JOINT GEOMETRY/FREQUENCY ANALYTICITY OF FIELDS SCATTERED BY PERIODIC LAYERED MEDIA*

MATTHEW KEHOE AND DAVID P. NICHOLLS †

Abstract. The scattering of linear waves by periodic structures is a crucial phenomena in many branches of applied physics and engineering. In this paper we establish rigorous analytic results necessary for the proper numerical analysis of a class of High–Order Perturbation of Surfaces/Asymptotic Waveform Evaluation (HOPS/AWE) methods for numerically simulating scattering returns from periodic diffraction gratings. More specifically, we prove a theorem on existence and uniqueness of solutions to a system of partial differential equations which model the interaction of linear waves with a periodic two–layer structure. Furthermore, we establish joint analyticity of these solutions with respect to both geometry and frequency perturbations. This result provides hypotheses under which a rigorous numerical analysis could be conducted on our recently developed HOPS/AWE algorithm.

Key words. High–Order Perturbation of Surfaces Methods; Layered media; Linear wave scattering; Helmholtz equation; Diffraction gratings.

AMS subject classifications. 65N35, 78A45, 78B22

1. Introduction.

The scattering of linear waves by periodic structures is a central model in many problems of scientific and engineering interest. Examples arise in areas such as geophysics [67, 8], imaging [51], materials science [28], nanoplasmonics [64, 47, 24], and oceanography [10]. In the case of nanoplasmonics there are many such topics, for instance, extraordinary optical transmission [23], surface enhanced spectroscopy [50], and surface plasmon resonance (SPR) biosensing [31, 33, 45, 35]. In all of these physical problems it is necessary to approximate scattering returns in a fast, robust, and highly accurate fashion.

The most popular approaches to solving these problems numerically in the engineering literature are *volumetric* methods. These include formulations based on the Finite Difference [43], Finite Element [34], Discontinuous Galerkin [30], Spectral Element [20], and Spectral Methods [29, 9, 66]. However, these methods suffer from the requirement that they discretize the full volume of the problem domain which results in an unnecessarily large number of degrees of freedom for a periodic *layered* structure. There is also the additional difficulty of approximating far–field boundary conditions explicitly [7].

For these reasons, *surface* methods are an appealing alternative, and we advocate the use of Boundary Integral Methods (BIM) [17, 40, 65] or High–Order Perturbation of Surfaces (HOPS) Methods [48, 49, 11, 12, 13, 57, 59]. Regarding the latter, we mention the classical Methods of Operator Expansions [48, 49] and Field Expansions [11, 12, 13], as well as the stabilized Method of Transformed Field Expansions [57, 59]. All of these surface methods are greatly advantaged over the volumetric algorithms discussed above primarily due to the greatly reduced number of degrees of freedom that they require. Additionally the *exact* enforcement of the far–field boundary conditions is assured for both BIM and HOPS approaches. Consequently, these approaches are a favorable alternative and are becoming more widely used by practitioners.

There has been a large amount of not only rigorous analysis of systems of partial differential equations which model these scattering phenomena, but also careful design

*D.P.N. gratefully acknowledges support from the National Science Foundation through Grants No. DMS–1813033 and DMS–2111283.

†Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607 (mkehoe5@uic.edu, davidn@uic.edu)

of numerical schemes to simulate solutions of these. Most of these results utilize either Integral Equation techniques or weak formulations of the volumetric problem, each of which lead to a variety of natural numerical implementations. We recommend the Habilitationsschrift of T. Arens [3] as a definitive reference for periodic layered media problems in two and three dimensions. In particular, we refer the interested reader to Chapter 1 which discusses in great detail the state-of-the-art in uniqueness and existence results for scattering problems on biperiodic structures. For the two dimensional problem we further refer the reader to the work of Petit [62]; Bao, Cowsar, and Masters [5]; and Wilcox [68]. In three dimensions, results on the Helmholtz equation can be found in Abboud and Nedelec [1]; Bao [4]; Bao, Dobson, and Cox [6]; and Dobson [22]. In the context of Maxwell's equations, we point out the work of Chen and Friedman [16], and Dobson and Friedman [21]. Of course the field has progressed from these classical contributions in a number of directions, and survey volumes like [5] give further details.

The previous work most closely related to the current contribution is that of Kirsch [38] on smoothness properties of the pressure field scattered by an acoustically soft two–dimensional periodic surface. More specifically, it was demonstrated that not only is this field continuous and differentiable with respect to a sufficiently small boundary deformation, but it is also *analytic* with respect to illumination frequency and angle of incidence, up to poles induced by the Rayleigh singularities (Wood Anomalies) which does not violate our theory. We generalize these results in a number of important ways. In addition, in contrast to their rather theoretical operator–theoretic approach using results from Kato's classical work [36], our method of proof is quite explicit and results in a stable and highly accurate numerical scheme which we discuss in [37].

Oftentimes in applications it is important to consider families of gratings interrogated over a range of illumination frequencies. An example of this is the computation of the Reflectivity Map, R, which records the energy scattered by a layered structure with interface shaped by $z = g(x)$ and illuminated by radiation of frequency ω (see, e.g., [42]). Taking the point of view that this configuration is simply one in a family with interface

$$z = \varepsilon f(x), \quad \varepsilon \in \mathbf{R},$$

illuminated by radiation of frequency

$$\omega = \underline{\omega} + \delta\underline{\omega}, \quad \delta \in \mathbf{R},$$

where $\underline{\omega}$ is a distinguished frequency of interest, our novel High–Order Perturbation of Surfaces/Asymptotic Waveform Evaluation (HOPS/AWE) method [53, 37] is a compelling numerical algorithm. In short, this scheme studies a *joint* Taylor expansion of the solutions of the scattering problem in both ε and δ. Upon insertion of this expansion into relevant governing equations, the resulting recursions can be solved up to a prescribed number of Taylor orders *once* and then simply summed for (ε, δ) many times. Clearly, this is a most efficient and accurate method for approximating $R = R(\varepsilon, \delta)$, as we have demonstrated in our previous work [53, 37], provided that this joint expansion can be justified. The point of the current contribution is to provide this justification in the language of rigorous analysis (see Theorem 4.7). Not only is this of intrinsic interest, but it also provides hypotheses and estimates as the starting point for a rigorous numerical analysis of our HOPS/AWE scheme (see, e.g., [60] for a possible path) for this problem.

We begin this program by assuming that ε and δ are sufficiently small. However, we have demonstrated in [58, 61] for a closely related problem concerning Laplace's equation, the domain of analyticity in ε is not merely a small disk centered at the origin in the complex plane, but rather a neighborhood of the *entire* real axis. We suspect that an analogous analysis can be conducted in the current setting and we intend to pursue this in future work. By contrast, as pointed out in [38], the domain of analyticity in δ is bounded by the presence of the Rayleigh singularities. We believe that a similar analysis may prove fruitful in verifying that the domain of analyticity can be extended right up to this limit which is supported by our numerics [37].

The paper is organized as follows: In Section 2 we summarize the equations which govern the propagation of linear waves in a two–dimensional periodic structure, and in Section 2.1 we discuss how the outgoing wave conditions can be exactly enforced through the use of Transparent Boundary Conditions. Then in Section 3 we restate our governing equations in terms of interfacial quantities via a Non–Overlapping Domain Decomposition phrased in terms of Dirichlet–Neumann Operators (DNOs). In Section 4 we discuss our analyticity result with a general theory in Section 4.1 and our specific result in Section 4.2. This requires a study of analyticity of the data in Section 4.3 and an investigation of the flat–interface situation in Section 4.4. We conclude with the final piece required for the general theory: The analyticity of Dirichlet–Neumann Operators (Section 6). We accomplish this by first establishing analyticity of the underlying fields (Section 5) requiring a special change of variables specified in Section 5.1. With this we demonstrate the analyticity of the scattered field in Sections 5.2 and 5.3. Given these theorems, we prove the analyticity of the DNOs in Section 6.

2. The Governing Equations.

An example of the geometry we consider is displayed in Figure 1: a y–invariant, doubly layered structure with a periodic interface

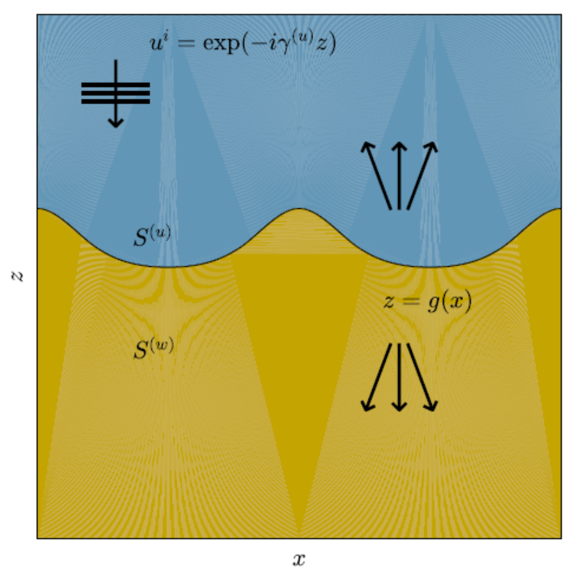

Fig. 1: A two-layer structure with a periodic interface, $z = g(x)$, separating two material layers, $S^{(u)}$ and $S^{(w)}$, illuminated by plane–wave incidence.

separating the two materials. The interface is specified by the graph of the function $z = g(x)$ which is d–periodic so that $g(x+d) = g(x)$. Dielectrics occupy both domains where an insulator (with refractive index n^u) fills the region above the graph $z = g(x)$

$$S^{(u)} := \{z > g(x)\},$$

and a second material (with index of refraction n^w) occupies

$$S^{(w)} := \{z < g(x)\}.$$

The superscripts are chosen to conform to the notation of the authors in previous work [52, 55]. The structure is illuminated from above by monochromatic plane–wave incident radiation of frequency ω and wavenumber $k^u = n^u\omega/c_0 = \omega/c^u$ (c_0 is the speed of light) aligned with the grooves

$$\underline{\mathbf{E}}^i(x,z,t) = \mathbf{A}e^{-i\omega t + i\alpha x - i\gamma^u z}, \quad \underline{\mathbf{H}}^i(x,z,t) = \mathbf{B}e^{-i\omega t + i\alpha x - i\gamma^u z},$$
$$\alpha := k^u \sin(\theta), \quad \gamma^u := k^u \cos(\theta).$$

We consider the reduced incident fields

$$\mathbf{E}^i(x,z) = e^{i\omega t}\underline{\mathbf{E}}^i(x,z,t), \quad \mathbf{H}^i(x,z) = e^{i\omega t}\underline{\mathbf{H}}^i(x,z,t),$$

where the time dependence $\exp(-i\omega t)$ has been factored out. As shown in [62], the reduced electric and magnetic fields, like the reduced scattered fields, are α–quasiperiodic due to the incident radiation. To close the problem, we specify that the scattered radiation is "outgoing," upward propagating in $S^{(u)}$ and downward propagating in $S^{(w)}$.

It is well known (see, e.g., Petit [62]) that in this two–dimensional setting, the time–harmonic Maxwell equations decouple into two scalar Helmholtz problems which govern the Transverse Electric (TE) and Transverse Magnetic (TM) polarizations. We define the invariant (y) direction of the scattered (electric or magnetic) field by $\tilde{u} = \tilde{u}(x,z)$ and $\tilde{w} = \tilde{w}(x,z)$ in $S^{(u)}$ and $S^{(w)}$, respectively. The incident radiation in the upper field is denoted by $\tilde{u}^i(x,z)$.

Following our previous work [53] we further factor out the phase $\exp(i\alpha x)$ from the fields $\tilde{u}$ and $\tilde{w}$

$$u(x,z) = e^{-i\alpha x}\tilde{u}(x,z), \quad w(x,z) = e^{-i\alpha x}\tilde{w}(x,z),$$

which, we note, are d–periodic. In light of all of this, we are led to seek outgoing, d–periodic solutions of

$$\Delta u + 2i\alpha\partial_x u + (\gamma^u)^2 u = 0, \qquad z > g(x), \tag{2.1a}$$
$$\Delta w + 2i\alpha\partial_x w + (\gamma^w)^2 w = 0, \qquad z < g(x), \tag{2.1b}$$
$$u - w = \zeta, \qquad z = g(x), \tag{2.1c}$$
$$\partial_N u - i\alpha(\partial_x g)u - \tau^2\left[\partial_N w - i\alpha(\partial_x g)w\right] = \psi, \qquad z = g(x), \tag{2.1d}$$

where $N := (-\partial_x g, 1)^T$. The Dirichlet and Neumann data are

$$\zeta(x) := -e^{-i\gamma^u g(x)}, \tag{2.1e}$$
$$\psi(x) := (i\gamma^u + i\alpha(\partial_x g))e^{-i\gamma^u g(x)}, \tag{2.1f}$$

and

$$\tau^2 = \begin{cases} 1, & \text{TE}, \\ (k^u/k^w)^2 = (n^u/n^w)^2, & \text{TM}, \end{cases}$$

where $k^w = n^w \omega / c_0 = \omega / c^w$ and $\gamma^w = k^w \cos(\theta)$.

2.1. Transparent Boundary Conditions. The Rayleigh expansions, which are derived through separation of variables [62], are the periodic, upward/downward propagating solutions of (2.1a) and (2.1b). In order to truncate the bi–infinite problem domain to one of finite size we use these to define Transparent Boundary Conditions. For this we choose values a and b such that

$$a > |g|_\infty, \quad -b < -|g|_\infty,$$

and define the artificial boundaries $\{z = a\}$ and $\{z = -b\}$. In $\{z > a\}$ the Rayleigh expansions tell us that upward propagating solutions of (2.1a) are

$$u(x,z) = \sum_{p=-\infty}^{\infty} \hat{a}_p e^{i\tilde{p}x + i\gamma_p^u z}, \tag{2.2}$$

while downward propagating solutions of (2.1b) in $\{z < -b\}$ can be expressed as

$$w(x,z) = \sum_{p=-\infty}^{\infty} \hat{d}_p e^{i\tilde{p}x - i\gamma_p^w z},$$

where, for $p \in \mathbf{Z}$ and $q \in \{u, w\}$,

$$\tilde{p} := \frac{2\pi p}{d}, \quad \alpha_p := \alpha + \tilde{p}, \quad \gamma_p^q := \begin{cases} \sqrt{(k^q)^2 - \alpha_p^2}, & p \in \mathcal{U}^q, \\ i\sqrt{\alpha_p^2 - (k^q)^2}, & p \notin \mathcal{U}^q, \end{cases} \tag{2.3}$$

and

$$\mathcal{U}^q := \{p \in \mathbf{Z} \mid \alpha_p^2 < (k^q)^2\},$$

which are the propagating modes in the upper and lower layers. With these we can define the Transparent Boundary Conditions in the following way: we first rewrite (2.2) as

$$u(x,z) = \sum_{p=-\infty}^{\infty} \left(\hat{a}_p e^{i\gamma_p^u a}\right) e^{i\tilde{p}x + i\gamma_p^u (z-a)} = \sum_{p=-\infty}^{\infty} \hat{\xi}_p e^{i\tilde{p}x + i\gamma_p^u (z-a)},$$

and observe that,

$$u(x,a) = \sum_{p=-\infty}^{\infty} \hat{\xi}_p e^{i\tilde{p}x} =: \xi(x),$$

and

$$\partial_z u(x,a) = \sum_{p=-\infty}^{\infty} (i\gamma_p^u) \hat{\xi}_p e^{i\tilde{p}x} =: T^u[\xi(x)],$$

which defines the order–one Fourier multiplier T^u. From this we state that upward–propagating solutions of (2.1a) satisfy the Transparent Boundary Condition at $z = a$

$$\partial_z u(x,a) - T^u[u(x,a)] = 0, \quad z = a. \tag{2.4}$$

A similar calculation leads to the Transparent Boundary Condition at $z = -b$

$$\partial_z w(x,-b) - T^w[w(x,-b)] = 0, \quad z = -b, \tag{2.5}$$

where

$$T^w[\psi(x)] := \sum_{p=-\infty}^{\infty} (-i\gamma_p^w)\hat{\psi}_p e^{i\tilde{p}x}.$$

We note that these conditions enforce the Upward and Downward Propagating Conditions described by Arens [3].

With these we now state the full set of governing equations as

$$\Delta u + 2i\alpha\partial_x u + (\gamma^u)^2 u = 0, \quad z > g(x), \tag{2.6a}$$
$$\Delta w + 2i\alpha\partial_x w + (\gamma^w)^2 w = 0, \quad z < g(x), \tag{2.6b}$$
$$u - w = \zeta, \quad z = g(x), \tag{2.6c}$$
$$\partial_N u - i\alpha(\partial_x g)u - \tau^2 [\partial_N w - i\alpha(\partial_x g)w] = \psi, \quad z = g(x), \tag{2.6d}$$
$$\partial_z u(x,a) - T^u[u(x,a)] = 0, \quad z = a, \tag{2.6e}$$
$$\partial_z w(x,-b) - T^w[w(x,-b)] = 0, \quad z = -b, \tag{2.6f}$$
$$u(x+d,z) = u(x,z), \tag{2.6g}$$
$$w(x+d,z) = w(x,z). \tag{2.6h}$$

3. A Non–Overlapping Domain Decomposition Method.

We now rewrite our governing equations (2.6) in terms of *surface* quantities via a Non–Overlapping Domain Decomposition Method [46, 19, 18]. For this we define

$$U(x) := u(x,g(x)), \quad \tilde{U}(x) := -\partial_N u(x,g(x)),$$
$$W(x) := w(x,g(x)), \quad \tilde{W}(x) := \partial_N w(x,g(x)),$$

where u is a d–periodic solution of (2.6a) and (2.6e), and w is a d–periodic solution of (2.6b) and (2.6f). In terms of these, our full governing equations (2.6) are equivalent to the pair of boundary conditions, (2.6c) and (2.6d),

$$U - W = \zeta, \tag{3.1a}$$
$$-\tilde{U} - (i\alpha)(\partial_x g)U - \tau^2 \left[\tilde{W} - (i\alpha)(\partial_x g)W\right] = \psi. \tag{3.1b}$$

This set of two equations and four unknowns can be closed by noting that the pairs $\{U, \tilde{U}\}$ and $\{W, \tilde{W}\}$ are connected, e.g., by Dirichlet–Neumann Operators (DNOs), which [59] showed are well–defined under the hypotheses presently listed.

Definition 3.1. *Given an integer $s \geq 0$, if $g \in C^{s+2}$ then the unique solution of*

$$\Delta u + 2i\alpha\partial_x u + (\gamma^u)^2 u = 0, \quad z > g(x), \tag{3.2a}$$
$$u = U, \quad z = g(x), \tag{3.2b}$$
$$\partial_z u(x,a) - T^u[u(x,a)] = 0, \quad z = a, \tag{3.2c}$$
$$u(x+d,z) = u(x,z), \tag{3.2d}$$

defines the upper layer DNO

$$G : U \to \tilde{U}. \tag{3.3}$$

DEFINITION 3.2. *Given an integer $s \geq 0$, if $g \in C^{s+2}$ then the unique solution of*

$$\Delta w + 2i\alpha\partial_x w + (\gamma^w)^2 w = 0, \quad z < g(x), \tag{3.4a}$$
$$w = W, \quad z = g(x), \tag{3.4b}$$
$$\partial_z w(x, -b) - T^w[w(x, -b)] = 0, \quad z = -b, \tag{3.4c}$$
$$w(x + d, z) = w(x, z). \tag{3.4d}$$

defines the lower layer DNO

$$J : W \to \tilde{W}. \tag{3.5}$$

The interfacial reformulation of our governing equations (3.1) now becomes

$$\mathbf{AV} = \mathbf{R}, \tag{3.6}$$

where

$$\mathbf{A} = \begin{pmatrix} I & -I \\ G + (\partial_x g)(i\alpha) & \tau^2 J - \tau^2(\partial_x g)(i\alpha) \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} U \\ W \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} \zeta \\ -\psi \end{pmatrix}. \tag{3.7}$$

4. Joint Analyticity of Solutions.

There are many possible ways to analyze (3.6) rigorously. Following our recent work [37], we select a jointly perturbative approach based on two assumptions:

1. Boundary Perturbation: $g(x) = \varepsilon f(x)$, $\varepsilon \in \mathbf{R}$,
2. Frequency Perturbation: $\omega = (1 + \delta)\underline{\omega} = \underline{\omega} + \delta\underline{\omega}$, $\delta \in \mathbf{R}$.

Remark 4.1. At inception one typically assumes that these perturbation parameters, ε and δ, are quite small and we can certainly begin there. However, we will show that these only need be *sufficiently* small (e.g., characterized by the C^2 norm of f for the domain of analyticity in ε) but not necessarily tiny. Furthermore, following the methods devised in [58, 61] for the related problem of analytic continuation of DNOs associated to Laplace's equation, we fully expect that the neighborhood of analyticity in ε contains the *entire* real axis. Beyond this we note that the domain of analyticity in δ is bounded by the Rayleigh singularities as discussed in [38]. However, it is possible that an extension of the approach in [58, 61] may deliver a rigorous justification of our numerical observations in [37] that the region of analyticity in δ extends right up to the limit imposed by the Rayleigh singularities. Verifying each of these predictions is a goal of current research by the authors.

The frequency perturbation has the following important consequences

$$k^q = \omega/c^q = (1 + \delta)\underline{\omega}/c^q =: (1 + \delta)\underline{k}^q = \underline{k}^q + \delta\underline{k}^q, \quad q \in \{u, w\},$$
$$\alpha = k^u \sin(\theta) = (1 + \delta)\underline{k}^u \sin(\theta) =: (1 + \delta)\underline{\alpha} = \underline{\alpha} + \delta\underline{\alpha},$$
$$\gamma^q = k^q \cos(\theta) = (1 + \delta)\underline{k}^q \cos(\theta) =: (1 + \delta)\underline{\gamma}^q = \underline{\gamma}^q + \delta\underline{\gamma}^q, \quad q \in \{u, w\}.$$

This, in turn, delivers

$$\alpha_p = \alpha + \tilde{p} = \underline{\alpha} + \delta\underline{\alpha} + \tilde{p} =: \underline{\alpha}_p + \delta\underline{\alpha}.$$

We now pursue this perturbative approach to establish the existence, uniqueness, and analyticity of solutions to (3.6). To accomplish this we will presently show the joint analytic dependence of $\mathbf{A} = \mathbf{A}(\varepsilon, \delta)$ and $\mathbf{R} = \mathbf{R}(\varepsilon, \delta)$ upon ε and δ, and then appeal to the regular perturbation theory for linear systems of equations outlined in [54] to discover the analyticity of the unique solution $\mathbf{V} = \mathbf{V}(\varepsilon, \delta)$. More precisely, we view (3.6) as

$$\mathbf{A}(\varepsilon, \delta)\mathbf{V}(\varepsilon, \delta) = \mathbf{R}(\varepsilon, \delta),$$

establish the analyticity of $\mathbf{A}$ and $\mathbf{R}$ so that

$$\{\mathbf{A}, \mathbf{R}\}(\varepsilon, \delta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\{\mathbf{A}_{n,m}, \mathbf{R}_{n,m}\}\varepsilon^n\delta^m, \tag{4.1}$$

and seek a solution of the form

$$\mathbf{V}(\varepsilon, \delta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathbf{V}_{n,m}\varepsilon^n\delta^m, \tag{4.2}$$

which we will show converges in a function space. To pursue this we insert (4.2) and (4.1) into (3.6) and find, at each perturbation order (n, m), that we must solve

$$\begin{aligned}\mathbf{A}_{0,0}\mathbf{V}_{n,m} = \mathbf{R}_{n,m} &- \sum_{\ell=0}^{n-1}\mathbf{A}_{n-\ell,0}\mathbf{V}_{\ell,m} - \sum_{r=0}^{m-1}\mathbf{A}_{0,m-r}\mathbf{V}_{n,r} \\ &- \sum_{\ell=0}^{n-1}\sum_{r=0}^{m-1}\mathbf{A}_{n-\ell,m-r}\mathbf{V}_{\ell,r}.\end{aligned} \tag{4.3}$$

A brief inspection of the formulas for $\mathbf{A}$ and $\mathbf{R}$, (3.7), reveals that

$$\mathbf{A}_{0,0} = \begin{pmatrix} I & -I \\ G_{0,0} & \tau^2 J_{0,0} \end{pmatrix}, \tag{4.4a}$$

$$\begin{aligned}\mathbf{A}_{n,m} &= \begin{pmatrix} 0 & 0 \\ G_{n,m} & \tau^2 J_{n,m} \end{pmatrix} \\ &\quad + \delta_{n,1}\left\{1 + \delta_{m,1}\right\}(\partial_x f)(i\underline{\alpha})\begin{pmatrix} 0 & 0 \\ 1 & -\tau^2 \end{pmatrix}, \qquad n \neq 0 \text{ or } m \neq 0,\end{aligned} \tag{4.4b}$$

$$\mathbf{R}_{n,m} = \begin{pmatrix} \zeta_{n,m} \\ -\psi_{n,m} \end{pmatrix}, \tag{4.4c}$$

where $\delta_{n,m}$ is the Kronecker delta function. Formulas for the terms $\{\zeta_{n,m}, \psi_{n,m}\}$ can be found in [37] or by using the recursions described in Section 4.3. The terms $G_{n,m}$ and $J_{n,m}$ are the (n, m)–th corrections of the DNOs G and J, respectively, in a Taylor series expansion of each jointly in ε and δ. This is explained in Section 6, together with precise estimates of the coefficients, $G_{n,m}$ and $J_{n,m}$, in the appropriate Sobolev spaces. Finally, in Section 4.4 we utilize expressions for the flat–interface DNOs, $G_{0,0}$ and $J_{0,0}$, to investigate the mapping properties of the linearized operator, $\mathbf{A}_{0,0}$, and its inverse.

4.1. A General Analyticity Theory. Given these estimates, existence, uniqueness, and analyticity of solutions can be deduced in a rather straightforward fashion using the following result from one of the authors' previous papers [54] (Theorem 3.2). This result uses multi–index notation [25], in particular

$$\tilde{\varepsilon} := \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_M \end{pmatrix}, \quad \tilde{n} := \begin{pmatrix} n_1 \\ \vdots \\ n_M \end{pmatrix},$$

and the convention

$$\sum_{\tilde{n}=0}^{\infty} A_{\tilde{n}} \, \tilde{\varepsilon}^{\tilde{n}} = \sum_{n_1=0}^{\infty} \cdots \sum_{n_M=0}^{\infty} A_{n_1,\dots,n_M} \varepsilon_1^{n_1} \cdots \varepsilon_M^{n_M}.$$

THEOREM 4.2. *Given two Banach spaces, $\tilde{X}$ and $\tilde{Y}$, suppose that:*

1. $\mathbf{R}_{\tilde{n}} \in \tilde{Y}$ *for all* $\tilde{n} \geq 0$, *and there exist M–multi–indexed constants* $\tilde{C}_R > 0$, $\tilde{B}_R > 0$,
$$\tilde{C}_R = \begin{pmatrix} C_{R,1} \\ \vdots \\ C_{R,M} \end{pmatrix}, \quad \tilde{B}_R^{\tilde{n}} = \begin{pmatrix} B_{R,1}^{n_1} \\ \vdots \\ B_{R,M}^{n_M} \end{pmatrix},$$
such that
$$\|\mathbf{R}_{\tilde{n}}\|_{\tilde{Y}} \leq \tilde{C}_R \tilde{B}_R^{\tilde{n}},$$
2. $\mathbf{A}_{\tilde{n}} : \tilde{X} \to \tilde{Y}$ *for all* $\tilde{n} \geq 0$, *and there exist M–multi–indexed constants* $\tilde{C}_A > 0$, $\tilde{B}_A > 0$ *such that*
$$\|\mathbf{A}_{\tilde{n}}\|_{\tilde{X} \to \tilde{Y}} \leq \tilde{C}_A \tilde{B}_A^{\tilde{n}},$$
3. $\mathbf{A}_0^{-1} : \tilde{Y} \to \tilde{X}$, *and there exists a constant* $C_e > 0$ *such that*
$$\left\|\mathbf{A}_0^{-1}\right\|_{\tilde{Y} \to \tilde{X}} \leq C_e.$$

Then the equation (3.6) has a unique solution,

$$\mathbf{V}(\tilde{\varepsilon}) = \sum_{\tilde{n}=0}^{\infty} \mathbf{V}_{\tilde{n}} \tilde{\varepsilon}^{\tilde{n}}, \tag{4.5}$$

and there exist M–multi–indexed constants $\tilde{C}_V > 0$ *and* $\tilde{B}_V > 0$ *such that*

$$\|\mathbf{V}_{\tilde{n}}\|_{\tilde{X}} \leq \tilde{C}_V \tilde{B}_V^{\tilde{n}},$$

for all $\tilde{n} \geq 0$ *and any*

$$\tilde{C}_V \geq 2C_e \tilde{C}_R, \quad \tilde{B}_V \geq \max\left\{\tilde{B}_R, 2\tilde{B}_A, 4C_e \tilde{C}_A \tilde{B}_A\right\},$$

enforced componentwise. This implies that, for any M–multi–indexed constant $0 \leq \tilde{\rho} < 1$, *(4.5), converges for all* $\tilde{\varepsilon}$ *such that* $B\tilde{\varepsilon} < \tilde{\rho}$, *i.e.,* $\tilde{\varepsilon} < \tilde{\rho}/B$.

Remark 4.3. In the current context we will use this result in the case $M = 2$ and

$$\tilde{\varepsilon} = \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix}, \quad \tilde{n} = \begin{pmatrix} n \\ m \end{pmatrix}, \quad \tilde{\rho} = \begin{pmatrix} \rho \\ \sigma \end{pmatrix}.$$

4.2. Analyticity of Solutions to the Two–Layer Problem. To state our theorem precisely we briefly define and recall classical properties of the L^2–based Sobolev spaces, H^s, of laterally periodic functions [40]. We know that any d–periodic L^2 function can be expressed in a Fourier series as

$$\mu(x) = \sum_{p=-\infty}^{\infty} \hat{\mu}_p e^{i\tilde{p}x}, \quad \hat{\mu}_p = \frac{1}{d}\int_0^d \mu(x) e^{-i\tilde{p}x}\,dx,$$

[40]. We define the symbol $\langle \tilde{p} \rangle^2 := 1 + |\tilde{p}|^2$ so that laterally periodic norms for surface and volumetric functions are defined by

$$\|\mu\|_{H^s}^2 := \sum_{p=-\infty}^{\infty} \langle \tilde{p} \rangle^{2s} |\hat{\mu}_p|^2,$$

and

$$\|u\|_{H^s}^2 := \sum_{\ell=0}^{s} \sum_{p=-\infty}^{\infty} \langle \tilde{p} \rangle^{2(s-\ell)} \int_0^a |\hat{u}_p(z)|^2 \; dz = \sum_{\ell=0}^{s} \sum_{p=-\infty}^{\infty} \langle \tilde{p} \rangle^{2(s-\ell)} \|\hat{u}_p\|_{L^2(0,a)}^2,$$

respectively. With these we define the laterally d–periodic Sobolev spaces H^s as the L^2 functions for which $\|\cdot\|_{H^s}$ is finite. For our present use we define the vector–valued spaces for $s \geq 0$

$$X^s := \left\{ \mathbf{V} = \begin{pmatrix} U \\ W \end{pmatrix} \middle| \, U, W \in H^{s+3/2}([0,d]) \right\},$$

and

$$Y^s := \left\{ \mathbf{R} = \begin{pmatrix} \zeta \\ -\psi \end{pmatrix} \middle| \, \zeta \in H^{s+3/2}([0,d]), \psi \in H^{s+1/2}([0,d]) \right\}.$$

These have the norms

$$\|\mathbf{V}\|_{X^s}^2 = \left\| \begin{pmatrix} U \\ W \end{pmatrix} \right\|_{X^s}^2 := \|U\|_{H^{s+3/2}}^2 + \|W\|_{H^{s+3/2}}^2,$$

$$\|\mathbf{R}\|_{Y^s}^2 = \left\| \begin{pmatrix} \zeta \\ -\psi \end{pmatrix} \right\|_{Y^s}^2 := \|\zeta\|_{H^{s+3/2}}^2 + \|\psi\|_{H^{s+1/2}}^2.$$

In addition to these function spaces we also require the following three results from the classical theory of Sobolev spaces [2, 44] and elliptic partial differential equations [41, 26, 27, 25]. (See also [56, 32] in the context of HOPS methods.)

LEMMA 4.4. *Given an integer $s \geq 0$ and any $\eta > 0$, there exists a constant $\mathcal{M} = \mathcal{M}(s)$ such that if $f \in C^s([0,d])$ and $u \in H^s([0,d] \times [0,a])$ then*

$$\|fu\|_{H^s} \leq \mathcal{M} |f|_{C^s} \|u\|_{H^s}, \tag{4.6}$$

and if $\tilde{f} \in C^{s+1/2+\eta}([0,d])$ and $\tilde{u} \in H^{s+1/2}([0,d])$ then

$$\left\|\tilde{f}\tilde{u}\right\|_{H^{s+1/2}} \leq \mathcal{M} \left|\tilde{f}\right|_{C^{s+1/2+\eta}} \|\tilde{u}\|_{H^{s+1/2}}. \tag{4.7}$$

THEOREM 4.5. *Given an integer $s \geq 0$, if $F \in H^s([0,d]) \times [0,a])$, $U \in H^{s+3/2}([0,d])$, $P \in H^{s+1/2}([0,d])$, then the unique solution of*

$$
\begin{aligned}
&\Delta u(x,z) + 2i\underline{\alpha}\partial_x u(x,z) + (\underline{\gamma}^u)^2 u(x,z) = F(x,z), && 0 < z < a,\\
&u(x,0) = U(x,0), && z = 0,\\
&\partial_z u(x,a) - T_0^u[u(x,a)] = P(x), && z = a,\\
&u(x+d,z) = u(x,z),
\end{aligned}
$$

satisfies

$$
\|u\|_{H^{s+2}} \leq C_e \left\{ \|F\|_{H^s} + \|U\|_{H^{s+3/2}} + \|P\|_{H^{s+1/2}} \right\}, \tag{4.8}
$$

for some constant $C_e > 0$ where $T_0^u = i\underline{\gamma}_D^u$ corresponds to the $\delta = 0$ scenario.

LEMMA 4.6. *Given an integer $s \geq 0$, if $F \in H^s([0,d]) \times [0,a])$, then $(a-z)F \in H^s([0,d]) \times [0,a])$ and there exists a positive constant $Z_a = Z_a(s)$ such that*

$$
\|(a-z)F\|_{H^s} \leq Z_a \|F\|_{H^s}.
$$

We now state our main result.

THEOREM 4.7. *Given an integer $s \geq 0$, if $f \in C^{s+2}([0,d])$ then the equation (3.6) has a unique solution, (4.2). Furthermore, there exist constants $B, C, D > 0$ such that*

$$
\|\mathbf{V}_{n,m}\|_{X^s} \leq C B^n D^m,
$$

for all $n, m \geq 0$. This implies that for any $0 \leq \rho, \sigma < 1$, (4.2) converges for all ε such that $B\varepsilon < \rho$, i.e., $\varepsilon < \rho/B$ and all δ such that $D\delta < \sigma$, i.e., $\delta < \sigma/D$.

Proof. As mentioned above, our strategy is to invoke Theorem 4.2 and thus we must verify its hypotheses. To begin, we consider the spaces

$$
\tilde{X} = X^s, \quad \tilde{Y} = Y^s.
$$

In Section 4.3 we will show that the vector $\mathbf{R}_{n,m}$, consisting of $\zeta_{n,m}$ and $\psi_{n,m}$, is bounded in Y^s for any $s \geq 0$ provided that $f \in C^{s+2}([0,d])$. (This implies that the $\mathbf{R}_{n,m}$ satisfies the estimates of Item 1 in Theorem 4.2.)

Then in Section 6 we show that the operators $G_{n,m}$ and $J_{n,m}$ in the Taylor series expansions of the DNOs satisfy appropriate bounds provided that $f \in C^{s+2}([0,d])$. With this, it is clear that the $\mathbf{A}_{n,m}$ satisfy the estimates of Item 2 in Theorem 4.2.

Finally, in Section 4.4 we show that the estimates and mapping properties of $\mathbf{A}_{0,0}^{-1}$ for Item 3 in Theorem 4.2 hold. ∎

4.3. Analyticity of the Surface Data. To establish the analyticity of the Dirichlet and Neumann data obeying suitable estimates, we begin by defining

$$
\mathcal{E}(x;\varepsilon,\delta) := e^{-i(1+\delta)\underline{\gamma}^u \varepsilon f(x)},
$$

and note that we can write (2.1e) and (2.1f) as

$$
\begin{aligned}
\zeta(x) &= \zeta(x;\varepsilon,\delta) = -\mathcal{E}(x;\varepsilon,\delta),\\
\psi(x) &= \psi(x;\varepsilon,\delta) = \left\{ i(1+\delta)\underline{\gamma}^u + i(1+\delta)\underline{\alpha}(\varepsilon\partial_x f) \right\} \mathcal{E}(x;\varepsilon,\delta).
\end{aligned}
$$

We will now demonstrate that the function $\mathcal{E}$ is jointly analytic in ε and δ, and subject to appropriate estimates, which clearly demonstrates the joint analytic dependence of the data, $\zeta(x;\varepsilon,\delta)$ and $\psi(x;\varepsilon,\delta)$.

LEMMA 4.8. *Given any integer $s \geq 0$, if $f \in C^{s+2}([0,d])$ then the function $\mathcal{E}(x;\varepsilon,\delta)$ is jointly analytic in ε and δ. Therefore*

$$\mathcal{E}(x;\varepsilon,\delta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathcal{E}_{n,m}(x)\varepsilon^n \delta^m, \tag{4.9}$$

and, for constants $C_{\mathcal{E}}, B_{\mathcal{E}}, D_{\mathcal{E}} > 0$,

$$\|\mathcal{E}_{n,m}\|_{H^{s+3/2}} \leq C_{\mathcal{E}} B_{\mathcal{E}}^n D_{\mathcal{E}}^m, \tag{4.10}$$

for all $n, m \geq 0$.

Proof. We begin by observing the classical fact that the composition of jointly (real) analytic functions is also jointly (real) analytic [39] so that (4.9) holds, and move to expressions and estimates for the $\mathcal{E}_{n,m}$. By evaluating at $\varepsilon = 0$ we find that

$$\mathcal{E}(x;0,\delta) = 1,$$

so that

$$\mathcal{E}_{0,m}(x) = \begin{cases} 1, & m = 0, \\ 0, & m > 0. \end{cases}$$

For $\varepsilon > 0$ we use the straightforward computation

$$\partial_{\varepsilon}\mathcal{E} = \left\{-i(1+\delta)\underline{\gamma}^u f\right\}\mathcal{E},$$

and the expansion (4.9) to learn that, for $m = 0$,

$$\mathcal{E}_{n+1,0} = \left(\frac{-i\underline{\gamma}^u f}{n+1}\right)\mathcal{E}_{n,0}, \tag{4.11}$$

and, for $m > 0$,

$$\mathcal{E}_{n+1,m} = \left(\frac{-i\underline{\gamma}^u f}{n+1}\right)\left\{\mathcal{E}_{n,m} + \mathcal{E}_{n,m-1}\right\}. \tag{4.12}$$

We work by induction in n and begin by establishing (4.10) at $n = 0$ for all $m \geq 0$. This is immediate as

$$\|\mathcal{E}_{0,0}\|_{H^{s+3/2}} = 1, \quad \|\mathcal{E}_{0,m}\|_{H^{s+3/2}} = 0.$$

We now assume (4.10) for all $n < \bar{n}$ and all $m \geq 0$, and seek this estimate in the case $n = \bar{n}$ and all $m \geq 0$. For this we conduct another induction on m, and for $m = 0$ we use (4.11) (together with Lemma 4.4 with $\tilde{s} = s + 1$) to discover

$$\begin{aligned} \|\mathcal{E}_{\bar{n},0}\|_{H^{s+3/2}} &\leq \mathcal{M}\left(\frac{\left|\underline{\gamma}^u\right| |f|_{C^{s+3/2+\eta}}}{\bar{n}}\right)\|\mathcal{E}_{\bar{n}-1,0}\|_{H^{s+3/2}} \\ &\leq \mathcal{M}\left(\frac{\left|\underline{\gamma}^u\right| |f|_{C^{s+2}}}{\bar{n}}\right) C_{\mathcal{E}} B_{\mathcal{E}}^{\bar{n}-1} \leq C_{\mathcal{E}} B_{\mathcal{E}}^{\bar{n}}, \end{aligned}$$

provided that

$$B_{\mathcal{E}} \geq \mathcal{M} \left|\underline{\gamma}^u\right| |f|_{C^{s+2}} \geq \mathcal{M} \left(\frac{\left|\underline{\gamma}^u\right| |f|_{C^{s+2}}}{\bar{n}} \right).$$

Finally, we assume the estimate (4.10) for $n = \bar{n}$ and $m < \bar{m}$, and use (4.12) to learn that

$$\begin{aligned} \left\|\mathcal{E}_{\bar{n},\bar{m}}\right\|_{H^{s+3/2}} &\leq \mathcal{M} \left(\frac{\left|\underline{\gamma}^u\right| |f|_{C^{s+3/2+\eta}}}{\bar{n}} \right) \left\{ \left\|\mathcal{E}_{\bar{n}-1,\bar{m}}\right\|_{H^{s+3/2}} + \left\|\mathcal{E}_{\bar{n}-1,\bar{m}-1}\right\|_{H^{s+3/2}} \right\} \\ &\leq \mathcal{M} \left(\frac{\left|\underline{\gamma}^u\right| |f|_{C^{s+2}}}{\bar{n}} \right) C_{\mathcal{E}} \left\{ B_{\mathcal{E}}^{\bar{n}-1} D_{\mathcal{E}}^{\bar{m}} + B_{\mathcal{E}}^{\bar{n}-1} D_{\mathcal{E}}^{\bar{m}-1} \right\} \\ &\leq C_{\mathcal{E}} B_{\mathcal{E}}^{\bar{n}} D_{\mathcal{E}}^{\bar{m}}, \end{aligned}$$

provided that

$$\mathcal{M} \left(\frac{\left|\underline{\gamma}^u\right| |f|_{C^{s+2}}}{\bar{n}} \right) \leq \frac{B_{\mathcal{E}}}{2}, \quad \mathcal{M} \left(\frac{\left|\underline{\gamma}^u\right| |f|_{C^{s+2}}}{\bar{n}} \right) \leq \frac{B_{\mathcal{E}} D_{\mathcal{E}}}{2},$$

which can be accomplished, e.g., with

$$B_{\mathcal{E}} \geq 2\mathcal{M} \left|\underline{\gamma}^u\right| |f|_{C^{s+2}} \geq 2\mathcal{M} \left(\frac{\left|\underline{\gamma}^u\right| |f|_{C^{s+2}}}{\bar{n}} \right), \quad D_{\mathcal{E}} \geq 1,$$

and we are done. □

With Lemma 4.8 it is straightforward to prove the following analyticity result for the Dirichlet and Neumann data.

LEMMA 4.9. *Given any integer $s \geq 0$, if $f \in C^{s+2}([0,d])$ then the functions $\zeta(x;\varepsilon,\delta)$ and $\psi(x;\varepsilon,\delta)$ are jointly analytic in ε and δ. Therefore*

$$\{\zeta, \psi\}(x;\varepsilon,\delta) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \{\zeta_{n,m}, \psi_{n,m}\}(x) \varepsilon^n \delta^m \tag{4.13}$$

and, for constants $C_\zeta, B_\zeta, D_\zeta > 0$, and $C_\psi, B_\psi, D_\psi > 0$,

$$\left\|\zeta_{n,m}\right\|_{H^{s+3/2}} \leq C_\zeta B_\zeta^n D_\zeta^m, \quad \left\|\psi_{n,m}\right\|_{H^{s+1/2}} \leq C_\psi B_\psi^n D_\psi^m, \tag{4.14}$$

for all $n, m \geq 0$.

4.4. Invertibility of the Flat–Interface Operator. The final hypothesis to be verified in order to invoke Theorem 4.2 is the existence and mapping properties of the linearized (flat–interface) operator $\mathbf{A}_{0,0}$. In our previous work [37] we showed that

$$\mathbf{A}_{0,0} = \begin{pmatrix} I & -I \\ G_{0,0} & \tau^2 J_{0,0} \end{pmatrix}, \tag{4.15}$$

where

$$G_{0,0} = -i\gamma_D^u, \quad J_{0,0} = -i\gamma_D^w, \tag{4.16}$$

are order–one Fourier multipliers defined by

$$(4.17)\qquad G_{0,0}[U] = \sum_{p=-\infty}^{\infty} (-i\gamma_p^u)\hat{U}_p e^{i\tilde{p}x}, \quad J_{0,0}[W] = \sum_{p=-\infty}^{\infty} (-i\gamma_p^w)\hat{W}_p e^{i\tilde{p}x}.$$

Lemma 4.10. *The linear operator $A_{0,0}$ maps X^s to Y^s boundedly, is invertible, and its inverse maps Y^s to X^s boundedly.*

Proof. We begin by defining the operator

$$\Delta := G_{0,0} + \tau^2 J_{0,0} = (-i\gamma_D^u) + \tau^2(-i\gamma_D^w),$$

which has Fourier symbol

$$\hat{\Delta}_p = (-i\gamma_p^u) + \tau^2(-i\gamma_p^w),$$

and noting that there exist positive constants C_G, C_J, and C_Δ such that

$$\left|-i\gamma_p^u\right| \leq C_G \langle\tilde{p}\rangle, \quad \left|-i\gamma_p^w\right| \leq C_J \langle\tilde{p}\rangle, \quad \left|\hat{\Delta}_p\right| \leq C_\Delta \langle\tilde{p}\rangle.$$

Importantly, provided that $n^u \neq n^w$, it is not difficult to establish the crucial fact that $\hat{\Delta}_p \neq 0$. Finally, one can also find a positive constant $C_{\Delta^{-1}}$ such that

$$\left|\frac{1}{\hat{\Delta}_p}\right| \leq C_{\Delta^{-1}} \langle\tilde{p}\rangle^{-1}.$$

With this it is a simple matter to realize that Δ^{-1} exists and that

$$\Delta : H^{s+3/2} \to H^{s+1/2}, \quad \Delta^{-1} : H^{s+1/2} \to H^{s+3/2}.$$

Next, we write generic elements of X^s and Y^s as

$$\mathbf{V} = \begin{pmatrix} U \\ W \end{pmatrix} \in X^s, \quad \mathbf{R} = \begin{pmatrix} \zeta \\ -\psi \end{pmatrix} \in Y^s.$$

Using the definitions of the norms of X^s and Y^s, and the facts

$$2ab \leq a^2 + b^2, \quad \|A+B\|^2 \leq (\|A\| + \|B\|)^2,$$

we find that

$$\begin{aligned}
\|\mathbf{A}_{0,0}\mathbf{V}\|_{Y^s}^2 &= \|U - W\|_{H^{s+3/2}}^2 + \left\|G_{0,0}U + \tau^2 J_{0,0}W\right\|_{H^{s+1/2}}^2 \\
&\leq 2\|U\|_{H^{s+3/2}}^2 + 2\|W\|_{H^{s+3/2}}^2 + C_G^2\|U\|_{H^{s+3/2}}^2 \\
&\quad + \tau^2 C_G C_J (\|U\|_{H^{s+3/2}}^2 + \|W\|_{H^{s+3/2}}^2) + C_J^2\tau^4\|W\|_{H^{s+3/2}}^2 \\
&\leq \max\{2, C_G^2, \tau^2 C_G C_J, \tau^4 C_J^2\}\left(\|U\|_{H^{s+3/2}}^2 + \|W\|_{H^{s+3/2}}^2\right) \\
&= \max\{2, C_G^2, \tau^2 C_G C_J, \tau^4 C_J^2\}\|\mathbf{V}\|_{X^s}^2,
\end{aligned}$$

so that $\mathbf{A}_{0,0}$ does indeed map X^s to Y^s boundedly. We define the operator

$$\mathbf{B} := \Delta^{-1}\begin{pmatrix} \tau^2 J_{0,0} & I \\ -G_{0,0} & I \end{pmatrix},$$

and note that

$$\mathbf{B}\mathbf{A}_{0,0} = \mathbf{A}_{0,0}\mathbf{B} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

so that the inverse of $\mathbf{A}_{0,0}$ exists and $\mathbf{A}_{0,0}^{-1} = \mathbf{B}$. Furthermore, as above,

$$\begin{aligned}\left\|\mathbf{A}_{0,0}^{-1}\mathbf{R}\right\|_{X^s}^2 &= \left\|\Delta^{-1}(\tau^2 J_{0,0}\zeta - \psi)\right\|_{H^{s+3/2}}^2 + \left\|\Delta^{-1}(-G_{0,0}\zeta - \psi)\right\|_{H^{s+3/2}}^2 \\ &\leq C_{\Delta^{-1}}^2 \tau^4 C_J^2 \left\|\zeta\right\|_{H^{s+3/2}}^2 + C_{\Delta^{-1}}^2 \tau^2 C_J (\left\|\zeta\right\|_{H^{s+3/2}}^2 + \left\|\psi\right\|_{H^{s+1/2}}^2) \\ &\quad + C_{\Delta^{-1}}^2 C_G^2 \left\|\zeta\right\|_{H^{s+3/2}}^2 + C_{\Delta^{-1}}^2 C_G (\left\|\zeta\right\|_{H^{s+3/2}}^2 + \left\|\psi\right\|_{H^{s+1/2}}^2) \\ &\quad + 2C_{\Delta^{-1}}^2 \left\|\psi\right\|_{H^{s+1/2}}^2 \\ &\leq C_{\Delta^{-1}}^2 \max\{2, C_G, C_G^2, \tau^2 C_J, \tau^4 C_J^2\} \left(\left\|\zeta\right\|_{H^{s+3/2}}^2 + \left\|\psi\right\|_{H^{s+1/2}}^2\right) \\ &= C_{\Delta^{-1}}^2 \max\{2, C_G, C_G^2, \tau^2 C_J, \tau^4 C_J^2\} \left\|\mathbf{R}\right\|_{Y^s}^2,\end{aligned}$$

and $\mathbf{A}_{0,0}^{-1}$ maps Y^s to X^s boundedly. ∎

5. Analyticity of the Scattered Fields. At this point we establish the analyticity of the fields which define the DNOs, G and J, though, for brevity, we restrict our attention to the one in the upper layer, G, and note that the considerations for the lower layer DNO, J, are largely the same.

5.1. Change of Variables and Formal Expansions. For our rigorous demonstration we appeal to the Method of Transformed Field Expansions (TFE) [56, 59] which begins with a domain–flattening change of variables (the σ–coordinates of oceanography [63] and the C–method of the dynamical theory of gratings [15, 14]) to the governing equations, (3.2),

$$x' = x, \quad z' = a\left(\frac{z - g(x)}{a - g(x)}\right). \tag{5.1}$$

With this we can rewrite the DNO problem, (3.2), in terms of the transformed field

$$u'(x', z') := u\left(x', \left(\frac{a - g(x')}{a}\right) z' + g(x')\right),$$

as (upon dropping primes)

$$\begin{aligned} &\Delta u + 2i\alpha\partial_x u + (\gamma^u)^2 u = F(x,z), && 0 < z < a, && (5.2a)\\ &u(x,0) = U(x), && z = 0, && (5.2b)\\ &\partial_z u(x,a) - T^u[u(x,a)] = P(x), && z = a, && (5.2c)\\ &u(x+d,z) = u(x,z), && && (5.2d)\end{aligned}$$

(Delete) where $T_0^u = i\gamma_{\perp D}^u$ (corresponding to the $\delta = 0$ scenario), and the DNO itself, (3.3), as

$$G(g)[U] = -\partial_z u(x,0) + H(x). \tag{5.3}$$

The forms for $\{F, P, H\}$ have been derived and reported in [59] and, for brevity, we do not repeat them here.

Following our HOPS/AWE philosophy we assume the joint boundary/frequency perturbation

$$g(x) = \varepsilon f(x), \quad \omega = \underline{\omega} + \delta\underline{\omega} = (1+\delta)\underline{\omega},$$

and study the effect of this on (5.2) and (5.3). These become

$$\Delta u + 2i\underline{\alpha}\partial_x u + (\underline{\gamma}^u)^2 u = \tilde{F}(x,z), \qquad 0 < z < a, \tag{5.4a}$$

$$u(x,0) = U(x), \qquad z = 0, \tag{5.4b}$$

$$\partial_z u(x,a) - T_0^u[u(x,a)] = \tilde{P}(x), \qquad z = a, \tag{5.4c}$$

$$u(x+d,z) = u(x,z), \tag{5.4d}$$

and

$$G(\varepsilon f)[U] = -\partial_z u(x,0) + \tilde{H}(x), \tag{5.5}$$

where $\tilde{F}, \tilde{P}, \tilde{H} = \mathcal{O}(\varepsilon) + \mathcal{O}(\delta)$. More specifically,

$$\begin{aligned}\tilde{F} = & -\varepsilon \mathrm{div}\,[A_1(f)\nabla u] - \varepsilon^2 \mathrm{div}\,[A_2(f)\nabla u] - \varepsilon B_1(f)\nabla u - \varepsilon^2 B_2(f)\nabla u \\ & - 2i\underline{\alpha}\delta\partial_x u - \delta^2(\underline{\gamma}^u)^2 u - 2\delta(\underline{\gamma}^u)^2 u \\ & - 2i\varepsilon S_1(f)\underline{\alpha}\partial_x u - 2i\varepsilon S_1(f)\underline{\alpha}\delta\partial_x u - \varepsilon S_1(f)\delta^2(\underline{\gamma}^u)^2 u \\ & - 2\varepsilon S_1(f)\delta(\underline{\gamma}^u)^2 u - \varepsilon S_1(f)(\underline{\gamma}^u)^2 u \\ & - 2i\varepsilon^2 S_2(f)\underline{\alpha}\partial_x u - 2i\varepsilon^2 S_2(f)\underline{\alpha}\delta\partial_x u - \varepsilon^2 S_2(f)\delta^2(\underline{\gamma}^u)^2 u \\ & - 2\varepsilon^2 S_2(f)\delta(\underline{\gamma}^u)^2 u - \varepsilon^2 S_2(f)(\underline{\gamma}^u)^2 u,\end{aligned} \tag{5.6}$$

and

$$\tilde{P} = -\frac{1}{a}(\varepsilon f(x)) T^u\left[u(x,a)\right] + \left(T^u - T_0^u\right)\left[u(x,a)\right], \tag{5.7}$$

and

$$\tilde{H} = \varepsilon(\partial_x f)\partial_x u(x,0) + \varepsilon \frac{f}{a} G(\varepsilon f)[U] - \varepsilon^2 \frac{f(\partial_x f)}{a}\partial_x u(x,0) - \varepsilon^2(\partial_x f)^2 \partial_z u(x,0). \tag{5.8}$$

It is not difficult to see that the forms for the A_j, B_j, and S_j are

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.9a}$$

$$A_1(f) = \begin{pmatrix} A_1^{xx} & A_1^{xz} \\ A_1^{zx} & A_1^{zz} \end{pmatrix} = \frac{1}{a}\begin{pmatrix} -2f & -(a-z)(\partial_x f) \\ -(a-z)(\partial_x f) & 0 \end{pmatrix}, \tag{5.9b}$$

$$A_2(f) = \begin{pmatrix} A_2^{xx} & A_2^{xz} \\ A_2^{zx} & A_2^{zz} \end{pmatrix} = \frac{1}{a^2}\begin{pmatrix} f^2 & (a-z)f(\partial_x f) \\ (a-z)f(\partial_x f) & (a-z)^2(\partial_x f)^2 \end{pmatrix}, \tag{5.9c}$$

and

$$B_1(f) = \begin{pmatrix} B_1^x \\ B_1^z \end{pmatrix} = \frac{1}{a}\begin{pmatrix} \partial_x f \\ 0 \end{pmatrix}, \quad B_2(f) = \begin{pmatrix} B_2^x \\ B_2^z \end{pmatrix} = \frac{1}{a^2}\begin{pmatrix} -f(\partial_x f) \\ -(a-z)(\partial_x f)^2 \end{pmatrix}, \tag{5.10}$$

and

$$S_0 = 1, \quad S_1(f) = -\frac{2}{a} f, \quad S_2(f) = \frac{1}{a^2} f^2. \tag{5.11}$$

At this point we posit the expansions

$$u(x,z;\varepsilon,\delta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} u_{n,m}(x,z)\varepsilon^n\delta^m, \quad G(\varepsilon,\delta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} G_{n,m}\varepsilon^n\delta^m,$$

and, upon insertion into (5.4) and (5.5), we find

$$\Delta u_{n,m} + 2i\underline{\alpha}\partial_x u_{n,m} + (\underline{\gamma}^u)^2 u_{n,m} = \tilde{F}_{n,m}(x,z), \qquad 0 < z < a, \tag{5.12a}$$

$$u_{n,m}(x,0) = U_{n,m}(x), \qquad z = 0, \tag{5.12b}$$

$$\partial_z u_{n,m}(x,a) - T_0^u[u_{n,m}(x,a)] = \tilde{P}_{n,m}(x), \qquad z = a, \tag{5.12c}$$

$$u_{n,m}(x+d,z) = u_{n,m}(x,z), \tag{5.12d}$$

and

$$G_{n,m}(f) = -\partial_z u_{n,m}(x,0) + \tilde{H}_{n,m}(x). \tag{5.13}$$

The formulas for $\tilde{F}_{n,m}$, $\tilde{P}_{n,m}$ and $\tilde{H}_{n,m}$ can be readily derived from (5.6), (5.7), and (5.8) giving

$$\begin{aligned}
\tilde{F}_{n,m} = &-\operatorname{div}\left[A_1(f)\nabla u_{n-1,m}\right] - \operatorname{div}\left[A_2(f)\nabla u_{n-2,m}\right] \\
&- B_1(f)\nabla u_{n-1,m} - B_2(f)\nabla u_{n-2,m} \\
&- 2i\underline{\alpha}\partial_x u_{n,m-1} - (\underline{\gamma}^u)^2 u_{n,m-2} - 2(\underline{\gamma}^u)^2 u_{n,m-1} \\
&- 2iS_1(f)\underline{\alpha}\partial_x u_{n-1,m} - 2iS_1(f)\underline{\alpha}\partial_x u_{n-1,m-1} - S_1(f)(\underline{\gamma}^u)^2 u_{n-1,m-2} \\
&- 2S_1(f)(\underline{\gamma}^u)^2 u_{n-1,m-1} - S_1(f)(\underline{\gamma}^u)^2 u_{n-1,m} \\
&- 2iS_2(f)\underline{\alpha}\partial_x u_{n-2,m} - 2iS_2(f)\underline{\alpha}\partial_x u_{n-2,m-1} - S_2(f)(\underline{\gamma}^u)^2 u_{n-2,m-2} \\
&- 2S_2(f)(\underline{\gamma}^u)^2 u_{n-2,m-1} - S_2(f)(\underline{\gamma}^u)^2 u_{n-2,m},
\end{aligned} \tag{5.14}$$

and

$$\tilde{P}_{n,m} = -\frac{1}{a} f(x) \sum_{r=0}^{m} T_{m-r}^u\left[u_{n-1,r}(x,a)\right] + \sum_{r=0}^{m-1} T_{m-r}^u\left[u_{n,r}(x,a)\right], \tag{5.15}$$

and

$$\begin{aligned}
\tilde{H}_{n,m} = &(\partial_x f)\partial_x u_{n-1,m}(x,0) + \frac{f}{a} G_{n-1,m}(f)[U] - \frac{f(\partial_x f)}{a}\partial_x u_{n-2,m}(x,0) \\
&- (\partial_x f)^2 \partial_z u_{n-2,m}(x,0).
\end{aligned} \tag{5.16}$$

5.2. Geometric Analyticity of the Upper Field. To prove our joint analyticity result we begin by stating the single, geometric, analyticity result for the field u under boundary perturbation, ε, alone. This was essentially established in [56] but we present it here for completeness.

THEOREM 5.1. *Given any integer $s \geq 0$, if $f \in C^{s+2}([0,d])$ and $U_{n,0} \in H^{s+3/2}([0,d])$ such that*

$$\|U_{n,0}\|_{H^{s+3/2}} \leq K_U B_U^n, \tag{5.17}$$

for constants $K_U, B_U > 0$, then $u_{n,0} \in H^{s+2}([0,d] \times [0,a])$ and

$$\|u_{n,0}\|_{H^{s+2}} \leq K B^n, \tag{5.18}$$

for constants $K, B > 0$.

To establish this we work by induction and the key estimate is the following Lemma.

LEMMA 5.2. *Given an integer $s \geq 0$, if $f \in C^{s+2}([0,d])$ and*

$$\|u_{n,0}\|_{H^{s+2}} \leq K B^n, \quad \forall n < \overline{n}, \tag{5.19}$$

for constants $K, B > 0$, then there exists a constant $\overline{C} > 0$ such that

$$\max\left\{ \left\| \tilde{F}_{\overline{n},0} \right\|_{H^s}, \left\| \tilde{P}_{\overline{n},0} \right\|_{H^{s+1/2}} \right\} \leq K\overline{C} \left\{ |f|_{C^{s+2}} B^{\overline{n}-1} + |f|^2_{C^{s+2}} B^{\overline{n}-2} \right\}. \tag{5.20}$$

Proof. [Lemma 5.2] We begin with $\tilde{F}_{\overline{n},0}$ and note that from (5.14), (5.9), (5.10), and (5.11) we have

$$\begin{aligned}
\|\tilde{F}_{\overline{n},0}\|^2_{H^s} \leq{}& \|A_1^{xx}\partial_x u_{\overline{n}-1,0}\|^2_{H^{s+1}} + \|A_1^{xz}\partial_z u_{\overline{n}-1,0}\|^2_{H^{s+1}} + \|A_1^{zx}\partial_x u_{\overline{n}-1,0}\|^2_{H^{s+1}} \\
&+ \|A_1^{zz}\partial_z u_{\overline{n}-1,0}\|^2_{H^{s+1}} + \|A_2^{xx}\partial_x u_{\overline{n}-2,0}\|^2_{H^{s+1}} + \|A_2^{xz}\partial_z u_{\overline{n}-2,0}\|^2_{H^{s+1}} \\
&+ \|A_2^{zx}\partial_x u_{\overline{n}-2,0}\|^2_{H^{s+1}} + \|A_2^{zz}\partial_z u_{\overline{n}-2,0}\|^2_{H^{s+1}} + \|B_1^{x}\partial_x u_{\overline{n}-1,0}\|^2_{H^{s}} \\
&+ \|B_1^{z}\partial_z u_{\overline{n}-1,0}\|^2_{H^{s}} + \|B_2^{x}\partial_x u_{\overline{n}-2,0}\|^2_{H^{s}} + \|B_2^{z}\partial_z u_{\overline{n}-2,0}\|^2_{H^{s}} \\
&+ \|2S_1 i\underline{\alpha}\partial_x u_{\overline{n}-1,0}\|^2_{H^{s}} + \|S_1(\underline{\gamma}^u)^2 u_{\overline{n}-1,0}\|^2_{H^{s}} + \|2S_2 i\underline{\alpha}\partial_x u_{\overline{n}-2,0}\|^2_{H^{s}} \\
&+ \|S_2(\underline{\gamma}^u)^2 u_{\overline{n}-2,0}\|^2_{H^{s}}.
\end{aligned}$$

We now estimate each of these by applying Lemmas 4.4 and 4.6. We begin with

$$\begin{aligned}
\|A_1^{xx}\partial_x u_{\overline{n}-1,0}\|_{H^{s+1}} &= \|-(2/a)f\partial_x u_{\overline{n}-1,0}\|_{H^{s+1}} \\
&\leq (2/a)\mathcal{M}|f|_{C^{s+1}}\|u_{\overline{n}-1,0}\|_{H^{s+2}} \\
&\leq (2/a)\mathcal{M}|f|_{C^{s+1}} K B^{\overline{n}-1},
\end{aligned}$$

and in a similar fashion

$$\begin{aligned}
\|A_1^{xz}\partial_z u_{\overline{n}-1,0}\|_{H^{s+1}} &= \|-((a-z)/a)(\partial_x f)\partial_z u_{\overline{n}-1,0}\|_{H^{s+1}} \\
&\leq (Z_a/a)\mathcal{M}|\partial_x f|_{C^{s+1}}\|u_{\overline{n}-1,0}\|_{H^{s+2}} \\
&\leq (Z_a/a)\mathcal{M}|f|_{C^{s+2}} K B^{\overline{n}-1}.
\end{aligned}$$

Also,

$$\begin{aligned}
\|A_1^{zx}\partial_x u_{\overline{n}-1,0}\|_{H^{s+1}} &= \|-((a-z)/a)(\partial_x f)\partial_x u_{\overline{n}-1,0}\|_{H^{s+1}} \\
&\leq (Z_a/a)\mathcal{M}|\partial_x f|_{C^{s+1}}\|u_{\overline{n}-1,0}\|_{H^{s+2}} \\
&\leq (Z_a/a)\mathcal{M}|f|_{C^{s+2}} K B^{\overline{n}-1},
\end{aligned}$$

and we recall that $A_1^{zz} \equiv 0$. Moving to the second order

$$\begin{aligned}
\|A_2^{xx}\partial_x u_{\overline{n}-2,0}\|_{H^{s+1}} &= \|(1/a^2)f^2\partial_x u_{\overline{n}-2,0}\|_{H^{s+1}} \\
&\leq (1/a^2)\mathcal{M}^2|f|^2_{C^{s+1}}\|u_{\overline{n}-2,0}\|_{H^{s+2}} \\
&\leq (1/a^2)\mathcal{M}^2|f|^2_{C^{s+1}} K B^{\overline{n}-2}.
\end{aligned}$$

Also,

$$\begin{aligned}
\|A_2^{xz}\partial_z u_{\overline{n}-2,0}\|_{H^{s+1}} &= \|((a-z)/a^2)f(\partial_x f)\partial_x u_{\overline{n}-2,0}\|_{H^{s+1}} \\
&\leq (Z_a/a^2)\mathcal{M}^2|f|_{C^{s+1}}|\partial_x f|_{C^{s+1}}\|u_{\overline{n}-2,0}\|_{H^{s+2}} \\
&\leq (Z_a/a^2)\mathcal{M}^2|f|^2_{C^{s+2}} K B^{\overline{n}-2},
\end{aligned}$$

and

$$\begin{aligned}\|A_2^{zx}\partial_x u_{\overline{n}-2,0}\|_{H^{s+1}} &= \|((a-z)/a^2)f(\partial_x f)\partial_z u_{\overline{n}-2,0}\|_{H^{s+1}}\\ &\le (Z_a/a^2)\mathcal{M}^2|f|_{C^{s+1}}|\partial_x f|_{C^{s+1}}\|u_{\overline{n}-2,0}\|_{H^{s+2}}\\ &\le (Z_a/a^2)\mathcal{M}^2|f|^2_{C^{s+2}}KB^{\overline{n}-2},\end{aligned}$$

and

$$\begin{aligned}\|A_2^{zz}\partial_z u_{\overline{n}-2,0}\|_{H^{s+1}} &= \|((a-z)^2/a^2)(\partial_x f)^2\partial_z u_{\overline{n}-2,0}\|_{H^{s+1}}\\ &\le (Z_a^2/a^2)\mathcal{M}^2|\partial_x f|^2_{C^{s+1}}\|u_{\overline{n}-2,0}\|_{H^{s+2}}\\ &\le (Z_a^2/a^2)\mathcal{M}^2|f|^2_{C^{s+2}}KB^{\overline{n}-2}.\end{aligned}$$

Next for the B_1 terms

$$\begin{aligned}\|B_1^{x}\partial_x u_{\overline{n}-1,0}\|_{H^{s}} &= \|(1/a)(\partial_x f)\partial_x u_{\overline{n}-1,0}\|_{H^{s}}\\ &\le (1/a)\mathcal{M}|\partial_x f|_{C^{s}}\|u_{\overline{n}-1,0}\|_{H^{s+1}}\\ &\le (1/a)\mathcal{M}|f|_{C^{s+1}}KB^{\overline{n}-1},\end{aligned}$$

and $B_1^z \equiv 0$. Moving to the second order

$$\begin{aligned}\|B_2^{x}\partial_x u_{\overline{n}-2,0}\|_{H^{s}} &= \|(-1/a^2)f(\partial_x f)\partial_x u_{\overline{n}-2,0}\|_{H^{s}}\\ &\le (1/a^2)\mathcal{M}^2|f|_{C^{s}}|\partial_x f|_{C^{s}}\|u_{\overline{n}-2,0}\|_{H^{s+1}}\\ &\le (1/a^2)\mathcal{M}^2|f|^2_{C^{s+1}}KB^{\overline{n}-2},\end{aligned}$$

and

$$\begin{aligned}\|B_2^{z}\partial_z u_{\overline{n}-2,0}\|_{H^{s}} &= \|(-1/a^2)(a-z)(\partial_x f)^2\partial_z u_{\overline{n}-2,0}\|_{H^{s}}\\ &\le (Z_a/a^2)\mathcal{M}^2|\partial_x f|^2_{C^{s}}\|u_{\overline{n}-2,0}\|_{H^{s+1}}\\ &\le (Z_a/a^2)\mathcal{M}^2|f|^2_{C^{s+1}}KB^{\overline{n}-2}.\end{aligned}$$

To address the S_0, S_1, S_2 terms we have

$$\begin{aligned}\|2S_1 i\underline{\alpha}\partial_x u_{\overline{n}-1,0}\|_{H^{s}} &= \|(-4/a)i\underline{\alpha}f\partial_x u_{\overline{n}-1,0}\|_{H^{s}}\\ &\le (4/a)\underline{\alpha}\mathcal{M}|f|_{C^{s}}\|u_{\overline{n}-1,0}\|_{H^{s+1}}\\ &\le (4/a)\underline{\alpha}\mathcal{M}|f|_{C^{s}}KB^{\overline{n}-1},\end{aligned}$$

and

$$\begin{aligned}\|S_1(\underline{\gamma}^u)^2 u_{\overline{n}-1,0}\|_{H^{s}} &= \|(-2/a)(\underline{\gamma}^u)^2 f u_{\overline{n}-1,0}\|_{H^{s}}\\ &\le (2/a)(\underline{\gamma}^u)^2\mathcal{M}|f|_{C^{s}}\|u_{\overline{n}-1,0}\|_{H^{s}}\\ &\le (2/a)(\underline{\gamma}^u)^2\mathcal{M}|f|_{C^{s}}KB^{\overline{n}-1},\end{aligned}$$

and

$$\begin{aligned}\|2S_2 i\underline{\alpha}\partial_x u_{\overline{n}-2,0}\|_{H^{s}} &= \|(2/a^2)i\underline{\alpha}f^2\partial_x u_{\overline{n}-2,0}\|_{H^{s}}\\ &\le (2/a^2)\underline{\alpha}\mathcal{M}^2|f|^2_{C^{s}}\|u_{\overline{n}-2,0}\|_{H^{s+1}}\\ &\le (2/a^2)\underline{\alpha}\mathcal{M}^2|f|^2_{C^{s}}KB^{\overline{n}-2},\end{aligned}$$

and

$$\begin{aligned}\|S_2(\underline{\gamma}^u)^2 u_{\overline{n}-2,0}\|_{H^s} &= \|(1/a^2)(\underline{\gamma}^u)^2 f^2 u_{\overline{n}-2,0}\|_{H^s} \\ &\leq (1/a^2)(\underline{\gamma}^u)^2 \mathcal{M}^2 |f|^2_{C^s} \|u_{\overline{n}-2,0}\|_{H^s} \\ &\leq (1/a^2)(\underline{\gamma}^u)^2 \mathcal{M}^2 |f|^2_{C^s} K B^{\overline{n}-2}.\end{aligned}$$

We satisfy the estimate for $\|\tilde{F}_{\overline{n},0}\|_{H^s}$ provided that we choose

$$\overline{C} > \max\left\{ \left(\frac{3 + 2Z_a + 4\underline{\alpha} + 2(\underline{\gamma}^u)^2}{a}\right)\mathcal{M}, \left(\frac{2 + 3Z_a + Z_a^2 + 2\underline{\alpha} + (\underline{\gamma}^u)^2}{a^2}\right)\mathcal{M}^2 \right\}.$$

The estimate for $\tilde{P}_{\overline{n},0}$ follows from an elementary estimate on the order–one Fourier multiplier T_0^u

$$\begin{aligned}\|\tilde{P}_{\overline{n},0}\|_{H^{s+1/2}} &= \| -(1/a) f T_0^u \left[u_{\overline{n}-1,0}\right] \|_{H^{s+1/2}} \\ &\leq (1/a)\mathcal{M}|f|_{C^{s+1/2+\eta}} \|T_0^u \left[u_{\overline{n}-1,0}\right]\|_{H^{s+1/2}} \\ &\leq (1/a)\mathcal{M}|f|_{C^{s+1/2+\eta}} C_{T_0^u} \|u_{\overline{n}-1,0}\|_{H^{s+3/2}} \\ &\leq (1/a)\mathcal{M}|f|_{C^{s+1/2+\eta}} C_{T_0^u} K B^{\overline{n}-1},\end{aligned}$$

and provided that

$$\overline{C} > (1/a)\mathcal{M} C_{T_0^u},$$

we are done. □

With this information, we can now prove Theorem 5.1.

Proof. [Theorem 5.1] We proceed by induction in n and at order $n = 0$ and $m = 0$ Theorem 4.5 guarantees a unique solution such that

$$\|u_{0,0}\|_{H^{s+2}} \leq C_e \|U_{0,0}\|_{H^{s+3/2}}.$$

So we choose $K \geq C_e \|U_{0,0}\|_{H^{s+3/2}}$. We now assume the estimate (5.18) for all $n < \overline{n}$ and study $u_{\overline{n},0}$. From Theorem 4.5 we have a unique solution satisfying

$$\|u_{\overline{n},0}\|_{H^{s+2}} \leq C_e \{ \|\tilde{F}_{\overline{n},0}\|_{H^s} + \|U_{\overline{n},0}\|_{H^{s+3/2}} + \|\tilde{P}_{\overline{n},0}\|_{H^{s+1/2}} \},$$

and appealing to the hypothesis (5.17) and Lemma 5.2 we find

$$\|u_{\overline{n},0}\|_{H^{s+2}} \leq C_e \{ K_U B_U^{\overline{n}} + 2K\overline{C}\left[|f|_{C^{s+2}} B^{\overline{n}-1} + |f|^2_{C^{s+2}} B^{\overline{n}-2} \right] \}.$$

We are done provided we choose $K \geq 3C_e K_U$ and

$$B > \max\left\{ B_U, 6C_e\overline{C}|f|_{C^{s+2}}, \sqrt{6C_e\overline{C}}|f|_{C^{s+2}} \right\}.$$ □

Analogous results hold in the lower field which we record here for completeness.

THEOREM 5.3. *Given any integer $s \geq 0$, if $f \in C^{s+2}([0,d])$ and $W_{n,0} \in H^{s+3/2}([0,d])$ such that*

$$\|W_{n,0}\|_{H^{s+3/2}} \leq K_W B_W^n,$$

for constants $K_W, B_W > 0$*, then* $w_{n,0} \in H^{s+2}([0,d] \times [-b,0])$ *and*

$$\|w_{n,0}\|_{H^{s+2}} \leq K B^n,$$

for constants $K, B > 0$*.*

5.3. Joint Analyticity of the Upper Field. We can now proceed to prove our main result concerning joint analyticity of the transformed field.

THEOREM 5.4. *Given any integer* $s \geq 0$*, if* $f \in C^{s+2}([0,d])$ *and* $U_{n,m} \in H^{s+3/2}([0,d])$ *such that*

$$\|U_{n,m}\|_{H^{s+3/2}} \leq K_U B_U^n D_U^m, \tag{5.21}$$

for constants $K_U, B_U, D_U > 0$*, then* $u_{n,m} \in H^{s+2}([0,d] \times [0,a])$ *and*

$$\|u_{n,m}\|_{H^{s+2}} \leq K B^n D^m, \tag{5.22}$$

for constants $K, B, D > 0$*.*

As before, we establish this result by induction.

LEMMA 5.5. *Given an integer* $s \geq 0$*, if* $f \in C^{s+2}([0,d])$ *and*

$$\|u_{n,m}\|_{H^{s+2}} \leq K B^n D^m, \quad \forall n \geq 0, m < \overline{m}, \tag{5.23}$$

for constants $K, B, D > 0$ *then there exists a constant* $\overline{C} > 0$ *such that*

$$\begin{aligned}
\max\{\|\tilde{F}_{n,\overline{m}}\|_{H^s}, \|\tilde{P}_{n,\overline{m}}\|_{H^{s+1/2}}\} \leq K\overline{C}\Big\{ & B^n D^{\overline{m}-1} + B^n D^{\overline{m}-2} + |f|_{C^{s+2}} B^{n-1} D^{\overline{m}} + \\
& |f|_{C^{s+2}} B^{n-1} D^{\overline{m}-1} + |f|_{C^{s+2}} B^{n-1} D^{\overline{m}-2} + |f|_{C^{s+2}}^2 B^{n-2} D^{\overline{m}} + \\
& |f|_{C^{s+2}}^2 B^{n-2} D^{\overline{m}-1} + |f|_{C^{s+2}}^2 B^{n-2} D^{\overline{m}-2} \Big\}.
\end{aligned}$$

Proof. [Lemma 5.5] We begin with $\tilde{F}_{n,\overline{m}}$ and note that from (5.14), (5.9), (5.10), and (5.11) we have

$$\begin{aligned}
\|\tilde{F}_{n,\overline{m}}\|_{H^s}^2 \leq\ & \|A_1^{xx}\partial_x u_{n-1,\overline{m}}\|_{H^{s+1}}^2 + \|A_1^{xz}\partial_z u_{n-1,\overline{m}}\|_{H^{s+1}}^2 + \|A_1^{zx}\partial_x u_{n-1,\overline{m}}\|_{H^{s+1}}^2 \\
& + \|A_1^{zz}\partial_z u_{n-1,\overline{m}}\|_{H^{s+1}}^2 + \|A_2^{xx}\partial_x u_{n-2,\overline{m}}\|_{H^{s+1}}^2 + \|A_2^{xz}\partial_z u_{n-2,\overline{m}}\|_{H^{s+1}}^2 \\
& + \|A_2^{zx}\partial_x u_{n-2,\overline{m}}\|_{H^{s+1}}^2 + \|A_2^{zz}\partial_z u_{n-2,\overline{m}}\|_{H^{s+1}}^2 + \|B_1^{x}\partial_x u_{n-1,\overline{m}}\|_{H^{s}}^2 \\
& + \|B_1^{z}\partial_z u_{n-1,\overline{m}}\|_{H^{s}}^2 + \|B_2^{x}\partial_x u_{n-2,\overline{m}}\|_{H^{s}}^2 + \|B_2^{z}\partial_z u_{n-2,\overline{m}}\|_{H^{s}}^2 \\
& + \|2i\underline{\alpha}\partial_x u_{n,\overline{m}-1}\|_{H^s}^2 + \|(\underline{\gamma}^u)^2 u_{n,\overline{m}-2}\|_{H^s}^2 + \|2(\underline{\gamma}^u)^2 u_{n,\overline{m}-1}\|_{H^s}^2 \\
& + \|2S_1 i\underline{\alpha}\partial_x u_{n-1,\overline{m}}\|_{H^s}^2 + \|2S_1 i\underline{\alpha}\partial_x u_{n-1,\overline{m}-1}\|_{H^s}^2 + \|S_1(\underline{\gamma}^u)^2 u_{n-1,\overline{m}-2}\|_{H^s}^2 \\
& + \|2S_1(\underline{\gamma}^u)^2 u_{n-1,\overline{m}-1}\|_{H^s}^2 + \|S_1(\underline{\gamma}^u)^2 u_{n-1,\overline{m}}\|_{H^s}^2 + \|2S_2 i\underline{\alpha}\partial_x u_{n-2,\overline{m}}\|_{H^s}^2 \\
& + \|2S_2 i\underline{\alpha}\partial_x u_{n-2,\overline{m}-1}\|_{H^s}^2 + \|S_2(\underline{\gamma}^u)^2 u_{n-2,\overline{m}-2}\|_{H^s}^2 \\
& + \|2S_2(\underline{\gamma}^u)^2 u_{n-2,\overline{m}-1}\|_{H^s}^2 + \|S_2(\underline{\gamma}^u)^2 u_{n-2,\overline{m}}\|_{H^s}^2.
\end{aligned}$$

We now estimate each of these by applying Lemmas 4.4 and 4.6. We begin with

$$\begin{aligned}
\|A_1^{xx}\partial_x u_{n-1,\overline{m}}\|_{H^{s+1}} &= \|-(2/a) f \partial_x u_{n-1,\overline{m}}\|_{H^{s+1}} \\
&\leq (2/a)\mathcal{M}|f|_{C^{s+1}}\|u_{n-1,\overline{m}}\|_{H^{s+2}} \\
&\leq (2/a)\mathcal{M}|f|_{C^{s+1}} K B^{n-1} D^{\overline{m}},
\end{aligned}$$

and in a similar fashion

$$\begin{aligned}\|A_1^{xz}\partial_z u_{n-1,\overline{m}}\|_{H^{s+1}} &= \|-((a-z)/a)(\partial_x f)\partial_z u_{n-1,\overline{m}}\|_{H^{s+1}} \\ &\le (Z_a/a)\mathcal{M}|\partial_x f|_{C^{s+1}}\|u_{n-1,\overline{m}}\|_{H^{s+2}} \\ &\le (Z_a/a)\mathcal{M}|f|_{C^{s+2}} K B^{n-1} D^{\overline{m}}.\end{aligned}$$

Also,

$$\begin{aligned}\|A_1^{zx}\partial_x u_{n-1,\overline{m}}\|_{H^{s+1}} &= \|-((a-z)/a)(\partial_x f)\partial_x u_{n-1,\overline{m}}\|_{H^{s+1}} \\ &\le (Z_a/a)\mathcal{M}|\partial_x f|_{C^{s+1}}\|u_{n-1,\overline{m}}\|_{H^{s+2}} \\ &\le (Z_a/a)\mathcal{M}|f|_{C^{s+2}} K B^{n-1} D^{\overline{m}},\end{aligned}$$

and we recall that $A_1^{zz} \equiv 0$. Moving to the second order

$$\begin{aligned}\|A_2^{xx}\partial_x u_{n-2,\overline{m}}\|_{H^{s+1}} &= \|(1/a^2) f^2 \partial_x u_{n-2,\overline{m}}\|_{H^{s+1}} \\ &\le (1/a^2)\mathcal{M}^2|f|^2_{C^{s+1}}\|u_{n-2,\overline{m}}\|_{H^{s+2}} \\ &\le (1/a^2)\mathcal{M}^2|f|^2_{C^{s+1}} K B^{n-2} D^{\overline{m}}.\end{aligned}$$

Also,

$$\begin{aligned}\|A_2^{xz}\partial_z u_{n-2,\overline{m}}\|_{H^{s+1}} &= \|((a-z)/a^2) f(\partial_x f)\partial_x u_{n-2,\overline{m}}\|_{H^{s+1}} \\ &\le (Z_a/a^2)\mathcal{M}^2|f|_{C^{s+1}}|\partial_x f|_{C^{s+1}}\|u_{n-2,\overline{m}}\|_{H^{s+2}} \\ &\le (Z_a/a^2)\mathcal{M}^2|f|^2_{C^{s+2}} K B^{n-2} D^{\overline{m}},\end{aligned}$$

and

$$\begin{aligned}\|A_2^{zx}\partial_x u_{n-2,\overline{m}}\|_{H^{s+1}} &= \|((a-z)/a^2) f(\partial_x f)\partial_z u_{n-2,\overline{m}}\|_{H^{s+1}} \\ &\le (Z_a/a^2)\mathcal{M}^2|f|_{C^{s+1}}|\partial_x f|_{C^{s+1}}\|u_{n-2,\overline{m}}\|_{H^{s+2}} \\ &\le (Z_a/a^2)\mathcal{M}^2|f|^2_{C^{s+2}} K B^{n-2} D^{\overline{m}},\end{aligned}$$

and

$$\begin{aligned}\|A_2^{zz}\partial_z u_{n-2,\overline{m}}\|_{H^{s+1}} &= \|((a-z)^2/a^2)(\partial_x f)^2\partial_z u_{n-2,\overline{m}}\|_{H^{s+1}} \\ &\le (Z_a^2/a^2)\mathcal{M}^2|\partial_x f|^2_{C^{s+1}}\|u_{n-2,\overline{m}}\|_{H^{s+2}} \\ &\le (Z_a^2/a^2)\mathcal{M}^2|f|^2_{C^{s+2}} K B^{n-2} D^{\overline{m}}.\end{aligned}$$

Next for the B_1 terms

$$\begin{aligned}\|B_1^{x}\partial_x u_{n-1,\overline{m}}\|_{H^{s}} &= \|(1/a)(\partial_x f)\partial_x u_{n-1,\overline{m}}\|_{H^{s}} \\ &\le (1/a)\mathcal{M}|\partial_x f|_{C^{s}}\|u_{n-1,\overline{m}}\|_{H^{s+1}} \\ &\le (1/a)\mathcal{M}|f|_{C^{s+1}} K B^{n-1} D^{\overline{m}},\end{aligned}$$

and $B_1^z \equiv 0$. Moving to the second order

$$\begin{aligned}\|B_2^{x}\partial_x u_{n-2,\overline{m}}\|_{H^{s}} &= \|(-1/a^2) f(\partial_x f)\partial_x u_{n-2,\overline{m}}\|_{H^{s}} \\ &\le (1/a^2)\mathcal{M}^2|f|_{C^{s}}|\partial_x f|_{C^{s}}\|u_{n-2,\overline{m}}\|_{H^{s+1}} \\ &\le (1/a^2)\mathcal{M}^2|f|^2_{C^{s+1}} K B^{n-2} D^{\overline{m}},\end{aligned}$$

and

$$\begin{aligned}\|B_2^z \partial_z u_{n-2,\overline{m}}\|_{H^s} &= \|(-1/a^2)(a-z)(\partial_x f)^2 \partial_z u_{n-2,\overline{m}}\|_{H^s} \\ &\leq (Z_a/a^2)\mathcal{M}^2 |\partial_x f|_{C^s}^2 \|u_{n-2,\overline{m}}\|_{H^{s+1}} \\ &\leq (Z_a/a^2)\mathcal{M}^2 |f|_{C^{s+1}}^2 K B^{n-2} D^{\overline{m}}.\end{aligned}$$

To address the S_0, S_1, S_2 terms we have

$$\begin{aligned}\|2i\underline{\alpha}\partial_x u_{n,\overline{m}-1}\|_{H^s} &\leq 2\underline{\alpha}\|u_{n,\overline{m}-1}\|_{H^{s+1}} \\ &\leq 2\underline{\alpha} K B^n D^{\overline{m}-1},\end{aligned}$$

and

$$\begin{aligned}\|(\underline{\gamma}^u)^2 u_{n,\overline{m}-2}\|_{H^s} &\leq (\underline{\gamma}^u)^2 \|u_{n,\overline{m}-2}\|_{H^s} \\ &\leq (\underline{\gamma}^u)^2 K B^n D^{\overline{m}-2},\end{aligned}$$

and

$$\begin{aligned}\|2(\underline{\gamma}^u)^2 u_{n,\overline{m}-1}\|_{H^s} &\leq 2(\underline{\gamma}^u)^2 \|u_{n,\overline{m}-1}\|_{H^s} \\ &\leq 2(\underline{\gamma}^u)^2 K B^n D^{\overline{m}-1},\end{aligned}$$

and

$$\begin{aligned}\|2S_1 i\underline{\alpha}\partial_x u_{n-1,\overline{m}}\|_{H^s} &= \|(-4/a) i\underline{\alpha} f \partial_x u_{n-1,\overline{m}}\|_{H^s} \\ &\leq (4/a)\underline{\alpha}\mathcal{M}|f|_{C^s}\|u_{n-1,\overline{m}}\|_{H^{s+1}} \\ &\leq (4/a)\underline{\alpha}\mathcal{M}|f|_{C^s} K B^{n-1} D^{\overline{m}},\end{aligned}$$

and

$$\begin{aligned}\|2S_1 i\underline{\alpha}\partial_x u_{n-1,\overline{m}-1}\|_{H^s} &= \|(-4/a) i\underline{\alpha} f \partial_x u_{n-1,\overline{m}-1}\|_{H^s} \\ &\leq (4/a)\underline{\alpha}\mathcal{M}|f|_{C^s}\|u_{n-1,\overline{m}-1}\|_{H^{s+1}} \\ &\leq (4/a)\underline{\alpha}\mathcal{M}|f|_{C^s} K B^{n-1} D^{\overline{m}-1},\end{aligned}$$

and

$$\begin{aligned}\|S_1(\underline{\gamma}^u)^2 u_{n-1,\overline{m}-2}\|_{H^s} &= \|(-2/a)(\underline{\gamma}^u)^2 f u_{n-1,\overline{m}-2}\|_{H^s} \\ &\leq (2/a)(\underline{\gamma}^u)^2\mathcal{M}|f|_{C^s}\|u_{n-1,\overline{m}-2}\|_{H^s} \\ &\leq (2/a)(\underline{\gamma}^u)^2\mathcal{M}|f|_{C^s} K B^{n-1} D^{\overline{m}-2},\end{aligned}$$

and

$$\begin{aligned}\|2S_1(\underline{\gamma}^u)^2 u_{n-1,\overline{m}-1}\|_{H^s} &= \|(-4/a)(\underline{\gamma}^u)^2 f u_{n-1,\overline{m}-1}\|_{H^s} \\ &\leq (4/a)(\underline{\gamma}^u)^2\mathcal{M}|f|_{C^s}\|u_{n-1,\overline{m}-1}\|_{H^s} \\ &\leq (4/a)(\underline{\gamma}^u)^2\mathcal{M}|f|_{C^s} K B^{n-1} D^{\overline{m}-1},\end{aligned}$$

and

$$\begin{aligned}\|S_1(\underline{\gamma}^u)^2 u_{n-1,\overline{m}}\|_{H^s} &= \|(-2/a)(\underline{\gamma}^u)^2 f u_{n-1,\overline{m}}\|_{H^s} \\ &\leq (2/a)(\underline{\gamma}^u)^2\mathcal{M}|f|_{C^s}\|u_{n-1,\overline{m}}\|_{H^s} \\ &\leq (2/a)(\underline{\gamma}^u)^2\mathcal{M}|f|_{C^s} K B^{n-1} D^{\overline{m}},\end{aligned}$$

and

$$\begin{aligned}\|2S_2 i\underline{\alpha}\partial_x u_{n-2,\overline{m}}\|_{H^s} &= \|(2/a^2)i\underline{\alpha}f^2\partial_x u_{n-2,\overline{m}}\|_{H^s}\\ &\leq (2/a^2)\underline{\alpha}\mathcal{M}^2|f|_{C^s}^2\|u_{n-2,\overline{m}}\|_{H^{s+1}}\\ &\leq (2/a^2)\underline{\alpha}\mathcal{M}^2|f|_{C^s}^2 KB^{n-2}D^{\overline{m}},\end{aligned}$$

and

$$\begin{aligned}\|2S_2 i\underline{\alpha}\partial_x u_{n-2,\overline{m}-1}\|_{H^s} &= \|(2/a^2)i\underline{\alpha}f^2\partial_x u_{n-2,\overline{m}-1}\|_{H^s}\\ &\leq (2/a^2)\underline{\alpha}\mathcal{M}^2|f|_{C^s}^2\|u_{n-2,\overline{m}-1}\|_{H^{s+1}}\\ &\leq (2/a^2)\underline{\alpha}\mathcal{M}^2|f|_{C^s}^2 KB^{n-2}D^{\overline{m}-1},\end{aligned}$$

and

$$\begin{aligned}\|S_2(\underline{\gamma}^u)^2 u_{n-2,\overline{m}-2}\|_{H^s} &= \|(1/a^2)(\underline{\gamma}^u)^2 f^2 u_{n-2,\overline{m}-2}\|_{H^s}\\ &\leq (1/a^2)(\underline{\gamma}^u)^2\mathcal{M}^2|f|_{C^s}^2\|u_{n-2,\overline{m}-2}\|_{H^s}\\ &\leq (1/a^2)(\underline{\gamma}^u)^2\mathcal{M}^2|f|_{C^s}^2 KB^{n-2}D^{\overline{m}-2},\end{aligned}$$

and

$$\begin{aligned}\|2S_2(\underline{\gamma}^u)^2 u_{n-2,\overline{m}-1}\|_{H^s} &= \|(2/a^2)(\underline{\gamma}^u)^2 f^2 u_{n-2,\overline{m}-1}\|_{H^s}\\ &\leq (2/a^2)(\underline{\gamma}^u)^2\mathcal{M}^2|f|_{C^s}^2\|u_{n-2,\overline{m}-1}\|_{H^s}\\ &\leq (2/a^2)(\underline{\gamma}^u)^2\mathcal{M}^2|f|_{C^s}^2 KB^{n-2}D^{\overline{m}-1},\end{aligned}$$

and

$$\begin{aligned}\|S_2(\underline{\gamma}^u)^2 u_{n-2,\overline{m}}\|_{H^s} &= \|(1/a^2)(\underline{\gamma}^u)^2 f^2 u_{n-2,\overline{m}}\|_{H^s}\\ &\leq (1/a^2)(\underline{\gamma}^u)^2\mathcal{M}^2|f|_{C^s}^2\|u_{n-2,\overline{m}}\|_{H^s}\\ &\leq (1/a^2)(\underline{\gamma}^u)^2\mathcal{M}^2|f|_{C^s}^2 KB^{n-2}D^{\overline{m}}.\end{aligned}$$

We satisfy the estimate for $\|\tilde{F}_{n,\overline{m}}\|_{H^s}$ provided that we choose

$$\begin{aligned}\overline{C} > \max\Bigg\{&\left(2\underline{\alpha}+3(\underline{\gamma}^u)^2\right), \left(\frac{3+2Z_a+8\underline{\alpha}+8(\underline{\gamma}^u)^2}{a}\right)\mathcal{M},\\ &\left(\frac{2+3Z_a+Z_a^2+4\underline{\alpha}+4(\underline{\gamma}^u)^2}{a^2}\right)\mathcal{M}^2\Bigg\}.\end{aligned}$$

The estimate for $\tilde{P}_{n,\overline{m}}$ follows from the mapping properties of T^u,

$$\begin{aligned}\left\|\tilde{P}_{n,\overline{m}}\right\|_{H^{s+1/2}} &= \left\|-\frac{1}{a}f(x)\sum_{r=0}^{\overline{m}} T^u_{\overline{m}-r}\left[u_{n-1,r}\right] + \sum_{r=0}^{\overline{m}-1} T^u_{\overline{m}-r}\left[u_{n,r}\right]\right\|_{H^{s+1/2}}\\ &\leq (1/a)\mathcal{M}|f|_{C^{s+1/2+\eta}}\sum_{r=0}^{\overline{m}}\left\|T^u_{\overline{m}-r}\left[u_{n-1,r}\right]\right\|_{H^{s+1/2}} + \sum_{r=0}^{\overline{m}-1}\left\|T^u_{\overline{m}-r}\left[u_{n,r}\right]\right\|_{H^{s+1/2}}\\ &\leq (1/a)\mathcal{M}|f|_{C^{s+1/2+\eta}} C_{T^u}\sum_{r=0}^{\overline{m}}\|u_{n-1,r}\|_{H^{s+3/2}} + C_{T^u}\sum_{r=0}^{\overline{m}-1}\|u_{n,r}\|_{H^{s+3/2}}\\ &\leq (1/a)\mathcal{M}|f|_{C^{s+1/2+\eta}} C_{T^u}KB^{n-1}\left(\frac{D^{\overline{m}+1}-1}{D-1}\right) + C_{T^u}KB^n\left(\frac{D^{\overline{m}}-1}{D-1}\right),\end{aligned}$$

and provided that $D > 2$ and

$$\overline{C} > \max\left\{(1/a)\mathcal{M}C_{T^u}D, C_{T^u}D\right\}$$

we are done. □

With this information, we can now prove Theorem 5.4.

Proof. [Theorem 5.4] We proceed by induction in m and at order $m = 0$ Theorem 5.1 guarantees a unique solution such that

$$\|u_{n,0}\|_{H^{s+2}} \leq KB^n, \quad \forall n \geq 0.$$

We now assume the estimate (5.22) for all $n, m < \overline{m}$ and study $u_{n,\overline{m}}$. From Theorem 4.5 we have a unique solution satisfying

$$\|u_{n,\overline{m}}\|_{H^{s+2}} \leq C_e\{\|\tilde{F}_{n,\overline{m}}\|_{H^s} + \|U_{n,\overline{m}}\|_{H^{s+3/2}} + \|\tilde{P}_{n,\overline{m}}\|_{H^{s+1/2}}\},$$

and appealing to the hypothesis (5.21) and Lemma 5.5 we find

$$\begin{aligned} \|u_{n,\overline{m}}\|_{H^{s+2}} \leq C_e \Bigg\{ & K_U B_U^n D_U^{\overline{m}} + 2K\overline{C}\Bigg(B^n D^{\overline{m}-1} + B^n D^{\overline{m}-2} + |f|_{C^{s+2}} B^{n-1} D^{\overline{m}} + \\ & |f|_{C^{s+2}} B^{n-1} D^{\overline{m}-1} + |f|_{C^{s+2}} B^{n-1} D^{\overline{m}-2} + |f|_{C^{s+2}}^2 B^{n-2} D^{\overline{m}} + \\ & |f|_{C^{s+2}}^2 B^{n-2} D^{\overline{m}-1} + |f|_{C^{s+2}}^2 B^{n-2} D^{\overline{m}-2} \Bigg) \Bigg\}. \end{aligned}$$

We are done provided we choose $K \geq 9C_e K_U$ and

$$\begin{aligned} B &> \max\left\{B_U, 18C_e\overline{C}|f|_{C^{s+2}}, \sqrt{18C_e\overline{C}|f|_{C^{s+2}}}\right\}, \\ D &> \max\left\{1, D_U, 18C_e\overline{C}, \sqrt{18C_e\overline{C}}\right\}. \end{aligned}$$

□

As before, a similar analysis will establish the joint analyticity of the lower field which we now record.

THEOREM 5.6. *Given any integer $s \geq 0$, if $f \in C^{s+2}([0,d])$ and $W_{n,m} \in H^{s+3/2}([0,d])$ such that*

$$\|W_{n,m}\|_{H^{s+3/2}} \leq K_W B_W^n D_W^m,$$

for constants $K_W, B_W, D_W > 0$, then $w_{n,m} \in H^{s+2}([0,d] \times [-b,0])$ and

$$\|w_{n,m}\|_{H^{s+2}} \leq KB^n D^m,$$

for constants $K, B, D > 0$.

6. Analyticity of the Dirichlet–Neumann Operators. Now that we have established the joint analyticity of the upper field u we move to establishing the analyticity of the upper layer DNO, $G(g) = G(\varepsilon f)$. To begin we give a recursive estimate of the $\tilde{H}_{n,m}$ appearing in (5.16).

LEMMA 6.1. *Given an integer $s \geq 0$, if $f \in C^{s+2}([0,d])$ and*

$$\|u_{n,m}\|_{H^{s+2}} \leq K B^n D^m, \quad \|G_{n,m}\|_{H^{s+1/2}} \leq \tilde{K} \tilde{B}^n \tilde{D}^m, \quad \forall\ n < \bar{n}, m \geq 0, \tag{6.1}$$

for constants $K, B, D, \tilde{K}, \tilde{B}, \tilde{D} > 0$ where $\tilde{K} \geq K, \tilde{B} \geq B, \tilde{D} \geq D$, then there exists a constant $\tilde{C} > 0$ such that

$$\|\tilde{H}_{\bar{n},m}\|_{H^{s+1/2}} \leq \tilde{K}\tilde{C} \left\{ |f|_{C^{s+2}} \tilde{B}^{n-1} \tilde{D}^m + |f|^2_{C^{s+2}} \tilde{B}^{n-2} \tilde{D}^m \right\}. \tag{6.2}$$

Proof. [Lemma 6.1] From (5.16) we estimate

$$\begin{aligned}
\|\tilde{H}_{\bar{n},m}\|_{H^{s+1/2}} \leq\ & \mathcal{M} |\partial_x f|_{C^{s+1/2+\eta}} \|\partial_x u_{\bar{n}-1,m}(x,0)\|_{H^{s+1/2}} \\
& + \frac{1}{a} \mathcal{M} |f|_{C^{s+1/2+\eta}} \|G_{\bar{n}-1,m}(f)[U]\|_{H^{s+1/2}} \\
& + \frac{1}{a} \mathcal{M}^2 |f|_{C^{s+1/2+\eta}} |\partial_x f|_{C^{s+1/2+\eta}} \|\partial_x u_{\bar{n}-2,m}(x,0)\|_{H^{s+1/2}} \\
& + \mathcal{M}^2 |\partial_x f|^2_{C^{s+1/2+\eta}} \|\partial_z u_{\bar{n}-2,m}(x,0)\|_{H^{s+1/2}}.
\end{aligned}$$

This gives

$$\begin{aligned}
\|\tilde{H}_{\bar{n},m}\|_{H^{s+1/2}} \leq \tilde{K} \Big\{ & \mathcal{M} |f|_{C^{s+2}} \tilde{B}^{\bar{n}-1} \tilde{D}^m + \frac{1}{a} \mathcal{M} |f|_{C^{s+2}} \tilde{B}^{\bar{n}-1} \tilde{D}^m \\
& + \frac{1}{a} \mathcal{M}^2 |f|^2_{C^{s+2}} \tilde{B}^{\bar{n}-2} \tilde{D}^m + \mathcal{M}^2 |f|^2_{C^{s+2}} \tilde{B}^{\bar{n}-2} \tilde{D}^m \Big\},
\end{aligned}$$

and we are done provided

$$\tilde{C} \geq \left(1 + \frac{1}{a}\right) \max\{\mathcal{M}, \mathcal{M}^2\}.$$

□

We now have everything we need to prove the analyticity of the upper layer DNO.

THEOREM 6.2. *Given any integer $s \geq 0$, if $f \in C^{s+2}([0,d])$ and $U_{n,m} \in H^{s+3/2}([0,d])$ such that*

$$\|U_{n,m}\|_{H^{s+3/2}} \leq K_U B_U^n D_U^m,$$

for constants $K_U, B_U, D_U > 0$, then $G_{n,m} \in H^{s+1/2}([0,d])$ and

$$\|G_{n,m}\|_{H^{s+1/2}} \leq \tilde{K} \tilde{B}^n \tilde{D}^m, \tag{6.3}$$

for constants $\tilde{K}, \tilde{B}, \tilde{D} > 0$.

Proof. [Theorem 6.2] As before, we work by induction in n. At $n = 0$ we have from (5.13) that

$$G_{0,m} = -\partial_z u_{0,m}(x,0),$$

and from Theorem 5.4 we have

$$\|G_{0,m}\|_{H^{s+1/2}} = \|\partial_z u_{0,m}(x,0)\|_{H^{s+1/2}} \leq \|u_{0,m}\|_{H^{s+2}} \leq K D^m.$$

So we choose $\tilde{K} \geq K$ and $\tilde{D} \geq D$. We now assume $\tilde{B} \geq B$ and the estimate (6.3) for all $n < \bar{n}$; from (5.13) we have

$$\|G_{\bar{n},m}(f)[U]\|_{H^{s+1/2}} \leq \|\partial_z u_{\bar{n},m}(x,0)\|_{H^{s+1/2}} + \|\tilde{H}_{\bar{n},m}(x)\|_{H^{s+1/2}}.$$

Using the inductive hypothesis, Lemma 6.1, and Theorem 5.4 we have

$$\|G_{\overline{n},m}(f)[U]\|_{H^{s+1/2}} \leq K B^{\overline{n}} D^m + \tilde{K}\tilde{C}\left\{ |f|_{C^{s+2}} \tilde{B}^{\overline{n}-1}\tilde{D}^m + |f|^2_{C^{s+2}} \tilde{B}^{\overline{n}-2}\tilde{D}^m \right\}.$$

We are done provided $\tilde{K} \geq 2K$ and

$$\tilde{B} \geq \max\left\{ B, 4\tilde{C}|f|_{C^{s+2}}, 2\sqrt{\tilde{C}}|f|_{C^{s+2}} \right\}.$$

□

Finally, a similar approach will give the joint analyticity of the DNO in the lower field.

THEOREM 6.3. *Given any integer $s \geq 0$, if $f \in C^{s+2}([0,d])$ and $W_{n,m} \in H^{s+3/2}([0,d])$ such that*

$$\|W_{n,m}\|_{H^{s+3/2}} \leq K_W B_W^n D_W^m,$$

for constants $K_W, B_W, D_W > 0$, then $J_{n,m} \in H^{s+1/2}([0,d])$ and

$$\|J_{n,m}\|_{H^{s+1/2}} \leq \tilde{K}\tilde{B}^n\tilde{D}^m, \tag{6.4}$$

for constants $\tilde{K}, \tilde{B}, \tilde{D} > 0$.

Remark 6.4. For the *parametric*, (ε, δ), analyticity we investigate in this paper, the smoothness we assume of the interface, $f(x) \in C^{s+2}$, $s \geq 0$, is sufficient to justify the transformation (5.1) and all of the steps we have taken. We note that our TFE approach equivalently states the DNO in terms of the transformed field, u' (rather than u), thereby delivering the analyticity result (Theorem 6.2). However, this is not the only result one could ponder. For instance, an interesting query is the (joint) smoothness of the DNO with respect to parameters and spatial variable, x. For instance, based upon our results in [58], we expect that mandating that f be analytic would deliver spatial analyticity of the DNO. Additionally, one could investigate the smoothness of the *untransformed* field, u, which would require the inversion of (5.1) and an accounting of its regularity. We leave these fascinating and important follow–on questions for future work.

Acknowledgments. D.P.N. gratefully acknowledges support from the National Science Foundation through grants No. DMS–1813033 and No. DMS–2111283.

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