbb{D} & \xrightarrow{\varrho} & \mathbb{T} \\ \downarrow 0 \oplus \pi & & \\ \mathbb{Z} & & \end{array} \quad (2.26)$$

Let $\sigma(\Gamma)$ be the intersection of positive quadrant in $\mathbb{D}$ and the kernel K of ϱ as in the paragraph after (2.23). Fixing a vector in the interior of $\sigma(\Gamma)$ gives us a decomposition

$$K_{\mathbb{R}}(\varrho) \cong K_{\mathbb{R}}(\varrho_{\bullet}) \oplus \mathbb{R}$$

such that $\sigma(\Gamma)$ is a cone containing the ray $0 \oplus \mathbb{R}_{\geq 0}$. Therefore, $\mathcal{M}_{g,k}^{\log}(X_{\emptyset}, A)_{\Gamma}$ is virtually a main stratum if and only if $K_{\bullet} = 0$, where $K_{\bullet} = \text{Ker}(\varrho_{\bullet})$ is the kernel of $\varrho_{\bullet}$. Intuitively, as we will show in Sect. 4, (the toric variety Y_{σ} corresponding to the cone σ inside) K corresponds to partial smoothings of the nodal map that might end up in a different fiber of the total space $\mathcal{Z}$ of the semi-stable degeneration, and $K_{\bullet}$ is the subset corresponding to smoothings inside $\mathcal{Z}_0$. Therefore, if $K_{\bullet} = 0$, it means there are no further smoothings of the nodal map inside $\mathcal{Z}_0$; thus, we are in the top stratum.

3 Proof of the main theorem

In this section, by following and adjusting the steps of the proof [7, Thm. 1.3], we prove Theorem 1.3.

Throughout this section, for the cases that involve the smoothing of $X_\emptyset$, let $\pi: \mathcal{Z} \rightarrow \Delta$ be a Kähler semistable degeneration as in (1.1) with compact fibers and the SNC central fiber

$$\mathcal{Z}_0 := \pi^{-1}(0) = X_\emptyset := \bigcup_{i \in \mathcal{I}} X_i.$$

Assume X is an $(n+1)$ -dimensional smooth complex variety and $D \subset X$ is a smooth complex divisor. For any $p \in D$, let U be an open set around p with local coordinates $(x_0, \dots, x_n)$ given by a chart map

$$\varphi: U \rightarrow \mathbb{C}^{n+1}$$

such that $D \cap U \equiv \varphi^{-1}(x_0 = 0)$. We call such a pair (U, φ) a D -compatible chart around p . A D -compatible chart (U, φ) induces a holomorphic trivialization

$$d_{\mathcal{N}_X D} \varphi: \mathcal{N}_X D|_{U \cap D} \rightarrow \varphi(U \cap D) \times \mathbb{C} \subset \mathbb{C}^{n+1}. \quad (3.1)$$

Before we state the next proposition, we need to elaborate on the topological aspects of the Gromov convergence [18] and setup the notation. For more details, we refer to [7, Sec. 3.1]. Suppose

$$(f_a \in \overline{\mathcal{M}}_{g,k}(\mathcal{Z}_{\lambda_a}, A))_{a \in \mathbb{N}}, \quad f_a = (u_{a,v'}, C_{a,v'} = (\Sigma_{v'}, \vec{z}_{v'}))_{v' \in \mathbb{V}'}, \quad (3.2)$$

is a sequence of stable maps with a fixed decorated dual graph $\Gamma' = \Gamma(\mathbb{V}', \mathbb{E}', \mathbb{L}')$ that Gromov converges to the stable map

$$f = (u_v, C_v)_{v \in \mathbb{V}} \in \overline{\mathcal{M}}_{g,k}(\mathcal{Z}_0, A) \quad (3.3)$$

with the decorated dual graph $\Gamma = \Gamma(\mathbb{V}, \mathbb{E}, \mathbb{L})$. Then, (for a sufficiently large) all the marked domains

$$(\Sigma_a = \bigcup_{v' \in \mathbb{V}'} \Sigma_{a,v'}, \vec{z}_a = \bigcup_{v' \in \mathbb{V}'} \vec{z}_{v'})$$

are smoothly isomorphic to a fixed marked domain $(\Sigma', \vec{z}')$ and the domain $\Sigma = \bigcup_{v \in \mathbb{V}} \Sigma_v$ of f is obtained by collapsing a set of embedded curves away from the marked and nodal points in Σ' . In other words, there is a continuous degeneration map

$$\gamma: \Sigma' \rightarrow \Sigma$$

that sends the marked points and nodal points of Σ' to the (corresponding) marked and nodal points of Σ , and collapses some embedded circles $\{\gamma_e\}_{e \in \mathbb{E}^{\text{cut}}}$ in Σ' to the new nodal points $\{q_e\}_{e \in \mathbb{E}^{\text{cut}}}$ in Σ . The map γ gives us a similarly denoted map

$$\gamma^*: \Gamma \rightarrow \Gamma'. \quad (3.4)$$

We have

$$\mathbb{E} \cong \mathbb{E}' \cup \mathbb{E}^{\text{cut}}, \quad \mathbb{L} = \mathbb{L}',$$

such that $\gamma^*|_{\mathbb{E}' \cup \mathbb{E}}$ and $\gamma^*|_{\mathbb{L}}$ are isomorphisms and

$$\gamma^*: \mathbb{E}^{\text{cut}} \cup \mathbb{V} \rightarrow \mathbb{V}'.$$

For every $v \in \mathbb{V}$, there exists a unique $v' \in \mathbb{V}'$ and a connected component U_v of $\Sigma_{v'} - \{\gamma_e\}_{e \in \mathbb{E}^{\text{cut}}}$ such that Σ_v is obtained by collapsing the boundary circles of the closure of U_v .

The goal is to, after possibly passing to a subsequence, find a set of meromorphic sections $\{\zeta_{v,i}\}_{v \in \mathbb{V}}$ that enhances (3.3) to a log map $f_{\log} \in \mathcal{M}_{g,k}^{\log}(\mathcal{Z}_0, A)_\Gamma$.

Definition 3.1 With notation as in (3.2) and (3.3), let

$$\zeta_{v,i} \in \Gamma_{\text{mero}}(\Sigma_v, u_v^* \mathcal{N}_i) \quad \forall v \in \mathbb{V}, i \in I_v,$$

be a set of meromorphic sections. We say (3.2) log-Gromov converges to

$$f_{\log} := (u_v, \zeta_v = (\zeta_{v,i})_{i \in I_v}, C_v)_{v \in \mathbb{V}} \quad (3.5)$$

if for each $v \in \mathbb{V}$ and $i \in I_v$, with $v' = \gamma^*(v) \in \mathbb{V}'$, the sequence $(u_{a,v'})_{a \in \mathbb{N}}$ is asymptotic to $\zeta_{v,i}$ in the normal direction to X_i in the following sense. For every $p \in U_v$, there exists an X_i -compatible holomorphic chart (U, φ) around $u_v(p) \in \mathcal{Z}$ and a sequence of non-zero complex numbers $(t_{a,v,i})_{a \in \mathbb{N}}$ (converging to zero) such that

$$(\text{uniformly}) \lim_{a \rightarrow \infty} t_{a,v,i}^{-1} x_0 \circ \varphi \circ u_{a,v'}|_K = x_0 \circ (d_{\mathcal{N}_i} \varphi(\zeta_{v,i}|_K)) \quad (3.6)$$

on any compact subset $K \subset U_v$.

Note that, since f_a Gromov converges to the stable map f , we have

$$\lim_{a \rightarrow \infty} x_i \circ \varphi \circ u_{a,v'}|_K = x_i \circ \varphi \circ u_v|_K \quad \forall i = 0, \dots, n. \quad (3.7)$$

uniformly on compact sets.

Remark 3.2 For two sequences of non-zero complex numbers $(t_a)_{a \in \mathbb{N}}$ and $(t'_a)_{a \in \mathbb{N}}$, we write

$$(t_a)_{a \in \mathbb{N}} \sim (t'_a)_{a \in \mathbb{N}} \quad \text{if} \quad \lim_{a \rightarrow \infty} t_a/t'_a = 1. \quad (3.8)$$

The right-hand side of (3.8) defines an equivalence relation on the set of such sequences and we denote the equivalence class of a sequence $(t_a)_{a \in \mathbb{N}}$ by $[(t_a)_{a \in \mathbb{N}}]$. For an equivalence class $[(t_a)_{a \in \mathbb{N}}]$ and $t \in \mathbb{C}^*$, the equation

$$t[(t_a)_{a \in \mathbb{N}}] := [(tt_a)_{a \in \mathbb{N}}]$$

is well-defined and defines an action of $\mathbb{C}^*$ on the set of equivalence classes. Moreover, the operation of point-wise multiplication/division between such sequences

$$(t_a)_{a \in \mathbb{N}} \cdot (t'_a)_{a \in \mathbb{N}} = (t_a t'_a)_{a \in \mathbb{N}}$$

descends to a well-defined multiplication/division operation between the equivalence classes. Condition (3.6) and the equivalence class of the rescaling sequence $[(t_{a,v,i})_{a \in \mathbb{N}}]$, over each irreducible component of the domain, are independent of the choice of the X_i -compatible local chart (φ, U) ; if the limit (3.6) holds in one chart, it will hold in every other chart for the same $\zeta_{v,i}$ because the involving functions are holomorphic. It is also clear from (3.6) that if $(t_{a,v,i})_{a \in \mathbb{N}}$ is a sequence of rescaling parameters associated to $\zeta_{v,i}$ and $(t'_{a,v,i})_{a \in \mathbb{N}}$ is a rescaling sequence associated to $c\zeta_{v,i}$, for any $c \in \mathbb{C}^*$, then

$$c[(t'_{a,v,i})_{a \in \mathbb{N}}] = [(t_{a,v,i})_{a \in \mathbb{N}}].$$

The relation between the sets of rescaling parameters $\{t_{a,v,i}\}$ in Definition 3.1 and $\{\lambda_a\}$ in (3.2) plays an important role in the rest of this chapter.

Lemma 3.3 *After passing to a subsequence, every sequence (3.2) log Gromov converges and the limit is unique up to equivalence. More specifically, given (3.2), after passing to a subsequence, the limiting holomorphic map f admits meromorphic sections $\{\zeta_{v,i}\}_{v \in \mathbb{V}, i \in I_v}$ as in Definition 3.1; furthermore,*

1. *these meromorphic sections are unique up to multiplication by a constant in $\mathbb{C}^*$;*

2. $\zeta_{v,i}$ has no pole/zero in $\Sigma_v - q_v$,
3. $\zeta_{v,i}$ has a zero/pole of order $s_{e,i}$ at q_e , for all $e \in \mathbb{E}_v$, $i \in I_v$;
4. for each $e \in \mathbb{E}$, the vector $s_e = (s_{e,i})_{i \in I_e}$ defined as in (2.11) belongs to $\mathbb{Z}_\bullet^{I_e}$;
5. for each $v \in \mathbb{V}$, the product

$$\bigotimes_{i \in I_v} \zeta_{v,i} \otimes \bigotimes_{i \in \mathcal{I} - I_v} u_v^* \xi_{I_v, i}$$

is a constant section.

Proof Except items (4) and (5), the rest directly follow from applying [7, Prp 3.10 and Lmm 3.13] to the SNC divisor $\mathcal{Z}_0 \subset \mathcal{Z}$. Also, with respect to the decomposition $\mathbb{E} \cong \mathbb{E}' \cup \mathbb{E}^{\text{cut}}$, by [7, Lmm 3.13], $s_e = 0$ for all $e \in \mathbb{E}' \subset \mathbb{E}$. Therefore, in order to prove (4), we can restrict to edges in $\mathbb{E}^{\text{cut}}$. We prove (4) and (5) by following and adjusting the details of [7, Lmm 3.13]. First, let us recall the setup used in [7, Sec 3.4] that we will also use in the rest of this section. For a sufficiently large, the domain $\Sigma'_a \cong \Sigma'$ of f_a is obtained from the nodal domain Σ of f in the following way. There exist

- a sequence of complex structures $j_a = (j_{a,v})_{v \in \mathbb{V}}$ on the nodal domain $\Sigma = (\Sigma_v)_{v \in \mathbb{V}}$ of f ,
- a sequence of local $j_{a,v}$ -holomorphic coordinates $z_{a,e}: \Delta_e \rightarrow \mathbb{C}$ around $q_e \in \Sigma_v$, for all $v \in \mathbb{V}$ and $e \in \mathbb{E}_v^{\text{cut}}$, and
- a sequence of non-zero complex numbers $(\varepsilon_{a,e})_{e \in \mathbb{E}^{\text{cut}}}$ converging to zero,

such that

1. $(\Sigma_a, \vec{z}_a)$ is isomorphic to the smoothing of $(\Sigma, \vec{z}, j_a = (j_{a,v})_{v \in \mathbb{V}})$ defined by

$$z_{a,e} z_{a,e} = \varepsilon_{a,e} \quad \forall e \in \mathbb{E}^{\text{cut}}, \quad (3.9)$$

2. the sequence $(j_{a,v})_{a \in \mathbb{N}}$ C^∞ -converges to j_v for all $v \in \mathbb{V}$,
3. the sequence $(z_{a,e})_{a \in \mathbb{N}}$ C^∞ -converges to z_e , where $z_e: \Delta_e \rightarrow \mathbb{C}$ is some fixed local j_v -holomorphic coordinate around $q_e \in \Sigma_v$, for all $v \in \mathbb{V}$ and $e \in \mathbb{E}_v^{\text{cut}}$.

With notation as above, for each $e \in \mathbb{E}^{\text{cut}}$, the union

$$A_e := \Delta_e \cup \Delta_e = \{(z_e, z_e) \in \Delta_e \times \Delta_e : z_e z_e = 0\}$$

is a neighborhood of q_e in Σ . We orient each circle $\partial \Delta_e$ in the direction of the counter-clockwise rotation in $\Delta_e \subset \mathbb{C}$. By (3.9), the neck region

$$A_{a,e} = \{(z_{a,e}, z_{a,e}) \in \Delta_e \times \Delta_e : z_{a,e} z_{a,e} = \varepsilon_{a,e}\}$$

in Σ_a is a cylinder with two (oppositely oriented) boundary circles

$$\partial A_{a,e} \cong \partial \Delta_e \quad \text{and} \quad \partial A_{a,e} \cong \partial \Delta_e. \quad (3.10)$$

For sufficiently large a , $s_{e,i}$ is equal to the winding number of $u_a|_{\partial A_{a,e}}$ around the divisor X_i ; see the proof of [7, Lmm 3.13].

If $I_e = \{i_1, \dots, i_k\}$, there exists a sufficiently small neighborhood U around $u(q_e) \in \mathcal{Z}$ with coordinates $(x_1, \dots, x_{n+1})$ such that

$$U \cap X_{i_j} = (x_j \equiv 0) \quad \forall j \in \{1, \dots, k\}$$

and the projection map $\pi: \mathcal{Z} \rightarrow \mathbb{C}$ has the form

$$(x_1, \dots, x_{n+1}) \rightarrow \prod_{j=1}^k x_j. \quad (3.11)$$

By (3.6) and (3.7), for sufficiently large a ,

$$s_{\varepsilon, i_j} = \text{winding number around } X_{i_j} \text{ of } x_j \circ u_a|_{\partial A_{a, \varepsilon}}.$$

Therefore, by (3.11) and since u_a has image in $\mathcal{Z}_{\lambda_a}$, we have $\sum_{j=1}^k s_{\varepsilon, i_j} = 0$; i.e. $s_{\varepsilon} \in \mathbb{Z}_{\bullet}^{I_{\varepsilon}}$. Proof of (5) is similar. Since the sections are holomorphic, it is enough to prove (5) on a sufficiently small open set around any point in Σ_v . Fix $p \in U_v$ and a local coordinate z on a sufficiently small compact disk K around it. If $I_v = \{i_1, \dots, i_k\}$, there exists a sufficiently small neighborhood U around $u(p) \in (X_{I_v} - \partial X_{I_v}) \subset \mathcal{Z}$ with coordinates $(x_1, \dots, x_{n+1})$ such that

$$U \cap X_{i_j} = (x_j \equiv 0) \quad \forall j \in \{1, \dots, k\}.$$

and the projection map $\pi: \mathcal{Z} \rightarrow \mathbb{C}$ has the form

$$(x_1, \dots, x_{n+1}) \rightarrow \prod_{j=1}^k x_{i_j}. \quad (3.12)$$

On $\mathcal{Z}$, the product $\otimes_{i \in \mathcal{I}} \xi_i$ is a section of the trivial line bundle

$$\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0) \cong \mathcal{Z} \times \mathbb{C}. \quad (3.13)$$

Here $\cong$ means that the line bundle $\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)$ is trivial and a choice of trivialization is fixed. Recall from (1.5) that, when a smoothing $\mathcal{Z}$ is given, we choose ξ_i such that the projection of $\otimes_{i \in \mathcal{I}} \xi_i$ to the $\mathbb{C}$ -factor in (3.13) is equal to π . Therefore, by (3.12), we can choose the local coordinates so that $\xi_{i_j}|_U = x_j$ for all $j \in \{1, \dots, k\}$. By (3.6) and the assumption above, for large a ,

$$\begin{aligned} \lambda_a &= \bigotimes_{i \in \mathcal{I}} u_{a, v'}|_K^* \xi_i = \bigotimes_{i_j \in I_v} x_j \circ \varphi \circ u_{a, v'}|_K \otimes \bigotimes_{i \in \mathcal{I} - I_v} u_{a, v'}|_K^* \xi_i \\ &\approx \bigotimes_{i_j \in I_v} t_{a, v, i_j} \zeta_{v, i_j} \otimes \bigotimes_{i \in \mathcal{I} - I_v} u_{a, v'}|_K^* \xi_i. \end{aligned}$$

Since

$$\lim_{a \rightarrow \infty} u_{a, v'}|_K^* \xi_i = u_v|_K^* \xi_i \quad \forall i \in \mathcal{I} - I_v,$$

we conclude that

$$\lim_{a \rightarrow \infty} \frac{\lambda_a}{\prod_{i \in I_v} t_{a, v, i}} = c_v$$

for some non-zero constant (section) $c_v \in \mathbb{C}^*$ and thus

$$\bigotimes_{i \in I_v} \zeta_{v, i} \otimes \bigotimes_{j \in \mathcal{I} - I_v} u_v^* \xi_{I_v, j} = c_v.$$

□

Corollary 3.4 *After passing to a subsequence, every sequence (3.2) has a unique limit (3.5) which belongs to $\mathcal{M}_{g, k}^{plog}(\mathcal{Z}_0, A)_{\Gamma}$.*

Proof On each Σ_v , we rescale one of $\zeta_{v,i}$ (and thus the corresponding sequence $\{t_{a,v,i}\}$) such that

$$\bigotimes_{i \in I_v} \zeta_{v,i} \otimes \bigotimes_{j \in \mathcal{I} - I_v} u_v^* \xi_{I_v, j} = 1 \quad \forall v \in \mathbb{V}.$$

Then, by Lemma 3.3, $f_{\log}$ satisfies all the properties of Definition 2.2. Note that we will then have

$$\lim_{a \rightarrow \infty} \frac{\lambda_a}{\prod_{i \in I_v} t_{a,v,i}} = 1. \quad (3.14)$$

We will use (3.14) in the proof of the main result below. $\square$

Remark 3.5 Since $\mathcal{Z}_0 \subset \mathcal{Z}$ is an SNC divisor, and $A \cdot \mathcal{Z}_0 = 0$, let $\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}, \mathcal{Z}_0, A)$ denote the relative (log) moduli space defined in [7] with trivial contact data (with $\mathcal{Z}_0$) at the marked points. By [7, Prp 3.14], we already know that the unique limit $f_{\log}$ in (3.5) belongs to $\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}, \mathcal{Z}_0, A)$. The linear map ϱ in (2.14) is the same as the linear map ϱ in [7, (2.28)] but it has a different domain and target. In the following diagram, the first row is (2.14), the second row is [7, (2.28)], and the vertical maps are the natural inclusion maps.

$$\begin{array}{ccc} \mathbb{Z}^{\mathbb{E}} \oplus \left(\bigoplus_{v \in \mathbb{V}} \mathbb{Z}^{I_v} \right)^* & \xrightarrow{\varrho} & \bigoplus_{e \in \mathbb{E}} \mathbb{Z}_{\bullet}^{I_e} \\ \downarrow & & \downarrow \\ \mathbb{Z}^{\mathbb{E}} \oplus \bigoplus_{v \in \mathbb{V}} \mathbb{Z}^{I_v} & \xrightarrow{\varrho} & \bigoplus_{e \in \mathbb{E}} \mathbb{Z}^{I_e}. \end{array}$$

The following proposition shows that (3.5) actually belongs to $\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}_0, A)_{\Gamma}$. Since $s_e \in \mathbb{Z}_{\bullet}^{I_e}$, by (2.23), the kernel of the second row is the same as the kernel of the first row; thus, Condition (C1) of Definition 2.10 is the same as Condition (1) of [7, Dfn 2.8]. However, the cokernels in each row and thus the groups $\mathcal{G}$ are different. In order to distinguish the notation, let us denote the group associated to Γ in [7, Dfn 2.8.(2)] by $\mathcal{G}^{\text{rel}}$. The commutative diagram above induces a homomorphism $\mathcal{G} \rightarrow \mathcal{G}^{\text{rel}}$, but this homomorphism does not need to be injective or surjective.

Proposition 3.6 Suppose (3.2) is a sequence of stable maps in $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}^*, A)$ that log-Gromov converges to (3.5) in the sense of Definition 3.1. Then (3.5) represents an element of $\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}_0, A)$.

Proof By Corollary 3.4, we already know that $f_{\log}$ in (3.5) is a pre-log map. By Remark 3.5 above (over $\mathbb{R}$), we also know that $f_{\log}$ satisfies Condition (C1) of Definition 2.10. More precisely, by the definition of ϱ after (2.14), every element in the kernel of ϱ over $\mathbb{R}$ is a vector

$$\bigoplus_{e \in \mathbb{E}} \lambda_e \oplus \bigoplus_{v \in \mathbb{V}} s_v \in \mathbb{R}^{\mathbb{E}} \oplus \left(\bigoplus_{v \in \mathbb{V}} \mathbb{R}^{I_v} \right)^*$$

satisfying

$$s_{v_2(e)} - s_{v_1(e)} = \lambda_e s_e \quad e \in \mathbb{E},$$

which is exactly the part (b) of Condition (C1). Note that the equality above is independent of the choice of orientation on e . Both sides will flip sign if we flip the orientation on e . The

positivity condition on s_v and s_e is the same as the positivity condition in [7, Dfn 2.8], and is achieved in the following way. The desired vector $\bigoplus_{e \in \mathbb{E}} \lambda_e \oplus \bigoplus_{v \in \mathbb{V}} s_v$ satisfying (C1) is explicitly derived from the data appearing in the convergence argument above (also listed in the bullets below for the rest of the proof) in the following way. In [7, Sec 3.4], as $a \rightarrow \infty$, the vectors s_v are obtained from the rescaling parameters by

$$s_{a,v} = (-\log t_{a,v,i})_{i \in I_v} \in \mathbb{R}^{I_v} \quad \forall a \in \mathbb{N}, v \in \mathbb{V},$$

and the real numbers λ_e are obtained from

$$\lambda_{a,e} = -\log(\varepsilon_{a,e}),$$

where $\varepsilon_{a,e}$ is the gluing parameters of the domain Σ_a at the node q_e . The details of this process also appear in the second half of the proof below. Since $t_{a,v,i}, \varepsilon_{a,e} \rightarrow 0$ as $a \rightarrow \infty$, the resulting vector $\bigoplus_{e \in \mathbb{E}} \lambda_e \oplus \bigoplus_{v \in \mathbb{V}} s_v$ will have positive coefficients. Furthermore, in the current context, by (3.14), we additionally have

$$\sum_{i \in I_v} \log t_{a,v,i} - \sum_{j \in I_{v'}} \log t_{a,v,j} \approx \log(\lambda_a) - \log(\lambda_a) = 0.$$

In the limit, this implies that

$$\bigoplus_{v \in \mathbb{V}} s_v \in \left(\bigoplus_{v \in \mathbb{V}} \mathbb{R}^{I_v} \right)^* \subset \bigoplus_{v \in \mathbb{V}} \mathbb{R}^{I_v}$$

It just remain to show that $f_{\log}$ also satisfies Condition (C2) of Definition 2.10.

The proof uses the relation between the following parameters:

- the parameters $\{\lambda_a\}_{a \in \mathbb{N}}$ in (3.2);
- the local holomorphic coordinates $z_{a,e}: \Delta_e \rightarrow \mathbb{C}$ and $z_{e'}: \Delta_{e'} \rightarrow \mathbb{C}$ around the nodal points (see the proof of Lemma 3.3);
- the local coordinates gluing parameters $\{\varepsilon_{a,e}\}_{a \in \mathbb{N}, e \in \mathbb{E}^{\text{cut}}}$ in (3.9),
- the rescaling parameters $\{t_{a,v,i}\}_{a \in \mathbb{N}, v \in \mathbb{V}, i \in I_v}$ in (3.6);
- and, the leading order terms

$$0 \neq \eta_{e,i} \in \mathcal{N}_i|_{u(q_e)}$$

of $f_{\log}$ on $\Delta_{e'}$ defined before (2.19) with respect to $z_{e'}$.

By [7, Prp 3.15], for every oriented edge $e' \in \mathbb{E}^{\text{cut}}$ that goes from v_1 to v_2 , and $i \in I_e$,

1. if $i \in I_{v_1}$ and $i \notin I_{v_2}$, we have

$$\lim_{a \rightarrow \infty} t_{a,v_1,i} \varepsilon_{a,e}^{s_{e,i}} = \frac{\eta_{e,i}}{\eta_{e',i}}; \quad (3.15)$$

2. if $i \in I_{v_1} \cap I_{v_2}$, we have

$$\lim_{a \rightarrow \infty} \frac{t_{a,v_1,i} \varepsilon_{a,e}^{s_{e,i}}}{t_{a,v_2,i}} = \frac{\eta_{e,i}}{\eta_{e',i}}. \quad (3.16)$$

Additionally, by (3.14), we have

$$\lim_{a \rightarrow \infty} \frac{\lambda_a}{\prod_{i \in I_v} t_{a,v,i}} = 1.$$

Lemma 3.7 We can choose the coordinates $\{z_e\}_{e \in \mathbb{E}^{\text{cut}}}$ and $\{z_{a,e}\}_{e \in \mathbb{E}^{\text{cut}}, a \in \mathbb{N}}$ satisfying (3.9) and item 3 after that, and rescalings of $\{\zeta_{v,i}\}_{v \in \mathbb{V}, i \in I_v}$ and $(t_{a,v,i})_{a \in \mathbb{N}, v \in \mathbb{V}, i \in I_v}$ such that

$$t_{a,v_1,i} \varepsilon_{a,e}^{s_{e,i}} = t_{a,v_2,i} \quad \forall i \in I_{v_1} \cap I_{v_2}, \quad a \gg 1, \quad (3.17)$$

$$t_{a,v_1,i} \varepsilon_{a,e}^{s_{e,i}} = 1 \quad \forall i \in I_{v_1} - I_{v_2}, \quad a \gg 1, \quad (3.18)$$

$$\prod_{i \in I_{v_1}} t_{a,v_1,i} = \prod_{i \in I_{v_2}} t_{a,v_2,i} \quad \forall v_1, v_2 \in \mathbb{V}, \quad a \gg 1. \quad (3.19)$$

Proof Throughout the proof we assume that the domains Σ_a are smooth; i.e. Γ' is a one vertex graph, $\mathbb{V}' = \{v'\}$, and thus $\mathbb{E} = \mathbb{E}^{\text{cut}}$. The argument in general reduces to this case by focusing on each component of Σ_a ; see the adjustments at end of the proof of [7, Prp 3.14]. We modify a given choice of

$$\{z_e\}_{e \in \mathbb{E}^{\text{cut}}}, \quad \{z_{a,e}\}_{e \in \mathbb{E}^{\text{cut}}, a \in \mathbb{N}}, \quad \{\zeta_{v,i}\}_{v \in \mathbb{V}, i \in I_v}, \quad (t_{a,v,i})_{a \in \mathbb{N}, v \in \mathbb{V}, i \in I_v},$$

to another set satisfying (3.17)–(3.19).

First, it follows from (3.15), (3.16), and (3.14), that

$$\prod_{i \in I_e} \frac{\eta_{e,i}}{\eta_{e,i}} = 1 \quad \forall e \in \mathbb{E};$$

i.e.

$$\left(\frac{\eta_{e,i}}{\eta_{e,i}} \right)_{i \in I_e} \in (\mathbb{C}^*)_{\bullet}^{I_e}.$$

Fix an orientation O on $\mathbb{E}$, and choose some branch

$$\eta = \bigoplus_{e \in O} \eta_e \in \bigoplus_{e \in \mathbb{E}} \mathbb{C}_{\bullet}^{I_e}, \quad \eta_e = (-\log(\eta_{e,i}/\eta_{e,i}))_{i \in I_e} \in \mathbb{C}_{\bullet}^{I_e} \quad \forall e \in O,$$

of the multi-valued function $\log$. By (3.14), for each $v \in \mathbb{V}$ and any $i \in I_v$, we can replace $\{t_{a,v,i}\}_{a \in \mathbb{N}}$ with another equivalent (in the sense of (3.8)) sequence such that

$$\lambda_a = \prod_{i \in I_v} t_{a,v,i} \quad \forall a \gg 1.$$

Then we will have

$$(t_{a,v,i})_{v \in \mathbb{V}, i \in I_v} \in \left(\prod_{v \in \mathbb{V}} \mathbb{C}_{\bullet}^{I_v} \right)^* \quad \forall a \gg 1.$$

By (3.15), (3.16), and definition of ϱ in (2.14) (via the chosen orientation O), we can choose the branches

$$\xi_a = ((-\log(\varepsilon_{a,e}))_{e \in \mathbb{E}}, (-\log(t_{a,v,i}))_{v \in \mathbb{V}, i \in I_v}) \in \mathbb{C}^{\mathbb{E}} \oplus \left(\bigoplus_{v \in \mathbb{V}} \mathbb{C}_{\bullet}^{I_v} \right)^* \quad \forall a \in \mathbb{N}$$

so that

$$\lim_{a \rightarrow \infty} \varrho_{\mathbb{C}}(\xi_a) = \eta.$$

By [7, Lmm 3.21] applied to $\varrho_{\mathbb{C}}$, there exists a sequence

$$(\xi'_a)_{a \in \mathbb{N}} \subset \mathbb{C}^{\mathbb{E}} \oplus \left(\bigoplus_{v \in \mathbb{V}} \mathbb{C}^{I_v} \right)^*$$

such that $\varrho_{\mathbb{C}}(\xi_a - \xi'_a) = 0$ for all $a \in \mathbb{N}$ and the limit $\lim_{a \rightarrow \infty} \xi'_a = \xi'$ exists. Taking the exponential of ξ' and ξ'_a , we find elements

$$((\alpha_e)_{e \in \mathbb{E}}, (\alpha_{v,i})_{v \in \mathbb{V}, i \in I_v}), ((\alpha_{a,e})_{e \in \mathbb{E}}, (\alpha_{a,v,i})_{v \in \mathbb{V}, i \in I_v})_{a \in \mathbb{N}} \in (\mathbb{C}^*)^{\mathbb{E}} \times \left(\prod_{v \in \mathbb{V}} (\mathbb{C}^*)^{I_v} \right)^*$$

such that

$$\lim_{a \rightarrow \infty} ((\alpha_{a,e})_{e \in \mathbb{E}}, (\alpha_{a,v,i})_{v \in \mathbb{V}, i \in I_v}) = ((\alpha_e)_{e \in \mathbb{E}}, (\alpha_{v,i})_{v \in \mathbb{V}, i \in I_v})$$

and

$$\frac{(\alpha_{a,v_1,i}^{-1} t_{a,v_1,i}) (\alpha_{a,e}^{-1} \varepsilon_{a,e})^{s_{e,i}}}{(\alpha_{a,v_2,i}^{-1} t_{a,v_2,i})} = 1 \quad \forall i \in I_{v_1} \cap I_{v_2}, a \in \mathbb{N}, \quad (3.20)$$

$$(\alpha_{a,v_1,i}^{-1} t_{a,v_1,i}) (\alpha_{a,e}^{-1} \varepsilon_{a,e})^{s_{e,i}} = 1 \quad \forall i \in I_{v_1} - I_{v_2}, a \in \mathbb{N}, \quad (3.21)$$

$$\alpha_a^{-1} \lambda_a := \prod_{i \in I_{v_1}} \alpha_{a,v_1,i}^{-1} t_{a,v_1,i} = \prod_{i \in I_{v_2}} \alpha_{a,v_2,i}^{-1} t_{a,v_2,i} \quad \forall v_1, v_2 \in \mathbb{V}, a \in \mathbb{N}. \quad (3.22)$$

By (3.20)–(3.22), for a sufficiently large, replacing

- $\{z_e\}_{e \in O}$ with $\{\alpha_e^{-1} z_e\}_{e \in O}$,
- $\{z_{a,e}\}_{e \in O}$ with $\{\alpha_{a,e}^{-1} z_{a,e}\}_{e \in O}$,
- $\{\varepsilon_{a,e}\}_{e \in \mathbb{E}}$ with $\{\alpha_{a,e}^{-1} \varepsilon_{a,e}\}_{e \in \mathbb{E}}$,
- $(t_{a,v,i})_{v \in \mathbb{V}, i \in I_v}$ with $(\alpha_{a,v,i}^{-1} t_{a,v,i})_{v \in \mathbb{V}, i \in I_v}$, and
- $(\zeta_{v,i})_{v \in \mathbb{V}, i \in I_v}$ with $(\alpha_{v,i} \zeta_{v,i})_{v \in \mathbb{V}, i \in I_v}$,

we get a new set of representatives satisfying (3.17), (3.18), and Definition 2.4.(1) with

$$\lambda(f_{\log}) := \prod_{i \in I_v} \alpha_{v,i} = \lim_{a \rightarrow \infty} \alpha_a \quad \text{for any } v \in \mathbb{V}.$$

□

In order to finish the proof of Proposition 3.6, by the lemma above, the modified set gives us a pre-log map equivalent to $f_{\log}$ that satisfies

$$\frac{\eta_{e,i}}{\eta_{e,i}} = 1 \quad \forall e \in \mathbb{E}^{\text{cut}}, i \in I_e.$$

By (2.9), we conclude that $\text{ob}_{\Gamma}(f_{\log}) = 1 \in \mathcal{G}$.

□

Proposition 3.6 applies to a sequence of stable maps in $\mathcal{Z}^*$. For the proof of Theorem 1.3, we also need to consider sequences in $\overline{\mathcal{M}}_{g,k}^{\log}(X_{\emptyset}, A)$ itself. In this case, the ambient smoothing $\mathcal{Z}$ is not needed and $X_{\emptyset}$ can be any d-semistable SNC variety. Suppose

$$f_{a,\log} := (u_{a,v'}, \zeta_{a,v'} = (\zeta_{a,v',i})_{i \in I_{v'}}, C_{v'})_{v' \in \mathbb{V}'} \quad a \in \mathbb{N} \quad (3.23)$$

is a sequence of log maps in $\overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A)$. Since the set of possible decorated graphs for each fixed (g, k, A) is finite, after passing to a subsequence, we may assume that (1) all $f_{a,\log}$ have the same decorated dual graph $\Gamma' = \Gamma(\mathbb{V}', \mathbb{E}', \mathbb{L}')$, and (2) the underlying stable maps $f_a \in \overline{\mathcal{M}}_{g,k}(X_\emptyset, A)$ Gromov-converge to a stable map

$$f = (u_v, C_v)_{v \in \mathbb{V}} \in \overline{\mathcal{M}}_{g,k}(X_\emptyset, A)$$

with dual graph $\Gamma = \Gamma(\mathbb{V}, \mathbb{E}, \mathbb{L})$. As in Lemma 3.3, we can find meromorphic sections $\{\zeta_{v,i}\}_{v \in \mathbb{V}, i \in I_v}$ such that

$$f_{\log} := (u_v, \zeta_v = (\zeta_{v,i})_{i \in I_v}, C_v)_{v \in \mathbb{V}} \in \mathcal{M}_{g,k}^{\text{plog}}(X_\emptyset, A)_\Gamma. \quad (3.24)$$

Repeating the proof of Proposition 3.6, with $\{\lambda(f_{a,\log})\}_{a \in \mathbb{N}}$ as in Definition 2.4.(1) in place of $\{\lambda_a\}_{a \in \mathbb{N}}$ in (3.2) yields the following.

Proposition 3.8 *Suppose (3.23) is a sequence of stable log maps in $\overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A)$ that log-Gromov converges to (3.24). The limit (3.24) is unique up to equivalence and represents an element of $\overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A)$.*

Proof of Theorem 1.3 Similarly to the classical case, consider the sequential convergence topologies on $\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}, A)$ or just $\overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A)$ given by Propositions 3.6 and 3.8: a subset W of the moduli space is closed if every sequence in W has a subsequence with a log-Gromov limit in W . Note that as in [27, Sec 5.1], we must show that convergence with respect to the topology defined above is equivalent to log-Gromov convergence. Since the forgetful map $\iota: \overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A) \rightarrow \overline{\mathcal{M}}_{g,k}(X_\emptyset, A)$ is finite-to-one and log-Gromov convergence is a lift of the classical Gromov convergence, this property follows from the corresponding statement for the Gromov convergence topology on $\overline{\mathcal{M}}_{g,k}(X_\emptyset, A)$. In other words, the five axioms⁸ in [27, Lmm 5.6.4] lift to sequences in $\overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A)$.

Suppose $W \subset \overline{\mathcal{M}}_{g,k}(X_\emptyset, A)$ or $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}, A)$ is closed and let $W' = \iota^{-1}(W)$. Let $(f_a, \log)_{a \in \mathbb{N}}$ be any sequence in W' . Its image $(f_a = \iota(f_a, \log))_{a \in \mathbb{N}}$ in W has a subsequence, still denoted by $(f_a)_{a \in \mathbb{N}}$, that Gromov converges to some $f \in W$. On the other hand, by Proposition 3.6 or 3.8, $(f_a, \log)_{a \in \mathbb{N}}$ has a subsequence that log-Gromov converges to some $f_{\log} \in \overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A)$ or $\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}, A)$. By definition, we have $\iota(f_{\log}) = f$, i.e. $f_{\log} \in W'$. Therefore, W' is closed. We conclude that ι is continuous.

Let $f_{\log}$ be an arbitrary log map in $\overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A)$ with the decorated dual graph Γ and $f = \iota(f_{\log})$ be the underlying stable map in $\overline{\mathcal{M}}_{g,k}(X_\emptyset, A)$. Let $(U_a)_{a \in \mathbb{N}}$ be a shrinking basis for the (metrizable) topology of $\overline{\mathcal{M}}_{g,k}(X_\emptyset, A)$ or $\overline{\mathcal{M}}_{g,k}(\mathcal{Z}, A)$ around f . By [7, Lmm 2.15], every stable map f admits at most finitely many log lifts $f_{\log}$, each of which is uniquely specified by the vector decorations on the nodes of its dual graph (i.e. the contact data s_e at the nodes q_e). Recall from the proof of Lemma 3.3 that, for a sufficiently large, by the classical gluing theorem, the domain of every map f' in U_a is obtained from the nodal domain Σ of f by gluing the nodes in a standard way. Furthermore, the image of f' is C^0 -close to the image of f . The dual graph Γ' of f' is a contraction of Γ in the sense of (3.4). With these identifications, if $f'_{\log}$ is a log lift of f' in U_a , by its decoration type, we mean

- the vector decorations s_e at its nodes q_e , together with
- the winding⁹ number of f' around X_i along the circles ∂A_e (see 3.10) on every neck A_e obtained from gluing the node q_e of the domain of f ; see the proof of Lemma 3.3.

⁸ Even though [27, Sec 5.1] is about the genus 0 moduli spaces, the statements used here are valid in all genus.

⁹ Contact points with X_∂ are among the nodal points and are away from the neck region.

Thus, we say $f'_{\log}$ has the same decoration type as $f_{\log}$ if

1. at every node of the domain of $f'_{\log}$ the vector decoration s_e is the same as the vector decoration at the corresponding node of $f_{\log}$,
2. on every neck A_e the winding number of f' around X_i along the circle ∂A_e is the same as the tangency order $s_{e,i}$ for f .

For a sufficiently large, define U'_a be the set of elements $f'_{\log}$ in $\overline{\mathcal{M}}^{\log}_{g,k}(X_{\emptyset}, A)$ or $\overline{\mathcal{M}}^{\log}_{g,k}(\mathcal{Z}, A)$ whose image f' under ι lies in U_a and $f'_{\log}$ has the same decoration type as $f_{\log}$. By (1) and (2) above, the restriction of ι to U'_a is one-to-one. We show that U'_a is open. Let $(f_{b,\log})_{b \in \mathbb{N}}$ be a sequence in the complement of U'_a that log-Gromov converges to $f'_{\log}$. After possibly passing to a subsequence, we can assume that the underlying sequence of stable maps $(f_b)_{b \in \mathbb{N}}$ lies either in U_a or its complement U_a^c . In the latter case, by definition, $f'_{\log}$ belongs to the complement of U'_a . In the former case, the decoration type of $f'_{\log}$ (with respect to $f_{\log}$) will be the same as the decoration type of $f_{b,\log}$ which is, by definition, different from the decoration type of $f_{\log}$. Therefore, $f'_{\log}$ belongs to the complement of U'_a . We conclude that U'_a is open. Furthermore, it is easy to see that $(U'_a)_{a \in \mathbb{N}}$ is a shrinking basis for the topology of $\overline{\mathcal{M}}^{\log}_{g,k}(X_{\emptyset}, A)$ or $\overline{\mathcal{M}}^{\log}_{g,k}(\mathcal{Z}, A)$ at f . Therefore, the log-Gromov topology on $\overline{\mathcal{M}}^{\log}_{g,k}(X_{\emptyset})$ or $\overline{\mathcal{M}}^{\log}_{g,k}(\mathcal{Z}, A)$ is first-countable.

Hausdorffness is the consequence of the uniqueness of the limit. If Y is a first-countable topological space and has the property that every convergent sequence has a unique limit then Y is Hausdorff. Finally, compactness of $\overline{\mathcal{M}}^{\log}_{g,k}(X_{\emptyset}, A)$ is the consequence of the existence of the limit. $\square$

4 Comments on deformation theory

In this section, we first calculate the expected dimension of each stratum

$$\mathcal{M}^{\log}_{g,k}(X_{\emptyset}, A)_{\Gamma} \subset \overline{\mathcal{M}}^{\log}_{g,k}(X_{\emptyset}, A)$$

and thus identify the virtually main components of $\overline{\mathcal{M}}^{\log}_{g,k}(X_{\emptyset}, A)$. We then describe the deformation-obstruction exact sequence at any log curve.

First, let us review the notion of logarithmic tangent bundle and set up the notation.

Let X be a smooth holomorphic manifold and $D \subset X$ be a normal crossings divisor. Around every point $p \in X$ there exists a chart $\varphi: U \rightarrow \mathbb{C}^n$ with coordinates $(x_1, \dots, x_n)$, with $n = \dim_{\mathbb{C}} X$, such that

$$\varphi(D \cap U) \equiv (x_1 \cdots x_k = 0) \subset \mathbb{C}^n \quad \text{for some } 0 \leq k \leq n.$$

In such coordinates, the sheaf $\mathcal{T}X$ of holomorphic sections of the complex tangent bundle TX is generated by

$$\partial_{x_1}, \dots, \partial_{x_n}$$

and the log tangent sheaf $\mathcal{T}X(-\log D)$ is the sub-sheaf generated by

$$\partial_{x_1}^{\log} := x_1 \partial_{x_1}, \dots, \partial_{x_k}^{\log} := x_k \partial_{x_k}, \quad \partial_{x_{k+1}}, \dots, \partial_{x_n}.$$

It is dual to the sheaf $\Omega_X^1(\log D)$ of meromorphic 1-forms with at most simple poles along D_i . Since $\mathcal{T}X(-\log D)$ is locally free, it is the sheaf of holomorphic sections of a holomorphic

vector bundle $TX(-\log D)$. The inclusion $TX(-\log D) \subset TX$ gives rise to a holomorphic homomorphism

$$\iota: TX(-\log D) \longrightarrow TX$$

which is an isomorphism away from D .

Now, suppose $X_\emptyset = \bigcup_{i \in \mathcal{I}} X_i$ is an SNC Kähler variety. For each $i \in \mathcal{I}$, let $TX_i(-\log \partial X_i)$ denote the logarithmic tangent bundle of the pair $(X_i, \partial X_i = X_\partial)$ defined above. If $X_\emptyset$ is d-semistable, then (it follows from [2, Thm 5.9] that) $X_\emptyset$ admits a natural holomorphic vector bundle $T^{\log} X_\emptyset$ such that

$$T^{\log} X_\emptyset|_{X_i} = TX_i(-\log \partial X_i).$$

In other words, the collection of logarithmic tangent bundles

$$\{TX_i(-\log \partial X_i)\}_{i \in \mathcal{I}}$$

naturally glue along the singular locus $X_\emptyset$ to define a vector bundle over $X_\emptyset$ that plays the role of tangent bundle for the central fiber. If $\mathcal{Z}$ is a semistable smoothing of $\mathcal{Z}_0 = X_\emptyset$ as in (1.1), then there is a short exact sequence

$$0 \longrightarrow T^{\log} X_\emptyset \longrightarrow T\mathcal{Z}(-\log X_\emptyset)|_{X_\emptyset} \longrightarrow \mathcal{O}_{X_\emptyset} \longrightarrow 0$$

meaning that the logarithmic normal bundle of $X_\emptyset$ in $\mathcal{Z}$ is the trivial line bundle $\mathcal{O}_{X_\emptyset}$; see [22, Prp 5.3, Exa 3.6]. Furthermore, via the holomorphic homomorphisms

$$T\mathcal{Z}(-\log X_\emptyset) \longrightarrow T\mathcal{Z} \quad \text{and} \quad T\mathbb{C}(-\log 0) \longrightarrow T\mathbb{C}$$

the derivative map

$$d\pi: T\mathcal{Z} \longrightarrow T\mathbb{C}$$

lifts to a surjective log derivative map

$$d^{\log} \pi: T\mathcal{Z}(-\log X_\emptyset) \longrightarrow T\mathbb{C}(-\log 0)$$

whose kernel over $\lambda \neq 0$ is $T\mathcal{Z}_\lambda$ and over $\lambda = 0$ is $T^{\log} \mathcal{Z}_0$. In this sense, $T^{\log} X_\emptyset$ can be considered as the smooth limit of $T\mathcal{Z}_\lambda$ when λ converges to 0. In local coordinates $x = (x_0, \dots, x_n)$ such that $\pi: \mathcal{Z} \longrightarrow \Delta$ is given by $x \longrightarrow z = x_0 \cdots x_k$, we have

$$d^{\log} \pi|_{\mathcal{Z}_0} (h_1 \partial_{x_0}^{\log} + \cdots + h_k \partial_{x_k}^{\log} + h_{k+1} \partial_{x_{k+1}} + \cdots + h_n \partial_{x_n}) = \left(\sum_{i=0}^k h_i \right) \partial_z^{\log} |_{z=0}. \quad (4.1)$$

Given $\mathcal{M}_{g,k}^{\log}(X_\emptyset, A)_\Gamma$, with notation as Remark 2.13, recall that both K and $K_\bullet$ are free $\mathbb{Z}$ -modules and

$$\dim K_{\mathbb{R}} = \text{rank } K = \dim K_{\bullet, \mathbb{R}} + 1 = \text{rank } K_\bullet + 1. \quad (4.2)$$

Lemma 4.1 *For any admissible decorated dual graph Γ , the expected complex dimension of $\mathcal{M}_{g,k}^{\log}(X_\emptyset, A)_\Gamma$ is*

$$c_1^{T^{\log} X_\emptyset}(A) + (n-3)(1-g) + k - \text{rank } K_\bullet. \quad (4.3)$$

Proof For each $i \in \mathcal{I}$,

$$\partial X_i := \bigcup_{j \in \mathcal{I} - i} X_{ij}$$

is an SNC divisor in X_i . For the given Γ , fix an arbitrary choice of indices $\{i_v \in I_v\}_{v \in \mathbb{V}}$. For each $v \in \mathbb{V}$, let Γ_v be the one-vertex graph $\{v\}$ with the labeling $I_v - \{i_v\} \subset \mathcal{I} - \{i_v\}$. Also, using the identification map $\mathbb{Z}_{\bullet}^{\mathcal{I}} \cong \mathbb{Z}^{\mathcal{I} - \{i_v\}}$ in (2.6), let $\mathfrak{s}_v$ be the set of contact vectors in $\mathbb{Z}^{\mathcal{I} - \{i_v\}}$ at the nodal points $\mathbb{E}_v$ together with the trivial contact vectors at the marked points $\vec{z}_v$. For every $f = (u_v, \zeta_v, C_v)_{v \in \mathbb{V}} \in \mathcal{M}_{g,k}^{\text{plog}}(X_{\emptyset}, A)_{\Gamma}$, by forgetting the i_v -th meromorphic section ζ_{v,i_v} in the v -th component (u_v, ζ_v, C_v) , we obtain element of the relative log space

$$\mathcal{M}_{g_v, \mathfrak{s}_v}^{\log}(X_{i_v}, \partial X_{i_v}, A_v)_{\Gamma_v} = \mathcal{M}_{g_v, \mathfrak{s}_v}^{\text{plog}}(X_{i_v}, \partial X_{i_v}, A_v)_{\Gamma_v}$$

constructed in [7]. Therefore, f belongs to the fiber product space

$$\times_{v \in \mathbb{V}} \mathcal{M}_{g_v, \mathfrak{s}_v}^{\log}(X_{i_v}, \partial X_{i_v}, A_v)_{\Gamma_v} \quad (4.4)$$

where the fiber product is over the evaluation maps into $X_{I_e} \times X_{I_e}$ at the pairs of nodal points (q_e, q_e) for all $e \in \mathbb{E}$. Then, ob_{Γ} in (2.22) is a map

$$\text{ob}_{\Gamma}: \mathcal{M}_{g,k}^{\text{plog}}(X_{\emptyset}, A)_{\Gamma} \cong \times_{v \in \mathbb{V}} \mathcal{M}_{g_v, \mathfrak{s}_v}^{\log}(X_{i_v}, \partial X_{i_v}, A_v)_{\Gamma_v} \longrightarrow \mathcal{G}(\Gamma)$$

such that

$$\mathcal{M}_{g,k}^{\log}(X_{\emptyset}, A)_{\Gamma} = \text{ob}_{\Gamma}^{-1}(1).$$

By [8, Prp 4.8], the complex expected dimension of each $\mathcal{M}_{g_v, \mathfrak{s}_v}^{\log}(X_{i_v}, \partial X_{i_v}, A_v)_{\Gamma_v}$ is

$$c_1^{TX_{i_v}(-\log \partial X_{i_v})}(A_v) + (n-3)(1-g_v) + k_v + |\mathbb{E}_v| - |I_v - \{i_v\}|, \quad \text{where } k_v = |\vec{z}_v|.$$

Since

$$c_1^{TX_{i_v}(-\log \partial X_{i_v})}(A_v) = c_1^{T^{\log} X_{\emptyset}}(A_v) \quad \forall v \in \mathbb{V},$$

the expected dimension of the fiber product (4.4) is equal to

$$\begin{aligned} & \sum_{v \in \mathbb{V}} (c_1^{T^{\log} X_{\emptyset}}(A_v) + (n-3)(1-g_v) + k_v + |\mathbb{E}_v| - |I_v| + 1) - \sum_{e \in \mathbb{E}} (n - |I_e| + 1) \\ &= c_1^{T^{\log} X_{\emptyset}}(A) + (n-3)(1-g) + k - |\mathbb{E}| - \sum_{v \in \mathbb{V}} (|I_v| - 1) + \sum_{e \in \mathbb{E}} (|I_e| - 1). \end{aligned} \quad (4.5)$$

By (2.14),

$$\text{rank } K - \dim_{\mathbb{C}}(\mathcal{G}) = |\mathbb{E}| + 1 + \sum_{v \in \mathbb{V}} (|I_v| - 1) - \sum_{e \in \mathbb{E}} (|I_e| - 1).$$

The identity (4.3) follows from the second equation in (4.5), the last equation, and (4.2). $\square$

Definition 4.2 We say an admissible decorated dual graph Γ is a main graph if

$$K_{\bullet} = \text{Ker}(\varrho_{\bullet}) = 0 \subset \mathbb{D}_{\bullet}, \quad (4.6)$$

Which is equivalent to $K = \text{Ker}(\varrho) \cong \mathbb{Z} \subset \mathbb{D}$.

The set of such Γ is the same as rigid configurations (rigid tropical map) considered in [3].

Corollary 4.3 *The moduli space $\overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A)$ has the correct expected dimension and the virtually main strata of $\overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A)$ correspond to main graphs.*

Unlike when the target space is smooth, often, $\overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A)$ has many virtually main strata, and they contribute differently to the VFC of $\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}_\lambda, A)$. The degeneration formula (5.3) describes the weights.

Over a smooth target X , the deformation-obstruction long exact sequence at a stable marked curve $f = [u, C = (\Sigma, \vec{z})] \in \mathcal{M}_{g,k}(X, A)$ (with smooth domain) is the sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{aut}(C) &\xrightarrow{\delta} \operatorname{Def}(u) \longrightarrow \operatorname{Def}(f) \longrightarrow \operatorname{Def}(C) \\ &\xrightarrow{\delta} \operatorname{Obs}(u) \longrightarrow \operatorname{Obs}(f) \longrightarrow 0, \end{aligned} \quad (4.7)$$

where

$$\begin{aligned} \operatorname{Aut}(C) &= H_{\vec{\partial}}^0(\Sigma, T\Sigma(-\log z)), & \operatorname{Def}(C) &= H_{\vec{\partial}}^1(\Sigma, T\Sigma(-\log z)), \\ \operatorname{Def}(u) &= H_{\vec{\partial}}^0(\Sigma, u^*TX), & \operatorname{Obs}(u) &= H_{\vec{\partial}}^1(\Sigma, u^*TX), \end{aligned} \quad (4.8)$$

and $T\Sigma(-\log z)$ is the logarithmic tangent bundle associated to the marked-points divisor $z \subset \Sigma$. Alternatively, we may replace $T\Sigma(-\log z)$ and TX with the corresponding sheaves of holomorphic sections $\mathcal{T}\Sigma(-\log z)$ and $\mathcal{T}X$, respectively, and use Čech cohomology. A similar description is feasible when Σ is nodal; see below. Furthermore, if u is an immersion with normal bundle

$$N_u = u^*TX/du(T\Sigma)$$

and there are no marked points, then

$$\operatorname{Def}(f) = H_{\vec{\partial}}^0(\Sigma, N_u) \quad \operatorname{Obs}(f) = H_{\vec{\partial}}^1(\Sigma, N_u).$$

If $\operatorname{Obs}(f) = 0$, then a small neighborhood $B(f)$ of f in $\mathcal{M}_{g,k}(X, A)$ is a smooth orbifold of the expected dimension (1.3); see [20, Sec 24.1] and [10, Rmk 6.2.1]. In the following, we outline the generalization of this setup to the case of analytical log maps.

Remark 4.4 In (4.7), if u is not an immersion or there are marked points, then the cokernel sheaf

$$\mathcal{N}_u = \frac{u^*\mathcal{T}X}{du(\mathcal{T}\Sigma(-\log z))} \quad (4.9)$$

admits a decomposition

$$\mathcal{N}_u = \mathcal{N}_u^{\text{free}} \oplus \mathcal{N}_u^{\text{tor}}$$

into a direct sum of a torsion free sheaf with the associated holomorphic vector bundle N_u and a skyscraper sheaf such that

$$\operatorname{Def}(f) = H_{\vec{\partial}}^0(\Sigma, N_u) \oplus H^0(\Sigma, \mathcal{N}_u^{\text{tor}}) \quad \text{and} \quad \operatorname{Obs}(f) = H_{\vec{\partial}}^1(\Sigma, N_u);$$

see [37, pp 284–285].

Remark 4.5 In the algebraic language, the cohomology groups in (4.8) are described as

$$\begin{aligned} \operatorname{Aut}(C) &= \operatorname{Hom}(\Omega_{\Sigma}^1(\log z), \mathcal{O}_{\Sigma}), & \operatorname{Def}(C) &= \operatorname{Ext}^1(\Omega_{\Sigma}^1(\log z), \mathcal{O}_{\Sigma}), \\ \operatorname{Def}(u) &= \operatorname{Hom}(u^*\Omega_X^1, \mathcal{O}_{\Sigma}), & \operatorname{Obs}(u) &= \operatorname{Ext}^1(u^*\Omega_X^1, \mathcal{O}_{\Sigma}). \end{aligned}$$

Lemma 4.6 Associated with any pre-log map $f \equiv (f_v \equiv (u_v, \zeta_v, C_v))_{v \in \mathbb{V}}$ (with notation as in 2.10), there exists a natural holomorphic¹⁰ homomorphism derivative map

$$T^{\log} \Sigma(-\log z) \xrightarrow{d(u, \zeta)} u^* T^{\log} X_{\emptyset}$$

that generalizes the derivative map in (4.9).

Proof The map $d(u, \zeta)$ is defined in the following way. Suppose $p \in \Sigma_v$ is not a marked or nodal point. Then $u_v(p) \in X_{I_v} - \partial X_{I_v}$ and $\zeta_{v,i}(p) \neq 0, \infty$ for all $i \in I_v$. If $I_v = \{i_1, \dots, i_k\}$, a neighborhood V of $u_v(p)$ in $X_{\emptyset}$ can be identified with a neighborhood U of 0 in the affine variety

$$(x_1 \dots x_k = 0) \subset \mathbb{C}^{n+1}$$

so that

$$(U \cap \{x_a = 0\}) \cong (V \cap X_{i_a}) \quad \forall a = 1, \dots, k.$$

The coordinates x_a also give local trivializations

$$\mathcal{N}_{X_{I_v-i_a}} X_{I_v} |_{(X_{I_v} \cap V)} \cong (X_{I_v} \cap V) \times \mathbb{C}, \quad \forall a = 1, \dots, k; \quad (4.10)$$

see (3.1). The sections $\xi_{I_v,i}$ in (2.3) (which are unique up to scaling) give (unique up to scaling) trivializations

$$\mathcal{O}_{X_{I_v}}(X_{I_v+i}) |_{X_{I_v} \cap V} \cong (X_{I_v} \cap V) \times \mathbb{C} \quad \forall i \in \mathcal{I} - I_v. \quad (4.11)$$

Using (4.11), the given trivialization in (2.1) thus gives us a trivialization

$$\bigotimes_{i \in I_v} \mathcal{N}_{X_{I_v-i}} X_{I_v} |_{(X_{I_v} \cap V)} \cong (X_{I_v} \cap V) \times \mathbb{C}. \quad (4.12)$$

We can choose the coordinates $x_1, \dots, x_k$ so that the product of the trivializations in (4.10) is equal to the trivialization in (4.12). In such coordinates,

$$T^{\log} X_{\emptyset} |_{(V \cap X_{I_v})} = \left\{ \sum_{a=1}^k h_a \partial_{x_a}^{\log} + \sum_{a=k+1}^{n+1} h_a \partial_{x_a} \mid h_1 + \dots + h_k = 0 \right\}; \quad (4.13)$$

see (4.1). For a local coordinate w on an open set Δ around p , with $p = (w = 0)$, define

$$d(u, \zeta)(\partial_w) = \sum_{a=1}^k \frac{\partial \zeta_{v,i_a}}{\partial w} \partial_{x_a}^{\log} + \sum_{a=k+1}^{n+1} \frac{\partial (x_a \circ u_v)}{\partial w} \partial_{x_a}. \quad (4.14)$$

Here $\partial \zeta_{v,i} / \partial w$ is defined using the local trivialization (4.10). It is clear from the definition that (4.14) is invariant under constant rescalings of $\zeta_{v,i}$. Also, it follows from Definition 2.2(a) and (4.12) that (4.14) satisfies (4.13). If p is one of the marked points, then the local generator of $T^{\log} \Sigma(-\log z)$ is $\partial_w^{\log} = w \partial w$. Therefore, we define

$$d(u, \zeta)(\partial_w^{\log}) = w d(u, \zeta)(\partial_w).$$

It is easy to check that

$$d_p(u, \zeta): T_p^{\log} \Sigma(-\log z) \longrightarrow T_{u_v(p)}^{\log} X_{\emptyset}$$

¹⁰ i.e. the restriction of $d(u, \zeta)$ to each irreducible component of Σ is a holomorphic homomorphism.

is independent of the choice of local coordinates w and $x_1, \dots, x_{n+1}$ used in driving (4.14) and $d_p(u, \zeta)$ is the zero homomorphism at the marked points.

Now suppose $q_e \in \Sigma$ is a nodal point obtained from attaching Σ_v at q_e to $\Sigma_{v'}$ at $q_{e'}$. Then

$$u(q_e) = u_v(q_e) = u_{v'}(q_e) \in X_{I_e} - \partial X_{I_e}, \quad \text{with } I_e = I_v \cup I_{v'}.$$

Fix local coordinates w_e on Δ_e around q_e and $w_{e'}$ on $\Delta_{e'}$ around $q_{e'}$. Then, as in (4.13), $T^{\log} \Sigma(-\log z)$ is generated by $\partial_{w_e}^{\log}$ around q_e and $\partial_{w_{e'}}^{\log}$ around $q_{e'}$ satisfying

$$\partial_{w_e}^{\log} |_{q_e} = -\partial_{w_{e'}}^{\log} |_{q_{e'}}. \quad (4.15)$$

As before, if $I_e = \{i_1, \dots, i_k\}$, a neighborhood V of $u(q_e)$ in $X_\emptyset$ can be identified with a neighborhood U of 0 in the affine variety

$$(x_1 \dots x_k = 0) \subset \mathbb{C}^{n+1}$$

such that

$$(U \cap \{x_a = 0\}) = (V \cap X_{i_a}), \quad \forall a = 1, \dots, k.$$

As before, the coordinates x_a also give local trivializations (4.10) on $X_{I_v} \cap V$ and $X_{I_{v'}} \cap V$ and the sections $\{\xi_{I_v, i}\}_{i \in I - I_e}$ and $\{\xi_{I_{v'}, i}\}_{i \in I - I_e}$ give compatible (over X_{I_e}) local trivializations

$$\begin{aligned} \mathcal{O}_{X_{I_v}}(X_{I_v+i})|_{X_{I_v} \cap V} &\cong (X_{I_v} \cap V) \times \mathbb{C} \quad \text{and} \\ \mathcal{O}_{X_{I_{v'}}}(X_{I_{v'}+i})|_{X_{I_{v'}} \cap V} &\cong (X_{I_{v'}} \cap V) \times \mathbb{C} \quad \forall i \in I - I_e. \end{aligned} \quad (4.16)$$

Using (4.16), the given trivialization in (2.1) thus gives us compatible trivializations

$$\left(\bigotimes_{i \in I_v} \mathcal{N}_{X_{I_v-i}} X_{I_v} \otimes \bigotimes_{i \in I_e - I_v} \mathcal{O}_{X_{I_v}}(X_{I_v+i}) \right) |_{(X_{I_v} \cap V)} \cong (X_{I_v} \cap V) \times \mathbb{C}. \quad (4.17)$$

We can choose the coordinates $x_1, \dots, x_k$ and local trivializations

$$\begin{aligned} \mathcal{O}_{X_{I_v}}(X_{I_v+i})|_{X_{I_v} \cap V} &\cong (X_{I_v} \cap V) \times \mathbb{C} \quad \forall i \in I_e - I_v, \\ \mathcal{O}_{X_{I_{v'}}}(X_{I_{v'}+i})|_{X_{I_{v'}} \cap V} &\cong (X_{I_{v'}} \cap V) \times \mathbb{C} \quad \forall i \in I_e - I_{v'} \end{aligned} \quad (4.18)$$

so that

$$\zeta_{I_v, i_a} = x_a \quad \forall i_a \in I_e - I_v \quad \text{and} \quad \zeta_{I_{v'}, i_a} = x_a \quad \forall i_a \in I_e - I_{v'},$$

with respect to (4.18), and the trivializations

$$\begin{aligned} \bigotimes_{i \in I_v} \mathcal{N}_{X_{I_v-i}} X_{I_v} |_{(X_{I_v} \cap V)} &\cong (X_{I_v} \cap V) \times \mathbb{C}, \\ \bigotimes_{i \in I_{v'}} \mathcal{N}_{X_{I_{v'}-i}} X_{I_{v'}} |_{(X_{I_{v'}} \cap V)} &\cong (X_{I_{v'}} \cap V) \times \mathbb{C}, \end{aligned} \quad (4.19)$$

obtained from (4.17) and (4.18) coincide with the product trivializations given by (4.10). Then, define

$$d(u, \zeta)(\partial_{w_e}^{\log}) = \sum_{i_a \in I_v} w_e \frac{\partial \zeta_{v, i_a}}{\partial w_e} \partial_{x_a}^{\log} + \sum_{i_a \in I_e - I_v} w_e \frac{\partial (x_a \circ u_v)}{\partial w_e} \partial_{x_a}^{\log} + \sum_{a=k+1}^{n+1} w_e \frac{\partial (x_a \circ u_v)}{\partial w_e} \partial_{x_a} \quad (4.20)$$

and

$$d(u, \zeta)(\partial_{w_\epsilon}^{\log}) = \sum_{i_a \in I_V} w_\epsilon \frac{\frac{\partial \zeta_{v, i_a}}{\partial w_\epsilon}}{\zeta_{i_a}} \partial_{x_a}^{\log} + \sum_{i_a \in I_e - I_V} w_\epsilon \frac{\frac{\partial (x_a \circ u_v)}{\partial w_\epsilon}}{x_a \circ u_v} \partial_{x_a}^{\log} + \sum_{a=k+1}^{n+1} w_\epsilon \frac{\partial (x_a \circ u_v)}{\partial w_\epsilon} \partial_{x_a}^{\log}. \quad (4.21)$$

Putting w_{ϵ_v} and w_ϵ equal to zero in (4.20) and (4.21), respectively, we get

$$d_{q_{\epsilon_v}}(u, \zeta)(\partial_{w_{\epsilon_v}}^{\log}) = \sum_{a=1}^k s_{\epsilon_v, i_a} \partial_{x_a}^{\log} \quad \text{and} \quad d_{q_\epsilon}(u, \zeta)(\partial_{w_\epsilon}^{\log}) = \sum_{a=1}^k s_{\epsilon, i_a} \partial_{x_a}^{\log}.$$

By Definition 2.4.3 and (4.15), the map

$$d_{q_\epsilon}(u, \zeta): T_{q_{\epsilon_v}}^{\log} \Sigma(-\log z) \longrightarrow T_{u(q_\epsilon)}^{\log} X_\emptyset$$

is well-defined. $\square$

Similarly to (4.7), for every stable log marked curve $f \equiv [f_v \equiv (u_v, \zeta_v, C_v)]_{v \in \mathbb{V}} \in \overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A)$, one can use the short exact sequence of sheaves

$$T^{\log} \Sigma(-\log z) \xrightarrow{d(u, \zeta)} u^* T^{\log} X_\emptyset \longrightarrow \mathcal{N}_{u, \zeta} = \frac{u^* T^{\log} X_\emptyset}{d(u, \zeta)(T^{\log} \Sigma(-\log z))} \quad (4.22)$$

to study the deformations-obstructions of f . It can also be deduced from (4.22) that the expected $\mathbb{C}$ -dimension of $\overline{\mathcal{M}}_{g,k}(X_\emptyset, A)$ is

$$c_1^{T^{\log} X_\emptyset}(A) + (n-3)(1-g) + k.$$

Remark 4.7 An element $\xi \in \text{Def}(u, \zeta)$ is a continuous section of $u^* T^{\log} X_\emptyset$ such that $\xi_v = \xi|_{\Sigma_v}$ is a holomorphic section of the vector bundle $u_v^* T X_i(-\log \partial X_i)$ for all $v \in \mathbb{V}$ and any $i \in I_v$. While the map $d(u, \zeta)$ is defined for arbitrary pre-log map, by the continuity of ξ at the nodes, the deformation space $\text{Def}(u, \zeta)$ only consists of those infinitesimal deformations of (u, ζ) that preserve Conditions (C1) and (C2) in Definition 2.10.

5 The degeneration formula

In this section, we describe an explicit (degeneration) formula for the contributions of the virtually main components of $\overline{\mathcal{M}}_{g,k}^{\log}(Z_0, A)$ in Corollary 4.3 to its (hypothetical) VFC. Assume $X_\emptyset = \bigcup_{i \in \mathcal{I}} X_i$ is a d -semistable SNC Kähler variety. For each $g, k \in \mathbb{N}$ and $A \in H_2(X_\emptyset, \mathbb{Z})$, the moduli space $\overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A)$ decomposes into a union of (virtually) main components

$$\overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A) = \bigcup_{\text{main } \Gamma} \overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A)_\Gamma. \quad (5.1)$$

By Definition 4.2 and (4.2), if Γ is a main graph, up to scaling, there is a unique pair of functions $(s: \mathbb{V} \rightarrow \mathbb{R}^{\mathcal{I}}, \lambda: \mathbb{E} \rightarrow \mathbb{R}_+)$ satisfying Definition 2.10.(C1). The condition (4.6)

implies that the image of the dual $\mathbb{Z}$ -linear map

$$\mathbb{T}^\vee \xrightarrow{\varrho_\bullet^\vee} \mathbb{D}_\bullet^\vee \quad (5.2)$$

is a sub-lattice of finite index. Let

$$m(\Gamma) := |\mathbb{D}_\bullet^\vee / \text{Im}(\varrho_\bullet^\vee)| \in \mathbb{Z}_+.$$

The degeneration formula (5.3) below proposes that the decomposition (5.1) gives rise to a similar but weighted decomposition formula for VFCs in a semistable degeneration.

Expectation 5.1 (Prospect Degeneration Formula) Let $\pi : \mathcal{Z} \longrightarrow \Delta$ be a semistable degeneration as in (1.1). Then, for every $A \in H_2(\mathcal{Z}_0, \mathbb{Z})$ and $g, k \in \mathbb{N}$, we have

$$[\overline{\mathcal{M}}_{g,k}(\mathcal{Z}_\lambda, A)]^{\text{VFC}} = \sum_{\text{main } \Gamma} \frac{m(\Gamma)}{|\mathfrak{Aut}(\Gamma)|} [\overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A)_\Gamma]^{\text{VFC}}, \quad (5.3)$$

where $|\mathfrak{Aut}(\Gamma)|$ is the order of the automorphism group of the decorated dual graph Γ .

The equality (5.3) should be thought of as an equality of Čech cohomology classes in $\overline{\mathcal{M}}_{g,k}^{\log}(\mathcal{Z}, A)$ in the sense of [28, Rmk 8.2.4]. The formula (5.3) and the decomposition formula [3, Thm 1.2] are both a sum over the same set of combinatorial data, but with different coefficients; see Remark 5.5 below. The difference in the coefficients shows that an algebraic log map is finer¹¹ than what is defined in this paper.

Lemma 5.2 *In the case of a basic degeneration, i.e. $\mathcal{I} = \{1, 2\}$, this formula coincides with the Jun Li's formula [25, Theorem 3.15].*

Proof It is easy to see that the only decorated graphs with $K_\bullet(\Gamma) = 0$ are bipartite graphs with one set of vertices $\mathbb{V}_1 \subset \mathbb{V}$ indexed by $\{1\}$ and the opposite set $\mathbb{V}_2 \subset \mathbb{V}$ indexed by $\{2\}$. Let us orient the edges to go from $\mathbb{V}_1$ to $\mathbb{V}_2$. Write $s_e = (-m_e, m_e) \neq 0 \in \mathbb{Z}_\bullet^2$ for all $e \in \mathbb{E}_{\mathbb{V}_1, \mathbb{V}_2}$. For each $v \in \mathbb{V}$, since $|I_v| = 1$, we have $\mathbb{Z}_\bullet^{I_v} = 0$. For each $e \in \mathbb{E}$, we have $I_e = \{1, 2\}$. By picking the second coordinate, we can identify $\mathbb{Z}_\bullet^{I_e}$ with $\mathbb{Z}$. Therefore, we have

$$\varrho_\bullet : \mathbb{D}_\bullet \cong \mathbb{Z}^\mathbb{E} \longrightarrow \mathbb{T} \cong \mathbb{Z}^\mathbb{E}, \quad (\lambda_e)_{e \in \mathbb{E}} \longrightarrow (\lambda_e m_e)_{e \in \mathbb{E}}.$$

We conclude that $m(\Gamma) = \prod_{e \in \mathbb{E}} m_e$ as in Jun Li's formula. $\square$

Remark 5.3 In the case of a basic degeneration, $\overline{\mathcal{M}}_{g,k}^{\log}(X_\emptyset, A)_\Gamma$ more or less coincides with the relative compactification (see [7, Sec 4.1]), and the degeneration formula is the same as the one stated in [14, Thm 1.1]. The analytic tools needed for the proof are contained in [14] and [6].

Unlike the basic case above, a main decorated dual graph for an SNC variety with non-trivial threefold (and higher) strata may have components mapped into a stratum X_I with $|I| \geq 2$; see Sect. 6.

A main step in establishing (5.3) is to prove a gluing theorem for smoothing the nodes of a log map f as in Definition 2.10 to get maps in $\mathcal{Z}_\lambda$ with $\lambda \neq 0$. The space of gluing parameters

¹¹ We mean there should be a (surjective) map from (the topological space underlying) AGCS's moduli space to the moduli space (set) defined in this paper.

for a fixed log map f with the decorated dual graph Γ is a sufficiently small neighborhood of the origin in

$$\begin{aligned} \mathcal{N}_\Gamma = & \left\{ ((\varepsilon_e)_{e \in \mathbb{E}}, (t_{v,i})_{v \in \mathbb{V}, i \in I_v}) \in \mathbb{C}^\mathbb{E} \times \prod_{v \in \mathbb{V}} \mathbb{C}^{I_v} : \prod_{i \in I_v} t_{v,i} = \prod_{i \in I_{v'}} t_{v',i} \right. \\ & \text{and } \varepsilon_e^{s_{e,i}} t_{v,i} = t_{v',i} \forall v, v' \in \mathbb{V}, e \in \mathbb{E}_{v,v'}, i \in I_e, e \text{ s.t. } s_{e,i} \geq 0 \left. \right\} \\ & \subset \mathbb{C}^\mathbb{E} \times \prod_{v \in \mathbb{V}} \mathbb{C}^{I_v}. \end{aligned} \quad (5.4)$$

In (5.4), if $i \in I_e - I_v$, by $t_{v,i}$ we mean 1. The complex numbers ε_e are the gluing parameters for the nodes of Σ and $t_{v,i}$ are the parameters for pushing u_v out in the direction of $\zeta_{v,i}$. The common value $\lambda = \prod_{i \in I_v} t_{v,i}$ describes the fiber Z_λ that will contain the glued map. In other words, the projection map $\pi: \mathcal{N}_\Gamma \rightarrow \mathbb{C}$ induced by $\pi: \mathcal{Z} \rightarrow \mathbb{C}$ is the map

$$((\varepsilon_e)_{e \in \mathbb{E}}, (t_{v,i})_{v \in \mathbb{V}, i \in I_v}) \rightarrow \lambda = \prod_{i \in I_v} t_{v,i}.$$

Let

$$\varrho^\vee: \mathbb{T}^\vee \rightarrow \mathbb{D}^\vee$$

denote the dual of (2.14). The image of $\varrho^\vee$ is a finite index sub-lattice of

$$\mathbb{K}^\perp = \{\eta \in \mathbb{D}^\vee : \eta(x) = 0 \forall x \in K\} \subset \mathbb{D}^\vee.$$

Define

$$m_{\text{red}}(\Gamma) = |\mathbb{K}^\perp / \text{Im}(\varrho^\vee)| \in \mathbb{Z}_+.$$

By Definition 4.2, if Γ is a main graph, then $\mathbb{K} = \text{Ker}(\varrho) \cong \mathbb{Z}$. Thus, the projection map

$$0 \oplus \pi: \mathbb{K} \cong \mathbb{Z} \rightarrow \mathbb{Z}$$

in (2.26) has the form $r \rightarrow m_\sigma r$ for some positive integer m_σ .

Lemma 5.4 *The space of gluing parameters $\mathcal{N}_\Gamma$ is a possibly non-irreducible and non-reduced affine toric sub-variety of $\mathbb{C}^\mathbb{E} \times \prod_{v \in \mathbb{V}} \mathbb{C}^{I_v}$ that is isomorphic to $m_{\text{red}}(\Gamma)$ copies of the irreducible reduced affine toric variety $Y_{\sigma(\Gamma)}$ (c.f. the paragraph after 2.23), counting with multiplicities. In particular, if Γ is a main graph, then $Y_\sigma \cong \mathbb{C}$ and $\pi: \mathcal{N}_\Gamma \rightarrow \mathbb{C}$ is a map of degree*

$$m(\Gamma) = m_{\text{red}}(\Gamma) \cdot m_\sigma,$$

where m_σ is the degree of $\pi: Y_\sigma \rightarrow \mathbb{C}$.

Proof Proof of the first claim is identical to the proof of [7, Prp 5.7]. We skip it here. For the second part, if Γ is a main graph, from the dual of the commutative diagram (2.26), we get the exact sequence

$$0 \rightarrow \text{ker}(\varrho^\vee) \rightarrow \text{Ker}(\varrho_\bullet^\vee) \xrightarrow{\delta} \mathbb{Z}^\vee \cong \mathbb{Z} \rightarrow \frac{\mathbb{D}^\vee}{\text{Im}(\varrho^\vee)} \rightarrow \frac{\mathbb{D}_\bullet^\vee}{\text{Im}(\varrho_\bullet^\vee)} \cong \frac{\mathbb{Z}}{m(\Gamma)\mathbb{Z}}.$$

The image of δ can not be the entire $\mathbb{Z}$. Therefore, it should be zero, and we get the exact sequence

$$\mathbb{Z} \longrightarrow \frac{\mathbb{D}^\vee}{\text{Im}(\varrho^\vee)} \longrightarrow \frac{\mathbb{Z}}{m(\Gamma)\mathbb{Z}}.$$

On one hand, the degree of $\pi: \mathcal{N}_\Gamma \longrightarrow \mathbb{C}$ is the index of the first inclusion map above which, by the exactness of the sequence, is equal to $m(\Gamma)$. On the other hand, the decomposition

$$\frac{\mathbb{D}^\vee}{\text{Im}(\varrho^\vee)} = \frac{\mathbb{D}^\vee}{K^\perp} \times \frac{K^\perp}{\text{Im}(\varrho^\vee)}$$

shows that $m(\Gamma)$ is the product of m_σ and $m_{\text{red}}(\Gamma)$. $\square$

Remark 5.5 Except for the coefficients $m(\Gamma)$, the degeneration formula (5.3) coincides with [3, Thm 1.2]. For instance, in the example of a basic degeneration (Lemma 5.2), with notation as above, since Γ is connected, we have

$$(\mathbb{Z}^\vee)^* = \{(t, 0)_v\}_{v \in \mathbb{V}_1} \oplus \{(0, t)_v\}_{v \in \mathbb{V}_2} : t \in \mathbb{Z} \cong \mathbb{Z}$$

and

$$\varrho: \mathbb{D} = \mathbb{Z}^\mathbb{E} \oplus (\mathbb{Z}^\vee)^* \cong \mathbb{Z}^\mathbb{E} \oplus \mathbb{Z} \longrightarrow \mathbb{T} = \bigoplus_{e \in \mathbb{E}} \mathbb{Z} \cdot (-1, 1) \cong \mathbb{Z}^\mathbb{E},$$

$$\varrho((\lambda_e)_{e \in \mathbb{E}}, t) = \bigoplus_{e \in \mathbb{E}} (\lambda_e m_e - t).$$

Let

$$\ell = \text{LCM}(m_e : e \in \mathbb{E}) \quad \forall e \in \mathbb{E}.$$

We get

$$\mathbb{Z} \cong \text{Ker}(\varrho) = \{((\lambda_e)_{e \in \mathbb{E}}, t) : t = \lambda_e m_e\} = \mathbb{Z} \cdot ((\ell_e)_{e \in \mathbb{E}}, \ell).$$

Since $0 \oplus \pi: \mathbb{D} \rightarrow \mathbb{Z}$ maps $((\ell_e)_{e \in \mathbb{E}}, \ell)$ to $\ell \in \mathbb{Z}$, we conclude that

$$m_\sigma = \text{LCM}(m_e : e \in \mathbb{E}) \quad \text{and} \quad m_{\text{red}}(\Gamma) = \frac{\prod_{e \in \mathbb{E}} m_e}{\text{LCM}(m_e : e \in \mathbb{E})}.$$

The multiplicity m_τ in [3, Thm 1.2] is the integer m_σ in Lemma 5.4; the relation between [3, Thm 1.2] and Jun Li's formula is explained in [23].

6 Rational curves in a pencil of cubic surfaces

In this section, we re-study the example of the degeneration of degree 3 rational curves in a pencil of cubic surfaces, originally studied in [3, Sec 5]. As our calculations show, we can easily identify the space of such log maps without blowing up the triple intersection.

Let P be a homogenous cubic polynomial in $x_0, \dots, x_3$ and

$$\mathcal{Z}' = \{(\lambda, [x_0, x_1, x_2, x_3]) \in \mathbb{C} \times \mathbb{P}^3 : x_1 x_2 x_3 = \lambda P(x_0, x_1, x_2, x_3)\} \subset \mathbb{C} \times \mathbb{P}^3.$$

Let $\pi': \mathcal{Z}' \longrightarrow \mathbb{C}$ be the projection map to the first factor. For a generic P and $\lambda \neq 0$, $\pi'^{-1}(\lambda)$ is a smooth cubic hypersurface (divisor) in $\mathbb{P}^3$. For $\lambda = 0$, $\pi'^{-1}(0)$ is the SNC variety

$$X'_\emptyset = \{0\} \times (X'_1 \cup X'_2 \cup X'_3) \subset \{0\} \times \mathbb{P}^3 \quad \text{with}$$

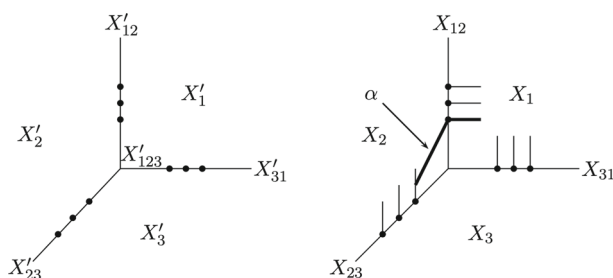


Fig. 4 On left, the central fiber of $\mathcal{Z}'$ with its 9 singular points. On right, the central fiber of $\mathcal{Z}$ with its 9 exceptional curves

$$X'_i \equiv (x_i = 0) \approx \mathbb{P}^2 \quad \forall i \in \{1, 2, 3\}, \quad X'_{ij} \equiv X'_i \cap X'_j \approx \mathbb{P}^1 \quad \forall i, j \in \{1, 2, 3\}, \quad i \neq j.$$

However, the total space $\mathcal{Z}'$ of π' is not smooth at the 9 points of

$$\mathcal{Z}'_{\text{sing}} \equiv \{0\} \times (X'_\emptyset \cap (P=0)) \subset X_\emptyset, \quad \text{where } X'_\emptyset = X'_{12} \cup X'_{13} \cup X'_{23} \subset \mathbb{P}^3.$$

A small Kähler resolution $\mathcal{Z}$ of $\mathcal{Z}'$ can be obtained by blowing up each singular point on X'_{ij} in either¹² X'_i or X'_j . The map π' then induces a projection $\pi: \mathcal{Z} \rightarrow \Delta$ and defines a semistable degeneration. Every fiber of π over $\mathbb{C}^*$ is a smooth cubic surface. The central fiber $\pi^{-1}(0)$ is the SNC variety $X_\emptyset \equiv X_1 \cup X_2 \cup X_3$ with 3 smooth components (i.e. $N=3$), each a blowup of $\mathbb{P}^2$ at some number of points. If each singular point on X'_{ij} is blown up in X'_i with $i < j$, then $\mathcal{Z}$ is obtained from $\mathcal{Z}'$ through two global blowups of $\mathbb{C} \times \mathbb{P}^3$ and is thus projective; see Fig. 4.

For the count of degree 1 and degree 2 rational curves in $\mathcal{Z}_\lambda$, it can be shown that all the limiting curves are of the classical type (i.e. they do not pass through the triple intersection). For example, the broken curve α in Fig. 4 is one of the 27 degree 1 rational curves in the limit. For each $\lambda \neq 0$, the moduli space $\overline{\mathcal{M}}_{0,2}(\mathcal{Z}_\lambda, [3])$ of 2-marked genus 0 degree 3 maps in $\mathcal{Z}_\lambda$ is of the (expected) complex dimension 4. In degree 3, for generic λ , there are 84 such curves passing through 2 generic points of $\mathcal{Z}_\lambda$ at the marked points. In the limiting SNC variety $X_\emptyset$, assuming that the two point constraints move to X_1 and X_2 , 81 of these 84 maps can be identified among the maps that do not intersect X_{123} . There is, however, a new type of main graph Γ contributing to the degeneration formula (5.3) that has no analogue in the Jun Li's formula. We are going to describe this Γ , identify the space of log maps $\mathcal{M}_{0,2}^{\log}(X_\emptyset, [3])_\Gamma$, and calculate the coefficient $m(\Gamma)$.

Let Γ be the graph with the set of vertices $\mathbb{V} = \{v_0, v_1, v_2, v_3\}$ and the set of edges $\mathbb{E} = \{e_1, e_2, e_3\}$ such that e_i connects v_0 and v_i , for all $i = 1, 2, 3$. Choose the orientations $\vec{e}_i$ to end at v_0 , for all $i = 1, 2, 3$, and assume

$$\begin{aligned} I_{v_0} = \mathcal{I} = \{1, 2, 3\}, \quad s_{\vec{e}_1} = (-2, 1, 1) \in \mathbb{Z}_\bullet^3, \quad s_{\vec{e}_2} = (1, -2, 1) \in \mathbb{Z}_\bullet^3, \\ s_{\vec{e}_3} = (1, 1, -2) \in \mathbb{Z}_\bullet^3, \quad I_{v_i} = \{i\}, \quad A_{v_i} = [1] \in H_2(X_i, \mathbb{Z}), \quad \forall i = 1, 2, 3, \end{aligned}$$

where, for each $i = 1, 2, 3$, $[1] \in H_2(X_i, \mathbb{Z})$ is the pre-image in X_i of the class of a line in X'_i away from the blow-up points. The two legs corresponding to the two marked points are attached to v_1 and v_2 ; see Fig. 5. A log curve with this dual graph is made of

- a line $\ell_i = \text{Im}(u_{v_i})$ in $X'_i \cong \mathbb{P}^2$ passing through the point¹³ X'_{123} for each $i \in \{1, 2, 3\}$, and

¹² Not all choices result in a Kähler configuration.

¹³ The choice of resolution and the exceptional curves are irrelevant in the following calculations.

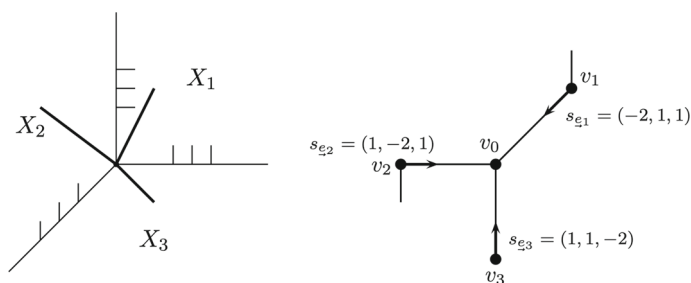


Fig. 5 Dual graph Γ and the image of a map belonging to $\overline{\mathcal{M}}_{0,2}(X_\emptyset, [3])_\Gamma$ in $X_\emptyset$

- a log tuple

$$\left(u_{v_0}, \{\zeta_j\}_{j \in \{1,2,3\}}, \Sigma_{v_0} \cong \mathbb{P}^1, q_{v_0} = \{q_{e_j}\}_{j \in \{1,2,3\}}\right)$$

such that

1. u_{v_0} is the constant map onto X_{123} , and
2. each ζ_j is a meromorphic section of the trivial bundle

$$u_{v_0}^* \mathcal{N}_j \cong \Sigma_{v_0} \times \mathbb{C}$$

with a zero of order 2 at q_{e_j} and poles of order 1 at $\{q_{e_k}\}_{k \in \{1,2,3\} - \{j\}}$.

The function $s: \mathbb{V} \rightarrow \mathbb{R}^3$ given by

$$s_{v_1} = (3, 0, 0), \quad s_{v_2} = (0, 3, 0), \quad s_{v_3} = (0, 0, 3), \quad \text{and} \quad s_{v_0} = (1, 1, 1)$$

satisfies (C1) of Definition 2.10 and is the unique such function up to rescaling. Therefore, Γ is a main dual graph (i.e. $K_\bullet = 0$ or $K \cong \mathbb{Z}$). Since the domain and the target of the injective map (2.14) are 6-dimensional and its kernel is 1-dimensional, we conclude that the obstruction group $\mathcal{G}$ is 1-dimensional. In fact, it is isomorphic to $\mathbb{C}^*$.

For each set I , let $\mathbb{Z}^I/\mathbb{Z}$ denote the quotient by the diagonal subgroup. We have

$$\mathbb{Z}^I/\mathbb{Z} \cong (\mathbb{Z}_\bullet^I)^\vee.$$

The dual map

$$\mathbb{T}^\vee = \bigoplus_{i \in \mathcal{I}} \left(\frac{\mathbb{Z}^{I_{e_i}}}{\mathbb{Z}} \right) \cong \left(\frac{\mathbb{Z}^3}{\mathbb{Z}} \right)^3 \xrightarrow{\varrho^\vee} \mathbb{D}_\bullet^\vee \cong (\mathbb{Z}^\mathbb{E})^\vee \oplus \left(\frac{\mathbb{Z}^{I_{v_0}}}{\mathbb{Z}} \right) \cong \mathbb{Z}^3 \oplus \left(\frac{\mathbb{Z}^3}{\mathbb{Z}} \right)$$

in (5.2) is given by

$$\begin{aligned} \varrho^\vee([\eta_1], [\eta_2], [\eta_3]) \\ = \left((-2\eta_{11} + \eta_{12} + \eta_{13}), (\eta_{21} - 2\eta_{22} + \eta_{23}), (\eta_{31} + \eta_{32} - 2\eta_{33}), -([\eta_1] + [\eta_2] + [\eta_3]) \right), \end{aligned}$$

where $\eta_i = [\eta_{i1}, \eta_{i2}, \eta_{i3}] \in \frac{\mathbb{Z}^3}{\mathbb{Z}}$, for any $i \in \{1, 2, 3\}$. It is straightforward to check that

$$\text{Im}(\varrho^\vee) = \left\{ (a, b, c, [x, y, z]) \in \mathbb{Z}^3 \oplus \frac{\mathbb{Z}^3}{\mathbb{Z}} : a + b + c \equiv x + y + z \pmod{3} \right\}.$$

Therefore, the quotient group $\mathbb{D}_\bullet^\vee/\text{Im}(\varrho^\vee)$ is isomorphic to $\mathbb{Z}_3$ and is generated by the class of $(1, 0, 0, [0, 0, 0])$; i.e. $m(\Gamma) = 3$.

Remark 6.1 In this example

$$m(\Gamma) = 3, \quad m_\sigma = 3, \quad m_{\text{red}}(\Gamma) = 1.$$

In the light of Remark 5.5, this explains why our coefficient $m(\Gamma) = 3$ coincides with the one calculated in [3, Section 5].

In the pre-log space $\mathcal{M}_{0,2}^{\text{log}}(X_\emptyset, [3])_\Gamma$, the three lines ℓ_1, ℓ_2, ℓ_3 are allowed to be any line passing through the point X_{123} with some slope in $\mathbb{C}^*$. However, the condition $\text{ob}_\Gamma(f) \in \mathcal{G} \cong \mathbb{C}^*$ in Definition 2.10.(C2) puts a restriction on the set of lines ℓ_1, ℓ_2, ℓ_3 that give rise to a log curve.

For each $i \in \{1, 2, 3\}$, the line ℓ_i is the completion of the image of a map of the form

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{C}^3, \quad z \longmapsto (x_{ij}(z))_{j=1,2,3} \subset \mathbb{C}^3, \quad x_{ii} = 0, \\ &\text{and } x_{ij}(z) = a_{ij}z, \quad a_{ij} \in \mathbb{C}^*, \quad \forall j \neq i. \end{aligned}$$

It follows from definition of $\text{ob}_\Gamma(f)$ in (2.21) that $\text{ob}_\Gamma(f) = 1$ if and only if for any set of 3 distinct points $q_{e_1}, q_{e_2}, q_{e_3} \in \Sigma_{v_0} = \mathbb{P}^1$ and local coordinates $z_{e_1}, z_{e_2}, z_{e_3}$ around them, respectively, there exists a set of meromorphic sections $(\zeta_i)_{i \in \{1,2,3\}}$ of $\mathbb{P}^1 \times \mathbb{C}$ (holomorphic away from $q_{e_1}, q_{e_2}, q_{e_3}$) such that the product $\zeta_1 \zeta_2 \zeta_3$ is a constant section and

$$\zeta_i(z_{e_j}) = a_{ji} z_{e_j}^{-1} \quad \forall i \in \{1, 2, 3\}, j \neq i.$$

A straightforward calculation shows that this is possible if and only if

$$\frac{a_{12}}{a_{13}} \frac{a_{31}}{a_{32}} \frac{a_{23}}{a_{21}} = -1,$$

i.e. the product of the slopes of ℓ_1, ℓ_2, ℓ_3 (in a certain order) is -1 . In other words,

$$\text{ob}_\Gamma(f) = -\frac{a_{12}}{a_{13}} \frac{a_{31}}{a_{32}} \frac{a_{23}}{a_{21}} \in \mathbb{C}^*.$$

In the degeneration formula (5.3), imposing two generic point constraints in X_1 and X_2 on the image of the two marked points fixes ℓ_1 and ℓ_2 . Then the slope condition above fixes ℓ_3 . Therefore, since $m(\Gamma) = 3$ and $\mathfrak{Aut}(\Gamma) = 1$, the contribution of such a star-shaped log map to the GW count of degree 3 rational curves in a smooth cubic surface passing through two generic points is 3. Together with the other 81 classical-type curves, we recover the 2-point degree 3 genus 0 GW invariant of cubic surface which is 84.

We finish with some comments on Question (3) in Page 3. After removing the trivial component $u_{v_0}: \Sigma_{v_0} \longrightarrow X_{123}$, the moduli space $\mathcal{M}_{0,2}^{\text{log}}(X_\emptyset, [3])_\Gamma$ decomposes into the relative spaces

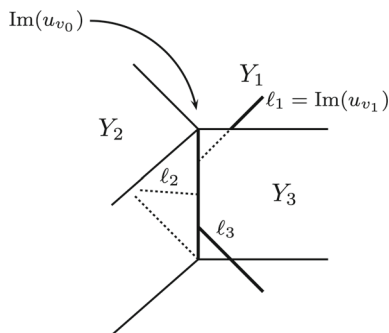
$$\begin{aligned} &\mathcal{M}_{0,((0,0),(1,1))}(X_1, X_{1;\partial}, [1]), \quad \mathcal{M}_{0,((0,0),(1,1))}(X_2, X_{2;\partial}, [1]), \\ &\text{and } \mathcal{M}_{0,(1,1)}(X_3, X_{3;\partial}, [1]). \end{aligned}$$

Thus, one might still hope to be able to get a decomposition formula in a situation like this. However, in higher dimensions and higher degrees, there seems to be no obvious way to get such a decomposition. The following two examples highlight the issue even further.

Example 6.2 Consider the family $\mathcal{Y} = \mathcal{Z} \times \mathbb{P}^1 \longrightarrow \mathbb{C}$, where $\mathcal{Z}$ is as above and π is the lift of the projection map $\pi: \mathcal{Z} \longrightarrow \mathbb{C}$. Let $Y_I = X_I \times \mathbb{P}^1$, for all $\emptyset \neq I \subset \{1, 2, 3\}$. Consider the same dual graph but with $k = 3$ (i.e. with a third marked point on Σ_{v_3}),

$$A_{v_0} = [0, 1] \in H_2(Y_{123}, \mathbb{Z}) \cong H_2(\{\text{point}\} \times \mathbb{P}^1, \mathbb{Z}) \cong \{0\} \times \mathbb{Z},$$

Fig. 6 The image of a map belonging to $\overline{\mathcal{M}}_{0,3}(Y_\emptyset, [3, 1])_\Gamma$ in $Y_\emptyset$



and

$$A_{v_i} = [1, 0] \in H_2(Y_i, \mathbb{Z}) \cong H_2(X_i \times \mathbb{P}^1, \mathbb{Z}) \cong H_2(X_i, \mathbb{Z}) \times \mathbb{Z} \quad \forall i \in \{1, 2, 3\};$$

see Fig. 6.

The moduli space $\overline{\mathcal{M}}_{0,3}(Y_\emptyset, [3, 1])_\Gamma$ is complex 8 dimensional with the same contributing factor $m(\Gamma) = 3$ to (5.3). A smooth fiber $\mathcal{Y}_\lambda$ of $\mathcal{Y}$ is the product of the smooth cubic surface $\mathcal{Z}_\lambda$ and $\mathbb{P}^1$. Let

$$\text{GW}_{0,3}^{\mathcal{Y}_\lambda}(\text{pt}, \text{pt}, \alpha \times \text{pt}) \quad (6.1)$$

be the number of bi-degree $[3, 1]$ rational curves in $\mathcal{Y}_\lambda$ with two point constraints and

$$\alpha \times \text{pt} \in H_2(\mathcal{Y}_\lambda, \mathbb{Z}) = H_2(\mathcal{Z}_\lambda \times \mathbb{P}^1, \mathbb{Z}),$$

where α is the homology class of the smoothing of the limiting line shown in Fig. 4-Right. Since $m(\Gamma) = 3$ as before, and there is a unique Γ -type log map in $Y_\emptyset$ with those constraints, we conclude that the contribution of Γ -type curves to (6.1) is again 3. In examples like this, where there is a non-constant map u_v in a stratum X_{I_v} with $|I_v| \neq 1$, for any decomposition of $\overline{\mathcal{M}}_{g,k}(X_\emptyset, A)_\Gamma$ into a fiber product of relative spaces, either (1) u_v has to be considered in one of the relative moduli spaces (which normally results in relative spaces with $s \notin \mathbb{N}^N$), or (2) (Σ_v, u_v) should be removed while its non-trivial GW contribution affects the matching conditions of the remaining parts. The first idea is a motivation behind studying punctured log GW invariants [4].

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