Discrete Applied Mathematics 324 (2023) 93–98

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Probabilistic chip-collecting games with modulo winning conditions

Joshua Harrington [a], Xuwen Hua [b], Xufei Liu [c], Alex Nash [d], Rodrigo Rios [e], Tony W.H. Wong [f,*]

[a] *Department of Mathematics, Cedar Crest College, United States of America*
[b] *Department of Mathematics, Pomona College, United States of America*
[c] *Department of Industrial Engineering, Georgia Institute of Technology, United States of America*
[d] *Department of Mathematics, Dickinson College, United States of America*
[e] *Department of Mathematics, Florida Atlantic University, United States of America*
[f] *Department of Mathematics, Kutztown University of Pennsylvania, United States of America*

ARTICLE INFO

Article history:
Received 25 January 2022
Accepted 30 August 2022
Available online xxxx

Keywords:
Probabilistic game
Random walk

ABSTRACT

Let a, b, and n be integers with $0 < a < b < n$. In a certain two-player probabilistic chip-collecting game, Alice tosses a coin to determine whether she collects a chips or b chips. If Alice collects a chips, then Bob collects b chips, and vice versa. A player is announced the winner when they have accumulated a number of chips that is a multiple of n. In this paper, we settle two conjectures from the literature related to this game.

1. Introduction

In a probabilistic chip-collecting game introduced by Wong and Xu [4], Alice and Bob take turns to toss a coin with Alice tossing first, which determines independently whether the player collects a chips or b chips. The winner of the game is the first player to accumulate n chips. Some variations of this game have been considered by Leung and Thanatipanonda [2,3] and Harrington et al. [1]. The versions of the game that were considered by Harrington et al. removed the independence of the chip collecting process, so that if Alice collects a chips, then Bob collects b chips, and vice versa. In one of these versions, called the *modulo dependent game*, a player is announced the winner when they have accumulated a number of chips that is a multiple of n.

For $a < b < n$, the modulo dependent game can be treated as a random walk on $\mathbb{Z}_n \times \mathbb{Z}_n$, where the number of chips accumulated by each player is recorded as an ordered pair (x, y) and each move is represented by either $(+a, +b)$ or $(+b, +a)$. Since Alice always collects chips first, for any $y \in \mathbb{Z}_n$ and $x \in \mathbb{Z}_n \setminus \{0\}$, positions $(0, y)$ and $(x, 0)$ are called the *winning positions* of Alice and Bob, respectively, and a random walk on $\mathbb{Z}_n \times \mathbb{Z}_n$ that starts from $(0, 0)$ *terminates* upon landing on any winning position. A position $(x, y) \in \mathbb{Z}_n \times \mathbb{Z}_n$ is said to be *reachable* if there exists a random walk that lands on (x, y) after leaving the starting position $(0, 0)$. As established by Harrington et al. [1], (a, a) and (b, b) are never reachable in $\mathbb{Z}_n \times \mathbb{Z}_n$. They further conjectured the following statement, for which we provide a proof in Section 2.

Theorem 1.1. *Every position in $\mathbb{Z}_n \times \mathbb{Z}_n \setminus \{(a, a), (b, b)\}$ is reachable if and only if $a \not\equiv 2b \pmod n$, $2a \not\equiv b \pmod n$, and $b^2 - a^2$ is relatively prime to n.*

* Corresponding author.
E-mail addresses: joshua.harrington@cedarcrest.edu (J. Harrington), xhaa2019@mymail.pomona.edu (X. Hua), xliu725@gatech.edu (X. Liu), nasha@dickinson.edu (A. Nash), riosr2018@fau.edu (R. Rios), wong@kutztown.edu (T.W.H. Wong).

https://doi.org/10.1016/j.dam.2022.08.031
0166-218X/

The modulo dependent game can naturally be extended to a variation that allows Alice and Bob to having different winning conditions. In particular, Harrington et al. considered a variation of the game where Alice wins by collecting a multiple of m chips and Bob wins by collecting a multiple of n chips. This game can be recognized as a random walk on $\mathbb{Z}_m \times \mathbb{Z}_n$, where $a < b < \min\{m, n\}$. Although this variation was not studied by Harrington et al. they did present the following conjecture.

Conjecture 1.2. *Let $m \mid n$. If all winning positions are of the form $(0, y)$, then $m \mid (b^2 - a^2)$.*

In Section 3, we will prove the following theorem, which establishes Conjecture 1.2.

Theorem 1.3. *In the modulo dependent game with parameters a, b, m, and n such that $\gcd(a, b, m, n) = 1$, all reachable winning positions are of the form $(0, y)$ if and only if $m \mid (b^2 - a^2)$ and $m \mid \gcd(a, b)\gcd(m, n)$.*

As a corollary to Theorem 1.3, in the modulo dependent with parameters a, b, m, and n, notice that Bob's winning probability is 0 if and only if $m \mid (b^2 - a^2)$ and $m \mid \gcd(a, b)\gcd(m, n)$.

2. Proof of Theorem 1.1

Proof. If every position in $\mathbb{Z}_n \times \mathbb{Z}_n \setminus \{(a, a), (b, b)\}$ is reachable, then $(1, 0)$ is reachable. In other words, $(ai + bj, aj + bi) = (1, 0)$ for some integers i and j. By adding or subtracting the two coordinates, we have $(a + b)(i + j) \equiv (b - a)(j - i) \equiv 1 \pmod n$, thus $\gcd(a + b, n) = \gcd(b - a, n) = 1$. Hence, $b^2 - a^2$ is relatively prime to n. To establish the remaining necessary conditions, we proceed with a proof by contrapositive. If $a \equiv 2b \pmod n$, then the position $(2b, 3b)$ can only be reached from $(0, 2b)$ or (b, b), so $(2b, 3b)$ is not reachable. Similarly, if $b \equiv 2a \pmod n$, then the position $(2a, 3a)$ is not reachable.

To prove the sufficient condition, let $\mathbf{q}_{i,j} = (ia + j(a + b), ib + j(a + b))$, where $i, j \in \mathbb{Z}$. Since $\gcd(b - a, n) = \gcd(a + b, n) = 1$, every position in $\mathbb{Z}_n \times \mathbb{Z}_n$ can be expressed in the form of $\mathbf{q}_{i,j}$ for some $0 \le i, j \le n - 1$. Furthermore, $\gcd(k(a + b), n) \le k < n$ and $\gcd(k(b - a), n) \le k < n$ for all $1 \le k < n$, thus

$$k(a + b) \not\equiv 0 \pmod n \text{ and } k(b - a) \not\equiv 0 \pmod n. \quad (1)$$

As a result, $2a \not\equiv 2b \pmod n$, which implies that every position $(x, x) \in \mathbb{Z}_n \times \mathbb{Z}_n \setminus \{(a, a), (b, b)\}$ is reachable by Harrington et al. [1, Theorem 3.6]. Hence, it remains to show that $\mathbf{q}_{i,j}$ is reachable for all $1 \le i \le n - 1$ and $0 \le j \le n - 1$.

We will prove by induction on j that $\mathbf{q}_{1,j}$ is reachable for all $0 \le j \le n - 1$. First, the position $\mathbf{q}_{1,0} = (a, b)$ is reachable, and the position $\mathbf{q}_{1,1}$ is reachable by the sequence of moves

$$\mathbf{q}_{1,0} = (a, b) \xrightarrow{(+b,+a)} (a + b, a + b) \xrightarrow{(+a,+b)} \mathbf{q}_{1,1}.$$

Now, assume that for some $1 \le j \le n - 2$, $\mathbf{q}_{1,j'}$ is reachable for all $0 \le j' \le j$. We proceed by considering the following cases.

Case 1: $\mathbf{q}_{1,j}$ is not a winning position.

Case 1(*a*): $\mathbf{q}_{2,j}$ is not a winning position.
The position $\mathbf{q}_{1,j+1}$ is reachable by the sequence of moves

$$\mathbf{q}_{1,j} \xrightarrow{(+a,+b)} \mathbf{q}_{2,j} \xrightarrow{(+b,+a)} \mathbf{q}_{1,j+1}.$$

Case 1(*b*): $\mathbf{q}_{2,j}$ is a winning position.
Since $\mathbf{q}_{2,j} = (2a + j(a + b), 2b + j(a + b))$, with a simple calculation, we have $\mathbf{q}_{2,j} \in \{(0, 2b - 2a), (2a - 2b, 0)\}$. Hence, $\mathbf{q}_{0,j+1} \in \{(b - a, b - a), (a - b, a - b)\}$, which does not intersect with $\{(a, a), (b, b)\}$ since $a \not\equiv 2b \pmod n$ and $b \not\equiv 2a \pmod n$. Therefore, $\mathbf{q}_{1,j+1}$ is reachable by the sequence of moves

$$\mathbf{q}_{1,j} \xrightarrow{(+b,+a)} \mathbf{q}_{0,j+1} \xrightarrow{(+a,+b)} \mathbf{q}_{1,j+1}.$$

Case 2: $\mathbf{q}_{1,j}$ is a winning position.
Since $\mathbf{q}_{1,j} = (a + j(a + b), b + j(a + b))$, with a simple calculation, we have $\mathbf{q}_{1,j} \in \{(0, b - a), (a - b, 0)\}$. Hence, $\mathbf{q}_{1,j-1} \in \{(-a - b, -2a), (-2b, -a - b)\}$.

Case 2(*a*): $\mathbf{q}_{1,j-1}$ is not a winning position.
Note that $b - 2a \not\equiv 0 \pmod n$ and $a - 2b \not\equiv 0 \pmod n$ by the given conditions, and $2b - 2a \not\equiv 0 \pmod n$ by (1). Hence, $\mathbf{q}_{2,j-1} \in \{(-b, b - 2a), (a - 2b, -a)\}$, $\mathbf{q}_{3,j-1} \in \{(a - b, 2b - 2a), (2a - 2b, b - a)\}$, and $\mathbf{q}_{2,j} \in \{(a, 2b - a), (2a - b, b)\}$ are not winning positions. Therefore, $\mathbf{q}_{1,j+1}$ is reachable by the sequence of moves

$$\mathbf{q}_{1,j-1} \xrightarrow{(+a,+b)} \mathbf{q}_{2,j-1} \xrightarrow{(+a,+b)} \mathbf{q}_{3,j-1} \xrightarrow{(+b,+a)} \mathbf{q}_{2,j} \xrightarrow{(+b,+a)} \mathbf{q}_{1,j+1}.$$

Case 2(*b*): $2a \equiv 0 \pmod n$ and $\mathbf{q}_{1,j-1} = (-a-b, 0)$.
Note that $j > 1$ since $\mathbf{q}_{1,1-1} = (a, b) \neq (-a-b, 0)$. Also note that $-2a-2b \equiv -2b \not\equiv -2a \equiv 0 \pmod n$ and $-a-2b \equiv a-2b \not\equiv 0 \pmod n$. Therefore, $\mathbf{q}_{1,j+1}$ is reachable by the sequence of moves

$$\mathbf{q}_{1,j-2} = (-2a-2b, -a-b) \xrightarrow{(+a,+b)} (-a-2b, -a) \xrightarrow{(+a,+b)} (-2b, b-a) \xrightarrow{(+a,+b)} (a-2b, 2b-a) \xrightarrow{(+b,+a)} (a-b, 2b) \xrightarrow{(+b,+a)} (a, a+2b) \xrightarrow{(+b,+a)} \mathbf{q}_{1,j+1}.$$

Case 2(*c*): $2b \equiv 0 \pmod n$ and $\mathbf{q}_{1,j-1} = (0, -a-b)$.
Note that $j > 1$ since $\mathbf{q}_{1,1-1} = (a, b) \neq (0, -a-b)$. Also note that $-2a-2b \equiv -2a \not\equiv -2b \equiv 0 \pmod n$ and $-2a-b \equiv -2a+b \not\equiv 0 \pmod n$. Therefore, $\mathbf{q}_{1,j+1}$ is reachable by the sequence of moves

$$\mathbf{q}_{1,j-2} = (-a-b, -2a-2b) \xrightarrow{(+a,+b)} (-b, -2a-b) \xrightarrow{(+a,+b)} (a-b, -2a) \xrightarrow{(+a,+b)} (2a-b, b-2a) \xrightarrow{(+b,+a)} (2a, b-a) \xrightarrow{(+b,+a)} (2a+b, b) \xrightarrow{(+b,+a)} \mathbf{q}_{1,j+1}.$$

Having shown that $\mathbf{q}_{1,j}$ is reachable for all $0 \le j \le n-1$, we will now prove by induction on i that $\mathbf{q}_{i,j}$ is reachable for all $2 \le i \le n-1$ and $0 \le j \le n-1$. Assume that for some $1 \le i \le n-2$, $\mathbf{q}_{i,j}$ is reachable for all $0 \le j \le n-1$. If $\mathbf{q}_{i,j}$ is not a winning position, then $\mathbf{q}_{i+1,j}$ is reachable by the move

$$\mathbf{q}_{i,j} \xrightarrow{(+a,+b)} \mathbf{q}_{i+1,j}.$$

Otherwise, if $\mathbf{q}_{i,j}$ is a winning position, i.e., $\mathbf{q}_{i,j} = (ia + j(a+b), ib + j(a+b)) \in \{(0, i(b-a)), (i(a-b), 0)\}$, then we proceed by considering the following cases.

Case 1: $\mathbf{q}_{i,j-1}$ is not a winning position.

Case 1(*a*): $\mathbf{q}_{i+1,j-1}$ is not a winning position.
By (1), $(i+1)(b-a) \not\equiv 0 \pmod n$. Hence, $\mathbf{q}_{i+2,j-1} \in \{(a-b, (i+1)(b-a)), ((i+1)(a-b), b-a)\}$ is not a winning position. Therefore, $\mathbf{q}_{i,j+1}$ is reachable by the sequence of moves

$$\mathbf{q}_{i,j-1} \xrightarrow{(+a,+b)} \mathbf{q}_{i+1,j-1} \xrightarrow{(+a,+b)} \mathbf{q}_{i+2,j-1} \xrightarrow{(+b,+a)} \mathbf{q}_{i+1,j}.$$

Case 1(*b*): $\mathbf{q}_{i+1,j-1}$ is a winning position.
Since $\mathbf{q}_{i+1,j-1} \in \{(-b, -a+i(b-a)), (-b+i(a-b), -a)\}$, we have $\mathbf{q}_{i+1,j-1} \in \{(-b, 0), (0, -a)\}$. Then $\mathbf{q}_{i,j-2} \in \{(-2a-2b, -a-2b), (-2a-b, -2a-2b)\}$ and $\mathbf{q}_{i+2,j-2} \in \{(-2b, -a), (-b, -2a)\}$.

Case 1(*b*)(*i*): $\mathbf{q}_{i,j-2}$ and $\mathbf{q}_{i+2,j-2}$ are not winning positions.
Note that $\mathbf{q}_{i+1,j-2} \in \{(-a-2b, -a-b), (-a-b, -2a-b)\}$, $\mathbf{q}_{i+3,j-2} \in \{(a-2b, b-a), (a-b, b-2a)\}$, and $\mathbf{q}_{i+2,j-1} \in \{(a-b, b), (a, b-a)\}$ are not winning positions. Therefore, $\mathbf{q}_{i,j+1}$ is reachable by the sequence of moves

$$\mathbf{q}_{i,j-2} \xrightarrow{(+a,+b)} \mathbf{q}_{i+1,j-2} \xrightarrow{(+a,+b)} \mathbf{q}_{i+2,j-2} \xrightarrow{(+a,+b)} \mathbf{q}_{i+3,j-2} \xrightarrow{(+b,+a)} \mathbf{q}_{i+2,j-1} \xrightarrow{(+b,+a)} \mathbf{q}_{i+1,j}.$$

Case 1(*b*)(*ii*): $\mathbf{q}_{i+2,j-2} = (-2b, -a)$ is a winning position, i.e., $2b \equiv 0 \pmod n$ and $\mathbf{q}_{i+2,j-2} = (0, -a)$.
Since $b < n$ and n divides $2b$, we have $n = 2b$, which is an even number. This implies that $n > 3$, thus $-3a-b \equiv -3(a+b) \not\equiv 0 \pmod n$ by (1). Moreover, $-2a-b \equiv -2a+b \not\equiv 0 \pmod n$ by the given conditions, and $-2a \not\equiv 0 \pmod n$ since $a < b = \frac{n}{2}$. Therefore, $\mathbf{q}_{i+1,j}$ is reachable by the sequence of moves

$$\mathbf{q}_{i,j-3} = (-3a-b, -2a-b) \xrightarrow{(+a,+b)} (-2a-b, -2a) \xrightarrow{(+a,+b)} (-a-b, b-2a) \xrightarrow{(+a,+b)} (-b, -2a) \xrightarrow{(+a,+b)} (a-b, b-2a) \xrightarrow{(+b,+a)} (a, b-a) \xrightarrow{(+b,+a)} (a+b, b) \xrightarrow{(+b,+a)} \mathbf{q}_{i+1,j}.$$

Case 1(*b*)(*iii*): $\mathbf{q}_{i+2,j-2} = (-b, -2a)$ is a winning position, i.e., $2a \equiv 0 \pmod n$ and $\mathbf{q}_{i+2,j-2} = (-b, 0)$.
Since $a < n$ and n divides $2a$, we have $n = 2a$, which is an even number. This implies that $n > 3$, thus $-a-3b \equiv -3(a+b) \not\equiv 0 \pmod n$ by (1). Moreover, $-a-2b \equiv a-2b \not\equiv 0 \pmod n$ by the given conditions, and $-2b \not\equiv 0 \pmod n$ since $\frac{n}{2} = a < b < n$. Therefore, $\mathbf{q}_{i+1,j}$ is reachable by the sequence of moves

$$\mathbf{q}_{i,j-3} = (-a-2b, -a-3b) \xrightarrow{(+a,+b)} (-2b, -a-2b) \xrightarrow{(+a,+b)} (a-2b, -a-b)$$

$$\xrightarrow{(+a,+b)} (-2b,-a) \xrightarrow{(+a,+b)} (a-2b, b-a) \xrightarrow{(+b,+a)} (a-b, b)$$
$$\xrightarrow{(+b,+a)} (a, a+b) \xrightarrow{(+b,+a)} \mathbf{q}_{i+1,j}.$$

Case 1(*b*)(*iv*): $\mathbf{q}_{i,j-2} = (-2a-2b, -a-2b)$ is a winning position, i.e., $a+2b \equiv 0 \pmod n$ and $\mathbf{q}_{i,j-2} = (-a, 0)$. Note that $n > 3$; otherwise, $a = 1$ and $b = 2$ by $a < b < n$, which contradicts that $a+2b \equiv 0 \pmod n$. By (1), $-2a-b \equiv -3(a+b) \not\equiv 0 \pmod n$. Therefore, $\mathbf{q}_{i+1,j}$ is reachable by the sequence of moves

$$\mathbf{q}_{i,j-3} = (-2a-b, -a-b) \xrightarrow{(+a,+b)} (-a-b, -a) \xrightarrow{(+a,+b)} (-b, b-a)$$
$$\xrightarrow{(+a,+b)} (a-b, 2b-a) \xrightarrow{(+a,+b)} (2a-b, 3b-a) = (a-3b, b-2a)$$
$$\xrightarrow{(+b,+a)} (a-2b, b-a) \xrightarrow{(+b,+a)} (a-b, b) \xrightarrow{(+b,+a)} \mathbf{q}_{i+1,j}.$$

Case 1(*b*)(*v*): $\mathbf{q}_{i,j-2} = (-2a-b, -2a-2b)$ is a winning position, i.e., $2a+b \equiv 0 \pmod n$ and $\mathbf{q}_{i,j-2} = (0, -b)$. Note that $n > 3$; otherwise, $a = 1$ and $b = 2$ by $a < b < n$, which contradicts that $2a+b \equiv 0 \pmod n$. By (1), $-a-2b \equiv -3(a+b) \not\equiv 0 \pmod n$. Therefore, $\mathbf{q}_{i+1,j}$ is reachable by the sequence of moves

$$\mathbf{q}_{i,j-3} = (-a-b, -a-2b) \xrightarrow{(+a,+b)} (-b, -a-b) \xrightarrow{(+a,+b)} (a-b, -a)$$
$$\xrightarrow{(+a,+b)} (2a-b, b-a) \xrightarrow{(+a,+b)} (3a-b, 2b-a) = (a-2b, b-3a)$$
$$\xrightarrow{(+b,+a)} (a-b, b-2a) \xrightarrow{(+b,+a)} (a, b-a) \xrightarrow{(+b,+a)} \mathbf{q}_{i+1,j}.$$

Case 2: $\mathbf{q}_{i,j-1} = (-a-b, (i-1)b-(i+1)a)$ is a winning position, i.e., $(i-1)b-(i+1)a \equiv 0 \pmod n$ and $\mathbf{q}_{i,j-1} = (-a-b, 0)$. If $2b \equiv 0 \pmod n$, then $\mathbf{q}_{i+1,j-1} = (-b, b) = (b, b) \in \{\mathbf{q}_{0,j'} : 0 \le j' \le n-1\}$. This implies that $i = n-1$, violating the bound given in the induction assumption. Hence, $2b \not\equiv 0 \pmod n$.

Case 2(*a*): $a + 2b \not\equiv 0 \pmod n$.
The position $\mathbf{q}_{i+1,j}$ is reachable by the sequence of moves

$$\mathbf{q}_{i,j-2} = (-2a-2b, -a-b) \xrightarrow{(+a,+b)} (-a-2b, -a) \xrightarrow{(+a,+b)} (-2b, b-a)$$
$$\xrightarrow{(+a,+b)} (a-2b, 2b-a) \xrightarrow{(+b,+a)} (a-b, 2b) \xrightarrow{(+b,+a)} \mathbf{q}_{i+1,j}.$$

Case 2(*b*): $a + 2b \equiv 0 \pmod n$.
Note that $n > 3$; otherwise, $a = 1$ and $b = 2$ by $a < b < n$, which contradicts that $a+2b \equiv 0 \pmod n$. By (1), $-2a-b \equiv -3(a+b) \not\equiv 0 \pmod n$. Moreover, $-2a \equiv -a+2b \not\equiv 0 \pmod n$ by the given conditions. Therefore, $\mathbf{q}_{i+1,j}$ is reachable by the sequence of moves

$$\mathbf{q}_{i,j-3} = (-2a-b, -2a-2b) \xrightarrow{(+a,+b)} (-a-b, -2a-b) \xrightarrow{(+a,+b)} (-b, -2a)$$
$$\xrightarrow{(+a,+b)} (a-b, b-2a) \xrightarrow{(+a,+b)} (2a-b, 2b-2a) \xrightarrow{(+b,+a)} (2a, 2b-a)$$
$$\xrightarrow{(+b,+a)} (2a+b, 2b) \xrightarrow{(+b,+a)} \mathbf{q}_{i+1,j}.$$

Case 3: $\mathbf{q}_{i,j-1} = ((i-1)a-(i+1)b, -a-b)$ is a winning position, i.e., $(i-1)a-(i+1)b \equiv 0 \pmod n$ and $\mathbf{q}_{i,j-1} = (0, -a-b)$. If $2a \equiv 0 \pmod n$, then $\mathbf{q}_{i+1,j-1} = (a, -a) = (a, a) \in \{\mathbf{q}_{0,j'} : 0 \le j' \le n-1\}$. This implies that $i = n-1$, violating the bound given in the induction assumption. Hence, $2a \not\equiv 0 \pmod n$.

Case 3(*a*): $2a + b \not\equiv 0 \pmod n$.
The position $\mathbf{q}_{i+1,j}$ is reachable by the sequence of moves

$$\mathbf{q}_{i,j-2} = (-a-b, -2a-2b) \xrightarrow{(+a,+b)} (-b, -2a-b) \xrightarrow{(+a,+b)} (a-b, -2a)$$
$$\xrightarrow{(+a,+b)} (2a-b, b-2a) \xrightarrow{(+b,+a)} (2a, b-a) \xrightarrow{(+b,+a)} \mathbf{q}_{i+1,j}.$$

Case 3(*b*): $2a + b \equiv 0 \pmod n$.
Note that $n > 3$; otherwise, $a = 1$ and $b = 2$ by $a < b < n$, which contradicts that $2a+b \equiv 0 \pmod n$. By (1), $-a-2b \equiv -3(a+b) \not\equiv 0 \pmod n$. Moreover, $-2b \equiv 2a-b \not\equiv 0 \pmod n$ by the given conditions. Therefore, $\mathbf{q}_{i+1,j}$ is reachable by the sequence of moves

$$\mathbf{q}_{i,j-3} = (-2a-2b, -a-2b) \xrightarrow{(+a,+b)} (-a-2b, -a-b) \xrightarrow{(+a,+b)} (-2b, -a)$$
$$\xrightarrow{(+a,+b)} (a-2b, b-a) \xrightarrow{(+a,+b)} (2a-2b, 2b-a) \xrightarrow{(+b,+a)} (2a-b, 2b)$$
$$\xrightarrow{(+b,+a)} (2a, a+2b) \xrightarrow{(+b,+a)} \mathbf{q}_{i+1,j}. \quad \square$$

3. Proof of Theorem 1.3

Proof. Let $d = \gcd(a, b)$ and $\delta = \gcd(m, n)$, and further let $a = da_0$, $b = db_0$, and $n = \delta n_0$ for some integers a_0, b_0, and n_0. Note that $\gcd(d, \delta) = 1$ since $\gcd(m, n, a, b) = 1$.

Suppose that $m \mid (b^2 - a^2)$ and $m \mid \gcd(a, b)\gcd(m, n)$. Then $m = d\delta/c$ for some $c \mid d$, and $(d\delta/c) \mid d^2(b_0^2 - a_0^2)$ implies that $\delta \mid cd(b_0^2 - a_0^2)$. Since $\gcd(c, \delta) = \gcd(d, \delta) = 1$, we have $\delta \mid (b_0^2 - a_0^2)$. Let $\delta = \delta^+\delta^-$, where $\delta^+ \mid (b_0 + a_0)$ and $\delta^- \mid (b_0 - a_0)$. Then $b_0 = s\delta^- + a_0$ for some integer s. Moreover, $\gcd(a_0, \delta^-) = 1$ since $\gcd(a_0, b_0) = 1$.

We will now show that if $(x_0, 0)$ is a reachable winning position, then $x_0 \equiv 0 \pmod m$. For any reachable position $(ai + bj, aj + bi)$ with $aj + bi \equiv 0 \pmod n$, we have $da_0j + d(s\delta^- + a_0)i = t\delta n_0$ for some integer t. Rearranging the terms, we have $da_0(j + i) = \delta^-(-dsi + t\delta^+n_0)$, so $\delta^- \mid (j + i)$ since $\gcd(da_0, \delta^-) = 1$.

As a result, $\delta^+\delta^- \mid (b_0 + a_0)(j + i)$, so $\delta \mid (a_0i + b_0j + a_0j + b_0i)$. Recalling that $n \mid (aj + bi)$, we have $\delta \mid (a_0j + b_0i)$. Consequently, $\delta \mid (a_0i + b_0j)$, which implies that $d\delta \mid (ai + bj)$. Therefore, $x_0 = ai + bj \equiv 0 \pmod m$, thus proving the sufficient condition for all reachable winning positions being of the form $(0, y)$.

To prove the necessary condition, we assume that all reachable winning positions are of the form $(0, y)$. First, consider the case when $m = a + b$. Then $m \mid (b^2 - a^2)$ trivially. Moreover, $d \mid m$ and $\delta \mid m$, which implies that $d\delta \mid m$ since $\gcd(d, \delta) = 1$. Hence, $m = \ell d\delta$ for some positive integer ℓ, or equivalently, $\delta = (a_0+b_0)/\ell$. Assume by way of contradiction that $\ell > 1$.

Let k be the smallest positive integer such that (ka, kb) is a reachable winning position. Then $ka = \mathrm{lcm}(a, m) = \mathrm{lcm}(a, a + b) = a(a_0 + b_0)$, implying that $k = a_0 + b_0$. Thus $\delta < k$, so the positions $(\delta a, \delta b + um)$ are reachable for all $u \geq 0$ by the following sequence of moves:

$$\underbrace{(0,0) \xrightarrow{(+a,+b)} (a,b) \xrightarrow{(+a,+b)} (2a,2b) \xrightarrow{(+a,+b)} \cdots \xrightarrow{(+a,+b)} (\delta a, \delta b)}_{\delta \text{ times of } (+a,+b)}$$

$$\left.\begin{array}{l} \xrightarrow{(+b,+a)} ((\delta-1)a, (\delta-1)b + m) \xrightarrow{(+a,+b)} (\delta a, \delta b + m) \\ \qquad\vdots \\ \xrightarrow{(+b,+a)} ((\delta-1)a, (\delta-1)b + um) \xrightarrow{(+a,+b)} (\delta a, \delta b + um). \end{array}\right\} \begin{array}{l} u \text{ times of } (+b,+a) \\ \text{and } (+a,+b) \end{array}$$

Since $\delta = \gcd(m, n)$, there exist positive integers u and v such that $\delta b = vn - um$. Hence, $(\delta a, \delta b + um)$ is a reachable winning position of the form $(x, 0)$ where $x \not\equiv 0 \pmod m$, which is a contradiction. Therefore, $\ell = 1$ and $m = \gcd(a, b)\gcd(m, n)$.

It remains to consider the case when $m \neq a+b$. For each positive integer r, let $\mathcal{D}_r = \{\mathbf{p}_{r,i} = (a(r-i)+bi, ai+b(r-i)) : 0 \leq i \leq r\}$. Note that $b - a \not\equiv 0 \pmod m$, so for any positive integer r and $0 \leq i \leq r$, $a(r-i)+bi$ and $a(r-i-1)+b(i+1)$ are not both congruent to 0 modulo m. In other words, $\mathbf{p}_{r,i}$ and $\mathbf{p}_{r,i+1}$ are not both winning positions. As a result, if both $\mathbf{p}_{r,i}$ and $\mathbf{p}_{r,i+1}$ are reachable positions, then at least one of the moves

$$\mathbf{p}_{r,i} \xrightarrow{(+b,+a)} \mathbf{p}_{r+1,i+1} \text{ and } \mathbf{p}_{r,i+1} \xrightarrow{(+a,+b)} \mathbf{p}_{r+1,i+1}$$

is valid, implying that $\mathbf{p}_{r+1,i+1}$ is reachable.

Note that $\mathbf{p}_{1,0}$, $\mathbf{p}_{1,1}$, $\mathbf{p}_{2,0}$, $\mathbf{p}_{2,1}$, and $\mathbf{p}_{2,2}$ are all reachable. Let $r \geq 2$ such that all positions in $\{\mathbf{p}_{r,i} : \sigma \leq i \leq \tau\}$ are reachable for some $0 \leq \sigma < \sigma + 2 \leq \tau \leq r$. Repeatedly applying the previous argument, we see that all positions in

$$\{\mathbf{p}_{r+1,i} : \sigma + 1 \leq i \leq \tau\} \cup \{\mathbf{p}_{r+2,i} : \sigma + 2 \leq i \leq \tau\} \cup \{\mathbf{p}_{r+3,i} : \sigma + 3 \leq i \leq \tau\} \tag{2}$$

are reachable. Furthermore, we claim that $\mathbf{p}_{r+2,\sigma+1}$, $\mathbf{p}_{r+2,\tau+1}$, $\mathbf{p}_{r+3,\sigma+1}$, $\mathbf{p}_{r+3,\sigma+2}$, $\mathbf{p}_{r+3,\tau+1}$, and $\mathbf{p}_{r+3,\tau+2}$ are also reachable, and we provide the proof below.

If $\mathbf{p}_{r,\sigma}$ is a winning position, then $\mathbf{p}_{r,\sigma} = (0, y_0)$ for some integer y_0. Hence, $\mathbf{p}_{r,\sigma+1} = (b - a, y_0 + a - b)$, $\mathbf{p}_{r+1,\sigma+1} = (b, y_0 + a)$, and $\mathbf{p}_{r+2,\sigma+1} = (a + b, y_0 + a + b)$ are all reachable non-winning positions, which further implies that both $\mathbf{p}_{r+3,\sigma+1}$ and $\mathbf{p}_{r+3,\sigma+2}$ are reachable.

On the other hand, if $\mathbf{p}_{r,\sigma}$ is not a winning position, then $\mathbf{p}_{r+1,\sigma}$ is reachable. Now, if $\mathbf{p}_{r+1,\sigma}$ is a winning position, then $\mathbf{p}_{r+1,\sigma} = (0, y_1)$ for some integer y_1. Hence, both $\mathbf{p}_{r+1,\sigma+1} = (b - a, y_1 + a - b)$ and $\mathbf{p}_{r+2,\sigma+1} = (b, y_1 + a)$ are reachable non-winning positions, thus both $\mathbf{p}_{r+3,\sigma+1}$ and $\mathbf{p}_{r+3,\sigma+2}$ are also reachable. Otherwise, if $\mathbf{p}_{r+1,\sigma}$ is not a winning position, then $\mathbf{p}_{r+2,\sigma}$ and $\mathbf{p}_{r+2,\sigma+1}$ are reachable. Recalling from (2) that $\mathbf{p}_{r+2,\sigma+2}$ is also reachable, it follows that both $\mathbf{p}_{r+3,\sigma+1}$ and $\mathbf{p}_{r+3,\sigma+2}$ are also reachable. Similar arguments will show that $\mathbf{p}_{r+2,\tau+1}$, $\mathbf{p}_{r+3,\tau+1}$, and $\mathbf{p}_{r+3,\tau+2}$ are all reachable, thus concluding our proof for the claim.

Since $\mathbf{p}_{r,i} = \mathbf{p}_{r,i'}$ if $i' = i + \mathrm{lcm}(m, n)$, the positions in $\mathcal{D}_r$ are periodic, meaning that as long as $\tau - \sigma \geq \mathrm{lcm}(m, n)$, we have $\{\mathbf{p}_{r,i} : \sigma \leq i \leq \tau\} = \mathcal{D}_r$. From the claim above, we observe that if all positions in $\{\mathbf{p}_{r,i} : \sigma \leq i \leq \tau\}$ are reachable for some $0 \leq \sigma < \sigma + 2 \leq \tau \leq r$, then all positions in

$$\{\mathbf{p}_{r+1,i} : \sigma + 1 \leq i \leq \tau\} \cup \{\mathbf{p}_{r+2,i} : \sigma + 1 \leq i \leq \tau + 1\} \cup \{\mathbf{p}_{r+3,i} : \sigma + 1 \leq i \leq \tau + 2\}$$

are also reachable. Applying the claim repeatedly, we know that all positions in

$$\{\mathbf{p}_{r+3w+1,i} : \sigma + w + 1 \le i \le \tau + 2w\} \cup \{\mathbf{p}_{r+3w+2,i} : \sigma + w + 1 \le i \le \tau + 2w + 1\}$$
$$\cup\{\mathbf{p}_{r+3w+3,i} : \sigma + w + 1 \le i \le \tau + 2w + 2\}$$

are reachable for all positive integers w. Hence, for all $w > \text{lcm}(m, n)$, all positions in $\mathcal{D}_{r+3w+1} \cup \mathcal{D}_{r+3w+2} \cup \mathcal{D}_{r+3w+3}$ are reachable. Moreover, since $\mathcal{D}_r = \mathcal{D}_{r'}$ if $r' = r + \text{lcm}(m, n)$, we conclude that every position $(ai + bj, aj + bi)$ is reachable.

From this, we see that if $i = \text{lcm}(m, n) - a$ and $j = b$, then $(ai + bj, aj + bi) = (b^2 - a^2, 0)$ is reachable. Based on the assumption that all winning positions are of the form $(0, y)$, we have $m \mid (b^2-a^2)$. Similarly, letting $i = n$ and $j = \text{lcm}(m, n)$, we know that both $(ai+bj, aj+bi) = (an, 0)$ and $(aj+bi, ai+bj) = (bn, 0)$ are reachable. Again, since all winning positions are of the form $(0, y)$, we have $m \mid an$ and $m \mid bn$. This implies that $m \mid \gcd(a, b)n$, thus $m \mid \gcd(a, b)\gcd(m, n)$. □

Acknowledgment

These results are based on work supported by the National Science Foundation under grant numbered DMS-1852378.

References

[1] J. Harrington, K. Karhadkar, M. Kohutka, T. Stevens, T.W.H. Wong, Two dependent probabilistic chip-collecting games, Discrete Appl. Math. 288 (2021) 74–86.
[2] H.H. Leung, T. Thanatipanonda, A probabilistic two-pile game, J. Integer Seq. 22 (2019) 19.4.8.
[3] H.H. Leung, T. Thanatipanonda, Game of pure chance with restricted boundary, Discrete Appl. Math. 283 (2020) 613–625.
[4] T.W.H. Wong, J. Xu, A probabilistic take-away game, J. Integer Seq. 21 (2018) 18.6.3.