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Inspired by the log Gromov–Witten (or GW) theory of Gross–Siebert/Abramovich–Chen, we introduce a geometric notion of log J–holomorphic curve relative to a simple normal crossings symplectic divisor defined by Tehrani–McLean–Zinger (2018). Every such moduli space is characterized by a second homology class, genus and contact data. For certain almost complex structures, we show that the moduli space of stable log J–holomorphic curves of any fixed type is compact and metrizable with respect to an enhancement of the Gromov topology. In the case of smooth symplectic divisors, our compactification is often smaller than the relative compactification and there is a projection map from the latter onto the former. The latter is constructed via expanded degenerations of the target. Our construction does not need any modification of (or any extra structure on) the target. Unlike the classical moduli spaces of stable maps, these log moduli spaces are often virtually singular. We describe an explicit toric model for the normal cone (ie the space of gluing parameters) to each stratum in terms of the defining combinatorial data of that stratum. In an earlier preprint, we introduced a natural set up for studying the deformation theory of log (and relative) curves and obtained a logarithmic analogue of the space of Ruan–Tian perturbations for these moduli spaces. In a forthcoming paper, we will prove a gluing theorem for smoothing log curves in the normal direction to each stratum. With some modifications to the theory of Kuranishi spaces, the latter will allow us to construct a virtual fundamental class for every such log moduli space, and define relative GW invariants without any restriction.

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1 Introduction

Studying pairs of a smooth variety X and a normal crossings (or NC) divisor[1] $D \subset X$ has a rich history in complex algebraic geometry. For example, studying such pairs is central to the minimal model program and to the construction of moduli spaces in algebraic geometry. By a celebrated theorem of Hironaka (1964), given a singular variety Y, there is a smooth "blowup" X of Y such that the preimage of the singular locus of Y is an NC divisor $D \subset X$. Therefore, the study of such pairs is also important toward the study of singularities. Curves are (Poincaré) dual objects to divisors. Moduli spaces of curves in X that intersect D in some particular ways are fundamental tools for understanding the geometry of (X,D).

In the last 40 years, analogues of these notions have been defined in the symplectic category and have led to significant advances in our understanding of symplectic manifolds. In the 1980s, Gromov combined the rigidity of algebraic geometry with the flexibility of the smooth category and initiated the use of J–holomorphic curves as a generalization of holomorphic curves in symplectic geometry. The use of J–holomorphic curve techniques has led to numerous connections with algebraic geometry, string theory, and to the appearance of symplectic divisors (as the dual objects) in various contexts. The latter includes relations with complex line bundles (see Donaldson [8]), relative Gromov–Witten (or GW) theory (see Ionel and T Parker [21], A Li and Ruan [23] and B Parker [40]), degeneration formulas for GW invariants (see Ionel and T Parker [22], A Li and Ruan [23], B Parker [37] and Tehrani and Zinger [48]), topological study of singularities (see McLean [31]), symplectic cohomology and mirror symmetry of complements $X \setminus D$ (see Auroux [6] and Ganatra and Pomerleano [15]), and classification of symplectic log Calabi–Yau 4–manifolds (see T Li and Mak [26]). A smooth symplectic divisor is simply a symplectic submanifold of real codimension two. Topological notions of NC symplectic divisors and varieties were recently introduced by McLean, Zinger and the author in [45; 46; 47].

While most applications of J–holomorphic curves in symplectic topology have so far concerned smooth symplectic manifolds, or pairs (X,D) of a smooth manifold and a smooth symplectic divisor, recent developments in symplectic topology and the existing rich structures in algebraic geometry (some of which are listed above) suggest the need for constructing and studying moduli spaces of J–holomorphic curves relative to

[1] Curves and divisors are, respectively, subvarieties of dimension 1 and codimension 1 over the ground field.

an arbitrary NC symplectic divisor from the analytical perspective. In this paper we introduce an explicit and efficient compactification of moduli spaces of J–holomorphic curves relative to an arbitrary simple normal crossings (SNC) symplectic divisor. In upcoming papers [11; 12], we will set up the analytic framework needed for constructing a (virtual) fundamental class, and define relative GW invariants. In particular, in [11], we will define a notion of semipositive pair that allows a direct construction of relative GW invariants via perturbed J–holomorphic maps as in Ruan and Tian [42]. In [43], based on these log moduli spaces, we outline an explicit degeneration formula that relates the GW invariants of smooth fibers to the GW invariants of central fiber, in a semistable degeneration with an SNC central fiber. It is worth mentioning that even in the case of smooth divisors, our compactification is different and smaller than the well-known relative compactification in Ionel and Parker [21], J Li [24] and A Li and Ruan [23].

We begin by setting up the most commonly used notation and recalling some of the known facts about the classical and relative moduli spaces of closed J–holomorphic curves. Therefore, experts may skip to Section 1.3, where the main question is explained.

1.1 Classical stable maps and GW invariants

For X a smooth manifold, $g,k \in \mathbb{N}$, $A \in H_2(X;\mathbb{Z})$, and an almost complex structure J on X,[2] a (nodal) k–marked genus-g degree-A J–holomorphic map into X is a tuple $(u,\Sigma,\mathfrak{j},z^1,\dots,z^k)$, where

- $(\Sigma,\mathfrak{j})$ is a connected nodal Riemann surface of arithmetic genus g with k distinct ordered marked points $z^1,\dots,z^k$ away from the nodes,
- $u\colon(\Sigma,\mathfrak{j})\to(X,J)$ is a continuous and componentwise smooth map satisfying the Cauchy–Riemann equation

$$\bar\partial u = \tfrac{1}{2}(du + J\,du\circ\mathfrak{j}) = 0 \tag{1-1}$$

  on each smooth component, and
- the map u represents the homology class A.

Two such tuples

$$(u,\Sigma,\mathfrak{j},z^1,\dots,z^k) \quad\text{and}\quad (u',\Sigma',\mathfrak{j}',w^1,\dots,w^k)$$

are equivalent if there exists a biholomorphic isomorphism $h\colon(\Sigma,\mathfrak{j})\to(\Sigma',\mathfrak{j}')$ such

[2] That is, J is a real-linear endomorphism of TX lifting the identity map satisfying $J^2 = -\mathrm{id}_{TX}$.

that $h(z^a) = w^a$ for all $a = 1, \dots, k$ and $u = u' \circ h$. Such a tuple is called stable if the group of self-automorphisms is finite. Let $\overline{\mathcal{M}}_{g,k}(X, A, J)$ (or simply $\overline{\mathcal{M}}_{g,k}(X, A)$ when J is fixed in the discussion) denote the space of equivalence classes of stable k–marked genus-g degree-A J–holomorphic maps into X. Such an equivalence class is called a marked J–holomorphic curve.

By a celebrated theorem[3] of Gromov [16, Theorem 1.5.B], for every smooth closed (ie compact and without boundary) symplectic manifold (X, ω), g, k, A as above, and an almost complex structure J compatible[4] with ω (or taming ω), the moduli space $\overline{\mathcal{M}}_{g,k}(X, A, J)$ has a natural sequential convergence topology, called the Gromov topology, which is compact, Hausdorff, and furthermore metrizable. The symplectic structure only gives an energy bound which is needed for establishing the compactness, and the precise choice of that, up to deformation, is not important. If $\overline{\mathcal{M}}_{g,k}(X, A)$ has an oriented orbifold structure of expected real dimension

$$2\, c_1^{TX}(A) + (n-3)(1-g) + k, \tag{1-2}$$

GW invariants are obtained by the integration of appropriate cohomology classes against its fundamental class. These numbers are independent of J and only depend on the deformation equivalence class of ω. These allow the formulation of symplectic analogues of enumerative questions from algebraic geometry, as well-defined invariants of symplectic manifolds. However, in general, such moduli spaces can be highly singular. This issue is known as the transversality problem. Fortunately, it has been shown (see[5] eg [25; 14; 27; 18; 30; 33]) that $\overline{\mathcal{M}}_{g,k}(X, A)$ still carries a rational homology class, called virtual fundamental class (or VFC); integration of cohomology classes against the VFC gives rise to GW invariants.

1.2 Relative stable maps

Given a symplectic manifold (X, ω) and a closed submanifold $D \subset X$, we say $D \subset X$ is a symplectic submanifold if $\omega|_D$ is a symplectic structure. A (smooth) symplectic divisor is a symplectic submanifold of real codimension 2. For such D (or a smooth divisor in complex algebraic geometry), relative GW theory (virtually) counts J–holomorphic curves in X with a fixed contact order $\mathbf{s} = (s_1, \dots, s_k) \in \mathbb{N}^k$ with D. In this theory, we

[3] And its subsequent refinements; see the remarks before Theorem 3.3.

[4] That is, $\omega(\cdot, J\cdot)$ is a metric.

[5] It is beyond the scope of this paper to list all the related literature.

require J to be also compatible with D in the following sense. First, we require D to be J–holomorphic, ie $J(TD) = TD$. This implies, for example, that every J–holomorphic map to X from a smooth domain is either mapped into D or intersects D positively in a finite set of points. Furthermore, we need to at least require J to be integrable to the first order in the normal direction to D, in the sense that

$$N_J(v_1, v_2) \in T_xD \quad \text{for all } x \in D,\ v_1, v_2 \in T_xX, \tag{1-3}$$

where $N_J \in \Gamma(X, \Omega_X^{2} \otimes TX)$ is the Nijenhuis tensor of J, satisfying

$$N_J(u, v) = [u, v] + J[u, Jv] + J[Ju, v] - [Ju, Jv] \quad \text{for all } u, v \in TX.$$

This ensures that certain operators are complex linear (see (4-7)), and certain sequences of almost complex structures on the normal bundle $\mathcal{N}_XD$ converge to a standard one (see Lemma 3.5). The space $\mathcal{J}(X, D, \omega)$ of ω–tame and D–compatible almost complex structures J on X is again nonempty and contractible. For every $J \in \mathcal{J}(X, D, \omega)$ and $s = (s_1, \dots, s_k) \in \mathbb{N}^k$, with

$$\sum_{a=1}^{k} s_a = A \cdot D, \tag{1-4}$$

let $\mathcal{M}_{g,s}(X, D, A) \subset \mathcal{M}_{g,k}(X, A)$ (in the stable range) be the subspace of k–marked degree-A genus-g J–holomorphic curves $[u, \Sigma, j, z^1, \dots, z^k]$ such that Σ is smooth and u has a tangency of order s_a at z^a with D. In particular, by (1-4),

$$u^{-1}(D) = \{z^1, \dots, z^k\}.$$

The subset of marked points z^a with $s_a = 0$ corresponds to the classical marked points of the classical GW theory with image away from D. The relative compactification $\overline{\mathcal{M}}^{\mathrm{rel}}_{g,s}(X, D, A)$ of $\mathcal{M}_{g,s}(X, D, A)$, constructed in [24] in the algebraic case, and in [21; 23] in the symplectic case, includes stable nodal curves with components mapped into X or an expanded degeneration[6] of that, so that the contact order s still makes sense; we will review this construction in Section 4.1.

1.3 J–holomorphic maps relative to SNC divisors

In [45; 46], with McLean and Zinger, we defined topological notions of symplectic normal crossings divisor and variety and showed that they are equivalent, in a suitable

[6] A normal crossings variety made of X and finite copies of the $\mathbb{P}^1$–bundle $\mathbb{P}_XD = \mathbb{P}(\mathcal{N}_XD \oplus \mathcal{O}_D)$ over D.

sense, to the desired rigid notions. For $N\in\mathbb{N}$, let

$$[N]=\{1,\dots,N\}.$$

In particular, $[0]=\emptyset$. A simple normal crossings (or SNC) symplectic divisor $D=\bigcup_{i\in[N]}D_i$ in (X,ω) is a transverse union of smooth symplectic divisors $\{D_i\}_{i\in[N]}$ in X such that all the strata

$$D_I\equiv\bigcap_{i\in I}D_i \quad \text{for all } I\subset[N]$$

are symplectic, and the symplectic orientation of D_I coincides with its "intersection" orientation for all $I\subset[N]$; see [45, Definition 2.1]. For

$$J\in\mathcal{J}(X,D;\omega)=\bigcap_{i\in[N]}\mathcal{J}(X,D_i;\omega),$$

we similarly define $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)$ (in the stable range) to be the space of equivalence classes of degree-A J–holomorphic maps from a k–marked genus-g connected smooth domain Σ into X of contact order $\mathfrak{s}$ with D, for which

$$\mathfrak{s}\equiv(s_a\equiv(s_{ai})_{i\in[N]})_{a\in[k]}\in(\mathbb{N}^N)^k;$$

each vector s_a records the intersection numbers of the a^{th} marked point z_a with the divisors $\{D_i\}_{i\in[N]}$, and

$$\text{(1-5)}\quad u^{-1}(D)\subset\{z^1,\dots,z^k\}, \qquad \text{or equivalently} \quad A\cdot D_i=\sum_{a=1}^{k}s_{ai} \quad \text{for all } i\in[N].$$

Because of the tangency conditions, it follows from (1-2) that the expected real dimension of $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)$ is equal to

$$\text{(1-6)}\quad 2\,c_1^{TX}(A)+(n-3)(1-g)+k-A\cdot D=2\,c_1^{TX(-\log D)}(A)+(n-3)(1-g)+k,$$

where $TX(-\log D)$ is the log tangent bundle associated to the deformation equivalence class of (X,D,ω), defined in [46, (8)]. In the holomorphic case, the log tangent sheaf is the sheaf of holomorphic tangent vector fields in TX whose restriction to each D_i is tangent to D_i. The definition in the symplectic case is similar but depends[7] on some auxiliary data. The similarity between the left-hand sides of (1-6) and (1-2) shows the importance of considering the log tangent bundle in the study of relative moduli spaces.

[7] The deformation equivalence class of complex vector bundle $TX(-\log D)$ is independent of the auxiliary data.

The main goal is:

(?) To construct a natural geometric compactification $\overline{\mathcal{M}}_{g;s}(X,D;A)$ of $\mathcal{M}_{g;s}(X,D;A)$ so that the definition of the contact vector s naturally extends to every element of $\overline{\mathcal{M}}_{g;s}(X,D;A)$, and $\overline{\mathcal{M}}_{g;s}(X,D;A)$ is (virtually) smooth enough to admit a natural class of cobordant Kuranishi structures of the expected real dimension (1-6).

We refer to [44; 30] for the technical terms in (?). If D is smooth, the well-known relative compactification $\overline{\mathcal{M}}^{\mathrm{rel}}_{g;s}(X,D;A)$ has (or is expected[8] to have) these nice properties.

In the algebraic category, every (algebraic) NC variety $D\subset X$ defines a natural "fine saturated log structure" on X; see [2] for a review of log geometry and log moduli spaces associated to NC pairs (X,D). Then the log GW theory of [1] and [17] constructs a good compactification with a perfect obstruction theory for every fine saturated log variety X. Unlike in [24], the algebraic log compactification does not require any expanded degeneration of the target. Instead, it uses the extra log structure on X (and various log structures on the domains) to keep track of the contact data for the curves that have image inside the support of the log structure (ie D).

Since the classical GW invariants are invariants of the deformation equivalence class of the underlying symplectic structure, it is interesting and important to generalize the results of [1; 17] to (or find an analogue of them for) the symplectic category, ie to construct log GW invariants as invariants of the symplectic deformation equivalence class of (X,D). With such a construction, the flexibility of symplectic topology can be used in certain situations to define log GW invariants as an actual count of J–holomorphic curves with tangency conditions, at the expense of deforming J or the Cauchy–Riemann equation (to avoid working with VFC); see [42; 11]. Moreover, in the case of moduli spaces of holomorphic curves with boundary on Lagrangian submanifolds, it is sometimes easier to work with an analytical construction of moduli spaces of J–holomorphic maps.

On the analytical side, in [36; 40; 39] and several other related papers, Brett Parker uses his enriched almost Kähler category of "exploded manifolds", defined in [34], to construct such a compactification relative to an almost Kähler NC divisor and address (?). His approach can be considered as a direct translation/generalization of the algebraic log GW theory involving some non-Hausdorff spaces, analytical sheaves,

[8] See [48] for an overview of the analytical approaches of [21; 23].

and a richer cohomology theory [38]. His approach has close ties to tropical geometry. In [20], Eleny Ionel approaches (?), by considering expanded degenerations similar to [21]. Nevertheless, the main motivation behind the log GW theory of Gross–Siebert–Abramovich–Chen, the exploded theory of Parker, and the current paper is that considering spaces and maps enriched with certain log structures is a better idea for addressing (?) in the general case. In particular, all these logarithmic approaches lead to similar "degeneration formulas" (the authors of [4] call it an "invariance property") relating the moduli spaces in smooth fibers and the SNC central fiber of an arbitrary semistable degeneration; see [4; 37; 11].

1.4 Log compactification and the main result

In this paper, for an arbitrary SNC symplectic divisor $D \subset (X,\omega)$ and certain $J \in \mathcal{J}(X,D,\omega)$, we construct a "minimal geometric compactification"

$$\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A) \tag{1-8}$$

that does not require any modification of the target (or the nodal domains). For its connection to the algebraic log maps, and the appearance of various log structures[9] throughout the construction, we call our maps/curves log J–holomorphic maps/curves. For $J \in \mathcal{J}(X,D,\omega)$, a (nodal) log J–holomorphic map into $(X, D = \bigcup_{i\in[N]} D_i)$ of contact type

$$\mathfrak{s} \equiv \big(s_a \equiv (s_{ai})_{i\in[N]}\big)_{a=1}^{k} \in (\mathbb{Z}^N)^k,$$

with the marked nodal domain $(\Sigma, \mathfrak{j}, \vec{z}) = \bigcup_{v\in\mathbb{V}} (\Sigma_v, \mathfrak{j}_v, \vec{z}_v)$, is a collection of tuples

$$u^{\log} \equiv \big((u_v\colon \Sigma_v \to D_{I_v}, \vec{z}_v), ([\zeta_{v,i}])_{i\in I_v}\big)_{v\in\mathbb{V}}$$

over smooth components of Σ such that

- $u \equiv (u_v)_{v\in\mathbb{V}}\colon (\Sigma,\mathfrak{j},\vec{z}) \to (X,J)$ is a k–marked J–holomorphic nodal map in the classical sense,
- for each $v \in \mathbb{V}$, $I_v \subset [N]$ is the maximal subset such that $\mathrm{Im}(u_v) \subset D_{I_v}$,
- for each $v \in \mathbb{V}$ and any $i \in I_v$, $[\zeta_{v,i}]$ is the $\mathbb{C}^*$–equivalence class[10] of a nontrivial meromorphic section $\zeta_{v,i}$ of the holomorphic[11] line bundle $u_v^*\mathcal{N}_X D_i$,
- the contact order vectors in $\mathbb{Z}^N$, defined in (2-14) and (2-15), are the opposite of each other at the nodal points,

[9] Such as the use of log tangent bundle in the deformation theory of log J–holomorphic curves.

[10] $\mathbb{C}^*$ acts by multiplication on the set of meromorphic sections.

[11] Since $\dim_{\mathbb{C}} \Sigma_v = 1$, the pullback line bundle $u_v^*\mathcal{N}_X D_i$ is holomorphic.

- every point in Σ with a nontrivial contact vector is either a marked point or a nodal point, and the contact order vector at z_a is the predetermined vector $s_a \in \mathbb{Z}^N$,
- there exists a vector-valued function $s\colon V \to \mathbb{R}^N$ such that for all $v \in V$, $s_v = s(v) \in \mathbb{R}_+^{I_v} \times \{0\}^{[N]-I_v}$, and for all $v, v' \in V$, $s_v - s_{v'}$ is a positive multiple of the contact order vector of any nodal point on Σ_v connected to $\Sigma_{v'}$, and
- a certain group (a complex torus) element associated to $u_{\log}$, defined in (2-32), is equal to 1.

See Definition 2.8 for more details. Two marked log maps are equivalent if one is a "reparametrization" of the other. A marked log map is stable if it has a finite "automorphism group". For $g, k \in \mathbb{N}$, $A \in H_2(X;\mathbb{Z})$, and $s \in (\mathbb{Z}^N)^k$, we denote the space of equivalence classes of stable k–marked degree-A genus-g log maps of contact type s by

$$\overline{\mathcal{M}}^{\log}_{g,s}(X,D,A).$$

Such an equivalence class is called a log curve. There is a natural forgetful map

$$\overline{\mathcal{M}}^{\log}_{g,s}(X,D,A) \to \overline{\mathcal{M}}_{g,k}(X,A),$$
$$(u_v\colon \Sigma_v \to D_{I_v}, \zeta_v, ([\zeta_{v,i}])_{i\in I_v})_{v\in V} \to (u_v\colon \Sigma_v \to X, \zeta_v)_{v\in V}.$$

Given $s \in (\mathbb{Z}^N)^k$, it turns out that for every k–marked stable nodal curve f in $\overline{\mathcal{M}}_{g,k}(X,A)$, there exist at most finitely many log curves $f_{\log} \in \overline{\mathcal{M}}^{\log}_{g,s}(X,D,A)$ (with distinct decorations on the dual graph) lifting f; see Lemma 2.15. Furthermore, $f_{\log}$ is stable if and only if f is stable (and the automorphism groups are often the same).

In the integrable case and in comparison with the algebraic approach, we conjecture the following statement:

Conjecture 1.1 *In the complex algebraic setting, for any choice of combinatorial data $\Gamma = (g, s, A)$ and the natural log structure on X associated to D, there is a stratified finite-to-one surjective map from the underlying space of the log moduli space $\mathcal{M}(X/\mathrm{pt}, \Gamma)$ in [4] to $\overline{\mathcal{M}}^{\log}_{g,s}(X,D,A)$, which is one-to-one over the main stratum $\mathcal{M}_{g,s}(X,D,A)$.*

In particular, this conjecture says that the group element (2-32), mentioned in the final bulleted condition above, is the only noncombinatorial obstruction for liftability of a nodal map (with correct combinatorial properties) to a log map (with the canonical log structures on X corresponding to D). It is likely that we need to allow certain "nonsaturated" curves in $\mathcal{M}(X/\mathrm{pt};\Gamma)$ for the conjecture to be true, or the projection map will not be surjective. The projection map conjectured above behaves like a normalization map between varieties (eg unfolding self-intersections). Based on a comparison of the coefficients of the degeneration formula in [4] with our degeneration formula outlined in [43], we think that the degree of the projection map on each stratum should be the multiplicity m_Γ in (5-14).

Similarly, in comparison with the Brett Parker approach in [36], under certain assumptions on the almost complex structure J, we expect the following statement.

Conjecture 1.2 *With respect to the exploded structure associated to an almost Kähler SNC divisor $D \subset X$, for any choice of combinatorial data $\Gamma = (g;s;A)$, the "smooth part" map gives a finite-to-one surjective map from the moduli stack in [36] to $\overline{\mathcal{M}}^{\log}_{g;s}(X;D;A)$.*

We postpone a careful comparison of the moduli spaces constructed in this paper and those arising from [1; 17] and [36] to a future paper.

Approaching (?), we face some new challenges that are not present in the case of the classical and relative stable maps. Unlike the smooth case, it is not a priori clear whether every SNC symplectic divisor $D \subset (X;\omega)$ admits a compatible almost complex structure. Furthermore, even if $\mathcal{J}(X;D;\omega) \neq \emptyset$, it is not clear whether it is contractible (or even connected). In order to address this issue, in [45], we consider the space[12] $\mathrm{Symp}(X;D)$ of all symplectic forms on X such that a given transverse configuration $D = \bigcup_{i\in[N]} D_i$ is an SNC symplectic divisor in $(X;\omega)$. Consequently, instead of focusing on a particular ω, we consider the connected component of symplectic forms in $\mathrm{Symp}(X;D)$ which are deformation equivalent to ω. With $\mathcal{J}(X;D;\omega)$ as before, let

$$\mathcal{J}(X;D) = \bigcup_{\omega\in \mathrm{Symp}(X;D)} \mathcal{J}(X;D;\omega)$$

be the space of all D–compatible pairs $(\omega;J)$. We then define a space of almost Kähler auxiliary data $\mathrm{AK}(X;D)$ consisting of tuples $(\omega;\mathcal{R};J)$ where $\omega \in \mathrm{Symp}(X;D)$, $\mathcal{R}$ is an "ω–regularization" for D in X, and J is ω–tame and $\mathcal{R}$–compatible (which we will simply call $(\mathcal{R};\omega)$–compatible) almost complex structure on X; see Section 3.2 or [45, page 8]. Roughly speaking, a regularization is a compatible set of symplectic

[12] In [45], this space is denoted by $\mathrm{Symp}^+(X;D)$.

identifications of neighborhoods of $\{D_I\}_{I\subset[N]}$ in their normal bundles with neighborhoods of them in X; see [45, Definition 2.12]. A regularization serves as a replacement for holomorphic defining equations in holomorphic manifolds. These regularizations are also the auxiliary data that we need to define the log tangent bundle $TX(-\log D)$. For every $(\omega,\mathcal{R},J)\in\mathrm{AK}(X,D)$, we have $(\omega,J)\in\mathcal{J}(X,D)$. Therefore, $\mathrm{AK}(X,D)$ is essentially a nice subset of $\mathcal{J}(X,D)$ consisting of those almost complex structures that are of some specified type in a sufficiently small neighborhood of D. These special almost complex structures are similar to the almost complex structures with translational symmetry considered in [23] and in SFT [10]. By [45, Theorem 2.13], the forgetful map

$$\mathrm{AK}(X,D)\to\mathrm{Symp}(X,D),\quad (\omega,\mathcal{R},J)\to\omega, \tag{1-9}$$

is a weak homotopy equivalence. This implies that any invariant of the deformation equivalence classes in $\mathrm{AK}(X,D)$ is an invariant of the symplectic deformation equivalence class of (X,D,ω). In particular, by restricting to the subclass $\mathrm{AK}(X,D)$, the last statement in (?) follows from constructing Kuranishi structures for families.

The main goal of this paper is to prove the following compactness result, addressing the first part of (?). We will address the rest in subsequent papers. We will briefly outline our approach to the deformation theory and gluing in Sections 5.1 and 5.2.

Definition 1.3 A continuous function $f\colon M\to N$ between two topological spaces is a local embedding if for all $x\in M$ there is an open neighborhood U of x such that $f|_U\colon U\to N$ is an embedding.

By Smirnov's theorem, every paracompact, Hausdorff, and locally metrizable space is metrizable. Therefore, if $f\colon M\to N$ is a local embedding from a compact Hausdorff space M to a compact metrizable space N, then M is metrizable.

Theorem 1.4 *Assume X is a compact symplectic manifold and $D=\bigcup_{i\in[N]}D_i\subset X$ is an SNC symplectic divisor. If $(\omega,\mathcal{R},J)\in\mathrm{AK}(X,D)$ or if (X,D,ω,J) is Kähler, then for every $A\in H_2(X;\mathbb{Z})$, $g,k\in\mathbb{N}$ and $\mathfrak{s}\in(\mathbb{Z}^N)^k$, the Gromov sequential convergence topology on $\overline{\mathcal{M}}_{g,k}(X,A)$ lifts to a compact Hausdorff sequential convergence topology on $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$ so that the natural forgetful map*

$$\colon\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)\to\overline{\mathcal{M}}_{g,k}(X,A) \tag{1-10}$$

is a local embedding. In particular, $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$ is metrizable. If $g=0$, then (1-10) is a global embedding.

In other words, the open sets of $\overline{M}^{\log}_{g;s}.X; D; A/$ are the components of the intersection of open sets in $\overline{M}_{g;k}.X; A/$ with the image of $\overline{M}^{\log}_{g;s}.X; D; A/$.

Remark 1.5 Except for the proof of Proposition 3.15, every other statement in the proof of Theorem 1.4 is stated and proved for arbitrary $.!; J/ 2 J .X; D/$. We expect the local statement of Proposition 3.15, and thus Theorem 1.4, to be true for a larger class of almost Kähler structures that are weakly homotopy equivalent to $Symp.X; D/$, which includes both AK(X,D) and the space of Kähler structures. If D is smooth, a sig-nificantly simpler version of Proposition 3.15 is sufficient for proving Proposition 3.14, and thus Theorem 1.4 for arbitrary $.!; J/ 2 J .X; D/$; see Remark 3.16. Nevertheless, by the argument around (1-9), the subclass $AK.X; D/$ is ideal for defining GW-type invariants and the holomorphic case is sufficient for most of the interesting examples and calculations.

Remark 1.6 While $\overline{M}^{\log}_{g;s}.X; D; A/$ is defined for arbitrary $s 2 .Z^N/^k$ satisfying the second identity in (1-5), and the compactness result holds for every such s, the resulting moduli spaces do not have some of the nice properties unless $s 2 .N^N/^k$; eg the (virtual) main stratum $M_{g;s}.X; D; A/$ would be empty if any of the s_{ai} were negative. For $s 2 .N^N/^k$, by Lemma 5.5, the expected dimension of $\overline{M}^{\log}_{g;s}.X; D; A/$ is equal to (1-6), and the only stratum with the top expected dimension is $M_{g;s}.X; D; A/$. As pointed out to the author by M Gross, the case where s_{ai} could be negative is called "punctured curves" in the work-in-progress [3]. One feature of these punctured curves is that the moduli spaces may not carry a VFC, as even in the unobstructed case the moduli space may have irreducible components of different dimension.

If D is smooth, we show in Proposition 4.5 that there is a surjective projection map

$$\overline{M}^{\mathrm{rel}}_{g;s}.X; D; A/ ! \quad \overline{M}^{\log}_{g;s}.X; D; A/:$$

This is as expected, since our notion of log J–holomorphic curve involves more C– quotients on the set of meromorphic sections than in the relative case. In the algebraic case, [5, Theorem 1.1] shows that an algebraic analogue of this projection map induces an equivalence of the virtual fundamental classes. We expect the same to hold for invariants/VFCs arising from our log moduli spaces.

Approaching the rest of (?), the transversality issue aside, log moduli spaces constructed in this paper are often virtually singular in the sense that the (virtual) normal cone of each stratum is not necessarily an orbibundle. More precisely, $\overline{\mathcal M}^{\log}_{g,s}(X,D,A)$ admits a stratification

$$\overline{\mathcal M}^{\log}_{g,s}(X,D,A)=\bigcup_{\Gamma}\mathcal M_{g,s}(X,D,A)_{\Gamma},$$

where Γ runs over all the possible "decorated dual graphs"; see Definition 2.12. For any f in $\mathcal M_{g,s}(X,D,A)_{\Gamma}$, the natural process of describing a neighborhood of f in $\overline{\mathcal M}^{\log}_{g,s}(X,D,A)$ is by first describing a neighborhood U of f in $\mathcal M_{g,s}(X,D,A)_{\Gamma}$, and then extending that, by a "gluing" theorem of smoothing the nodes, to a neighborhood of the form $U\times\mathcal N^0_{\Gamma}$ for f in $\overline{\mathcal M}^{\log}_{g,s}(X,D,A)$, where $\mathcal N^0_{\Gamma}$ is a neighborhood of the origin in an affine subvariety $\mathcal N_{\Gamma}\subset\mathbb C^m$ for some $m\in\mathbb N$. In this situation, we say that $\mathcal N_{\Gamma}$ is the normal cone to $\mathcal M_{g,s}(X,D,A)_{\Gamma}$, or it is the space of gluing parameters. In the case of classical stable maps, $\mathcal N_{\Gamma}$ is isomorphic to $\mathbb C^{E}$, where E is the set of edges of Γ (or nodes of the nodal domain). Unlike in the classical case, for the log (or relative) moduli spaces, $\mathcal N_{\Gamma}$ could be reducible, and the normalization of $\mathcal N_{\Gamma}$ might be singular as well; see Example 5.6. Nevertheless, we show that $\mathcal N_{\Gamma}$ is (isomorphic to some finite copy of) an affine toric variety that can be explicitly described in terms of Γ. More precisely, let V and E be the set of vertices and edges of Γ, respectively. For each $v\in V$, $I_v\subset[N]$ is the maximal subset such that the image of the v^{th} component of f lies in D_{I_v}. Similarly, for each $e\in E$, $I_e\subset[N]$ is the maximal subset such that the image of the e^{th} node lies in D_{I_e}. In (2-26), associated to every such Γ, we construct a $\mathbb Z$–linear map

$$\varrho:=\varrho(\Gamma)=\mathbb Z^{E}\oplus\bigoplus_{v\in V}\mathbb Z^{I_v}\to T(\Gamma)=\bigoplus_{e\in E}\mathbb Z^{I_e}\tag{1-11}$$

so that $\mathcal N_{\Gamma}$ is isomorphic to (some finite copy of) the toric variety associated to a maximal convex rational polyhedral cone in $\operatorname{Ker}(\varrho)\otimes\mathbb R$. Moreover, the group element mentioned in the final bulleted condition on page 997 (ie in the definition of a log map) is an element of the Lie group $\mathbb G(\Gamma)$ with the Lie algebra $\operatorname{Coker}(\varrho)\otimes\mathbb C$. In other words, $\operatorname{Ker}(\varrho)$ gives the deformation space in the normal direction and $\operatorname{Coker}(\varrho)$ gives an obstruction for the smoothability of such maps.

1.5 Outline

In Section 2.1, we review the definition and properties of $\mathfrak i$–operators. The $\mathfrak i$–operator $\mathfrak i_{\mathcal N_X D}$ on the normal bundle $\mathcal N_X D$ described in Lemma 2.1 plays a key role in defining the basic building blocks of relative and log maps. In Section 2.2, we set up our notation for the decorated dual graph of nodal maps. The $\mathbb Z$–linear map (1-11) is defined in terms of such decorated dual graphs. In Section 2.3, we define the moduli spaces of log J–holomorphic curves and provide several examples to highlight their features. This is done in two steps: first, in Definition 2.4, we define a straightforward notion of prelog map. Then in Definition 2.8, we impose two nontrivial conditions on such a prelog map to define a log map. The proof of Theorem 1.4 relies on Gromov's compactness result for the underlying stable maps. In Section 3.1, we review the Gromov compactness theorem and set up the notation for the proof of Theorem 1.4. In Section 3.2, we state a log enhancement of the Gromov compactness theorem. Proof of the main result is done in multiple steps in Sections 3.3 and 3.4. The main step of the proof is Proposition 3.15, which compares the limiting behavior of the rescaling and gluing parameters. In the case of smooth divisors, we compare the relative and the log compactifications of the same combinatorial type in Section 4.2. We review the construction of relative compactification in Section 4.1. In Section 5.1, we outline a Fredholm setup for studying the deformation theory of log J–holomorphic maps, and draw some conclusions. This setup is extended to perturbed log maps and discussed in detail in [11]. In Section 5.2, we explicitly describe the space of gluing parameters of any fixed type €, and identify it with an explicit affine toric variety.

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2 Log pseudoholomorphic maps

In this section, we construct the moduli spaces of log J–holomorphic curves relative to an arbitrary SNC symplectic divisor defined in [45]. This is done by first introducing a notion of prelog J–holomorphic map, which only involves a matching condition of contact orders at the nodes. We then define a Z–linear map between certain Z–modules associated to the dual graph of such a prelog map, which encodes the essential deformation/obstruction data for defining and studying log maps.

Let us start with some well-known facts about almost complex structures. Let (X,ω) be a smooth symplectic manifold and J be an ω–tame almost complex structure on X. Let ∇ be the Levi-Civita connection of the metric $\langle u,v\rangle = \frac{1}{2}(\omega(u,Jv)+\omega(v,Ju))$ and let

$$\hat{\nabla}_v\xi = \nabla_v\xi - \tfrac{1}{2}J(\nabla_v J)\xi = \tfrac{1}{2}(\nabla_v\xi - J\nabla_v(J\xi)) \quad \text{for all } v\in TX,\ \xi\in\Gamma(X,TX) \tag{2-1}$$

be the associated Hermitian connection. The Hermitian connection $\hat{\nabla}$ coincides with ∇ if and only if (X,ω,J) is Kähler, ie $\nabla J\equiv 0$. The torsion T of the modified $\mathbb{C}$–linear connection

$$\tilde{\nabla}_v\xi = \hat{\nabla}_v\xi - A(\xi)v, \qquad A(\xi) = \tfrac{1}{4}(\nabla_{J\xi}J + J\nabla_{\xi}J) \tag{2-2}$$

for all $v\in TX$ and $\xi\in\Gamma(X,TX)$, is related to the Nijenhuis tensor (1-3) by

$$T_{\tilde{\nabla}}(v,w) = -\tfrac{1}{4}N_J(v,w) \quad \text{for all } v,w\in TX. \tag{2-3}$$

If J is ω–compatible, $\hat{\nabla}$ coincides with $\tilde{\nabla}$. See [29, Chapter 3.1 and Appendix C] for details.

2.1 Almost complex structures and $\bar{\partial}$–operators

Suppose M is a smooth manifold, i_M is an almost complex structure on M, and $(L,i_L)\to M$ is a complex vector bundle. Let

$$\Omega^{1,0}_{M,i_M} = \{\eta\in T^*M\otimes_{\mathbb{R}}\mathbb{C} : \eta\circ i_M = i\eta\},\quad \Omega^{0,1}_{M,i_M} = \{\eta\in T^*M\otimes_{\mathbb{R}}\mathbb{C} : \eta\circ i_M = -i\eta\} \tag{2-4}$$

be the bundles of $\mathbb{C}$–linear and $\mathbb{C}$–antilinear 1–forms on M, where i is the unit imaginary number in $\mathbb{C}$. Given a smooth function $f\colon M\to\mathbb{C}$, (2-4) gives a decomposition of df into $\mathbb{C}$–linear and $\mathbb{C}$–antilinear parts ∂f and $\bar{\partial} f$, respectively. A $\bar{\partial}$–operator on (L,i_L) is a complex linear operator

$$\bar{\partial}\colon \Gamma(M,L)\to\Gamma(M,\Omega^{0,1}_{M,i_M}\otimes_{\mathbb{C}}L) \tag{2-5}$$

such that

$$\bar{\partial}(f\xi) = \bar{\partial}f\otimes\xi + f\,\bar{\partial}\xi \quad \text{for all } f\in C^{\infty}(M;\mathbb{C}),\ \xi\in\Gamma(M;L).$$

Given a complex linear connection ∇ on $(L,i_L)\to(M,i_M)$, the $(0,1)$–part

$$\nabla^{(0,1)} = \tfrac{1}{2}(\nabla + i_L\nabla\circ i_M) \tag{2-6}$$

of ∇ is a $\bar\partial$–operator, which we denote by $\bar\partial_\nabla$. Every $\bar\partial$–operator is the associated $\bar\partial$–operator of some $\mathbb{C}$–linear connection ∇ as above. The connection, however, is not uniquely determined. Every two connections ∇ and ∇' differ by a global $\mathrm{End}(L)$–valued 1–form α, ie $\nabla' = \nabla + \alpha$. If ∇ and ∇' are complex linear connections on (L, i_L) with $\nabla' = \nabla + \alpha$, then

$$\bar\partial_{\nabla'} = \bar\partial_\nabla + \alpha^{(0,1)};$$

where $\alpha^{(0,1)}$ is the $(0,1)$–part of α in the decomposition (2-4). In particular, $\bar\partial_{\nabla'} = \bar\partial_\nabla$ whenever α is of $(1,0)$–type.

By [50, Lemma 2.2], corresponding to every $\bar\partial$–operator (2-5) there exists a unique almost complex structure $\mathcal{J} = \mathcal{J}_{\bar\partial}$ on the total space of L, such that

(1) the projection $\pi\colon L \to M$ is an $(i_M, \mathcal{J})$–holomorphic map (ie $d\pi + i\,d\pi\,\mathcal{J} = 0$),
(2) the restriction of $\mathcal{J}$ to the vertical tangent bundle $TL^{\mathrm{ver}} \cong \pi^*L \subset TL$ agrees with i_L, and
(3) the map $u\colon M \to L$ corresponding to a section $\zeta \in \Gamma(M; L)$ is $(\mathcal{J}, i_M)$–holomorphic if and only if $\bar\partial\zeta = 0$.

Suppose (X, ω) is a symplectic manifold, D is a symplectic submanifold, and J is an ω–tame almost complex structure on X such that $J(TD) = TD$. The last condition implies that J induces a complex structure $i_{\mathcal{N}_X D}$ on (the fibers of) the normal bundle

(2-7) $$\pi\colon \mathcal{N}_X D \equiv TX|_D / TD \to D.$$

Under the isomorphism

$$\mathcal{N}_X D \cong TD^{\perp} = \{u \in TX|_D : \omega(u, v) = 0 \text{ for all } v \in TD\};$$

$i_{\mathcal{N}_X D}$ is the same as the restriction to $TD^{\perp}$ of J. Let J_D denote the restriction of J to TD.

Lemma 2.1 Suppose (X, ω) is a symplectic manifold, D is a symplectic submanifold, J is an ω–tame almost complex structure on X such that $J(TD) = TD$, and ∇ is the $\mathbb{C}$–linear connection associated to (ω, J) in (2-2). Then the $\bar\partial$–operator

$$\bar\partial_\nabla \equiv \nabla^{(0,1)}\colon \Gamma(X; TX) \to \Gamma(X; \Omega^{0,1}_{X,J} \otimes_{\mathbb{C}} TX) \text{ in}$$

(2-6) descends to a $\bar\partial$–operator

(2-8) $$\bar\partial_{\mathcal{N}_X D}\colon \Gamma(D; \mathcal{N}_X D) \to \Gamma(D; \Omega^{0,1}_{D,J_D} \otimes_{\mathbb{C}} \mathcal{N}_X D)$$

on $(\mathcal{N}_X D, i_{\mathcal{N}_X D}) \to (D, J_D)$.

Proof We need to show that $\bar\partial_\nabla$ maps $\Gamma(D;TD)$ to $\Gamma(D;\Omega^{0,1}_{D,J_D}\otimes_{\mathbb{C}} TD)$. Let ∇ and ∇^D be the Levi-Civita connections of the metrics associated to (ω,J) and $(\omega|_{TD},J_D)$ on X and D, respectively. Then

$$\nabla\zeta=\nabla^D\zeta+\nabla^N\zeta \qquad \text{for all } \zeta\in\Gamma(D;TD),$$

with

$$\nabla^N\zeta\in\Gamma(D;\Omega^1_D\otimes TD^\perp).$$

Similarly, let $\tilde\nabla$ and $\tilde\nabla^D$ be the Chern connections on TX and TD associated to ∇ and ∇^D, respectively, as in (2-1). It follows from (2-1) that

$$\tilde\nabla\zeta=\tilde\nabla^D\zeta+\tilde\nabla^N\zeta \qquad \text{for all } \zeta\in\Gamma(D;TD), \tag{2-9}$$

where

$$\tilde\nabla^N\zeta=\tfrac12\big(\nabla^N\zeta-J\nabla^N(J\zeta)\big)\in\Gamma(D;\Omega^1_D\otimes_{\mathbb{C}}TD^\perp).$$

Let $\hat\nabla$ and $\hat\nabla^D$ be the modifications of $\tilde\nabla$ and $\tilde\nabla^D$ as in (2-2), respectively. By (2-2) and (2-9), we also have

$$\hat\nabla_\xi\zeta=\hat\nabla^D_\xi\zeta+\hat\nabla^N_\xi\zeta \qquad \text{for all } \xi,\zeta\in\Gamma(D;TD), \tag{2-10}$$

where

$$\hat\nabla^N\zeta=\tilde\nabla^N\zeta-A^N(\zeta)\in\Gamma(D;\Omega^1_D\otimes_{\mathbb{C}}TD^\perp),\quad A^N(\zeta)=\tfrac14\big(\nabla_{J\zeta}J+J\nabla_\zeta J\big)^N,$$

$$(\nabla^N J)\xi:=\nabla^N(J\xi)-J\nabla^N\xi.$$

From (2-3), (2-10), and

$$N_J(\xi,\zeta)=N_{J_D}(\xi,\zeta)\in TD \qquad \text{for all } \xi,\zeta\in\Gamma(D;TD),$$

we conclude that

$$\begin{aligned}\hat\nabla^N_\xi\zeta-\hat\nabla^N_\zeta\xi&=(\hat\nabla_\xi\zeta-\hat\nabla_\zeta\xi)-(\hat\nabla^D_\xi\zeta-\hat\nabla^D_\zeta\xi)\\&=(\hat\nabla_\xi\zeta-\hat\nabla_\zeta\xi-[\xi,\zeta])-(\hat\nabla^D_\xi\zeta-\hat\nabla^D_\zeta\xi-[\xi,\zeta])\\&=T_{\hat\nabla}(\xi,\zeta)-T_{\hat\nabla^D}(\xi,\zeta)=0.\end{aligned}$$

In other words,

$$\hat\nabla^N_\xi\zeta=\hat\nabla^N_\zeta\xi \qquad \text{for all } \xi,\zeta\in\Gamma(D;TD).$$

From the last identity we get

$$\hat\nabla^N_{J\xi}\zeta+J\hat\nabla^N_\xi\zeta=\hat\nabla^N_\zeta(J\xi)+J\hat\nabla^N_\zeta\xi=\hat\nabla^N_\zeta(J\xi)-\hat\nabla^N_\zeta(J\xi)=0 \qquad \text{for all } \xi,\zeta\in\Gamma(D;TD).$$

Therefore,

$$\begin{aligned}\tilde\nabla^{(0,1)}\xi &= \tfrac12(\nabla\xi + J\nabla_J\xi)\\ &= \tfrac12(\nabla\xi^D + J\nabla_J\xi^D) + \tfrac12(\nabla\xi^N + J\nabla_J\xi^N)\\ &= \tfrac12(\nabla\xi^D + J\nabla_J\xi^D)\\ &= \nabla^{D,(0,1)}\xi \in \Gamma(D, T_D) \quad \text{for all } \xi \in \Gamma(D, T_D). \end{aligned}$$ □

Remark 2.2 The term $A(\xi)v$ in (2-2) is $\mathbb{C}$–linear in ξ and $\mathbb{C}$–antilinear in v. It vanishes if J is ω–compatible. Therefore,

$$\tilde\nabla^{0,1} = \nabla^{0,1} - A(\cdot).$$

2.2 Decorated dual graphs

Let $\Gamma = \Gamma(V, E, L)$ be a graph with set of vertices V, edges E, and legs L; the latter, also called flags or roots, are half-edges that have a vertex at one end and are open at the other end. Let $\vec{E}$ be the set of edges with an orientation. Given an oriented edge $\vec{e} \in \vec{E}$, let $\overleftarrow{e}$ denote the same edge e with the opposite orientation. For each $\vec{e} \in \vec{E}$, let $v_1(\vec{e})$ and $v_2(\vec{e})$ in V denote the starting and ending points of the arrow, respectively. For $v, v' \in V$, let $E_{v,v'}$ denote the subset of edges between the two vertices and $\vec{E}_{v,v'}$ denote the subset of oriented edges from v to v'. For every $v \in V$, let $\vec{E}_v$ denote the subset of oriented edges starting from v.

A genus labeling of Γ is a function $g\colon V \to \mathbb{N}$. An ordering of the legs of Γ is a bijection $a\colon L \to \{1, \dots, |L|\}$. If a decorated graph Γ is connected, the arithmetic genus of Γ is

$$g = g_\Gamma = \sum_{v \in V} g_v + \operatorname{rank} H_1(\Gamma; \mathbb{Z}), \tag{2-11}$$

where $H_1(\Gamma; \mathbb{Z})$ is the first homology group of the underlying topological space of Γ. Figure 1, left, illustrates a labeled graph with 2 legs.

Such decorated graphs Γ characterize different topological types of nodal marked surfaces

$$(\Sigma, \vec{z} = (z^1, \dots, z^k))$$

in the following way. Each vertex $v \in V$ corresponds to a smooth[13] component Σ_v of Σ with genus g_v. Each edge $e \in E$ corresponds to a node q_e obtained by connecting Σ_v

[13]We mean a smooth closed oriented surface.

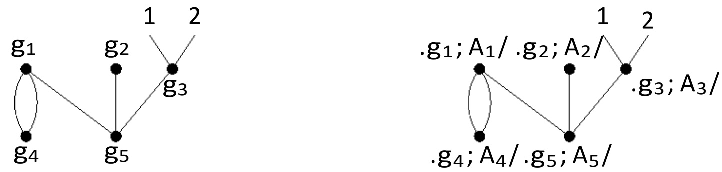

Figure 1: Left, a labeled graph Γ representing elements of $\overline{\mathcal{M}}_{g,2}$. Right, a labeled graph Γ representing elements of $\overline{\mathcal{M}}_{g,2}(X,A)$.

and $\Sigma_{v'}$ at the points $q_{\vec{e}} \in \Sigma_v$ and $q_{\overleftarrow{e}} \in \Sigma_{v'}$, where $e \in E_{v,v'}$ and $\vec{e}$ is an orientation on e with $v_1(\vec{e}) = v$. The last condition uniquely specifies $\vec{e}$ unless e is a loop connecting v to itself. Finally, each leg $l \in L$ connected to the vertex v_l corresponds to a marked point $z_{a^l} \in \Sigma_{v_l}$ disjoint from the connecting nodes. If Σ is connected, then g_Γ is the arithmetic genus of Σ. Thus we have

$$(\Sigma, \mathbf{z}) = \coprod_{v \in V} (\Sigma_v, \mathbf{z}_v, q_v)/\sim, \qquad q_{\vec{e}} \sim q_{\overleftarrow{e}} \text{ for all } e \in E, \tag{2-12}$$

where

$$\mathbf{z}_v = \mathbf{z} \cap \Sigma_v \quad \text{and} \quad q_v = \{q_{\vec{e}} : \vec{e} \in \vec{E}_v\} \quad \text{for all } v \in V.$$

In this situation, we say Γ is the dual graph of $(\Sigma, \mathbf{z})$. We treat q_v as an unordered set of marked points on Σ_v. If we fix an ordering on the set q_v, we denote the ordered set by $\vec{q}_v$.

A complex structure j on Σ is a set of complex structures $(j_v)_{v \in V}$ on its components. By a (complex) marked nodal curve, we mean a marked nodal real surface together with a complex structure $(\Sigma, j, \mathbf{z})$. Figure 2 illustrates a nodal curve with $(g_1, g_2, g_3, g_4, g_5) = (0, 2, 0, 1, 0)$ corresponding to Figure 1, left.

Similarly, for nodal marked surfaces mapping into a topological space X, we consider similar decorated graphs where the vertices carry an additional degree labeling

$$A: V \to H_2(X, \mathbb{Z}), \quad v \to A_v,$$

recording the homology class of the image of the corresponding component. Figure 1, right, illustrates a dual graph associated to a marked nodal map over the graph on the left.

Assume $D = \bigcup_{i \in [N]} D_i \subset X$ is an SNC symplectic divisor, $(\omega, J) \in \mathcal{J}(X, D)$, and (Σ, j) is a connected smooth complex curve. Then every J–holomorphic map $u: (\Sigma, j) \to (X, J)$ has a well-defined depth $I \subset [N]$, which is the maximal subset of $[N]$ such that $\mathrm{Image}(u) \subset D_I$. In particular, any map u intersecting D in a discrete set is of depth $I = \emptyset$. We say a point $x \in \Sigma$ is of depth I if D_I is the minimal stratum

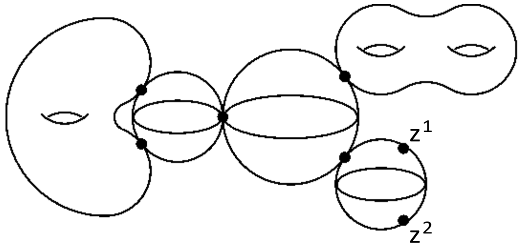

Figure 2: A nodal curve in $\overline{\mathcal{M}}_{4,2}$.

containing $u(x)$. Let $\mathcal{P}(N)$ be the set of subsets of $[N]$. The dual graph of (u,Σ) carries additional labelings

(2-13) $$I\colon V\sqcup E\longrightarrow \mathcal{P}(N), \quad v\longrightarrow I_v \ \text{for all } v\in V, \quad e\longrightarrow I_e \ \text{for all } e\in E,$$

recording the depths of smooth components and nodes of Σ.

2.3 Log moduli spaces

Assume $D=\bigcup_{i\in[N]} D_i\subset X$ is an SNC symplectic divisor, $(\omega,J)\in \mathcal{J}(X,D)$, and $u\colon(\Sigma,\mathfrak{j})\longrightarrow(X,J)$ is a J–holomorphic map of depth $I\subset[N]$ with smooth domain. Then, for every $i\in[N]-I$, the function

(2-14) $$\mathrm{ord}^i_u\colon\Sigma\longrightarrow\mathbb{N}, \quad \mathrm{ord}^i_u(x)=\mathrm{ord}_x(u,D_i),$$

recording the contact order of u with D_i at x is well-defined. For every $i\in I$, let $u^*\bar\partial_{\mathcal{N}_X D_i}$ be the pullback of the $\bar\partial$–operator $\bar\partial_{\mathcal{N}_X D_i}$ associated to (J,D_i) in (2-8). Since every $\bar\partial$–operator over a complex curve is integrable, $u^*\bar\partial_{\mathcal{N}_X D_i}$ defines a holomorphic structure on $u^*\mathcal{N}_X D_i$; see [29, Remark C.1.1]. The holomorphic line bundles

$$(u^*\mathcal{N}_X D_i,\, u^*\bar\partial_{\mathcal{N}_X D_i}) \quad \text{for all } i\in I$$

play a key role in the definition of the log moduli space below. Let $\Gamma_{\mathrm{mero}}(\Sigma,u^*\mathcal{N}_X D_i)$ be the space of nontrivial meromorphic sections of $u^*\mathcal{N}_X D_i$ with respect to $u^*\bar\partial_{\mathcal{N}_X D_i}$; $\mathbb{C}^*$ acts on $\Gamma_{\mathrm{mero}}(\Sigma,u^*\mathcal{N}_X D_i)$ by multiplication. We denote the $\mathbb{C}^*$–equivalence class of a section

$$\zeta\in\Gamma_{\mathrm{mero}}(\Sigma,u^*\mathcal{N}_X D_i)$$

by $[\zeta]$. The function

(2-15) $$\mathrm{ord}_{[\zeta]}\colon\Sigma\longrightarrow\mathbb{Z}, \quad \mathrm{ord}_{[\zeta]}(x)=\mathrm{ord}_x(\zeta),$$

recording the vanishing order of ζ at x (which is negative if ζ has a pole at x) is well-defined.

A log J–holomorphic tuple $(u,[\zeta];\Sigma,\mathfrak{j},w)$ consists of a smooth (closed) connected curve $(\Sigma,\mathfrak{j})$, ℓ distinct points $w=\{w^1,\dots,w^\ell\}$ on Σ, a $(J,\mathfrak{j})$–holomorphic map $u\colon(\Sigma,\mathfrak{j})\to(X,J)$ of depth $I\subset[N]$, and

$$[\zeta]\equiv([\zeta_i])_{i\in I}\in\prod_{i\in I}\big(\Gamma_{\mathrm{mero}}(\Sigma,u^*\mathcal{N}_XD_i)/\mathbb{C}^*\big) \tag{2-16}$$

such that

$$\mathrm{ord}_{u;[\zeta]}(x)\neq 0 \implies x\in w \quad \text{for all } x\in\Sigma, \tag{2-17}$$

where the vector-valued order function

$$\mathrm{ord}_{u;[\zeta]}(x)=\big((\mathrm{ord}^i_u(x))_{i\in[N]-I};(\mathrm{ord}_{[\zeta_i]}(x))_{i\in I}\big)\in\mathbb{Z}^N \quad \text{for all } x\in\Sigma$$

is defined via (2-14) and (2-15).

In particular, if u is of degree $A\in H_2(X;\mathbb{Z})$, then (2-17) implies

$$(A\cdot D_i)_{i\in[N]}=\sum_{w^a\in w}\mathrm{ord}_{u;[\zeta]}(w^a)\in\mathbb{Z}^N. \tag{2-18}$$

Remark 2.3 For every J–holomorphic map $u\colon(\Sigma,\mathfrak{j})\to(X,J)$ with smooth domain, ℓ distinct points $w^1,\dots,w^\ell$ in Σ, and $s_1,\dots,s_\ell\in\mathbb{Z}$, if $\mathrm{Im}(u)\subset D_i$, then up to $\mathbb{C}^*$–action there exists at most one meromorphic section $\zeta_i\in\Gamma_{\mathrm{mero}}(\Sigma,u^*\mathcal{N}_XD_i)$ with zeros/poles of order s_a at w^a (and nowhere else).

Definition 2.4 Let $D=\bigcup_{i\in[N]}D_i\subset X$ be an SNC symplectic divisor, let $(\omega,J)\in\mathcal{J}(X,D)$, and let

$$C\equiv(\Sigma,\mathfrak{j},\vec{z})=\Big(\coprod_{v\in V}C_v=(\Sigma_v,\mathfrak{j}_v,\vec{z}_v,q_v)\Big)\Big/\sim,\qquad q_{\vec{e}}\sim q_{\overleftarrow{e}}\quad\text{for all } \vec{e}\in\vec{E},$$

be a k–marked connected nodal curve with smooth components C_v and dual graph $\Gamma=\Gamma(V,E,L)$ as in (2-12). A prelog J–holomorphic map of contact type $\mathfrak{s}=(s_a)_{a=1}^k\in(\mathbb{Z}^N)^k$ from C to X is a collection

$$f\equiv\big(f_v=(u_v,[\zeta_v];C_v)\big)_{v\in V} \tag{2-19}$$

such that

1. for each $v\in V$, $(u_v,[\zeta_v])\equiv(([\zeta_{v,i}])_{i\in I_v};\Sigma_v,\mathfrak{j}_v,z_v\cup q_v)$ is a log J–holomorphic tuple,
2. $u_v(q_{\vec{e}})=u_{v'}(q_{\overleftarrow{e}})\in X$ for all $\vec{e}\in\vec{E}_{v,v'}$,

(3) $s_{\vec e} = \mathrm{ord}_{u_v;v}(q_{\vec e}) = -\mathrm{ord}_{u_{v'};v'}(q_{\overleftarrow e}) = -s_{\overleftarrow e}$ for all $v,v'\in V$ and $e\in E_{v,v'}$, (4) $\mathrm{ord}_{u_v;v}(z^a) = s_a$ for all $v\in V$ and $z^a\in z_v$.

In other words, a prelog map is a nodal J–holomorphic map with a bunch of meromorphic sections on each smooth component, opposite contact orders at the nodes, and prescribed contact orders at the marked points.

Remark 2.5 For every $v\in V$ and $\vec e\in \vec E_v$, let

(2-20) $$s_{\vec e} = (s_{\vec e;i})_{i\in[N]} = \big((\mathrm{ord}^i_{u_v}(q_{\vec e}))_{i\in[N]-I_v};\ (\mathrm{ord}_{[\zeta_{v;i}]}(q_{\vec e}))_{i\in I_v}\big)\in\mathbb{Z}^N$$

be the contact order data at the nodal point $q_{\vec e}\in\Sigma_v$. For $\vec e\in\vec E_{v,v'}$, if u_v and $u_{v'}$ have image in D_{I_v} and $D_{I_{v'}}$, respectively, by condition (2) above, we have

$$u(q_e) = u_v(q_{\vec e}) = u_{v'}(q_e)\in D_{I_v}\cap D_{I_{v'}} = D_{I_v\cup I_{v'}};$$

ie $I_e \supset I_v\cup I_{v'}$. If $i\in[N]-I_v\cup I_{v'}$, by (2-14) we have

$$s_{\vec e;i},\ s_{\overleftarrow e;i}\ge 0.$$

Therefore, by condition (3) above, they are both zero, ie

(2-21) $$I_e = I_v\cup I_{v'}\quad\text{and}\quad s_{\vec e}\in\mathbb{Z}^{I_e}\times\{0\}^{[N]-I_e}\subset\mathbb{Z}^N\qquad\text{for all } \vec e\in\vec E_{v,v'}.$$

The dual graph Γ of every prelog map in Definition 2.8 carries an additional decoration $s_{\vec e}\in\mathbb{Z}^N$ for all $\vec e\in\vec E$, which records the contact order of $(u_v,[\zeta_v])$ at the nodal point $q_{\vec e}\in\Sigma_v$ for every $\vec e\in\vec E_v$; see Figure 3. The set L of legs of Γ is also decorated with the vector-valued contact order function

$$\mathfrak{s}\colon L\to\mathbb{Z}^N,\quad l\to s_l;$$

recording the contact vector at the marked point z^{a_l} corresponding to l.

Two prelog maps $(u,[\zeta],C)\equiv(u_v,[\zeta_v],C_v)_{v\in V}$ and $(\tilde u,[\tilde\zeta],\tilde C)\equiv(\tilde u_v,[\tilde\zeta_v],\tilde C_v)_{v\in V}$ with isomorphic decorated dual graphs Γ as in Definition 2.4 are equivalent if there exists a biholomorphic identification

(2-22) $$h\colon C\to\tilde C\equiv\big(h_v\colon(\Sigma_v,j_v)\to(\tilde\Sigma_{h(v)},\tilde j_{h(v)})\big)_{v\in V}$$

such that

$$h(\mathbf{z}^a)=\tilde z^a\qquad\text{for all } a=1,\dots,k;$$
$$\tilde u\circ h=u;$$
$$[\tilde\zeta_{h(v);i}\circ h_v]=[\zeta_{v;i}]\qquad\text{for all } v\in V,\ i\in I_v.$$

A prelog map f is stable if the group of self-equivalences $\mathrm{Aut}(f)$ is finite. By Remark 2.3, a prelog map is stable if and only if the underlying nodal marked J–holomorphic map is stable. Clearly, the automorphism group of a prelog map is a subgroup of the automorphism group of the underlying nodal marked J–holomorphic map. Example 2.18 below illustrates some rare cases when the two groups are different. The equivalence class of a prelog map is called a prelog curve. For every such Γ, we denote the space of k–marked degree-A prelog J–holomorphic curves with dual graph Γ and contact pattern s by

(2-23) $$\mathcal{M}^{\mathrm{plog}}_{g,s}(X,D,A)_\Gamma.$$

If Γ has only one vertex v with $I = I_v$, then

$$\mathcal{M}_{g,s}(X,D,A)_I \cong \mathcal{M}^{\mathrm{plog}}_{g,s}(X,D,A)_\Gamma$$

is simply the space of equivalence classes of genus-g degree-A k–marked log J–holomorphic tuples with an ordering on the marked points and contact type s.

In $g = 0$, the forgetful map

(2-24) $$\mathcal{M}_{0,s}(X,D,A)_I \to \mathcal{M}_{0,k}(D_I,A), \quad [u,[\zeta],\Sigma,j,\vec{z}] \mapsto [u,\Sigma,j,\vec{z}],$$

into the (virtual) main stratum of moduli space of k–marked degree-A J–holomorphic curves into D_I gives an identification of two sets. That is because for every degree $d \in \mathbb{Z}$ holomorphic line bundle $L \to \mathbb{P}^1$, every set of distinct points $z^1,\ldots,z^k \in \mathbb{P}^1$, and every set of integers $m_1,\ldots,m_k$ such that $m_1+\cdots+m_k = d$, up to the action of $\mathbb{C}^*$, there always exists exactly one meromorphic section of L with poles/zeros of order m_i at z^i. In the higher genus case, however, the (virtual) normal bundle of this embedding is the direct sum of $|I|$ copies of the dual of the Hodge bundle (ie tangent space of $\mathrm{Pic}^0(\Sigma)$ at the trivial line bundle); see Lemma 5.2.

Example 2.6 If D is smooth, ie $N = 1$, a (pre)log map with smooth domain of depth $\emptyset$ is just a J–holomorphic map u with image not into D, $u^{-1}(D) \subset \vec{z}$, and

$$s = (\mathrm{ord}_{z^a}(u,D))_{a\in[k]} \in \mathbb{N}^k$$

as in the definition of the relative moduli spaces in (4-4). Thus there exists a one-to-one correspondence between the virtual main stratum of the moduli space of relative J–holomorphic curves of contact order s, and the space of depth $\emptyset$ (pre)log curves of the same contact pattern. Also, a depth-$\{1\}$ (pre)log J–holomorphic curve with smooth domain is represented by a J–holomorphic map $u\colon(\Sigma,j) \to (D, J|_{TD})$ and a meromorphic section ζ of $u^*\mathcal{N}_X D$ such that Σ includes the set of zeros and poles of ζ, and

$$\mathfrak{s} = (\mathrm{ord}_{z^a}(\zeta))_{a\in[k]} \in \mathbb{Z}^k$$

as in the definition of the relative moduli spaces. The definitions, however, become different if we consider maps with nodal domain.

For some decorated dual graphs Γ, the expected dimension of $\mathcal{M}^{\mathrm{plog}}_{g,\mathfrak{s}}(X,D,A)_\Gamma$, calculated via (5-4) and the matching conditions at the nodes, could be bigger than or equal to the expected dimension of the (virtual) main stratum $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)$ (something that we do not want to happen); see the following example. In order for a nodal prelog curve to be in the limit of the (virtual) main stratum, there are other global combinatorial and noncombinatorial obstructions that we are going to describe next. Of course, as in the classical case, we might get prelog curves satisfying these conditions that do not belong to the closure of the main stratum.

Example 2.7 Let $X = \mathbb{P}^2$ with projective coordinates $[x_1;x_2;x_3]$ and $D = D_1 \cup D_2$ (thus $N = 2$) be a transverse union of two hyperplanes (lines). For

$$g = 0;\quad \mathfrak{s} = ((3,2),(0,1)) \in (\mathbb{N}^2)^2 \quad\text{and}\quad A = [3] \in H_2(X,\mathbb{Z}) \cong \mathbb{Z};$$

we have that $\mathcal{M}_{0,\mathfrak{s}}(X,D,[3])$ is a manifold of complex dimension 4. If $D_1 = (x_1 = 0)$ and $D_2 = (x_2 = 0)$, every element in $\mathcal{M}_{0,\mathfrak{s}}(X,D,[3])$ is equivalent to a holomorphic map of the form

$$[z;w] \to [z^3; z^2w; a_3z^3 + a_2z^2w + a_1zw^2 + a_0w^3]. \tag{2-25}$$

Let Γ be the dual graph with three vertices v_1, v_2, v_3, and two edges e_1, e_2 connecting v_1 to v_3 and v_2 to v_3, respectively. Furthermore, choose the orientations $\vec{e}_1$ and $\vec{e}_2$ to end at v_3, and assume

$$I_{v_1} = I_{v_2} = \emptyset;\quad I_{v_3} = \{1,2\};\quad s_{\vec{e}_1} = (2,1);\quad s_{\vec{e}_2} = (1,1);\quad A_{v_1} = [2];\quad A_{v_2} = [1].$$

See Figure 3. Note that u_{v_3} is map of degree 0 from a sphere with three special points, two of which are the nodes connecting Σ_{v_3} to Σ_{v_1} and Σ_{v_2}, and the other one is the first marked point z^1 with contact order $(3,2)$. The second marked point with contact order $(0,1)$ lies on Σ_{v_1}. A simple calculation shows that $\mathcal{M}^{\mathrm{plog}}_{0,\mathfrak{s}}(X,D,[3])_\Gamma$ is also a manifold of complex dimension 4. The image of u_2 (dashed curve) could be any line different from D_1 and D_2 passing through D_{12}, and every such u_1 is equivalent to a holomorphic map of the form

$$[z;w] \to [z^2; zw; a_2z^2 + a_1zw + a_0w^2]. \qquad \square$$

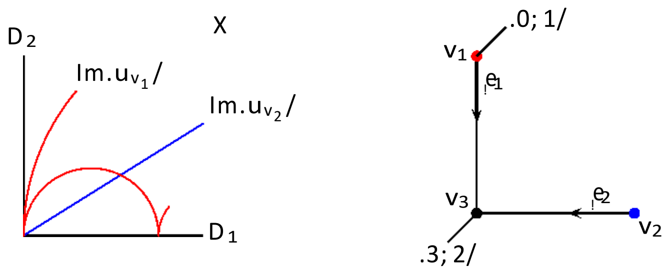

Figure 3: A 2–marked genus-0 nodal degree-3 prelog map in $\mathbb{P}^3$ relative to two lines. The dashed curve is a line. The dotted curve is a conic. They are connected by a ghost bubble that maps to D_{12}.

Corresponding to the decorated dual graph $\Gamma = \Gamma(V;E;L)$ of a prelog map as in Definition 2.4 and an arbitrary orientation $\mathrm{O} = \{\vec{e}\}_{e\in E} \subset \vec{E}$ on the edges, we define a homomorphism of $\mathbb{Z}$–modules

$$\delta = \delta(\Gamma)\colon \mathbb{Z}^E \oplus \bigoplus_{v\in V} \mathbb{Z}^{I_v} \xrightarrow{\delta=\delta_{\mathrm{O}}} \mathcal{T} = \mathcal{T}(\Gamma) = \bigoplus_{e\in E} \mathbb{Z}^{I_e} \tag{2-26}$$

in the following way. For every $e \in E$, let

$$\delta(1_e) = s_{\vec{e}} \in \mathbb{Z}^{I_e}, \tag{2-27}$$

where 1_e is the generator of $\mathbb{Z}^e$ in $\mathbb{Z}^E$ and $\vec{e}$ is the chosen orientation on e in O. In particular, $\delta(1_e) = 0$ for any e with $I_e = \emptyset$. Similarly, for every $v \in V$ and $i \in I_v$, let $1_{v;i}$ be the generator of the i^{th} factor in $\mathbb{Z}^{I_v}$, and define

$$\delta(1_{v;i}) = \delta_{v;i} \in \bigoplus_{e\in E} \mathbb{Z}^{I_e} \tag{2-28}$$

to be the vector which has $1_{e;i} \in \mathbb{Z}^{I_e} \subset \mathbb{Z}^N$ in the e^{th} factor if $v = v_1(\vec{e})$ and e is not a loop, which has $-1_{e;i} \in \mathbb{Z}^{I_e}$ in the e^{th} factor if $v = v_2(\vec{e})$ and e is not a loop, and which is zero otherwise. This is well-defined by the first equality in (2-21). Let

$$\begin{gathered} \mathfrak{f} = \mathfrak{f}(\Gamma) = \operatorname{image}(\delta); \quad \mathcal{K} = \mathcal{K}(\Gamma) = \operatorname{Ker}(\delta); \\ \mathcal{CK} = \mathcal{CK}(\Gamma) = \mathcal{T}/\mathfrak{f} = \operatorname{coker}(\delta). \end{gathered} \tag{2-29}$$

By Definition 2.4(3), the $\mathbb{Z}$–modules $\mathfrak{f}$, $\mathcal{K}$, and $\mathcal{CK}$ are independent of the choice of orientation O on E and are invariants of the decorated graph Γ. In particular,

$$\mathcal{K} = \Big\{ \big((\lambda_e)_{e\in E}; (s_v)_{v\in V}\big) \in \mathbb{Z}^E \oplus \bigoplus_{v\in V} \mathbb{Z}^{I_v} : s_{v'} - s_v = \lambda_e s_{\vec{e}} \ \text{ for all } v, v' \in V;\ \vec{e} \in \vec{E}_{v',v} \Big\}. \tag{2-30}$$

Here, via the first identity in (2-21) and the inclusion $\mathbb{Z}^{I_v}\subset\mathbb{Z}^{I_v}\oplus\{0\}^{I_e-I_v}\cong\mathbb{Z}^{I_e}$, we think of s_v as a vector also in $\mathbb{Z}^{I_e}$ for all $e\in E_v$. For any field $\mathbb{F}$ of characteristic zero, let

(2-31) $$D_{\mathbb{F}}=D\otimes_{\mathbb{Z}}\mathbb{F};\quad T_{\mathbb{F}}=T\otimes_{\mathbb{Z}}\mathbb{F};\quad \mathfrak{f}_{\mathbb{F}}=\mathfrak{f}\otimes_{\mathbb{Z}}\mathbb{F};\\ K_{\mathbb{F}}=K\otimes_{\mathbb{Z}}\mathbb{F}\quad\text{and}\quad CK_{\mathbb{F}}=CK\otimes_{\mathbb{Z}}\mathbb{F}$$

be the corresponding $\mathbb{F}$–vector spaces and $\varrho_{\mathbb{F}}\colon D_{\mathbb{F}}\to T_{\mathbb{F}}$ be the corresponding $\mathbb{F}$–linear map. Via the exponentiation map, let

$$\exp(\mathfrak{f}_{\mathbb{C}})\subset\prod_{e\in E}(\mathbb{C}^*)^{I_e}$$

be the subgroup corresponding to the sub-Lie algebra $\mathfrak{f}_{\mathbb{C}}\subset T_{\mathbb{C}}$, and denote the quotient group by

$$\mathbb{G}=\mathbb{G}(\mathfrak{f})=\prod_{e\in E}(\mathbb{C}^*)^{I_e}\Big/\exp(\mathfrak{f}_{\mathbb{C}})=\frac{\prod_{e\in E}(\mathbb{C}^*)^{I_e}}{\exp\big(\varrho_{\mathbb{C}}\big((\mathbb{C})^E\oplus\bigoplus_{v\in V}(\mathbb{C})^{I_v}\big)\big)}.$$

In the following, we will construct a map

(2-32) $$\mathcal{M}^{\mathrm{plog}}_{g,s}(X,D,A)_\Gamma\xrightarrow{\ \mathrm{ob}_\Gamma\ }\mathbb{G}(\Gamma),$$

which will be used in the definition of log moduli spaces.

Given a prelog map $f\equiv f_v=(u_v,[\Sigma_v],C_v)_{v\in V}$ as in Definition 2.4, fix an arbitrary set of representatives

(2-33) $$\zeta_v=(\zeta_{v,i})_{i\in I_v}\in\Gamma_{\mathrm{mero}}(\Sigma_v,u_v^*N_XD_{I_v})\qquad\text{for all } v\in V.$$

For each $v\in V$ and $\vec e\in E_v$, let $z_{\vec e}$ be an arbitrary holomorphic coordinate in a sufficiently small disk $\Delta_{\vec e}$ around the nodal point $(z_{\vec e}=0)=q_{\vec e}\in\Sigma_v$. By (2-15), for every $v\in V$, $\vec e\in E_v$ and $i\in I_v$, in a local holomorphic trivialization

$$u^*N_XD_i|_{\Delta_{\vec e}}\cong N_XD_i|_{u(q_e)}\times\Delta_e,$$

we have

(2-34) $$\zeta_{v,i}(z_{\vec e})=z_{\vec e}^{s_{\vec e,i}}\,\tilde\zeta_{v,i}(z_{\vec e})$$

such that

$$0\neq\tilde\zeta_{v,i}(0)\equiv\eta_{\vec e,i}\in N_XD_i|_{u(q_e)}$$

is independent of the choice of the trivialization. Similarly, by [48, (6.1)], for every $v\in V$, $\vec e\in E_v$ and $i\in I_e-I_v$, the map u_v has a well-defined $s_{\vec e,i}^{\text{th}}$ derivative

(2-35) $$\eta_{\vec e,i}\in N_XD_i|_{u(q_e)}$$

(with respect to the coordinate $z_{\vec e}$) in the normal direction to D_i at the nodal marked point $q_{\vec e}$.

With the choice of orientation $\mathbb{O} \equiv \{\vec e\}_{e\in E} \subset \hat E$ on the edges as before, since $\beta_{\vec e;i} \neq 0$ for all $\vec e \in \hat E$ and $i \in I_e$, the tuples

(2-36) $$\beta_e = (\beta_{\vec e;i}/\beta_{\overleftarrow{e};i})_{i\in I_e} \in (\mathbb{C}^*)^{I_e} \qquad \text{for all } \vec e \in \mathbb{O}$$

give rise to an element

(2-37) $$(\beta_e)_{e\in E} \in \prod_{e\in E} (\mathbb{C}^*)^{I_e}.$$

The action of the subgroup $\exp(\mathfrak{f}_{\mathbb{C}})$ on β corresponds to rescalings of (2-33) and change of coordinates in (2-34); ie the class $\text{ob}_\Gamma(f) = [\beta]$ of β in

$$G = \prod_{e\in E} (\mathbb{C}^*)^{I_e} \Big/ \exp(\mathfrak{f}_{\mathbb{C}})$$

is independent of the choice of representatives in (2-33) and local coordinates in (2-34). If f and f' are equivalent with respect to a reparametrization $h: \Sigma' \to \Sigma$ as in (2-22), the respective associated group elements β and β' would be the same with respect to any h–symmetric choice of holomorphic coordinates $\{z_{\vec e}\}_{\vec e \in \hat E}$. Therefore,

(2-38) $$\text{ob}_\Gamma : [f] \mapsto [\beta] \in G$$

is well-defined. By definition, $\text{ob}_\Gamma([f]) = 1$ if and only if there exists a choice of representatives $\{\zeta_{v;i}\}_{i\in I_v;\, v\in V}$ and local coordinates $\{z_{\vec e}\}_{\vec e \in \hat E}$ such that

$$\beta_{\vec e} = \beta_{\overleftarrow{e}} \qquad \text{for all } e \in E.$$

Definition 2.8 Let $D = \bigcup_{i\in[N]} D_i \subset X$ be an SNC symplectic divisor and $(\omega, J) \in \mathcal{J}(X, D)$. A *log J–holomorphic map* is a prelog J–holomorphic map f with the decorated dual graph Γ such that

(1) there exist functions

$$s: V \to \mathbb{R}^N, \quad v \mapsto s_v, \qquad \text{and} \qquad \lambda: E \to \mathbb{R}_+, \quad e \mapsto \lambda_e,$$

such that

(a) $s_v \in \mathbb{R}_+^{I_v} \times \{0\}^{[N]-I_v}$ for all $v \in V$,

(b) $s_{v_2(\vec e)} - s_{v_1(\vec e)} = \lambda_e s_{\vec e}$ for every $\vec e \in \hat E$;

(2) $\text{ob}_\Gamma(f) = 1 \in G(\Gamma)$.

Condition (1)(b) is well-defined because of Definition 2.4(3). If (2) holds, we say that the prelog map f is G–unobstructed. Condition (2) is independent of the choice of orientation $\mathfrak{O}$ on E used to define ob_Γ.

Remark 2.9 A nodal map in the relative compactification (when D is smooth) with image in an expanded degeneration $X[m]$ comes with a partial ordering of the smooth components of the domain, such that the components mapped into X have order 0 and those mapped into the r^{th} copy of $\mathbb{P}_X D$ are of order r; see Section 4.1. In the compactification process, a component sinking faster into D results in a component with higher order. From our perspective, the vector-valued function $s\colon V \to \mathbb{R}^N$ in condition (1) is a generalization of this partial ordering to the SNC case with $\mathbb{R}^N$ instead of $\mathbb{Z}$; see Lemma 4.3. From the tropical perspective of [4, Definition 2.5.3], condition (1) is equal to the existence of a tropical map from a tropical curve associated to Γ into $\mathbb{R}^N_{\geq 0}$. This condition puts a big restriction on the set of contact vectors $s_{\vec{e}}$. For example, if $I_v, I_{v'} = \emptyset$, then for any other $v'' \in V$ and oriented edges $\vec{e} \in \vec{E}_{v,v''}$ and $\vec{e}' \in \vec{E}_{v',v''}$, the contact vectors $s_{\vec{e}}$ and $s_{\vec{e}'}$ should be positively proportional. Condition (2) has no explicit equivalent in [1; 17; 36; 20], but it is related to the slope condition at each node in [20].

Remark 2.10 The discussion above includes $\mathbb{R}^N$–valued functions, all of them denoted by s, on the set of vertices, oriented edges and legs of a decorated dual graph Γ, which play different roles and should not be confused. The contact orders $\mathbf{s} = (s_1, \dots, s_k)$ at the legs (marked points) are fixed for a moduli space (they are independent of Γ) and define a function $s\colon L \to \mathbb{Z}^N$. The contact orders $(s_{\vec{e}})_{\vec{e} \in \vec{E}}$ at nodal points define a function $s\colon \vec{E} \to \mathbb{Z}^N$ and are part of the decoration of Γ. Finally the function $s\colon V \to \mathbb{R}^N$ (and $\lambda\colon E \to \mathbb{R}_+$) is not part of the defining data of a log map. We only require the latter to exist in order for a prelog map to define a log map.

Example 2.11 Example 2.7 does not satisfy Definition 2.8(1). Since $I_{v_1} = I_{v_2} = \emptyset$, we should have $s_{v_1} = s_{v_2} = (0, 0)$. Then condition (1)(b) requires $s_{\vec{e}_1} = (2, 1)$ and $s_{\vec{e}_1} = (1, 1)$ to be positive multiples of s_{v_3}, which is impossible. A straightforward calculation shows that the line component u_{v_2} in any limit of (2-25) with a component u_{v_1} as in Figure 3 should lie in D_1. Then the function $s\colon V \to \mathbb{R}^2$ given by $s_{v_1} = (0, 0)$, $s_{v_2} = (1, 0)$ and $s_{v_3} = (2, 1)$ satisfies Definition 2.8(1).

The following definition lists the combinatorial properties of an admissible decorated dual graph.

Definition 2.12 For a fixed SNC symplectic divisor $D=\bigcup_{i\in[N]} D_i$ in X, given $g,k\in\mathbb{N}$, $A\in H_2(X;\mathbb{Z})$ and $s\in(\mathbb{Z}^N)^k$, we denote by $\mathrm{DG}(g,s,A)$ the set of (stable) connected dual graphs $\Gamma=\Gamma(V,E,L)$ with k legs and

(a) a genus decoration of total genus g,

(b) a degree decoration of total degree A,

(c) an ordering $L\to\{1,\ldots,k\}$,

(d) set decorations $I\colon V,E\to\mathcal{P}(N)$ satisfying $I_e=I_v\cup I_{v'}$ for all $v,v'\in V$ and $e\in E_{v,v'}$, and

(e) a vector decoration on the set $\vec{E}$ of oriented edges, $\vec{e}\to s_{\vec{e}}\in\mathbb{Z}^{I_e}\subset\mathbb{Z}^N$, satisfying

$$s_{\vec{e}}+s_{\overleftarrow{e}}=0\quad\text{for all }\vec{e}\in\vec{E};$$

such that condition (1) of Definition 2.8 holds and

$$(A\cdot D_i)_{i\in[N]}=\sum_{\vec{e}\in\vec{E}_v} s_{\vec{e}}+\sum_{\substack{l\in L\\ v_l=v}} s_l\quad\text{for all } v\in V. \tag{2-39}$$

$\mathrm{DG}(g,s,A)$ is the set of possible combinatorial types of stable connected genus-g k–marked degree-A log curves of contact type s. Note that the defining conditions of $\mathrm{DG}(g,s,A)$ do not capture Definition 2.8(2); the latter is a noncombinatorial condition. Example 2.13 below illustrates a legitimate Γ such that the moduli space of prelog curves of type Γ has an expected dimension larger than the expected dimension of the (virtual) main stratum. Then, imposing condition (2) of Definition 2.8 would reduce the dimension to less than the expected dimension of the (virtual) main stratum.

For every $\Gamma\in\mathrm{DG}(g,s,A)$, define

$$\mathcal{M}_{g,s}(X,D,A)_\Gamma=\mathrm{ob}_\Gamma^{-1}(1)\subset\mathcal{M}^{\mathrm{plog}}_{g,s}(X,D,A)_\Gamma \tag{2-40}$$

to be the stratum of log J–holomorphic curves of type Γ. We then define the moduli space of genus-g degree-A stable nodal log J–holomorphic curves of contact type s to be the union

$$\overline{\mathcal{M}}^{\log}_{g,s}(X,D,A)=\bigcup_{\Gamma\in\mathrm{DG}(g,s,A)}\mathcal{M}_{g,s}(X,D,A)_\Gamma. \tag{2-41}$$

Example 2.13 Let

$$X=\mathbb{P}^3,\quad D_1\cup D_2=\mathbb{P}^2\cup\mathbb{P}^2,\quad A=2d\in H_2(X;\mathbb{Z})\cong\mathbb{Z},\quad g=(d-1)^2,$$
$$s=((1,0),\ldots,(1,0),(0,1),\ldots,(0,1))\in(\mathbb{Z}^2)^{4d},$$

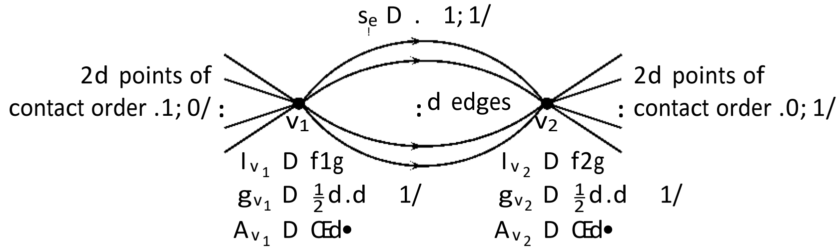

Figure 4: A decorated graph in $DG(g=(d-1)^2, s, A=2d[\ell])$, corresponding to two generic degree-d curves in D_1 and D_2, intersecting at d points along D_{12}.

and let $\Gamma \in DG(g,s,A)$ be the decorated dual graph illustrated in Figure 4. Note that the function $s: V \to \mathbb{R}^2$ given by $s_{v_1}=(1,0)$ and $s_{v_2}=(0,1)$ satisfies Definition 2.8(1). Every element of $M^{plog}_{g,s}(X,D,A)_\Gamma$ is supported on two generic degree-d plane curves in D_1 and D_2 intersecting at d points along D_{12}. By (5-4) and Definition 2.4(2), the expected $\mathbb{C}$–dimensions of $M_{g,s}(X,D,A)$ and $M^{plog}_{g,s}(X,D,A)_\Gamma$ are $8d$ and $9d-2$, respectively.

Orient each edge so that $v_1(\vec{e}_i) = v_1$ for all $i=1,\dots,d$. Then ƒ = %(D) in (2-29) is generated by the vectors $s_{\vec e_1},\dots,s_{\vec e_d}$, ${}_{v_1} = {}_{v_1;1}$ and ${}_{v_2} = {}_{v_2;2}$, so that the only relation is

$$ {}_{v_1} + {}_{v_2} + (s_{\vec e_1} + \dots + s_{\vec e_d}) = 0. $$

We conclude that the obstruction group $G(\Gamma)$ is complex $(d-1)$–dimensional. Therefore, the subset of log curves

$$ M_{g,s}(X,D,A)_\Gamma \subset M^{plog}_{g,s}(X,D,A)_\Gamma $$

is of the expected $\mathbb{C}$–dimension $(9d-2)-(d-1)=8d-1<8d$. □

Remark 2.14 By Remark 2.3, for every k–marked stable nodal curve $f \in \overline{M}_{g,k}(X,A)$ with dual graph Γ, fixing $s \in (\mathbb{Z}^N)^k$ and the vector decoration $\{s_{\vec e}\}_{\vec e \in \vec E}$ as in Definition 2.12(e), there exists at most one log curve $f_{\log} \in \overline{M}^{\log}_{g,s}(X,D,A)$ with orders s_i at z^i and $s_{\vec e}$ at $q_{\vec e}$ lifting f. Furthermore, $f_{\log}$ is stable if and only if f is stable.

Lemma 2.15 Given $f \in \overline{M}_{g,k}(X,A)$ with the dual graph Γ and $s \in (\mathbb{Z}^N)^k$, the set of possible vector decorations $\{s_{\vec e}\}_{\vec e \in \vec E}$ as in Definition 2.12 satisfying (2-39), and thus the set of possible log lifts of f, is finite.

Proof Since $s_{\vec e,i} = \pm s_{e,i}$ for all $e \in E$ and $i \in I_e$, it is sufficient to show that the set of possible values for $\{s_{\vec e,i}\}_{\vec e \in \vec E; i \in I_e}$ is bounded from above. Fix $i \in [N]$. If $v \in V$ and $e \in E_v$ such that $i \in I_e \subset I_v$, then $s_{\vec e,i}$ and $s_{e,i}$ are uniquely determined by the tangency order of u_v with D_i. Therefore, we can restrict to the subset $V_i \subset V$ of all vertices v such that $i \in I_v$, and the edges between them. Given a decoration $\{s_{\vec e}\}_{\vec e \in \vec E}$ as in Definition 2.12 satisfying (2-39), let $\vec E_i$ be the subset of oriented edges $\vec e$ such that $e \in E_{v,v'}$ for some $v, v' \in V_i$ and $s_{\vec e,i} > 0$. Let Γ_i be the oriented graph made of V_i and the oriented edges in $\vec E_i$. By condition (1)(b) in Definition 2.8, Γ_i does not have any oriented loop. Therefore, $\vec E_i$ defines a partial order on V_i. Let $v \in V_i$ be a maximal vertex. There is no oriented edge in $\vec E_i$ pointing toward v. Therefore, for every $\vec e \in \vec E_v$, either $s_{\vec e,i} = 0$ or $s_{\vec e,i} > 0$. The identity

$$A_v \cdot D_i = \sum_{\vec e \in \vec E_v} s_{\vec e,i} + \sum_{l \in L;\, v_l = v} s_{l,i}$$

puts an upper bound on $\{s_{\vec e,i}\}_{\vec e \in \vec E_v}$. Moving down in the partial order on V_i we get upper bounds on other $s_{\vec e,i}$. □

Lemma 2.16 For every genus-0 k–marked stable nodal map f in $\overline{\mathcal{M}}_{0,k}(X,A)$ with dual graph Γ and a fixed s, there exists at most one vector decoration $\{s_{\vec e}\}_{\vec e \in \vec E}$ as in Definition 2.12(e) satisfying (2-39). In particular, the forgetful map

$$\overline{\mathcal{M}}^{\log}_{0,s}(X,D,A) \to \overline{\mathcal{M}}_{0,k}(X,A)$$

is an embedding (of sets).

Proof Assume that there are two different decorations $\{s_{\vec e}\}_{\vec e \in \vec E}$ and $\{s'_{\vec e}\}_{\vec e \in \vec E}$ as in Definition 2.12 satisfying (2-39). There is some $i \in [N]$ such that $\{s_{\vec e,i}\}_{\vec e \in \vec E}$ and $\{s'_{\vec e,i}\}_{\vec e \in \vec E}$ are different. Since $g = 0$, Γ is a tree and the subset of edges $E' \subset E$ where $s_{\vec e,i} \neq s'_{\vec e,i}$ determines a subtree of that. In particular, there exists a vertex $v \in V$ that is connected to only one edge $e^0 \in E'$. Orient e^0 so that v is the starting point. Then, by (2-39),

$$A_v \cdot D_i = s_{\vec e^0,i} + \sum_{\vec e \in \vec E_v - \{\vec e^0\}} s_{\vec e,i} + \sum_{\substack{l \in L \\ v_l = v}} s_{l,i} \neq s'_{\vec e^0,i} + \sum_{\vec e \in \vec E_v - \{\vec e^0\}} s'_{\vec e,i} + \sum_{\substack{l \in L \\ v_l = v}} s'_{l,i} = A_v \cdot D_i;$$

a contradiction. □

Example 2.17 below describes a situation where f has different lifts but the automorphism groups of f and its lifts are the same. Example 2.18 describes a situation where f has different lifts and some of them have smaller automorphism groups.

Example 2.17 Let $X = \mathbb{P}^2$, let $D = \mathbb{P}^1$ be a hyperplane (line), and $p_1; p_2; p_3; p_4$ be four distinct points in D. Let $u_{v_1}\colon \Sigma_{v_1} \cong \mathbb{P}^1 \to D$ be a degree-one map and ζ_{v_1} be a meromorphic section of $u_{v_1}^*\mathcal{N}_X D$ with two poles of orders 1 and 2 at $q_{\underline{e}_1} = u_{v_1}^{-1}(p_1)$ and $q_{\underline{e}_2} = u_{v_1}^{-1}(p_2)$, respectively, and a zero of order 4 at $z^1 = u_{v_1}^{-1}(p_3)$. Similarly, let $u_{v_2}\colon \Sigma_{v_2} \to D$ be a degree-one map and ζ_{v_2} be a meromorphic section of $u_{v_2}^*\mathcal{N}_X D$ with two zeros of orders 1 and 2 at $q_{e_1} = u_{v_2}^{-1}(p_1)$ and $q_{e_2} = u_{v_2}^{-1}(p_2)$, respectively, and a pole of order 2 at $q_{\underline{e}_3} = u_{v_2}^{-1}(p_4)$. Finally, let $u_{v_3}\colon \Sigma_{v_3} \to X$ be a smooth conic with a tangency of order 2 with D at $q_{e_3} = u_{v_3}^{-1}(p_4)$. The tuple

$$u_{\log} \equiv (u_{v_3}; (u_{v_2}; \zeta_{v_2}); (u_{v_1}; \zeta_{v_1}))$$

with the nodal 1–marked domain

$$(\Sigma; z^1) = (\Sigma_{v_1}; z^1; q_{\underline{e}_1}; q_{\underline{e}_2}) \sqcup (\Sigma_{v_2}; q_{e_1}; q_{e_2}; q_{\underline{e}_3}) \sqcup (\Sigma_{v_3}; q_{e_3})/\sim; \qquad q_{\underline{e}} \sim q_e \quad \text{for all } e \in \{e_1; e_2; e_3\};$$

defines an element of $\overline{\mathcal{M}}^{\log}_{1;(4)}(X; D; [4])$. Let $u'_{\log}$ be a similar tuple with the roles of p_1 and p_2 reversed, ie $u = (u_{v_1}; u_{v_2}; u_{v_3})$ remains the same but ζ_{v_1} and ζ_{v_2} exchange their orders at the preimages of p_1 and p_2. Therefore, $[u_{\log}; \Sigma; z^1]$ and $[u'_{\log}; \Sigma; z^1]$ are different lifts of the same 1–marked stable curve $[u; \Sigma; z^1]$ in $\overline{\mathcal{M}}_{1;1}(X; [4])$. Note that e_1 and e_2 form a loop in Γ. In this example, the two vector decorations corresponding to $(u_{\log}; \Sigma; z^1)$ and $(u'_{\log}; \Sigma; z^1)$ yield isomorphic decorated dual graphs Γ. In other words, the forgetful map

$$\mathcal{M}_{1;(4)}(X; D; [4])_{\Gamma} \to \overline{\mathcal{M}}_{1;1}(X; [4])$$

is a double-covering of its image. □

Example 2.18 Assume $u\colon \Sigma \to D \subset X$ is a stable map, where Σ is the genus-1 nodal curve made of two copies of $\mathbb{P}^1$, say $\mathbb{P}^1_1$ and $\mathbb{P}^1_2$, attached at 0 and ∞, and $u_i = u|_{\mathbb{P}^1_i}\colon \mathbb{P}^1_i \to D$ for $i = 1; 2$ is a double-covering of some rational curve $C_i \subset D$, with $u_i(z^{-1}) = u_i(z)$; ie

$$u_i(0) = u_i(\infty) = x \in C_1 \cap C_2 \subset D.$$

Further, assume $\mathcal{N}_X D|_{C_1} \cong \mathcal{O}(2)$ and $\mathcal{N}_X D|_{C_2} \cong \mathcal{O}(-2)$. The automorphism group of the stable map $f \equiv (u; \Sigma)$ is $\mathbb{Z}_2$. Since $u_1^*\mathcal{N}_X D \cong \mathcal{O}(4)$ and $u_2^*\mathcal{N}_X D \cong \mathcal{O}(-4)$, there are two possible ways to lift f to a log map $f_{\log} \in \overline{\mathcal{M}}^{\log}_{1;\emptyset}(X; D; 2(C_1 + C_2))$. The holomorphic section ζ_1 of $u_1^*\mathcal{N}_X D$ can be chosen to have zeros of orders $(3; 1)$, $(2; 2)$ or $(1; 3)$ at $(0; \infty)$. In the middle case, the automorphism group of $f_{\log}$ is $\mathbb{Z}_2$. In the remaining two cases, the two lifts are equivalent with respect to the reparametrization map

$$h\colon \Sigma \to \Sigma, \qquad h|_{\mathbb{P}^1_i}(z) = z^{-1} \quad \text{for } i = 1,2;$$

and their equivalence class defines a single element of $\overline{\mathcal{M}}^{\log}_{1;\mathfrak{s}}(X;D;2(C_1+C_2))$ with the trivial automorphism group.

In Section 3, for J as in the statement of Theorem 1.4, we will lift the Gromov convergence topology to a compact sequential convergence topology on (2-41) such that the forgetful map (1-10) is a continuous local embedding. It follows that the lifted topology is also metrizable. If $g>0$, globally, (1-10) behaves like an immersion. If $\mathfrak{s} \in (\mathbb{N}^N)^k$ then by Lemma 5.5 below, $\overline{\mathcal{M}}^{\log}_{g;\mathfrak{s}}(X;D;A)$ is a compact space of expected real dimension

$$2\,c_1^{TX(-\log D)}(A) + (\dim_{\mathbb{C}} X - 3)(1-g) + k. \tag{2-42}$$

In subsequent papers we will construct Kuranishi-type charts of dimension (2-42) around every point of $\overline{\mathcal{M}}^{\log}_{g;\mathfrak{s}}(X;D;A)$.

The following example describes the log compactification of the moduli space of lines in $\mathbb{P}^2$ relative to a transverse union of two hyperplanes (lines). The same example is studied in [35], where Parker uses tropical geometry to describe Ionel's compactification in [20] and compare it with his construction.

Example 2.19 Suppose that $X = \mathbb{P}^2$ with projective coordinates $[x_1;x_2;x_3]$, and let $D_1 = (x_1=0)$, $D_2 = (x_2=0)$, $D = D_1 \cup D_2$, $A = [1] \in H_2(\mathbb{P}^2;\mathbb{Z}) \cong \mathbb{Z}$ and $\mathfrak{s} = ((1,0),(0,1))$. Then, as we show below, the moduli space

$$\overline{\mathcal{M}}^{\log}_{0;\mathfrak{s}}(X;D;[1]) \tag{2-43}$$

can be identified[14] with $B_{\mathrm{pt}_1;\mathrm{pt}_2}\mathbb{P}^2_{\mathrm{dual}}$ (two-point blowup of $\mathbb{P}^2$), where $\mathbb{P}^2_{\mathrm{dual}}$ is the dual space of lines in $X = \mathbb{P}^2$, pt_1 is the point corresponding to the line D_1, and pt_2 is the point corresponding to the line D_2. Let E_1 and E_2 be the exceptional curves of $B_{\mathrm{pt}_1;\mathrm{pt}_2}\mathbb{P}^2_{\mathrm{dual}}$ and let L be the proper transform of the line connecting pt_1 and pt_2. Any line in X not passing through D_{12} intersects D_1 and D_2 at two disjoint points z^1 and z^2, respectively. By (2-14),

$$\mathrm{ord}(z^1) = (1,0) \quad \text{and} \quad \mathrm{ord}(z^2) = (0,1).$$

[14] The identification is a homeomorphism with respect to the topology that we describe in Section 3.

This gives an identification of

$$\mathcal{M}_{0;s}(X;D;[1]) \cong \overline{\mathcal{M}}^{\log}_{0;s}(X;D;[1])$$

with $B_{pt_1;pt_2}\mathbb{P}^2 - (E_1 \cup E_2 \cup L)$. Every other log map $(u;[\zeta])$ with smooth domain in (2-43) is either of depth $\{1\}$ or of depth $\{2\}$ with two marked points z^1 and z^2 of the corresponding orders. Those of depth $\{1\}$ are given by an isomorphism $u\colon \mathbb{P}^1 \xrightarrow{\cong} D_1$ and a holomorphic section ζ of $N_X D_1 \cong \mathcal{O}_{\mathbb{P}^1}(1)$ such that ζ has a simple zero at the marked point z^1 and $z^1 \neq z^2 = u^{-1}(D_2)$. Such $[\zeta]$ is uniquely determined by $u(z^1) \in D_1 \cong \mathbb{P}$. Therefore, via the identification

$$E_1 \cong \mathbb{P}H_0(N_X D_1) \cong \mathbb{P}^1,$$

such maps correspond to $E_1 - \{E_1 \cap L\} \cong \mathbb{C}$. Similarly, the maps of depth $\{2\}$ with smooth domain correspond to $E_2 - \{E_2 \cap L\} \cong \mathbb{C}$. For other log maps f in (2-43), z^1 and z^2 are mapped to the point D_{12} and thus live on a "ghost bubble" $u_2\colon \mathbb{P}^1 \to X$, with $\mathrm{Im}(u_2) = D_{12}$. This ghost bubble and the nontrivial map $u_1\colon \mathbb{P}^1 \to X$ are attached to each other at nodal points $z^3 \in \mathrm{Dom}(u_2)$ and $z^0 \in \mathrm{Dom}(u_1)$. By definition, the meromorphic section $\zeta = (\zeta_1;\zeta_2)$ defining the log map $(u_2;[\zeta] = ([\zeta_1];[\zeta_2]))$ is a meromorphic section of the trivial bundle $u_2^* N_X D_{12} \cong \mathbb{P}^1 \times \mathbb{C}^2$ such that

$$\mathrm{ord}_{z^1}(\zeta) = (1;0) \quad \text{and} \quad \mathrm{ord}_{z^2}(\zeta) = (0;1).$$

Since $u_2^* N_X D_{12}$ is trivial, we should have $\mathrm{ord}_{z^3}(\zeta) = (-1;-1)$, and these restrictions specify a unique $(\mathbb{C}^*)^2$–class $[\zeta]$. There are thus three possibilities for f:

(1) u_1 is of depth $\emptyset$. In this case, by Definition 2.4(3), u_1 specifies an element of $\mathcal{M}_{0;((1;1))}(X;D;[1])$ and we get an identification of such curves $f = [u_1;(u_2;[\zeta])]$ in (2-43) with the points of $L - \{L \cap E_1; L \cap E_2\}$. The associated decorated dual graph Γ is made of two vertices v_1 and v_2 corresponding to u_1 and u_2, with $I_{v_1} = \emptyset$ and $I_{v_2} = \{1;2\}$, connected by an edge e with $I_e = \{1;2\}$ and $s_{\vec{e}} = \pm(1;1)$ (depending on the choice of orientation). The group $G(\Gamma)$ in this case is trivial and the function $s\colon V \to \mathbb{R}^2$ in Definition 2.8(1) can be taken to be $s_{v_1} = (0;0)$ and $s_{v_2} = (1;1)$.

(2) u_1 is of depth $\{1\}$. In this case u_1 comes with a holomorphic section ζ_0 of $\mathcal{O}_{\mathbb{P}^1}(1)$ as before. Since $\mathrm{ord}(z^0) = (1;1)$, by Definition 2.4(3), ζ^0 should be zero at z^0 and this uniquely determines $[\zeta_0]$. This unique element $f = [(u_1;[\zeta_0]);(u_2;[\zeta])]$ corresponds to the point $E_1 \cap L$. The associated decorated dual graph Γ is made of two vertices v_1 and v_2 corresponding to u_1 and u_2, with $I_{v_1} = \{1\}$ and $I_{v_2} = \{1;2\}$, connected by an edge e with $I_e = \{1;2\}$ and $s_{\vec{e}} = \pm(1;1)$ (depending on the choice of orientation).

The group $G(\Gamma)$ in this case is trivial and the function $s\colon V\to\mathbb{R}^2$ in Definition 2.8(1) can be taken to be $s_{v_1}=(1,0)$ and $s_{v_2}=(2,1)$.

(3) u_1 is of depth $\{2\}$ Similarly, there is a unique such map which corresponds to the point $E_2\in L$.

2.4 Forgetful maps

In this section, we show that the process of forgetting some of the smooth components of an SNC divisor $D=\bigcup_{i\in[N]}D_i$ gives us a forgetful map between the corresponding log moduli spaces. The results are not used in the rest of the paper. While (1-10) is not always an embedding, the map (2-47) below is an embedding. This embedding can be used to reduce certain arguments to the case of smooth divisors.

Let $D=\bigcup_{i\in[N]}D_i\subset X$ be an SNC symplectic divisor, $(\omega,J)\in\mathcal{J}(X,D)$, $g,k\in\mathbb{N}$,

$$s=(s_a=(s_{ai})_{i\in[N]})_{a=1}^{k}\in(\mathbb{Z}^N)^k \tag{2-44}$$

and $\Gamma\in\mathrm{DG}(g,s,A)$. Given $I\subset[N]$, let

$$s|_I=(s_a=(s_{ai})_{i\in I})_{a=1}^{k}\in(\mathbb{Z}^I)^k;\qquad D|_I=\bigcup_{i\in I}D_i;$$

and let $\Gamma|_I\in\mathrm{DG}(g,s|_I,A)$ be the decorated dual graph with the same set of vertices and edges, but with the reduced set of decorations

$$\begin{aligned} I_v^c&=I_v\cap I &&\text{for all } v\in V;\\ I_e^c&=I_v\cap I &&\text{for all } e\in E;\\ s_{\vec e}^c&=(s_{\vec e,i})_{i\in I}\in\mathbb{Z}^I &&\text{for all } \vec e\in\vec E.\end{aligned}$$

Define

$$\varpi_{[N],I}\colon \overline{\mathcal{M}}^{\mathrm{plog}}_{g,s}(X,D,A)_\Gamma\to\overline{\mathcal{M}}^{\mathrm{plog}}_{g,s|_I}(X,D|_I,A)_{\Gamma|_I} \tag{2-45}$$

to be the (well-defined) forgetful map obtained by removing the meromorphic sections

$$(\zeta_{v,i})_{i\in I_v\cap([N]-I)}\ \text{in}$$

(2-19) for all $v\in V$.

Lemma 2.20 The map $\varpi_{[N],I}$ defined in (2-45) above sends $\overline{\mathcal{M}}_{g,s}(X,D,A)_\Gamma\subset\overline{\mathcal{M}}^{\mathrm{plog}}_{g,s}(X,D,A)_\Gamma$ to $\overline{\mathcal{M}}_{g,s|_I}(X,D|_I,A)_{\Gamma|_I}\subset\overline{\mathcal{M}}^{\mathrm{plog}}_{g,s|_I}(X,D|_I,A)_{\Gamma|_I}$.

Proof Fix an orientation $\mathfrak{O}$ on E. With notation as in (2-26), the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^E \oplus \bigoplus_{v\in V} \mathbb{Z}^{I_v} & \xrightarrow{\varrho} & \bigoplus_{e\in E} \mathbb{Z}^{I_e} \\ \big\downarrow \mathrm{pr}_D & & \big\downarrow \mathrm{pr}_T \\ \mathbb{Z}^E \oplus \bigoplus_{v\in V} \mathbb{Z}^{I_v^c} & \xrightarrow{\varrho'} & \bigoplus_{e\in E} \mathbb{Z}^{I_e^c} \end{array}$$

where pr_D and pr_T are the obvious projection maps and ϱ and ϱ' are defined via $\mathfrak{O}$, induces a group homomorphism $\mathrm{pr}_{[N],I}\colon G(\Gamma) \to G(\Gamma|_I)$ such that

$$\mathrm{pr}_{[N],I}(\mathrm{ob}_\Gamma(f)) = \mathrm{ob}_{\Gamma|_I}(\iota_{[N],I}(f)) \quad \text{for all } f \in \mathcal{M}^{\mathrm{plog}}_{g,s}(X,D,A)_\Gamma.$$

Therefore, $\mathrm{ob}_\Gamma(f)=1$ implies $\mathrm{ob}_{\Gamma|_I}(\iota_{[N],I}(f))=1$. □

Taking the union over all Γ, we obtain the stratified forgetful map

$$\iota_{[N],I}\colon \overline{\mathcal{M}}^{\log}_{g,s}(X,D,A) \to \overline{\mathcal{M}}^{\log}_{g,s|_I}(X,D|_I,A).$$

For example, the $I=\varnothing$ case of (2-45) is the map (1-10) into the underlying moduli space of stable maps; moreover,

(2-46) $$\iota_{[N],I'} = \iota_{I;I'} \circ \iota_{[N],I}\colon \overline{\mathcal{M}}^{\log}_{g,s}(X,D,A) \to \overline{\mathcal{M}}^{\log}_{g,s|_{I'}}(X,D|_{I'},A)$$

for all $I'\subset I\subset [N]$. For s as in (2-44), let $s_i = s|_{\{i\}} = (s_{ai})_{a=1}^k \in (\mathbb{Z})^k$ for all $i\in[N]$, and define

(2-47) $$\iota_{[N],1} = \prod_{i\in[N]} \iota_{[N],\{i\}}\colon \overline{\mathcal{M}}^{\log}_{g,s}(X,D,A) \to \prod_{i\in[N]} \overline{\mathcal{M}}^{\log}_{g,s_i}(X,D_i,A),$$

where the right-hand side is the fiber product of

$$\{\iota_{\{i\},\varnothing}\colon \overline{\mathcal{M}}^{\log}_{g,s_i}(X,D_i,A) \to \overline{\mathcal{M}}_{g,k}(X,A)\}_{i\in[N]}.$$

The map $\iota_{[N],1}$ is well-defined by (2-46) and it is an embedding[15] by Remark 2.14. As the following example shows, this embedding can be proper (ie not an equality).

Example 2.21 In Example 2.13, the obstruction groups $G(\Gamma|_{\{1\}})$ and $G(\Gamma|_{\{2\}})$ associated to $\Gamma|_{\{1\}}$ and $\Gamma|_{\{2\}}$ are trivial. Therefore, for an element of the right-hand side in (2-47), the corresponding sections $\zeta_{v_1;1}$ and $\zeta_{v_2;2}$ can be arbitrary (modulo the combinatorial conditions imposed by Definitions 2.4 and 2.8). On the other hand, for

[15] By the results of Section 3, the maps $\iota_{[N],I}$ and thus $\iota_{[N],1}$ are continuous.

such a pair $(v_{1,1}, v_{2,2})$ to define an element of the left-hand side, the corresponding group element in the nontrivial group G has to be the identity. Therefore, the restriction

$$_{\{1,2\},1}\colon M_{g,s}(X,D,A)_\Gamma \to \prod_{i=1,2} M_{g,s_i}(X,D_i,A)_{\Gamma|_{\{i\}}}$$

of (2-47) to $\overline{M}^{\log}_{g,s}(X,D,A)_\Gamma$ is not an isomorphism.

3 Compactness

In this section, after a quick review of the convergence problem for the Deligne–Mumford space and for the classical moduli spaces of J–holomorphic curves, we slightly rephrase and prove Theorem 1.4 in several steps. The main step of the proof is Proposition 3.15, which relates the sequence of "gluing" and "rescaling" parameters, when a sequence of J–holomorphic curves breaks into two pieces with at least one of them mapped into D.

3.1 Classical Gromov convergence

Definition 3.1 Given a k–marked genus g (possibly not stable) nodal surface $C \equiv (\Sigma, \mathfrak{z})$ with dual graph Γ, a cutting configuration with dual graph Γ' is a set of disjoint embedded circles

$$\{\gamma_e\}_{e\in E(\Gamma'/\Gamma)} \subset \Sigma,$$

away from the nodes and marked points, such that the nodal marked surface $(\Sigma', \mathfrak{z}')$ obtained by pinching every γ_e into a node q_e has dual graph Γ'.

Thus, a cutting configuration corresponds to a continuous map

$$\varphi\colon C \to C',$$

called a γ–degeneration[16] in what follows, onto a k–marked genus-g nodal surface C' with dual graph Γ' such that $\mathfrak{z}' = \varphi(\mathfrak{z})$, the preimage of every node of Σ is either a node in Σ' or a circle in γ, and the restriction

$$\varphi\colon \Sigma\setminus\gamma \to \Sigma'\setminus\varphi(\gamma) = \{q_e\}_{e\in E(\Gamma'/\Gamma)}$$ is a

diffeomorphism. Let

$$\varphi\colon \Gamma' \to \Gamma \tag{3-1}$$

be the map corresponding to φ between the dual graphs. We have

$$E(\Gamma') \cong E(\Gamma) \cup E(\Gamma'/\Gamma) \quad \text{and} \quad L(\Gamma') \cong L(\Gamma)$$

[16] It is called a deformation in [41].

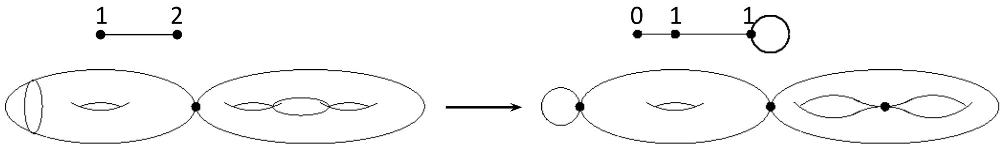

Figure 5: Left: a 1–nodal curve of genus 3 and a cutting set made of two circles. Right: the resulting pinched curve.

such that $j_{E(\Gamma)/E(\Gamma')}$ and $j_{L(\Gamma')}$ are isomorphisms and :

$$E(\Gamma'/\Gamma)\to V(\Gamma)$$

sends the edge e corresponding to γ_e to v if $\gamma_e\subset\Sigma_v$. For every $v'\in V(\Gamma')$ there exists a unique $v\in V(\Gamma)$ and a connected component $U_{v'}$ of $\Sigma_v\setminus\{\gamma_e\}_{e\in E(\Gamma'/\Gamma)}$ such that $\Sigma'_{v'}\subset\Sigma'$ is obtained by collapsing the boundaries of $\mathrm{cl}(U_{v'})$ (here cl means closure). This identification determines the surjective map

$$\wp\colon V(\Gamma')\to V(\Gamma),\qquad v'\to v. \tag{3-2}$$

From another perspective, a cutting configuration corresponds to expanding each vertex $v\in V(\Gamma)$ into a subgraph $\Gamma_v\subset\Gamma'$ (sometimes, this involves just adding more loops to the existing graph) with the set of vertices and edges

$$V(\Gamma'_v)=\wp^{-1}(v)\quad\text{and}\quad E(\Gamma'_v)=\wp^{-1}(v)\cap E(\Gamma'/\Gamma).$$

Moreover, $g_v=g_{\Gamma'_v}$, the ordering of marked points is as before, and

$$A_v=\sum_{v'\in V(\Gamma'_v)}A_{v'}. \tag{3-3}$$

Figure 5 illustrates a cutting configuration over a 1–nodal curve of genus 3, and the corresponding dual graphs.

A sequence $\{\varphi_a\colon C_a\to C_0\}_{a\in\mathbb{N}}$ of degenerations of marked nodal curves is called monotonic if $\Gamma(C_a)\approx\Gamma$ for some fixed Γ and the induced maps $\wp_a\colon\Gamma'\to\Gamma$ are all the same. In this situation, the underlying marked nodal surfaces are isomorphic, ie

$$(C_a,\mathfrak{z}_a)\approx((\Sigma,\mathfrak{j}_a,\mathfrak{z}),\mathcal{C})\quad\text{for all } a\in\mathbb{N}, \tag{3-4}$$

for some fixed marked surface $(\Sigma,\mathfrak{z})$ with dual graph Γ and cutting configuration $\mathcal{C}$. In the following, we let Σ^* denote the complement of the set of nodes

$$\{q_e\}_{e\in E(\Gamma'/\Gamma)}\subset\Sigma'.$$

Definition 3.2 [41, Definition 13.3] A sequence $\{C_a \equiv (\Sigma_a, j_a, \vec{z}_a)\}_{a\in\mathbb{N}}$ of genus-g k–marked nodal curves *monotonically converges* to $C' \equiv (\Sigma', j', \vec{z}\,')$ if there exist a sequence of cutting configurations $\mathcal{C}_a$ on C_a of type Γ', and a monotonic sequence $\varphi_a\colon C_a \to C'$ of $\mathcal{C}_a$–degenerations, such that the sequence $(\varphi_a|_{\Sigma_a\setminus\mathcal{C}_a})_* j_a$ converges to $j'|_{\Sigma'^*}$ in the C^∞–topology.[17]

By [41, Section 13], the topology underlying the holomorphic orbifold structure of $\overline{\mathcal{M}}_{g,k}$ is equivalent to the sequential DM–convergence topology: a sequence $\{C_a\}_{a\in\mathbb{N}}$ of genus-g k–marked stable nodal curves DM–converges to C' if a subsequence monotonically converges to C'. The following result, known as Gromov's compactness theorem [16, Theorem 1.5.B], describes a convergence topology on $\overline{\mathcal{M}}_{g,k}(X, A, J)$ which is compact and metrizable; see [32], [19, Theorem 1.2], [49, Theorem 0.1] and [29, Chapter 5] for further details. In the special case of Deligne–Mumford space, Gromov convergence is equivalent to the DM–convergence discussed above.

Theorem 3.3 *Let (X, ω) be a compact symplectic manifold, $\{J_a\}_{a\in\mathbb{N}}$ be a sequence of ω–tame almost complex structures on X converging in the C^∞–topology to J, and*

$$\{f_a \equiv (u_a, C_a \equiv (\Sigma_a, j_a, \vec{z}_a))\}_{a\in\mathbb{N}}$$

be a sequence of stable J_a–holomorphic maps of bounded (symplectic) area into X. After passing to a subsequence, still denoted by $\{f_a\}_{a\in\mathbb{N}}$, there exists a unique (up to automorphism) stable J–holomorphic map

$$f' \equiv (u', C' \equiv (\Sigma', j', \vec{z}\,'))$$

such that $\{C_a\}_{a\in\mathbb{N}}$ monotonically converges to C', and such that

(1) *we can choose the $\mathcal{C}_a$–degeneration maps $\varphi_a\colon \Sigma_a \to \Sigma'$ of the monotonic convergence so that the restriction*

$$u_a|_{\Sigma_a\setminus\mathcal{C}_a} \circ \varphi_a^{-1}|_{\Sigma'^*}$$

converges uniformly with all derivatives to $u'|_{\Sigma'^*}$ over compact sets;

(2) *with the dual graphs $\Gamma \cong \Gamma(C_a)$ and $\Gamma' = \Gamma(C')$ as in the definition of monotonic sequences,*

$$\lim_{a\to\infty} u_a(\gamma_{a;e}) = u'(q_e) \qquad \text{for all } e \in E(\Gamma'/\Gamma)$$

(3) *the symplectic area of f' coincides with the symplectic area of f_a for all $a \in \mathbb{N}$.*

[17] Uniform convergence on compact sets with all derivatives.

It follows from the properties (1) and (3) that for every $v'\in\Gamma'$, with $U_{a;v'}\subset\Sigma_a$ as in the definition of $\varphi_a(v')$,

$$\lim_{a\to\infty}\int_{\Sigma_{v'}} u_a^*\omega=\int_{\mathrm{cl}(U_{a|v'})} u'^*\omega.$$

Moreover, the stronger identity (3-3) holds. With respect to the identification of the domains and degeneration maps

$$(\varphi_a\colon\Sigma_a\to\Sigma')\cong(\varphi\colon\Sigma\to\Sigma')$$

as in (3-4), property (2) implies that the sequence $(u_a\colon\Sigma\to X)_{a\in\mathbb N}$ C^0–converges to $u\circ\varphi$.

Assume $D\subset X$ is an SNC symplectic divisor, $(\omega,J)\in\mathcal J(X,D)$, and

$$\{f_a\equiv(u_a,\zeta_a,C_a\equiv(\Sigma_a,j_a,\vec z_a))\}_{a\in\mathbb N} \tag{3-5}$$

is a sequence of stable log maps in $\overline{\mathcal M}^{\log}_{g,\mathfrak s}(X,D,A)$. After passing to a subsequence, we may assume that all the maps in (3-5) have the same decorated dual graph $\Gamma(V,E,L)$, and that the underlying sequence of stable maps

$$\{h_a\equiv(u_a,C_a\equiv(\Sigma_a,j_a,\vec z_a))\}_{a\in\mathbb N} \tag{3-6}$$

in $\overline{\mathcal M}_{g,k}(X,A)$ (with the same domain) Gromov converges to the stable map

$$h\equiv(u,C\equiv(\Sigma,j,\vec z))\in\overline{\mathcal M}_{g,k}(X,A)$$

as in Theorem 3.3. Then, in order to prove Theorem 1.4, (for J as in the statement of the theorem) after passing to a further subsequence, we prove that h lifts to a unique log map $f\in\overline{\mathcal M}^{\log}_{g,\mathfrak s}(X,D,A)$. The meromorphic sections that lift h to the log map f are specified in Section 3.2. We first prove that f is a prelog map in Lemma 3.13; the proof works for arbitrary $(\omega,J)\in\mathcal J(X,D)$. Then, in Proposition 3.14, we prove that f satisfies the conditions of Definition 2.8. Since there are only finitely many possible log lifts of a stable map f with different decorations on the dual graph, it follows with little effort that (1-10) is a continuous local embedding.

3.2 Log-Gromov convergence

In this section, first, we recall some basic structures associated to smooth/SNC symplectic divisors. Then we state the definition of log-Gromov convergence and a convergence result from which Theorem 1.4 will be deduced.

Let $D\subset(X,\omega)$ be a smooth symplectic divisor, $J\in\mathcal{J}(X,D;\omega)$, and let $\mathfrak{i}_{\mathcal{N}_XD}$ be the induced complex structure on $\mathcal{N}_XD$. Let $J_{X;D}$ be the almost complex structure on $\mathcal{N}_XD$ induced by the $\bar\partial$–operator $\bar\partial_{\mathcal{N}_XD}$ associated to $(\mathcal{N}_XD,\mathfrak{i}_{\mathcal{N}_XD})$ as in the end of Section 2.1. Fix a compatible pair of Hermitian metric ρ and Hermitian connection ∇ on $\mathcal{N}_XD$. Such a connection ∇ defines a 1–form α_∇ on $\mathcal{N}_XD-D$, whose restriction to each fiber $\mathcal{N}_XD|_p-\{p\}\cong\mathbb{C}^*$ is the 1–form $d\theta$ with respect to the polar coordinates (r,θ) determined by $\rho=r^2$ and the complex structure $\mathfrak{i}_{\mathcal{N}_XD}$. Recall from Section 2.1 that the connection ∇ gives a splitting

$$T\mathcal{N}_XD\cong TD\oplus\mathcal{N}_XD \tag{3-7}$$

such that $J_{X;D}$ is equal to J_D on the first summand and to $\mathfrak{i}_{\mathcal{N}_XD}$ on the second one. By the symplectic neighborhood theorem [28, Theorem 3.30], for $\mathcal{N}'_XD$ sufficiently small there exists a diffeomorphism

$$\Psi\colon\mathcal{N}'_XD\longrightarrow X \tag{3-8}$$

from a neighborhood of D in $\mathcal{N}_XD$ onto a neighborhood of D in X such that $\Psi(x)=x$, the isomorphism

$$\mathcal{N}_XD|_x=T_x^{\mathrm{ver}}\mathcal{N}_XD\longrightarrow T_x\mathcal{N}_XD\xrightarrow{d_x\Psi}T_xX\longrightarrow\frac{T_xX}{T_xD}=\mathcal{N}_XD|_x \tag{3-9}$$

is the identity map for every $x\in D$, and

$$\Psi^*\omega=\omega_{X;D}=(\omega|_D)+\tfrac{1}{2}d(\rho\,\alpha_\nabla). \tag{3-10}$$

The last property is not needed for many of the arguments in Section 3.2. In the language of [45, Definition 2.9], the tuple $\mathcal{R}=(\rho,\nabla,\Psi)$ is called an ω–regularization. If $\Psi^*J=J_{X;D}$, then the tuple $(\omega,\mathcal{R},J)$ is an element of $\mathrm{AK}(X,D)$ mentioned in (1-9). In general, if $D=\bigcup_{i\in[N]}D_i$ is an SNC symplectic divisor in (X,ω), a system of regularizations for D in X is a collection of smooth embeddings

$$\Psi_I\colon\mathcal{N}'_XD_I\longrightarrow X,\quad I\subset[N],$$

from open neighborhoods $\mathcal{N}'_XD_I\subset\mathcal{N}_XD_I$ of D_I such that

- $\Psi_I|_{D_I}=\mathrm{id}_{D_I}$,
- $d\Psi_I$ induces the identity map on $\mathcal{N}_XD_I\cong\bigoplus_{i\in I}\mathcal{N}_XD_i|_{D_I}$, and
- $\Psi_I(\mathcal{N}_{I;I'}\cap\mathrm{Dom}(\Psi_I))=D_{I'}\cap\mathrm{Im}(\Psi_I)$ for all $I'\subset I\subset[N]$.

Here,

$$\mathcal{N}_{I;I'} \colon \equiv \bigoplus_{i\in I-I'} \mathcal{N}_X D_i|_{D_I} \longrightarrow D_I$$

is the normal bundle of D_I in $D_{I'}$. The last identity implies that the derivative $d\Psi_I$ induces an isomorphism of split vector bundles

$$\mathfrak{D}\Psi_{I;I'} \colon \mathcal{N}_{I;I'}|_{\mathcal{N}_{I;I'}\cap \mathrm{Dom}(\Psi_I)} \longrightarrow \mathcal{N}_X D_{I'}|_{D_{I'}\cap \mathrm{Im}(\Psi_I)}. \tag{3-11}$$

See [46, Section 2.2]. A regularization for D in X is a system of regularizations for D in X as above satisfying the compatibility conditions

$$\mathrm{Dom}(\Psi_I) = \mathfrak{D}\Psi_{I;I'}^{-1}(\mathrm{Dom}(\Psi_{I'})),$$
$$\Psi_I = \Psi_{I'} \circ \mathfrak{D}\Psi_{I;I'}|_{\mathrm{Dom}(\Psi_I)} \quad \text{for all } I' \subset I \subset [N].$$

Definition 3.4 [46, Definition 2.9] An ω–regularization for D in X consists of a choice of Hermitian structure $(\rho_{I;i}, \mathfrak{i}_{I;i}, \nabla^{(I;i)})$ on $\mathcal{N}_X D_i|_{D_I}$ for all $i \in I \subset [N]$, together with a regularization for D in X as above so that

$$\Psi_I^*\omega = (\omega|_{D_I}) + \frac{1}{2}\sum_{i\in I} d(\rho_{I;i}\,\nabla^{(I;i)}) \qquad \text{for all } I \subset [N];$$

and (3-11) is an isomorphism of split Hermitian vector bundles for all $I' \subset I \subset [N]$.

Finally, an element of $\mathcal{AK}(X,D)$ is a tuple $(\omega, \mathcal{R}, J)$, where $\mathcal{R}$ is an ω–regularization as in Definition 3.4, and

$$\Psi_I^* J = \pi_I^*(J|_{TD_I}) \oplus \bigoplus_{i\in I} \mathfrak{i}_{I;i}$$

with respect to the decomposition (3-7). The main reason for restricting to $\mathcal{AK}(X,D)$ or the integrable almost complex structures in Theorem 1.4 is that in the proof of Proposition 3.15, for any $p \in D_I$, we need J to be $(\mathbb{C}^*)^I$–equivariant in a neighborhood of p with respect to a (local) $(\mathbb{C}^*)^I$–action that preserves D and fixes D_I.

For any $c \in \mathbb{R}_{>0}$, define

$$\mathcal{N}_X D(c) = \{v \in \mathcal{N}_X D : |v| < c\}.$$

For any $t \in \mathbb{C}$, define

$$\begin{aligned} &R_t \colon \mathcal{N}_X D \longrightarrow \mathcal{N}_X D, && R_t(v) = tv \quad \text{for all } v \in \mathcal{N}_X D; \\ &\Psi_t = \Psi \circ R_t \colon R_t^{-1}(\mathcal{N}_X' D) \longrightarrow X, && J_t = \Psi_t^* J. \end{aligned} \tag{3-12}$$

Note that if $\Psi^*J = J_{X;D}$, then $J_t \equiv J_{X;D}$ is independent of t. The following lemma is an expansion of the sentence after [21, equation (6.5)].

Lemma 3.5 For J satisfying (1-3), we have

$$\lim_{t\to 0} J_t|_{\overline{N_X D}(c)} = J_0 := J_{X;D}|_{\overline{N_X D}(c)} \quad \text{for all } c\in\mathbb{R}_{>0},$$

uniformly with all derivatives.

Proof In order to simplify the notation, let us forget about Ψ and think of J as an almost complex structure on $N_X^0 D$ itself; then $J|_D = J_{X;D}|_D$, and $J_t = R_t^* J$ for every $t\in\mathbb{C}$. Via (3-7), we decompose J into four components

$$J_v(\xi) = \big(J_v^{11}(\xi_1)+J_v^{21}(\xi_2)\big)\oplus\big(J_v^{12}(\xi_1)+J_v^{22}(\xi_2)\big)$$
$$\text{for all } x\in D,\ v\in N_X D|_x \text{ and } \xi=\xi_1\oplus\xi_2\in (TD\oplus N_X D)|_v,$$

where, for example, J^{11} is the component which maps the horizontal subspace TD to itself. Identifying ξ_1 and ξ_2 with the corresponding vectors in T_xD and $N_X D|_x$, respectively, we get

$$(J_t)_v(\xi) = \big(J_{tv}^{11}(\xi_1)+J_{tv}^{21}(t\xi_2)\big)\oplus\big(\tfrac{1}{t}J_{tv}^{12}(\xi_1)+J_{tv}^{22}(\xi_2)\big).$$

On each compact set $\overline{N_X D}(c)$, the first summand uniformly converges to $J_D(\xi_1)$, and $J_{tv}^{22}(\xi_2)$ uniformly converges to $\mathfrak{i}_{N_X D}(\xi_2)$ (with all derivatives). Finally, the term

$$\tfrac{1}{t}J_{tv}^{12}(\xi_1)$$

C^∞–converges to the normal part of (a multiple of) $N_J(v, J\xi_1)$, which is zero by (1-3); see Remark 4.2. □

For any (continuous) map $u\colon\Sigma\to N_X D$, let

$$\underline{u} = \pi\circ u\colon\Sigma\to D$$

denote its projection to D. Then u is equivalent to a section $\zeta\in\Gamma(\Sigma,\underline{u}^*N_X D)$ in the sense that

(3-13) $$u(x) = \zeta(x)\in N_X D|_{\underline{u}(x)} \quad \text{for all } x\in\Sigma.$$

We will use this correspondence repeatedly in the following arguments. In particular, by (1)–(3) on page 1004, u is $J_{X;D}$–holomorphic if and only if $\underline{u}$ is J_D–holomorphic and $\bar\partial_{N_X D}\zeta = 0$.

Definition 3.6 With (X, D, ω, J, Φ) as above (ie D is smooth), let
$$f_a \equiv (u_a, C_a \equiv (\Sigma_a, j_a, \vec z_a))_{a\in\mathbb N}$$
be a sequence of stable maps with smooth domain in $\mathcal M_{g,s}(X,D,A)$ that Gromov converges, considered as a sequence in $\overline{\mathcal M}_{g,k}(X,A)$, to the marked nodal map
$$f_\infty \equiv (u_v, C_v \equiv (\Sigma_v, j_v, \vec z_v))_{v\in V} \in \overline{\mathcal M}_{g,k}(X,A)$$
with dual graph $\Gamma = \Gamma(V,E,L)$ and nodal domain $\Sigma = \bigcup_{v\in V}\Sigma_v$. With notation as in (2-12), (4-8) and Theorem 3.3, for each $v\in V_1$ we say $(u_a)_{a\in\mathbb N}$ is asymptotic to
$$\zeta_v \in \Gamma_{\mathrm{mero}}(\Sigma_v, u_v^* N_X D)$$
on Σ_v in the normal direction to D if there exists a sequence of nonzero complex numbers $(t_{a,v})_{a\in\mathbb N}$ satisfying

(3-14) (uniformly) $\lim_{a\to\infty} \Phi^{-1}_{t_{a,v}}(u_a \circ \varphi_a|_K) = \zeta_v|_K$ in

the sense of (3-13), for every compact set $K \subset \Sigma_v - q_v$.

Proposition 3.10 below shows that, after passing to a subsequence, the limiting J–holomorphic map f_∞ always admits such meromorphic sections ζ_v, and that they are unique up to multiplication by a constant in $\mathbb C^*$. Since $d\Phi$ in (3-9) is supposed to be the identity map on $N_X D$, (3-14) does not depend on the particular choice of Φ in (3-8).

Definition 3.7 Let $D \subset X$ be an SNC symplectic divisor, $(\omega, J) \in \mathcal J(X,D)$, and
$$f_a \equiv (u_{a,v}, \zeta_{a,v} \equiv (\zeta_{a,v,i})_{i\in I_v}, C_{a,v} \equiv (\Sigma_v, j_{a,v}, \vec z_v))_{v\in V,\, a\in\mathbb N} \in \mathcal M^{\log}_{g,s}(X,D,A)$$
be a sequence of stable log maps in $\overline{\mathcal M}^{\log}_{g,s}(X,D,A)$[18] with a fixed decorated dual graph $\Gamma = \Gamma(V,E,L)$. We say this sequence log-Gromov converges to the log (resp. prelog) map
$$f' = (u_{v'}, \zeta_{v'} = (\zeta_{v',i})_{i\in I_{v'}}, C_{v'})_{v'\in V'}$$
in $\overline{\mathcal M}^{\log}_{g,s}(X,D,A)$ (resp. in $\mathcal M^{\mathrm{plog}}_{g,s}(X,D,A)_{\Gamma'}$) with the decorated dual graph $\Gamma' = \Gamma(V',E',L')$ if the underlying sequence of stable maps in $\overline{\mathcal M}_{g,k}(X,A)$ Gromov converges to the underlying marked nodal map

(3-15) $$\underline{f'} = (u_{v'}, C_{v'})_{v'\in V'} \in \overline{\mathcal M}_{g,k}(X,A)$$

with nodal domain $\Sigma' = \bigcup_{v'\in V'}\Sigma_{v'}$, and the following hold. With $\mathcal W\colon V' \longrightarrow V$

[18] More precisely, they represent equivalence classes of elements in $\overline{\mathcal M}^{\log}_{g,s}(X,D,A)$.

as in (3-2) and notation as in Theorem 3.3, for each $v \in V$ and $v' \in V'$ with $\pi(v') = v$,

- if $i \in I_{v'} \setminus I_v$, then $(u_{a;v})_{a\in\mathbb{N}}$ is asymptotic to $\zeta_{v';i}$ on $\Sigma_{v'}$ in the normal direction to D_i in the sense of Definition 3.6;
- if $i \in I_v$, there exists a sequence $(t_{a;v';i})_{a\in\mathbb{N}} \in \mathbb{C}^*$ such that for every compact set $K \subset \Sigma_{v'} \setminus (z_{v'} \cup q_{v'})$, the sequence $t_{a;v';i}^{-1}\,\zeta_{a;v;i} \circ \varphi_a|_K$ uniformly converges to $\zeta_{v';i}|_K$.

Theorem 3.8 *Assume that $D \subset X$ is an SNC symplectic divisor, that $(\omega, \mathcal{R}, J) \in \mathrm{AK}(X, D)$ for some regularization $\mathcal{R}$ or J is integrable, and that*

$$\{ f_a \equiv (u_{a;v}, [\zeta_{a;v}] = ([\zeta_{a;v;i}])_{i\in I_v}, C_{a;v} = (\Sigma_v, j_{a;v}, z_v))_{v\in V} \}_{a\in\mathbb{N}} \tag{3-16}$$

is a sequence of log maps in $\overline{\mathcal{M}}^{\log}_{g,s}(X, D, A)$. After passing to a subsequence, there exists a unique (up to reparametrization) log map

$$f' = (u_{v'}, [\zeta_{v'}] = ([\zeta_{v';i}])_{i\in I_{v'}}, C_{v'})_{v'\in V'} \tag{3-17}$$

such that (3-16) log-Gromov converges to (3-17) in the sense of Definition 3.7.

We break the proof of Theorem 3.8 into smaller steps. The main steps are proved in the subsequent sections.

For two sequences of nonzero complex numbers $(t_a)_{a\in\mathbb{N}}$ and $(t'_a)_{a\in\mathbb{N}}$, we write

$$(t_a)_{a\in\mathbb{N}} \sim (t'_a)_{a\in\mathbb{N}} \quad \text{if } \lim_{a\to\infty} \frac{t_a}{t'_a} = 1. \tag{3-18}$$

The right-hand side of (3-18) defines an equivalence relation on the set of such sequences and we denote the equivalence class of a sequence $(t_a)_{a\in\mathbb{N}}$ by $[(t_a)_{a\in\mathbb{N}}]$. For an equivalence class $[(t_a)_{a\in\mathbb{N}}]$ and $t \in \mathbb{C}^*$, the equation

$$t[(t_a)_{a\in\mathbb{N}}] := [(t\,t_a)_{a\in\mathbb{N}}]$$

is well-defined and defines an action of $\mathbb{C}^*$ on the set of equivalence classes. Moreover, the operation of pointwise multiplication/division between such sequences

$$(t_a)_{a\in\mathbb{N}} (t'_a)^{\pm1}_{a\in\mathbb{N}} = (t_a t'^{\pm1}_a)_{a\in\mathbb{N}}$$

descends to a well-defined multiplication/division operation between the equivalence classes.

The next proposition corresponds to [21, Proposition 6.6].

Remark 3.9 There is a minor issue in the proof of [21, Proposition 6.6]. In [21, equation (6.13)], the authors use the intermediate value theorem to find the right rescaling parameter $t = t_m$. However, the energy function used there is not necessarily continuous in t. For example, applying their argument to the example where $X = C_1 \times C_2$ is the product of two curves, the divisor V is $\{p\} \times C_2$ for some point $p \in C_1$, and the sequence of curves is $\{\{p_i\} \times C_2\}_{i=1}^{\infty}$, with $\lim_{i\to\infty} p_i = p$.

Proposition 3.10 As in Definition 3.6 (ie D is smooth), let

$$\{f_a \equiv (u_a, C_a \equiv (\Sigma_a, j_a, \mathfrak{z}_a))\}_{a\in\mathbb{N}} \tag{3-19}$$

be a sequence of stable maps with smooth domain in $\mathcal{M}_{g,\mathfrak{s}}(X, D, A)$ that Gromov converges, considered as a sequence in $\overline{\mathcal{M}}_{g,k}(X, A)$, to the marked nodal map

$$f \equiv (u_v, C_v \equiv (\Sigma_v, j_v, \mathfrak{z}_v))_{v\in V} \in \overline{\mathcal{M}}_{g,k}(X, A).$$

After passing to a subsequence (which we still denote by $\mathbb{N}$), for each $v \in V_1$ there exists a unique

$$[\zeta_v] \in \Gamma_{\mathrm{mero}}(\Sigma_v, u_v^* \mathcal{N}_X D)/\mathbb{C}^*$$

such that $(u_a)_{a\in\mathbb{N}}$ is asymptotic to ζ_v on Σ_v in the normal direction to D in the sense of Definition 3.6. Furthermore, ζ_v has no pole/zero in $\Sigma_v - (q_v \cup \mathfrak{z}_v)$, and it has a zero of order s_i at z^i for all $z^i \in \mathfrak{z}_v$.

Proof For every fixed $K \Subset \Sigma_v - q_v$, by Theorem 3.3, the sequence

$$\tilde u_{a;K} = \pi \circ u_{a;K} \colon K \to D \quad \text{with } u_{a;K} \equiv \Psi^{-1} \circ u_a \circ \phi_a^{-1}|_K \colon K \to \mathcal{N}'_X D \text{ for all } a \gg 1$$

converges uniformly with all derivatives to $u_v|_K$, and we have that

$$u_{a;K}(z) = \zeta_{a;K}(z)$$

for some nontrivial smooth section $\zeta_{a;K} \in \Gamma(K, \tilde u_{a;K}^* \mathcal{N}_X D)$ in the sense of (3-13), so that the sequence $\zeta_{a;K}$ converges uniformly with all derivatives (with respect to a connection ∇) to 0. Choose $(t_{a;v;K})_{a\gg 1}$ so that

$$\|t_{a;v;K}^{-1}\zeta_{a;K}\|_{L^\infty(K)} = c_K \quad \text{for all } a \gg 1 \tag{3-20}$$

for some arbitrary nonzero constant c_K. Then, by [29, Theorem 4.1.1] (after passing to a subsequence), the sequence

$$(\Psi_{t_{a;v;K}}^{-1} \circ u_v \circ \phi_a^{-1}|_K)_{a\gg 1}$$

of $J_{t_{a;v;K}}$–holomorphic maps in $N_X D(c_K)$ converges uniformly with all derivatives to a unique $J_{X;D}$–holomorphic map

$$u_{\infty;K}\colon K \to N_X D(c_K).$$

By (3-20), property (3) on page 1004, and since $\pi\circ u_{a;v}$ converges to $u_v|K$, we have

$$\pi\circ u_{\infty;K} = u_v|K \quad\text{and}\quad u_{\infty;K} = \zeta_{v;K}$$

for some unique nontrivial $\bar\partial_{N_X D}$–holomorphic section $\zeta_{v;K}$ of $u_v^* N_X D|_K$. Since $\zeta_{a;K}$ is nonzero away from $\mathbf{z}_a\cap\varphi_a(K)$, $\zeta_{v;K}$ is nonzero away from $\mathbf{z}_v\cap K$.

Let

$$K_1\subset K_2\subset\cdots$$

be a sequence exhausting $\Sigma_v - q_v$. For each K_i, let $\zeta_{v;K_i}$ and c_{K_i} respectively be the section and constant corresponding to K_i in the argument above. Choose a reference point $p\in K_1$ and fix a nonzero vector $v_p\in N_X D|_{u_v(p)}$. For each i, we can equally rescale c_{K_i} and $(t_{a;v;K_i})_{a\gg1}$ by a constant number in $\mathbb{C}^*$ so that $\zeta_{v;K_i}(p)$ becomes equal to v_p. Then, by the uniqueness of the limiting section, we get

$$\zeta_{v;K_i} = \zeta_{v;K_{i+1}}|_{K_i} \qquad \text{for all } i\in\mathbb{N}.$$

Therefore, the equation

$$\zeta_v(x) := \zeta_{v;K_i}(x) \qquad \text{for all } x\in\Sigma_v - q_v;\ i\in\mathbb{N} \text{ such that } x\in K_i$$

defines a holomorphic section of $u_v^* N_X D|_{\Sigma_v - q_v}$ such that (3-14) holds. Moreover,

$$(t_{a;v;K_i})_{a\in\mathbb{N}} \sim (t_{a;v;K_j})_{a\in\mathbb{N}} \qquad \text{for all } i,j\in\mathbb{N}.$$

It remains to show that ζ_v has at most finite-order poles at the nodes and $\mathrm{ord}_{z^i}(\zeta_v) = s_i$ for all $z^i\in\mathbf{z}_v$.

For any marked point $z^i\in\mathbf{z}_v$, let $\Delta_i\subset\Sigma_v$ be a sufficiently small disk around z^i that contains no other marked point or nodal point. For a sufficiently large, the order of vanishing of u_a at z^i_a is equal to the winding number of

$$\Phi^{-1}_{t_{a;v;\partial\Delta_i}}\circ u_a\circ\varphi_a^{-1}|_{\partial\Delta_i} \qquad \text{for all } a\gg 1$$

around D. With $K=\Delta_i$ in (3-14), these numbers are the same for $a\gg1$ and they are equal to the winding number of $u_{\infty;\Delta_i}|_{\partial\Delta_i}$ around D. The latter is equal to the order of ζ_v at z^i. We conclude that the contact orders stay the same at the marked points.

Similarly, for any nodal point $q_{\vec{e}}\in\Sigma_v$, with $\vec{e}\in\vec{\mathbb{E}}$ and $v_1(\vec{e}) = v$, let $\Delta\subset\Sigma_v$ be a sufficiently small disk around $q_{\vec{e}}$ that contains no other marked point or nodal point.

Choose a compact set $K \subset \Sigma_v \setminus q_v$ so that one of its boundary circles coincides with $\partial\Delta$. Since the convergence in (3-14) is uniform, the winding numbers of

$$\Phi\big(t_{a;v;K}^{-1}\, \zeta_a \circ \phi_a^{-1}|_{\partial\Delta_x}\big) \quad \text{for all } a \gg 1$$

around D are the same as the winding number of $u_{1;K}|_{\partial\Delta_x}$ around D. The latter is equal to the order of ζ_v at q_e. We conclude that ζ_v extends to a meromorphic section at q_e. □

Remark 3.11 The sections ζ_v and the equivalence class of the rescaling sequence $[(t_{a;v})_{a\in\mathbb{N}}]$ are independent of the choice of Φ. It is also clear from (3-14) that if $(t_{v;a})_{a\in\mathbb{N}}$ is a rescaling sequence associated to ζ_v and $(t'_{v;a})_{a\in\mathbb{N}}$ is a rescaling sequence associated to $c\zeta_v$ for any $c \in \mathbb{C}^*$, then

$$[(t'_{v;a})_{a\in\mathbb{N}}] = c^{-1}[(t_{v;a})_{a\in\mathbb{N}}]. \tag{3-21}$$

The following is the analogue of Proposition 3.10 for a sequence of stable log maps with smooth domain and image in D.

Corollary 3.12 If D is smooth, consider a sequence

$$\{f_a \equiv (u_a, \zeta_a, C_a \equiv (\Sigma_a, j_a, \vec{z}_a))\}_{a\in\mathbb{N}} \tag{3-22}$$

of representatives of stable log maps with smooth domain in $\mathcal{M}_{g,\mathfrak{s}}(X, D, A)_{\{1\}}$ such that the underlying sequence of stable J_D–holomorphic maps

$$\{f_a \equiv (u_a, C_a \equiv (\Sigma_a, j_a, \vec{z}_a))\}_{a\in\mathbb{N}} \text{ Gromov} \tag{3-23}$$

converges, as a sequence in $\mathcal{M}_{g,k}(D, \overline{A})$, to the nodal map

$$f \equiv (u_v, C_v \equiv (\Sigma_v, j_v, \vec{z}_v))_{v\in V} \in \overline{\mathcal{M}}_{g,k}(D, A).$$

With notation as in (2-12), (4-8) and Theorem 3.3, after passing to a subsequence (whose index we still denote by $\mathbb{N}$), for every $v \in V$ there exists a unique

$$[\zeta_v] \in \Gamma_{\mathrm{mero}}(\Sigma_v, u_v^*\mathcal{N}_X D)/\mathbb{C}^*$$

and a unique equivalence class of sequences of nonzero complex numbers $[(t_{a;v})_{a\in\mathbb{N}}]$ such that

$$\lim_{a\to\infty} t_{a;v}^{-1}\, \zeta_a \circ \phi_a^{-1}|_K = \zeta_v|_K \tag{3-24}$$

for any compact set $K \subset \Sigma_v \setminus q_v$. Furthermore, $[\zeta_v]$ only depends on the sequence of equivalence classes $([\zeta_a])_{a\in\mathbb{N}}$, it has no pole/zero in $\Sigma_v \setminus (q_v \cup z_v)$, and it has a zero/pole of the same order s_i at z^i for all $z^i \in \vec{z}_v$.

Proof If (3-24) holds for a sequence $(\phi_a; t_{a;v})_{a\in\mathbb{N}}$, then it also holds for any other simultaneous reparametrization $(t_a\phi_a; t_a t_{a;v})_{a\in\mathbb{N}}$. Therefore, (3-24) only depends on the sequence of equivalence classes $([\phi_a])_{a\in\mathbb{N}}$. Every map in the sequence (3-22) corresponds to a $J_{X;D}$–holomorphic map in

$$\mathcal{M}_{g;s}(N_XD; D; A).$$

Choose the representatives ϕ_a so that their image in N_XD lie in an arbitrarily small compact neighborhood[19] of D. Replacing $(X; D; \omega; J)$ with $(N_XD; D; \omega_{X;D}; J_{X;D})$ and Ψ with the identity map in Proposition 3.10, we get the desired result. □

From Proposition 3.10 and Corollary 3.12 we derive the following conclusion.

Lemma 3.13 Let $D\subset X$ be an SNC symplectic divisor, $(\omega; J)\in\mathcal{J}(X; D)$, and

$$f_a\equiv\big(u_{a;v};[\zeta_{a;v}]=([\zeta_{a;v;i}])_{i\in I_v}; C_{a;v}=(\Sigma_v; j_{a;v}; \vec{z}_v)\big)_{v\in V,\ a\in\mathbb{N}} \tag{3-25}$$

be a sequence of stable log maps in $\overline{\mathcal{M}}^{\log}_{g;s}(X; D; A)$. After passing to a subsequence, there exists a unique prelog map

$$f'=\big(u_{v'};[\zeta_{v'}]=([\zeta_{v';i}])_{i\in I_{v'}}; C_{v'}\big)_{v'\in V'} \tag{3-26}$$

such that (3-25) log-Gromov converges to (3-26) in the sense of Definition 3.7.

Proof First, we apply Gromov convergence to the underlying sequence of stable maps. Then, running through all D_i and $v\in V$ one at a time, applying Proposition 3.10 (with $D\equiv D_i$) to the sequence

$$(u_{a;v}; C_{a;v})_{a\in\mathbb{N}}$$

whenever $i\notin I_v$, and Corollary 3.12 to the sequence

$$(u_{a;v}; \zeta_{a;v;i}; C_{a;v})_{a\in\mathbb{N}}$$

whenever $i\in I_v$, we obtain f'. We need to show that f' satisfies the conditions of Definition 2.4. The first condition is obviously satisfied.

Continuity The matching condition (2) of Definition 2.4 is about the continuity of the underlying stable map f' and already holds by Gromov compactness.

[19] So we can still apply the Gromov convergence theorem. We can also use the compact manifold $\mathbb{P}_XD$ in (4-2) instead of N_XD with the symplectic form $\omega_{X;D}\equiv(\omega|_D)+\epsilon\, d(\alpha\rho/(1+\rho))$, where $\epsilon>0$ is a sufficiently small constant. Then, for t sufficiently small, by interpolating between $J_t|_{\Phi^{-1}(N'D)}$ and $J_{X;D}|_{\mathbb{P}_XD}$, we can construct a family of almost complex structures $\tilde{J}_t$ on $\mathbb{P}_XD$ so that $\tilde{J}_t$ converges to $J_{X;D}$; see [21, Proposition 6.6].

Contact orders at the nodes In order to show that the condition (3) of Definition 2.4 is satisfied, let us first fix some notation. Since

$$s_{\vec e} = -s_{\overleftarrow{e}} \iff s_{\vec e;i} = -s_{\overleftarrow{e};i} \in \mathbb{Z} \quad \text{for all } i \in [N];$$

it is enough to show that condition (3) is satisfied relative to each smooth component D_i; ie we may assume D is smooth. In the context/notation[20] of Proposition 3.10, for every $v, v' \in V$ and any node $q_e = (q_{\vec e}, q_{\overleftarrow{e}})$, $e \in E_{v,v'}$, connecting Σ_v and $\Sigma_{v'}$, let $\Delta_{\vec e} \subset \Sigma_v$ be a sufficiently small disk around $q_{\vec e}$ (not containing any other marked point or nodal point), $\Delta_{\overleftarrow{e}} \subset \Sigma_{v'}$ be a sufficiently small disk around $q_{\overleftarrow{e}}$, and $A_e = \Delta_{\vec e} \cup \Delta_{\overleftarrow{e}}$ be the resulting neighborhood of q_e in Σ. We orient each circle $\partial\Delta_{\vec e}$ in the direction of the counterclockwise rotation in $\Delta_{\vec e} \subset \mathbb{C}$. For each $e \in E$, $A_{a;e} = \varphi_a^{-1}(A_e)$ is a cylinder in Σ_a with two (oppositely oriented) boundaries

$$\partial A_{a;\vec e} = \varphi_a^{-1}(\partial\Delta_{\vec e}) \quad \text{and} \quad \partial A_{a;\overleftarrow{e}} = \varphi_a^{-1}(\partial\Delta_{\overleftarrow{e}}) \tag{3-27}$$

such that $u_a|_{A_{a;e}}$ does not intersect D for $a \gg 1$. Since $u_a|_{A_{a;e}}$ is continuous and does not intersect D, the winding numbers of u_a around D on the two boundary circles of the annulus $A_{a;e}$ (if oriented compatibly) are the same. But $\partial A_{a;\vec e}$ and $\partial A_{a;\overleftarrow{e}}$ are the boundary circles of the annulus $A_{a;e}$ with opposite orientations, therefore the winding numbers of

$$u_a|_{\partial A_{a;\vec e}} \quad \text{and} \quad u_a|_{\partial A_{a;\overleftarrow{e}}}$$

are opposites of each other. If $v \in V_1$, by the proof of Proposition 3.10

$$s_{\vec e} := \mathrm{ord}_{q_{\vec e}} \mathcal{u}_v = \text{winding number of } (u_a|_{\partial A_{a;\vec e}}) \quad \text{for all } a \gg 1.$$

Similarly, if $v \in V_0$, then

$$s_{\vec e} := \mathrm{ord}_{q_{\vec e}}(u_v; D) = \text{winding number of } (u_a|_{\partial A_{a;\vec e}}) \quad \text{for all } a \gg 1.$$

Therefore,

$$s_{\vec e} = -s_{\overleftarrow{e}} \quad \text{for all } v, v' \in V,\ e \in E_{v,v'}. \tag{3-28}$$

The same conclusion holds in the case of Corollary 3.12 (since it is a corollary of Proposition 3.10).

The contact-order condition (3) in Definition 2.4 follows, for every $e \in E(\Gamma_0/\Gamma)$, from equation (3-28). For each $\vec e \in \vec E = \vec E(\Gamma) \sqcup \vec E^0 = \vec E(\Gamma_0)$, with $v = v_1(\vec e) \in V$, the

[20] Note that the notation used for the limiting map in Proposition 3.10 is different than that in the statement of Lemma 3.13.

nodal point $q_{\vec{e}}$ is a marked point for (u_v, C_v). For such e, by the last statements in Proposition 3.10 and Corollary 3.12, the contact order $s_{\vec{e}}$ remains unchanged in the limiting process. Therefore, the contact-order condition (3) in Definition 2.4 follows, for every $e \in E - E_0$, from the corresponding condition on $(f_a)_{a \in \mathbb{N}}$.

Contact orders at the marked points. Finally, condition (4) in Definition 2.4 follows from the corresponding statements in Proposition 3.10 and Corollary 3.12. □

In order to prove Theorem 3.8 (and thus Theorem 1.4), it just remains to prove the following proposition.

Proposition 3.14 *If, further, $(\omega, \mathcal{R}, J) \in \mathcal{AK}(X, D)$ for some regularization $\mathcal{R}$ or if J is integrable, then the prelog J–holomorphic map f' in (3-26) satisfies conditions (1) and (2) of Definition 2.8.*

We prove Proposition 3.14 in Section 3.4. The proof uses a fine comparison result between the rescaling parameters $(t_{a;v',i})_{a \in \mathbb{N}}$ corresponding to the sections $\zeta_{v';i}$ for all $v' \in V'$ and $i \in I_{v'}$, and the "gluing parameters" of the nodes. We expect Proposition 3.15 and thus Proposition 3.14 to be true for a larger class of almost Kähler structures containing $\mathcal{AK}(X, D)$ and the space of Kähler structures.

3.3 Local behavior of convergence

Proposition 3.14 is essentially a consequence of Proposition 3.15 below, which relates the sequence of rescaling parameters $(t_{a;v',i})_{a \in \mathbb{N}}$ corresponding to the sections $\{\zeta_{v';i}\}_{i \in I_{v'}, v' \in V'}$ in Lemma 3.13 to the "gluing parameters" at the nodes and the ratios of leading-order coefficients $0 \neq \eta_{\vec{e}';i} \in \mathcal{N}_X D_i|_{u'(q_{\vec{e}'})}$ in (2-36). We use the natural log of these parameters to cook up the map required in condition (1) of Definition 2.8.

Let us start with a local picture of what is happening in Lemma 3.13 with respect to any smooth component of D. Suppose D is a smooth symplectic divisor in (X, ω) and $J \in \mathcal{J}(X, D; \omega)$. Fix a regularization $\Phi\colon \mathcal{N}_X^{\mathbb{C}} D \to X$ as in (3-8). Let Δ_1 and Δ_2 be compact discs of some fixed sufficiently small radius ϵ around $0 \in \mathbb{C}$ with coordinates z_1 and z_2. For $i = 1, 2$, let $\{z_{i;a}\}_{a \in \mathbb{N}}$ be a sequence of complex coordinates[21] on Δ_i converging to z_i uniformly with all derivatives.

[21] More precisely, $z_{i;a}\colon \Delta_i \to \mathbb{C}$ is a sequence of smooth functions converging to the function $z_i\colon \Delta_i \to \mathbb{C}$ in C^∞–topology.

Local case 1 For a sequence of complex numbers $(\varepsilon_a)_{a\in\mathbb{N}}$ converging to zero, suppose $u_a\colon A_a \to \mathrm{Im}(\Phi)\subset X$, where

$$A_a = \{(z_{1,a}, z_{2,a}) \mid z_{1,a}z_{2,a} = \varepsilon_a,\ z_{1,a}\in\Delta_1,\ z_{2,a}\in\Delta_2\}\subset\Delta_1\times\Delta_2 \tag{3-29}$$

with $a\in\mathbb{N}$ is a sequence of J–holomorphic maps that Gromov converges to the nodal map

$$u_1(z_1)\colon \Delta_1 \to X,\quad u_2(z_2)\colon \Delta_2\to D,\qquad x = u_1(0) = u_2(0)\in D.$$

In other words, for any $\delta > 0$,

(a) the sequence of J–holomorphic maps

$$u_a(z_{1,a}) \equiv u_a(z_{1,a}, \varepsilon_a/z_{1,a})\colon A_a \supset \{z_{1,a}\in\mathbb{C} \mid |\varepsilon_a|/\delta \le |z_{1,a}| \le \delta\} \to X$$

converges uniformly with all derivatives on the compact set

$$\{(z_{1,a}, z_{2,a})\in A_a \mid \delta \le |z_{1,a}|\} \cong \{z_{1,a}\in\mathbb{C} \mid \delta\le |z_{1,a}| \le \delta\}$$

to $u_1|_{\{z_1\in\mathbb{C} \mid \delta\le|z_1|\le\delta\}}$, (b)

the reparametrization

$$u_a(z_{2,a}) \equiv u_a(\varepsilon_a/z_{2,a}, z_{2,a})\colon A_a \supset \{z_{2,a}\in\mathbb{C} \mid |\varepsilon_a|/\delta \le |z_{2,a}| \le \delta\} \to X$$

converges uniformly with all derivatives on the compact set

$$\{(z_{1,a}, z_{2,a})\in A_a \mid \delta \le |z_{2,a}|\} \cong \{z_{2,a}\in\mathbb{C} \mid \delta\le |z_{2,a}| \le \delta\}$$

to $u_2|_{\{z_2\in\mathbb{C} \mid \delta\le|z_2|\le\delta\}}$, and

(c) we do not get any bubbling in between the two maps (ie the energy in between shrinks to zero with δ).

Furthermore, suppose that

(1) u_1 has a tangency order of $s > 0$ with D at $z_1 = 0$, and
(2) there exists a meromorphic section ζ of $u_2^*N_XD$ with (only) a pole of order s at the origin, and a sequence of complex numbers $(t_a)_{a\in\mathbb{N}}$ converging to zero such that $t_a^{-1}\Phi^{-1}(u_a(z_{2,a}))$ converges to $\zeta(z_2)$ uniformly with all derivatives on any compact set $\{z_2\in\mathbb{C} \mid \delta\le|z_2|\le\delta\}\subset\Delta_2$.

Let $0\neq\zeta_2\in N_XD|_x$ be the leading coefficient of ζ with respect to the coordinate z_2 as in (2-34), and $0\neq\zeta_1\in N_XD|_x$ be the s^{th} derivative of u_1 in the normal direction to D at 0 with respect to the coordinate z_1 as in (2-35). Proposition 3.15 below shows that there is an explicit relation between the sequence of gluing parameters $(\varepsilon_a)_{a\in\mathbb{N}}$, the sequence of rescaling parameters $(t_a)_{a\in\mathbb{N}}$, and the ratio $\zeta_2/\zeta_1\in\mathbb{C}$.

Local case 2 Similarly, consider the situation where the sequence of J–holomorphic maps $\{u_a\}_{a\in\mathbb{N}}$ in (3-29) Gromov converges to the nodal map

$$(u_1\colon \Sigma_1 \to D,\ u_2\colon \Sigma_2 \to D), \qquad x = u_1(0) = u_2(0) \in D;$$

with the following property: there exist meromorphic sections $\zeta_1(z_1)$ and $\zeta_2(z_2)$ of $u_1^*\mathcal{N}_X D$ and $u_2^*\mathcal{N}_X D$, respectively, such that

$$\operatorname{ord}_0(\zeta_1) = s, \qquad \operatorname{ord}_0(\zeta_2) = -s$$

and, for $i = 1, 2$, there exists a sequence of complex numbers $(t_{i;a})_{a\in\mathbb{N}}$ converging to zero such that $t_{i;a}^{-1}\Phi^{-1}(u_a(z_{i;a}))$ converges to $\zeta_i(z_i)$ uniformly with all derivatives on any compact set $\{z_i : |z_i| \neq 0\} \subset \Sigma_i$. With ζ_1 and ζ_2 as before, the following proposition also shows that there is a similar relation between the sequence of gluing parameters $(\varepsilon_a)_{a\in\mathbb{N}}$, rescaling parameters $(t_{i;a})_{a\in\mathbb{N}}$, and the ratio $\zeta_2/\zeta_1 \in \mathbb{C}$.

Proposition 3.15 *With notation as above, if in addition $(\omega, \mathcal{R}, J) \in \mathrm{AK}(X, D)$ for some regularization $\mathcal{R}$ or if J is integrable, in local case 1 we have*

$$\lim_{a\to\infty} \frac{\varepsilon_a^s}{t_a} = \frac{\zeta_2}{\zeta_1} \tag{3-30}$$

in local case 2 we have

$$\lim_{a\to\infty} \frac{t_{1;a}\,\varepsilon_a^s}{t_{2;a}} = \frac{\zeta_2}{\zeta_1}; \tag{3-31}$$

Note that the situation in (3-31) reduces to the situation in (3-30) after a rescaling of the sequence $\{u_a\}_{a\in\mathbb{N}}$ via $(t_{1;a})_{a\in\mathbb{N}}$. For the rescaled sequence we will have $(t_a = t_{2;a}/t_{1;a})_{a\in\mathbb{N}}$. We prove Proposition 3.15 in the next section. The proof of this proposition is the only place where we use the extra assumption on J in the statement of Theorem 1.4, but we expect this proposition, and thus Theorem 1.4, to be true for a larger class of almost complex structures that contains $\mathcal{J}(X, D)$ and holomorphic structures.

Remark 3.16 It is easy to see that the limit conditions in (3-30) and (3-31) are independent of Φ, the representatives ζ_1 and ζ_2, and the local coordinates z_1 and z_2. For example, in (3-31), substituting z_2 with ϱz_2 and ζ_2 with $\beta\zeta_2$ for some $\varrho, \beta \in \mathbb{C}^*$ changes ζ_2 on the right-hand side of (3-31) to $\varrho^s\beta\zeta_2$, changes ε_a and $t_{2;a}$ on the left-hand side of (3-31) to $\varrho\varepsilon_a$ and $\beta^{-1}t_{2;a}$, respectively, and has no effect on the other terms. Thus it affects both sides of (3-31) equally. It is also clear that (3-30) and (3-31) only depend on the equivalence classes $[(\varepsilon_a)_{a\in\mathbb{N}}]$, $[(t_a)_{a\in\mathbb{N}}]$, $[(t_{1;a})_{a\in\mathbb{N}}]$ and $[(t_{2;a})_{a\in\mathbb{N}}]$.

Remark 3.17 In the case of smooth divisors, a significantly simpler version of (3-31) suffices for proving Proposition 3.14. Instead of (3-31), in order to get the partial order in Lemma 4.3 we only need to prove that

$$\lim_{a\to\infty} \frac{t_{1,a}}{t_{2,a}} = \begin{cases} \frac{\ 2}{\ 1} & \text{if } s = 0; \\ \infty & \text{if } s > 0. \end{cases} \tag{3-32}$$

The equalities in (3-32) can be proved without the extra restriction on J. Thus, if D is smooth, Theorem 1.4 holds for arbitrary $(\omega, J) \in \mathcal{J}(X, D)$.

Proof of Proposition 3.15 The proof below is by constructing a modified sequence of J–holomorphic maps in $N_X D$.

Let $(R = (\rho; r; \perthousand); i_{N_X D}; \nabla_{N_X D}; J_{X;D})$ be as in the beginning of Section 3.2. If $(\omega; R; J) \in AK(X; D)$, then $\perthousand J = J_{X;D}$. If J is holomorphic, we consider a holomorphic chart $(z_1; \dots; z_n)$ around $x = u_1(0) = u_2(0) \in D$ such that $D = (z_1 = 0)$. Then, replacing the rescaling procedure in the proof below with holomorphic rescaling of z_1, the same proof works for the holomorphic case.

Assume $\perthousand J = J_{X;D}$. Note that $J_{X;D}$ is $\mathbb{C}$–invariant. Since the argument is local, in order to simplify the notation let us forget about $\perthousand$ and think of $\{u_a\}_{a\in\mathbb{N}}$ as a sequence of $J_{X;D}$–holomorphic maps into $N_X D$ itself.

Assume that we are in the situation of local case 1. For each $a \in \mathbb{N}$, let

$$\ _a = \frac{\epsilon_a^s}{t_a}.$$

Claim 1 There is no subsequence $(a_1; a_2; \dots)$ of $\mathbb{N}$ such that

$$\lim_{i\to\infty} \ _{a_i} = 0 \text{ or } \infty.$$

Thus, we conclude that there is $M > 0$ such that $M^{-1} < |\ _a| < M$ for all $a \in \mathbb{N}$.

Claim 2 For any subsequence $(a_1; a_2; \dots)$ such that the limit

$$\lim_{i\to\infty} \ _{a_i} =$$

exists, $= \ _2/\ _1$.

This implies that (3-30) holds over all of $\mathbb{N}$.

In order to prove these claims, we first construct two new sequences of J–holomorphic maps. For $a \in \mathbb{N}$, define

$$u_{1;a}\colon A_a \to N_X D, \quad u_{1;a}(z_{1;a}, z_{2;a}) = z_{1;a}^{-s}\, u_a(z_{1;a}, z_{2;a}), \tag{3-33}$$

$$u_{2;a}\colon A_a \to N_X D, \quad u_{2;a}(z_{1;a}, z_{2;a}) = z_{1;a}^{-s}\,\lambda_a\, u_a(z_{1;a}, z_{2;a}), \tag{3-34}$$

where the multiplications on the right-hand sides are with respect to the complex structure $\mathfrak{i}_{N_X D}$ on $N_X D$. By (1)–(3) on page 1004, both (3-33) and (3-34) are sequences of $J_{X;D}$–holomorphic maps in $N_X D$.

We will also use the following fact. For any $c > 0$, there exists a sufficiently small $\epsilon_c > 0$ such that

$$\omega_c = (\omega|_D) + \tfrac{1}{2}\epsilon_c\, d\nabla r$$

tames $J_{X;D}$ on $\overline{N_X D}(c)$. For any compact 2–dimensional domain Σ and smooth map $u\colon \Sigma \to \overline{N_X D}(c)$, let

$$\omega_c(u) = \int_\Sigma u^*\omega_c$$

denote the symplectic area of u.

In order to prove Claim 1, we separate the problem into two cases. In the first and second parts below, we consider the cases where the limit is ∞ or zero, respectively. In each case, we apply Gromov convergence to the auxiliary sequences in (3-33) and (3-34) to get a contradiction if the limit is ∞ or 0.

Proof of Claim 1, part 1 After passing to a subsequence, suppose

$$\lim_{a\to\infty} \lambda_a = \infty. \tag{3-35}$$

By (a) on page 1040 and the previous paragraph, for any $0 < r < \iota$, the sequence $\{u_{1;a}(z_{1;a})\}_{a\in\mathbb{N}}$ restricted to $r \le |z_1| \le \iota$ (and its preimages in A_a) converges uniformly with all derivatives to the $J_{X;D}$–holomorphic map

$$u_{1;\infty;1}(z_1) = z_1^{-s}\, u_1(z_1).$$

By definition of η_1, the function $u_{1;\infty;1}(z_1)$ extends to $z_1 = 0$ with $u_{1;\infty;1}(0) = \eta_1 \in N_X D|_x$, where $x = u_1(0) = u_2(0) \in D$. By assumptions (b) and (2) on page 1040, equation (3-35), and since

$$z_{1;a}^s = \varepsilon_a^s\, z_{2;a}^s;$$

the sequence $\{u_{1;a}(z_{2;a})\}_{a\in\mathbb{N}}$ restricted to $r\le|z_2|\le\iota$ (and its preimages in A_a) converges uniformly with all derivatives to the $J_{X;D}$–holomorphic map

$$u_{1;\infty;2}(z_2)=u_2(z_2)\in D.$$

This obviously extends to the entire Σ_2 with $u_{1;\infty;2}(0)=x$. The following subclaim shows that the sequence $\{u_{1;a}\}_{a\in\mathbb{N}}$ is bounded in between, so that Gromov convergence applies.

Subclaim There exists a sufficiently large $c>0$ such that

$$\text{(3-36)}\qquad \mathrm{Im}(u_{1;a})\subset N_X\overline{D}(c)\quad\text{and}\quad \omega_c(u_{1;a})\le c\qquad\text{for all } a\in\mathbb{N}.$$

Proof of subclaim Suppose (3-36) does not hold. Then (after passing to a subsequence), by assumptions (a)–(c) on page 1040, for any $c>1$ there exists a sequence $\{r_a\}_{a\in\mathbb{N}}$ with

$$\text{(3-37)}\qquad \lim_{a\to\infty} r_a=\infty$$

and

$$\text{(3-38)}\qquad \max_{(z_{1;a},z_{2;a})\in A_a} \big(|r_a^{-1}z_{1;a}\cdot u_a(z_{1;a},z_{2;a})|,\ \omega_c(z_{1;a};r_a^{-1}z_{1;a}\cdot u_a(z_{1;a},z_{2;a}))\big)=c$$

for all $a\ge 1$. Let

$$\tilde u_{1;a}\colon A_a\to Z,\quad \tilde u_{1;a}(z_{1;a},z_{2;a})=r_a^{-1}z_{1;a}\cdot u_a(z_{1;a},z_{2;a}).$$

Then:

- By (a) on page 1040 and equation (3-37), for any $0<r<\iota$, the rescaled sequence $\{\tilde u_{1;a}(z_{1;a})\}_{a\in\mathbb{N}}$ restricted to $r\le|z_1|\le\iota$ converges uniformly with all derivatives to the $J_{X;D}$–holomorphic map

$$\tilde u_{1;\infty;1}(z_1)=\overline{u}_1(z_1)\in D;$$

  where $\overline{u}_1$ is the image of u_1 in D.

- By assumptions (b) and (2) on page 1040, the sequence $\{\tilde u_{1;a}(z_{2;a})\}_{a\in\mathbb{N}}$ restricted to $r\le|z_2|\le\iota$ still converges uniformly with all derivatives to the $J_{X;D}$–holomorphic map

$$\tilde u_{1;\infty;2}(z_2)=u_2(z_2)\in D.$$

By (3-38), (the proof of) the Gromov convergence theorem in [29] applies[22] to the sequence $\{\tilde u_{1;a}\}_{a\in\mathbb N}$. In the limit we get a bubble domain Σ_1 with $\bullet_1$ and $\bullet_2$ at the two ends and at least one closed bubble in between (because of (3-38)), and a continuous $J_{X;D}$–holomorphic map

$$\tilde u_{1;1}\colon \Sigma_1 \to Z$$

such that

$$\tilde u_{1;1}|_{\bullet_1} = \tilde u_{1;1;1} \quad \text{and} \quad \tilde u_{1;1}|_{\bullet_2} = \tilde u_{1;1;2}.$$

Any nontrivial bubble would have trivial image in D, thus its image lives in $\overline{N}_X D.c/|_x$. This is impossible since the latter is open and there are no marked points to stabilize such a bubble. □

Going back to the proof of Claim 1, part 1, by (3-36), (the proof of) the Gromov convergence theorem in [29] applies to the sequence $\{u_{1;a}\}_{a\in\mathbb N}$. In the limit we get a bubble domain Σ_1 with $\bullet_1$ and $\bullet_2$ at the two ends and possibly some closed bubbles in between, and a continuous $J_{X;D}$–holomorphic map

$$u_{1;1}\colon \Sigma_1 \to Z$$

such that

$$u_{1;1}|_{\bullet_1} = u_{1;1;1} \quad \text{and} \quad u_{1;1}|_{\bullet_2} = u_{1;1;2}.$$

Since

$$u_{1;1;1}(0) \neq u_{1;1;2}(0),$$

Σ_1 should include at least one nontrivial bubble. Such a nontrivial bubble would have trivial image in D, thus its image lives in $\overline{N}_X D.c/|_x$. This is impossible since the latter is a domain in $\mathbb C$ and there are no marked points to stabilize such a bubble. □

Proof of Claim 1, part 2 After passing to a subsequence, suppose

$$\lim_{a\to 1} \,{}_a = 0. \tag{3-39}$$

By assumptions (b) and (2) on page 1040, since

$$u_{2;a}(z_{1;a}; z_{2;a}) = z_{1;a}^{s}\, {}_a\, u_a(z_{1;a}; z_{2;a}) = z_{2;a}^{s}\, t_a^{-1}\, u_a(z_{1;a}; z_{2;a});$$

[22] Gromov convergence applies because on the open ends of A_a we already know that $\{\tilde u_{1;a}\}_{a\in\mathbb N}$ uniformly converges to $\tilde u_{1;1;1}$ and $\tilde u_{1;1;2}$, and in the middle the sequence is bounded with bounded energy.

for any $0 < r < ı$ the sequence $\{u_{2;a}(z_{2;a})\}_{a\in\mathbb{N}}$ restricted to $r \le |z_2| \le ı$ (and its preimages in A_a) converges uniformly with all derivatives to the $J_{X;D}$–holomorphic map

$$u_{2;1;2}(z_2) = z_2^{s}(z_2):$$

By definition of $_2$, the function $u_{2;1;2}(z_2)$ extends to $z_2 = 0$ with $u_{2;1;2}(0) = {}_2$. On the other hand, by (a) on page 1040 and (3-39), the sequence $\{u_{2;a}(z_{1;a})\}_{a\in\mathbb{N}}$ restricted to $r \le |z_1| \le ı$ (and its preimages in A_a) converges uniformly with all derivatives to the $J_{X;D}$–holomorphic map

$$u_{2;1;1}(z_1) = u_1(z_1) = :$$

This obviously extends to the entire $\bullet_1$ with $u_{2;1;1}(0) = x$. By a similar argument as in the previous case, the inequality

$$u_{2;1;1}(0) \neq u_{2;1;2}(0)$$

leads to a contradiction. This finishes the proof of Claim 1. □

Proof of Claim 2 After passing to a subsequence, suppose

$$\lim_{a\to\infty} {}_a = \neq 0: \text{Then,}$$

going back to the proof of Claim 1, part 1, since

$$z_{1;a}^{s} = \varepsilon_a^{s} z_{2;a}^{s} \quad \text{for all } a \in \mathbb{N};$$

the sequence $\{u_{1;a}(z_{2;a})\}_{a\in\mathbb{N}}$ restricted to $r \le |z_2| \le ı$ converges uniformly with all derivatives to the $J_{X;D}$–holomorphic map

$$u_{1;1;2}(z_2) = {}^{1} z^{s}{}_{2}(z_2):$$

This extends to the entire $\bullet_2$ with $u_{1;1;2}(0) = {}_2$. By a similar argument as in the proof of Claim 1, part 1, if

$$u_{1;1;1}(0) \neq u_{1;1;2}(0);$$

we get a contradiction. Therefore,

$${}_1 = u_{1;1;1}(0) = u_{1;1;2}(0) = {}^{1}{}_2;$$

in other words, $= {}_2/{}_1$. □

This finishes the proof of Proposition 3.15 in the local case 1.

For the local case 2, repeat the exact same proof with

$$u_{1;a}\colon A_a \to N_X D, \quad u_{1;a}(z_{1;a}, z_{2;a}) = z_{1;a}^{s}\, t_{1;a}^{-1}\, u_a(z_{1;a}, z_{2;a}),$$
$$u_{2;a}\colon A_a \to N_X D, \quad u_{2;a}(z_{1;a}, z_{2;a}) = z_{1;a}^{s}\, \lambda_a\, u_a(z_{1;a}),$$

in place of (3-33) and (3-34), respectively, where

$$\lambda_a = \frac{t_{1;a}\,\epsilon_a^{s}}{t_{2;a}} \quad \text{for all } a \in \mathbb{N}.$$

This finishes the proof of Proposition 3.15 under the assumption $\tilde{J} = J_{X;D}$. □

Remark 3.18 For arbitrary J on $N_X D$, define

$$\mathcal{Z} = \{(t,v) \in \mathbb{C} \times N_X D \mid t^s v \in N^0_X D\}, \qquad \mathcal{Z}^* = \{(t,v) \in \mathcal{Z} \mid t \in \mathbb{C}^*\},$$

and

$$F\colon \mathcal{Z}^* \to \mathbb{C}^* \times N_X D^*, \qquad F(t,v) = (t, t^s v).$$

Let $J_{\mathcal{Z}} = F^*(i \times J)$, where i is the standard almost complex structure on $\mathbb{C}$ and $i \times J$ is the product almost complex structure on the target. By an argument similar to Lemma 3.5, the almost complex structure $J_{\mathcal{Z}}$ on $\mathcal{Z}^*$ extends to a (similarly denoted) almost complex structure on all of $\mathcal{Z}$ satisfying

$$J_{\mathcal{Z}}|_{\{0\} \times N_X D \,\cup\, \mathbb{C} \times D} \equiv i \times J_{X;D}. \tag{3-40}$$

Similarly, for every $a \in \mathbb{N}$, let

$$\mathcal{Z}_a = \{(t,v) \in \mathbb{C} \times N_X D \mid t^s \epsilon_a^{-1} v \in N^0_X D\}, \qquad \mathcal{Z}_a^* = \{(t,v) \in \mathcal{Z}_a \mid t \in \mathbb{C}^*\},$$

and define

$$F_a\colon \mathcal{Z}_a^* \to \mathbb{C}^* \times N_X D^*, \qquad F_a(t,v) = (t, \epsilon_a^{-1} t^s v). \tag{3-41}$$

For each $a \in \mathbb{N}$, let $J_a = F_a^*(i \times J)$. By Lemma 3.5 and the previous paragraph, for each $a \in \mathbb{N}$, the almost complex structure J_a on $\mathcal{Z}_a^*$ extends to a (similarly denoted) almost complex structure on the entire $\mathcal{Z}_a$ satisfying (3-40).

For $a \in \mathbb{N}$, define

$$u_{1;a}\colon A_a \to \mathcal{Z}, \qquad u_{1;a}(z_{1;a}, z_{2;a}) = (z_{1;a}, z_{1;a}^{s}\, u_a(z_{1;a}, z_{2;a})), \tag{3-42}$$
$$u_{2;a}\colon A_a \to \mathcal{Z}_a, \qquad u_{2;a}(z_{1;a}, z_{2;a}) = (z_{1;a}, z_{1;a}^{s}\, \lambda_a\, u_a(z_{1;a}, z_{2;a})). \tag{3-43}$$

By definition, (3-33) is a sequence of $J_{\mathcal{Z}}$–holomorphic maps in $\mathcal{Z}$ and (3-34) is a sequence of J_a–holomorphic maps in $\mathcal{Z}_a$. In principle, one may try the proof above by replacing (3-33) and (3-34) with (3-42) and (3-43), respectively. However, multiplication by $_a^{-1}$ in (3-41) and by r_a^{-1} in (3-38) have adverse effects on the almost complex structure, making it hard to apply Gromov convergence.

3.4 Proof of Proposition 3.14 and Theorem 1.4

Going back to the setup of Proposition 3.14, first assume that the dual graph Γ of f_a in (3-25) is made of only one vertex $V=\{v\}$ — in other words, restrict to the v^{th} component of the sequence $(f_a)_{a\in\mathbb{N}}$ in (3-25) — and fix a set of representatives

$$(\cdot_{a;v;i})_{i\in I_v}$$

for $[\ ]_{a,v}$. For each $v'\in V_0$ and $i\in I_{v'}$ fix a representative $_{v';i}$ of the $\mathbb{C}$–equivalence class $[\ ]_{v';i}$ in Lemma 3.13, and a sequence of rescaling parameters $(t_{a;v';i})_{a\in\mathbb{N}}$ satisfying Proposition 3.10 or Corollary 3.12, depending on whether $i\notin I_v$ or $i\in I_v$, respectively.

By the surjectivity of the classical gluing theorem of J–holomorphic maps (see for instance [13, Section 7]), for a sufficiently large, the domain $\Sigma_a\cong\Sigma$ of (the stable map underlying) f_a can be obtained from the nodal domain Σ_0 of (the stable map underlying) f in the following way. There exist

- a sequence of complex structures $\mathfrak{j}^0_a=(j_{v';a})_{v'\in V_0}$ on the nodal domain $\Sigma_0=(\Sigma_{v'})_{v'\in V_0}$ of the stable nodal map (f) in (3-15),
- a sequence of local $j_{v';a}$–holomorphic coordinates $z_{\vec{e}';a}\colon\Delta_{\vec{e}'}\to\mathbb{C}$ around $q_{\vec{e}'}\in\Sigma_{v'}$ for all $v'\in V_0$ and $\vec{e}'\in\vec{E}'_{v'}$, and
- a sequence of nonzero complex numbers $(\varepsilon_{e';a})_{e'\in E'}$ converging to zero

such that

(1) $(\Sigma_a, j_a, \vec{z}_a)$ is isomorphic to the smoothing of $(\Sigma_0, \mathfrak{j}^0_a=(j_{v';a})_{v'\in V_0})$ defined by

$$z_{\vec{e}';a}\,z_{\overleftarrow{e}';a}=\varepsilon_{e';a}\quad\text{for all } e'\in E'; \tag{3-44}$$

(2) the sequence $(j_{v';a})_{a\in\mathbb{N}}$ C^∞–converges to $j_{v'}$ for all $v'\in V_0$, and

(3) the sequence $(z_{\vec{e}';a})_{a\in\mathbb{N}}$ C^∞–converges to $z_{\vec{e}'}$, where $z_{\vec{e}'}\colon\Delta_{\vec{e}'}\to\mathbb{C}$ is some fixed local $j_{v'}$–holomorphic coordinate around $q_{\vec{e}'}\in\Sigma_{v'}$ for all $v'\in V_0$ and $\vec{e}'\in\vec{E}'_{v'}$.

We will use this standard presentation of (Σ_a, j_a) in the proof of Proposition 3.14 and Theorem 1.4.

Remark 3.19 For $\delta > 0$ sufficiently small, let

$$\Sigma_{\vec{e}^0;a}(\delta) = \{x \in \Sigma_{\vec{e}^0} \mid |z_{\vec{e}^0;a}(x)| < \delta\} \quad \text{for all } \vec{e}^0 \in E^0,\ a \geq 1$$

and

$$A_{e^0;a} = \{z_{\vec{e}^0;a} z_{\overleftarrow{e}^0;a} = \varepsilon_{e^0;a} \mid z_{\vec{e}^0;a} \in \Sigma_{\vec{e}^0}(2\varepsilon_{e^0;a}),\ z_{\overleftarrow{e}^0;a} \in \Sigma_{\overleftarrow{e}^0}(2\varepsilon_{e^0;a})\} \subset \Sigma_a$$

for all $e^0 \in E^0$, $a \geq 1$. Then, with respect to the identification of the domains in (1), the Σ_a–degeneration maps

$$\varphi_a \colon \Sigma_a \to \Sigma^0$$

can be taken to be the identity on the complement of $\bigcup_{e^0 \in E^0} A_{e^0;a}$ and some "nice" degeneration map

$$A_{e^0;a} \to \Sigma_{\vec{e}^0}(2\varepsilon_{e^0;a}) \cup \Sigma_{\overleftarrow{e}^0}(2\varepsilon_{e^0;a})$$

on the neck region.

For each $\vec{e}^0 \in \vec{E}^0$ and $i \in I_{e^0}$, let

$$0 \neq \eta_{\vec{e}^0;i} \in \mathcal{N}_X D_i|_{u^0(q_{\vec{e}^0})}$$

be the leading coefficient term in (2-36) with respect to $z_{\vec{e}^0}$ (and $\zeta_{v^0;i}$ if $i \in I_{v^0}$). By Proposition 3.15, for every $v_1^0, v_2^0 \in V^0$ and $\vec{e}^0 \in \vec{E}^0_{v_1^0,v_2^0}$ we have

$$\lim_{a \to \infty} \frac{t_{a;v_1^0;i}\, \varepsilon_{e^0;a}^{s_{\vec{e}^0;i}}}{t_{a;v_2^0;i}} = \frac{\eta_{\overleftarrow{e}^0;i}}{\eta_{\vec{e}^0;i}} \quad \text{for all } i \in I_{v_1^0} \cap I_{v_2^0}; \tag{3-45}$$

$$\lim_{a \to \infty} t_{a;v_1^0;i}\, \varepsilon_{e^0;a}^{s_{\vec{e}^0;i}} = \frac{\eta_{\vec{e}^0;i}}{\eta_{\overleftarrow{e}^0;i}} \quad \text{for all } i \in I_{v_1^0} - I_{v_2^0}. \tag{3-46}$$

The following proposition shows that, for a sufficiently large, we can adjust the choices involved to get equality at each a.

Proposition 3.20 There exists a choice of coordinates $\{z_{\vec{e}^0}\}_{\vec{e}^0 \in \vec{E}^0}$ and $\{z_{\vec{e}^0;a}\}_{\vec{e}^0 \in \vec{E}^0;\, a \in \mathbb{N}}$ satisfying (3-44) and item (3) after that, and a choice of representatives $\zeta_{v^0;i}$ and $(t_{a;v^0;i})_{a \in \mathbb{N}}$ for $[\zeta_{v^0;i}]$ and $[(t_{a;v^0;i})_{a \in \mathbb{N}}]$, respectively, such that

$$t_{a;v_1^0;i}\, \varepsilon_{e^0;a}^{s_{\vec{e}^0;i}} = t_{a;v_2^0;i} \quad \text{for all } i \in I_{v_1} \cap I_{v_2},\ a \geq 1; \tag{3-47}$$

$$t_{a;v_1^0;i}\, \varepsilon_{e^0;a}^{s_{\vec{e}^0;i}} = 1 \quad \text{for all } i \in I_{v_1^0} - I_{v_2^0},\ a \geq 1. \tag{3-48}$$

The proof of Proposition 3.20 uses the following lemma with the linear map

$$\varrho_{\mathbb{C}} \colon \mathbb{C}^{E^0} \oplus \bigoplus_{v^0 \in V^0} \mathbb{C}^{I_{v^0}} \to \bigoplus_{e^0 \in E^0} \mathbb{C}^{I_{e^0}}$$

defined in (2-26). We will use Proposition 3.20 to construct maps

$$s\colon V' \to \mathbb{R}^N, \quad v' \to s_{v'} \qquad \text{and} \qquad \epsilon\colon E' \to \mathbb{R}_C, \quad e' \to \epsilon_{e'}$$

satisfying condition (1), and also to show that the limit satisfies condition (2) of Definition 2.8.

Lemma 3.21 Assume $f\colon \mathbb{C}^n \to \mathbb{C}^m$ is a complex-linear map and $(\alpha_a)_{a\in\mathbb{N}} \subset \mathbb{C}^n$ is a sequence such that

$$\lim_{a\to\infty} f(\alpha_a) = \beta. \tag{3-49}$$

Then there exists a convergent sequence $(\alpha'_a)_{a\in\mathbb{N}} \subset \mathbb{C}^n$ (ie there exists $\alpha'_0 \in \mathbb{C}^n$ such that $\lim_{a\to\infty}\alpha'_a = \alpha'_0$) such that $f(\alpha_a - \alpha'_a) = 0$ for all $a \in \mathbb{N}$.

Proof Since $\mathrm{Im}(f) \subset \mathbb{C}^m$ is closed, (3-49) implies that $\beta \in \mathrm{Im}(f)$. Let $\beta = f(\alpha)$. Fix an affine subspace[23] $H \subset \mathbb{C}^n$ passing through α and transverse[24] to the hyperplane $f^{-1}(\beta) \subset \mathbb{C}^n$. By (3-49), there exists $M \in \mathbb{N}$ such that H is transverse to $f^{-1}(f(\alpha_a))$ for all $a > M$. Then the sequence given by $\alpha'_a = f^{-1}(f(\alpha_a)) \cap H$ if $a > M$, and $\alpha'_a = \alpha_a$ if $a \le M$, has the desired properties. □

Proof of Proposition 3.20 Throughout the proof we assume $\mathrm{l}_v = \emptyset$; for $\mathrm{l}_v \neq \emptyset$, the argument reduces to $\mathrm{l}_v = \emptyset$ by considering the associated sequence of maps in $\mathcal{N}_X = \mathrm{l}_v$. We modify a given set of representatives to another set satisfying (3-47) and (3-48). Assuming $\mathrm{l}_v = \emptyset$, fix an orientation O on E', and choose some branch

$$\eta = \bigoplus_{\vec{e}'\in O} \eta_{e'} \in \bigoplus_{e'\in E'} \mathbb{C}^{\mathrm{l}_{e'}}, \qquad \eta_{e'} = \left(\log \frac{\epsilon'_{\vec{e}';i}}{\epsilon'_{\overleftarrow{e}';i}}\right)_{i\in \mathrm{l}_{e'}} \in \mathbb{C}^{\mathrm{l}_{e'}} \quad \text{for all } \vec{e}' \in O$$

of the multivalued function log. By (3-45)–(3-46) and the definition of ϖ_C in (2-26) (via the chosen orientation O), for all $a \in \mathbb{N}$ we can choose the branches

$$\eta_a = \big((\log \epsilon_{e';a})_{e'\in E'}, (\log t_{a;v';i})_{v'\in V', i\in \mathrm{l}_{v'}}\big) \in \mathbb{C}^{E'} \oplus \bigoplus_{v'\in V'} \mathbb{C}^{\mathrm{l}_{v'}} \text{ so}$$

that

$$\lim_{a\to\infty} \varpi_C(\eta_a) = \eta.$$

By Lemma 3.21 applied to ϖ_C, there exists a sequence

$$(\eta'_a)_{a\in\mathbb{N}} \subset \mathbb{C}^{E'} \oplus \bigoplus_{v'\in V'} \mathbb{C}^{\mathrm{l}_{v'}}$$

---

[23] A shifted linear subspace.

[24] Assuming f is not trivial; otherwise, the lemma is obvious.

such that $\%_{\mathbb{C}}(a\ {}_0)_a = 0$ for all $a\in\mathbb{N}$ and $\lim_{a\to\infty} 0_a = 0$. Taking the exponential of ${}_a^{\mathbb{C}}$ and 0, we conclude that there exist

$$\big((\rho_{e^0})_{e^0\in E^0};(\rho_{v^0;i})_{v^0\in V^0,\,i\in I_{v^0}}\big) \quad\text{and}\quad \big((\rho_{e^0;a})_{e^0\in E^0},(\rho_{a;v^0;i})_{v^0\in V^0,\,i\in I_{v^0}}\big)_{a\in\mathbb{N}}$$

in $(\mathbb{C}^*)^{E^0}\times\prod_{v^0\in V^0}(\mathbb{C}^*)^{I_{v^0}}$ such that

$$\lim_{a\to\infty}\big((\rho_{e^0;a})_{e^0\in E^0};(\rho_{a;v^0;i})_{v^0\in V^0,\,i\in I_{v^0}}\big) = \big((\rho_{e^0})_{e^0\in E^0};(\rho_{v^0;i})_{v^0\in V^0,\,i\in I_{v^0}}\big)$$

and

$$\frac{(\rho_{v_1^0;i}^{-1}t_{a;v_1^0;i})\,(\rho_{e^0;a}^{-1}\varepsilon_{e^0;a})^{s_{\vec{e}^0;i}}}{(\rho_{v_2^0;i}^{-1}t_{a;v_2^0;i})}=1 \quad \text{for all } i\in I_{v_1^0}\cap I_{v_2^0};\ a\in\mathbb{N}; \tag{3-50}$$

$$(\rho_{v_1^0;i}^{-1}t_{a;v_1^0;i})\,(\rho_{e^0;a}^{-1}\varepsilon_{e^0;a})^{s_{\vec{e}^0;i}}=1 \quad \text{for all } i\in I_{v_1^0}-I_{v_2^0};\ a\in\mathbb{N}. \tag{3-51}$$

By (3-50) and (3-51), for a sufficiently large, replacing

$\{z_{\vec{e}^0}\}_{\vec{e}^0\in\mathbb{O}}$ with $\{\rho_{e^0}^{-1}z_{\vec{e}^0}\}_{\vec{e}^0\in\mathbb{O}}$,

$\{z_{\vec{e}^0;a}\}_{\vec{e}^0\in\mathbb{O}}$ with $\{\rho_{e^0;a}^{-1}z_{\vec{e}^0;a}\}_{\vec{e}^0\in\mathbb{O}}$,

$\{\varepsilon_{e^0;a}\}_{e^0\in E^0}$ with $\{\rho_{e^0;a}^{-1}\varepsilon_{e^0;a}\}_{e^0\in E^0}$,

$(t_{a;v^0;i})_{v^0\in V^0,\,i\in I_{v^0}}$ with $(\rho_{a;v^0;i}^{-1}t_{a;v^0;i})_{v^0\in V^0,\,i\in I_{v^0}}$, and

$(\nu_{v^0;i})_{v^0\in V^0,\,i\in I_{v^0}}$ with $(\rho_{v^0;i}\nu_{v^0;i})_{v^0\in V^0,\,i\in I_{v^0}}$,

we get a new set of representatives satisfying (3-47) and (3-48). In particular, the limits in (3-45) and (3-46) can be set to be equal to 1. □

Proof of Proposition 3.14 First, assume that the dual graph Γ of f_a in (3-25) is made of only one vertex $V=\{v\}$ and fix a set of representatives

$$(\rho_{a;v;i})_{i\in I_v}$$

for $[\mathfrak{a}_{a;v}]$. By Propositions 3.15 and 3.20, we can choose the coordinates $\{z_{\vec{e}^0}\}_{\vec{e}^0\in E^0}$ and $\{z_{\vec{e}^0;a}\}_{\vec{e}^0\in E^0;\,a\in\mathbb{N}}$, and the representatives $\nu_{v^0;i}$ and $(t_{a;v^0;i})_{a\in\mathbb{N}}$ so that (3-47) and (3-48) hold. For each $v^0\in V_0$ and $i\in I_{v^0}-I_v$, note that $(t_{a;v^0;i})_{a\in\mathbb{N}}$ converges to 0; therefore,

$$-\log|t_{a;v^0;i}| > 0 \quad \text{for all } v^0\in V;\ i\in I_{v^0}-I_v;\ a\gg 1;$$

and it converges to infinity. Choose a sequence of positive vectors $s_v^a=(s^a_{v;i})_{i\in I_v}\in\mathbb{R}_+^{I_v}$ such that

$$s^a_{v;i}-\log|t_{a;v^0;i}| > -1 \quad \text{for all } v^0\in V^0;\ i\in I_v. \tag{3-52}$$

With these choices, for $a \gg 1$ the functions $s_a\colon V_0 \to \mathbb{R}^N$ defined by

(3-53) $s_{a;v'} = \big((s_{a;i} - \log|t_{a;v';i}|)_{i\in I_v};\ (-\log|t_{a;v';i}|)_{i\in I_{v'} - I_v}\big) \in \mathbb{R}_+^{I_{v'}}$ for all $v' \in V'$;

and $\alpha\colon E' \to \mathbb{R}_+$ defined by

$$\alpha_{e'} = -\log|\varepsilon_{e';a}| \quad \text{for all } e' \in E',$$

satisfy condition (1) of Definition 2.8. By (3-45)–(3-46), f' also satisfies condition (2) of Definition 2.8.

For general ε, by the definition of $\sim$ in (2-37), we can choose a set of representatives

$$(\varphi_{a;v;i})_{a\in\mathbb{N};\, v\in V;\, i\in I_v}$$

and coordinates $(z_{\vec e;a} = \lambda_{a;\vec e}\, z_{\vec e})_{a\in\mathbb{N};\, \vec e\in \vec E}$ such that the leading coefficients $\eta_{\vec e;i;a}$ in (2-36) satisfy

(3-54) $$\eta_{\vec e;i;a} = \eta_{e;i;a} \quad \text{for all } e \in E,\ i \in I_e,\ a \in \mathbb{N}.$$

Let $\lambda_{a;e} = \lambda_{a;\vec e}\,\lambda_{a;\overleftarrow{e}}$, for all $e \in E$. For each $v \in V$, choose representatives

$$(\varphi_{v';i})_{v'\in \mathbb{V}(v);\, i\in I_{v'}} \qquad \text{and} \qquad (t_{a;v';i})_{v'\in\mathbb{V}(v);\, i\in I_{v'};\, a\in\mathbb{N}}$$

so that (3-47) and (3-48) hold. By (3-54), we have

(3-55) $$\lim_{a\to\infty} \frac{t_{a;v'_1;i}\,\lambda_{e;a}^{s_{\vec e;i}}}{t_{a;v'_2;i}} = 1 \quad \text{for all } i \in I_{v'_1} \cap I_{v'_2};$$

(3-56) $$\lim_{a\to\infty} t_{a;v'_1;i}\,\lambda_{e;a}^{s_{\vec e;i}} = 1 \quad \text{for all } i \in I_{v'_1} - I_{v'_2}.$$

With an argument similar to the proof of Proposition 3.20, we can choose these representatives so that further,

(3-57) $$t_{a;v'_1;i}\,\lambda_{e;a}^{s_{\vec e;i}} = t_{a;v'_2;i} \quad \text{for all } i \in I_{v'_1} \cap I_{v'_2},\ a \in \mathbb{N};$$

(3-58) $$t_{a;v'_1;i}\,\lambda_{e;a}^{s_{\vec e;i}} = 1 \quad \text{for all } i \in I_{v'_1} - I_{v'_2},\ a \in \mathbb{N}.$$

Also choose the functions $s_a\colon V \to \mathbb{R}^N$ and $\alpha\colon E \to \mathbb{R}_+$ satisfying condition (1) of Definition 2.8 so that (3-52) holds and

$$\alpha_e - \log\lambda_{e;a} > 1 \quad \text{for all } e \in E,\ a \gg 1.$$

Then, similarly to (3-53), for $a \gg 1$, the extended functions $s_a^{\mathrm{new}}\colon V_0 \to \mathbb{R}^N$ given by

(3-59) $s^{\mathrm{new}}_{a;v'} = \big((s_{a;i} - \log|t_{a;v';i}|)_{i\in I_v};\ (-\log|t_{a;v';i}|)_{i\in I_{v'} - I_v}\big) \in \mathbb{R}_+^{I_{v'}}$

for all $v'\in V'$, $v=\varphi(v')$, and $\ell^a\colon E'\to\mathbb{R}_+$ given by

$$\ell^a_e=-\log(\iota_{e;a})\qquad \text{if } e\in E\subset E';$$
$$\ell^a_{e'}=-\log|\varepsilon_{e';a}|\qquad \text{if } e'\in E'/E;$$

satisfy condition (1) of Definition 2.8. By (3-45)–(3-46) applied to E'/E, the assumption (3-54), and (3-57)–(3-58), f also satisfies condition (2) of Definition 2.8. □

Proof of Theorem 1.4 As in the classical case, consider the sequential convergence topology on $\overline{\mathcal{M}}^{\log}_{g;s}(X;D;A)$ given by Definition 3.7: a subset W of $\overline{\mathcal{M}}^{\log}_{g;s}(X;D;A)$ is closed if every sequence in W has a subsequence with a log-Gromov limit in W. Note that as in [29, Section 5.1], we must show that convergence with respect to the topology defined above is equivalent to log-Gromov convergence. Since the forgetful map $\overline{\mathcal{M}}^{\log}_{g;s}(X;D;A)\to\overline{\mathcal{M}}_{g;k}(X;A)$ is finite-to-one and log-Gromov convergence is a lift of the classical Gromov convergence, this property follows from the corresponding statement for the Gromov convergence topology on $\overline{\mathcal{M}}^{\log}_{g;k}(X;A)$. In other words, the five axioms[25] in [29, Lemma 5.6.4] lift to sequences in $\overline{\mathcal{M}}^{\log}_{g;s}(X;D;A)$.

Suppose that $W\subset\overline{\mathcal{M}}_{g;k}(X;A)$ is closed and let $W'=\iota^{-1}(W)$. Let $(f_a)_{a\in\mathbb{N}}$ be any sequence in W'. Its image $(h_a=\iota(f_a))_{a\in\mathbb{N}}$ in W has a subsequence, still denoted by $(h_a)_{a\in\mathbb{N}}$, that Gromov converges to some $h\in W$. On the other hand, by Theorem 3.8, $(f_a)_{a\in\mathbb{N}}$ has a subsequence that log-Gromov converges to some $f\in\overline{\mathcal{M}}^{\log}_{g;s}(X;D;A)$. By Definition 3.7, we have $\iota(f)=h$, ie $f\in W'$. Therefore, W' is closed. We conclude that ι is continuous.

Let f be an arbitrary log map in $\overline{\mathcal{M}}^{\log}_{g;s}(X;D;A)$ with the decorated dual graph Γ and let $h=\iota(f)$ be the underlying stable map in $\overline{\mathcal{M}}_{g;k}(X;A)$. Let $(U_a)_{a\in\mathbb{N}}$ be a shrinking basis for the (metrizable) topology of $\overline{\mathcal{M}}_{g;k}(X;A)$ around h. Recall from Lemma 2.15 that every stable map h admits at most finitely many log lifts f, each of which is uniquely specified by the vector decorations on the nodes of its dual graph (ie the contact data s_e at the nodes q_e). Furthermore, by Lemma 2.16, such a lift is unique if the genus is zero. As we explained before Remark 3.19, for a sufficiently large, by the classical gluing theorem the domain of every map h' in U_a is obtained from the nodal domain Σ of h by gluing the nodes in a standard way. Furthermore, the image of h' is C^0–close to the image of h. The dual graph Γ' of h' is a contraction of Γ in the sense[26]

[25] Even though [29, Section 5.1] is about the genus-0 moduli spaces, the statements used here are valid in all genera.

[26] Their roles are reversed here.

of (3-1). With these identifications, if f' is a log lift of h' in U_a, by its decoration type we mean (1) the vector decorations s_e at its nodes q_e, together with (2) the winding number[27] of h' around D_i along the circles ∂A_e (see (3-27)) on every neck A_e obtained from gluing the node q_e of the domain of h; see the proof of Lemma 3.13. Thus, f' has the same decoration type as f if (1) at every node of the domain of f' the vector decoration s_e is the same as the vector decoration at the corresponding node of f, and (2) on every neck A_e the winding number of h' around D_i along the circle ∂A_e is the same as the tangency order $s_{e;i}$ for f.

For a sufficiently large, define U'_a to be the set of elements f' in $\overline{M}^{\log}_{g;s}(X; D; A)$ whose image h' under lies in U_a and such that f' has the same decoration type as f. By Remark 2.14, the restriction of to U'_a is one-to-one. We show that U'_a is open. Let $(f_b)_{b\in\mathbb{N}}$ be a sequence in the complement of U'_a that log-Gromov converges to f'. After possibly passing to a subsequence, we can assume that the underlying sequence of stable maps $(h_b)_{b\in\mathbb{N}}$ lies either in U_a or its complement U_a^c. In the latter case, by Definition 3.7, f' belongs to the complement of U'_a. In the former case, the decoration type of f' (with respect to f) will be the same as the decoration type of f_b which is, by definition, different from the decoration type of f. Therefore, f' belongs to the complement of U'_a. We conclude that U'_a is open. Furthermore, it is easy to see that $(U'_a)_{a\in\mathbb{N}}$ is a shrinking basis for the topology of $\overline{M}_{g;s}(X; D; A)$ at f. Therefore, the log-Gromov topology of $\overline{M}_{g;s}(X; D; A)$ is first-countable.

Hausdorffness is the consequence of uniqueness of the limit in Theorem 3.8. If Y is a first-countable topological space and has the property that every convergent sequence has a unique limit, then Y is Hausdorff. Finally, compactness of $\overline{M}^{\log}_{g;s}(X; D; A)$ is the consequence of the existence of the limit in Theorem 3.8. □

4 Log vs relative compactification

In Section 4.1, following the description in [48], we review the construction of the relative moduli spaces for smooth symplectic divisors in [23; 21]. In Section 4.2, we show that the natural forgetful map from the relative compactification to our log compactification is onto.

First, let us recall some relevant facts from Section 2.1. Suppose $D \subset (X; \omega)$ is a smooth symplectic divisor, J is an ω–tame almost complex structure on X such

[27] Contact points with D_i are among the marked/nodal points and are away from the neck region.

that $J(TD)=TD$, and $\bar\partial_{N_XD}$ is the $\bar\partial$–operator in Lemma 2.1. With notation as in Section 2.1, choose a Hermitian connection ∇^N on (N_XD, i_{N_XD}) so that $\bar\partial_{N_XD}=\bar\partial_{\nabla^N}$. The connection ∇^N gives a splitting of the exact sequence

(4-1) $$0\longrightarrow N_XD\longrightarrow T(N_XD)\xrightarrow{d\pi} TD\longrightarrow 0$$

of vector bundles over N_XD, which restricts to the canonical splitting over the zero section and is preserved by the multiplication by $\mathbb{C}^*$; see [48, Section 4.1]. Let

(4-2) $$\mathbb{P}_XD=\mathbb{P}(N_XD\oplus D\times\mathbb{C}),\qquad D_0=\mathbb{P}(0\oplus D\times\mathbb{C})\quad\text{and}\quad D_\infty=\mathbb{P}(N_XD\oplus 0)\subset\mathbb{P}_XD.$$

The splitting of (4-1) extends to a splitting of the exact sequence

$$0\longrightarrow T^{\mathrm{vrt}}(\mathbb{P}_XD)\longrightarrow T(\mathbb{P}_XD)\xrightarrow{d\pi} TD\longrightarrow 0,$$

where $\pi\colon\mathbb{P}_XD\to D$ is the bundle projection map induced by (2-7); this splitting restricts to the canonical splittings over $D_0\approx D_\infty\approx D$ and is preserved by the multiplication by $\mathbb{C}^*$. Via this splitting, the almost complex structure J_D and the complex structure i_{N_XD} in the fibers of π induce an almost complex structure $J_{X;D}$ on $\mathbb{P}_XD$, which restricts to J_D on D_0 and D_∞, and is preserved by the $\mathbb{C}^*$–action. In fact, $J_{X;D}|_{N_XD}$ is the almost complex structure $J_{\bar\partial_{N_XD}}$ associated to $\bar\partial_{N_XD}$ described in items (1)–(3) of page 1004 and is independent of the choice of ∇^N. By property (1), the projection $\pi\colon\mathbb{P}_XD\to D$ is $(J_D,J_{X;D})$–holomorphic. By (3), there is a one-to-one correspondence between the space of $J_{X;D}$–holomorphic maps $u\colon(\Sigma,\mathfrak{j})\to(\mathbb{P}_XD,J_{X;D})$ (not mapped into $D_{X;0}$ and $D_{X;\infty}$) and tuples (u_D,ζ) where $u_D\colon(\Sigma,\mathfrak{j})\to(D,J_D)$ is a J_D–holomorphic map into D and ζ is a nontrivial meromorphic section of $u_D^*N_XD$ with respect to the holomorphic structure defined by $u_D^*\bar\partial_{N_XD}$.

4.1 Relative compactification

Let (X,ω) be a smooth symplectic manifold, $D\subset X$ be a smooth symplectic divisor, and $J\in\mathcal{J}(X,D,\omega)$. With notation as in (4-2), for each $m\in\mathbb{N}$ let

$$X[m]=(X\sqcup\{1\}\times\mathbb{P}_XD\sqcup\cdots\sqcup\{m\}\times\mathbb{P}_XD)/\sim,\quad\text{where}$$

$$D\sim\{1\}\times D_\infty\quad\text{and}\quad\{r\}\times D_0\sim\{r+1\}\times D_\infty\quad\text{for all } r=1,\dots,m-1;$$

see Figure 6. This is a basic (ie there are no triple or higher intersections) SNC variety, which is smoothable to (a symplectic manifold deformation equivalent to) X itself.

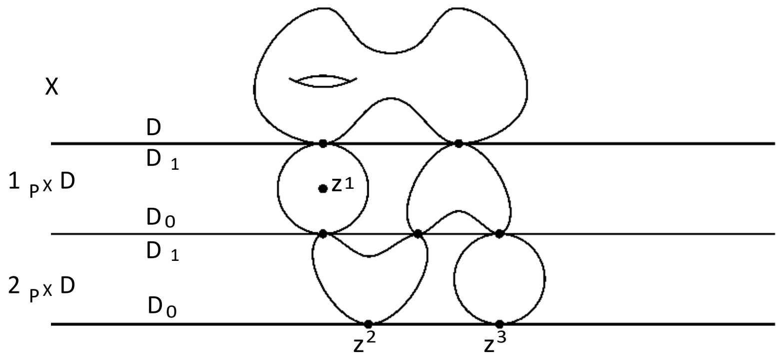

Figure 6: A relative map with $k = 3$ and $s = (0; 2; 2)$ into the expanded degeneration $X[2]_\infty$.

There exists a continuous projection map $\pi_m\colon X[m] \to X$ which is the identity on X and on each $P_X D$. We denote by J_m the almost complex structure on $X[m]$ such that

$$J_m|_X = J_X \quad \text{and} \quad J_m|_{\{r\}\times P_X D} = J_{X,D} \quad \text{for all } r = 1, \dots, m.$$

For each $(c_1, \dots, c_m) \in (\mathbb{C}^*)^m$, define $\Phi_{c_1,\dots,c_m}\colon X[m] \to X[m]$ by

$$\Phi_{c_1,\dots,c_m}(x) = \begin{cases} (r, [c_r v, w]) & \text{if } x = (r, [v, w]) \in \{r\} \times P_X D, \\ x & \text{if } x \in X. \end{cases}$$

This diffeomorphism is biholomorphic with respect to J_m and preserves the fibers of the projection $P_X D \to D$ and the sections D_0 and D_∞.

The moduli space of relative stable curves for (X, D) in [21, Section 7] is defined in the following way. With slight modification, we follow the description in [48]. Suppose that $k \in \mathbb{N}$, $A \in H_2(X; \mathbb{Z})$ and $s = (s_1, \dots, s_k) \in \mathbb{N}^k$ is a tuple satisfying

$$\sum_{a=1}^{k} s_a = A \cdot D. \tag{4-3}$$

A level-zero genus-g k–marked degree A relative J–holomorphic map into X of contact type s with D is simply a stable J–holomorphic map in $\overline{\mathcal{M}}_{g,k}(X, A)$ such that

$$u^{-1}(D) = \{z^1, \dots, z^k\} \quad \text{and} \quad \operatorname{ord}_{z^a}(u, D) = s_a \quad \text{for all } a = 1, \dots, k. \tag{4-4}$$

For $m \in \mathbb{Z}_+$, a level m k–marked relative J–holomorphic map of contact type s is a continuous map $u\colon \Sigma \to X[m]$ from a marked connected nodal curve $(\Sigma, j, \vec{z} = (z^1, \dots, z^k))$ such that

$u^{-1}(\{m\} \times D_0) = \{z^1, \dots, z^k\}$,

$$\mathrm{ord}_{z^a}(u;\{m\}\times D_0)=s_a \text{ for all } z^a\in u^{-1}(\{m\}\times D_0),\quad s_a=0 \text{ if and only if } z^a\notin u^{-1}(\{m\}\times D_0),$$

and the restriction of u to each irreducible component Σ_j of Σ is either

(1) a J–holomorphic map to X such that the set $u|_{\Sigma_j}^{-1}(D)$ consists of the nodes joining Σ_j to irreducible components of Σ mapped to $\{1\}\times\mathbb{P}_XD$, or

(2) a $J_{X,D}$–holomorphic map to $\{r\}\times\mathbb{P}_XD$ for some $r=1,\dots,m$ such that

  (a) the set $u|_{\Sigma_j}^{-1}(\{r\}\times D_1)$ consists of the nodes $q_{j,i}$ joining Σ_j to irreducible components of Σ mapped to $\{r-1\}\times\mathbb{P}_XD$ if $r>1$ and to X if $r=1$ and
$$\mathrm{ord}_{q_{j,i}}(u;D_1)=\begin{cases}\mathrm{ord}_{q_{i,j}}(u;D_0) & \text{if } r>1;\\ \mathrm{ord}_{q_{i,j}}(u;D) & \text{if } r=1;\end{cases}$$
  where $q_{i,j}\in\Sigma_{i,j}$ is the point identified with $q_{j,i}$,

  (b) if $r<m$, the set $u|_{\Sigma_j}^{-1}(\{r\}\times D_0)$ consists of the nodes joining Σ_j to irreducible components of Σ mapped to $\{r+1\}\times\mathbb{P}_XD$.

See Figure 6. The genus and the degree of such a map $u:\Sigma\to X[m]$ are the arithmetic genus of Σ and the homology class

$$A=[\pi_m\circ u]\in H_2(X;\mathbb{Z}).$$

Two tuples $(u,\Sigma,\mathfrak{j},\vec z)$ and $(u',\Sigma',\mathfrak{j}',\vec z\,')$ as above are equivalent if there exist a biholomorphic map $\varphi:(\Sigma,\mathfrak{j})\to(\Sigma',\mathfrak{j}')$ and $c_1,\dots,c_m\in\mathbb{C}^*$ such that

$$\varphi(z^a)=z'^a \quad\text{for all } a=1,\dots,k \qquad\text{and}\qquad u'=\Phi_{c_1,\dots,c_m}\circ u\circ\varphi:$$

A tuple as above is stable if it has finitely many automorphisms (self-equivalences).

If $A\in H_2(X;\mathbb{Z})$, $g,k\in\mathbb{N}$, and $s=(s_1,\dots,s_k)\in\mathbb{N}^k$ is a tuple satisfying (4-3), then the relative moduli space

$$\overline{\mathcal{M}}^{\mathrm{rel}}_{g,s}(X,D,A) \tag{4-5}$$

is the set of equivalence classes of such connected stable k–marked genus-g degree-A J–holomorphic maps into $X[m]$ for any $m\in\mathbb{N}$. If X is compact, the latter space has a natural compact Hausdorff topology.

Remark 4.1 In (4-3), we are allowing s_a to be zero for some $a\in\{1,\dots,k\}$. A marked point z with contact order 0 has image away from D (or D_0, D_1). Therefore, such points are ordinary marked points as in the classical moduli spaces of J–holomorphic curves. In the literature, marked points are usually divided into the classical part $(z^1,\dots,z^k)$ and the relative part $(z^{k+1},\dots,z^{k+\ell})$ so that $s_a=\mathrm{ord}_{z^{k+a}}(u,D)>0$ and $\sum_{a=1}^{\ell}s_a=A\cdot D$. Then the moduli space (4-5) is denoted by $\overline{\mathcal{M}}^{\mathrm{rel}}_{g,k,s}(X,D,A)$ with $s\in(\mathbb{Z}_+)^{\ell}$. This sort of separation works fine in the relative case, because there are only two types of points: in D or away from D. In the general SNC case $D=\bigcup_{i\in[N]}D_i$, however, there are 2^N types of points and it is notationally cumbersome (and useless) to divide points into separate groups based on their type.

Remark 4.2 Let (X,ω) be a smooth symplectic manifold, $D\subset X$ be a smooth symplectic divisor, and J be an ω–tame almost complex structure on X such that $J(TD)=TD$. If $u\colon(\Sigma,\mathfrak{j})\to(X,J)$ is J–holomorphic, the linearization of the Cauchy–Riemann operator (1-1) at u is given by

$$D_u\bar\partial\colon\Gamma(\Sigma,u^*TX)\to\Gamma(\Sigma,\Omega^{0,1}_{\Sigma,\mathfrak{j}}\otimes_{\mathbb{C}}u^*TX),\qquad D_u\bar\partial(\xi)=u^*\bar\partial^{\nabla}\xi+\tfrac{1}{4}N_J(\xi,du),\tag{4-6}$$

where ∇ is the $\mathbb{C}$–linear connection in (2-2) and $\bar\partial^{\nabla}$ is the associated $\bar\partial$–operator on $\Gamma(X,TX)$ in Lemma 2.1; see [29, Chapter 3.1]. The kernel of $D_u\bar\partial$ corresponds to infinitesimal deformations of u (over the fixed domain $(\Sigma,\mathfrak{j})$) and the cokernel of that is the obstruction space for integrating infinitesimal deformations to actual deformations.

If, furthermore, $\mathrm{Im}(u)\subset D$, then the linearization map $D_u\bar\partial$, defined in (4-6), satisfies

$$D_u\bar\partial(\Gamma(\Sigma,u^*TD))\subset\Gamma(\Sigma,\Omega^{0,1}_{\Sigma,\mathfrak{j}}\otimes_{\mathbb{C}}u^*TD),$$

because the restriction of $D_u\bar\partial$ to $\Gamma(\Sigma,u^*TD)$ is the linearization[28] of the $\bar\partial$–operator at u for the space of maps into D. Thus, $D_u\bar\partial$ descends to a first-order differential operator

$$D^{N_XD}_u\bar\partial\colon\Gamma(\Sigma,u^*N_XD)\to\Gamma(\Sigma,\Omega^{0,1}_{\Sigma,\mathfrak{j}}\otimes_{\mathbb{C}}u^*N_XD).\tag{4-7}$$

If $J\in\mathcal{J}(X,D,\omega)$, ie (1-3) holds, then the normal part of $N_J(\cdot,du)$ vanishes. From (4-6) and Lemma 2.1 we conclude that

$$D^{N_XD}_u\bar\partial=u^*\bar\partial_{N_XD}$$

is a complex linear operator. From another point of view, we can use (1-3) to show that a certain sequence of almost complex structures on the normal bundle N_XD converges to $J_{X,D}$; see Lemma 3.5.

[28] The linearization of (1-1) is independent of the choice of the connection at every J–holomorphic map.

4.2 Comparison

In this section, for the case where D is smooth (ie $N=1$ in Definition 2.8), we compare $\overline{\mathcal{M}}^{\mathrm{rel}}_{g,s}(X,D,A)$ and $\overline{\mathcal{M}}^{\log}_{g,s}(X,D,A)$. Proposition 4.5 shows that the latter is smaller and there is a projection map from the relative compactification onto the log compactification. This is expected, since the notion of nodal log curve involves more $\mathbb{C}$–quotients on the set of meromorphic sections. In the algebraic case, [5, Theorem 1.1] shows that an algebraic analogue of the projection map (4-13) induces an equivalence of virtual fundamental classes. We expect the same to hold for the invariants/VFC arising from our log compactification.

First, we start with a simple lemma that highlights the relation between Definition 2.8(1) and the layer structure in the relative compactification. In the following, when D is smooth ($N=1$), for a (pre)log map with the decorated dual graph $\Gamma(V,E,L)$ we define

$$V_i=\{v\in V \mid |I_v|=i\} \quad\text{and}\quad E_i=\{e\in E \mid |I_e|=i\} \quad \text{with } i=0,1; \tag{4-8}$$
$$E_{1,0}=\{e\in E \mid |I_e|=1,\ s_{\vec{e}}=0\};\quad E_{1,*}=\{e\in E \mid |I_e|=1,\ s_{\vec{e}}\neq 0\}.$$

Lemma 4.3 Let $D\subset(X,\omega)$ be a smooth symplectic divisor, $J\in\mathcal{J}(X,D,\omega)$, and

$$[f]=[(u_v,[\zeta_v],C_v)_{v\in V_1},(u_v,C_v)_{v\in V_0}]\in\mathcal{M}^{\mathrm{plog}}_{g,s}(X,D,A)_\Gamma \tag{4-9}$$

be a prelog J–holomorphic curve with dual graph $\Gamma(V,E,L)$. Then there exists a function $s\colon V\to\mathbb{R}_{\geq 0}$ satisfying Definition 2.8(1) if and only if the relations

(a) $v_1\approx v_2$ if v_1 and v_2 are connected and $s_{\vec{e}}=0$ for any $\vec{e}\in\vec{E}_{v_1,v_2}$, and

(b) $v_1\prec v_2$ if v_1 and v_2 are connected and $s_{\vec{e}}>0$ for any $\vec{e}\in\vec{E}_{v_1,v_2}$

are independent of the choice of $\vec{e}\in\vec{E}_{v_1,v_2}$ (ie they are well-defined), and generate a partial order $\preceq$ on V.

Note that for a classical edge e connecting $v_1,v_2\in V_0$, since $I_e=\emptyset$ by (2-21), we always have

$$s_{\vec{e}}=0\in\{0\}=\mathbb{R}^{\emptyset}\subset\mathbb{R}^{N=1}=\mathbb{R}.$$

Proof If (a) and (b) define a partial order $(V,\preceq)$, we construct $s\colon V\to\mathbb{R}$ satisfying Definition 2.8(1) in the following way. For every $v\in V_0$ define $s_v=0$. Let $V^{(1)}_{\min}$ be the subset of minimal vertices in V_1. For every $v\in V^{(1)}_{\min}$ define $s_v=1$. Having constructed $V^{(1)}_{\min},\dots,V^{(\ell)}_{\min}$, let $V^{(\ell+1)}_{\min}$ be the subset of minimal vertices in

$$V_1-(V^{(1)}_{\min}\cup\dots\cup V^{(\ell)}_{\min}).$$

For every $v\in V^{(\ell+1)}_{\min}$ define $s_v=\ell+1$. This function clearly satisfies Definition 2.8(1). Conversely, given such a function $s\colon V\to\mathbb{R}$ satisfying Definition 2.8(1), define $v_1\approx v_2$ (resp. $v_1\prec v_2$) if they are connected by a path and $s_{v_1}=s_{v_2}$ (resp. $s_{v_1}<s_{v_2}$). This is a partial order whose defining conditions match with (a) and (b). □

Lemma 4.4 With notation as in Lemma 4.3, the prelog curve f satisfies the properties of Definition 2.8(2) if and only if there exists a set of representatives $\{\zeta_v\}_{v\in V_1}$ such that

$$\zeta_v(q_{\vec e})=\zeta_{v'}(q_e)\quad\text{for all } v,v'\in V_1 \text{ and } e\in E_{v,v'} \text{ such that } s_{\vec e}=0. \tag{4-10}$$

Proof The last equation is well-defined by Definition 2.4(3). Then the homomorphism (2-26) (corresponding to some fixed orientation O on E) takes the form

$$\mathbb{Z}^{E_0}\oplus\mathbb{Z}^{E_1}\oplus\mathbb{Z}^{V_1}\xrightarrow{\varrho}\mathbb{Z}^{E_1}; \tag{4-11}$$

where $\varrho|_{\mathbb{Z}^{E_0}}\equiv 0$, $\varrho(1_e)=s_{\vec e}\in\mathbb{Z}$ for all $e\in E_1$, and

$$\varrho(1_v-1_{v;1})_e=\begin{cases}1_e & \text{if } v_1(\vec e)=v;\\ -1_e & \text{if } v_2(\vec e)=v;\\ 0 & \text{if } e \text{ is a loop or otherwise.}\end{cases}$$

Therefore, tensoring (4-11) with $\mathbb{C}^*$, the cokernel $CK_{\mathbb{C}^*}$ of $\varrho_{\mathbb{C}^*}$ is equal to the cokernel of the induced map

$$(\mathbb{C}^*)^{V_1}\xrightarrow{\varrho_{\mathbb{C}^*}}(\mathbb{C}^*)^{E_{1;0}}.$$

Fix an arbitrary set of representatives

$$(\zeta_v\in\mathcal{M}_{\mathrm{mero}}(\Sigma_v;u_v^*N_XD))_{v\in V_1}. \tag{4-12}$$

By (2-34) and (2-36), for every $e\in E_{1;0}$ with $v=v_1(\vec e)$ and $v'=v_2(\vec e)$, we have

$$\xi_e=\zeta_v(q_e)/\zeta_{v'}(q_{\vec e})\in\mathbb{C}^*.$$

Therefore,

$$x\equiv(\xi_e)_{e\in E_{1;0}}\in\prod_{e\in E_{1;0}}(\mathbb{C}^*)^{E_{1;0}}$$

is equal to $(1)_{e\in E_{1;0}}$ if and only if (4-10) holds. Since the cokernel of $\tilde{\partial}_C$ coincides with the cokernel of ∂_C, the element

$$[\eta] \in (\mathbb{C}^*)^{E}/\exp(\mathrm{im}(\partial_C))$$

in (2-38) is the identity element if and only if

$$[\tilde{\eta}] \in (\mathbb{C}^*)^{E_{1;0}}/\exp(\tilde{\partial}_C(\mathbb{C}^{V_1}))$$

is the identity element. The latter holds if and only if is there exists a rescaling of the sections $(\zeta_v)_{v\in V_1}$ for which (4-10) holds. □

Proposition 4.5 Let $D \subset (X,\omega)$ be a smooth symplectic divisor, $J \in \mathcal{J}(X,D;\omega)$, and $\mathbf{s} \in \mathbb{N}^k$. Then there exists a natural surjective map

$$\text{(4-13)} \qquad \Phi\colon \overline{\mathcal{M}}^{\mathrm{rel}}_{g,\mathbf{s}}(X,D;A) \to \overline{\mathcal{M}}^{\log}_{g,\mathbf{s}}(X,D;A).$$

Proof For each relative curve f, $\Phi(f)$ is the log curve obtained by forgetting those unstable $\mathbb{P}^1$–components of the domain which are isomorphically mapped to the trivial fibers of $\mathbb{P}_X D$, and restricting the equivalence class of each section defining a map into a $\mathbb{P}_X D$ to the equivalence classes of its restrictions to each connected component. The required function $s\colon V(\Gamma) \to \mathbb{R}_{\geq 0}$ in Definition 2.8(1) can be taken to be the one given by the layer structure of the relative moduli space. Moreover, by Lemma 4.4, $\Phi(f)$ satisfies (2-40) because a set of sections representing f have equal values at the nodes q_e with $I_e = \{1\}$ and $s_{e} = 0$.

Conversely, let f be any log map with dual graph Γ. By Corollary 5.4, we can assume that the function $s\colon V(\Gamma) \to \mathbb{R}_{\geq 0}$ in Definition 2.8(1) is integral. Furthermore, we take s so that $\max(s)$ is the smallest among all such s. For each connected component Σ_v of Σ in f with $I_v = \{1\}$, choose an arbitrary section ζ_v representing the equivalence class $[\zeta_v]$ in f. By Lemma 4.4, we can choose these sections to have equal values at the nodes q_e with $I_e = \{1\}$ and $s_e = 0$. Define a relative map $\tilde{f}$ whose restriction to Σ_v is the map corresponding to ζ_v into the s_v^{th} $\mathbb{P}_X D$ and such that disconnected nodes are connected by adding extra $\mathbb{P}^1$–components to the domain and by mapping them bijectively to the $\mathbb{P}^1$–fibers of $\mathbb{P}_X D$. Since $\max(s)$ is the smallest among all such s, there is at least one nontrivial component in each $\mathbb{P}_X D$ of the expanded degeneration $X[\max(s)]$; ie $\tilde{f}$ defines a stable map into $X[\max(s)]$. It is clear from the construction that $\Phi(\tilde{f}) = f$. □

Next, we give an example where the projection map (4-13) is nontrivial and both the relative and the log moduli spaces are smooth. The relative moduli space in this example is some blowup of the log moduli space.

Example 4.6 Let $X = \mathbb{P}^1$ and let $D = D_1 = \mathrm{pt}_1 \sqcup \mathrm{pt}_2$ (so $N = 1$) be the disjoint union of two points. Let $g = 0$, $k = 4$ and $A = [1] \in H_2(\mathbb{P}^1;\mathbb{Z}) \cong \mathbb{Z}$. Therefore $s = (0;0;1;1) \in \mathbb{N}^4$ (or a permutation of this) is the only option for the contact pattern. Then the relative moduli space $\overline{M}^{\mathrm{rel}}_{0;s}(X;D;[1])$ can be identified with a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at 4 points, while $\overline{M}^{\log}_{0;s}(X;D;[1])$ can be identified with a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at 2 (of those) points. The projection map in (4-13) corresponds to the blowdown of the two extra exceptional curves.

5 Comments on deformation theory and gluing

5.1 Deformation theory and the expected dimension

In this section, we outline a Fredholm setup for studying the deformation theory of log J–holomorphic maps and draw some conclusions. This setup is discussed in detail in [11], where it is also extended to log (J,ν)–holomorphic maps.

In the case of the classical moduli space of stable J–holomorphic curves $\overline{M}_{g,k}(X,A)$, for a J–holomorphic map $u\colon (\Sigma,j) \to (X,J)$ with smooth domain, the linearization $D_u\bar\partial$ of the Cauchy–Riemann equation in (4-6) is Fredholm. Therefore, the real vector spaces

$$\mathrm{Def}(u) = \ker(D_u\bar\partial) \quad \text{and} \quad \mathrm{Obs}(u) = \operatorname{coker}(D_u\bar\partial)$$

are finite-dimensional. The first space corresponds to infinitesimal deformations of u (over the fixed domain C) and the second one is the obstruction space for integrating the elements of $\mathrm{Def}(u)$ to actual deformations. In the nodal case, the kernel $\mathrm{Def}(u)$ of the similarly defined linearization map in [44, Section 6.3] corresponds to infinitesimal deformations of u in the stratum $\overline{M}_{g,k}(X,A)_\Gamma$. Deformations into $\overline{M}_{g,k}(X,A)$ correspond to gluing the nodes of the domain with gluing parameters from $\mathbb{C}^E$ and the gluing is virtually unobstructed, ie if $\mathrm{Obs}(u) = 0$, then for every sufficiently small smoothing (Σ',j') of the nodes of the domain (Σ,j), there exists a J–holomorphic map $u'\colon(\Sigma',j') \to (X,J)$ close to u; see [44, Theorem 6.3.5] for $\mathrm{Obs}(u) \neq 0$. In other words, moduli spaces $\overline{M}_{g,k}(X,A)$ are virtually smooth (orbifolds) and the "virtual normal cone" of the stratum $\overline{M}_{g,k}(X,A)_\Gamma$ is an (orbi)bundle of rank $|E|$. For the log moduli spaces defined in this paper, as (2-40) indicates, there are new obstructions for smoothability of nodal prelog curves. The claim is that, in addition to a logarithmic version of $D_u\bar\partial$, the deformation/obstruction is encoded in the combinatorial linear map (2-26).

In the setting of Theorem 1.4, suppose $(u,\Sigma,z^1,\dots,z^k)$ is an element of

$$\mathcal{M}_{g,s}(X,D,A)\subset\mathcal{M}_{g,k}(X,A)\quad\text{for some } s=(s_a=(s_{ai})_{i\in[N]})_{a\in[k]}\in(\mathbb{N}^N)^k,$$

ie Σ is smooth, $u^{-1}(D)=\{z^1,\dots,z^k\}$ and $\mathrm{ord}_{z^a}(u,D_i)=s_{ai}$ for all $i\in[N]$ and $a\in[k]$. For $(\omega,\mathcal{R},J)\in\mathrm{AK}(X,D)$, let $TX(-\log D)$ be the log tangent bundle in [46, (8)], and if J is integrable, let $TX(-\log D)$ be the usual holomorphic logarithmic tangent bundle. There is a natural complex linear homomorphism

$$\iota\colon TX(-\log D)\to TX$$

(covering id_X) that is an isomorphism away from D. This homomorphism induces similarly denoted maps

$$\iota_1\colon\Gamma(\Sigma,u^*TX(-\log D))\to\Gamma(\Sigma,u^*TX),$$
$$\iota_2\colon\Gamma(\Sigma,\Omega^{0,1}_{\Sigma,j}\otimes_{\mathbb{C}}u^*TX(-\log D))\to\Gamma(\Sigma,\Omega^{0,1}_{\Sigma,j}\otimes_{\mathbb{C}}u^*TX).$$

The following is one of the key steps in understanding the deformation theory of J–holomorphic maps relative to an SNC divisor; see [11, Section 5.1].

Theorem 5.1 [11] With notation as above, the linearization $D_u\bar\partial$ naturally lifts to a Fredholm linear map

(5-1) $$D^{\log}_u\bar\partial\colon W^{\ell,p}(\Sigma,u^*TX(-\log D))\to W^{\ell-1,p}(\Sigma,\Omega^{0,1}_{\Sigma,j}\otimes_{\mathbb{C}}u^*TX(-\log D))$$

such that $\iota_2\circ D^{\log}_u\bar\partial=D_u\bar\partial\circ\iota_1$ over the space of smooth sections. Furthermore, if $\mathrm{coker}(D^{\log}_u\bar\partial)=0$, the set of J–holomorphic maps (the marked domain is fixed) of contact type s close to u (in a suitable Banach manifold) forms an oriented smooth manifold of real dimension

(5-2) $$2\deg(u^*TX(-\log D))+\dim_{\mathbb{C}}X(1-g).$$

Note that (5-2) follows from Riemann–Roch and (5-1). Considering the deformations of the marked domain $(\Sigma,j,z^1,\dots,z^k)$, it follows from (5-2) that the expected dimension of $\mathcal{M}_{g,s}(X,D,A)$ is equal to the naive dimension count (1-6).

Next, consider a log map $f = (u, [\xi_i]_{i\in I}, \Sigma, z^1, \dots, z^k)$ in the stratum $\mathcal{M}_{g,s}(X, D, A)_I$, ie Σ is smooth, $u(\Sigma) \subset D_I$ for a nontrivial maximal subset $I \subset [N]$, $\mathrm{ord}_{z^a}(u, D_i) = s_{ai}$ for all $i \notin I$, and $\mathrm{ord}_{z^a}(\xi_i) = s_{ai}$ for all $i \in I$. Forgetting the meromorphic sections, by Remark 2.3, we get an inclusion map

$$\mathcal{M}_{g,s}(X, D, A)_I \hookrightarrow \mathcal{M}_{g,\bar{s}}(D_I, \bar{D}, A),$$
$$(u, [\xi_i]_{i\in I}, \Sigma, z^1, \dots, z^k) \to (u, \Sigma, z^1, \dots, z^k),$$

where

$$\bar{D} = \bigcup_{i \in S - I} D_{I \cup i} \subset D_I \quad \text{and} \quad \bar{s} = \big(\bar{s}_a = (s_{ai})_{i\in[N]-I}\big)_{a=1}^{k} \in (\mathbb{N}^{[N]-I})^k.$$

With $(D_I, \bar{D})$ in place of (X, D) in (5-1), deformation theory of $\mathcal{M}_{g,s}(D_I, \bar{D}, A)$ is given by the restriction of $D_u^{\log} \oplus \bar{\partial}$ to $TD_I(-\log \bar{D})$. It is worth mentioning that restricted to D_I, there is a natural isomorphism

$$TX(-\log D)|_{D_I} \cong TD_I(-\log \bar{D}) \oplus D_I \times \mathbb{C}^I. \tag{5-3}$$

Lemma 5.2 There exists a map $P_I = (P_{I;i})_{i\in I} : \mathcal{M}_{g,\bar{s}}(D_I, \bar{D}, A) \to (\mathrm{Pic}^0(\Sigma))^I$ such that

$$\mathcal{M}_{g,s}(X, D, A)_I = P_I^{-1}(\mathcal{O}^I).$$

In particular,

$$\mathcal{M}_{0,s}(X, D, A)_I = \mathcal{M}_{0,\bar{s}}(D_I, \bar{D}, A).$$

Here $\mathrm{Pic}^0(\Sigma)$ is the group of degree-0 holomorphic lines bundles on (Σ, j) and $\mathcal{O} \in \mathrm{Pic}^0(\Sigma)$ is the trivial holomorphic line bundle.

Proof For each $i \in I$, define

$$P_{I;i}([u, \Sigma, z^1, \dots, z^k]) = u^* \mathcal{N}_X D_i \otimes \mathcal{O}\Big(-\sum_{a=1}^{k} s_{ai} z^a\Big) \in \mathrm{Pic}^0(\Sigma),$$

where $\mathcal{O}\big(\sum_{a=1}^k s_{ai} z^a\big)$ is the line bundle corresponding to the divisor $\sum_{a=1}^k s_{ai} z^a$. Therefore,

$$P_{I;i}([u, \Sigma, z^1, \dots, z^k]) = \mathcal{O}$$

if and only if there exists a meromorphic section $\zeta_{I;i}$ of $u^* \mathcal{N}_X D_i$ with zeros/poles of order s_a and z^a (and nowhere else). □

We conclude that the deformation/obstruction theory of the stratum $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)_I$ is given by $D^{\log}_u\bar\partial$ on $\mathcal{M}_{g,\bar{\mathfrak{s}}}(D_I,\bar{D},A)$ and the linearization of $\mathcal{P}_I$. By (1-6) and Lemma 5.2, the expected real dimension of $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)_I$ is

$$2\big(c_1^{TX(-\log D)}(A)+(\dim_{\mathbb{C}}X-3)(1-g)+k-|I|\big). \tag{5-4}$$

Via the identification (5-3), the maps $D^{\log}_u\bar\partial$ on $\mathcal{M}_{g,\bar{\mathfrak{s}}}(D_I,\bar{D},A)$ and $\mathcal{P}_I$ can be combined into a single Fredholm operator as in (5-1); see [11, Section 5.2].

Moving to the nodal case, with notation as in (2-31), let

$$\Lambda(\Gamma)=K_{\mathbb{R}}\cap\Big(\mathbb{R}^{E}_{\geq 0}\oplus\bigoplus_{v\in V}\mathbb{R}^{I_v}_{\geq 0}\Big)\subset K_{\mathbb{R}} \tag{5-5}$$

be the cone of nonnegative elements in the kernel of $\varrho_{\mathbb{R}}\colon D_{\mathbb{R}}\to T_{\mathbb{R}}$. This cone is independent of the choice of the orientation $\mathcal{O}$ used to define (2-26); in fact, by (2-30),

$$\Lambda=\Big\{((\lambda_e)_{e\in E},(s_v)_{v\in V})\in\mathbb{R}^{E}_{\geq 0}\oplus\bigoplus_{v\in V}\mathbb{R}^{I_v}_{\geq 0}: s_v-s_{v'}=\lambda_e s_e \text{ for all } v,v'\in V \text{ and } e\in E_{v,v'}\Big\}. \tag{5-6}$$

The integral lattice underlying Λ coincides with the monoid $Q^{\vee}$ in [4, Section 2.3.9].

Lemma 5.3 For every $\Gamma\in\mathrm{DG}(g,\mathfrak{s},A)$, $\Lambda(\Gamma)$ is a top-dimensional strictly convex rational polyhedral cone in $K_{\mathbb{R}}(\Gamma)$.

Proof The functions s and λ in Definition 2.8(1) define an element m_+ of

$$K_{\mathbb{R}}\cap\Big(\mathbb{R}^{E}_{+}\oplus\bigoplus_{v\in V}\mathbb{R}^{I_v}_{+}\Big). \tag{5-7}$$

Since all of the coefficients in m_+ are positive, for any arbitrary $m\in K_{\mathbb{R}}$ there exists a sufficiently large $r>0$ such that $m+r\,m_+\in\Lambda$. We conclude that Λ is top-dimensional.

Since $\mathbb{R}^{E}_{\geq 0}\oplus\bigoplus_{v\in V}\mathbb{R}^{I_v}_{\geq 0}$ is a strictly convex rational polyhedral cone and $K_{\mathbb{R}}$ is an integrally defined subvector space, the intersection (5-7) is a strictly convex rational polyhedral cone. □

Corollary 5.4 By Lemma 5.3, the functions s and λ in Definition 2.8(1) can be chosen to be integral-valued.

In conclusion, with a setup similar to [44, Section 6.3], the deformation/obstruction theory of any stratum

$$\mathcal{M}_{g,\mathfrak{s}}(X,D,A)_{\Gamma}$$

around $f\equiv(u,[\Sigma,\mathfrak{j},z^1,\ldots,z^k])$ is given by (1) $D^{\log}_u\bar\partial$ and $\mathcal{P}_I$ for each smooth component Σ_v of Σ, and (2) the obstruction map (2-32).

Lemma 5.5 For any decorated dual graph $\Gamma \in \mathrm{DG}(g;s;A)$, the expected complex dimension of $\mathcal{M}_{g;s}(X;D;A)_\Gamma$ is

$$c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k - \dim_{\mathbb{R}} K_{\mathbb{R}}(\Gamma). \tag{5-8}$$

The only stratum with $\dim K_{\mathbb{R}}(\Gamma) = 0$ is $\mathcal{M}_{g;s}(X;D;A)$.

Proof The expected complex dimension of each component $\mathcal{M}_{g_v;s_v}(X;D;A_v)_{I_v}$ is, by (5-4), equal to

$$c_1^{TX(-\log D)}(A_v) + (n-3)(1-g_v) + k_v + \ell_v - |I_v|;$$

where $k_v = |\mathfrak{z}_v|$, $\ell_v = |q_v|$ and s_v is the set of contact order vectors at $\mathfrak{z}_v \cup q_v$. The prelog space $\mathcal{M}^{\mathrm{plog}}_{g;s}(X;D;A)_\Gamma$ in (2-32) is the fiber product of $\{\mathcal{M}_{g_v;s_v}(X;D;A_v)_{I_v}\}_{v\in V}$ over the evaluation maps at the nodal points,

$$\prod_{v\in V} \mathcal{M}_{g_v;s_v}(X;D;A_v)_{I_v} \to \prod_{e\in E} (D_{I_e} \times D_{I_e}).$$

Therefore, using (2-11), the expected complex dimension of $\mathcal{M}^{\mathrm{plog}}_{g;s}(X;D;A)_\Gamma$ is

$$\begin{aligned}\sum_{v\in V} \big(c_1^{TX(-\log D)}(A_v) + (n-3)(1-g_v) + k_v + \ell_v - |I_v|\big) - \sum_{e\in E} (n - |I_e|) \\ = c_1^{TX(-\log D)}(A) + (n-3)(1-g) + k - |E| - \sum_{v\in V} |I_v| + \sum_{e\in E} |I_e|.\end{aligned} \tag{5-9}$$

By (2-26),

$$\dim_{\mathbb{R}} K_{\mathbb{R}}(\Gamma) - \dim_{\mathbb{C}}(\mathcal{G}) = |E| + \sum_{v\in V} |I_v| - \sum_{e\in E} |I_e|.$$

By (2-32), the stratum $\mathcal{M}_{g;s}(X;D;A)_\Gamma$ is the preimage of the identity element under the map

$$\mathrm{ob}_\Gamma \colon \mathcal{M}^{\mathrm{plog}}_{g;s}(X;D;A)_\Gamma \to \mathcal{G}.$$

Therefore, the expected complex dimension of $\mathcal{M}_{g;s}(X;D;A)_\Gamma$ is equal to the difference of (5-9) and

$$\dim_{\mathbb{C}}(\mathcal{G}) = \dim_{\mathbb{R}} K_{\mathbb{R}}(\Gamma) - \Big(|E| + \sum_{v\in V} |I_v| - \sum_{e\in E} |I_e|\Big);$$

which is equal to (5-8).

By Definition 2.8(1) and (2-30), a function $(s;\ )$ as in Definition 2.8(1) gives us an element of $K_{\mathbb{R}}(\Gamma)$. This element is trivial only if $\Gamma = \{v\}$ is a one-vertex graph with no edge and $I_v = \emptyset$. This establishes the last claim. □

5.2 Gluing parameters

The last step in describing the deformation theory and establishing (?) is to prove a gluing theorem for smoothing the nodes (ie deformations normal to each stratum). In this section, we describe the space of gluing parameters for each $\Gamma \in \mathrm{DG}(g,s,A)$ and show that it is essentially an affine toric variety. We sketch our idea for the construction of gluing map and defer to a future work [12] for the details.

For a classical nodal J–holomorphic map with $|E|$ nodes, the space of gluing parameters is a neighborhood of the zero in $\mathbb{C}^E$. For a log map f as in (2-19), the gluing procedure involves a simultaneous smoothing of the nodes, together with pushing u_v out in the direction of ${}_{v,i}$ for some $v \in V$ and $i \in I_v$. Thus, a priori, the space of gluing parameters could be quite complicated and the log moduli spaces (2-41) are not always virtually smooth. For example, the log moduli space of Example 5.6 below has an A_1–singularity along some stratum. For the log moduli spaces, the space of gluing parameters along $\mathcal{M}_{g,s}(X,D,A)_\Gamma$ belongs to (a neighborhood of the origin in finitely many copies of) the affine toric variety $Y_{(\Gamma)}$ constructed from the toric fan $(\Gamma) \subset K_{\mathbb{R}}$. In other words, the kernel of (2-26) gives the gluing deformation and, by (2-40), its cokernel gives the obstruction space for smoothability of such prelog maps.

In the following example, we describe a tuple (X,D,g,s,A,Γ) where $\mathcal{M}_{g,s}(X,D,A)_\Gamma$ is a point and Y has an A_1–singularity at its center. In this example, the relative moduli space $\mathcal{M}_{g,s}(\overline{X},\overline{D},A)$ replaces the A_1–singularity with a small resolution of it.

Example 5.6 Suppose $X = \mathbb{P}^3$, $D \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth degree-2 hypersurface, and let

$$g = 1; \quad A = 2[H] \in H_2(\mathbb{P}^3;\mathbb{Z}) \cong \mathbb{Z}; \quad s = (0,0,4).$$

By [21, Lemma 4.2] and (5-8), both $\overline{\mathcal{M}}^{\mathrm{rel}}_{g,s}(X,D,A)$ and $\overline{\mathcal{M}}^{\log}_{g,s}(X,D,A)$ are of the expected complex dimension 7. Let $\mathcal{M}^{\mathrm{rel}}_{g,s}(X,D,A)_\Gamma$ be the stratum of maps in the expanded degeneration $X[2]$ with connected components:

- a degree-1 map $u_0\colon \mathbb{P}^1 \to X$ (a line) that intersects D at two distinct points (with multiplicity 1),
- a map $u_3\colon \mathbb{P}^1 \to \mathbb{P}_X D$ in the second layer $\{2\} \times \mathbb{P}_X D$ of $X[2]$ which is made of a degree-1 map $\underline{u}_3\colon \mathbb{P}^1 \to D$ and a meromorphic section of $\underline{u}_3^* \mathcal{N}_X D \cong \mathcal{O}_{\mathbb{P}^1}(2)$ with a zero of order 4 and 2 poles of order one, and

Figure 7: Left: a nodal 2–marked $g = 1$ relative map in $X[2]$. Right: the decorated dual graph of the image log map.

two maps $u_1, u_2 : \mathbb{P}^1 \to \mathbb{P}_X D$ in the first layer $\{1\} \times \mathbb{P}_X D$ of $X[2]$ carrying the first and the second marked point, respectively, which are degree-1 covers of fibers of $\mathbb{P}_X D$ connecting u_0 and u_4.

See the left-hand side of Figure 7. While the stratum $\mathcal{M}^{\mathrm{rel}}_{g,s}(X, D, A)_\Gamma$ is of virtual $\mathbb{C}$–codimension 2, by (5-8), its image

$$\mathcal{M}_{g,s}(X, D, A)_\Gamma = \pi(\mathcal{M}^{\mathrm{rel}}_{g,s}(X, D, A)_\Gamma)$$

in the log moduli space, given by the projection map π of Proposition 4.5 below, is of virtual $\mathbb{C}$–codimension 3.

In fact, with the labeling and the choice of orientation on the edges of the associated decorated dual graph Γ in Figure 7, right, we have

$$\varphi : \mathbb{Z}^E \oplus \bigoplus_{i \in [3]} \mathbb{Z}^{\mathrm{l}_{v_i}} \cong \mathbb{Z}^{\{e_1,e_2,e_3,e_4\}} \oplus \mathbb{Z}^{\{v_1,v_2,v_3\}} \to \bigoplus_{i \in [4]} \mathbb{Z}^{\mathrm{l}_{e_i}} \cong \mathbb{Z}^{\{e_1,e_2,e_3,e_4\}}$$

$$\varphi(1^{e_i}) = 1_{e_i} \quad \text{for all } i \in [4];$$

$$\varphi(1_{v_1}) = -1_{e_1} + 1_{e_3}; \quad \varphi(1_{v_2}) = -1_{e_2} + 1_{e_4}; \quad \varphi(1_{v_3}) = -1_{e_3} - 1_{e_4}.$$

Therefore,

$$\sigma = \ker(\varphi_{\mathbb{R}}) \cap (\mathbb{R}_{\geq 0}^{\{e_1,e_2,e_3,e_4\}} \oplus \mathbb{R}_{\geq 0}^{\{v_1,v_2,v_3\}})$$

is the cone generated by the set of 4 vectors

$$\xi_1 = 1_{v_3} + 1_{e_3} + 1_{e_4}; \qquad \xi_2 = 1_{v_1} + 1_{v_3} + 1_{e_1} + 1_{e_4};$$

$$\xi_3 = 1_{v_2} + 1_{v_3} + 1_{e_2} + 1_{e_3}; \quad \xi_4 = 1_{v_1} + 1_{v_2} + 1_{v_3} + 1_{e_1} + 1_{e_2}.$$

Since the only relation among ξ_i is $\xi_1 + \xi_4 = \xi_2 + \xi_3$, the associated toric variety Y is isomorphic to the 3–dimensional affine subvariety

$$(x_1 x_4 - x_2 x_3 = 0) \subset \mathbb{C}^4. \qquad \square$$

For every log curve $f\in\mathcal{M}_{g,s}(X,D,A)_\Gamma$ choose a representative

$$\big(u_v,\{f_{v,i}\}_{i\in I_v},C_v=(\Sigma_v,j_v,\vec{z}_v)\big)_{v\in V} \tag{5-10}$$

and a set of local coordinates $\{z_{\vec e}\}_{\vec e\in\vec E}$ around the nodes. Since f is G–unobstructed, by the definition of ζ in (2-37), we can choose $\zeta_{v,i}$ and $z_{\vec e}$ such that the leading coefficient vectors $\eta_{\vec e}$ in (2-36) satisfy

$$\eta_{\vec e}=\eta_{e}\quad\text{for all } e\in E. \tag{5-11}$$

For every $v\in V$ and $i\in[N]-I_v$, let $t_{v,i}=1$ in (5-12). Then the space of gluing parameters for f is a sufficiently small neighborhood of the origin in the complex subvariety

$$N_\Gamma=\Big\{\big((\varepsilon_e)_{e\in E},(t_{v,i})_{v\in V,i\in I_v}\big)\in\mathbb{C}^E\times\prod_{v\in V}\mathbb{C}^{I_v}:\ \varepsilon_{\vec e}^{s_{\vec e,i}}t_{v,i}=t_{v',i}\ \text{for all } v,v'\in V,\ e\in E_{v,v'},\ i\in I_e \text{ and } \vec e \text{ such that } s_{\vec e,i}\geq 0\Big\}\subset\mathbb{C}^E\times\prod_{v\in V}\mathbb{C}^{I_v}. \tag{5-12}$$

The complex numbers ε_e are the gluing parameters for the nodes of Σ and $t_{v,i}$ are the parameters for pushing u_v out in the direction of $\zeta_{v,i}$. In the gluing construction outlined below, given a set of representatives $(\{z_{\vec e}\}_{\vec e\in\vec E},\{f_{v,i}\}_{v\in V,i\in I_v})$ satisfying (5-11) and a sufficiently small

$$(\varepsilon,t)\equiv\big((\varepsilon_e)_{e\in E},(t_{v,i})_{v\in V,i\in I_v}\big)\in N_\Gamma,$$

we will construct a pregluing log map $f_{\varepsilon,t}$. Then we must show that there is an actual log J–holomorphic map "close" to it.

Let

$$\mathfrak{T}^\vee\equiv\bigoplus_{e\in E}\mathbb{Z}^{I_e}\xrightarrow{\ \varrho^\vee\ }\mathfrak{D}^\vee\equiv\mathbb{Z}^E\oplus\bigoplus_{v\in V}\mathbb{Z}^{I_v}$$

be the dual of the $\mathbb{Z}$–linear map ϱ associated to Γ in (2-26) (for a fixed choice of orientation O on E). With the kernel subspace $K=\ker(\varrho)\subset\mathfrak{D}$ as in (2-29), let

$$K^\perp=\{m\in\mathfrak{D}^\vee \mid \langle m,\kappa\rangle=0 \text{ for all } \kappa\in K\}\subset\mathfrak{D}^\vee.$$

Then $\mathrm{Im}(\varrho^\vee)\subset K^\perp$, with the finite quotient

$$K^\perp/\mathrm{image}(\varrho^\vee).$$

Proposition 5.7 The space of gluing parameters N_Γ in (5-12) is a possibly reducible and nonreduced affine toric subvariety of $\mathbb{C}^E \times \prod_{v\in V}\mathbb{C}^{I_v}$ that is isomorphic to $|K^\perp/\mathrm{Im}(\varrho_\Gamma)|$ copies of the irreducible reduced affine toric variety $Y_{\Sigma(\Gamma)}$ (with toric fan $\Sigma(\Gamma)$), counting with multiplicities.[29] Replacing $\{z_e\}_{e\in E}$ and $\{\mathfrak{f}_{v,i}\}_{v\in V,\, i\in I_v}$ with another choice satisfying (5-11) corresponds to a torus action on N_Γ.

Proof Let us start with some general facts about toric varieties. For $n\in\mathbb{Z}_+$, every vector $m\in\mathbb{Z}^n$ has a unique presentation $m = m_+ - m_-$ such that $m_+, m_- \in (\mathbb{Z}_{\geq 0})^n$. Every $m = (a_1,\dots,a_n)\in(\mathbb{Z}_{\geq 0})^n$ corresponds to the monomial

$$x^m \equiv x_1^{a_1}\cdots x_n^{a_n}\in\mathbb{C}[x_1,\dots,x_n].$$

For every arbitrary $m\in\mathbb{Z}^n$, the binomial corresponding to m is the expression

$$x^{m;\pm} \equiv x^{m_+} - x^{m_-} \in \mathbb{C}[x_1,\dots,x_n].$$

For example, if $m=0$, then $x^{m;\pm} = 1-1 = 0$. A binomial ideal[30] I in $\mathbb{C}[x_1,\dots,x_n]$ is an ideal generated by a finite set of binomials $x^{m_1;\pm},\dots,x^{m_\ell;\pm}$.

Suppose $K_\Sigma \cong \mathbb{Z}^\ell$ is a lattice and $\mathbb{Z}^n \to K_\Sigma$ is a surjective $\mathbb{Z}$–linear map. Let $\mathbb{R}^n \to K_{\Sigma,\mathbb{R}}$ be the corresponding $\mathbb{R}$–linear projection map and let σ_Σ be the image of the cone $\mathbb{R}^n_{\geq 0}$ in $K_{\Sigma,\mathbb{R}}$. Then the dual map $\iota\colon K^*_\Sigma \hookrightarrow \mathbb{Z}^n$ is an embedding and the dual of σ_Σ is the toric fan

$$\Sigma = K^*_{\mathbb{R}} \cap \iota^{-1}(\mathbb{R}_{\geq 0})^n.$$

In this situation, by [7, Proposition 1.1.9], the toric variety Y_Σ associated to the toric fan Σ is the zero set of the binomial ideal

$$I_\Sigma = \{x^{m;\pm} \mid m\in K^\perp \subset \mathbb{Z}^n\}. \tag{5-13}$$

With $\mathbb{Z}^n = \mathbb{Z}^E \oplus \bigoplus_{v\in V}\mathbb{Z}^{I_v}$, K as in (2-30) and $\Sigma = \Sigma(\Gamma)$ as in (5-5), the previous argument implies that $Y_{\Sigma(\Gamma)}$ is the zero set of the binomial ideal (5-13).

Let $I' \subset I$ be the binomial subideal generated by the elements of $\mathrm{Im}(\varrho_\Gamma) \subset K^\perp$. By definition of ϱ and (5-12), the space of gluing parameters N_Γ is the zero set (scheme) of I'. Therefore $Y_{\Sigma(\Gamma)} \subset N_\Gamma$. Note that $Y_{\Sigma(\Gamma)}$ is the Zariski closure of the irreducible subgroup

$$\{t\in(\mathbb{C}^*)^n \mid t^m = 1 \text{ for all } m\in K^\perp\} \subset (\mathbb{C}^*)^n$$

[29] We do not know of any example, arising from such dual graphs, for which the multiplicities are bigger than 1.

[30] For more general binomial ideals, see [9].

and $\mathcal{N}_\Gamma$ is the Zariski closure of possibly nonirreducible subgroup

(5-14) $$\{t\in(\mathbb{C}^*)^N \mid t^m=1 \text{ for all } m\in\operatorname{Im}(\varrho^\vee)\}\subset(\mathbb{C}^*)^N.$$

See [7, Definition 1.1.7]. Therefore, all the irreducible components of $\mathcal{N}_\Gamma$ are isomorphic to $Y(\Gamma)$. Since

(5-15) $$|I/I'|=m_\Gamma:=|K^\perp/\operatorname{Im}(\varrho^\vee)|;$$

$\mathcal{N}_\Gamma$ is isomorphic to m_Γ copies of $Y(\Gamma)$, counting with multiplicities. The last statement in Proposition 5.7 follows from the way subgroup (5-14) acts on (5-12). □

Example 5.8 Suppose $N=2$ and Γ is the decorated dual graph with two vertices $V=\{v_1,v_2\}$ and two edges e_1 and e_2 connecting them. Choose $\vec{e}_1$ and $\vec{e}_2$ to be the orientations starting at v_1. Suppose

$$I_{v_1}=\{1\},\quad I_{v_2}=\{2\},\quad s_{\vec{e}_1}=s_{\vec{e}_2}=(-2,2).$$

Then the linear map

$$\varrho\colon\mathbb{Z}^E\oplus\mathbb{Z}^{I_{v_1}}\oplus\mathbb{Z}^{I_{v_2}}\approx\mathbb{Z}_{e_1}\oplus\mathbb{Z}_{e_2}\oplus\mathbb{Z}_{v_1}\oplus\mathbb{Z}_{v_2}\to\mathbb{Z}^{\{1,2\}}_{e_1}\oplus\mathbb{Z}^{\{1,2\}}_{e_2}$$

is given by

$$\varrho(1_{e_1})=((-2,2)_{e_1},(0,0)_{e_2}),\qquad \varrho(1_{e_2})=((0,0)_{e_1},(-2,2)_{e_2}),$$
$$\varrho(1_{v_1})=((1,0)_{e_1},(1,0)_{e_2}),\qquad \varrho(1_{v_2})=((0,-1)_{e_1},(0,-1)_{e_2}).$$

It is straightforward to check that $\operatorname{Ker}(\varrho)$ is one-dimensional and is generated by

$$1_{e_1}+1_{e_2}+2\,1_{v_1}+2\,1_{v_2};$$

ie $Y(\Gamma)\cong\mathbb{C}^*$. On the other hand, $\mathcal{N}_\Gamma$ is the subvariety cut out by

$$\varepsilon_1^2=t_{v_2},\quad \varepsilon_2^2=t_{v_2},\quad \varepsilon_1^2=t_{v_1},\quad \varepsilon_2^2=t_{v_1}.$$

This is isomorphic to 2 copies of $\mathbb{C}^*$, the component $Y(\Gamma)$ is the image of $t\to(t,t,t^2,t^2)$ and the other one is the image of $t\to(t,-t,t^2,t^2)$. It is straightforward to see that

$$\operatorname{Ker}(\varrho)^\perp/\operatorname{Im}(\varrho^\vee)$$

is isomorphic to $\mathbb{Z}_2$ and is generated by the class of $[1_{e_1}-1_{\bar{e}_2}]$. □

Given a log J–holomorphic map $f\equiv(u,[\mathfrak{f}],\mathfrak{C},z^1,\ldots,z^k)$ in $\overline{\mathcal{M}}^{\log}_{g,\mathfrak{s}}(X,D,A)$ with nodal domain (2-12), a set of local coordinates $\{z_{\vec{e}}\}_{\vec{e}\in\vec{E}}$ around the nodes such that (5-11) holds, and a gluing parameter $(\varepsilon,t)\equiv((\varepsilon_e)_{e\in E},(t_{v,i})_{v\in V,i\in I_v})$ in (5-12), the gluing construction can/will be done in the following way.

Consider for example a node q_e connecting Σ_v and $\Sigma_{v'}$ with $\mathrm{ord}_{q_e}(u; D_i) = s_{e,i} > 0$. Then the log tuple on $\Sigma_{v'}$ includes a section $\zeta_{v',i}$ of $u_{v'}^* N_X D_i$ with a pole of order $s_{e,i}$ at the nodal point $q_e \in \Sigma_{v'}$. Near q_e, the map u_v has the product form

$$u_v(z_e) = (\eta_{e,i} z_e^{s_{e,i}}; \tilde{u}_v) \in \mathbb{C} \times D_i.$$

On the other hand, $\zeta_{v',i}$ has a local expansion $\zeta_{v',i}(z_e') = \eta'_{e,i} z_e'^{-s_{e,i}} + \cdots$. By (5-11) and (5-12), we have

$$\epsilon_e^{s_{e,i}} t_{v,i}\, \eta_{e,i} = t_{v',i}\, \eta'_{e,i} \tag{5-16}$$

at all the nodes, simultaneously. The smoothing of Σ is given by smoothing the nodes q_e via the equation $z_e z_e' = \epsilon_e$. The identity (5-16) means that the expression

$$\eta_{e,i}\, t_{v,i}\, z_e^{s_{e,i}} = \eta'_{e,i}\, t_{v',i}\, z_e'^{-s_{e,i}} \tag{5-17}$$

defines a function from the neck region into $N_X D_i$. We then construct the approximate-gluing log map $f^{\approx}_{\epsilon,t}$ in the following way. On each neck region — unlike in the classical gluing construction where the approximate-gluing map is defined to be constant — we define the approximate-gluing map to be (5-17) in the i^{th} direction. Away from the nodes, $f^{\approx}_{\epsilon,t}$ is defined to be the pushout[31] of u_v via the section $\sum_{i\in J} t_{v,i}\zeta_{v,i}$ on the v^{th} component. The latter is J–holomorphic due to some properties of $AK(X, D)$. In between the two regions, $f^{\approx}_{\epsilon,t}$ interpolates between the two maps. Then, with $D^{\log}_{\bar\partial}$ in place of $D_{\bar\partial}$ in [29, Chapter 10], an argument similar to the classical argument allows us to find a log J–holomorphic map close to $f^{\approx}$.

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