ENDS OF GRADIENT RICCI SOLITONS

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ABSTRACT. Self similar solutions to Ricci flows, called Ricci solitons, are important geometric objects. To address the question whether new solitons can be constructed from existing ones through connected sums, we are led to investigate the issue of connectedness at infinity for solitons. The paper provides a brief account of our work along this line as well as a new result. The new result says that an n-dimensional gradient shrinking Ricci soliton is necessarily connected at infinity if its scalar curvature is bounded above by $\frac{n}{3}$.

1. INTRODUCTION

1.1. **Gradient Ricci solitons.** In studying the singularities of Ricci flows, Hamilton [15] investigated Ricci solitons and signified their importance. The importance was further demonstrated in the work of Perelman [36, 37], where his classification result of three-dimensional gradient shrinking Ricci solitons played a crucial role in the affirmation of the Poincaré conjecture. Recall that a complete Riemannian manifold (M, g) is called a gradient Ricci soliton if the equation

$$\mathrm{Ric} + \mathrm{Hess}\,(f) = \lambda\, g$$

holds for some function f. Here, λ is a constant, Ric the Ricci curvature of (M, g) and Hess (f) the Hessian of f. It is customary to normalize the metric so that $\lambda = \frac{1}{2}$, 0 or $-\frac{1}{2}$ and the soliton is called shrinking, steady, and expanding, respectively.

The importance of Ricci solitons can be partially seen through a conjecture attributed to Hamilton, which asserts that the blow-ups around a type-I singularity point of a Ricci flow (see [9] for definition) always converge to (nontrivial) gradient shrinking Ricci solitons. Important progress toward the conjecture was made in [33, 39], where it was shown that blow-up limits must be gradient shrinking Ricci solitons. The nontriviality issue, which was raised by Cao [2], was later taken up by Enders, Müller and Topping [12], see also Cao and Zhang [5].

Two-dimensional shrinking Ricci solitons have been classified by Hamilton [15], which are either $\mathbb{R}^2$ or a quotient of $\mathbb{S}^2$. For the three-dimensional case, Perelman [37] made the breakthrough and concluded that a noncollapsed gradient shrinking Ricci soliton with bounded and non-negative curvature must be a quotient of the sphere $\mathbb{S}^3$, or $\mathbb{R}^3$, or $\mathbb{S}^2 \times \mathbb{R}$. The result was later shown to be true without any extra assumptions through the effort of [3, 34] by a different approach. See also [33, 35] for related works. A complete classification for gradient shrinking Ricci solitons remains open for dimension 4, though recent work [30, 31, 11] has shed some light on it. Presently, there is very limited information available concerning general gradient Ricci solitons in higher dimensions. An answer to the following basic question, an analogue to the classical Cheeger-Gromoll [6] splitting theorem for complete manifolds with non-negative Ricci curvature, is still missing.

Question 1.1. *If a complete gradient shrinking Ricci soliton M contains a line, is it isometric to the product of $\mathbb{R} \times N$ for some N?*

Examples in [13] show that the Ricci curvature of complete gradient shrinking Ricci solitons in general may change sign. Less ambitiously, one may ask if a complete gradient shrinking Ricci soliton must be a cylinder of the form $\mathbb{R} \times N$ for some compact manifold N if it is not connected at infinity. Of course, similar issues can also be raised for both steady and expanding Ricci solitons. A positive answer would imply that no new gradient Ricci solitons could be constructed from given ones via a connected sum and perturbation process. For steady Ricci solitons, we do have a satisfactory conclusion [27, 29].

Theorem 1.2. *Let (M, g, f) be a complete gradient steady Ricci soliton. Then either M is connected at infinity or M is isometric to $\mathbb{R} \times N$ for a compact Ricci flat manifold N.*

A similar result for expanding solitons holds as well [28].

Theorem 1.3. *Let (M, g, f) be a complete gradient expanding Ricci soliton of dimension n. Assume that the scalar curvature $S \geq -\frac{n-1}{2}$ on M. Then either M is connected at infinity or $M = \mathbb{R} \times N^{n-1}$, where N is a compact Einstein manifold.*

According to [38, 45], the scalar curvature of an n-dimensional gradient expanding Ricci soliton always satisfies

$$S \geq -\frac{n}{2}.$$

Note that in the case of M being a cylinder, its scalar curvature $S = -\frac{n-1}{2}$. This somewhat explains why the assumption $S \geq -\frac{n-1}{2}$. However, at this point it is unclear to us whether such an assumption is in fact superfluous for nontrivial expanding gradient Ricci solitons, though obviously there are Einstein manifolds with infinitely many ends.

The case of gradient shrinking Ricci solitons is more challenging and the issue has not been completely resolved yet. However, in the Kähler case, this has been settled in affirmative in [32].

Theorem 1.4. *Let (M, g, f) be a complete gradient Kähler shrinking Ricci soliton. Then (M, g) is connected at infinity, i.e., it has only one end.*

Under a suitable upper bound on the scalar curvature, we have a positive answer as well.

Theorem 1.5. *Let (M, g) be a complete gradient shrinking Ricci soliton. Assume that*

$$S \leq \frac{n}{3} \quad \text{on } M.$$

Then M has one end.

To prove these results, we cast gradient Ricci solitons as model smooth metric measure spaces with the Bakry-Émery curvature bounded below and apply a theory developed by Li and Tam [18]. The theory uses harmonic functions to detect the number of ends of a complete manifold.

1.2. **Smooth metric measure spaces.** A smooth metric measure space $\left(M, g, e^{-f}\, dv\right)$ is simply a Riemannian manifold (M, g) endowed with a weighted measure of the form $e^{-f}dv$ for a smooth function f on M, where dv is the Riemannian volume element induced by metric g. Such f is called the weight function. With respect to the weighted measure, one has the corresponding weighted Dirichlet energy $E_f(u) = \int_M |\nabla u|^2\, e^{-f}\, dv$. The associated Euler-Lagrange operator of $E_f(u)$ is the so-called weighted Laplacian Δ_f given by

$$\Delta_f u = \Delta u - \langle \nabla f, \nabla u\rangle.$$

A function u is called $f-$harmonic if $\Delta_f u = 0$. In terms of Δ_f, the Bochner formula can be written into

$$\frac{1}{2}\,\Delta_f |\nabla u|^2 = |\mathrm{Hess}(u)|^2 + \langle \nabla u, \nabla \Delta_f u\rangle + \mathrm{Ric}_f\,(\nabla u, \nabla u),$$

where

$$\mathrm{Ric}_f := \mathrm{Ric} + \mathrm{Hess}\,(f)\,.$$

The curvature quantity Ric_f is called the Bakry-Émery curvature [1] of the smooth metric measure space $(M, g, e^{-f}\, dv)$.

It was shown in [1] that if $\mathrm{Ric}_f \geq \frac{1}{2}$, then the following logarithmic Sobolev inequality holds

$$\int_M u^2\, \ln u^2\, e^{-f}\, dv \leq 4 \int_M |\nabla u|^2\, e^{-f}\, dv$$

for any compactly supported smooth function u on M satisfying

$$\int_M u^2\, e^{-f}\, dv = \int_M e^{-f}\, dv.$$

Specializing to the Euclidean space $\mathbb{R}^n$ with $f(x) = \frac{1}{4}\,|x|^2$, one then recovers the classical logarithmic Sobolev inequality due to L. Gross [14].

More generally, in recent work of Lott and Villani [25], and Sturm [40, 41], a lower bound for the Bakry-Émery curvature is interpreted as a convexity measurement for certain entropy functional on the space of probability measures over M with respect to the Wasserstein distance. Indeed, this point of view has enabled them to define “Ricci curvatures” for more general metric measure spaces.

The study of smooth manifolds with the Bakry-Émery curvature bounded below has been very active in recent years. Much effort has been directed toward establishing results parallel to the case that the Ricci curvature is bounded below. This has been largely successful when the weight function f is assumed to be bounded. We refer to the work of Lott [24], Wei and Wylie [44], and ourselves [27, 28] for further details. In fact, Lichenorewicz [23] has already observed that the classical Cheeger-Gromoll splitting theorem [6] continues to hold for manifolds with $\mathrm{Ric}_f \geq 0$ if the weight function f is bounded. Various results concerning volume comparison and analysis of the Laplacian have also been established. However, for nontrivial gradient Ricci solitons, the potential function is never bounded. In fact, it could be of linear growth in the steady case and of quadratic growth for both shrinking and expanding cases. We are thus led to study smooth metric measure spaces with the weight function of quadratic growth in [29]. The following version of Laplace comparison served as the starting point for many of the results there.

Proposition 1.6. *Let γ be a minimizing normal geodesic in a smooth metric measure space $(M, g, e^{-f}\,dv)$ with $p=\gamma(0)$ and $x=\gamma(r)$. If $\mathrm{Ric}_f \geq \lambda$, then*

$$\Delta d\left(p,x\right) \leq \frac{n-1}{r} - \frac{\lambda}{2} r + f'\left(r\right) - \frac{1}{r}\ln\left(\frac{J\left(p,r,\xi\right)}{r^{n-1}}\right) - \frac{1}{r}\left(f\left(r\right)-f\left(0\right)\right) \tag{1.1}$$

and

$$\Delta_f d\left(p,x\right) \leq \frac{n-1}{r} - \frac{\lambda}{2} r - \frac{1}{r}\ln\left(\frac{J\left(p,r,\xi\right)}{r^{n-1}}\right) - \frac{1}{r}\left(f\left(x\right)-f\left(p\right)\right). \tag{1.2}$$

Here, $f'(r)$ is the derivative of the function $f(\gamma(t))$ at $t=r$ and $J(p,r,\xi)$ the area element of metric g under geodesic polar coordinates centered at p in the direction $\xi=\gamma'(0)$. Notice that the right hand side of (1.2) only involves the values of f at the two end points of the geodesic γ.

1.3. **Li-Tam theory.** Recall that an end of a complete manifold M with respect to a compact smooth domain $\Omega\subset M$ is simply an unbounded component of $M\setminus\Omega$. The number of ends $e(M)$ of M is defined to be the maximal number of ends with respect to all such Ω. In [18], to each pair of ends E and F of M, they associate a harmonic function f on M. Such f is bounded with finite Dirichlet energy if both ends E and F are nonparabolic, that is, admitting a positive Green's function satisfying the Neumann boundary conditions. If only one of them is nonparabolic, then such f is positive but unbounded. In the case both E and F are parabolic, such f necessarily changes sign on M. However, it is of one sign on each end. Moreover, with one end fixed and the second ranging through all other ends of M, they demonstrated the resulting harmonic functions are linearly independent. Therefore, the question of bounding the number of ends $e(M)$ is reduced to one on estimating the dimension of the space spanned by those functions.

The theory was successfully applied in [18] to show that $e(M)$ is necessarily finite when the Ricci curvature of M is nonnegative outside a compact set. In [19, 20], it was also used to address the rigidity issue of the well-known estimate [8] that the bottom spectrum $\lambda_(M)\leq\frac{(n-1)^2}{4}$ for an $n-$dimensional complete manifold M with Ricci curvature bound $\mathrm{Ric}\geq-(n-1)$.

Theorem 1.7. *Let M be a complete manifold of dimension $n\geq 3$ with $\mathrm{Ric}\geq -\left(n-1\right)$. Assume that $\lambda_1\left(M\right)=\frac{(n-1)^2}{4}$. Then either M is connected at infinity or it splits as a warped product $M=\mathbb{R}\times N$ with $ds_M^2=dt^2+h^2\left(t\right)ds_N^2$, where N is compact and the function $h\left(t\right)=e^t$ if $n\geq 4$ and $h\left(t\right)=e^t$ or $h\left(t\right)=\cosh t$ if $n=3$.*

We shall refer to [17] for more applications of the theory. Recently, a variant of the theory was developed in [26] by considering instead the Schrödinger operator

$$L=\Delta-\sigma$$

with σ being a nonnegative but not identically zero smooth function on M. One nice feature is that all the resulting solutions u_i are positive.

Theorem 1.8. *Let (M,g) be a complete manifold and $E_1, E_2, \cdots, E_l$ the ends of M with respect to a compact smooth domain Ω of M with $l\geq 2$. Then for each* *end E_i, there exists a positive solution u_i to the equation $Lu_i = 0$ on M satisfying $0 < u_i \leq 1$ on $M \backslash E_i$ and*

$$\sup_M u_i = \limsup_{x \to E_i(\infty)} u_i(x) > 1.$$

Moreover, the functions $u_1, \cdots, u_l$ are linearly independent.

We remark that both Li-Tam theory and its variant continue to hold with the Laplacian replaced by the weighted Laplacian Δ_f for an arbitrary function f.

The paper is arranged as follows. In Sections 2 and 3, we summarize some existing results concerning connectedness at infinity for steady and expanding solitons, respectively. In Section 4, we focus on shrinking solitons. After briefly recalling some existing results, we give a proof of Theorem 1.5.

We would like to dedicate this paper to Professor Peter Li on the occasion of his seventieth birthday. We both have benefited enormously from his teaching, advice and friendship.

2. Ends of steady Ricci solitons

In this section, we consider gradient steady Ricci solitons and view them as smooth metric measure spaces with non-negative Bakry-Émery curvature. So let $\left(M, g, e^{-f}dv\right)$ be a smooth metric measure space with $\mathrm{Ric}_f \geq 0$. We assume that f satisfies

$$\left|f\right|(x) \leq \alpha r\left(x\right) + \beta,$$

where $r\left(x\right) := d\left(p, x\right)$ is the distance to a fixed point $p \in M$, and α, $\beta > 0$ are positive constants. The infimum value of all such α is defined to be the linear growth rate of f. The following result was established in [27].

Theorem 2.1. *Let $\left(M, g, e^{-f}dv\right)$ be a complete smooth metric measure space with $\mathrm{Ric}_f \geq 0$. Then the bottom spectrum of the weighted Laplacian Δ_f satisfies $\lambda_1\left(\Delta_f\right) \leq \frac{1}{4}a^2$, where a is the linear growth rate of f. Moreover, if $\lambda_1\left(\Delta_f\right) = \frac{1}{4}a^2$, then either M is connected at infinity or M is isometric to $\mathbb{R} \times N$ for some compact manifold N.*

Recall that on a gradient steady Ricci soliton $\left(M, g\right)$,

$$\mathrm{Ric} + \mathrm{Hess}\left(f\right) = 0.$$

After a suitable scaling of the metric g, the potential f and the curvature are related by (see [15])

$$\begin{aligned} \Delta f + S &= 0 \\ \left|\nabla f\right|^2 + S &= 1, \end{aligned}$$

where S is the scalar curvature of $\left(M, g\right)$. It is known by [7] that $S \geq 0$. In particular,

$$\left|\nabla f\right| \leq 1.$$

Therefore, f is of linear growth with growth rate $a \leq 1$. On the other hand,

$$\begin{aligned} \Delta_f e^{\frac{1}{2}f} &= \left(-\frac{1}{2}+\frac{1}{4}\left|\nabla f\right|^2\right) e^{\frac{1}{2}f} \\ &\leq \left(-\frac{1}{2}+\frac{1}{4}\right) e^{\frac{1}{2}f} \\ &= -\frac{1}{4}e^{\frac{1}{2}f}. \end{aligned}$$

This implies that $\lambda_1\left(\Delta_f\right) \geq \frac{1}{4}$. In conclusion, $a=1$ and $\lambda_1\left(\Delta_f\right)=\frac{1}{4}$. Applying the preceding theorem, one concludes the following.

Theorem 2.2. *Let (M,g,f) be a complete gradient steady Ricci soliton. Then either M is connected at infinity or M is isometric to $\mathbb{R}\times N$ for a compact Ricci flat manifold N.*

Theorem 2.1 is very much in the spirit of Theorem 1.7. In fact, the proof is also quite similar. In [29], a strengthened version, which has no counterpart in the unweighted case, was established. In the following, we use $\mathrm{Vol}_f\left(B\left(p,r\right)\right)$ to denote the weighted volume of the geodesic ball $B(p,r)$ centered at point p of radius r.

Theorem 2.3. *Let $\left(M,g,e^{-f}dv\right)$ be a smooth metric measure space with $\mathrm{Ric}_f\geq 0$. Assume that f has linear growth rate $a\geq 0$. Then the weighted volume entropy $h_f(M)$ of $\left(M,g,e^{-f}dv\right)$ satisfies*

$$h_f\left(M\right) := \limsup_{r\to\infty}\frac{\ln \mathrm{Vol}_f\left(B\left(p,r\right)\right)}{r}\leq a.$$

In the case $h_f\left(M\right)=a$, either M is connected at infinity or M splits as a metric product $M=\mathbb{R}\times N$, where N is compact.

Theorem 2.3 extends Theorem 2.1. Indeed, it holds true generally that

$$\lambda_1(\Delta_f)\leq\frac{h_f^2(M)}{4}$$

as shown in [19]. So $h_f(M)=a$ if $\lambda_1(\Delta_f)$ achieves its maximal value $\frac{a^2}{4}$.

The proof of Theorem 2.3 relies on the Laplace comparison (1.2) together with a Busemann function argument.

3. Ends of expanding Ricci solitons

In this section, we consider gradient expanding Ricci solitons. Recall that a gradient expanding Ricci soliton is a Riemannian manifold (M,g) such that

$$\mathrm{Ric}+\mathrm{Hess}\left(f\right)=-\frac{1}{2}g$$

for some function f. According to [15],

$$S+\left|\nabla f\right|^2=-f \tag{3.1}$$

by adding a suitable constant to f, where S denotes the scalar curvature of M. Also, taking trace of the soliton equation, we obtain

$$\Delta f+S=-\frac{n}{2},$$

where n is the dimension of M. On the other hand, by the maximum principle, it was proved in [38, 45] that

$$S \geq -\frac{n}{2}.$$

By using these facts, a direct computation gives

$$\Delta_f e^f = \left(\Delta_f (f) + |\nabla f|^2\right) e^f = (\Delta f)\, e^f = -\left(\frac{n}{2} + S\right) e^f.$$

Therefore, the following weighted Poincaré inequality holds.

Lemma 3.1. *Let (M, g, f) be a complete nontrivial gradient expanding Ricci soliton. Define $\sigma := S + \frac{n}{2}$. Then $\sigma > 0$ on M and*

$$\int_M \sigma \phi^2 e^{-f} \leq \int_M |\nabla \phi|^2 e^{-f}$$

for any $\phi \in C_0^\infty(M)$.

With the help of the above weighted Poincaré inequality, we obtained the following result in [28]. That weighted Poincaré inequalities can be used to deal with the issue of connectedness at infinity for Riemannian manifolds was first realized in [21].

Theorem 3.2. *Let (M, g, f) be a complete gradient expanding Ricci soliton. Assume that $S \geq -\frac{n-1}{2}$ on M. Then either M is connected at infinity or $M = \mathbb{R} \times N^{n-1}$, where N is a compact Einstein manifold.*

An important technical step of the proof is to show that each end of M must be $f-$nonparabolic. Once this is done, by Li-Tam theory, one obtains an $f-$harmonic function u with finite weighted energy if M is not connected at infinity. Combining the Bochner formula for function u with the weighted Poincaré inequality then forces the scalar curvature S to be constant $-\frac{n-1}{2}$. The result follows from there.

In the case of gradient expanding Kähler Ricci solitons, the restriction on the scalar curvature could be replaced by the properness of f [32].

Theorem 3.3. *Let (M, g, f) be a gradient expanding Kähler Ricci soliton. Assume that the potential f is proper. Then (M, g) has only one end.*

Again, since all ends of M are $f-$nonparabolic, if M has more than one end, then by Li-Tam theory, one obtains an $f-$harmonic function u with finite weighted energy. The crucial point now is that such u is necessarily pluriharmonic due to the fact that ∇f is a real holomorphic vector field. It follows that $\langle \nabla f, \nabla u \rangle = 0$. It is then easy to conclude that u must be constant if f is proper.

4. Ends of shrinking Ricci solitons

In this section, we focus on gradient shrinking Ricci solitons.

4.1. Preliminaries and existing results.

Recall that a gradient shrinking Ricci soliton is a manifold (M, g) for which a potential function f exists such that

$$\mathrm{Ric} + \mathrm{Hess}(f) = \frac{1}{2} g.$$

The potential f and the curvature are related as follows [15].

$$
\begin{aligned}
\Delta f + S &= \frac{n}{2} \\
|\nabla f|^2 + S &= f \\
\operatorname{Ric}(\nabla f) &= \frac{1}{2}\nabla S \\
\Delta_f S &= S - 2\,|\operatorname{Ric}|^2,
\end{aligned} \tag{4.1}
$$

where S is the scalar curvature of (M, g) and n the dimension of M. It is known by [7] that $S \geq 0$. In particular,

$$|\nabla f|^2 \leq f.$$

In fact, by [10],

$$S \geq \frac{C}{f} \text{ on } M \tag{4.2}$$

for some positive constant C unless the soliton is flat. Furthermore, according to [4, 16], there exists a point $p \in M$ and a constant $c(n)$ depending only on dimension n of M such that

$$\frac{1}{4}r^2(x) - c(n)\,r(x) \leq f(x) \leq \frac{1}{4}r^2(x) + c(n)\,r(x) \tag{4.3}$$

for all $x \in M$, where $r(x) = r(p, x)$ is the distance from p to x. It is also known by [4, 28] that the volume $\mathrm{V}(p, R) = \operatorname{Vol}(B(p, R))$ of the geodesic ball $B(p, R)$ must satisfy

$$c_0\, R \leq \mathrm{V}(p, R) \leq c(n)\ R^n \tag{4.4}$$

for $R > 1$, where the constant c_0 depends on n and $\mu(g) := \int_M e^{-f} dv < \infty$.

According to [29], both (4.3) and (4.4) hold more generally on complete smooth metric measure space $\left(M, g, e^{-f}dv\right)$ satisfying $\operatorname{Ric}_f \geq \frac{1}{2}$ and $|\nabla f|^2 \leq f$. Since such M shares many similar geometric properties with a complete manifold of non-negative Ricci curvature, one may ask if an analogous version of Cheeger-Gromoll splitting theorem [6] holds for M.

Question 4.1. *Let $\left(M, g, e^{-f}dv\right)$ be a complete smooth metric measure space satisfying $\operatorname{Ric}_f \geq \frac{1}{2}$ and $|\nabla f|^2 \leq f$. Is $(M, g) = \mathbb{R} \times N$ metrically for some smooth manifold N if M contains a line?*

Regarding this question, we have the following result [29].

Theorem 4.2. *Let $\left(M, g, e^{-f}dv\right)$ be a complete smooth metric measure space with $\operatorname{Ric}_f \geq \frac{1}{2}$. Assume that f satisfies the upper bound*

$$f(x) \leq \frac{1}{4} d(x, K)^2 + C \quad \text{on } M$$

for some constant $C > 0$ and compact subset $K \subset M$. Then there is no line passing through K, or otherwise M is isometric to $\mathbb{R} \times N$. In particular if $M \setminus K$ has at least two unbounded components, then $M = \mathbb{R} \times N$ for some compact manifold N.

Again, the proof relies on Laplace comparison theorem (1.2) and uses a Busemann function argument. It is unclear at this point whether the assumption on f holds true on a gradient shrinking Ricci soliton. So the question of connectedness at infinity for complete gradient shrinking Ricci solitons remains open in general. We have the following partial result from [26].

Theorem 4.3. *Let (M, g) be a complete gradient shrinking Ricci soliton with $\alpha < \infty$. Then the number of ends of M is bounded from above by $\Gamma(n, \alpha, \mu(g))$, a constant depending only on dimension n, $\mu(g)$ and α.*

Here
$$\alpha = \limsup_{R\to\infty} \frac{1}{\mathrm{V}(p,R)} \int_{B(p,R)} \left(S\, r^2\right)^{\frac{n-1}{2}}$$
and Perelman's entropy $\mu(g)$ is given by
$$\mu(g) = \int_M e^{-f}.$$
Since an asymptotically conical gradient shrinking Ricci soliton must have $\alpha < \infty$, the result in particular provides an effective estimate on the number of ends for such solitons. While it is possible to prove the theorem by using a variant of Li-Tam theory Theorem 1.8, we instead adapt the arguments from [42, 43] to show that for each large R the volume of $E \cap B(p, R)$ satisfies $\mathrm{V}\left(E \cap B(p, R)\right) \geq c\, R^n$ for some constant c for at least one half of the ends E of M. Note that for different R the choice of such set of ends E may be different. Nonetheless, the desired estimate on the number of ends follows as the total volume of the ball $B(p, R)$ is at most of $C(n)\, R^n$. We emphasize that the argument strongly depends on the Sobolev inequality from [22].

In the Kähler case, we have the following satisfactory result [32].

Theorem 4.4. *Let (M, g, f) be a complete gradient Kähler shrinking Ricci soliton. Then (M, g) is connected at infinity, i.e., it has only one end.*

For the proof, we consider the smooth metric measure space $(M, g, e^{-F}\, dv)$, where $F = -f$. A crucial step is to show that each end of M must be $F-$nonparabolic. Once this is done, Li-Tam theory gives an $F-$harmonic function u with finite weighted Dirichlet energy in the case M has more than one end. Using the fact that ∇F is a real holomorphic vector field, one concludes that u must be pluriharmonic and $\langle \nabla u, \nabla F\rangle = 0$. However, the properness of F forces u to be a constant. This contradiction implies the result.

In the preceding argument, the fact that each end is $F-$nonparabolic holds true for general gradient shrinking Ricci solitons. We now take advantage of this fact and establish a new result.

4.2. **A new result.** This subsection is devoted to a proof of Theorem 1.5. We first make some preparations. Let (M, g) be a complete noncompact gradient shrinking Ricci soliton of dimension n with more than one end. For a given constant $a > 0$ we consider the weight

$$F = -af. \tag{4.5}$$

According to Theorem 2.1 of [32], all ends of M must be F-nonparabolic. Recall that an end E of M is called F-nonparabolic if it admits a positive symmetric

Green's function for the weighted Laplacian

$$\Delta_F u = \Delta u + a \langle \nabla f, \nabla u \rangle$$

subject to the Neumann boundary conditions on ∂E.

For an end E_1 of M, denote by $E_2 = M \setminus E_1$. Then E_2 is another end of M with $E_1 \cup E_2 = M$. Since both E_1 and E_2 are F-nonparabolic, according to [18], there exists an F-harmonic function u on M or

$$\Delta u + a \langle \nabla f, \nabla u \rangle = 0 \text{ on } M \tag{4.6}$$

such that

$$\begin{aligned} 0 &< u < 1 \text{ on } M \\ \inf_{E_1} u &= 0 \text{ and } \sup_{E_2} u = 1. \end{aligned} \tag{4.7}$$

Moreover, u is obtained as a limit of the sequence u_i of F-harmonic functions on geodesic ball $B(x_0, R_i)$ satisfying the following Dirichlet boundary conditions.

$$\begin{aligned} u_i &= 0 \text{ on } \partial B(x_0, R_i) \cap E_1 \\ u_i &= 1 \text{ on } \partial B(x_0, R_i) \cap E_2. \end{aligned} \tag{4.8}$$

It is easily seen that

$$\int_M |\nabla u|^2 \, e^{-F} < \infty, \text{ hence } \int_M |\nabla u|^2 \, e^{a\,f} < \infty. \tag{4.9}$$

In the following, we will derive a sequence of estimates involving the function u. The standing assumption is that (M, g) is a complete gradient shrinking Ricci soliton with its scalar curvature

$$S \leq A$$

for some constant A.

Lemma 4.5. *For any constant $b < a$ we have*

$$\int_M |\nabla u|^2 \, e^{2bf} < \infty.$$

Proof. In view of (4.9) and (4.3), it suffices to verify the lemma for $b > \frac{a}{2}$. We first claim that the following Poincaré type inequality holds on M.

$$b^2 \int_M (f - c_0) \, \phi^2 e^{2bf} \leq \int_M |\nabla \phi|^2 \, e^{2bf} \tag{4.10}$$

for any $\phi \in C_0^\infty(M)$, where c_0 is a constant depending on n, b and A.

Indeed, a direct calculation implies that

$$\begin{aligned} \Delta_{(-2bf)} e^{-bf} &= \Delta e^{-bf} + 2b \left\langle \nabla f, \nabla e^{-bf} \right\rangle \\ &= \left(-b\Delta f - b^2 |\nabla f|^2 \right) e^{-bf} \\ &= -\left(b^2 f + \frac{n}{2} b - \left(b^2 + b \right) S \right) e^{-bf}, \end{aligned}$$

where in the last line we have used (4.1). Since $S \leq A$, we conclude

$$\Delta_{(-2bf)} e^{-bf} \leq -b^2 (f - c_0)\, e^{-bf}$$

for some constant c_0 depending on n, b and A. By [21], the existence of a positive function $v > 0$ satisfying

$$\Delta_{(-2bf)} v \leq -b^2 (f - c_0)\, v$$

implies the claimed Poincaré inequality (4.10).

Denote by $E_1(x_0, R) = E_1 \cap B(x_0, R)$. Let

$$\psi(x) = \begin{cases} 0 & \text{on } M \backslash E_1(x_0, R_0) \\ r(x) - R_0 & \text{on } E_1(x_0, R_0 + 1) \backslash E_1(x_0, R_0) \\ 1 & \text{on } E_1 \backslash E_1(x_0, R_0 + 1). \end{cases}$$

Applying (4.10) to $\phi = \psi\, u_i$ and using (4.8) we get

$$b^2 \int_M (f - c_0)\, \psi^2 u_i^2 e^{2bf} \leq \int_M |\nabla(\psi u_i)|^2 e^{2bf}. \tag{4.11}$$

Expanding the right side of (4.11) yields

$$\begin{aligned} \int_M |\nabla(\psi u_i)|^2 e^{2bf} &= \int_M |\nabla \psi|^2 u_i^2 e^{2bf} + \int_M |\nabla u_i|^2 \psi^2 e^{2bf} + \frac{1}{2} \int_M \left\langle \nabla u_i^2, \nabla \psi^2 \right\rangle e^{2bf} \\ &= \int_M |\nabla \psi|^2 u_i^2 e^{2bf} + \int_M |\nabla u_i|^2 \psi^2 e^{2bf} - \frac{1}{2} \int_M \left(\Delta_{(-2bf)} u_i^2 \right) \psi^2 e^{2bf} \\ &= \int_M |\nabla \psi|^2 u_i^2 e^{2bf} - \int_M u_i \left(\Delta_{(-2bf)} u_i \right) \psi^2 e^{2bf} \\ &= \int_M |\nabla \psi|^2 u_i^2 e^{2bf} - (2b - a) \int_M u_i \langle \nabla u_i, \nabla f \rangle \psi^2 e^{2bf}, \end{aligned}$$

where we have used that u_i is F-harmonic,

$$\Delta u_i = -a \langle \nabla f, \nabla u_i \rangle$$

in the last line.

We conclude from above that

$$\begin{aligned} \int_M |\nabla(\psi u_i)|^2 e^{2bf} &= \int_M |\nabla \psi|^2 u_i^2 e^{2bf} - \left(b - \frac{a}{2} \right) \int_M \left\langle \nabla u_i^2, \nabla f \right\rangle \psi^2 e^{2bf} \\ &= \int_M |\nabla \psi|^2 u_i^2 e^{2bf} \\ &\quad + \left(b - \frac{a}{2} \right) \int_M u_i^2 \left(\Delta f + \left\langle \nabla f, \nabla \psi^2 \right\rangle + 2b |\nabla f|^2 \right) e^{2bf} \\ &= \int_M |\nabla \psi|^2 u_i^2 e^{2bf} + \left(b - \frac{a}{2} \right) \int_M u_i^2 \left\langle \nabla f, \nabla \psi^2 \right\rangle e^{2bf} \\ &\quad + \left(b - \frac{a}{2} \right) \int_M u_i^2 \left(\frac{n}{2} - (1 + 2b)\, S + 2bf \right) e^{2bf}. \end{aligned}$$

Since $\nabla \psi$ has support in $E_1(x_0, R_0 + 1) \backslash E_1(x_0, R_0)$ and $u_i \to u$, there exists a constant c_1 depending only on n, b and A and a constant $C > 0$ independent of i such that

$$\int_M |\nabla(\psi u_i)|^2 e^{2bf} \leq 2b\left(b - \frac{a}{2}\right)\int_M (f + c_1)\, u_i^2 e^{2bf} + C. \tag{4.12}$$

Plugging this into (4.11) implies that

$$(a-b)\, b \int_M (f - c_2)\, \psi^2 u_i^2 e^{2bf} \leq C$$

for some constant c_2 depending only on n, b and A. This proves that

$$\int_{E_1} f\, u_i^2\, e^{2bf} \leq C$$

as f is proper. Using (4.12) again, one sees that

$$\int_{E_1} |\nabla u_i|^2 e^{2bf} \leq C.$$

Similar estimate holds on E_2 as well. The lemma follows by letting $i \to \infty$. □

We now prove a similar result for the hessian of u.

Lemma 4.6. *For any constant $b < a$ we have*

$$\int_M |u_{ij}|^2 e^{2bf} < \infty.$$

Proof. Again, we only need to consider $b > \frac{a}{2}$. Let us denote with

$$D(T) = \{f \leq T\}.$$

Define the cut-off function

$$\phi(x) = \begin{cases} 1 & \text{on } D(T) \\ T + 1 - f(x) & \text{on } D(T+1) \backslash D(T) \\ 0 & \text{on } M \backslash D(T+1). \end{cases}$$

Integrating by parts we get

$$\begin{aligned} \int_M |u_{ij}|^2 e^{2bf} \phi^2 &= -\int_M u_{ijj} u_i e^{2bf} \phi^2 - 2b \int_M u_{ij} u_i f_j e^{2bf} \phi^2 \\ &\quad -2\int_M u_{ij} u_i \phi_j e^{2bf} \phi. \end{aligned} \tag{4.13}$$

The second term on the right hand side can be estimated as

$$\begin{aligned} -2b \int_M u_{ij} u_i f_j e^{2bf} \phi^2 &\leq 2b \int_M |u_{ij}|\, |\nabla u|\, |\nabla f|\, e^{2bf} \phi^2 \\ &\leq \frac{1}{4}\int_M |u_{ij}|^2 e^{2bf} \phi^2 + 4b^2 \int_M |\nabla u|^2 |\nabla f|^2 e^{2bf} \phi^2. \end{aligned}$$

Since $|\nabla f|^2 \leq f$ and $b < a$, by Lemma 4.5 we conclude that

$$\int_M |\nabla u|^2 |\nabla f|^4 e^{2bf} < \infty. \tag{4.14}$$

Proceeding similarly for the last term in (4.13), we have

$$-2\int_M u_{ij}u_i\phi_j e^{2bf}\phi \leq \frac{1}{4}\int_M |u_{ij}|^2 e^{2bf}\phi^2 + C.$$

In view of these estimates, (4.13) becomes

$$\begin{aligned}\int_M |u_{ij}|^2 e^{2bf}\phi^2 &\leq -2\int_M u_{ijj}u_i e^{2bf}\phi^2 + C \\ &= -2\int_M \langle\nabla\Delta u, \nabla u\rangle e^{2bf}\phi^2 \\ &\quad -2\int_M \mathrm{Ric}\left(\nabla u, \nabla u\right) e^{2bf}\phi^2 + C.\end{aligned} \tag{4.15}$$

Integrating by parts and using (4.6) we have

$$\begin{aligned}\int_M \langle\nabla\Delta u, \nabla u\rangle e^{2bf}\phi^2 &= -\int_M (\Delta u)^2 e^{2bf}\phi^2 - 2b\int_M (\Delta u)\langle\nabla u, \nabla f\rangle e^{2bf}\phi^2 \\ &\quad -\int_M (\Delta u)\left\langle\nabla u, \nabla\phi^2\right\rangle e^{2bf} \\ &= \left(2ab - a^2\right)\int_M \langle\nabla u, \nabla f\rangle^2 e^{2bf}\phi^2 \\ &\quad +a\int_M \langle\nabla u, \nabla f\rangle\left\langle\nabla u, \nabla\phi^2\right\rangle e^{2bf}.\end{aligned}$$

Hence, by (4.14),

$$\left|\int_M \langle\nabla\Delta u, \nabla u\rangle e^{2bf}\phi^2\right| \leq C. \tag{4.16}$$

Therefore,

$$\int_M |u_{ij}|^2 e^{2bf}\phi^2 \leq 2\left|\int_M \mathrm{Ric}\left(\nabla u, \nabla u\right) e^{2bf}\phi^2\right| + C. \tag{4.17}$$

To bound the right side of (4.17), we use the soliton equation and Lemma 4.5 to get

$$\left|\int_M \mathrm{Ric}\left(\nabla u, \nabla u\right) e^{2bf}\phi^2\right| \leq \left|\int_M f_{ij}u_iu_j e^{2bf}\phi^2\right| + C. \tag{4.18}$$

However, by (4.6),

$$\langle\nabla\Delta u, \nabla u\rangle = -au_{ij}u_if_j - af_{ij}u_iu_j.$$

Hence, in view of (4.16),

$$\begin{aligned}
\left|\int_M f_{ij}u_iu_je^{2bf}\phi^2\right| &\leq \left|\int_M u_{ij}u_if_je^{2bf}\phi^2\right| + \frac{1}{a}\left|\int_M \left\langle \nabla\Delta u, \nabla u\right\rangle e^{2bf}\phi^2\right| \\
&\leq \left|\int_M u_{ij}u_if_je^{2bf}\phi^2\right| + C \\
&\leq \frac{1}{4}\int_M \left|u_{ij}\right|^2 e^{2bf}\phi^2 + \int_M \left|\nabla u\right|^2\left|\nabla f\right|^2 e^{2bf}\phi^2 + C \\
&\leq \frac{1}{4}\int_M \left|u_{ij}\right|^2 e^{2bf}\phi^2 + C,
\end{aligned}$$

where in the last line we have used (4.14). Therefore, we conclude from (4.18) that

$$\left|\int_M \operatorname{Ric}\left(\nabla u, \nabla u\right) e^{2bf}\phi^2\right| \leq \frac{1}{4}\int_M \left|u_{ij}\right|^2 e^{2bf}\phi^2 + C.$$

Combining with (4.17) implies

$$\int_M \left|u_{ij}\right|^2 e^{2bf}\phi^2 < \infty.$$

This proves the result. □

We also record the following estimate.

Lemma 4.7. *For any constant $b < a$ we have*

$$\int_M \left|\operatorname{Ric}\right|^2 \left|\nabla u\right|^2 e^{2bf} < \infty.$$

Proof. Define the cut-off function

$$\phi(x) = \begin{cases} 1 & \text{on } D(T) \\ T + 1 - f(x) & \text{on } D(T+1)\setminus D(T) \\ 0 & \text{on } M\setminus D(T+1) \end{cases}$$

Using the identity

$$\Delta_f S = S - 2\left|\operatorname{Ric}\right|^2$$

and Lemma 4.5, we have

$$2\int_M \left|\operatorname{Ric}\right|^2 \left|\nabla u\right|^2 \phi^2 e^{2bf} \leq -\int_M \left(\Delta_f S\right)\left|\nabla u\right|^2 \phi^2 e^{2bf} + C \tag{4.19}$$

as S is bounded. We now estimate the first term of the right hand side.

Integrating by parts we get

$$\begin{aligned}
-\int_M \left(\Delta_f S\right)\left|\nabla u\right|^2 e^{2bf}\phi^2 &= \int_M \left\langle \nabla S, \nabla\left(\left|\nabla u\right|^2 e^{2bf}\right)\right\rangle \phi^2 + \int_M \left\langle \nabla S, \nabla f\right\rangle \left|\nabla u\right|^2 e^{2bf}\phi^2 \\
&\quad + \int_M \left\langle \nabla S, \nabla \phi^2\right\rangle \left|\nabla u\right|^2 e^{2bf}.
\end{aligned}$$

After some simplifications, it becomes

$$
\begin{aligned}
(4.20)\quad -\int_M (\Delta_f S)\,|\nabla u|^2\, e^{2bf}\phi^2 &= 2\int_M (u_{ij}u_iS_j)\, e^{2bf}\phi^2 \\
&\quad + (2b+1)\int_M \langle \nabla S, \nabla f\rangle\, |\nabla u|^2\, e^{2bf}\phi^2 \\
&\quad + \int_M \left\langle \nabla S, \nabla \phi^2\right\rangle |\nabla u|^2\, e^{2bf}.
\end{aligned}
$$

Since

$$\nabla S = 2\mathrm{Ric}\,(\nabla f)\,,$$

it follows that

$$
\begin{aligned}
2u_{ij}u_iS_j &\le 4\,|u_{ij}|\,|\nabla u|\,|\nabla f|\,|\mathrm{Ric}| \\
&\le \frac{1}{4}\,|\mathrm{Ric}|^2\,|\nabla u|^2 + 16\,|u_{ij}|^2\,|\nabla f|^2\,.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(2b+1)\,\langle \nabla S, \nabla f\rangle\,|\nabla u|^2 &\le 2\,(2b+1)\,|\mathrm{Ric}|\,|\nabla u|^2\,|\nabla f|^2 \\
&\le \frac{1}{4}\,|\mathrm{Ric}|^2\,|\nabla u|^2 + 4\,(2b+1)^2\,|\nabla u|^2\,|\nabla f|^4
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\left\langle \nabla S, \nabla \phi^2\right\rangle\right|\,|\nabla u|^2 &\le 4\,|\mathrm{Ric}|\,|\nabla f|^2\,|\nabla u|^2\,\phi \\
&\le \frac{1}{4}\,|\mathrm{Ric}|^2\,|\nabla u|^2\,\phi^2 + 16\,|\nabla u|^2\,|\nabla f|^4\,.
\end{aligned}
$$

Plugging these estimates into (4.20) yields

$$
\begin{aligned}
(4.21)\qquad & - \int_M (\Delta_f S)\,|\nabla u|^2\, e^{2bf}\phi^2 \\
& \le \frac{3}{4}\int_M |\mathrm{Ric}|^2\,|\nabla u|^2\,\phi^2 e^{2bf} + 16\int_M |u_{ij}|^2\,|\nabla f|^2\, e^{2bf} \\
& \quad + \left(4\,(2b+1)^2 + 16\right)\int_M |\nabla u|^2\,|\nabla f|^4\, e^{2bf}.
\end{aligned}
$$

By (4.14) and Lemma 4.6, we have

$$\int_M |\nabla u|^2\,|\nabla f|^4\, e^{2bf} < C$$

and

$$\int_M |u_{ij}|^2\,|\nabla f|^2\, e^{2bf} \le C.$$

In conclusion,

$$-\int_M (\Delta_f S)\,|\nabla u|^2\, e^{2bf}\phi^2 \le \frac{3}{4}\int_M |\mathrm{Ric}|^2\,|\nabla u|^2\,\phi^2 e^{2bf} + C.$$

Plugging this in (4.19) we arrive at

$$\int_M |\mathrm{Ric}|^2 |\nabla u|^2 e^{2bf} < \infty.$$

This proves the lemma. $\square$

The following L^1 estimate is a direct consequence of the previous lemmas.

Lemma 4.8. *For any constant $b < a$ we have*

$$\int_M |\nabla u| e^{bf} + \int_M |u_{ij}| e^{bf} + \int_M |\mathrm{Ric}| |\nabla u| e^{bf} < \infty.$$

Proof. According to Lemma 4.5, for $\bar{b} = \frac{a+b}{2} < a$,

$$\int_M |\nabla u|^2 e^{2\bar{b}f} < \infty.$$

Now the Cauchy-Schwarz inequality together with (4.3) and (4.4) implies

$$\left(\int_M |\nabla u| e^{bf}\right)^2 \leq \left(\int_M |\nabla u|^2 e^{2\bar{b}f}\right)\left(\int_M e^{-(a-b)f}\right) < \infty.$$

The other two terms can be estimated similarly. $\square$

We also need the following technical estimate.

Lemma 4.9. *For all $b < a$, we have*

$$\begin{aligned} 2\int_M |\mathrm{Ric}|^2 |\nabla u| e^{bf} \quad \leq \quad & -(2b-a+1)\int_M \langle \nabla |\nabla u|, \nabla f\rangle S e^{bf} \\ & -\left(b^2+b\right)\int_M |\nabla f|^2 |\nabla u| S e^{bf} \\ & -\int_M \left((b+1)\Delta f - \frac{a}{2} - 1\right) |\nabla u| S e^{bf} \\ & -(1+a)\int_M \mathrm{Ric}(\nabla u, \nabla u) |\nabla u|^{-1} S e^{bf}. \end{aligned}$$

Proof. Let

$$D(T) = \{f \leq T\}$$

and define the cut-off function

$$\phi(x) = \begin{cases} 1 & \text{on } D(T) \\ T+1-f(x) & \text{on } D(T+1)\backslash D(T) \\ 0 & \text{on } M\backslash D(T+1). \end{cases}$$

From the soliton identity

$$\Delta_f S = S - 2|\mathrm{Ric}|^2,$$

we have

$$
\begin{aligned}
(4.22) \qquad 2\int_M |\mathrm{Ric}|^2 |\nabla u| e^{bf}\phi^2 &= -\int_M (\Delta S) |\nabla u| e^{bf}\phi^2 \\
&\quad + \int_M \langle \nabla S, \nabla f\rangle |\nabla u| e^{bf}\phi^2 \\
&\quad + \int_M S |\nabla u| e^{bf}\phi^2.
\end{aligned}
$$

Integrating by parts, we get

$$
\begin{aligned}
(4.23) \qquad & \int_M \langle \nabla S, \nabla f\rangle |\nabla u| e^{bf}\phi^2 \\
= & -\int_M S \left\langle \nabla f, \nabla \left(|\nabla u| e^{bf}\phi^2\right)\right\rangle - \int_M S (\Delta f) |\nabla u| e^{bf}\phi^2 \\
= & -\int_M S \Big(\langle \nabla |\nabla u|, \nabla f\rangle + b |\nabla f|^2 |\nabla u| + |\nabla u| \Delta f \Big) e^{bf}\phi^2 \\
& -\int_M S \left\langle \nabla f, \nabla \phi^2\right\rangle |\nabla u| e^{bf}.
\end{aligned}
$$

Plugging (4.23) into (4.22) yields

$$
\begin{aligned}
(4.24) \qquad & 2\int_M |\mathrm{Ric}|^2 |\nabla u| e^{bf}\phi^2 \\
& = -\int_M (\Delta S) |\nabla u| e^{bf}\phi^2 \\
& -\int_M S \Big(\langle \nabla |\nabla u|, \nabla f\rangle + b |\nabla f|^2 |\nabla u| + |\nabla u| \Delta f \Big) e^{bf}\phi^2 \\
& +\int_M S |\nabla u| e^{bf}\phi^2 \\
& -\int_M S \left\langle \nabla f, \nabla \phi^2\right\rangle |\nabla u| e^{bf}.
\end{aligned}
$$

On the other hand, by the Bochner formula,

$$
\begin{aligned}
(4.25) \qquad \frac{1}{2}\Delta |\nabla u|^2 &= |u_{ij}|^2 + \langle \nabla \Delta u, \nabla u\rangle \\
&\quad + \mathrm{Ric}(\nabla u, \nabla u) \\
&= |u_{ij}|^2 - a \langle \nabla (\langle \nabla u, \nabla f\rangle), \nabla u\rangle \\
&\quad + \mathrm{Ric}(\nabla u, \nabla u) \\
&= |u_{ij}|^2 - a u_{ij} u_i f_j - a f_{ij} u_i u_j \\
&\quad + \mathrm{Ric}(\nabla u, \nabla u) \\
&= |u_{ij}|^2 - \frac{a}{2}\left\langle \nabla |\nabla u|^2, \nabla f\right\rangle - \frac{a}{2}|\nabla u|^2 \\
&\quad + (a+1)\,\mathrm{Ric}(\nabla u, \nabla u).
\end{aligned}
$$

Together with Kato's inequality

$$
|\nabla |\nabla u||^2 \le |u_{ij}|^2,
$$

(4.25) can be rewritten into

$$\Delta\left|\nabla u\right|\geq -a\left\langle \nabla\left|\nabla u\right|,\nabla f\right\rangle -\frac{a}{2}\left|\nabla u\right|+(a+1)\operatorname{Ric}\left(\nabla u,\nabla u\right)\left|\nabla u\right|^{-1}.$$

Therefore,

$$\begin{aligned}
\Delta\left(\left|\nabla u\right|e^{bf}\right) &= \left(\Delta\left|\nabla u\right|\right)e^{bf}+\left|\nabla u\right|\Delta e^{bf}+2\left\langle \nabla\left|\nabla u\right|,\nabla e^{bf}\right\rangle \\
&\geq \left(-a\left\langle \nabla\left|\nabla u\right|,\nabla f\right\rangle -\frac{a}{2}\left|\nabla u\right|+(a+1)\operatorname{Ric}\left(\nabla u,\nabla u\right)\left|\nabla u\right|^{-1}\right)e^{bf} \\
&\quad +\left(b\Delta f+b^{2}\left|\nabla f\right|^{2}\right)\left|\nabla u\right|e^{bf} \\
&\quad +2b\left\langle \nabla\left|\nabla u\right|,\nabla f\right\rangle e^{bf}.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\Delta\left(\left|\nabla u\right|e^{bf}\right) &\geq (2b-a)\left\langle \nabla\left|\nabla u\right|,\nabla f\right\rangle e^{bf} \\
&\quad +\left(b\Delta f+b^{2}\left|\nabla f\right|^{2}-\frac{a}{2}\right)\left|\nabla u\right|e^{bf} \\
&\quad +(a+1)\operatorname{Ric}\left(\nabla u,\nabla u\right)\left|\nabla u\right|^{-1}e^{bf}.
\end{aligned}\tag{4.26}$$

Integration by parts implies

$$\begin{aligned}
-\int_M(\Delta S)\left|\nabla u\right|e^{bf}\phi^{2} &= \int_M\left\langle \nabla S,\nabla\left(\left|\nabla u\right|e^{bf}\right)\right\rangle \phi^{2}+\int_M\left\langle \nabla S,\nabla\phi^{2}\right\rangle \left|\nabla u\right|e^{bf} \\
&= -\int_M S\Delta\left(\left|\nabla u\right|e^{bf}\right)\phi^{2}-\int_M S\left\langle \nabla\left(\left|\nabla u\right|e^{bf}\right),\nabla\phi^{2}\right\rangle \\
&\quad +\int_M\left\langle \nabla S,\nabla\phi^{2}\right\rangle \left|\nabla u\right|e^{bf}.
\end{aligned}$$

In view of (4.26), it becomes

$$\begin{aligned}
&-\int_M(\Delta S)\left|\nabla u\right|e^{bf}\phi^{2} \\
\leq\ &-(2b-a)\int_M\left\langle \nabla\left|\nabla u\right|,\nabla f\right\rangle Se^{bf}\phi^{2} \\
&-\int_M\left(b\Delta f+b^{2}\left|\nabla f\right|^{2}-\frac{a}{2}\right)\left|\nabla u\right|Se^{bf}\phi^{2} \\
&-(1+a)\int_M\operatorname{Ric}\left(\nabla u,\nabla u\right)\left|\nabla u\right|^{-1}Se^{bf}\phi^{2} \\
&-\int_M S\left\langle \nabla\left(\left|\nabla u\right|e^{bf}\right),\nabla\phi^{2}\right\rangle +\int_M\left\langle \nabla S,\nabla\phi^{2}\right\rangle \left|\nabla u\right|e^{bf}.
\end{aligned}\tag{4.27}$$

Plugging (4.27) into (4.24) we conclude that

$$
\begin{aligned}
(4.28)\qquad & 2\int_M |\mathrm{Ric}|^2 |\nabla u|\, e^{bf}\phi^2 \\
\leq\; & -(2b-a+1)\int_M \langle \nabla |\nabla u|, \nabla f\rangle\, S e^{bf}\phi^2 \\
& -\left(b^2+b\right)\int_M |\nabla f|^2 |\nabla u|\, S e^{bf}\phi^2 \\
& -\int_M \left((b+1)\,\Delta f - \frac{a}{2} - 1\right)|\nabla u|\, S e^{bf}\phi^2 \\
& -(1+a)\int_M \mathrm{Ric}(\nabla u, \nabla u)\, |\nabla u|^{-1}\, S e^{bf}\phi^2 \\
& -\int_M S\left\langle \nabla\left(|\nabla u|\, e^{bf}\right), \nabla\phi^2\right\rangle + \int_M \left\langle \nabla S, \nabla \phi^2\right\rangle |\nabla u|\, e^{bf} \\
& -\int_M S\left\langle \nabla f, \nabla\phi^2\right\rangle |\nabla u|\, e^{bf}.
\end{aligned}
$$

Note that as $T\to\infty$,

$$
\left|\int_M S\left\langle \nabla f, \nabla\phi^2\right\rangle |\nabla u|\, e^{bf}\right| + \left|\int_M S\left\langle \nabla\left(|\nabla u|\, e^{bf}\right), \nabla\phi^2\right\rangle\right| \to 0.
$$

Indeed, this is because by Lemma 4.8,

$$
\int_M |\nabla u|\, e^{\bar{b}f} < \infty
$$

for $\bar{b} = \frac{a+b}{2}$. But

$$
\begin{aligned}
\int_M S\left|\left\langle \nabla f, \nabla\phi^2\right\rangle\right| |\nabla u|\, e^{bf} &\leq C\int_{D(T+1)\setminus D(T)} |\nabla f|^2 |\nabla u|\, e^{bf} \\
&\leq C\int_{D(T+1)\setminus D(T)} |\nabla u|\, e^{\bar{b}f}.
\end{aligned}
$$

Similarly, using that $\nabla S = 2\mathrm{Ric}(\nabla f)$ and Lemma 4.8 we obtain that

$$
\left|\int_M \left\langle \nabla S, \nabla\phi^2\right\rangle |\nabla u|\, e^{bf}\right| \to 0 \text{ as } T\to\infty.
$$

In conclusion, after letting $T\to\infty$ in (4.28), we obtain

$$
\begin{aligned}
(4.29)\qquad 2\int_M |\mathrm{Ric}|^2 |\nabla u|\, e^{bf} \leq\; & -(2b-a+1)\int_M \langle \nabla |\nabla u|, \nabla f\rangle\, S e^{bf} \\
& -\left(b^2+b\right)\int_M |\nabla f|^2 |\nabla u|\, S e^{bf} \\
& -\int_M \left((b+1)\,\Delta f - \frac{a}{2} - 1\right)|\nabla u|\, S e^{bf} \\
& -(1+a)\int_M \mathrm{Ric}(\nabla u, \nabla u)\, |\nabla u|^{-1}\, S e^{bf}.
\end{aligned}
$$

□

We are now ready to prove Theorem 1.5 which is restated below.

Theorem 4.10. *Let (M,g) be a complete gradient shrinking Ricci soliton of dimension $n\geq 4$. Assume that*

$$S\leq \frac{n}{3} \quad \text{on } M.$$

Then M has one end.

Proof. Assume by contradiction that M has at least two ends. Set $a=1$ and consider the F-harmonic function u with $F=-f$.

$$\textbf{Claim:} \qquad \frac{n}{3}\int_M S\left|\nabla u\right| \leq \int_M S^2\left|\nabla u\right|.$$

To verify the claim, we apply Lemma 4.9 with $a=1$ and $b=0$. Then,

$$\begin{aligned} 2\int_M \left|\mathrm{Ric}\right|^2\left|\nabla u\right| &\leq -\int_M\left(\Delta f-\frac{3}{2}\right)\left|\nabla u\right| S \\ &\quad -2\int_M \mathrm{Ric}\left(\nabla u,\nabla u\right)\left|\nabla u\right|^{-1} S. \end{aligned} \tag{4.30}$$

On the other hand, for any constant λ, we have

$$\begin{aligned} -2\mathrm{Ric}\left(\nabla u,\nabla u\right)\left|\nabla u\right|^{-1} S &= -2\left(R_{ij}-\lambda S g_{ij}\right)u_i u_j\left|\nabla u\right|^{-1} S \\ &\quad -2\lambda\left|\nabla u\right| S^2. \end{aligned} \tag{4.31}$$

The first term of the right hand side can be bounded as

$$\begin{aligned} -2\left(R_{ij}-\lambda S g_{ij}\right)u_i u_j\left|\nabla u\right|^{-1} S &\leq 2\left|R_{ij}-\lambda S g_{ij}\right|\left|\nabla u\right| S \\ &\leq 2\left|R_{ij}-\lambda S g_{ij}\right|^2\left|\nabla u\right| + \frac{1}{2}\left|\nabla u\right| S^2 \\ &= 2\left|\mathrm{Ric}\right|^2\left|\nabla u\right| + \left(2n\lambda^2-4\lambda+\frac{1}{2}\right)S^2\left|\nabla u\right|. \end{aligned}$$

Therefore,

$$-2\mathrm{Ric}\left(\nabla u,\nabla u\right)\left|\nabla u\right|^{-1} S\leq 2\left|\mathrm{Ric}\right|^2\left|\nabla u\right| + \left(2n\lambda^2-6\lambda+\frac{1}{2}\right)S^2\left|\nabla u\right|.$$

Optimizing the right hand side by taking $\lambda=\frac{3}{2n}$, we conclude

$$-2\mathrm{Ric}\left(\nabla u,\nabla u\right)\left|\nabla u\right|^{-1} S\leq 2\left|\mathrm{Ric}\right|^2\left|\nabla u\right| + \frac{n-9}{2n}S^2\left|\nabla u\right|. \tag{4.32}$$

Plugging into (4.30) and using that

$$\Delta f=\frac{n}{2}-S$$

we arrive at

$$0\leq \frac{3\left(n-3\right)}{2n}\int_M S^2\left|\nabla u\right| - \frac{n-3}{2}\int_M S\left|\nabla u\right|. \tag{4.33}$$

This verifies the claim.

Since the scalar curvature is assumed to satisfy $S \leq \frac{n}{3}$, it follows from the claim that all the inequalities are in fact equalities. In particular,

$$S = \frac{n}{3}$$

and

$$\left| \mathrm{Ric} - \frac{3}{2n} S g \right| = \frac{1}{2} S.$$

It follows that

$$|\mathrm{Ric}|^2 = \frac{n\,(n+3)}{36}.$$

However, for $n \geq 4$, this is an obvious contradiction to the identity

$$\Delta_f S = S - 2\,|\mathrm{Ric}|^2\,.$$

The theorem is proved. □

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