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NORMAL MEASURES ON LARGE CARDINALS

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Abstract. The space of normal measures on a measurable cardinal is naturally ordered by the Mitchell ordering. In the first part of this paper we show that the Mitchell ordering can be linear on a strong cardinal where the Generalised Continuum Hypothesis fails. In the second part we show that a supercompact cardinal at which the Generalised Continuum Hypothesis fails may carry a very large number of normal measures of Mitchell order zero.

Introduction

If κ is a measurable cardinal, then the *Mitchell ordering* on normal measures on κ is defined by $U \lhd U' \iff U \in \mathrm{Ult}(V, U')$. We review a few standard facts and definitions:

- If $U \lhd U'$ then $j_U(\kappa) < j_{U'}(\kappa) < (2^\kappa)^+$.
- The Mitchell ordering is well-founded and has height at most $(2^\kappa)^+$.
- The *Mitchell order* of κ is the height of the Mitchell ordering, and the *Mitchell order* of a normal measure is its height in the Mitchell ordering.
- A normal measure U has Mitchell order zero if and only if U concentrates on non-measurable cardinals, equivalently κ is not measurable in $\mathrm{Ult}(V, U)$. If κ is the least measurable cardinal then automatically all normal measures on κ have order zero.
- For any normal measure U on κ, there are at most 2^κ many normal measures U' with $U' \lhd U$.

If κ is a strong cardinal then the Mitchell ordering at κ has the maximal height and cardinality. More precisely there are 2^{2^κ} normal measures and the Mitchell order of κ is $(2^\kappa)^+$. See Fact 1.4 below for the proofs of these facts.

If V is the canonical inner model for a strong cardinal then the Mitchell ordering at each measurable κ is linear, and since GCH holds the height of the Mitchell ordering is at most κ^{++}. See Remark 1.8 below for a proof of linearity in this model.

The situation is less clear when κ is a supercompact cardinal, or when κ is strong and $2^\kappa > \kappa^+$, and this is largely what motivates the work in this paper. We note that:

- Recent work by Goldberg [17] shows that if the Ultrapower Axiom (UA) holds then the Mitchell ordering is always linear. UA holds in any inner

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model constructed by the known techniques, so we should expect that it is consistent with the existence of a supercompact cardinal.
- The argument that if κ is strong then the Mitchell order at κ has the maximal height and cardinality does not completely pin down the structure of the Mitchell ordering. In particular when $2^\kappa > \kappa^+$ the argument leaves open whether the ordering is linear, and the related question of how many measures of order zero there are.

This paper has two main parts. In Section 1 we investigate the Mitchell ordering at a strong cardinal κ, and show that:

- (Theorem 1.23) It is consistent that GCH fails at κ while the Mitchell ordering at κ is linear.
- (Theorem 1.26) It is consistent that GCH fails at κ while the Mitchell ordering at κ is a non-linear prewellordering.
- (Theorem 1.31) It is consistent that GCH holds at κ while the Mitchell ordering at κ is a non-linear prewellordering.

The proofs in Section 1 involve analysing extenders in canonical inner models and their generic extensions, using a small amount of inner model theory.

In Section 2 we investigate measures of order zero on a measurable cardinal κ in a setting where $2^\kappa > \kappa^+$. We show that:

- (Theorem 2.1) It is consistent that there exists a normal measure of order zero on κ whose associated ultrapower exhibits a surprising degree of closure.
- (Theorem 2.8) It is consistent that a measurable cardinal κ where GCH fails can have a very large (in a sense to be made precise) number of normal measures of order zero.
- (Theorem 2.10) It is consistent that there exists a supercompact cardinal, GCH fails at every measurable cardinal, and every measurable cardinal carries a very large number of normal measures of order zero.

The proofs in Section 2 involve lifting various ultrapower maps onto generic extensions.

For general background on large cardinals (including the basic facts about extenders) and forcing, we refer the reader to Kanamori's monograph [18] and the second author's survey [12]. Mitchell's Handbook paper [23] and Zeman's monograph [28] both cover all the inner model theory we will use.

A *prewellordering* is a preordering whose quotient ordering is a wellordering. The *level* of an element is its height in the quotient ordering, so that if $\alpha < \beta$ all elements on level α are below all elements on level β.

Investigating the number of measures on a measurable cardinal and the possible structures for the Mitchell ordering on those measures is a longstanding theme in set theory. Mitchell [21, 22] introduced the Mitchell ordering as part of his seminal work on canonical inner models with many measurable cardinals. Kunen and Paris [19] used forcing to show that a measurable cardinal can carry many normal measures of Mitchell order zero. Subsequently some relevant partial results were obtained by Apter, Cummings and Hamkins [2], Baldwin [6], Cummings [10, 11], Leaning and Ben-Neria [20], and Witzany [25] among others.

Questions about the number of measures of order zero on a measurable cardinal were largely settled by Friedman and Magidor [14]. Among their main results are that:

- ([14, Theorem 1]) It is consistent (modulo the existence of a measurable cardinal) that there is a measurable cardinal carrying any (reasonable) prescribed number of normal measures. More precisely: if κ is measurable, GCH holds and $1 \leq \alpha \leq \kappa^{++}$ then there is a cofinality-preserving generic extension in which κ is measurable and carries exactly α measures.
- ([14, Theorem 12]) It is consistent (modulo the existence of a $(\kappa+2)$-strong cardinal) that there is a measurable cardinal at which GCH fails and which carries exactly one normal measure. Subsequently Ben-Neria and Gitik [9] obtained the same conclusion from the optimal hypothesis, the existence of κ such that $o(\kappa) = \kappa^{++}$.

The question about the potential structure of the Mitchell ordering was completely settled by Ben-Neria [8, 7]. He showed, by a mixture of techniques from forcing and inner model theory, that any well-founded partial ordering may be realised as a Mitchell ordering.

- ([8, Theorem 7.5]) From an assumption weaker than $o(\kappa) = \kappa^+$, every "tame" partial ordering of size at most κ can be realised as the Mitchell ordering on κ.
- ([7, Theorem 1.1]) If V is a canonical inner model for a technical assumption about a measurable cardinal κ, involving extenders which overlap measurable cardinals above κ, then for every well-founded poset S with $|S| \leq \kappa$ there is a set-generic extension in which S is realised as the Mitchell ordering on normal measures at κ.
- ([7, Corollary 3.1]) If V is a canonical inner model for a global version of the hypothesis of ([7, Theorem 1.1]), then there is a class-generic extension in which every well-founded partial ordering may be realised as a Mitchell ordering on some measurable cardinal.

1. Measures on a strong cardinal

Our main result in this section is that consistently κ may be strong with GCH failing at κ and the Mitchell order at κ linear. In this model $2^\kappa = 2^{\kappa^+} = \kappa^{++}$ and $2^{\kappa^{++}} = \kappa^{+++}$. We'll also produce some models where κ is strong, GCH may either hold or fail at κ, and the Mitchell ordering at κ is a non-linear prewellordering.

1.1. Measures and extenders.

We begin by recalling a few standard facts about extenders and their generators. All the extenders we use will be *short*, that is we only consider (κ, λ)-extenders E where $\lambda < j_E(\kappa)$. It is easy to see that $E \notin \mathrm{Ult}(V, E)$ for any such extender E, because if $E \in \mathrm{Ult}(V, E)$ then $\mathrm{Ult}(V, E)$ can construct a surjection from $[\lambda]^{<\omega} \times {}^{[\kappa]^{<\omega}}\kappa$ onto $j_E(\kappa)$ in which (a, f) maps to $j_E(f)(a)$. This contradicts the inaccessibility of $j_E(\kappa)$ in $\mathrm{Ult}(V, E)$.

Suppose that E is a (κ, λ) extender and $j_E : V \to M = \mathrm{Ult}(V, E)$ is the associated ultrapower map. The *set of generators of* E is the set of $\alpha \in [\kappa, \lambda)$ such that $\alpha \notin \{j_E(f)(a) : f : [\kappa]^{<\omega} \to \kappa, a \in [\alpha]^{<\omega}\}$. It is easy to see that:

- κ is a generator.
- If A is the set of generators, then $M = \{j_E(f)(a) : f : [\kappa]^{<\omega} \to V, f \in V, a \in [A]^{<\omega}\}$.

- E is equivalent to a normal measure on κ (that is to say there is a normal measure U on κ with $j_U = j_E$) if and only if κ is the only generator.
- E is equivalent to a (κ, α)-extender if and only if the set of generators is a subset of α.

Now suppose that $\mathbb{P}$ is a forcing poset in V, G is $\mathbb{P}$-generic over V and there is $H \in V[G]$ such that H is $j_E(\mathbb{P})$-generic over V with $j_E[G] \subseteq H$. Then we may lift j_E to obtain an embedding $j^* : V[G] \to M[H]$. It is easy to see that:

- $M[H] = \{j^*(f)(a) : f : [\kappa]^{<\omega} \to V[G], f \in V[G], a \in [\lambda]^{<\omega}\}$.
- j^* is the ultrapower map formed from a (κ, λ) extender $E^* \in V[G]$ such that $E_a \subseteq E^*_a$ for all a.
- The set of generators of E^* is a subset of the set of generators of E.

We will be particularly interested in the possibility that κ is the only generator of E^*, so that E^* is equivalent to a normal measure.

We also need some information about the closure properties of extender ultrapowers. The following fact is well-known:

Fact 1.1. *Let E be a (κ, λ)-extender. The following are equivalent:*

- $\mathrm{Ult}(V, E)$ *is closed under κ-sequences.*
- ${}^{\kappa}\lambda \subseteq \mathrm{Ult}(V, E)$.

Proof. For the non-trivial direction, let $\vec{x} = (x_\alpha)_{\alpha<\kappa}$ be a sequence of elements of $\mathrm{Ult}(V, E)$ and let $x_\alpha = j_E(f_\alpha)(a_\alpha)$, where $a_\alpha \in [\lambda]^{n_\alpha}$ and $\mathrm{dom}(f_\alpha) = [\kappa]^{n_\alpha}$. By hypothesis $\vec{a} = (a_\alpha)_{\alpha<\kappa} \in \mathrm{Ult}(V, E)$. Let $\vec{F} = (f_\alpha)_{\alpha<\kappa}$, then $j_E(\vec{F}) \restriction \kappa = (j_E(f_\alpha))_{\alpha<\kappa} \in \mathrm{Ult}(V, E)$, so that easily $\vec{x} \in \mathrm{Ult}(V, E)$. □

For use later, we record a general fact about extenders which generalises a result of Friedman and Thompson [15, proof of Lemma 4].

Fact 1.2. *Let E be a (κ, λ)-extender such that the set of generators is contained in $j_E(h)(\kappa)$ for some function $h : \kappa \to \kappa$. Then κ is the only ordinal which lies in $j_E(C)$ for every club $C \subseteq \kappa$.*

Proof. Suppose for a contradiction that $\kappa < \alpha < j_E(\kappa)$ and $\alpha \in j_E(C)$ for every club $C \subseteq \kappa$. Let $\alpha = j_E(F)(\vec{\gamma})$ where $\vec{\gamma}$ is an increasing n-tuple of generators and $F : [\kappa]^n \to \kappa$. Let D be the club of $\beta < \kappa$ such that $h[\beta] \subseteq \beta$ and $F[[\beta]^n] \subseteq \beta$, so that by hypothesis $\alpha \in j_E(D)$. Since $\kappa < \alpha$, $j_E(h)(\kappa) < \alpha$ and in particular $\vec{\gamma} \in [\alpha]^n$. But then $j_E(F)(\vec{\gamma}) < \alpha$, contradicting the choice of F. □

Remark 1.3. It will turn out that all the extenders which are relevant for our purposes satisfy the hypotheses of Fact 1.2, see Remark 1.6(2). We note that the proof of a corresponding fact in our earlier paper [1, Subclaim 2.1.2] is incorrect, but in fact the extender discussed in that proof does satisfy the hypotheses of Fact 1.2 so the conclusion is sound. There do exist short extenders which do not satisfy the conclusion of Fact 1.2: for example if U is a normal measure on κ and j_{02} is the two-step iteration by U, then $j_{01}(\kappa) \in j_{02}(C)$ for every club $C \subseteq \kappa$.

Finally we recall some easy facts about strong cardinals. For the convenience of the reader we have sketched the proofs, which are adaptations of results by Solovay [24] about supercompact cardinals. Facts 1.4 and 1.5 are formulated in slightly different ways, because Fact 1.4 will be used to analyse measures in a universe where GCH may fail and Fact 1.5 will be used to analyse extenders in a model where GCH holds.

Fact 1.4. *Let κ be $(\kappa+2)$-strong. Then for every $A \in V_{\kappa+2}$ there is U a normal measure with $A \in \mathrm{Ult}(V,U)$. As a consequence:*

- *The height of the Mitchell ordering on the set of normal measures on κ is exactly $(2^\kappa)^+$.*
- *The cardinal κ carries exactly 2^{2^κ} normal measures.*
- *If the Mitchell ordering is linear then $2^{2^\kappa} = (2^\kappa)^+$.*

Proof. Let $\prec$ be a well-ordering of V_κ, and let $j : V \to M$ witness that κ is $(\kappa+2)$-strong. For each $\alpha < \kappa$, let a_α be the least element of $V_{\alpha+2}$ such that $a_\alpha \notin \mathrm{Ult}(V,u)$ for any normal measure u on α. Assume for a contradiction that the claim fails and let $A \in V_{\kappa+2}$ be the $j(\prec)$-minimal counterexample. By the agreement between V and M, $A = j(a)_\kappa$. Now let U be the normal measure derived from j and let $k : \mathrm{Ult}(V,U) \to M$ be given by $k : [F]_U \mapsto j(F)(\kappa)$, so that $j = k \circ j_U$. Routinely $V_{\kappa+1} \subseteq \mathrm{rge}(k)$ and so $A = j_U(a)(\kappa)$, contradicting the choice of A as a counterexample. □

Fact 1.5. *Let κ be strong and let GCH hold. Then for every $A \subseteq \kappa^{++}$ there is a (κ,κ^{++})-extender E such that $P(\kappa^+) \cup \{A\} \subseteq \mathrm{Ult}(V,E)$.*

Proof. Let $\prec$ be a well-ordering of V_κ, and let $j : V \to M$ witness that κ is λ-strong for some large λ. For each $\alpha < \kappa$ let a_α be least such that $a_\alpha \notin \mathrm{Ult}(V,e)$ for any (α,α^{++})-extender e. Let A be the $j(\prec)$-minimal counterexample to the claim, so that $A = j(a)(\kappa)$. Let E be the (κ,κ^{++})-extender derived from j and let $k : \mathrm{Ult}(V,E) \to M$ be given by $k : j_E(f)(a) \mapsto j(f)(a)$. As usual $P(\kappa^+) \subseteq \mathrm{Ult}(V,E)$ and $\kappa^{++} \subseteq \mathrm{rge}(k)$, so $A = j_E(a)(\kappa)$ for an immediate contradiction. □

1.2. **The minimal inner model with a strong cardinal.** All the constructions in this section will involve forcing over the minimal inner model with a strong cardinal. More explicitly we assume that $V = L[\vec{E}]$ where $\vec{E}$ is a coherent non-overlapping sequence of extenders in the sense of Mitchell's survey [23], further we assume that $o^{\vec{E}}(\kappa) = \infty$. It follows that:

- κ is strong.
- κ is largest with $o^{\vec{E}}(\kappa) > 0$.
- $o^{\vec{E}}(\alpha) < \kappa$ for all $\alpha < \kappa$.
- α is measurable if and only if $o^{\vec{E}}(\alpha) > 0$.
- GCH holds.
- For every regular $\lambda > \kappa$ and every $A \subseteq \lambda$, there is $\eta < \lambda^+$ such that $A \in \mathrm{Ult}(V, E(\kappa,\eta))$.
- For every regular $\lambda > \kappa$, $P(\lambda) \subseteq \mathrm{Ult}(V, E(\kappa,\zeta))$ for all $\zeta \geq \lambda^+$.
- For every generic extension $V[G]$ of V and every elementary embedding $i : V[G] \to N$ defined in $V[G]$, $i \restriction V$ is a uniquely determined normal iteration (that is to say an iteration with increasing critical points) of V via extenders on $\vec{E}$. The uniqueness is easy to see: coherence tells us exactly which extenders to apply in the course of the iteration. As a special case, if F is a (κ,λ)-extender in V then j_F is a normal iteration whose first extender is of the form $E(\kappa,\zeta)$.

We refer the reader to [23, Section 3] for a detailed discussion of this model.

In the sequel we will lighten the notation by writing $o(\alpha)$ for $o^{\vec{E}}(\alpha)$, E_ζ for $E(\kappa,\zeta)$, j_ζ for $j_{E(\kappa,\zeta)}$ and M_ζ for $\mathrm{Ult}(V, E(\kappa,\zeta))$.

The following remarks will be useful later:

Remark 1.6.

1. By coherence, $j_\zeta(o)(\kappa) = \zeta$.
2. Since the generators of E_ζ are contained in $\kappa + 1 + \zeta$ and $j_\zeta(o)(\kappa) = \zeta$, it follows that E_ζ satisfies the hypotheses of Fact 1.2.
3. If $\mathbb{P}$ is a forcing iteration of length κ where the iterands are non-trivial only at α with $o(\alpha) > 0$, then in $j_\zeta(\mathbb{P})$ the first non-trivial iterand past κ occurs past ζ. This holds because the image of $\vec{E}$ is non-overlapping.
4. If $i : V \to N$ is a normal iteration of length greater than one via extenders on $\vec{E}$, and E_ζ is the first extender applied, then the critical point at the second step is greater than ζ, so that M_ζ and N agree well past rank ζ. In particular $E_\zeta \notin N$.
5. For regular $\lambda > \kappa$ the generators of $E(\kappa, \lambda^+)$ are unbounded in λ^+. Otherwise $E(\kappa, \lambda^+)$ would be equivalent to a (κ, η)-extender F for some $\eta < \lambda^+$, and since $P(\lambda) \subseteq M_{\lambda^+}$ we would have $F \in M_{\lambda^+} = \mathrm{Ult}(V, F)$ for an immediate contradiction.

For our purposes we will need some detailed information about the extenders on $\vec{E}$ with critical point κ.

Lemma 1.7. *Let F be a (κ, κ^{++})-extender such that $P(\kappa^+) \subseteq \mathrm{Ult}(V, F)$. Then there is a unique ζ such that $\kappa^{++} \le \zeta < \kappa^{+++}$, the generators of E_ζ form a subset of κ^{++}, and $F = E_\zeta \restriction \kappa^{++}$. Conversely if $\kappa^{++} \le \zeta < \kappa^{+++}$, the generators of E_ζ form a subset of κ^{++}, and $F = E_\zeta \restriction \kappa^{++}$, then $P(\kappa^+) \subseteq \mathrm{Ult}(V, F)$.*

Proof. For the forward direction note that j_F is a normal iteration of V by extenders on $\vec{E}$ whose first step has critical point κ, and let E_ζ be the first extender applied. We claim that $\zeta \ge \kappa^{++}$: if $\zeta < \kappa^{++}$ then $E_\zeta \in \mathrm{Ult}(V, F)$, but as we noted above this is not possible in a normal iteration by extenders on $\vec{E}$.

Next we claim that the iteration for j_F only goes one step. Suppose for a contradiction that there is a second step, so that $j_F = i \circ j_\zeta$ where $\zeta < \mathrm{crit}(i) = \lambda$ say. Since κ is the largest measurable cardinal in V, $\lambda \le j_\zeta(\kappa)$, so $\lambda < i(\lambda) \le j_F(\kappa)$ and thus λ has the form $j_F(h)(a)$ for some $a \in [\kappa^{++}]^n$ and $h : [\kappa]^n \to \kappa$. Since $\kappa^{++} \le \zeta < \lambda = \mathrm{crit}(i)$, we have that $\lambda = i(j_\zeta(h)(a))$, which is not possible as $\lambda = \mathrm{crit}(i)$.

It follows that $j_F = j_\zeta$. Since the definition of the generators of an extender depends only on the associated ultrapower map, it follows that F and E_ζ have the same generators. Since F is a (κ, κ^{++})-extender, the generators of E_ζ form a subset of κ^{++}. Finally for every $a \in [\kappa^{++}]^{<\omega}$ and $X \subseteq [\kappa]^{|a|}$ we have $X \in F_a \iff a \in j_F(X) \iff a \in j_\zeta(X) \iff X \in (E_\zeta)_a$, so that $F = E_\zeta \restriction \kappa^{++}$.

For the converse let $\kappa^{++} \le \zeta < \kappa^{+++}$, assume that the generators of E_ζ form a subset of κ^{++}, and let $F = E_\zeta \restriction \kappa^{++}$. It follows from the facts listed at the start of this section that $P(\kappa^+) \subseteq M_\zeta$. Since the generators of E_ζ form a subset of κ^{++}, F is equivalent to E_ζ, and so in particular $M_\zeta = \mathrm{Ult}(V, F)$ and hence $P(\kappa^+) \subseteq \mathrm{Ult}(V, F)$. □

Remark 1.8. A similar argument to that for Lemma 1.7 lets us classify the normal measures on κ. If U is a normal measure then there is a unique $\zeta < \kappa^{++}$ such that $E(\kappa, \zeta)$ has κ as its only generator and U is equivalent to $E(\kappa, \zeta)$. It follows immediately from coherence that the Mitchell ordering of normal measures on κ is linear.

Lemma 1.9. *Let F be an extender of the type identified in Lemma 1.7. Then the generators of F are unbounded in κ^{++}, and $\mathrm{Ult}(V, F)$ is closed under κ-sequences.*

Proof. If the generators are a subset of η for $\eta < \kappa^{++}$ then F is equivalent to an extender which is coded by a subset of κ^+ and hence lies in $\mathrm{Ult}(V, F)$, but this is impossible. For the closure just note that since $P(\kappa^+) \subseteq \mathrm{Ult}(V, F)$ we have ${}^{\kappa}\kappa^{++} \subseteq \mathrm{Ult}(V, F)$, and appeal to Fact 1.1. □

It will follow from Theorem 1.23 that there are unboundedly many $\zeta \in [\kappa^{++}, \kappa^{+++})$ with the set of generators contained in κ^{++}, but this also follows from what we already proved. Appealing to Fact 1.5 we can build a Mitchell increasing sequence of extenders $(F_i)_{i<\kappa^{+++}}$ such that $P(\kappa^+) \subseteq \mathrm{Ult}(V, F_i)$ for all i, and by Lemma 1.7 these correspond to an unbounded set of ζ's as required.

It will be convenient later to have names for the class of ζ identified in Lemma 1.7 and its reflections to smaller critical points.

Definition 1.10. Let $\alpha \leq \kappa$ be regular. Then Y_α is the set of $\zeta < o(\alpha)$ such that $\zeta \geq \alpha^{++}$ and the generators of $E(\alpha, \zeta)$ form a subset of α^{++}.

It is easy to see that $Y_\alpha \subseteq \alpha^{+++}$: the point is that for $\zeta \geq \alpha^{+++}$ we have $P(\alpha^{++}) \subseteq M_\zeta$, so that $E(\alpha, \zeta)$ can not be equivalent to an (α, α^{++})-extender.

For completeness we will prove that most extenders E_ζ for $\zeta \in [\kappa^{++}, \kappa^{+++})$ are of a different character. Using arguments from Cummings' paper [11, Section 6], we will show that for most ζ in the interval $[\kappa^{++}, \kappa^{+++})$ the generators of E_ζ are unbounded in ζ, and for many such ζ the model M_ζ is not κ-closed. These results are not needed in the sequel and the impatient reader may skip to the start of Section 1.3.

We will use a standard fact which is a form of condensation for $L[\vec{E}]$. See [3, Fact 2.7] for a sketch of the argument, and [23, Theorem 3.24] for a much more general condensation lemma.

Fact 1.11. *Let $\theta > \kappa^+$ be a successor cardinal and let $X \prec L_\theta[\vec{E}]$ with $P(\kappa) \subseteq X$ and $\vec{E} \restriction \theta \in X$. Let the transitive collapse of X be $L_{\bar{\theta}}[\vec{F}]$. Then $E_\zeta = F(\kappa, \zeta)$ for all $\zeta < o^{\vec{F}}(\kappa)$.*

Lemma 1.12. *There is a club set $C \subseteq [\kappa^{++}, \kappa^{+++})$ such that:*

- *For every $\zeta \in C$, the generators of E_ζ form an unbounded subset of ζ.*
- *For every $\zeta \in \lim(C) \cap \mathrm{cof}(\omega)$, M_ζ is not ω-closed.*

Proof. Let θ be large enough. Build a continuous strictly increasing chain $(X_i)_{i<\kappa^{+++}}$ of elementary substructures of $L_\theta[\vec{E}]$ such that $\{\vec{E} \restriction \theta\} \cup H_{\kappa^{++}} \subseteq X_0$, $\kappa^{++} \subseteq X_0$, $|X_i| = \kappa^{++}$, $\delta_i = X_i \cap \kappa^{+++} \in X_{i+1} \cap \kappa^{+++}$. Let $C = \{\delta_i : i < \kappa^{+++}\}$, so that C is club in κ^{+++}.

It follows from Fact 1.11 that $E(\kappa, \kappa^{+++}) \in X_i$ and it collapses to $E(\kappa, \delta_i)$. By part 5 of Remark 1.6, the generators of $E(\kappa, \kappa^{+++})$ are unbounded in κ^{+++}. By elementarity the collapse of X_i believes that the generators of $E(\kappa, \delta_i)$ are unbounded in δ_i, and by the agreement between V and the collapse of X_i it is easy to check that this is true in V.

Now let i be limit with $\mathrm{cf}(i) = \omega$. For every $\eta < \kappa^{+++}$ we have $E(\kappa, \kappa^{+++}) \restriction \eta \in M_{\kappa^{+++}}$, and arguing as in the last paragraph we see that for all $\eta < \delta_i$ we have

$E(\kappa, \delta_i) \restriction \eta \in M_{\delta_i}$. Since $E(\kappa, \delta_i) \notin M_{\delta_i}$ and $\mathrm{cf}(\delta_i) = \omega$, we see that M_{δ_i} is not ω-closed. □

By similar (but simpler) arguments we may also analyse the normal measures on κ. In this case we just sketch the arguments.

Lemma 1.13. *For every normal measure on κ there is a unique $\zeta < \kappa^{++}$ such that E_ζ has κ as its only generator and U is the normal measure derived from j_ζ. The set of ζ such that E_ζ has κ as its only generator is non-stationary and unbounded in κ^{++}.*

Proof. Arguing exactly as in Lemma 1.7, if U is a normal measure then there is a unique ζ such that $j_U = j_\zeta$, and so E_ζ has only one generator and U is the measure derived from j_ζ. Since there is a Mitchell increasing κ^{++}-sequence of normal measures, there are cofinally many $\zeta < \kappa^{++}$ such that E_ζ has only one generator. Reflecting the properties of $E(\kappa, \kappa^{++})$ to a continuous κ^{++}-chain of models of cardinality κ^+, we see as in the proof of Lemma 1.12 that for almost every $\zeta < \kappa^{++}$ (modulo the club filter) the generators of E_ζ are unbounded in ζ. □

1.3. **Linear Mitchell ordering at a strong cardinal where GCH fails.** We will use essentially the same forcing construction that was used by Friedman and Magidor [14, Theorem 12] to get a model where GCH fails at a measurable κ and κ carries a unique normal measure. One small difference in the forcing construction is that we only force at measurable $\alpha < \kappa$: this will enable us to lift j_ζ for arbitrary $\zeta \geq \kappa^{++}$ by "spacing out" the support of the iteration on the j-side. To make this paper more self-contained we describe the construction in some detail and state its basic properties, but for proofs of these properties we will refer the reader to [14], together with papers by Friedman and Thompson [15] and Friedman and Honzik [13].

We need a certain parameter for the forcing construction which we choose by the same method as in [14]. Let $S = \kappa^{++} \cap \mathrm{cof}(\kappa^+)$. It is a standard fact (using fine structure) that in our ground model V there is a $\diamondsuit_S$ sequence $\langle S_\beta : \beta \in S \rangle$, which is definable without parameters in $H_{\kappa^{++}}$: moreover for every measurable $\alpha < \kappa$ the same formula defines a diamond sequence on $\alpha^{++} \cap \mathrm{cof}(\alpha^+)$ when we interpret it in $H(\alpha^{++})$.

For each $\gamma < \kappa^{++}$ let $T^\kappa_\gamma = \{\beta \in S : S_\beta = \{\gamma\}\}$, so that the sets T^κ_γ are pairwise disjoint stationary subsets of S, and let $\vec{T}^\kappa = (T^\kappa_\gamma)_{\gamma < \kappa^{++}}$. Similarly, for $\alpha < \kappa$ measurable we use the definable diamond sequence on $\alpha^{++} \cap \mathrm{cof}(\alpha^+)$ to define a sequence $\vec{T}^\alpha = (T^\alpha_\gamma)_{\gamma < \alpha^{++}}$ of disjoint stationary subsets of α^{++}. Now let $\zeta \geq \kappa^{++}$, so that $H(\kappa^{++}) \subseteq M_\zeta$. It is easy to see that $\vec{T}^\kappa = j_\zeta(\alpha \mapsto \vec{T}^\alpha)(\kappa)$, where the key point is that the sets T^κ_γ are stationary in V but are uniformly definable in M_ζ.

We can now describe the forcing construction. For α inaccessible, $\mathrm{Sacks}^*(\alpha, 1)$ is the set of closed subtrees of ${}^{<\alpha}2$ which satisfy the following uniform splitting condition: there is a club set $C \subseteq \alpha$ such that for singular $\beta \in C$ every point on level β has 2 immediate successors, while all points on the remaining levels have a unique successor. $\mathrm{Sacks}^*(\alpha, \alpha^{++})$ is the product of α^{++} copies of $\mathrm{Sacks}^*(\alpha, 1)$, taken with supports of size α.

It is easy to see that $\mathrm{Sacks}^*(\alpha, \alpha^{++})$ is α-closed, and that if $2^\alpha = \alpha^+$ then $\mathrm{Sacks}^*(\alpha, \alpha^{++})$ is α^{++}-cc. If $p \in \mathrm{Sacks}^*(\alpha, 1)$ and $s \in p$ then we can refine p to

$p[s]$, the subtree of sequences comparable with s: informally we say "thin p using s". Given a condition $q \in \text{Sacks}^*(\alpha, \alpha^{++})$, a set $S \subseteq \text{supp}(q)$ with $|S| < \alpha$ and an ordinal $\gamma < \alpha$, an (S, γ)-*thinning* of q is a refinement of q obtained by thinning $q(\beta)$ for each $\beta \in S$ using some point on level γ of $q(\beta)$. If D is a dense subset of $\text{Sacks}^*(\alpha, \alpha^{++})$ then q *reduces* D if there exists S and γ as above such that every (S, γ)-thinning of q reduces D: a fusion argument shows that if $\mathcal{D}$ is a family of α dense sets then the set of conditions which reduce all $D \in \mathcal{D}$ is dense. This readily implies that α^+ is preserved. For proofs of these properties see [13, Section 2.4].

At each measurable $\alpha \leq \kappa$, we will force with a two-step iteration $\text{Sacks}^*(\alpha, \alpha^{++}) * \dot{\text{Code}}(\alpha)$, where $\text{Code}(\alpha)$ is a version of Jensen coding defined using $\vec{T}^\alpha$. Let X^α_i for $i < \alpha^{++}$ be the generic subset of α added by $\text{Sacks}^*(\alpha, \alpha^{++})$ at coordinate i. The forcing poset $\text{Code}(\alpha)$ adds a club set $c^\alpha \subseteq \alpha^{++}$ which selectively destroys the stationarity of certain elements in the sequence $\vec{T}^\alpha$, so that both c^α and the sets X^α_i can be recovered from the set of i such that T^α_i is stationary. One subtle point is that membership of β in c^α or X^α_i is coded twice, by one entry in $\vec{T}^\alpha$ remaining stationary and another becoming non-stationary: this makes the coding more absolute. For our purposes all we need to know about $\text{Code}(\alpha)$ is that it is α^+-closed, adds no α^+-sequences of ordinals and has cardinality α^{++}.

The iteration is done with non-stationary supports. Let $\mathbb{P}_{\kappa+1}$ be the resulting forcing poset. It will be crucial later that for any $\zeta \geq \kappa^{++}$, $j_\zeta(\mathbb{P}_\kappa) \restriction \kappa + 1 = \mathbb{P}_{\kappa+1}$: this is true by the agreement between V and $\text{Ult}(V, E_\zeta)$, and the careful choice of the sequences $\vec{T}^\alpha$.

Let G be $\mathbb{P}_{\kappa+1}$-generic over V. Then we decompose G as $G_\kappa * g$ where G_κ is $\mathbb{P}_\kappa$-generic over V and g is $\text{Sacks}^*(\kappa, \kappa^{++}) * \dot{\text{Code}}(\kappa)$-generic over $V[G_\kappa]$. We further decompose g as $g_0 * g_1$, where g_0 is $\text{Sacks}^*(\kappa, \kappa^{++})$-generic over $V[G_\kappa]$ and g_1 is $\text{Code}(\kappa, \kappa^{++})$-generic over $V[G_\kappa][g_0]$.

We will use the following fact, which is proved by the same argument as [14, Lemma 14].

Fact 1.14. *For every $f \in (\kappa^\kappa)^{V[G]}$, there is $F \in V$ such that $|F(\alpha)| \leq \alpha^{++}$ and $f(\alpha) \in F(\alpha)$ for all α.*

Lemma 1.15. *For every normal measure U on κ in $V[G]$ there is a unique ζ such that $\kappa^{++} \leq \zeta < \kappa^{+++}$ and $j^{V[G]}_U$ is a lift of j_ζ. Moreover the set of generators of E_ζ is a subset of κ^{++}.*

Proof. Start by observing that since $2^\kappa = \kappa^{++}$ in $V[G]$, $\kappa^{++} < j^{V[G]}_U(\kappa) < \kappa^{+++}$. By our assumptions on V, $j^{V[G]}_U \restriction V$ is a normal iteration i of V by extenders on $\vec{E}$, and we let E_ζ be the first extender used. Clearly $\zeta < \kappa^{+++}$, for otherwise $i(\kappa) \geq j_\zeta(\kappa) > \zeta \geq \kappa^{+++}$, contradicting $i(\kappa) = j^{V[G]}_U(\kappa) < \kappa^{+++}$.

By the argument of [14, Lemma 18], the iteration i only goes for one step, that is to say $i = j_\zeta$. It follows that $\kappa^{++} \leq \zeta$, because if $\zeta < \kappa^{++}$ then by an easy counting argument $j_\zeta(\kappa) < \kappa^{++}$, contradicting $j_\zeta(\kappa) = j^{V[G]}_U(\kappa) \geq \kappa^{++}$.

To finish we verify that the set of generators of E_ζ is contained in κ^{++}. Note that κ^{++} is not a generator because, since M_ζ computes κ^{++} correctly, $j_\zeta(\alpha \mapsto \alpha^{++})(\kappa) = \kappa^{++}$. Suppose for a contradiction that E_ζ has generators greater than κ^{++}, and let η be the least one. So $\kappa^{++} < \eta < \zeta$ and η is not of the form $j_\zeta(f)(\vec{\alpha})$ for $f \in V$ with $f : \kappa^{<\omega} \to \kappa$ and $\vec{\alpha} \in (\kappa^{++})^{<\omega}$.

Since $\eta < \zeta < j_\zeta(\kappa) = j_U^{V[G]}(\kappa)$, $\eta = j_U^{V[G]}(h)(\kappa)$ for some $h \in V[G]$ with $h : \kappa \to \kappa$. By Fact 1.14 there is $H \in V$ such $h(\alpha) \in H(\alpha) \subseteq \kappa$ and $|H(\alpha)| \leq \alpha^{++}$ for all α. Let I be a function such that $I(\alpha)$ is a surjection from α^{++} onto $H(\alpha)$, so $\eta = j_\zeta(I)(\kappa)(\rho)$ for some $\rho < \kappa^{++}$, contradicting our hypothesis on η. □

At this point we are interested in lifting those extenders E_ζ with generators contained in κ^{++}. It turns out that under favourable circumstances, the lift will automatically be a normal measure.

Lemma 1.16. *Let $\kappa^{++} \leq \zeta < \kappa^{+++}$ and assume that the generators of E_ζ are a subset of κ^{++}. Let j^* be a lift of j_ζ onto $V[G]$ such that $j^*(G) \restriction \kappa + 1 = G$. Then j^* is an ultrapower map by a normal measure on κ in $V[G]$.*

Proof. Let j^* be such a lift of j_ζ onto $V[G]$, so that by the discussion at the start of this section $j^* = j_{E^*}$ for some (κ, ζ)-extender E^* in $V[G]$. It will suffice to show that κ is the only generator of E^*.

For each measurable α with $\alpha \leq \kappa$ and each $i < \alpha^{++}$, let X_i^α be the i^{th} Sacks generic function at α added by G. We define $f_i : \kappa \to \kappa$ for $i < \kappa^{++}$ as follows: if $\alpha < \kappa$ is measurable and $X_i^\kappa \restriction \alpha = X_j^\alpha$ for some $j < \alpha^{++}$ then $f_i(\alpha)$ is the least such j, otherwise $f_i(\alpha)$ is 0.

By the assumption on the agreement between G and $j^*(G)$, G and $j^*(G)$ add the same κ^{++}-sequence of subsets of κ at coordinate κ. Since $j^*(X_i^\kappa) \cap \kappa = X_i^\kappa$ and all the sets X_i^κ are distinct, we see that $j^*(f_i^\kappa)(\kappa) = i$ for all $i < \kappa^{++}$. Since the generators of E form a subset of κ^{++} it follows that κ is the only generator of E^* and so j^* is the ultrapower by a normal measure. □

Now we use techniques from [14] and from our prior work [1] to show that for every $\zeta \geq \kappa^{++}$ such that M_ζ is κ-closed, the embedding j_ζ has a unique lift to $V[G]$. This will be used to show that κ is still strong in $V[G]$ and to analyse the set of normal measures on κ in $V[G]$.

We start by quoting a slightly simplified version of the relevant part of [1, Lemma 2.30].

Lemma 1.17. *Let $j : V \to M$ be an embedding with critical point κ, let $I \subseteq \kappa$ with I an unbounded set of inaccessible cardinals and $\kappa \in j(I)$. Let $\mathbb{R}_\kappa$ be an iteration where the supports are non-stationary subsets of I, and the iterand at α is forced to be an α-closed forcing poset of cardinality less than $\min(I \setminus (\alpha + 1))$. Suppose that for every dense $D \subseteq j(\mathbb{R}_\kappa)$ with $D \in M$, there exists a sequence $(\mathcal{D}_\alpha)_{\alpha \in I}$ of families of dense subsets of $\mathbb{R}_\kappa$ such that $|\mathcal{D}_\alpha| < \min(I \setminus (\alpha + 1))$ and $D \in j(\mathcal{D})_\kappa$.*

Let G_κ be $\mathbb{R}_\kappa$-generic over V and let $G_\kappa * g$ be $j(\mathbb{R}_\kappa) \restriction \kappa + 1$-generic over M. Then there is a unique filter H such that H is $j(\mathbb{R}_\kappa)$-generic over M, $j[G_\kappa] \subseteq H$ and $H \restriction \kappa + 1 = G_\kappa * g$.

Lemma 1.18. *For every $\zeta \geq \kappa^{++}$ such that M_ζ is closed under κ-sequences, the embedding j_ζ has a unique lift to $V[G]$. Moreover the unique lift j^* has the property that $j^*(G) \restriction \kappa + 1 = G$.*

Proof. As we already noted, $j_\zeta(\mathbb{P}_\kappa) \restriction \kappa + 1 = \mathbb{P}_{\kappa+1}$. We claim that $V[G]$ and $M_\zeta[G]$ agree on $H(\kappa^{++})$. By the distributivity of the coding forcing, this amounts to showing that $V[G_\kappa][g_0]$ and $M_\zeta[G_\kappa][g_0]$ agree. But $G_\kappa * g_0$ is generic for κ^{++}-cc forcing of cardinality κ^{++}, and $H(\kappa^{++}) \subseteq M_\zeta$, so the agreement claim follows immediately.

To find a suitable choice for $j^*(G_\kappa)$ we use Lemma 1.17, with I the set of measurable cardinals less than κ. Recall that since $\vec{E}$ is non-overlapping, if α and β are successive points of I then $o(\alpha) < \beta$: we will use this to get the required bound on the cardinality of the set $\mathcal{D}_\alpha$.

To apply Lemma 1.17, we need only to verify the technical condition on dense subsets of $j(\mathbb{P}_\kappa)$. So let $D \in M_\zeta$ be dense in $j(\mathbb{P}_\kappa)$, so that $D = j_\zeta(d)(\vec{a})$ for some $\vec{a} \in [\zeta]^n$ and some function d from $[\kappa]^n$ to dense sets in $\mathbb{P}_\kappa$. Let $\mathcal{D}_\alpha = \{d(\beta) : \beta \in [o(\alpha)]^n\}$. As we noted above, it follows from $\vec{E}$ being non-overlapping that $|\mathcal{D}_\alpha| = o(\alpha) < \min(I \setminus (\alpha+1))$. By the coherence of $\vec{E}$, $j_\zeta(o)(\kappa) = \zeta$, so that easily $D = j_\zeta(d)(\vec{a}) \in j(\mathcal{D})_\kappa$. Appealing to Lemma 1.17 we get a generic object $j^*(G_\kappa)$ with $j^*(G_\kappa) \restriction \kappa + 1 = G_\kappa * g$ and $j_\zeta[G_\kappa] \subseteq j^*(G_\kappa)$, together with a lifted map $j^* : V[G_\kappa] \to M_\zeta[j^*(G_\kappa)]$.

We will first show that $j^*[g_0]$ generates a filter $j^*(g_0)$ which is generic over $M_\zeta[j^*(G_\kappa)]$ for $\text{Sacks}^*(j_\zeta(\kappa), j_\zeta(\kappa^{++}))$. Once this is done we can lift again to obtain $j^* : V[G_\kappa][g_0] \to M_\zeta[j^*(G_\kappa)][j^*(g_0)]$. At this point we are basically done, since g_1 is generic for forcing which is sufficiently distributive that there will be no problem showing that $j^*[g_1]$ generates a suitably generic filter $j^*(g_1)$.

To find $j^*(g_0)$ we use arguments which parallel those of Friedman and Honzik [13, Theorem 2.22] and [15, Lemma 5]. The main difference is that we are using Sacks forcing which only splits at singular levels: this only simplifies the arguments.

Since κ is inaccessible in $M_\zeta[j^*(G_\kappa)]$, conditions in $j^*(\text{Sacks}^*(\kappa, \kappa^{++}))$ have the property that no tree appearing in them splits at level κ. This is slightly simpler than the situation in [13, Theorem 2.22], where the splitting condition on Sacks forcing is different and there is splitting at coordinates in the range of j.

We claim that that for each $\alpha < j(\kappa^{++})$, there is a unique $x_\alpha : j_\zeta(\kappa) \to 2$ such that for all $\beta < j_\zeta(\kappa)$ and all $p \in g_0$ with $\alpha \in \text{supp}(j^*(p))$, $x_\alpha \restriction \beta \in j^*(p)(\alpha)$. For each $\gamma < \kappa$ the set of conditions in $\text{Sacks}^*(\kappa, \kappa^{++})$ with no splitting before γ is dense, so easily $x_\alpha \restriction \kappa$ is unique. By Fact 1.2, for every $\delta < j_\zeta(\kappa)$ there is a club set $E_\delta \subseteq \kappa$ such that $j_\zeta(E_\delta) \cap (\kappa, \delta] = \emptyset$: there is a dense set of conditions in $\text{Sacks}^*(\kappa, \kappa^{++})$ with splitting contained in E_δ, and since there is no splitting at level κ it follows that $x_\alpha \restriction [\kappa, \delta)$ is unique for all δ.

We let $j^*(g_0)$ be the set of conditions $q \in j^*(\text{Sacks}^*(\kappa, \kappa^{++}))$ such that for every $\alpha \in \text{supp}(q)$, x_α is a branch through $q(\alpha)$. Clearly $j^*[g_0] \subseteq j^*(g_0)$, and the remaining issue is to show that $j^*(g_0)$ is generic. Since M_ζ is closed under κ-sequences, this follows exactly as in [13, Theorem 2.22].

The claims about uniqueness of j^* follow by the arguments of [14], which we briefly rehearse here. Suppose $j' : V[G] \to M[j'(G)]$ is a lift of j_ζ. Clearly $G \restriction \kappa = j'(G) \restriction \kappa$. The coding forcing at κ codes its generic object and the Sacks generic object at κ in a way which is both upwards and downwards absolute, so that $j'(G) \restriction \kappa + 1 = G$. Since $j_\zeta[G_\kappa] = j'[G_\kappa] \subseteq j'(G_\kappa)$ and $j'(G) \restriction \kappa + 1 = G$, it follows from the uniqueness part of Lemma 1.17 that $j'(G_\kappa) = j^*(G_\kappa)$ and hence $j^* \restriction V[G_\kappa] = j' \restriction V[G_\kappa]$. Finally $j^*(g_0)$ and $j^*(g_1)$ are generated by $j^*[g_0] = j'[g_0]$ and $j^*[g_1] = j'[g_1]$, so $j^*(g) = j'(g)$ and hence $j^* = j'$.

□

Remark 1.19. With a bit more effort it is possible to show that j_ζ has a unique lift for every $\zeta \geq \kappa^{++}$. We have omitted this argument, since for our purposes it suffices to lift only those j_ζ for which M_ζ is closed under κ-sequences.

Remark 1.20. Recalling the definition of Y_κ from Definition 1.10, Lemmas 1.15, 1.16 and 1.18 give us a complete description of the normal measures on κ in $V[G]$ and their ultrapower maps. The ultrapower maps are exactly the unique lifts of j_ζ for $\zeta \in Y_\kappa$, and of course these maps determine the normal measures.

Lemma 1.21. *κ is strong in $V[G]$.*

Proof. Let θ be a regular cardinal greater than κ, and choose ζ much larger than θ such that $V_\theta \subseteq M_\zeta$ and M_ζ is closed under κ-sequences. Let $j^* : V[G] \to M_\zeta[j^*(G)]$ be the unique lift of j_ζ. Since $j^*(G) \restriction \kappa + 1 = G$, the next point in the support of $j^*(G)$ is greater than ζ, and the tail forcing $j(\mathbb{P}_{\kappa+1})/G$ is highly closed, we see that $V[G]$ and $M_\zeta[j^*(G)]$ agree to rank θ. □

Lemma 1.22. *The Mitchell ordering at κ is linear in $V[G]$.*

Proof. Let U and U' be normal measures on κ in $V[G]$. By Lemmas 1.7, 1.9, 1.15, 1.16, and 1.18, there are unique ordinals $\zeta, \zeta' \in Y_\kappa$ (in particular lying in the interval $[\kappa^{++}, \kappa^{+++})$) such that $j_U^{V[G]}$ is the unique lift of j_ζ, and $j_{U'}^{V[G]}$ is the unique lift of $j_{\zeta'}$. If $\zeta = \zeta'$ then $U = U'$, so assume that $\zeta < \zeta'$.

By coherence $E_\zeta \in M_{\zeta'}$. It is now routine to check that we may perform the argument for lifting $j_\zeta^{M_{\zeta'}}$ in $M_{\zeta'}[G]$, using the fact that $M_{\zeta'}[G]$ and $V[G]$ agree on $H(\kappa^{++})$. It follows that $U_\zeta \in M_{\zeta'}[G] = \mathrm{Ult}(V[G], U_{\zeta'})$. □

Putting these results together, we have proved:

Theorem 1.23. *It is consistent (relative to the existence of a strong cardinal) that there is a strong cardinal κ such that $2^\kappa = 2^{\kappa^+} = \kappa^{++}$, $2^{\kappa^{++}} = \kappa^{+++}$, and the Mitchell ordering at κ is linear.*

Using the same ideas we can arrange some other "close to linear" behaviours for the Mitchell ordering at a strong cardinal where GCH fails, by making the Mitchell ordering a prewellordering. It is in this setting that the very fine analysis from the proof of Lemma 1.15 really pays off. We illustrate the ideas with a model where the Mitchell ordering is a prewellordering with one measure at every odd level and two measures at every even level.

We begin by stating some easy properties of the sets Y_α from Definition 1.10.

- For every $\zeta \geq \kappa^{++}$ and regular $\eta < \kappa$, $Y_\eta^{M_\zeta} = Y_\eta$.
- For every $\zeta \geq \kappa^{++}$, $j_\zeta(\alpha \mapsto Y_\alpha)(\kappa) = Y_\kappa^{M_\zeta} = Y_\kappa \cap \zeta$.
- Let $\zeta \in Y_\kappa$ and let $\eta = \mathrm{ot}(Y_\kappa \cap \zeta)$. Then $j_\zeta(\alpha \mapsto \mathrm{ot}(Y_\alpha)) = \eta$.

Now let $\alpha \leq \kappa$ be inaccessible and define a variation $\mathrm{Sacks}'(\alpha)$ of Sacks forcing at α as follows. Conditions are uniformly splitting subtrees p of ${}^{<\alpha}2$ with the following splitting condition: there is a club set $C \subseteq \alpha$ such that the splitting levels of p are those $\beta \in C$ such that either β is singular, or β is regular and $\mathrm{ot}(Y_\beta)$ is even. By the properties of the sequence (Y_α) we have that for all $\zeta \in Y_\kappa$:

- $j_\zeta(\alpha \mapsto \mathrm{Sacks}'(\alpha))(\kappa) = \mathrm{Sacks}'(\kappa)$.
- $j_\zeta(\mathrm{Sacks}'(\kappa))$ has a splitting level at κ if and only if $\mathrm{ot}(Y_\kappa \cap \zeta)$ is even.

Now we modify the construction in the proof of Theorem 1.23, using $\mathrm{Sacks}^*(\alpha, \alpha^{++}) \times \mathrm{Sacks}'(\alpha)$ in place of $\mathrm{Sacks}^*(\alpha, \alpha^{++})$. Accordingly we modify the coding forcing, so that it codes the generic object for $\mathrm{Sacks}'(\alpha)$ in addition to the generic object for $\mathrm{Sacks}^*(\alpha, \alpha^{++})$ at each non-trivial stage α.

Now we work through the arguments for Theorem 1.23 making changes as necessary. The analogues of Fact 1.14, Lemma 1.15 and Lemma 1.16 have the same statements and proofs as before.

The key point is that normal measures in $V[G]$ correspond exactly to ordinals in Y_κ, so that if $\zeta \in Y_\kappa$ and $j_U^{V[G]}$ lifts j_ζ then the Mitchell order of U is precisely $\mathrm{ot}(Y_\kappa \cap \zeta)$. It is these considerations that motivated the choice of the splitting condition for the forcing $\mathrm{Sacks}'(\alpha)$.

Lemma 1.18 is modified as follows:

Lemma 1.24. *For every $\zeta \geq \kappa^{++}$ such that M_ζ is closed under κ-sequences, the embedding j_ζ has exactly one lift to $V[G]$ if $\mathrm{ot}(Y_\kappa \cap \zeta)$ is odd and exactly two lifts to $V[G]$ if $\mathrm{ot}(Y_\kappa \cap \zeta)$ is even. Every lift j^* of j_ζ has the property that $j^*(G) \restriction \kappa + 1 = G$.*

Proof. The argument is the same as in the proof of Lemma 1.18 up to the point where we construct $j^*(g_0)$, where in the current setting it is no longer true that $j^*(g_0)$ generates a generic filter. The issue is that the splitting condition has changed at level κ in the forcing at $j_\zeta(\kappa)$ on the j_ζ-side. To be more specific, the factor $j^*(\mathrm{Sacks}'(\kappa))$ has a splitting level at κ if $\mathrm{ot}(Y_\kappa \cap \zeta)$ is even. The case when $\mathrm{ot}(Y_\kappa \cap \zeta)$ is odd and there is no splitting at κ is the same as in Lemma 1.18, so we concentrate on the case when $\mathrm{ot}(Y_\kappa \cap \zeta)$ is even.

Let g_0' be the $\mathrm{Sacks}'(\kappa)$-generic added by the second factor in g_0. Using Fact 1.2 as in the proof of Lemma 1.18, for $i \in 2$ there is a unique function $x_i' : j_\zeta(\kappa) \to 2$ such that:

- $x_i'(\kappa) = 1$.
- $x_i' \restriction \beta \in j^*(p)$ for all $\beta \in j_\zeta(\kappa)$ and $p \in g_0'$.

Using the functions x_i' and the arguments from Lemma 1.18, we can now argue that there are exactly two generic filters containing $j^*[g_0]$. The functions x_α for $\alpha < j_\zeta(\kappa^{++})$ are constructed exactly as in the proof of Lemma 1.18. The two filters are each of the following form: the set of pairs (q, q') such that x_i' is a branch through q', and x_α is a branch through q_α for every α in the support of q. The argument for genericity is exactly as in the proof of Lemma 1.18. Since the x_α's are unique and x_i' is determined by the value $x_i'(\kappa)$, these filters represent the only two possibilities.

Each of the filters we just described is mutually generic with the filter generated by $j^*[g_1]$, and the rest of the argument proceeds as in the proof of Lemma 1.18, with each of the two choices for $j^*(g_0)$ giving a unique lift. □

With Lemma 1.24 in hand, we can prove the analogue of Lemma 1.21 exactly as before. Since $\mathrm{ot}(Y_\kappa) = \kappa^{+++}$ which is an even ordinal, it turns out that for every $\zeta \geq \kappa^{+++}$ there are two lifts for j_ζ, but this makes no difference in the argument. Finally we modify Lemma 1.22.

Lemma 1.25. *The Mitchell ordering at κ in $V[G]$ is a prewellordering of height κ^{+++} with exactly two measures at every even level and exactly one measure at every odd level.*

Proof. Let U and U' be normal measures on κ in $V[G]$. Exactly as in the proof of Lemma 1.22, there are unique ordinals $\zeta, \zeta' \in Y_\kappa$ such that $j_U^{V[G]}$ is a lift of j_ζ, and $j_{U'}^{V[G]}$ is a lift of $j_{\zeta'}$. As in the proof of Lemma 1.22, it is easy to see that $U \lhd U'$ if and only if $\zeta < \zeta'$.

This analysis, along with the coherence properties of the sets Y_α listed above, shows that if $j_U^{V[G]}$ is a lift of j_ζ then the Mitchell order of U is $\mathrm{ot}(Y_\kappa \cap \zeta)$. By Lemma 1.24 there are two possibilities for U when $\mathrm{ot}(Y_\kappa \cap \zeta)$ is even and only one possibility when $\mathrm{ot}(Y_\kappa \cap \zeta)$ is odd. □

We have proved:

Theorem 1.26. *It is consistent (relative to the existence of a strong cardinal) that there is a strong cardinal κ such that $2^\kappa = 2^{\kappa^+} = \kappa^{++}$, $2^{\kappa^{++}} = \kappa^{+++}$, and the Mitchell ordering at κ is a prewellordering with two measures at even levels and one measure at odd levels.*

Remark 1.27. It is possible to prove versions of Theorems 1.23 and 1.26 in a setting where the GCH fails more severely at a strong cardinal. Suppose that $\lambda = \mathrm{cf}(\lambda) > \kappa$, and there is $h : \kappa \to \kappa$ such that $j_\eta(h)(\kappa) = \lambda$ for all ordinals η with $\eta \geq \lambda$. This hypothesis is satisfied by (for example) λ of the form $\kappa^{+\alpha+1}$ for $\alpha < \kappa$, or the least inaccessible cardinal greater than κ.

Let $\lambda_\alpha = h(\alpha)$ for $\alpha < \kappa$ inaccessible, where we may assume that $\lambda_\alpha = \mathrm{cf}(\lambda_\alpha) > \alpha$. To prove a version of Theorem 1.23 with $2^\kappa = \lambda$ we will modify the construction by replacing $\mathrm{Sacks}^*(\alpha, \alpha^{++})$ by $\mathrm{Sacks}^*(\alpha, \lambda_\alpha)$, and modifying $\mathrm{Code}(\alpha)$ to add a subset of λ_α coding the λ_α sets added by $\mathrm{Sacks}^*(\alpha, \lambda_\alpha)$ plus itself. The argument now goes through with the following modifications:

- (Fact 1.14): For every $f \in (\kappa^\kappa)^{V[G]}$, there is $F \in V$ such that $|F(\alpha)| \leq \lambda_\alpha$ and $f(\alpha) \in F(\alpha)$ for all α.
- (Lemma 1.15) For every normal measure U on κ in $V[G]$ there is a unique ζ such that $\lambda \leq \zeta < \lambda^+$ and $j_U^{V[G]}$ is a lift of j_ζ. Moreover the set of generators of E_ζ is a subset of λ, and M_ζ is closed under κ-sequences.
- (Lemma 1.16) Let ζ be as in the last item and let j^* be a lift of j_ζ onto $V[G]$ such that $j^*(G) \restriction \kappa + 1 = G$. Then j^* is an ultrapower map by a normal measure on κ in $V[G]$.
- (Lemma 1.18) For every $\zeta \geq \lambda$ such that M_ζ is closed under κ-sequences, the embedding j_ζ has a unique lift to $V[G]$. Moreover the unique lift j^* has the property that $j^*(G) \restriction \kappa + 1 = G$.

To get a version of Theorem 1.26 with $2^\kappa = \lambda$, we additionally replace Y_α with the set of $\zeta < o(\alpha)$ such that $\zeta \geq \lambda_\alpha$ and the generators of $E(\alpha, \zeta)$ form a subset of λ_α.

Remark 1.28. By varying the construction we may also obtain versions of Theorem 1.26 with more general prewellorderings. Let $f : \kappa \to \kappa \setminus \{0\}$, change the support of the iteration to consist of inaccessible closure points of f, and modify the definition of $\mathrm{Sacks}'(\alpha)$ as follows: conditions are uniformly splitting subtrees of ${}^{<\alpha}\alpha$ such that for some club set $C \subseteq \alpha$ splitting occurs only on levels in C, nodes t on levels which are singular elements of C have successors $t^\frown i$ for $i \in 2$, nodes t on levels which are regular elements of C have successors $t^\frown i$ for $i \in f(\mathrm{ot}(Y_\beta))$.

Let $\zeta \in Y_\kappa$, so that lifts of j_ζ give rise to the measures of Mitchell order η where $\eta = \mathrm{ot}(Y_\kappa \cap \zeta)$. It is easy to see that j_ζ has exactly $j_\zeta(f)(\eta)$ distinct lifts, so there are $j_\zeta(f)(\eta)$ many measures of order η in $V[G]$.

1.4. **The Mitchell ordering at a strong cardinal where GCH holds.** Now we consider an extension of the same general kind as the one used by Friedman and Magidor to produce a measurable cardinal with exactly two normal measures [14, Theorem 1]. We will do a non-stationary support iteration forcing at each measurable $\alpha \leq \kappa$ with $\mathrm{Sacks}(\alpha, 1) * \dot{\mathrm{Code}}(\alpha)$, where $\mathrm{Sacks}(\alpha, 1)$ is defined using trees which split at almost every *non-measurable* inaccessible level and $\mathrm{Code}(\alpha)$ is designed to code the Sacks generic object and its own generic object. $\mathrm{Code}(\alpha)$ is defined using a sequence of disjoint subsets of $\alpha^+ \cap \mathrm{cof}(\alpha)$ which is uniformly definable in $H(\alpha^{++})$. We adopt the same notational conventions as in the last section.

We use the following covering fact (which is easily proved by modifying the proof of [14, Lemma 14]):

Fact 1.29. *For $f \in (\kappa^\kappa)^{V[G]}$, there is $F \in V$ such that $|F(\alpha)| \leq \alpha^+$ and $f(\alpha) \in F(\alpha)$ for all α.*

Lemma 1.30. *For every normal measure U on κ in $V[G]$ there is a unique $\zeta < \kappa^{++}$ such that $j_U^{V[G]}$ is a lift of j_ζ. Moreover E_ζ has κ as its only generator.*

Proof. The argument that $j_U^{V[G]}$ is a lift of j_ζ is exactly as before. Assume for a contradiction that E_ζ has at least two generators, so that if η is the second generator then η is the least ordinal not of the form $j_\zeta(f)(\kappa)$ for $f \in V$ with $f : \kappa \to \kappa$. Since $\eta < \zeta < j_\zeta(\kappa) = j_U^{V[G]}(\kappa)$, $\eta = j_U^{V[G]}(h)(\kappa)$ for some $h \in V[G]$ with $h : \kappa \to \kappa$. By Fact 1.29 there is $H \in V$ such that $h(\alpha) \in H(\alpha) \subseteq \kappa$ and $|H(\alpha)| \leq \alpha^+$ for all α. Let $I \in V$ be a function such that $I(\alpha)$ is a surjection from α^+ onto $H(\alpha)$.

Now $\eta = j_U^{V[G]}(h)(\kappa) \in j_U^{V[G]}(H)(\kappa) = j_\zeta(H)(\kappa)$, hence $\eta = j_\zeta(I)(\kappa)(\nu)$ for some $\nu < \kappa^+$. Let c_ν be the ν^{th} canonical function, so that $j_\zeta(c_\nu)(\kappa) = \nu$. Let $f \in V$ be defined by setting $f(\alpha) = I(\alpha)(c_\nu(\alpha))$. Then $j_\zeta(f)(\kappa) = j_\zeta(I)(\kappa)(j_E(c_\nu)(\kappa)) = j_E(I)(\kappa)(\nu) = \eta$, contradicting the choice of η as a generator. □

The arguments of the last section adapt readily to show that κ is strong in $V[G]$. For $\zeta > 0$ the embedding j_ζ will lift uniquely, but j_0 will have two distinct lifts: this is because κ is non-measurable in M_0 but measurable in M_ζ for $\zeta > 0$. The analysis of the Mitchell ordering works in exactly the same way as before.

We have proved:

Theorem 1.31. *It is consistent (relative to the existence of a strong cardinal) that there is a strong cardinal κ, GCH holds, there are exactly two normal measures on κ of order zero and exactly one normal measure of order ζ for $\zeta > 0$.*

It is now straightforward to prove other results of this kind by varying the splitting condition in the Sacks forcing, along the same lines as in Remark 1.28. We begin with a function $f : \kappa \to \kappa \setminus \{0\}$ in the ground model. In a suitable generic extension κ is strong, GCH holds, measures of order η arise from lifts of j_ζ for $\zeta \in Y_\kappa$ with $\mathrm{ot}(Y_\kappa \cap \zeta) = \eta$, and measures form a prewellordering with $j_\zeta(f)(\eta)$ measures of order η.

Remark 1.32. We can view Theorems 1.23, 1.26 and 1.31 as first steps towards a version for strong cardinals of Ben-Neria's theorem [7] that any reasonable partial ordering can be the Mitchell ordering of normal measures on a measurable cardinal. Of course such a result would have to incorporate the constraints imposed by Fact 1.4.

2. Measurable cardinals with many measures of order zero

We begin by observing that it is fairly easy to get a measurable cardinal with 2^{2^κ} measures of order zero if $2^\kappa = \kappa^+$. To see this we use an argument of the second author [10], which simplifies some results by Kunen and Paris [19].

Suppose that κ is measurable and $2^\kappa = \kappa^+$, let U be a measure of order zero on κ and let $j : V \to M$ be the ultrapower map. Let $\mathbb{P}$ be an Easton support iteration which adds a Cohen subset of α^{++} for every inaccessible $\alpha < \kappa$, and let G be $\mathbb{P}$-generic over V. Observe that:

- $V[G] \models {}^\kappa M[G] \subseteq M[G]$.
- $j(\mathbb{P})/G$ is $j(\kappa)$-cc forcing of cardinality $j(\kappa)$ in $M[G]$.
- $j(\mathbb{P})/G$ is κ^+-closed in $V[G]$.
- $|j(\kappa)| = \kappa^+$.
- The set of maximal antichains of $j(\mathbb{P})/G$ which lie in $M[G]$ has cardinality $j(\kappa)$ in $M[G]$, and hence has cardinality κ^+ in $V[G]$.
- Since $\mathbb{P} \times \mathbb{P}$ is κ-cc and κ is not measurable in M, κ is not measurable in $M[G]$.
- Every condition in $j(\mathbb{P})/G$ has two incompatible extensions.

Using these facts it is easy to work in $V[G]$ and build a perfect binary tree of height κ^+ of decreasing sequences from $j(\mathbb{P})/G$, such that any branch generates a $j(\mathbb{P})/G$-generic filter over $M[G]$. For each such filter H we may lift j to get $j : V[G] \to M[G][H]$, which is the ultrapower map by a measure U_H on κ in $V[G]$. By the agreement between $M[G]$ and $M[G][H]$ we see that κ is not measurable in $M[G][H]$, so that U_H is a measure of order zero.

This shows that in $V[G]$ the cardinal κ carries $2^{\kappa^+} = 2^{2^\kappa}$ measures of order zero. Since we could have prepared the universe by adding as many Cohen subsets of κ^+ as we please, preserving the measurability of κ and the hypothesis that $2^\kappa = \kappa^+$, the value of 2^{κ^+} in this construction may be arbitrarily large.

It is natural to ask about the situation where κ is measurable and $2^\kappa > \kappa^+$. In this setting the argument we just gave fails, because the ultrapower by a normal measure on κ is not closed under κ^+-sequences. In this section we will give a version of the argument which does work to produce 2^{2^κ} measures for $2^\kappa > \kappa^+$.

2.1. A measure of order zero with a highly closed ultrapower.

Theorem 2.1. *It is consistent (modulo GCH plus the existence of κ which is κ^+-supercompact) that there exist a cardinal κ with $2^\kappa = 2^{\kappa^+} = \kappa^{++}$ and a normal measure U on κ with the following properties:*

1. *κ is the least measurable cardinal, so that in particular the measure U has order zero.*
2. *${}^{\kappa^+} j_U(\kappa) \subseteq \mathrm{Ult}(V, U)$.*

We digress briefly to discuss some historical background. Answering a question raised by Apter, Woodin showed using Radin forcing [26] that consistently the least measurable cardinal κ can be κ^+-supercompact. Subsequently Apter and Shelah [5, 4] showed how to prove the same result using Easton support iteration. Woodin also showed using Easton support iteration [27] that consistently there can exist a normal measure U on κ such that ${}^{\kappa^+} j_U(\kappa) \subseteq \mathrm{Ult}(V, U)$.

As some context for Theorem 2.1, note that:

- For any normal measure U on a measurable cardinal κ, j_U is continuous at κ^+ and hence $j_U[\kappa^+] \notin \mathrm{Ult}(V,U)$. It follows that ${}^{\kappa^+}j_U(\kappa^+) \nsubseteq \mathrm{Ult}(V,U)$, so clause (2) in Theorem 2.1 is as close to being closed under κ^+-sequences as an ultrapower by a normal measure can come.
- If κ is measurable with $2^\kappa = \kappa^+$ and M is any inner model such that $P(\kappa^+) \subseteq M$, then κ is measurable in M. So assuming $2^\kappa = \kappa^+$, there can be no measure U of order zero on κ with the property that ${}^{\kappa^+}2 \subseteq \mathrm{Ult}(V,U)$.
- If κ is measurable with $2^\kappa = \kappa^+$ and U is a normal measure on κ, then $\mathrm{cf}(j_U(\kappa)) = \kappa^+$. It follows that ${}^{\kappa^+}j_U(\kappa) \nsubseteq \mathrm{Ult}(V,U)$, since $j_U(\kappa) > \kappa^+$ and $j_U(\kappa)$ is regular in $\mathrm{Ult}(V,U)$.

We begin by describing a forcing construction by Apter and Shelah [4]. Its properties are only outlined in [4], so we give the proofs in more detail with references to parallel arguments from [5]. Giving the arguments at this level of detail helps us verify that we can mix in Woodin's methods to prove Theorem 2.1.

2.2. **Killing measurability in a mild way.** Let $\gamma < \delta < \lambda$ with γ regular, δ inaccessible, λ inaccessible or the successor of a cardinal with cofinality above δ. Let GCH hold above δ. We will define a forcing poset $\mathbb{P}^0$ in V, and then forcing posets $\mathbb{P}^1$ and $\mathbb{P}^2$ in the extension by $\mathbb{P}^0$. The definitions of $\mathbb{P}^0$ and $\mathbb{P}^2$ depend only on γ and λ, but the definition of $\mathbb{P}^1$ also involves δ. All these posets preserve cardinals and cofinalities.

The key features of these posets are that $\mathbb{P}^0 * \mathbb{P}^1$ forces that δ is not measurable and that $2^\delta = \lambda$, while $\mathbb{P}^0 * (\mathbb{P}^1 \times \mathbb{P}^2)$ is equivalent to $\mathrm{Add}(\lambda, 1) \times \mathrm{Add}(\delta, \lambda)$. In our intended application in Section 2.3 there is a large cardinal κ and we set $\gamma = \omega$, $\delta = \kappa$, $\lambda = \kappa^{++}$. If κ has been suitably prepared, then $\mathbb{P}^0 * \mathbb{P}^1$ kills the measurability of κ and the GCH at κ, but forcing with $\mathbb{P}^2$ resurrects measurability because of the simple form of $\mathbb{P}^0 * (\mathbb{P}^1 \times \mathbb{P}^2)$.

$\mathbb{P}^0$ is the natural forcing to a add a non-reflecting stationary subset S of $\lambda \cap \mathrm{cof}(\gamma)$ via initial segments of the characteristic function. It is easy to see that $\mathbb{P}^0$ adds no $< \lambda$-sequences of ordinals.

Working in $V[S]$, let $\mathbb{P}^2$ be the natural forcing to add a club subset C of λ with C disjoint from S. It is a standard fact that $\mathbb{P}^0 * \dot{\mathbb{P}}^2$ is equivalent to $\mathrm{Add}(\lambda, 1)$.

It can be shown ([5, Lemma 1] or [4, Lemma 1]) that in $V[S]$ there is a sequence $\langle x_\alpha : \alpha \in S\rangle$ witnessing $\clubsuit_\lambda(S)$. That is to say x_α is cofinal in α, $\mathrm{ot}(x_\alpha) = \gamma$, and for every unbounded $A \subseteq \lambda$ there are stationarily many $\alpha \in S$ such that $x_\alpha \subseteq A$.

The following fact is standard (see [5, Lemma 2] or [4, Lemma 2] for a proof): If T is a set of limit ordinals such that $T \cap \alpha$ is non-stationary in α for all α, and $(y_\alpha)_{\alpha \in T}$ is a family of sets such that y_α is a cofinal subset of α for all α, then there exist $(z_\alpha)_{\alpha \in T}$ such that z_α is a final segment of y_α and the sets z_α are pairwise disjoint.

In our context this gives us the following facts:

- In the generic extension $V[S]$ by $\mathbb{P}^0$, for every $\eta < \lambda$ there exist $(z_\alpha)_{\alpha \in S \cap \eta}$ such that the sets z_α are pairwise disjoint final segments of the sets x_α for $\alpha \in S \cap \eta$.
- In the generic extension $V[S * C]$ by $\mathbb{P}^0 * \dot{\mathbb{P}}^2$ there exist $(z_\alpha)_{\alpha \in S}$ such that the sets z_α are pairwise disjoint final segments of the sets x_α for $\alpha \in S$. By distributivity, this sequence has the property that $(z_\alpha)_{\alpha \in S \cap \eta} \in V[S]$ for all $\eta < \lambda$.

$\mathbb{P}^1$ is defined in $V[S]$ as follows: conditions have the form $(w, \alpha, \bar{r}, Z)$ where

(1) $w \subseteq \lambda$ and $|w| < \delta$.
(2) $\alpha < \delta$.
(3) $\bar{r} = (r_i)_{i \in w}$ where $r_i : \alpha \to 2$ for each $i \in w$.
(4) $Z \subseteq \{x_\beta : \beta \in S\}$, and every $z \in Z$ is contained in w on a tail (so $|Z| < \delta$).

The ordering is as follows: $(w', \alpha', \bar{r}', Z') \leq (w, \alpha, \bar{r}, Z)$ iff

(1) $w \subseteq w'$.
(2) $\alpha \leq \alpha'$.
(3) $r'_i \restriction \alpha = r_i$ for $i \in w$.
(4) $Z \subseteq Z'$.
(5) If $z \in Z$ with $z \subseteq w$, and $\alpha \leq \eta < \alpha'$, then both $\{i \in z : r'_i(\eta) = 0\}$ and $\{i \in z : r'_i(\eta) = 1\}$ are cofinal in z.

Lemma 2.2 follows the same lines as the discussion following the definition of $\mathbb{P}^1_{\delta,\lambda}[S]$ in [5, pages 107 and 108]. We note that the conclusion given in that discussion is the γ-directed closure of $\mathbb{P}^1_{\delta,\lambda}[S]$, but the argument actually establishes δ-directed closure.

Lemma 2.2. *$\mathbb{P}^1$ is δ-directed closed.*

Proof. Let A be directed with $|A| < \delta$, and define a condition q as follows: $w^q = \bigcup_{p \in A} w^p$, $\alpha^q = \sup_{p \in A} \alpha^p$, $r^q_i = \bigcup_{p \in A, i \in w^p} r^p_i$ for $i \in w^q$, and $Z^q = \bigcup_{p \in A} Z^p$. It is routine that q is a condition and that the first four clauses in the definition of $q \leq p$ are satisfied for $p \in A$.

For the last clause let $z \in Z^p$ with $z \subseteq w^p$ and $\alpha^p \leq \eta < \alpha^q$, so that $\alpha^p \leq \eta < \alpha^r$ for some $r \in A$ where we may assume that $r \leq p$ by directedness. Then $w^p \subseteq w^r$, $r^q_i(\eta) = r^r_i(\eta)$ for all $i \in w^p$, and $\{i \in z : r^r_i(\eta) = 0\}$ and $\{i \in z : r^r_i(\eta) = 1\}$ are both cofinal in z. $\square$

Taken together, Lemmas 2.3 and 2.4 parallel [5, Lemma 3].

Lemma 2.3. *In the generic extension by $\mathbb{P}^1$, for each $i < \lambda$ let r^*_i be the union of r^p_i for p in the generic filter such that $i \in w^p$. Then $\mathrm{dom}(r^*_i) = \delta$.*

Proof. It is easy to see that the set of p with $i \in w^p$ is dense, so we assume that $i \in w^p$ and $\alpha^p < \eta < \delta$ and claim that there is $q \leq p$ with $\alpha^q = \eta$. We will set $w^q = w^p$ and $Z^q = Z^p$, so it remains to define $r^q_i \restriction [\alpha^p, \eta)$ for $i \in w^p$.

Let $S' = \{\beta \in S : x_\beta \in Z^p\}$, note that S' is a bounded subset of S and so we may choose tails y_β of x_β for $\beta \in S'$ so that the y_β's are disjoint. Then for $\alpha^p \leq \zeta < \eta$ and $\beta \in S'$, we may easily choose values of $r^q_i(\zeta)$ for $i \in y_\beta$ such that $\{i \in z : r^q_i(\zeta) = 0\}$ and $\{i \in z : r^q_i(\zeta) = 1\}$ are both cofinal in y_β, since there is no "interference" between different values of β. $\square$

Let $r^l_i = \{\alpha < \delta : r^*_i(\alpha) = l\}$ for $l \in 2$.

Lemma 2.4. *It is forced by $\mathbb{P}^1$ that δ is not measurable.*

Proof. Let p force that $\dot{D}$ is a measure on δ, where we may assume that p lies in the dense set of conditions where $z \subseteq w$ for all $z \in Z$. For all $i < \lambda$, choose $p_i \leq p$ to decide whether r^0_i or r^1_i is in $\dot{D}$ and arrange that $i \in w^{p_i}$, also that p_i is in the dense set described above.

Following the same lines as some familiar chain condition arguments, we will successively thin out the sequence of conditions p_i for i in the stationary set $\lambda \cap$

$\mathrm{cof}(\delta)$, eventually producing a stationary set $T \subseteq \lambda \cap \mathrm{cof}(\delta)$ such that $(p_i)_{i\in T}$ is a highly regular sequence of conditions:

- We may assume that $\sup(w^{p^i} \cap i)$ is constant with some value ρ for $i \in T$. This is possible by Fodor's lemma, since $|w^{p_i}| < \delta = \mathrm{cf}(i)$.
- We may assume that $w^{p_i} \cap i$ is constant with value w^* for $i \in T$. This is possible since $\rho^{<\delta} < \lambda$.
- Intersecting T with the club set of closure points of the function $i \mapsto \sup(w^{p_i})$, we may assume that $\sup(w^{p_i}) < j = \min(w^{p_j} \setminus j)$ for $i, j \in T$ with $i < j$. Note that the sets w^{p_i} for $i \in T$ form a head-tail-tail Δ-system with root w^*.
- We may assume that α^{p_i} is constant with value α for $i \in T$. This is possible because $\alpha^{p_i} < \delta$.
- We may assume that there is a sequence $\bar{r} = (r_k)_{k\in w^*}$ such that $r_k : \alpha \to 2$ and $r_k = r^{p_i}_k$ for all $k \in w^*$. This is possible because δ is inaccessible, so that in particular $|w^*| < \delta$ and $\alpha < \delta$.
- Finally, we may assume that there is $l < 2$ such that $p_i \Vdash r^l_i \in \dot{D}$ for all $i \in T$.

Note that $w^p \subseteq w^*$, and $r_i \restriction \alpha^p = r^p_i$ for all $i \in w^p$. Since the x_β's form a $\clubsuit_\lambda(S)$-sequence and T is unbounded in λ, we may find $\beta \in S$ such that $x_\beta \subseteq T$.

Define a condition q as follows:

- $w^q = \bigcup_{i\in x_\beta} w^{p_i}$. This is legal as $|x_\beta| = \gamma < \delta$. Note that since $i \in w^{p_i}$ we have that $x_\beta \subseteq w^q$.
- $\alpha^q = \alpha$.
- $r^q_k = r^{p_i}_k$ for $i \in x_\beta$ and $k \in w^{p_i}$. This makes sense as the r-parts of the p_i's agree on the root w^* of the Δ-system formed by w^{p_i} for $i \in T$, and also $x_\beta \subseteq T$.
- $Z^q = \bigcup_{i\in x_\beta} Z^{p_i} \cup \{x_\beta\}$. This makes sense as $|x_\beta| < \delta$.

Clearly q is a condition, so we verify that $q \le p$ and q forces that $\dot{D}$ fails to be a measure. For $q \le p$ we just check the last clause in the definition. Let $z \in Z^p$ (so that $z \subseteq w^p$) and $\alpha^p \le \eta < \alpha = \alpha^q$. By construction, for any $i \in x_\beta$ we have $r^q_k(\eta) = r^{p_i}_k(\eta)$ for all $k \in z$, and since $p_i \le p$ we are done.

Now consider the sets r^l_i for $i \in x_\beta$. We claim that q forces that $\bigcap_{i\in x_\beta} r^l_i \subseteq \alpha^q$. The point is that $x_\beta \in Z^q$ and $x_\beta \subseteq w^q$, so that for any $r \le q$ and any η with $\alpha^q \le \eta < \alpha^r$ we have that $\{i \in x_\beta : r^r_i(\eta) = 1 - l\}$ is unbounded in x_β, in particular $r \Vdash \eta \notin \bigcap_{i\in x_\beta} r^l_i$.

This is a contradiction since q also forces that the sets r^l_i for $i \in x_\beta$ are measure one for $\dot{D}$, and that $\dot{D}$ is a δ-complete measure. □

Lemma 2.5 parallels [5, Lemma 4].

Lemma 2.5. $\mathbb{P}^0 * (\dot{\mathbb{P}}^1 \times \dot{\mathbb{P}}^2)$ *is equivalent to* $\mathrm{Add}(\lambda, 1) \times \mathrm{Add}(\delta, \lambda)$.

Proof. As we already remarked, $\mathbb{P}^0 * \dot{\mathbb{P}}^2$ is equivalent to $\mathrm{Add}(\lambda, 1)$, so it will suffice to show that after forcing with $\mathbb{P}^0 * \dot{\mathbb{P}}^2$ the forcing poset $\mathbb{P}^1$ is equivalent to $\mathrm{Add}(\delta, \lambda)$. The key idea is to reorganise $\mathbb{P}^1$ using the fact that S becomes non-stationary after forcing with $\mathbb{P}^2$.

Since every initial segment of S (including S itself) is non-stationary, we may choose disjoint tails y_β of the sets x_β for $\beta \in S$. We claim that the set E of conditions $q \in \mathbb{P}^1$ such that for all $\beta \in S$ the set y_β is either contained in or disjoint from w^q is dense in $\mathbb{P}^1$. To see this let p be arbitrary, and define q as follows: $w^q = w^p \cup (\bigcup\{y_\beta : \beta \in S, y_\beta \cap w^p \neq \emptyset\})$, $\alpha^q = \alpha^p$, $\bar{r}^q$ is any sequence with $\bar{r}^q \restriction w^p = \bar{r}^p$, and $Z^q = Z^p$. Since the sets y_β are disjoint sets of size γ we have $|w^q| \leq \gamma \cdot |w^p| < \delta$, and easily $q \leq p$ with $q \in E$.

Now we define posets $\mathbb{Q}^0_\beta$ ($\beta \in S$) and $\mathbb{Q}^1$, and show that a condition $q \in E$ can essentially be decomposed into pieces which each lie in one of these posets. Each poset is a subset of $\mathbb{P}^1$ with the ordering inherited from $\mathbb{P}^1$.

- For each $\beta \in S$, $\mathbb{Q}^0_\beta$ is the set of conditions $(w, \alpha, \bar{r}, Z) \in \mathbb{P}^1$ such that $w = y_\beta$. For such a condition, if $x_\gamma \in Z$ then $\gamma = \beta$, so that $Z \subseteq \{x_\beta\}$.
- $\mathbb{Q}^1$ is the set of conditions $(w, \alpha, \bar{r}, Z) \in \mathbb{P}^1$ where $w \subseteq \lambda \setminus \bigcup_{\beta \in S} y_\beta$. For such a condition necessarily $Z = \emptyset$.

Let $q = (w^q, \alpha^q, \bar{r}^q, Z^q) \in E$. Since $q \in E$, w^q has the form $(\bigcup_{\beta \in s} y_\beta) \cup (w^q \setminus \bigcup_{y \in S} y_\beta)$ for some set $s \subseteq S$ with $|s| < \delta$. Since the sets y_β are disjoint, $Z^q \subseteq \{x_\beta : \beta \in s\}$. We define conditions $q^0_\beta \in \mathbb{Q}^0_\beta$ ($\beta \in S$) and $q^1 \in \mathbb{Q}^1$ as follows:

- q^0_β is the trivial condition for $\beta \notin s$.
- $q^0_\beta = (y_\beta, \alpha^q, \bar{r}^q \restriction y_\beta, Z^q \cap \{x_\beta\})$ for $\beta \in s$.
- $q^1 = (w^q \setminus \bigcup_{y \in S} y_\beta, \alpha^q, \bar{r}^q \restriction (w^q \setminus \bigcup_{y \in S} y_\beta), \emptyset)$.

It is routine to check that for $q, r \in E$, $r \leq q$ if and only if $r^0_\beta \leq q^0_\beta$ for all β and $r^1 \leq q^1$: the key point is that in clause (5) of the definition of $r \leq q$, if $z \in Z^q$ then $z = x_\beta$ where $y_\beta \subseteq w^q$, so that satisfying clause (5) for $r \leq q$ and this value of z amounts to satisfying clause (5) for $r^0_\beta \leq q^0_\beta$.

Let $\mathbb{Q}^*$ be the $< \delta$-support product of the posets $\mathbb{Q}^0_\beta$ together with $\mathbb{Q}^1$. We write elements of this poset in the form $((a_\beta)_{\beta \in S}, b)$ where $a_\beta \in \mathbb{Q}^0_\beta$, $b \in \mathbb{Q}^1$, and a_β is non-trivial for fewer than δ many values of β. Let E' be the subset of $\mathbb{Q}^*$ consisting of conditions $((a_\beta)_{\beta \in S}, b)$ such that, for some $\alpha < \delta$, we have $\alpha^b = \alpha$ and $\alpha^{a_\beta} = \alpha$ for all β such that a_β is non-trivial. It is easy to see that E' is dense in $\mathbb{Q}^*$.

We have shown that E is isomorphic to E'. Since δ is inaccessible and $|y_\beta| = \gamma < \delta$, the poset $\mathbb{Q}^0_\beta$ is δ-closed with cardinality δ, hence it is equivalent to $\mathrm{Add}(\delta, 1)$. Since clause (5) is irrelevant to the ordering of $\mathbb{Q}^1$, it is clear that $\mathbb{Q}^1$ is equivalent to $\mathrm{Add}(\delta, |\lambda \setminus \bigcup_{\beta \in S} y_\beta|)$. It follows that $\mathbb{P}^1$ is equivalent to $\mathrm{Add}(\delta, \lambda)$. □

Corollary 2.6. *It is forced by $\mathbb{P}^0$ that $\dot{\mathbb{P}}^1$ has the δ^+-chain condition.*

The following lemma will be used in the lifting argument for the proof of Theorem 2.1.

Lemma 2.7. *If $S * (G \times C)$ is $\mathbb{P}^0 * (\dot{\mathbb{P}}^1 \times \dot{\mathbb{P}}^2)$-generic and we rearrange it as $g \times h$ which is $\mathrm{Add}(\lambda, 1) \times \mathrm{Add}(\delta, \lambda)$-generic as in Lemma 2.5, then $h \restriction \eta \in V[S * G]$ for every $\eta < \lambda$.*

Proof. In the construction from the proof of Lemma 2.5, each of the λ coordinates in $\mathrm{Add}(\delta, \lambda)$ corresponds either to y_β for some $\beta \in S$, or to some point in $\lambda \setminus \bigcup_{\beta \in S} y_\beta$. It follows that to compute $h \restriction \eta$ from G we only need a proper initial segment of the sequence $(y_\beta)_{\beta \in S}$, and all such initial segments lie in $V[S]$ because forcing over $V[S]$ with $\mathbb{P}^2$ does not add any new $< \lambda$-sequences of ordinals. □

The point of Lemma 2.5 is that if δ happens to be a suitably prepared $< \lambda$-supercompact cardinal, we may kill its measurability with $\mathbb{P}^0 * \dot{\mathbb{P}}^1$ and then resurrect it by forcing with $\mathbb{P}^2$. We note that $\mathbb{P}^0 * \dot{\mathbb{P}}^1$ is $< \delta$-strategically closed, by an easy argument along the lines of [5, Lemma 5]; the key point is that $\mathbb{P}^0 * \dot{\mathbb{P}}^1$ embeds into $\mathbb{P}^0 * (\dot{\mathbb{P}}^1 \times \dot{\mathbb{P}}^2)$, which is equivalent to a δ-closed forcing poset by Lemma 2.5. In the sequel we write $\mathbb{P}^i_{\gamma,\delta,\lambda}$ for the version of $\mathbb{P}^i$ defined with parameters γ, δ and λ.

2.3. **Proof of Theorem 2.1.** We start by assuming that GCH holds and κ is κ^+-supercompact, and fix a supercompactness measure W on $P_\kappa \kappa^+$. Let $j : V \to N = \mathrm{Ult}(V, W)$ be the ultrapower map, so that by standard arguments:

- $\kappa^{++} = (\kappa^{++})^N < j(\kappa) < j(\kappa^+) < j(\kappa^{++}) < j(\kappa^{+++}) = \kappa^{+++}$.
- j is continuous at κ^{++}.

We force with an Easton support iteration $\mathbb{Q}$ of length $\kappa + 1$ defined as follows: at inaccessible $\alpha < \kappa$ we force with $\mathbb{P}^0_{\omega,\alpha,\alpha^{++}} * \dot{\mathbb{P}}^1_{\omega,\alpha,\alpha^{++}}$, and at κ we force with $\mathbb{P}^0_{\omega,\kappa,\kappa^{++}} * (\dot{\mathbb{P}}^1_{\omega,\kappa,\kappa^{++}} \times \dot{\mathbb{P}}^2_{\omega,\kappa,\kappa^{++}})$. Let g^i_α be the $\mathbb{P}^i$-generic object added at stage α, let G_κ be the generic object for the iteration up to stage κ, and let $G_{\kappa+1}$ be the generic object for the whole iteration $\mathbb{Q}$. By Lemma 2.4 the forcing at each inaccessible $\alpha < \kappa$ destroys the measurability of α, and since the remainder of the iteration is highly closed it follows that $\mathbb{Q}$ destroys the measurability of every cardinal less than κ.

The first stage of the construction is to lift j onto $V[G_{\kappa+1}]$, making sure that for the lifted version of j we have $\kappa^{++} \subseteq \{j(h)(\kappa) : h \in V[G_{\kappa+1}]\}$. As usual we will lift j by building a generic object for $j(\mathbb{Q})$ over N which contains $j[G_{\kappa+1}]$. The first few steps are routine:

- $j(\mathbb{Q})$ is an iteration such that $j(\mathbb{Q}) \restriction \kappa = \mathbb{Q} \restriction \kappa$, and $j(\mathbb{Q})_\kappa = \mathbb{P}^0_{\omega,\kappa,\kappa^{++}} * \dot{\mathbb{P}}^1_{\omega,\kappa,\kappa^{++}}$.
- $G_\kappa * g^0_\kappa * g^1_\kappa$ is $j(\mathbb{Q}) \restriction (\kappa + 1)$-generic over N.
- Since G_κ is generic for κ-cc forcing, g^0_κ is generic for forcing which adds no κ^+-sequences, and g^1_κ is generic for κ^+-cc forcing, it follows that $V[G_\kappa * g^0_\kappa * g^1_\kappa] \models {}^{\kappa^+}N[G_\kappa * g^0_\kappa * g^1_\kappa] \subseteq N[G_\kappa * g^0_\kappa * g^1_\kappa]$.
- In $N[G_\kappa * g^0_\kappa * g^1_\kappa]$, the part of $j(\mathbb{Q})$ between κ and $j(\kappa)$ is δ-closed forcing with $j(\kappa)$ maximal antichains, where δ is the least N-inaccessible greater than κ.
- In $V[G_\kappa * g^0_\kappa * g^1_\kappa]$, the part of $j(\mathbb{Q})$ between κ and $j(\kappa)$ is κ^{++}-closed forcing with κ^{++} maximal antichains.
- There is $H \in V[G_\kappa * g^0_\kappa * g^1_\kappa]$ such that H is generic over $N[G_\kappa * g^0_\kappa * g^1_\kappa]$ for the part of $j(\mathbb{Q})$ between κ and $j(\kappa)$.
- $V[G_\kappa * g^0_\kappa * g^1_\kappa] \models {}^{\kappa^+}N[G_\kappa * g^0_\kappa * g^1_\kappa * H] \subseteq N[G_\kappa * g^0_\kappa * g^1_\kappa * H]$.
- Since $\mathbb{Q}$ has Easton supports, $j[G_\kappa] \subseteq G_\kappa * g^0_\kappa * g^1_\kappa * H$, so j lifts in $V[G_\kappa * g^0_\kappa * g^1_\kappa]$ to $j : V[G_\kappa] \to N[G_\kappa * g^0_\kappa * g^1_\kappa * H]$.

Since $\mathbb{Q}_\kappa$ is equivalent in $V[G_\kappa]$ to $\mathrm{Add}(\kappa, \kappa^{++}) \times \mathrm{Add}(\kappa^{++}, 1)$ (as computed in $V[G_\kappa]$), we may as well assume that $\mathbb{Q}_\kappa = \mathrm{Add}(\kappa, \kappa^{++}) \times \mathrm{Add}(\kappa^{++}, 1)$. With this in mind we write the generic object at κ as $g \times g'$, where we know that $g \restriction \delta \times \eta \in V[G_\kappa][g^0_\kappa * g^1_\kappa]$ for all $\eta < \kappa^{++}$.

Our task is now to construct compatible generic objects $h \supseteq j[g]$ and $h' \supseteq j[g']$, where we note that by Easton's lemma mutual genericity between h and h' is automatic.

We note that $N[G_\kappa * g^0_\kappa * g^1_\kappa * H] = \{j(h)(j[\kappa^+]) : h \in V[G_\kappa], \mathrm{dom}(h) = (P_\kappa \kappa^+)^V\}$, in particular every element is represented by a function whose domain has cardinality κ^+. Since g' is generic for κ^{++}-closed forcing, it follows by standard arguments (see for example [12, Proposition 15.1]) that $j[g']$ generates a generic filter h' for $j(\mathrm{Add}(\kappa^{++}, 1))$ over $N[G_\kappa * g^0_\kappa * g^1_\kappa * H]$.

Before constructing h, we digress briefly to analyse dense subsets of $\mathrm{Add}(\kappa, \kappa^{++})^{V[G_\kappa]}$ working in the model $V[G_\kappa]$. By GCH, or alternatively using that κ is inaccessible, $|\mathrm{Add}(\kappa, \alpha)| \leq \kappa^+$ for all $\alpha < \kappa^{++}$. Now let D be a dense set, then we claim there is a function $f_D : \kappa^{++} \to \kappa^{++}$ such that for every $\alpha < \kappa^{++}$ and every $p \in \mathrm{Add}(\kappa, \alpha)^{V[G_\kappa]}$ there is $q \leq p$ with $q \in D \cap \mathrm{Add}(\kappa, f_D(\alpha))$: just choose an extension in D for every condition in $\mathrm{Add}(\kappa, \alpha)$, and use the observation that $|\mathrm{Add}(\kappa, \alpha)| \leq \kappa^+$ to choose β so large that all the extensions lie in $\mathrm{Add}(\kappa, \beta)$.

Let E_D be the club set of closure points of f_D and let $\gamma \in E_D$ with $\mathrm{cf}(\gamma) \geq \kappa$, where we note that $\mathrm{Add}(\kappa, \gamma) = \bigcup_{\alpha < \gamma} \mathrm{Add}(\kappa, \alpha)$. We claim that for every $p \in \mathrm{Add}(\kappa, \gamma)$ there is $q \leq p$ with $q \in D \cap \mathrm{Add}(\kappa, \gamma)$. To see this let $p \in \mathrm{Add}(\kappa, \gamma)$, so that $p \in \mathrm{Add}(\kappa, \alpha)$ for some $\alpha < \gamma$. By the definition of f_D there is $q \leq p$ with $q \in D \cap \mathrm{Add}(\kappa, f_D(\alpha))$, and since γ is a closure point of f_D we have $q \in D \cap \mathrm{Add}(\kappa, \gamma)$.

To construct $j(g)$ we enumerate the antichains of $j(\mathrm{Add}(\kappa, \kappa^{++})^{V[G_\kappa]})$ which lie in $N[G_\kappa * g^0_\kappa * g^1_\kappa * H]$. Since $j(\kappa^{++}) < \kappa^{+++}$ there are only κ^{++} such antichains, so we may enumerate them in $V[G_\kappa * g^0_\kappa * g^1_\kappa]$ as $(A_i)_{i<\kappa^{++}}$. We also enumerate the ordinals less than $j(\kappa)$ as $(\mu_i)_{i<\kappa^{++}}$.

Let D_i be the dense open set of conditions which extend some condition in A_i and let $D_i = j(d_i)([\mathrm{id}])$, where $d_i \in V[G_\kappa]$, $\mathrm{dom}(d_i) = (P_\kappa \kappa^+)^V$, and $d_i(x)$ is a dense subset of $\mathrm{Add}(\kappa, \kappa^{++})^{V[G_\kappa]}$ for all x. Let $E_i = \bigcap_x E_{d_i(x)}$, so that E_i is club in κ^{++}. By elementarity, if $\gamma \in j(E_i)$ and $N[G_\kappa * g^0_\kappa * g^1_\kappa * H] \models \mathrm{cf}(\gamma) \geq j(\kappa)$ then for every $p \in \mathrm{Add}(j(\kappa), \gamma)^{N[G_\kappa * g^0_\kappa * g^1_\kappa * H]}$ there is $q \leq p$ with $q \in D_i \cap \mathrm{Add}(j(\kappa), \gamma)^{N[G_\kappa * g^0_\kappa * g^1_\kappa * H]}$.

Now we work in $V[G_\kappa][g^0_\kappa * g^1_\kappa * g^2_\kappa]$ to build a decreasing κ^{++}-sequence $(f_i)_{i<\kappa^{++}}$ of conditions in $j(\mathrm{Add}(\kappa, \kappa^{++})^{V[G_\kappa]})$. The properties which we require are:

- The sequence $(f_i)_{i<\kappa^{++}}$ eventually meets each of the dense open sets D_i (this will ensure that the sequence generates a generic filter).
- For every $\eta < \kappa^{++}$ and $\nu < \kappa$, $\bigcup_{i<\kappa^{++}} f_i(j(\eta), \nu) = \bigcup g(\eta, \nu)$ (this will ensure that when we lift j, $j[g]$ is contained in the filter generated by $(f_i)_{i<\kappa^{++}}$).
- For every $\eta < \kappa^{++}$, $\bigcup_{i<\kappa^{++}} f_i(j(\eta), \kappa) = \mu_\eta$. This will ensure that when we lift j, if x_η is the η^{th} generic function added by g then $j(x_\eta)(\kappa) = \mu_\eta$: so every ordinal less than $j(\kappa)$ is of the form $j(x)(\kappa)$ for some $x : \kappa \to \kappa$. This idea was first used by Woodin [27] in a similar context, and marks the key step here, as we will see at the end of the proof.
- Every proper initial segment of $(f_i)_{i<\kappa^{++}}$ lies in $V[G_\kappa * g^0_\kappa * g^1_\kappa]$ (this, combined with the fact that $V[G_\kappa * g^0_\kappa * g^1_\kappa] \models {}^{\kappa^+}N[G_\kappa * g^0_\kappa * g^1_\kappa * H] \subseteq N[G_\kappa * g^0_\kappa * g^1_\kappa * H]$, will ensure that we can continue the construction for κ^{++} steps).

We start by setting f_0 equal to the empty condition. For j limit we let $f_j = \bigcup_{i<j} f_i$. Suppose that we have chosen f_i. We start by finding $\gamma \in E_i \cap \mathrm{cof}(\kappa)$ so large that $f_i \in \mathrm{Add}(j(\kappa), j(\gamma))$. Note that it follows from the choice of E_i that for every $p \in \mathrm{Add}(j(\kappa), j(\gamma))$ there is $q \leq p$ with $q \in D_i \cap \mathrm{Add}(j(\kappa), j(\gamma))$.

We first choose $f'_i \leq f_i$ such that $f'_i \in \mathrm{Add}(j(\kappa), j(\gamma))$, $f'_i(j(\eta), \kappa) = \mu_\eta$ for every $\eta < \gamma$, and $f'_i(j(\eta), \nu) = \bigcup g(\eta, \nu)$ for every $\eta < \gamma$ and $\nu < \kappa$. This is possible because $j : V[G_\kappa] \to N[G_\kappa * g^0_\kappa * g^1_\kappa * H]$ is defined in $V[G_\kappa * g^0_\kappa * g^1_\kappa]$, $V[G_\kappa * g^0_\kappa * g^1_\kappa] \models {}^{\kappa^+} N[G_\kappa * g^0_\kappa * g^1_\kappa * H] \subseteq N[G_\kappa * g^0_\kappa * g^1_\kappa * H]$, and $g \restriction \gamma \in V[G_\kappa * g^0_\kappa * g^1_\kappa]$. Then we choose $f_{i+1} \leq f'_i$ such that $f_{i+1} \in D_i \cap \mathrm{Add}(j(\kappa), j(\gamma))$. This completes the construction.

Now we take stock: let $V' = V[G_\kappa * g^0_\kappa * (g^1_\kappa \times g^2_\kappa)]$, then we have a lifted embedding $j' : V' \to N'$ such that j' witnesses κ is κ^+-supercompact, κ is not measurable in N' (in fact κ is the least measurable cardinal in V'), and $j'(\kappa) = \{j'(f)(\kappa) : f' : \kappa \to \kappa, f' \in V'\}$. As usual we may factor j' through the ultrapower $i' : V' \to M'$ by the normal measure U' induced by j', and we have $k' : M' \to N'$ with $k' \circ i' = j'$.

By construction $\mathrm{crit}(k') > j'(\kappa)$, so easily $i'(\kappa) = j'(\kappa)$ and $(V_{i'(\kappa)})^{M'} = (V_{i'(\kappa)})^{N'}$. Finally as $V' \models {}^{\kappa^+} N' \subseteq N'$ and ${}^{\kappa^+} i'(\kappa) \subseteq (V_{i'(\kappa)})^{N'}$, we see that $V' \models {}^{\kappa^+} i'(\kappa) \subseteq M'$ as required. Since κ is not measurable in M' the measure U' has order zero, concluding the proof of Theorem 2.1.

2.4. **Many measures of order zero.**

Theorem 2.8. *It is consistent (modulo the existence of a cardinal κ which is κ^+-supercompact) that for the least measurable cardinal κ, $2^\kappa = 2^{\kappa^+} = \kappa^{++}$ and κ carries $2^{\kappa^{++}}$ normal measures of order zero. The value of $2^{\kappa^{++}}$ may be taken arbitrarily large.*

Proof. We assume the conclusion of Theorem 2.1: that is to say κ is the least measurable cardinal, $2^\kappa = \kappa^{++}$, and there is a normal measure U on κ such that if $j : V \to M$ is the ultrapower map then κ is not measurable in M and ${}^{\kappa^+} j(\kappa) \subseteq M$. Let $\mathbb{P}$ be the Easton support iteration which adds a single Cohen subset of α^{+++} for every inaccessible $\alpha < \kappa$, and let G be $\mathbb{P}$-generic. Note that $V[G] \models$ "$j(\mathbb{P})/G$ is κ^{++}-closed forcing of size $j(\kappa)$".

Since $\mathbb{P}$ is κ-cc forcing of cardinality κ, a $\mathbb{P}$-name for a function from κ^+ to $j(\kappa)$ may be coded by a function from $\kappa \times \kappa^+$ to $j(\kappa)$. By hypothesis all such functions lie in M, so $V[G] \models {}^{\kappa^+} j(\kappa) \subseteq M[G]$. Since $\mathbb{P} \times \mathbb{P}$ is κ-cc in M and κ is not measurable in M, κ is not measurable in $M[G]$.

Working in $V[G]$, we may enumerate the maximal antichains of $j(\mathbb{P})/G$ which lie in $M[G]$ in order type κ^{++}. We will then build a complete binary tree of height κ^{++} of conditions in $j(P)/G$ such that every branch meets every maximal antichain of $j(\mathbb{P})/G$ which lies in $M[G]$. This is possible by adapting the argument from [10] given at the start of this section in the obvious way: the key point is that by what we have proved about $M[G]$ and $j(\mathbb{P})/G$, any $< \kappa^{++}$-sequence from $V[G]$ of conditions in $j(\mathbb{P})/G$ lies in $M[G]$ and therefore has a lower bound.

We may now construct $2^{\kappa^{++}}$ normal measures of order zero on κ in $V[G]$. The conclusion of Theorem 2.1 remains true if we add Cohen subsets of κ^{++} so we may make $2^{\kappa^{++}}$ as large as we wish.

□

Remark 2.9. We can prove versions of Theorems 2.1 and 2.8 with larger values for 2^κ. Assume that GCH holds and λ is either inaccessible or of the form μ^+ where $\mathrm{cf}(\mu) > \kappa$. Assume that κ is $< \lambda$-supercompact, that is to say there is $j : V \to M$ such that $\kappa = \mathrm{crit}(j)$, $j(\kappa) > \lambda$, and ${}^{<\lambda} M \subseteq M$. We may assume that:

- (λ inaccessible) j is the limit ultrapower by a tower of supercompactness measures $(U_\zeta)_{\kappa \leq \zeta < \lambda}$ where U_ζ is a measure on $P_\kappa \zeta$, so that $\lambda < j(\lambda) = \sup j[\lambda] < \lambda^+$.
- (λ is μ^+ for $\mathrm{cf}(\mu) > \kappa$) j is the ultrapower by a supercompactness measure on $P_\kappa \mu$, so that again $\lambda < j(\lambda) = \sup j[\lambda] < \lambda^+$.

It is now straightforward to modify the proofs and obtain cardinal and cofinality-preserving extensions where:

- (Theorem 2.1): κ is the least measurable cardinal, $2^\kappa = \lambda$, and there is a normal measure U of order zero on κ such that ${}^{<\lambda} j_U(\kappa) \subseteq \mathrm{Ult}(V, U)$.
- (Theorem 2.8): κ is the least measurable cardinal, $2^\delta = \lambda$ for all $\delta \in [\kappa, \lambda)$ and κ carries 2^λ normal measures of order zero. The value of 2^λ may be taken arbitrarily large.

We can also prove a global version of Theorem 2.8, in which there is a supercompact cardinal and every measurable cardinal satisfies the conclusion of that theorem.

Theorem 2.10. *It is consistent (relative to the existence of a supercompact cardinal) that there exists a supercompact cardinal, and that $2^\delta = 2^{\delta^+} = \delta^{++}$ and δ carries $2^{\delta^{++}}$ measures of order zero for every measurable cardinal δ.*

Proof. We start by assuming that GCH holds, κ is supercompact, and there are no inaccessible cardinals above κ. We do an Easton support iteration $\mathbb{P}$ of length $\kappa + 1$ which is non-trivial only at cardinals δ which are measurable in V. Given such δ we work in $V[G_\delta]$ to define $\mathbb{P}^0_\delta$, $\mathbb{P}^1_\delta$ and $\mathbb{P}^2_\delta$ as in Section 2.2 with $\gamma = \omega$ and $\lambda = \delta^{++}$.

- If δ is not δ^+-supercompact then we force over $V[G_\delta]$ with $\mathbb{P}^0_\delta * \dot{\mathbb{P}}^1_\delta$.
- If δ is δ^+-supercompact then we force over $V[G_\delta]$ with $\mathbb{P}^0_\delta * (\dot{\mathbb{P}}^1_\delta \times \dot{\mathbb{P}}^2_\delta)$.

Let V^* be the generic extension by the iteration $\mathbb{P}$. By the properties of $\mathbb{P}^0_\delta * \dot{\mathbb{P}}^1_\delta$, every cardinal δ which is measurable but not δ^+-supercompact in V is not measurable in $V[G_{\delta+1}]$, and so by the closure of the tail forcing is not measurable in V^*. Moreover if δ is a measurable cardinal in V^* then δ is certainly Mahlo in V, so that $(\mathbb{P} \restriction \delta) \times (\mathbb{P} \restriction \delta)$ is δ-cc and hence δ is measurable in V. In summary, if δ is measurable in V^* then δ must be δ^+-supercompact in V.

The proof of Theorem 2.1 shows that if δ is δ^+-supercompact in V, then δ is measurable and satisfies the conclusion of that theorem in $V[G_{\delta+1}]$, and hence by the closure of the tail forcing it also satisfies the conclusion in V^*. That is to say, in V^* we have that $2^\delta = 2^{\delta^+} = \delta^{++}$, and there is a normal measure U of order zero on δ such that ${}^{\delta^+} j_U(\delta) \subseteq \mathrm{Ult}(V^*, U)$.

We claim that κ is supercompact in V^*. Let $\lambda > \kappa^{++}$ with λ regular, and let $j : V \to M$ be the ultrapower by some supercompactness measure on $P_\kappa \lambda$. $\mathbb{P} \restriction \kappa$ is κ-cc forcing of size κ, the last stage of $\mathbb{P}$ is essentially $\mathrm{Add}(\kappa, \kappa^{++}) \times \mathrm{Add}(\kappa^{++}, 1)$, and (since there are no inaccessible cardinals above κ) the support of $j(\mathbb{P})$ is empty in the interval $(\kappa, \lambda]$. Standard arguments now allow us to lift the embedding j to the model V^*.

Working in V^* let $\mathbb{Q}$ be an Easton support iteration to add a Cohen subset of α^{+++} for every inaccessible $\alpha < \kappa$. Similar arguments to those given earlier in the proof show that after forcing with $\mathbb{Q}$ the cardinal κ is still supercompact, and no measurable cardinals are created or destroyed. Arguments as in the proof of

Theorem 2.8 show that every measurable cardinal δ now carries the desired number $2^{\delta^{++}}$ of measures of order zero. $\square$

Remark 2.11. Starting from GCH and a cardinal κ which is κ^{++}-supercompact, similar arguments produce a set model in which there is a proper class of strong cardinals and every measurable cardinal satisfies the conclusions of Theorem 2.8.

We finish with two natural questions raised by the use of κ^+-supercompactness as a hypothesis in Theorems 2.1 and 2.8. It is easy to see that these theorems require at least the strength of a measurable cardinal κ with $o(\kappa) = \kappa^{++}$, since by work of Gitik [16] this is the consistency strength of the failure of GCH at a measurable cardinal.

Question 2.12. What is the consistency strength of the assertion that there is a normal measure U on κ such that ${}^{\kappa^+} j_U(\kappa) \subseteq \mathrm{Ult}(V, U)$?

Question 2.13. What is the consistency strength of the assertion that $2^\kappa > \kappa^+$ and κ carries 2^{2^κ} normal measures of order zero?

Remark 2.14. After reading the first draft of this paper, Moti Gitik pointed out Theorem 2.8 can be proved from the optimal hypothesis that there exists κ with $o(\kappa) = \kappa^{++}$, resolving Question 2.13. He also noted that Theorem 2.1 does not require the full strength of the hypothesis that κ is κ^+-supercompact, and outlined a proof that some hypothesis at the level of superstrong cardinals is required.

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