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Covering systems with odd moduli

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ABSTRACT

The concept of a covering system was first introduced by Erdős in 1950. Since their introduction, a lot of the research regarding covering systems has focused on the existence of covering systems with certain restrictions on the moduli. Arguably, the most famous open question regarding covering systems is the odd covering problem. In this paper, we explore a variation of the odd covering problem, allowing a single odd prime to appear as a modulus in the covering more than once, while all other moduli are distinct, odd, and greater than 1. We also consider this variation while further requiring the moduli of the covering system to be square-free.

1. Introduction

In 1950, Erdős [1] introduced the concept of a covering system of the integers, which is defined as follows.

Definition 1.1. A *covering system of the integers* is a finite collection of congruences such that every integer satisfies at least one of the congruences in the collection.

Since their introduction, the existence of certain types of covering systems and their applications has been the investigation of many articles [2–7,11,13–15]. Of particular interest to the investigation of this paper is the speculative existence of an odd covering. The following is a well known question, originally asked by Erdős.

Question 1.2 *(The Odd Covering Problem).* Does there exist a covering system of the integers whose moduli are all odd, distinct, and greater than 1?

If the answer to this question is yes, then such a covering system would be called an *odd covering system of the integers*. Hough and Nielsen [12] showed in 2019 that if the moduli of a covering system are distinct and greater than 1, then at least one of the moduli of the system must be divisible by either 2 or 3.

If an odd covering exists such that all moduli of the system are square-free, then we refer to this as a *square-free odd covering system of the integers*. In 2005, Guo and Sun [8] showed that the lowest common multiple of all moduli in a square-free odd covering system (if it exists) must have at least 22 distinct prime factors.

In 2015, Harrington [10] posed the following question related to the odd covering problem.

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Question 1.3. Given an odd integer $n \geq 3$, does there exist a covering system of the integers that has n as a modulus at most twice, such that all other moduli are odd, distinct, and greater than 1?

Harrington answered this question in the affirmative when $n = 3$ in the same paper. The following more general question was later asked by Hammer, Harrington, and Marotta [9] in 2018.

Question 1.4. For any odd prime p, let t_p be the smallest nonnegative integer for which there exists a covering system of the integers that has p as a modulus exactly t_p times, such that all other moduli are odd, distinct, and greater than 1. Given a fixed odd prime p, what is t_p?

If an odd covering system of the integers exists, then the answer to Question 1.4 is that $t_p \leq 1$ for all odd primes p. Harrington's result yields $t_3 \leq 2$ and Hammer, Harrington, and Marotta showed that $t_5 \leq 3$. One may further ask a similar question regarding square-free odd coverings.

Question 1.5. For any odd prime p, let τ_p be the smallest nonnegative integer for which there exists a covering system of the integers that has p as a modulus exactly τ_p times, such that all other moduli are odd, distinct, square-free, and greater than 1. Given a fixed odd prime p, what is τ_p?

In Theorem 3.2 of Section 3, we establish that if $\tau_p \leq 2$ for some odd prime p, then there exists an odd covering of the integers. We also show in Section 3 that $\tau_7 \leq 6$. In Section 4, we show that $t_7 \leq 4$, $t_{11} \leq 7$, and $t_p \leq p - 5$ for all primes $p \geq 23$. We conclude our investigation in Section 5 by introducing a lemma that allows us to generalize the main results in Sections 3 and 4.

2. Condensed tree diagram and notation

We will adopt the notation used in [5]. Given lists $[r_1, r_2, \ldots, r_k]$ and $[m_1, m_2, \ldots, m_k]$ with $m_1, m_2 \ldots, m_k$ pairwise relatively prime positive integers, we let

$$([r_1, r_2, \ldots, r_k], [m_1, m_2, \ldots, m_k])$$

denote the congruence system $x \equiv r_j \pmod{m_j}$, $1 \leq j \leq k$. By the Chinese Remainder Theorem, this congruence system is equivalent to the congruence $x \equiv r \pmod{m}$, where $m = m_1 m_2 \cdots m_k$ and $r \in \{0, 1, \ldots, m-1\}$. Hence, we sometimes also refer to this congruence system as simply a congruence. With this notation, we provide the following example of a covering system.

$$\begin{gathered}([1],[2]),\ ([0,1],[2,3]),\ ([0,2],[2,3]),\ ([0,3],[2,9]),\\ ([0,0,0],[2,9,5]),\ ([0,0,1],[2,9,5]),\ ([0,0,2],[2,9,5]),\ ([0,0,3],[2,9,5]),\ ([0,0,4],[2,9,5]),\\ ([0,6,0],[2,9,5]),\ ([0,6,1],[2,9,5]),\ ([0,6,2],[2,9,5]),\ ([0,6,3],[2,9,5]),\ ([0,6,4],[2,9,5]).\end{gathered} \tag{1}$$

This is a covering system since we first cover all integers that are congruent to 1 modulo 2 with $([1],[2])$, then all integers congruent to 0 modulo 2 are split into congruence classes $0, 1, 2$ modulo 3, covered by $([0,0],[2,3])$, $([0,1],[2,3])$, $([0,2],[2,3])$. The congruences $([0,1],[2,3])$ and $([0,2],[2,3])$ are explicit in (1), while integers covered by $([0,0],[2,3])$ are further split into congruence classes $0, 3, 6$ modulo 9, covered by $([0,0],[2,9])$, $([0,3],[2,9])$, $([0,6],[2,9])$. Once again, the congruence $([0,3],[2,9])$ is explicit in (1), while integers covered by each of $([0,0],[2,9])$ and $([0,6],[2,9])$ are split into $0, 1, 2, 3, 4$ modulo 5.

The covering system given in (1), however, has many repeated moduli. To eliminate repeated moduli, we forgo some of the congruences inside a congruence system. For example, we forgo 0 (mod 2) in $([0,1],[2,3])$ and write it as $([1],[3])$, and we forgo 0 (mod 2) and 0 (mod 9) in $([0,0,0],[2,9,5])$ to get $([0],[5])$. When we forgo some congruences inside a congruence system, the set of integers covered by the new congruence system is a superset of the original one. Hence, the covering system given by (1) can be modified to obtain the following covering system.

$$\begin{gathered}([1],[2]),\quad ([1],[3]),\quad ([0,2],[2,3]),\quad ([3],[9]),\\ ([0],[5]),\quad ([0,1],[2,5]),\quad ([0,2],[3,5]),\quad ([0,0,3],[2,3,5]),\quad ([0,4],[9,5]),\\ ([0],[5]),\quad ([0,1],[2,5]),\quad ([0,2],[3,5]),\quad ([0,0,3],[2,3,5]),\quad ([0,6,4],[2,9,5]).\end{gathered} \tag{2}$$

Notice that several congruence systems are repeated in (2), thus we can simplify the covering system as follows.

$$\begin{gathered}([1],[2]),\quad ([1],[3]),\quad ([0,2],[2,3]),\quad ([3],[9]),\quad ([0],[5]),\\ ([0,1],[2,5]),\quad ([0,2],[3,5]),\quad ([0,0,3],[2,3,5]),\quad ([0,4],[9,5]),\quad ([0,6,4],[2,9,5]).\end{gathered}$$

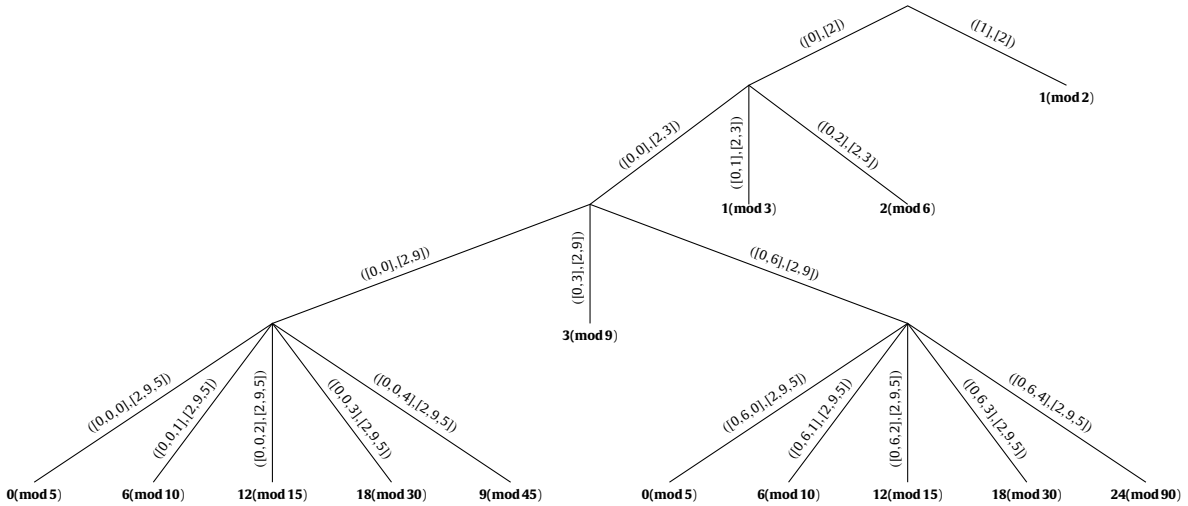

Fig. 1. A tree diagram of a covering system showing all moduli and residues.

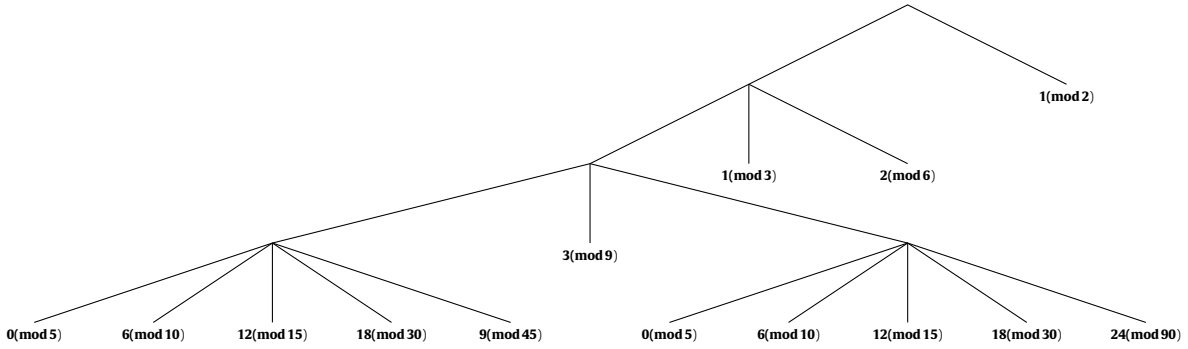

Fig. 2. A tree diagram of a covering system with congruences on the branches omitted.

This corresponds to the following set of congruences.

$$\begin{aligned} &1\,(\mathrm{mod}\,2), \quad 1\,(\mathrm{mod}\,3), \quad 2\,(\mathrm{mod}\,6), \quad 3\,(\mathrm{mod}\,9), \quad 0\,(\mathrm{mod}\,5), \\ &6\,(\mathrm{mod}\,10), \quad 12\,(\mathrm{mod}\,15), \quad 18\,(\mathrm{mod}\,30), \quad 9\,(\mathrm{mod}\,45), \quad 24\,(\mathrm{mod}\,90). \end{aligned} \tag{3}$$

To visualize the covering system given by (1), we represent the process of splitting integers into congruence classes using a tree diagram, as illustrated in Fig. 1. We refer to the node at the top as the *root*, and the nodes with no further branches below as the *leaves*. The congruences given at the leaves of this tree come from (3).

The branches immediately below a node in the tree diagram are called the *child branches* of that node. At each non-leaf node, the number of child branches is always given by a prime number p, and we refer to such a node as a *p-node*. Each p-node represents a certain subset S of the integers, and its child branches represent a partition of S into congruence classes modulo $p^\alpha m'$ for some positive integers α and m', where α is equal to the number of p-nodes along the path from the root and $p \nmid m'$. If the p-node represents the subset of integers satisfying the congruence $([r, r'], [p^{\alpha-1}, m'])$, then we arrange the congruences such that $([r, r'], [p^\alpha, m'])$, $([r+p^{\alpha-1}, r'], [p^\alpha, m'])$, $([r+2p^{\alpha-1}, r'], [p^\alpha, m'])$, $\ldots$, $([r+(p-1)p^{\alpha-1}, r'], [p^\alpha, m'])$ are placed from left to right among the child branches. In other words, the congruences on the branches are implied by the positioning of the branches and thus can be omitted from the diagram, as illustrated in Fig. 2.

Next, consider a fixed leaf in the tree diagram. Recall that the congruence is originally obtained by combining all congruences along the path from the root using the Chinese Remainder Theorem. We then choose a subset of these congruences along the path to form the congruence displayed at the leaf. Note that the residue in this congruence is implied by the modulus of the leaf since the factors in the modulus determine which congruences along the path were used. Hence, the residue at the leaf can also be omitted from the diagram, as illustrated in Fig. 3.

Continuing to simplify the tree diagram, if the moduli at several leaves branching from a node share the same factor m_0, then we can merge them to form a wedge, and we write $\{m_1, m_2, \ldots, m_k\} \times m_0$ to indicate 2^k moduli, given by the product of any subset of $\{m_1, m_2, \ldots, m_k\}$ together with m_0. These congruences are arranged so that their moduli increase in magnitude from left to right among the child branches. This allows us to obtain a condensed tree diagram in Fig. 4.

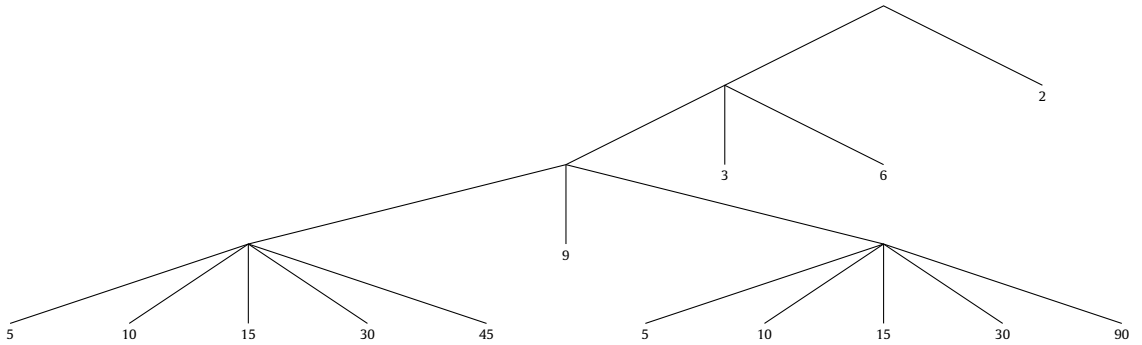

Fig. 3. A tree diagram of a covering system showing only the moduli.

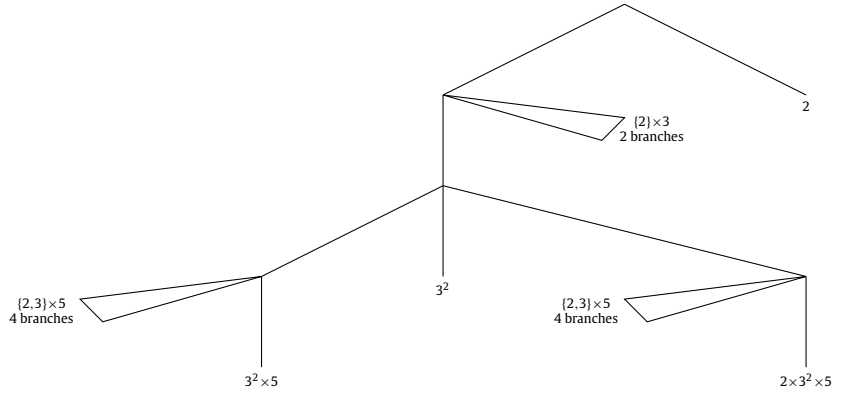

Fig. 4. A condensed tree diagram of the covering system given by (3).

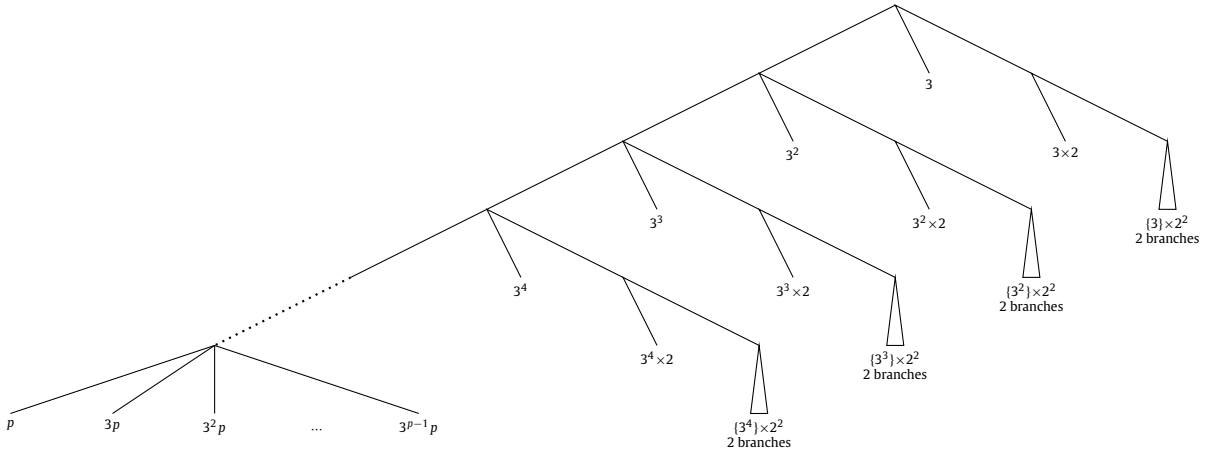

Fig. 5. A condensed tree diagram of a covering system whose smallest modulus is 3.

To illustrate a common technique used when building covering systems of the integers, we now turn our attention to a different system. Let $p > 3$ be a prime and consider the following tree diagram.

In Fig. 5, note that the structure of the subtree given in Fig. 6 is repeated for $i \in \{1, 2, \ldots, p-1\}$. As we consistently split the leftmost node to attach a repeated subtree, we call the leftmost branch from the root a "power branch" of the tree, and condense it as in Fig. 7. To convert Fig. 7 back to Fig. 5, the i-th repeated subtree is obtained by substituting every factor of 3 in the first repeated subtree with 3^i for $2 \leq i \leq p-1$.

We will use tree diagrams similar to Fig. 7 throughout the rest of this paper to denote various covering systems.

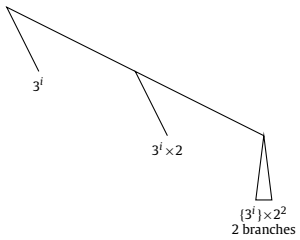

Fig. 6. Repeated subtree from Fig. 5.

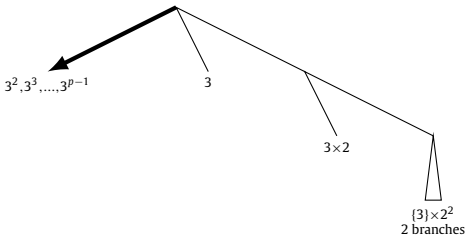

Fig. 7. A tree diagram using a "power branch" notation.

3. Investigating Question 1.5

To begin our investigation for square-free covering systems of the integers, we first provide the following lemma, originally given by Hammer, Harrington, and Marotta [9].

Lemma 3.1. *Let p be a prime and let r_1 and r_2 be distinct integers with $0 \le r_1, r_2 \le p-1$. Suppose $\mathcal{C}$ is a covering system of the integers such that $r_1 \pmod p \in \mathcal{C}$ and $r_2 \pmod p \notin \mathcal{C}$. Then there exists a covering system $\mathcal{C}'$ with the same moduli as those of $\mathcal{C}$ such that $r_2 \pmod p \in \mathcal{C}'$ and $r_1 \pmod p \notin \mathcal{C}'$. Furthermore, if $r \pmod p \in \mathcal{C}$ with $r \not\equiv r_1 \pmod p$, then $r \pmod p \in \mathcal{C}'$.*

Theorem 3.2. *Let $p \ge 3$ be a prime. If there exists a covering system of the integers such that all moduli are odd, square-free, and distinct except that p is used exactly twice as a modulus, then there exists an odd covering of the integers.*

Proof. Suppose that $\mathcal{C}_0$ is a covering system of the integers such that all moduli are odd, square-free, and distinct except that p is used exactly twice as a modulus. By Lemma 3.1, we may assume that $0 \pmod p$ and $1 \pmod p$ are congruences in $\mathcal{C}_0$. Let k be a nonnegative integer and let ℓ_ξ be nonnegative integers for each $2 \le \xi \le p-1$ such that

$$\mathcal{C}_0 = \{0 \pmod p, 1 \pmod p\} \cup \{r_j \pmod{m_j} : 1 \le j \le k\} \cup \bigcup_{\xi=2}^{p-1} \{r_{\xi,j} \pmod{pm_{\xi,j}} : 1 \le j \le \ell_\xi\},$$

where $p \nmid m_j$ for all $1 \le j \le k$, and $p \nmid m_{\xi,j}$ and $r_{\xi,j} \equiv \xi \pmod p$ for all $1 \le j \le \ell_\xi$ and $2 \le \xi \le p-1$.

We claim that for each $2 \le \xi \le p-1$, the set of congruences

$$\mathcal{C}_\xi = \{r_j \pmod{m_j} : 1 \le j \le k\} \cup \{r_{\xi,j} \pmod{m_{\xi,j}} : 1 \le j \le \ell_\xi\}$$

forms a covering system of the integers. To see this, consider any integer r. Let

$$m = \operatorname{lcm}\big(\{m_j : 1 \le j \le k\} \cup \{m_{\xi,j} : 1 \le j \le \ell_\xi\}\big)$$

and let n be an integer that satisfies the congruence system

$$\begin{cases} n \equiv \xi \pmod p, \\ n \equiv r \pmod m. \end{cases}$$

Since $\mathcal{C}_0$ is a covering system of the integers, n must satisfy one of the congruences in

$$\{r_j \pmod{m_j} : 1 \le j \le k\} \cup \{r_{\xi,j} \pmod{pm_{\xi,j}} : 1 \le j \le \ell_\xi\}.$$

As a result, if $n \equiv r_{\tilde{j}} \pmod{m_{\tilde{j}}}$ for some $1 \le \tilde{j} \le k$, then

$$r \equiv n \equiv r_{\tilde{j}} \pmod{m_{\tilde{j}}};$$

if $n \equiv r_{\xi,\tilde{j}} \pmod{pm_{\xi,\tilde{j}}}$ for some $1 \le \tilde{j} \le \ell_\xi$, then

$$r \equiv n \equiv r_{\xi,\tilde{j}} \pmod{m_{\xi,\tilde{j}}}.$$

This completes the proof of our claim.

Let q be an odd prime such that $q \nmid pm$. For each $0 \le i \le q-2$, $2 \le \xi \le p-2$, and $1 \le j \le \ell_\xi$, let $r_{\xi,i,j}$ be an integer that satisfies the congruence system

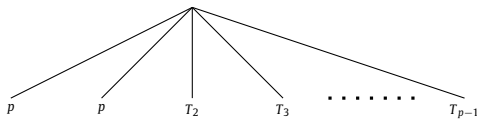

Fig. 8. The root of a tree diagram for $\mathcal{C}_0$.

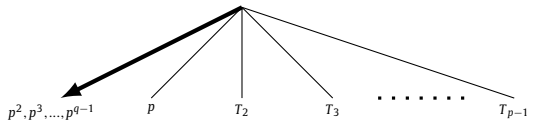

Fig. 9. The root of a tree diagram for $\mathcal{C}$.

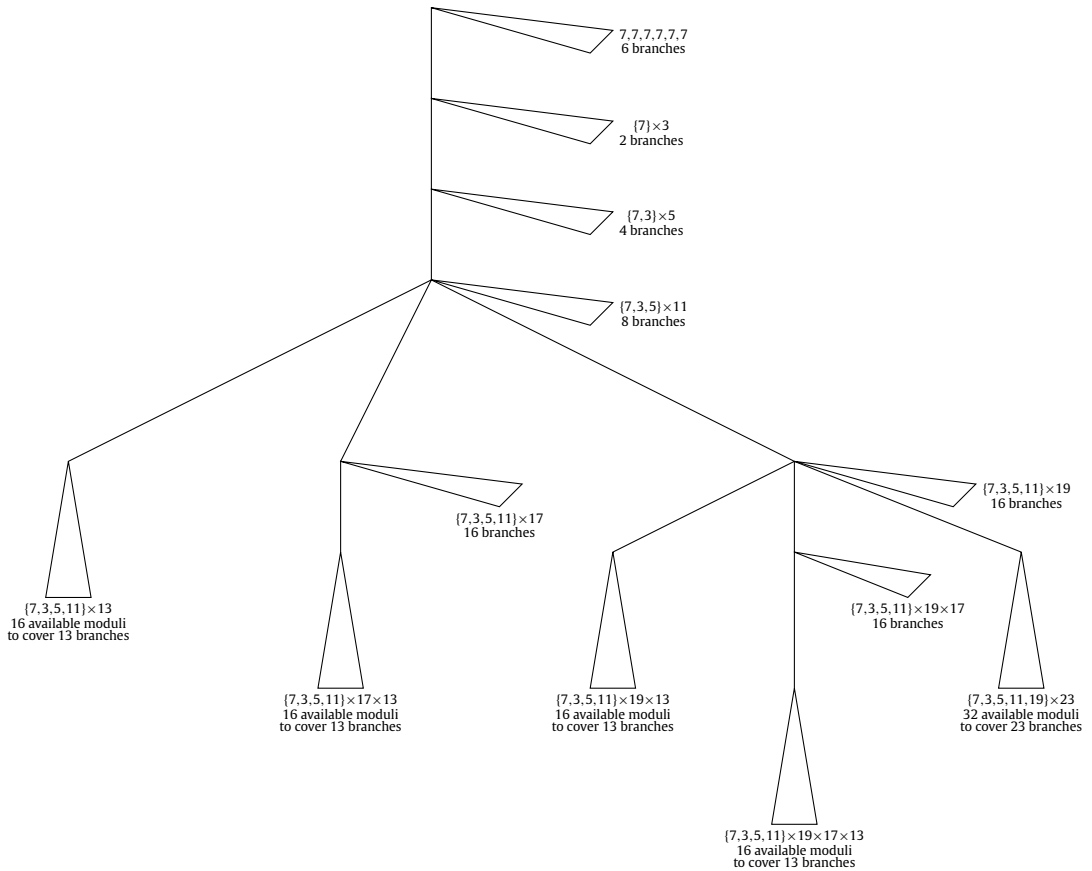

Fig. 10. Square-free odd covering with 7 used exactly six times as a modulus.

$$\begin{cases} r_{\xi,i,j} \equiv \xi \cdot p^i \pmod{p^{i+1}}, \\ r_{\xi,i,j} \equiv r_{\xi,j} \pmod{m_{\xi,j}}. \end{cases}$$

Moreover, for each $0 \le i \le q-1$, let s_i be an integer that satisfies the congruence system

$$\begin{cases} s_i \equiv 0 \pmod{p^i}, \\ s_i \equiv i \pmod{q}. \end{cases}$$

Now, we are ready to build an odd covering system based on the existence of $\mathcal{C}_0$. Consider

$$\begin{aligned} \mathcal{C} = &\{p^i \pmod{p^{i+1}} : 0 \le i \le q-2\} \cup \{s_i \pmod{p^i q} : 0 \le i \le q-1\} \\ &\cup \{r_j \pmod{m_j} : 1 \le j \le k\} \cup \bigcup_{\xi=2}^{p-1} \{r_{\xi,i,j} \pmod{p^{i+1} m_{\xi,j}} : 0 \le i \le q-2, 1 \le j \le \ell_\xi\}. \end{aligned}$$

It is easy to see that all moduli of $\mathcal{C}$ are odd and distinct. To show that $\mathcal{C}$ is a covering system of the integers, let N be an arbitrary integer, and let i_0 be the maximum integer such that $p^{i_0} \mid N$. If $0 \le i_0 \le q-2$, then either $N \equiv p^{i_0} \pmod{p^{i_0+1}}$, which is trivially covered by $\mathcal{C}$, or $N \equiv \xi \cdot p^{i_0} \pmod{p^{i_0+1}}$ for some $2 \le \xi \le p-1$, which is covered by

$$\{r_j \pmod{m_j} : 1 \le j \le k\} \cup \{r_{\xi,i_0,j} \pmod{p^{i_0+1} m_{\xi,j}} : 1 \le j \le \ell_\xi\}$$

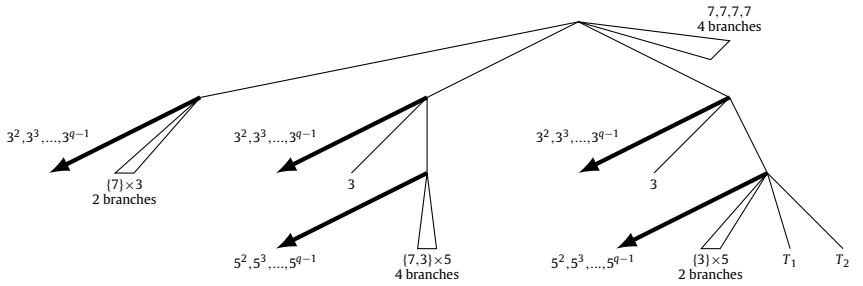

Fig. 11. Odd covering with 7 used exactly four times as a modulus.

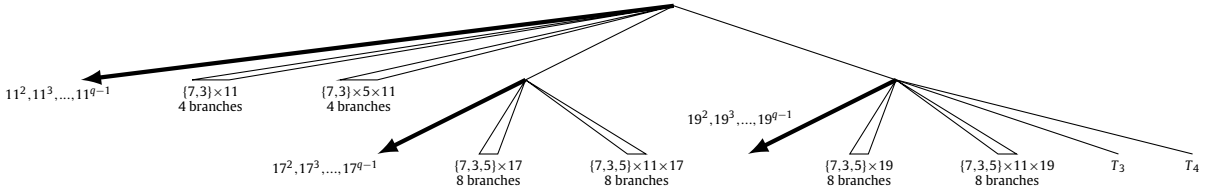

Fig. 12. T_1 in Fig. 11.

since C_ξ is a covering system of the integers. Lastly, if $i_0 \geq q-1$, then N is covered by $\{s_i \pmod{p^i q} : 0 \leq i \leq q-1\}$. □

Remark 3.3. The proof of Theorem 3.2 can be summarized as a transformation between two tree diagrams. Consider a tree diagram for C_0 as shown in Fig. 8. In this tree diagram, the root is a p-node, and for each $i \in \{2, 3, \ldots, p-1\}$, T_i denotes the subtree below the node that corresponds to the congruence $i \pmod p$. If we replace the leftmost branch by a power branch $p^2, p^3, \ldots, p^{q-1}$, as shown in Fig. 9, then the resultant tree diagram represents a covering system $\mathcal{C}$ with odd and distinct moduli that are greater than 1.

Recalling the definition of τ_p from Question 1.5, we note that Theorem 3.2 can be restated as follows: if $\tau_p \leq 2$ for some prime $p \geq 3$, then there is an odd covering system of the integers. Hence, to investigate the existence of an odd covering system, one possible approach is to give bounds on τ_p for an odd prime p. In the following, we show that $\tau_7 \leq 6$.

Theorem 3.4. *There exists a covering system of the integers such that all moduli are odd, square-free, and distinct except that* 7 *is used exactly six times as a modulus.*

Proof. Fig. 10 shows a tree diagram for a covering system of the integers such that all moduli are odd, square-free, and distinct except that 7 is used exactly six times as a modulus. In the tree diagram, note that at the wedge $\{7, 3, 5, 11\} \times 13$, there are 16 available moduli according to the discussions in Section 2, but there are only 13 branches since it is a 13-node. Thus, we are going to choose the 13 smallest moduli given by the product of subsets of $\{7, 3, 5, 11\}$ with 13 among the child branches. Similar situations occurs at a few other wedges, as indicated in the figure. □

4. Investigating Question 1.4

Recall the definition of t_p from Question 1.4. In this section, we establish upper bounds for t_p for all odd primes p. We begin by showing that $t_7 \leq 4$.

Theorem 4.1. *There exists a covering system of the integers such that all moduli are odd and distinct except that* 7 *is used exactly four times as a modulus.*

Proof. A tree diagram of such a covering system is given by Fig. 11, where subtrees T_1 to T_6 are given in Figs. 12 to 17. Here, $q > 19$ is a prime, and as discussed in Section 2, we note that the repeated moduli in the tree diagram give rise to the same congruences in the covering.

Please take special note that some power branches in Fig. 11 are nested into other power branches. For instance, the middle section of this tree contains a power branch with powers of 5, and this power branch is nested into the power branch with powers of 3 that appears above it. We further note that in the power branch $7^3, 7^4, \ldots, 7^{q-1}$ in Fig. 15, the repeated subtrees are obtained by substituting every factor of 7^2 (but not 7) in the first repeated subtree with 7^i for $3 \leq i \leq q-1$. □

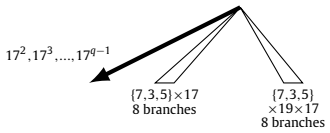

Fig. 13. T_3 in Fig. 12.

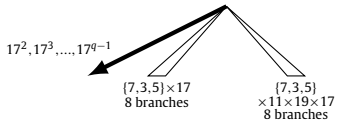

Fig. 14. T_4 in Fig. 12.

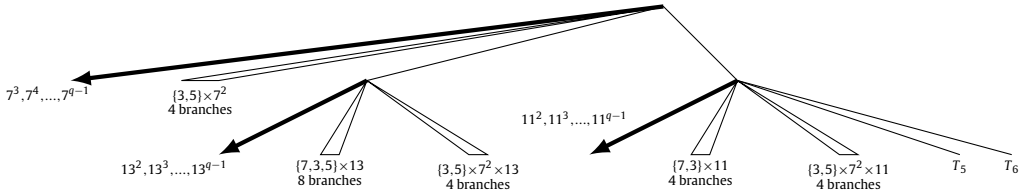

Fig. 15. T_2 in Fig. 11.

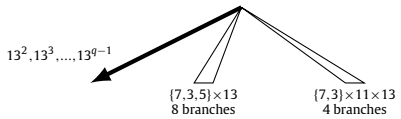

Fig. 16. T_5 in Fig. 15.

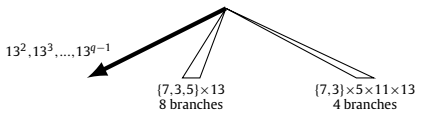

Fig. 17. T_6 in Fig. 15.

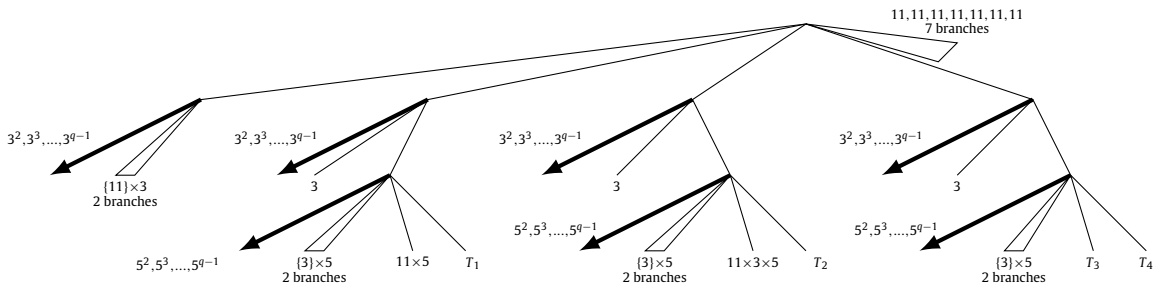

Fig. 18. Odd covering with 11 used exactly seven times as a modulus.

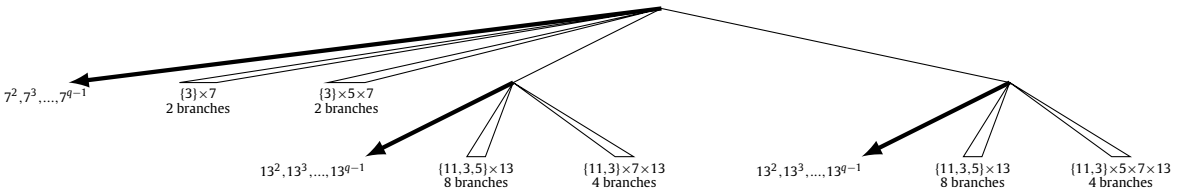

Fig. 19. T_1 in Fig. 18.

Next, we show that $t_{11} \leq 7$.

Theorem 4.2. *There exists a covering system of the integers such that all moduli are odd and distinct except that* 11 *is used exactly seven times as a modulus.*

Proof. A tree diagram of such a covering system is given by Fig. 18, where subtrees T_1 to T_4 are given in Figs. 19 to 22. Here, $q > 19$ is a prime. □

Lastly, we establish that $t_p \leq p - 5$ for all primes $p \geq 23$.

Theorem 4.3. *Let $p \geq 23$ be a prime. There exists a covering system of the integers such that all moduli are odd and distinct except that p is used exactly $p - 5$ times as a modulus.*

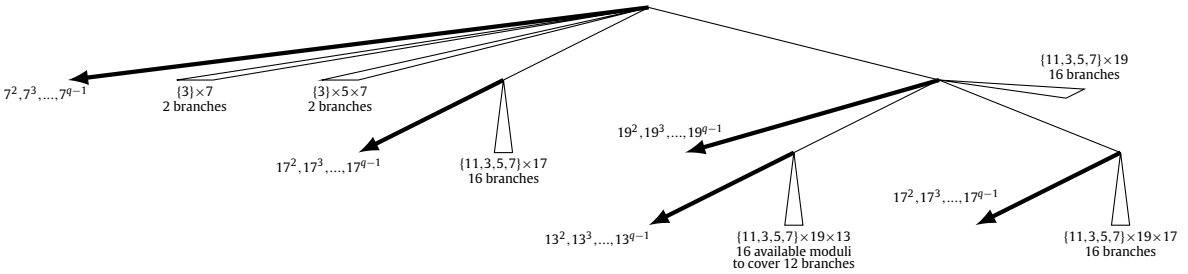

Fig. 20. T_2 in Fig. 18.

Fig. 21. T_3 in Fig. 18.

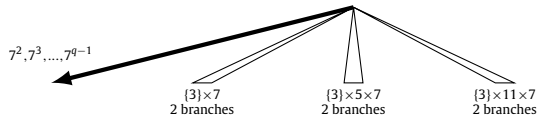

Fig. 22. T_4 in Fig. 18.

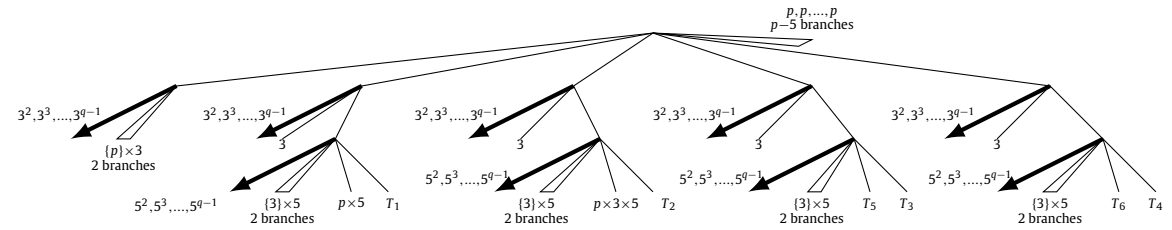

Fig. 23. Odd covering with p ($p \geq 23$) used exactly $p-5$ times as a modulus.

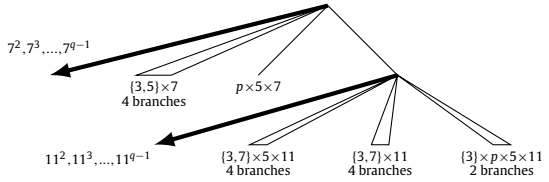

Fig. 24. T_1 in Fig. 23.

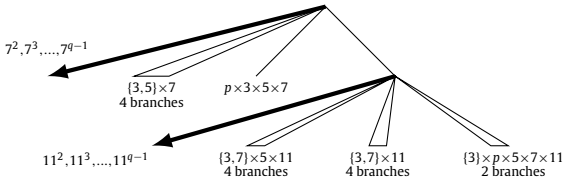

Fig. 25. T_2 in Fig. 23.

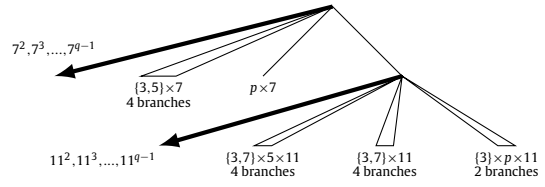

Fig. 26. T_3 in Fig. 23.

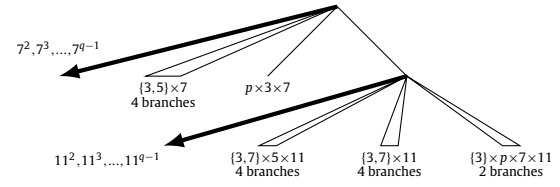

Fig. 27. T_4 in Fig. 23.

Proof. A tree diagram of such a covering system is given by Fig. 23, where subtrees T_1 to T_{11} are given in Figs. 24 to 34, respectively. Here, $q > 19$ and $q \neq p$ is a prime. □

5. Extension of results and concluding remarks

We can extend Theorems 3.4 and 4.2 by using the following lemma.

Lemma 5.1. *Let p be a prime and t be a positive integer such that $t \leq p$. Suppose that $\mathcal{C}_0$ is a covering system of the integers whose tree diagram is given in Figure 35, where all moduli are distinct except that p is used exactly $p-t$ times as a modulus, and no modulus of*

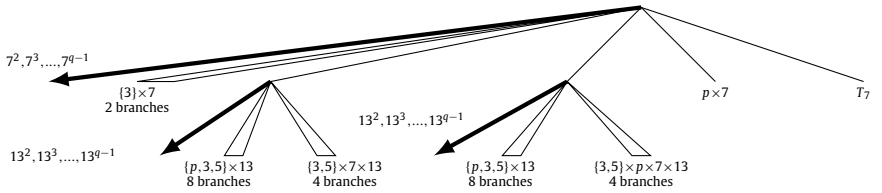

Fig. 28. T_5 in Fig. 23.

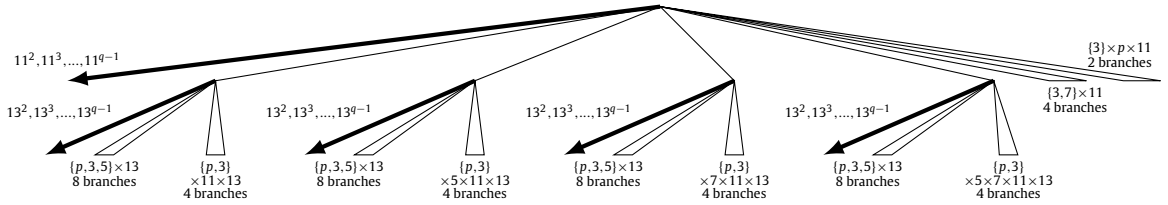

Fig. 29. T_7 in Fig. 28.

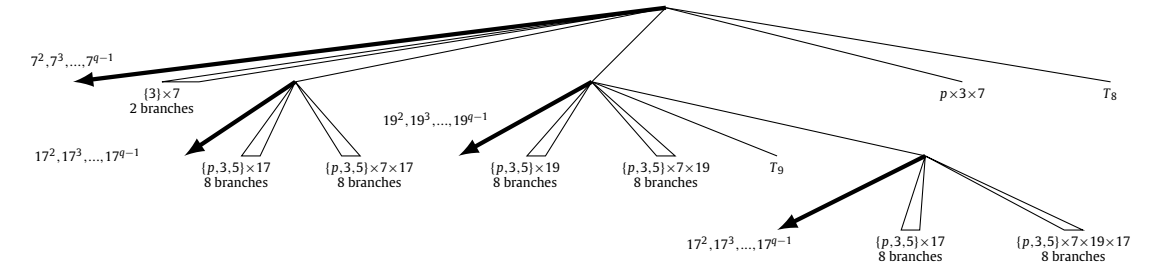

Fig. 30. T_6 in Fig. 23.

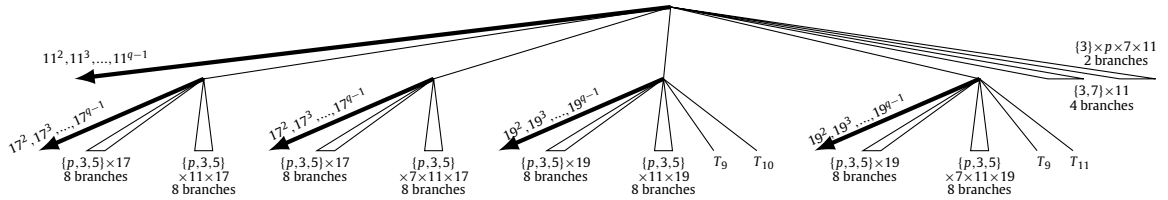

Fig. 31. T_8 in Fig. 30.

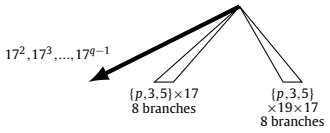

Fig. 32. T_9 in Fig. 30 and Fig. 31.

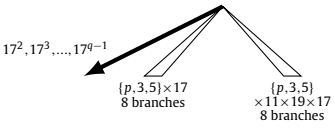

Fig. 33. T_{10} in Fig. 31.

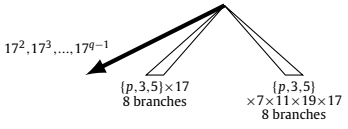

Fig. 34. T_{11} in Fig. 31.

$\mathcal{C}_0$ is divisible by p^2. Then for all primes $q > p$, there exists a covering system $\mathcal{C}$ of the integers such that all moduli are distinct except that q is used exactly $q - t$ times as a modulus, and no modulus of $\mathcal{C}$ is divisible by q^2. Furthermore, if all moduli of $\mathcal{C}_0$ are odd, then all moduli of $\mathcal{C}$ are odd; if all moduli of $\mathcal{C}_0$ are square-free, then all moduli of $\mathcal{C}$ are square-free.

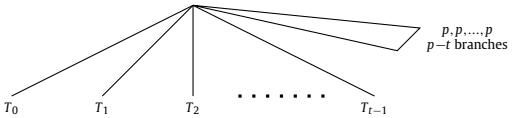

Fig. 35. The root of a tree diagram for $\mathcal{C}_0$.

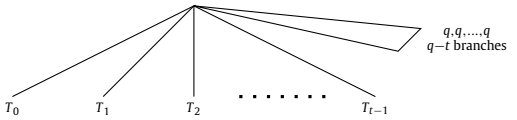

Fig. 36. The root of a tree diagram for $\mathcal{C}$.

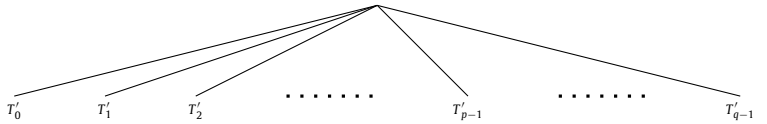

Fig. 37. A q-node in the tree diagram for $\mathcal{C}_0$.

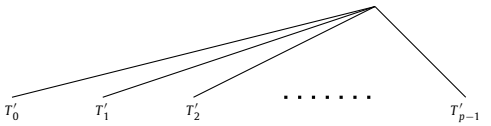

Fig. 38. A p-node in the tree diagram for $\mathcal{C}$.

Proof. In the tree diagram of $\mathcal{C}_0$ as shown in Fig. 35, the root is a p-node, and for each $i \in \{0, 1, \ldots, t-1\}$, T_i denotes the subtree below the node that corresponds to the congruence $i \pmod p$. Note that this is the only p-node in the tree since no modulus of $\mathcal{C}_0$ is divisible by p^2.

To obtain a tree diagram for $\mathcal{C}$, we are going to replace the root of $\mathcal{C}_0$ by a q-node as shown in Fig. 36. The only difficulty in this root replacement is that it may not be compatible with an existing q-node in the tree diagram for $\mathcal{C}_0$. To solve this issue, we replace every q-node in the tree diagram for $\mathcal{C}_0$ (Fig. 37) by a p-node (Fig. 38) and keep only the subtrees T'_i for $i \in \{0, 1, \ldots, p-1\}$. Lastly, if a modulus m in $\mathcal{C}_0$ satisfies $m = pr$ such that $p \nmid r$, then the resultant modulus in $\mathcal{C}$ is qr; if a modulus m in $\mathcal{C}_0$ satisfies $m = q^\alpha s$ for some positive integer α such that $q \nmid s$, then the resultant modulus in $\mathcal{C}$ is $p^\alpha s$. This completes our construction of a tree diagram for $\mathcal{C}$. □

Combining Lemma 5.1 with Theorem 3.4, we have the following result.

Corollary 5.2. *For all primes $p \geq 7$, $\tau_p \leq p - 1$.*

As for the bounds on t_p for primes $p \geq 3$, we summarize the results in the following table.

Table 1
Summary of results regarding Question 1.4.

p	Upper bound on t_p	Sources
3	$t_p \leq 2 = p - 1$	[10]
5	$t_p \leq 3 = p - 2$	[9]
7	$t_p \leq 4 = p - 3$	Theorem 4.1
$11 \leq p \leq 19$	$t_p \leq p - 4$	Theorem 4.2 & Lemma 5.1
$p \geq 23$	$t_p \leq p - 5$	Theorem 4.3

We observe from Table 1 that there exists a constant c such that for all sufficiently large primes p, $t_p \leq p - c$. For future investigation, it is certainly of interest to continue improving on the constant c. Nevertheless, it is of greater interest to investigate the following question.

Question 5.3. Does there exist a constant $0 \leq \epsilon < 1$ such that for all sufficiently large primes p, $t_p \leq \epsilon p$?

The existence of an odd covering system would imply that we can take $\epsilon = 0$ for all sufficiently large primes p, thus Question 5.3 provides a progressive approach to solving the odd covering problem.

Before we end this paper, we also provide a similar lemma to Lemma 5.1, which could be useful in the future.

Lemma 5.4. *Let p be a prime and t be a positive integer such that $t \leq p$. Let $\mathcal{C}_0$ be a covering system of the integers whose tree diagram is given in Figure 39, where all moduli are distinct except that p is used exactly $p - t$ times as a modulus. Let M be the least common multiple of all moduli of $\mathcal{C}_0$. Then for all odd primes $q \geq t$ that satisfies $q \nmid M$, there exists a covering system $\mathcal{C}$ of the integers such that*

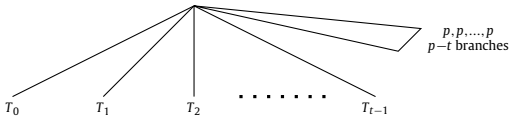

Fig. 39. The root of a tree diagram for $\mathcal{C}_0$.

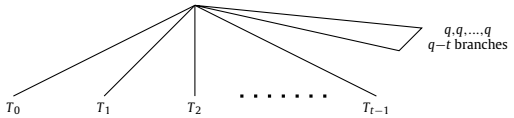

Fig. 40. The root of a tree diagram for $\mathcal{C}$.

all moduli are distinct except that q is used exactly $q-t$ times as a modulus. Furthermore, if all moduli of $\mathcal{C}_0$ are odd, then all moduli of $\mathcal{C}$ are odd; if all moduli of $\mathcal{C}_0$ are square-free, then all moduli of $\mathcal{C}$ are square-free.

Proof. In the tree diagram of $\mathcal{C}_0$ as shown in Fig. 39, the root is a p-node, and for each $i \in \{0, 1, \ldots, t-1\}$, T_i denotes the subtree below the node that corresponds to the congruence $i \pmod p$.

To obtain a tree diagram for $\mathcal{C}$, we are going to replace the root of $\mathcal{C}_0$ by a q-node as shown in Fig. 40. There are no existing q-nodes in the tree diagram for $\mathcal{C}_0$, so we do not need to perform any further replacement, and have already obtained a tree diagram for $\mathcal{C}$. Nonetheless, it is worth noting that if a modulus m in $\mathcal{C}_0$ satisfies $m = p^\alpha r$ for some positive integer α such that $p \nmid r$, then the resultant modulus in $\mathcal{C}$ is $qp^{\alpha-1}r$. □

Declaration of competing interest

We declare that there is no conflict of interests in submitting this manuscript for publication.

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