

MAKING AN H -FREE GRAPH k -COLORABLE

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ABSTRACT. We study the following question: how few edges can we delete from any H -free graph on n vertices in order to make the resulting graph k -colorable? It turns out that various classical problems in extremal graph theory are special cases of this question. For H any fixed odd cycle, we determine the answer up to a constant factor when n is sufficiently large. We also prove an upper bound when H is a fixed clique that we conjecture is tight up to a constant factor, and prove upper bounds for more general families of graphs. We apply our results to get a new bound on the maximum cut of graphs with a forbidden odd cycle in terms of the number of edges.

1. INTRODUCTION

All graphs we consider are finite, undirected and simple, unless otherwise specified. A graph is H -free if it does not contain H as a subgraph. For a collection \mathcal{H} of graphs, a graph is \mathcal{H} -free if it does not contain any graph in \mathcal{H} as a subgraph. The *girth* of a graph is the length of the shortest cycle, and it is infinite if the graph is a forest. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors needed to properly color the vertices of the graph so that no two adjacent vertices receive the same color.

A famous result of Erdős [19] states that there are graphs of arbitrarily large girth and chromatic number. While these graphs are locally sparse, they cannot be properly colored with few colors. We study here a slightly different local-global problem in graphs with a similar flavor: how resilient to being k -colorable can a graph be given a local constraint like a forbidden subgraph?

Precisely, for a graph G and a positive integer k , how few edges, which we denote by $h(G, k)$, can we delete from G in order to make the remaining subgraph k -colorable? For a graph G and positive integers n and k , let $h(n, k, H)$ be the maximum of $h(G, k)$ over all n -vertex graphs G which are H -free, that is, which do not contain H as a subgraph. We define $h(n, k, \mathcal{H})$ analogously for \mathcal{H} a family of forbidden subgraphs. Determining or estimating $h(n, k, H)$ is a very challenging problem. Special cases of this problem include several famous problems in extremal graph theory. For example, the case $k = 1$ is the classical Turán problem on the maximum number of edges an H -free graph on n vertices can have.

A longstanding conjecture of Erdős (he wrote in 1975 [23] that it was already old) would solve the case where H is a triangle and $k = 2$. This conjecture states that every triangle-free graph on n vertices can be made bipartite by deleting at most $n^2/25$ edges. If true, this conjectured bound is the best possible. This can be seen by considering a balanced blow-up of a cycle on five vertices. While there are many papers on this problem, the best known upper bound [28] is a little better than $n^2/18$. Solving another conjecture of Erdős, Sudakov [45] showed that any K_4 -free graph on n vertices can be made bipartite by removing at most $n^2/9$ edges. That is, $h(n, 2, K_4) \leq n^2/9$. This bound is tight, which can be seen by considering a balanced blow-up of a triangle. He deduced as a corollary that, if H is a fixed graph with $\chi(H) = 4$, then $h(n, 2, H) \leq (1 + o(1))n^2/9$. Sudakov further conjectured for $r > 4$ that the balanced complete $(r - 1)$ -partite graph on n vertices is

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the furthest from being bipartite over all K_r -free graphs, which would determine $h(n, 2, K_r)$ for $r > 3$. It was recently announced [38] that Hu, Lidický, Martins-Lopez, Norin, and Volec verified the case $r = 6$ of Sudakov’s conjecture and further proved the corresponding upper bound for all H of chromatic number 6.

The other extreme is determining the minimum k for which $h(n, k, H)$ is zero. This is the same as determining the maximum possible chromatic number that an H -free graph on n vertices can have. This problem is very old (see [19]), and closely related to estimating the Ramsey number $r(H, K_s)$. If $r(H, K_s) > n$ and $k \leq n/s$, then there is an H -free graph G on n vertices which does not contain an independent set of order s . In particular, in any k -coloring of the vertices, one of the color classes has at least $n/k \geq s$ vertices and must contain an edge, implying $h(n, k, H) > 0$. On the other hand, if H is connected, $k \geq (n/s) \cdot \log_2(2n)$, and $n > r(H, K_s)$, then by greedily picking out largest independent sets, one can properly k -color any H -free graph on n vertices. This bound on the number of colors can be deduced from the fact that the minimum possible independence number of an H -free graph on n vertices is a monotonically increasing and subadditive function of n .

In this paper, we will be primarily interested in the intermediate case, when $|H| \ll k \ll n$. Throughout the article, we use the notation Θ_x , Ω_x , O_x , and o_x to indicate that the implicit constant factors may depend on x .

Note that for any graph G on n vertices we have the simple bound $h(G, k) \leq h(K_n, k) \leq \binom{n}{2}/k \leq n^2/(2k)$ by considering a random k -partition of $V(G)$. Thus, we will be primarily interested in understanding how much of an improvement we can give over this straightforward bound for H -free graphs.

The following theorem gives an upper bound on $h(n, k, H)$ when H is a clique on r vertices.

Theorem 1.1. *For each integer $r \geq 3$ there is c_r such that for all positive integers n, k we have*

$$h(n, k, K_r) \leq c_r \frac{n^2}{k^{(r-1)/(r-2)}}.$$

We conjecture that Theorem 1.1 is sharp up to the constant factor c_r for n sufficiently large in terms of k , and prove this for $r = 3$.

For fixed odd cycles and n sufficiently large in k , we determine this function up to a constant factor. We first show an upper bound on $h(G, k)$ for graphs G of large odd girth.

Theorem 1.2. *For each positive integer r there is c_r such that the following holds. Let $\mathcal{H}_r = \{C_3, C_5, \dots, C_{2r+1}\}$ be the set of odd cycles of length at most $2r + 1$. Then,*

$$h(n, k, \mathcal{H}_r) < c_r \frac{n^2}{k^{r+1}}.$$

From this result on graphs of large odd girth, we can deduce a similar result for graphs with a single forbidden odd cycle.

Theorem 1.3. *For positive integers $n \geq k \geq 1$, and $r \geq 1$, we have $h(n, k, C_{2r+1}) = h(n, k, \mathcal{H}_r) + O_r(n^{3/2})$, where \mathcal{H}_r is the family of odd cycles of length at most $2r + 1$. In particular,*

$$h(n, k, C_{2r+1}) = O_r \left(\frac{n^2}{k^{r+1}} \right).$$

In the other direction, we prove the following lower bound which shows that Theorem 1.3 is tight up to the constant factor in r for n sufficiently large in terms of k . The proof leverages a construction of Alon and Kahale [8] of a family of pseudorandom graphs of large odd girth.

Theorem 1.4. *For each positive integer r there is $\alpha_r > 0$ such that the following holds. For each positive integer k , for each sufficiently large positive integer n , there is a graph G on n vertices with odd girth larger than $2r + 1$ and with $h(G, k) \geq \alpha_r n^2/k^{r+1}$.*

In the case H is a triangle, we determine up to a constant factor how large n has to be for the above to hold using a result about a semi-random variant of the triangle-free process [10, 33].

The *wheel* W_ℓ is the graph on $\ell + 1$ vertices consisting of an ℓ -cycle and an additional vertex adjacent to all of the vertices of the ℓ -cycle. For even wheels (when ℓ is even), we prove that $h(n, k, W_\ell)$ is asymptotically the same as $h(n, k, K_3)$. Combining the methods used in the proofs of Theorems 1.1 and 1.2, we prove the following upper bound for odd wheels, which we conjecture is tight up to the constant factor which depends on the length of the wheel.

Theorem 1.5. *For each positive integer r there is c_r such that if $n \geq k \geq 2$, then*

$$h(n, k, W_{2r+1}) \leq c_r \frac{n^2}{k^{2-1/(r+1)}}.$$

More broadly, we use the graph removal lemma in Section 4.1 to prove the following result, which shows that if H has a subgraph H' for which H has a homomorphism to H' , then $h(n, k, H)$ and $h(n, k, H')$ are close.

Theorem 1.6. *If a graph H has a subgraph H' such that there exists a homomorphism from H to H' , then*

$$h(n, k, H') \leq h(n, k, H) \leq h(n, k, H') + o(n^2).$$

The maximum cut of a graph G , denoted by $\text{Max-Cut}(G)$, is the maximum number of a bipartite subgraph. This well-studied graph parameter is the main focus of this paper through the identity $\text{Max-Cut}(G) = e(G) - h(G, 2)$, where $e(G)$ is the number of edges of G . It is a simple exercise to show that every graph G with m edges has $\text{Max-Cut}(G) \geq m/2$. Edwards [17, 18] proved that this bound can be improved to

$$\text{Max-Cut}(G) \geq \frac{m}{2} + \frac{-1 + \sqrt{8m+1}}{8},$$

which is sharp if $m = \binom{k}{2}$ for some positive integer k , as shown by taking $G = K_k$. Further results for intermediate values of m were established in [3, 7, 11].

There has been a lot of research on improving the lower order term in the Edwards bound for graphs with a fixed forbidden subgraph. Alon, Krivelevich, and Sudakov [4] showed that for H fixed and m significantly large, every H -free graph G with m edges satisfies $\text{Max-Cut}(G) \geq \frac{m}{2} + m^{1/2+\epsilon}$ for some $\epsilon = \epsilon(H) > 0$, and they conjectured that $1/2$ in the exponent can be replaced by $3/4$.

A case of particular interest is when H is a cycle. Solving a problem of Erdős, Alon [3] proved that every triangle-free graph G with m edges satisfies $\text{Max-Cut}(G) \geq \frac{m}{2} + cm^{4/5}$ for some positive constant c , and this is tight up to the constant factor c . More generally, Alon et al. [9] conjectured that for all $k \geq 3$, every C_k -free graph G with m edges satisfies $\text{Max-Cut}(G) \geq \frac{m}{2} + \Omega_k(m^{(k+1)/(k+2)})$. They verified their conjecture for k even, and showed that the conjectured bound is tight for $k \in \{4, 6, 10\}$.¹ Alon, Bollobás, Krivelevich, and Sudakov [4] observed that for odd k , a well-known construction of Alon [2, 7] gives a pseudorandom graph G with odd-girth greater than k , m edges, and $\text{Max-Cut}(G) = \frac{m}{2} + O_k(m^{(k+1)/(k+2)})$. This construction shows that if the Alon-Krivelevich-Sudakov conjecture is true, then the bound it gives is best possible for odd k . Recently, Zeng and Hou [47] proved that for fixed odd k , every C_k -free graph G with m edges satisfies $\text{Max-Cut}(G) \geq \frac{m}{2} + m^{(k+1)/(k+3)+o(1)}$. Using Theorem 1.3 and some additional tools, we prove the following result giving an improved bound.

Theorem 1.7. *If $k \geq 3$ is odd and $G = (V, E)$ is a C_k -free graph with m edges, then*

$$\text{Max-Cut}(G) \geq \frac{m}{2} + \Omega_k(m^{(k+5)/(k+7)}).$$

¹While [47] attributes this conjecture to [9], what is actually conjectured in [9] is that for even k the conjectured bound is tight.

Organization. We begin in Section 2.1 by developing tools that allow us to conclude some incidental results such as Proposition 2.4, a strengthening of Mantel’s theorem, and also give a foundation to prove Theorem 1.1 and Theorem 1.5. In Section 2.2, we give an upper bound on $h(n, k, K_r)$, the number of edges that must be removed from an arbitrary K_r -free graph on n vertices to guarantee the resulting subgraph is k -partite.

Subsequently in Section 3, we give upper bounds on $h(n, k, \mathcal{H})$ for $\mathcal{H} = \{C_3, \dots, C_{2r+1}\}$ and use this bound on graphs of large odd girth to obtain an upper bound on $h(n, k, C_{2r+1})$. Using a generalization of Alon’s construction of a family of pseudorandom graphs of large odd girth, we show that our bound is tight up to a constant factor depending on r . In Section 4, we leverage the above bounds to obtain associated bounds on $h(n, k, H)$ for more other forbidden subgraphs H .

We apply our results to the problem of bounding the Max- k -Cut of a graph, the size of the largest k -partite subgraph of a graph, noticing that $h(G, k) = e(G) - \text{Max-}k\text{-Cut}(G)$. We first give some simple lemmas to translate between bounds on maximum k -cuts and maximum l -cuts for $l < k$ in Section 5. This enables us in Section 6 to prove Theorem 1.7 giving a new lower bound on $\text{Max-Cut}(G)$ for graphs with a forbidden odd cycle. Finally, we conclude in Section 7 with some unresolved open questions.

2. CUTTING GRAPHS USING NEIGHBORHOODS

For a graph G and vertex subset $U \subset V(G)$, let $G[U]$ denote the induced subgraph of G with vertex set U . We let $e(G)$ denote the number of edges of G , and $e(U) = e(G[U])$ denote the number of edges with both vertices in U . For a vertex v of G , the neighborhood $N(v)$ is the set of vertices of G adjacent to v . The degree of v , which is $|N(v)|$, is denoted $d(v)$.

In this section, we study the following extremal problem in graph theory.

Question 2.1. Given a graph G , how many edges of G can we cover by the union of k neighborhoods of vertices of G ?

In understanding Question 2.1, we will build up a series of tools that will be useful in our subsequent analysis of $h(n, k, H)$ when H is a clique or an odd wheel. The methods we describe below are also of independent interest. In Section 2.1 we include a few applications of this analysis beyond our study of how far graphs are from k -colorable.

2.1. Covering edges with the union of neighborhoods. We tackle Question 2.1, denoting the relevant value $u(G, k)$, which is defined formally below.

Definition 2.2. For a graph G and positive integer k , let $u(G, k)$ be the maximum of $e\left(\bigcup_{i=1}^k N(v_i)\right)$ over all choices of vertices v_1, \dots, v_k of G .

We would like to understand how few edges can we leave uncovered by the union of k neighborhoods of vertices of a graph on n vertices.

Definition 2.3. Let $m(n, k)$ be the minimum of $e(G) - u(G, k)$ over all graphs G on n vertices. That is, $m(n, k)$ is the minimum r such that, for every graph G on n vertices, there are k vertices v_1, \dots, v_k such that at most r edges are not contained in the induced subgraph $G[N(v_1) \cup \dots \cup N(v_k)]$ whose vertex set is the union of the neighborhoods of v_1, \dots, v_k .

As an aside, we first observe that $m(n, 1) = \lfloor \frac{n^2}{4} \rfloor$. This is a strengthening of Mantel’s theorem, that every triangle-free graph on n vertices has at most $\lfloor \frac{n^2}{4} \rfloor$ edges, as it is easy to see that $u(G, 1) = 0$ if and only if G is triangle-free.

Proposition 2.4. *Every graph G on n vertices has a vertex whose neighborhood contains all but at most $\lfloor \frac{n^2}{4} \rfloor$ edges of G , and this bound is sharp. That is, $m(n, 1) = \lfloor \frac{n^2}{4} \rfloor$.*

Proof. The balanced complete bipartite graph on n vertices realizes $m(n, 1) \geq \lfloor \frac{n^2}{4} \rfloor$.

If $m(n, 1) > \lfloor \frac{n^2}{4} \rfloor$, then it would be realized by a graph G on n vertices and $m = \frac{n^2}{4} + t$ edges with t positive. A result of Moon and Moser (c.f. [40]) states that any graph on n vertices and m edges has at least

$$\frac{m(4m - n^2)}{3n} = \left(\frac{n^2}{4} + t \right) \frac{4t}{3n} = \frac{tn}{3} + \frac{4t^2}{3n}$$

triangles. Hence, a random vertex of G is in expectation at least

$$\frac{3}{n} \cdot \left(\frac{tn}{3} + \frac{4t^2}{3n} \right) > t$$

triangles, and hence there is a vertex v of G where $e(N(v)) > t$. The number of edges of G not in the neighborhood of v is an integer which is less than $m - t = \frac{n^2}{4}$, and hence $m(n, 1) \leq \lfloor \frac{n^2}{4} \rfloor$. ■

The following result yields a lower bound on $u(G, k)$ by considering a random choice of k vertices.

Lemma 2.5. *If $G = (V, E)$ is a graph on n vertices and k is a positive integer, then*

$$(2.1) \quad e(G) - u(G, k) \leq \sum_{u \in V} d(u) \left(1 - \frac{d(u)}{n} \right)^k - \sum_{\{u, w\} \in E} \left(1 - \frac{|N(u) \cup N(w)|}{n} \right)^k.$$

Proof. Pick k vertices $v_1, \dots, v_k \in V$ uniformly at random with repetition. Let $U = \bigcup_{i=1}^k N(v_i)$. An edge (u, w) of G is not in $G[U]$ if and only if v_1, \dots, v_k are in $V \setminus N(u)$ or $V \setminus N(w)$. By the inclusion-exclusion principle, the probability that not both u and w are in U is

$$(2.2) \quad \left(1 - \frac{d(u)}{n} \right)^k + \left(1 - \frac{d(w)}{n} \right)^k - \left(1 - \frac{|N(u) \cup N(w)|}{n} \right)^k.$$

Splitting up the sum and then summing the first two terms over vertices of G , we find that the expected value of $e(G) - e(U)$ is at most the right hand side of (2.1). Hence, there is a choice of v_1, \dots, v_k such that $e(G) - e(U)$ (and hence $e(G) - u(G, k)$) is at most the right hand side of (2.1). ■

We have no idea what the exact or asymptotic value of $m(n, k)$ is for any fixed $k \geq 2$. We will prove in general (using Lemma 2.5) that $m(n, k) \leq \frac{n^2}{ek}$, which, for k sufficiently large and n sufficiently large in terms of k , is within 20% of the lower bound that comes from considering an appropriate Erdős-Renyi random graph $G(n, p)$ with $p = c/k$. To see this, pick $c > 0$ to maximize $ce^{-c} - \frac{c}{2}e^{-2c}$. Note that a simple union bound shows that almost surely all of the linear-sized induced subgraphs of $G(n, p)$ have edge density $(1 + o(1))p$. This implies that the union of the neighborhood of any k vertices has size $(1 + o(1))(1 - (1 - p)^k)n$ and the induced subgraph will have edge density $(1 + o(1))p$. Thus

$$m(n, k) \geq (1 + o_k(1)) \left(ce^{-c} - \frac{c}{2}e^{-2c} \right) \frac{n^2}{k},$$

for n sufficiently large as a function of k , and with the $o_k(1)$ term tending to 0 as $k \rightarrow \infty$.

Corollary 2.6. *We have $m(n, k) \leq \frac{n^2}{ek}$. That is, for every graph G on n vertices, there are k vertices of G such that the induced subgraph on the union of the neighborhoods of these k vertices contains all but at most $\frac{n^2}{ek}$ edges of G .*

Proof. By Lemma 2.5, for any graph G on n vertices, we have $e(G) - u(G, k)$ is at most

$$(2.3) \quad \sum_{u \in V} d(u) \left(1 - \frac{d(u)}{n} \right)^k,$$

since the second sum in (2.1) is non-negative. The function $f(x) = x(1 - \frac{x}{n})^k$ has derivative $f'(x) = (1 - \frac{(k+1)x}{n})(1 - \frac{x}{n})^{k-1}$, which is non-negative for $x \leq \frac{n}{k+1}$ and is non-positive if $\frac{n}{k+1} < x \leq n$. Thus, $f(x)$ for $x \leq n$ is maximized at $x = \frac{n}{k+1}$. Hence, for $x \leq n$ we have

$$f(x) \leq f\left(\frac{n}{k+1}\right) = \frac{1}{k+1} \left(1 - \frac{1}{k+1}\right)^k n = \frac{1}{k} \left(1 - \frac{1}{k+1}\right)^{k+1} n \leq \frac{n}{ek}.$$

Applying this with $x = d(u)$ in (2.3) gives the desired inequality. \blacksquare

Lemma 2.7. *If a graph $G = (V, E)$ has disjoint vertex subsets U_1, \dots, U_r , then there is a partition of $V = W_1 \sqcup \dots \sqcup W_r$ with $U_i \subset W_i$ for $1 \leq i \leq r$ and the number $\sum_{i=1}^r e(W_i) - e(U_i)$ of edges that are in a $G[W_i]$ but not $G[U_i]$ is at most $(e(G) - e(U_1 \cup \dots \cup U_r)) / r$.*

Proof. Let $X = U_1 \cup \dots \cup U_r$. For each vertex $v \in V \setminus X$, randomly add v to one of the r sets U_i . This gives a random partition of V into r sets. Let W_i denote the part which is a superset of U_i . Each edge with not both of its vertices in X has a probability $1/r$ that both of its vertices end up in the same part of the random partition. Hence, by linearity of expectation, the expected number of edges of G with not both its vertices in X that end up in the same part of the random partition is $(e(G) - e(X))/r$. So there is such a partition with at most $(e(G) - e(X))/r$ edges with not both its vertices in X and which lie in the same part. \blacksquare

We remark that the above probabilistic proof of Lemma 2.7 can be made deterministic by greedily assigning the vertices to the part that it has the fewest edges to.

Lemma 2.8. *If $G = (V, E)$ is a graph with disjoint vertex subsets V_1, \dots, V_t and s is a positive integer, then*

$$h(G, st) \leq \frac{1}{st} (e(G) - e(V_1 \cup \dots \cup V_t)) + \sum_{i=1}^t h(G[V_i], s).$$

Proof. For each $1 \leq i \leq t$, there is a partition $V_i = U_{i1} \sqcup \dots \sqcup U_{is}$ so that we can make each of the s vertex subsets U_{ij} independent sets by removing at most $h(G[V_i], s)$ total edges. By Lemma 2.7, we can grow the st disjoint subsets $\{U_{ij}\}$ for $1 \leq i \leq t$ and $1 \leq j \leq s$ into a partition of V with st parts which adds at most $\frac{1}{st}(e(G) - e(V_1 \cup \dots \cup V_t))$ edges that are internal to the parts. Deleting these additional edges, we obtain a vertex partition of G into st parts from which we deleted at most $\frac{1}{st}(e(G) - e(V_1 \cup \dots \cup V_t)) + \sum_{i=1}^t h(G[V_i], s)$ edges in order to make it st -partite. \blacksquare

We primarily focus on the case where we take disjoint vertex subsets V_1, \dots, V_k such that each V_i is contained in the neighborhood of some vertex v_i . This yields a bound on $e(G) - e(V_1 \cup \dots \cup V_k)$ and an associated bound on $h(G, k)$.

Corollary 2.9. *If $G = (V, E)$ is a graph on n vertices and k is a positive integer, there are disjoint vertex subsets V_1, \dots, V_k such that each $V_i \subset N(v_i)$ for some vertex $v_i \in V$ and the number of edges of G not in $G[V_1 \cup \dots \cup V_k]$ is at most $\frac{n^2}{ek}$.*

Proof. Apply Lemma 2.6 to obtain vertices $v_1, \dots, v_k \in V$ such that for $U = \bigcup_{i=1}^k N(v_i)$, $e(G) - e(U) \leq n^2/(ek)$. Define $V_1 = N(v_1)$ and, for $i \geq 2$, define $V_i = N(v_i) \setminus \bigcup_{j < i} N(v_j)$, so $V_i \subset N(v_i)$. The sets V_1, \dots, V_k are disjoint and satisfy $V_1 \cup \dots \cup V_k = U$, so the corollary clearly follows. \blacksquare

Via Corollary 2.9 and Lemma 2.8, we have the following immediate corollary.

Corollary 2.10. *If $G = (V, E)$ is a graph on n vertices and s and t are positive integers, then there are disjoint vertex subsets V_1, \dots, V_t such that each V_i is contained in the neighborhood of some vertex v_i and*

$$h(G, st) \leq \frac{n^2}{es^2t^2} + \sum_{i=1}^t h(G[V_i], s).$$

2.2. K_r -Free Graphs. The above results are helpful tools to bound $h(n, k, H)$ for a variety of families of forbidden subgraphs H . Here, we consider the case $H = K_r$ with $r \geq 3$. As a first application, Corollary 2.10 immediately enables us to give an upper bound on $h(n, k, K_3)$.

Proposition 2.11. *Any triangle-free graph on n vertices can be made k -partite for $k \leq n$ by deleting at most n^2/ek^2 edges, so $h(n, k, K_3) \leq \frac{n^2}{ek^2}$.*

Proof. Let G be a triangle-free graph on n vertices. Applying Corollary 2.10 with $t = k$ and $s = 1$ implies that we can find vertices v_1, \dots, v_k and disjoint vertex subsets V_1, \dots, V_k with $V_i \subset N(v_i)$ such that

$$h(G, k) \leq \frac{n^2}{ek^2} + \sum_{i=1}^k h(G[V_i], 1).$$

Since G is triangle-free, $N(v_i)$ is an independent set and thus so is V_i , which implies that $h(G[V_i], 1) = 0$. Therefore, $h(G, k) \leq \frac{n^2}{ek^2}$. Since this holds for all triangle-free graphs G on n vertices, we obtain the desired bound on $h(n, k, K_3)$. \blacksquare

It is helpful in further applications if the V_i are not much larger than their average size. This can be obtained by furthering partitioning the large sets V_i obtained in Corollary 2.9.

Corollary 2.12. *If $G = (V, E)$ is a graph on n vertices and $t \leq n$ is a positive integer, then there are disjoint vertex subsets U_1, \dots, U_{2t} such that each U_j satisfies $|U_j| \leq \frac{n}{t}$ and is contained in the neighborhood of some vertex u_j , and $e(G) - e(U_1 \cup \dots \cup U_{2t}) \leq \frac{n^2}{et}$.*

Proof. By Corollary 2.9, there are vertex subsets V_1, \dots, V_t , each a subset of a vertex neighborhood, where $V_1 \cup \dots \cup V_t$ contains all but at most n^2/et edges of G . Arbitrarily partition each V_i into sets U_j of size $\lfloor n/t \rfloor$, including if needed one set of size less than $\lfloor n/t \rfloor$. Thus, we obtain a sets U_j of size $\lfloor n/t \rfloor$ and b sets U_j of size strictly smaller than $\lfloor n/t \rfloor$, where $b \leq t$. If $a < t$, then $a + b < 2t$. Otherwise, $a \geq t$, and t of these sets of size $\lfloor n/t \rfloor$ together have $t \lfloor n/t \rfloor > n - t$ elements, so less than t elements are not in these t sets. The remaining $a + b - t$ sets each have at least one element, so $a + b - t < t$ or equivalently $a + b < 2t$. We can add additional empty sets to make $2t$ total sets U_j , each of size at most n/t , and with $U_j \subset V_i \subset N(v_i)$ for some i . \blacksquare

For a graph H and vertex v , let H_v denote the induced subgraph of H formed by deleting v . We prove the following recursive upper bound on $h(n, k, H)$.

Lemma 2.13. *If s, t, n are positive integers, H is a graph, and v is a vertex of H so that H_v has no isolated vertices, then*

$$h(n, 2st, H) \leq 2t \cdot h(n/t, s, H_v) + \frac{n^2}{2est^2}.$$

Proof. Let G be an H -free graph on n vertices. By Corollary 2.12, there are $2t$ disjoint vertex subsets U_1, \dots, U_{2t} such that each U_i satisfies $|U_i| \leq \frac{n}{t}$ and is contained in the neighborhood of some vertex u_i . Further, we can pick the U_i so that the number of edges of G not in $G[U_1 \cup \dots \cup U_{2t}]$ is at most $\frac{n^2}{et}$. As G is H -free, then for all $u_i \in V(G)$, the induced subgraph $G[N(u_i)]$ is H_v -free for any vertex $v \in H$. Thus, for each i , $G[U_i]$ can be made s -partite by removing at most $h(|U_i|, s, H_v) \leq h(n/t, s, H_v)$ edges in $G[U_i]$.

For $1 \leq i \leq 2t$, we label the s independent sets (after removing edges as above) which partition U_i as W_{i1}, \dots, W_{is} . By Lemma 2.7, we can then grow $\{W_{ij}\}_{1 \leq i \leq 2t, 1 \leq j \leq s}$ to a partition of $V(G)$ by adding at most $\frac{1}{2st} (e(G) - e(U_1 \cup \dots \cup U_{2t})) \leq \frac{1}{2st} \cdot \frac{n^2}{et} = \frac{n^2}{2est^2}$ edges internal to the parts. Deleting these edges, we obtain the upper bound

$$h(n, 2st, H) \leq 2t \cdot h(n/t, s, H_v) + \frac{n^2}{2est^2}.$$

Applying Lemma 2.13 and induction on r , we can bound the number of edges to remove from a K_r -free graph G on n vertices so that the resulting subgraph is k -colorable. We first establish a bound on $h(n, k, K_r)$ when k is a perfect $(r-2)$ nd power of an even integer. ■

Lemma 2.14. *For positive integers n, k , and $r \geq 3$ so that $k = t^{r-2}$ for t even, we have*

$$h(n, k, K_r) \leq \alpha_r \cdot \frac{n^2}{k^{(r-1)/(r-2)}},$$

where $\alpha_r = \frac{5 \cdot 4^{r-3} - 2}{3e}$.

Proof. The proof is by induction on r . We proved the base case $r = 3$ in Proposition 2.11. Let $s = k/t = t^{r-3}$. By the inductive hypothesis, we know that for all positive integers n_0 ,

$$h(n_0, s, K_{r-1}) \leq \alpha_{r-1} \cdot \frac{n_0^2}{s^{(r-2)/(r-3)}}.$$

Note that the induced subgraph formed by deleting a vertex of K_r is K_{r-1} . By Lemma 2.13 (with parameter $t/2$ instead of t) and the above inequality (with $n_0 = 2n/t$), we have

$$h(n, k, K_r) \leq \frac{2n^2}{est^2} + t \cdot h\left(\frac{2n}{t}, s, K_{r-1}\right) \leq \frac{2n^2}{est^2} + t\alpha_{r-1} \frac{(2n/t)^2}{s^{(r-2)/(r-3)}} = \alpha_r \cdot \frac{n^2}{k^{(r-1)/(r-2)}},$$

where the equality uses $\alpha_r = \frac{2}{e} + 4\alpha_{r-1}$. This completes the proof. ■

Lemma 2.14 establishes Theorem 1.1 when k is a perfect $(r-2)$ nd power of an even integer. The following result is Theorem 1.1 with an explicit constant factor.

Theorem 2.15. *For positive integers n, k , and $r \geq 3$, we have*

$$h(n, k, K_r) \leq \frac{5}{3} \cdot 4^{r-3} \cdot \frac{n^2}{k^{(r-1)/(r-2)}}.$$

Proof. Proposition 2.11 handles the case $r = 3$, so we may assume $r \geq 4$. Recall that any graph on n vertices can be made k -partite by removing at most $n^2/(2k)$ edges. Thus, if $k \leq (2r)^{r-2}$, as $4^{r-3} \geq r \geq \frac{1}{2}k^{1/(r-2)}$, we have the desired inequality. We therefore suppose that $k > (2r)^{r-2}$.

Let ℓ be the largest even perfect $(r-2)$ -power which is at most k , so $\ell = \left(2 \left\lfloor \frac{k^{1/(r-2)}}{2} \right\rfloor\right)^{r-2}$. By monotonicity and $\ell \leq k$, we have $h(n, k, K_r) \leq h(n, \ell, K_r)$. Applying Lemma 2.14,

$$h(n, \ell, K_r) \leq \alpha_r \frac{n^2}{\ell^{(r-1)/(r-2)}}, \quad \text{where } \alpha_r = \frac{5 \cdot 4^{r-3} - 2}{3e}.$$

Since $k > (2r)^{r-2}$, it follows that $k \leq \left(\frac{2(r+1)}{2r}\right)^{r-2} \ell = \left(1 + \frac{1}{r}\right)^{r-2} \ell$. It follows that

$$k^{(r-1)/(r-2)} \leq \left(1 + \frac{1}{r}\right)^{r-1} \ell^{(r-1)/(r-2)} \leq e \ell^{(r-1)/(r-2)}.$$

Substituting, we obtain

$$h(n, k, K_r) \leq \frac{5}{3} \cdot 4^{r-3} \cdot \frac{n^2}{k^{(r-1)/(r-2)}}. \quad \blacksquare$$

3. ODD CYCLE-FREE GRAPHS

In this section we study how few edges we can remove from any graph on n vertices with a fixed forbidden odd cycle to make it k -colorable.

3.1. Upper bounds for odd cycles. We begin by tackling a simpler problem, bounding how far a graph of large odd girth is from being k -colorable. That is, we first consider $h(n, k, \mathcal{H})$ where \mathcal{H} is the family of odd cycles of length at most $2r + 1$. We later show that this number is close to $h(n, k, C_{2r+1})$.

We prove that graphs of large odd girth contain an independent set B with relatively many edges incident to B . We repeatedly apply this to pull out k disjoint independent sets B_1, \dots, B_k such that the remaining induced subgraph contains relatively few edges. By Lemma 2.7, we can grow these k independent sets into a k -partition of the vertex set so that few edges are internal to the parts.

The following definitions will be helpful.

Definition 3.1. Given a graph $G = (V, E)$ and a vertex $v \in V$, the i th neighborhood of v , denoted by $N_i(v)$, is the set of vertices in V of distance exactly i from v .

For example, $N_0(v) = \{v\}$, $N_1(v) = N(v)$, and $N_2(v)$ is the set of vertices in $V \setminus (\{v\} \cup N(v))$ that have a neighbor in $N(v)$. For a vertex subset T , let $N(T)$ denote the set of vertices in $V \setminus T$ adjacent to at least one vertex in T . For a graph G and vertex subsets S and T , let $e(S, T)$ be the number of pairs in $S \times T$ that are edges of G .

Definition 3.2. For a graph $G = (V, E)$ and $S \subset V$, let $D(S) = e(S, V)$ be the sum of the degrees of vertices in S .

Note that $D(S)$ counts the edges contained in S twice and the edges with exactly one endpoint in S once. It is a useful measure of the number of edges that contain a vertex in S .

We first show that for any graph $G = (V, E)$ of large odd girth and any subset $S \subset V$, there is an independent set $B \subset S$ with poor edge expansion into S . Removing such B and its neighborhood and iteratively applying the argument will give an independent set A (the union of the B 's) with comparatively large $D(A)$. In the following lemmas, we take a graph $G = (V, E)$ on n vertices, r a positive integer, and a fixed $S \subset V$. We let

$$x := \left(\frac{|S|n}{D(S)} \right)^{1/r}.$$

Lemma 3.3. *Let $G = (V, E)$ be a graph of odd girth larger than $2r + 1$ and $S \subset V$. There exists an independent set $B \subset S$ such that*

$$D(B) \geq \frac{D(N(B) \cap S)}{x + 1}.$$

Proof. Pick $v \in V, u \in S$ uniformly at random, so

$$\text{Prob}(u \in N_1(v)) = \frac{\mathbb{E}_{u \in S}[d(u)]}{n}.$$

If $u \in N_1(v) \cap S$, then it contributes $d(u)$ to $D(N_1(v) \cap S)$. Therefore,

$$\mathbb{E}_v[D(N_1(v) \cap S)] = \mathbb{E}_v \left[\sum_{u \in S} d(u) \text{Prob}(u \in N_1(v)) \right] = \sum_{u \in S} \mathbb{E}_v \left[d(u) \cdot \frac{d(u)}{n} \right] = \sum_{u \in S} \frac{d(u)^2}{n}.$$

Hence, by picking $v \in V$ such that $D(N_1(v) \cap S)$ is maximized,

$$D(N_1(v) \cap S) \geq \sum_{u \in S} \frac{d(u)^2}{n} \stackrel{(*)}{\geq} \frac{(\sum_{u \in S} d(u))^2}{n|S|} = \frac{D(S)^2}{n|S|},$$

where $(*)$ follows by the Cauchy-Schwarz inequality. Let $N_i = N_i(v) \cap S$. Note N_1, \dots, N_r are all independent sets. Indeed, if some N_i for $1 \leq i \leq r$ contained an edge $e = (v_1, v_2)$, then $v \rightarrow \dots \rightarrow v_1 \xrightarrow{e} v_2 \rightarrow \dots \rightarrow v$ is an odd walk of length $2i + 1$ and thus contains an odd cycle of length at most $2r + 1$. This contradicts G having odd girth larger than $2r + 1$.

We next study the growth of $D(N_i)$. If $D(N_2) < xD(N_1)$, then

$$D(N(N_1) \cap S) \leq |N_1| + D(N_2) < |N_1| + xD(N_1) \leq (x+1)D(N_1),$$

which implies that

$$D(N_1) > \frac{D(N(N_1) \cap S)}{x+1},$$

so we can take $B = N_1$ to satisfy the lemma. Else, for $i = 2, \dots, r-1$, if $D(N_j) \geq xD(N_{j-1})$, for all $2 \leq j < i$, and $D(N_i) < xD(N_{i-1})$, we similarly find that

$$D(N(N_i) \cap S) \leq D(N_{i-1}) + D(N_{i+1}) < \frac{D(N_i)}{x} + xD(N_i),$$

and hence

$$D(N_i) > \frac{D(N(N_i) \cap S)}{x+1/x} \geq \frac{D(N(N_i) \cap S)}{x+1},$$

so we can take $B = N_i$ to satisfy the lemma. If none of N_1, \dots, N_{r-1} satisfy the conditions of the lemma statement as a subset $B \subset S$, then

$$D(N_r) \geq x^{r-1} \frac{D(S)^2}{|S|n} = \frac{D(S)}{x}.$$

This implies that we can pick $B = N_r = N_r(v) \cap S$ to satisfy the lemma. ■

We can use this lemma to establish a more helpful result in the same direction.

Lemma 3.4. *Let $G = (V, E)$ be a graph of odd girth larger than $2r+1$ and $S \subset V$. There exists an independent set $A \subset S$ such that $D(A) \geq D(S)/8x$.*

Proof. We use Lemma 3.3 to pull out independent sets one at a time. By deleting their neighborhoods and repeating, we construct a large independent set A which is the union of these independent sets and show that A has the desired properties.

Let $V_1 = V$ and $S_1 = S$. We apply Lemma 3.3 to obtain an independent set $B_1 \subset S_1$ such that

$$D(B_1) \geq \frac{D(N(B_1) \cap S_1)}{x_1 + 1}, \quad x_1 = \left(\frac{|S_1|n}{D(S_1)} \right)^{1/r}.$$

We repeatedly apply Lemma 3.3 to $G[V_i]$ and S_i , letting

$$V_i = V_{i-1} \setminus (B_{i-1} \cup N(B_{i-1})), \quad S_i = S_{i-1} \cap V_i.$$

At each iteration, we obtain an independent set $B_i \subset S_i$ such that

$$D(B_i) \geq \frac{D(N(B_i) \cap S_i)}{x_i + 1}, \quad x_i = \left(\frac{|S_i|n}{D(S_i)} \right)^{1/r}.$$

By construction, by step i we have deleted B_{i-1} and its neighbor set $N(B_{i-1})$, so $\bigcup_{j < i} B_j$ is an independent set. We continue the construction described above as long as $D(S_i) \geq D(S)/2$. Suppose we construct s independent sets in total through this process. Then $D(S_s) < D(S)/2$, but $D(S_i) \geq D(S)/2$ for $i < s$. Let $A = \bigcup_{i=1}^s B_i$ be the resulting large independent set. We can bound

$$x_i = \left(\frac{|S_i|n}{D(S_i)} \right)^{1/r} \leq \left(\frac{|S|n}{D(S_i)} \right)^{1/r} \leq \left(\frac{|S|n}{D(S)/2} \right)^{1/r} \leq 2^{1/r} x.$$

From above, we have

$$D(B_i) \geq \frac{D(N(B_i) \cap S_i)}{x_i + 1} \stackrel{(*)}{\geq} \frac{D(N(B_i) \cap S_i) + D(B_i)}{x_i + 2} \geq \frac{D(N(B_i) \cap S_i) + D(B_i)}{2x + 2}.$$

where (*) follows since $D(B_i) \geq D(B_i)/1$ and if $x \geq a/b, c/d$ then $x \geq (a+c)/(b+d)$. This allows us to bound $D(A)$ as

$$(3.1) \quad D(A) = \sum_{i=1}^s D(B_i) \geq \sum_{i=1}^s \frac{D(N(B_i) \cap S_i) + D(B_i)}{2x+2} = \frac{D(A) + D(N(A) \cap S)}{2x+2},$$

where the last equality follows since A is the union of the disjoint sets B_1, \dots, B_s , $N(A) \cap S$ is the union of the disjoint sets $N(B_1) \cap S_1, \dots, N(B_s) \cap S_s$, and D is additive on the union of disjoint sets. Further,

$$\begin{aligned} D(S) &= D(A) + D(N(A) \cap S) + D(S \setminus (A \cup N(A))) \\ &= D(A) + D(N(A) \cap S) + D(S_s), \end{aligned}$$

and, as $D(S_s) < D(S)/2$,

$$(3.2) \quad D(A) + D(N(A) \cap S) \geq \frac{D(S)}{2}.$$

Combining (3.1) and (3.2) gives the desired bound: $D(A) \geq D(S)/(4x+4) \geq D(S)/8x$. \blacksquare

We next use this lemma to obtain an upper bound on $h(G, k)$ for graphs G with no short odd cycles.

Proof of Theorem 1.2. We iteratively apply Lemma 3.4 to obtain k disjoint independent sets which are each incident to many edges. Let $S_1 = V$, and let A_1 be a subset of S_1 with the properties guaranteed by Lemma 3.4. Proceed for k iterations, letting $S_i = V \setminus \bigcup_{j=1}^{i-1} A_j$, to obtain $A_i \subset S_i$ per Lemma 3.4 with large $D(A_i)$. By construction, $S_{k+1} = V \setminus \bigcup_{i=1}^k A_i$, the sets A_1, \dots, A_k are independent, and for each A_i ,

$$D(A_i) \geq \frac{D(S_i)}{8x_i}, \quad x_i = \left(\frac{n|S_i|}{D(S_i)} \right)^{\frac{1}{r}}.$$

By Lemma 2.7, we can assign each $v \in S_{k+1}$ to one of the A_i and can make the graph k -partite by deleting the at most $D(S_{k+1})/k$ edges in parts.

By construction, we have the recursive upper bound

$$D(S_{i+1}) = D(S_i) - D(A_i) \leq \left(1 - \frac{1}{8x_i} \right) D(S_i).$$

Let $\delta_i := \frac{D(S_i)}{n^2}$. Since $|S_i| \leq n$, the above relation yields the recursive inequality

$$(3.3) \quad \delta_{i+1} \leq \delta_i \left(1 - \frac{1}{8} \delta_i^{1/r} \right).$$

Inequality (3.3) implies that if $\delta_l > \varepsilon/2$ for some $\varepsilon > 0$, then

$$\delta_{l+1} - \delta_l < \frac{-\delta_l}{8} \left(\frac{\varepsilon}{2} \right)^{1/r}.$$

If $\delta_i \leq \varepsilon \leq 1$ and $\delta_j > \varepsilon/2$ for $j > i$, then

$$\delta_j - \delta_i = \sum_{l=i}^j (\delta_{l+1} - \delta_l) < \sum_{l=i}^j \frac{-\delta_l}{8} \left(\frac{\varepsilon}{2} \right)^{1/r} < \frac{(j-i)\delta_i}{16} \left(\frac{\varepsilon}{2} \right)^{1/r}.$$

Thus, for $j - i \geq 8(2/\varepsilon)^{1/r}$,

$$\delta_j - \delta_i \leq \frac{-8(2/\varepsilon)^{1/r} \delta_i}{16} \left(\frac{\varepsilon}{2} \right)^{1/r} = \frac{-\delta_i}{2},$$

which yields $\delta_j \leq \delta_i/2 \leq \varepsilon/2$.

Note that $\delta_1 \leq 1$. Let

$$u = \left\lceil r \log_2 \left(\frac{k \ln 2}{32r} \right) \right\rceil - 1.$$

We show that $\delta_{k+1} \leq 1/2^u$. If $u < 0$, then we have $\delta_{k+1} \leq \delta_1 \leq 1 \leq 1/2^u$. So we can suppose that $u \geq 0$. Using the above bound on the decay of δ_i , letting $\varepsilon = 2^{-i}$ for $i = 0, 1, \dots, u-1$, we note that $\delta_j \leq 1/2^u$ for $j := 8 \sum_{i=0}^{u-1} \left\lceil \left(\frac{2}{2^{-i}} \right)^{1/r} \right\rceil$, since

$$j = 8 \sum_{i=0}^{u-1} \left\lceil \left(\frac{2}{2^{-i}} \right)^{1/r} \right\rceil = 16 \sum_{i=0}^{u-1} \left(2^{1/r} \right)^{i+1} = 16 \cdot 2^{1/r} \cdot \left(\frac{2^{(u+1)/r} - 1}{2^{1/r} - 1} \right) \stackrel{(*)}{<} \frac{32(2^{(u+1)/r})}{(\ln 2)/r} \leq \frac{32r \left(\frac{\ln 2}{32r} \cdot k \right)}{\ln 2} < k + 1,$$

where $(*)$ follows since $e^x \geq 1 + x$ for all x (so $2^{1/r} - 1 = e^{\ln 2/r} - 1 > \ln 2/r$). For this choice of u we have that

$$\delta_{k+1} \leq \frac{1}{2^u} \leq 2^{1 - \lceil r \log_2(\frac{\ln 2}{8r} k) \rceil} \leq 4 \left(\frac{k \ln 2}{8r} \right)^{-r} = \frac{4 \left(\frac{8r}{\ln 2} \right)^r}{k^r}.$$

This gives the desired bound on $h(G, k)$ with $c_r = 4(8r/\ln 2)^r$:

$$h(G, k) \leq \frac{D(S_{k+1})}{k} \leq \frac{\delta_{k+1} n^2}{k} \leq \frac{c_r n^2}{k^{r+1}} < \frac{4(12r)^r n^2}{k^{r+1}}.$$

■

Theorem 1.2 gives a bound on $h(G, k)$ when G has odd girth larger than $2r + 1$, which, as we show in the next subsection, is tight up to a factor depending only on r . Our goal is to understand a less constrained family of graphs, those with a single fixed forbidden odd cycle. To do this, the following lemma shows that if a graph has a forbidden odd cycle C_{2r+1} , then we can delete a small number of edges to get rid of the next shorter odd cycles.

Lemma 3.5. *If a graph $G = (V, E)$ on n vertices is C_{2r+1} -free, then G can be made to have odd girth larger than $2r + 1$ by removing $O_r(n^{3/2})$ edges.*

Proof. For each odd integer $1 < \ell < 2r + 1$, fix a maximal collection \mathcal{C}_ℓ of edge-disjoint copies of C_ℓ in G . Suppose we have t_ℓ such copies. To remove all copies of C_ℓ from G , we must delete at least t_ℓ edges (at least one edge in each edge-disjoint C_ℓ). If we delete all ℓt_ℓ edges in these cycles, the resulting graph is C_ℓ -free. Thus, the minimum number of edges to delete to make the graph C_ℓ -free is within a factor ℓ of the size of any maximal collection of edge-disjoint copies of C_ℓ .

Consider a random subset $U \subset V$ formed by including each element with probability $p = 1/\ell$ independently of the other vertices. Call an edge of a cycle in \mathcal{C}_ℓ *special* if both of its vertices are in U and no other vertex of the cycle is in U . The probability that a given edge of a cycle in \mathcal{C}_ℓ is special is $p^2(1-p)^{\ell-2} > 1/(e\ell^2)$. Hence, by linearity of expectation, the expected number of special edges is at least $\ell t_\ell/(e\ell^2) = t_\ell/(e\ell)$. So we can fix a subset U with at least $t_\ell/(e\ell)$ special edges.

Let $2d := 2r + 3 - \ell$, so $d \in [2, r]$ is an integer as ℓ is odd. As the graph G is C_{2r+1} -free, there is no cycle of length $2d$ of special edges. Indeed, suppose there is a cycle C' of length $2d$ of special edges, and let e be an edge of this cycle. Edge e is by definition in a cycle C'' of length ℓ with none of its other vertices in U . So gluing together C' and C'' and deleting the common edge e , we obtain a cycle of length $(2r + 3 - \ell) + \ell - 2 = 2r + 1$, contradicting the assumption that G is C_{2r+1} -free.

Recall that the *extremal number* $\text{ex}(n, H)$ is the maximum number of edges an H -free graph on n vertices can have. So the number of special edges is at most

$$\text{ex}(|U|, C_{2d}) \leq \text{ex}(n, C_{2d}) \leq 8(d-1)n^{1+1/d},$$

where the last bound is due to Verstraëte [46]². Hence, $t_\ell/(e\ell) \leq 8(d-1)n^{1+1/d}$, or equivalently, $lt_\ell \leq \ell^2 e 8(d-1)n^{1+1/d}$. So the number of edges we can delete from G to make the resulting subgraph have odd girth larger than $2r+1$ is at most

$$\sum_{3 \leq \ell \leq 2r-1, \ell \text{ odd}} lt_\ell \leq \sum_{d=2}^r \ell^2 e 8(d-1)n^{1+1/d} \leq \sum_{d=2}^r 32e r^2 d n^{1+1/d} \leq 100r^4 n^{3/2}.$$

■

From Lemma 3.5 and Theorem 1.2, we immediately obtain Theorem 1.3.

We remark that Conlon, Fox, Sudakov, and Zhao [16] recently improved the bound in Lemma 3.5 for $r=2$ to $o(n^{3/2})$.

3.2. Lower bounds for odd cycles. In this subsection, we give a construction which shows that Theorem 1.3 is tight for sufficiently large n to within a factor only depending on the length of the forbidden odd cycle. The construction is based on a construction of Alon [2] (see the discussions in [8] and [37]) of a rather dense pseudorandom graph of large odd girth.

Definition 3.6. An (n, d, λ) -graph is a d -regular graph G on n vertices such that the second largest in absolute value eigenvalue has magnitude at most λ .

To discuss the properties of this construction, we first recall the expander mixing lemma, a classical result in spectral graph theory. An early version is due to Alon and Chung [5].

Lemma 3.7 (Expander Mixing Lemma). *If $G = (V, E)$ is an (n, d, λ) -graph and $A, B \subset V$, then*

$$\left| e(A, B) - \frac{d}{n}|A||B| \right| \leq \lambda \sqrt{|A||B|}.$$

Note that if a graph G on $n \leq n'$ vertices is \mathcal{H} -free and no graph in \mathcal{H} has isolated vertices, by adding $n' - n$ dummy vertices we can obtain a graph G' on n' vertices, where $h(n', k, \mathcal{H}) \geq h(G, k)$. Since this holds for all such G on n vertices, we have the following observation, which will be relevant to our future discussion of blow-ups of a fixed graph.

Proposition 3.8. *Given a family of graphs \mathcal{H} , if no graph in \mathcal{H} has isolated vertices and $n' \geq n$, then*

$$h(n', k, \mathcal{H}) \geq h(n, k, \mathcal{H}).$$

The results which follow proceed towards the goal of showing that $h(n, k, C_{2r+1}) = \Omega_r(n^2/k^{r+1})$.

We first note that for arbitrary $G = (V, E)$, we can compute $h(G[t], k)$ in terms of $h(G, k)$, where $G[t]$ is the t -blow-up of G , the graph on $t|V|$ vertices given by the lexicographic product of G with an empty graph on t vertices.

Lemma 3.9. *Let $G[t]$ be the t -blow-up of $G = (V, E)$. Then,*

$$h(G[t], k) = t^2 h(G, k).$$

Proof. By taking the blow-up of any vertex partition of G into k parts, we see that

$$h(G[t], k) \leq t^2 h(G, k).$$

To complete the proof, we next show the reverse inequality. Consider a vertex partition P of $G[t]$ into k parts. Consider a copy of G in $G[t]$ with exactly one vertex in each of the $|V|$ parts of order t . Each such copy has at least $h(G, k)$ of its edges inside parts of P . The number of such copies of G is $t^{|V|}$ and each edge of $G[t]$ is in exactly $t^{|V|-2}$ such copies of G . Thus, at least $h(G, k)t^{|V|}/t^{|V|-2} = t^2 h(G, k)$ edges of G must be inside parts of P . Hence, $h(G[t], k) \geq t^2 h(G, k)$. ■

²There is a long history of bounding the extremal number of even cycles, including by Erdős [20], Bondy and Simonovits [13], and most recently improvements by Pikhurkho [40] and further by Bukh and Jiang [14].

It will be helpful to introduce the following definition.

Definition 3.10. A family \mathcal{H} of graphs is *closed under homomorphism* if for any $H \in \mathcal{H}$ and graph homomorphism $\phi : H \rightarrow H'$, graph H' is in \mathcal{H} .

We want to establish a result similar to Lemma 3.8 for t -blow-ups.

Lemma 3.11. *If \mathcal{H} is closed under homomorphism and t is a positive integer, then*

$$h(tn, k, \mathcal{H}) \geq t^2 h(n, k, \mathcal{H}).$$

Proof. Consider a graph G on n vertices which is \mathcal{H} -free. Its t -blow-up $G[t]$ is also \mathcal{H} -free. Hence, $h(G[t], k) \leq h(tn, k, \mathcal{H})$. Furthermore, by Lemma 3.9, we see that $h(G[t], k) = t^2 h(G, k)$ and therefore $t^2 h(G, k) \leq h(tn, k, \mathcal{H})$ holds for any \mathcal{H} -free graph G . By taking the maximum over the left hand side of this inequality, we obtain the desired inequality. \blacksquare

We use the following result, extending Alon's construction of a pseudorandom triangle-free graph which is as dense as possible to one of large odd girth [8, 37].

Lemma 3.12 (§3, [8]). *For each positive integer r there is c_r such that the following holds. For every integer $a \geq 2$ and $N = 2^{(2r+1)a}$, there is an (N, d, λ) -graph G of odd girth larger than $2r + 1$ such that $d \geq \frac{1}{8}N^{2/(2r+1)}$ and $\lambda \leq c_r \sqrt{d}$.*

We use this lemma to get a lower bound on $h(n, k, \mathcal{H})$ with $\mathcal{H} = \{C_3, C_5, \dots, C_{2r+1}\}$.

Proof of Theorem 1.4. Let $a = \lceil \log_2(2c_r k) \rceil$, where c_r is chosen as in Lemma 3.12, and $N = 2^{(2r+1)a}$. By Lemma 3.12, there is a (N, d, λ) -graph H with $d \geq \frac{1}{8}N^{2/(2r+1)}$ and $\lambda \leq c_r \sqrt{d}$. For $n \geq N$, let $t = \lfloor n/N \rfloor$ and let $n_0 = Nt$, so $n_0 \leq n$. Let $G = H[t]$ be the balanced t -blow up of H , so $|V(G)| = n_0$. Since H has odd girth larger than $2r + 1$, G also has odd girth larger than $2r + 1$. Let $\mathcal{H} = \{C_3, C_5, \dots, C_{2r+1}\}$. By Proposition 3.8 and Lemma 3.9, we have

$$h(n, k, \mathcal{H}) \geq h(n_0, k, \mathcal{H}) \geq h(G, k) = h(H[t], k) = t^2 h(H, k).$$

We give a lower bound on $h(H, k)$ which implies the desired lower bound on $h(n, k, \mathcal{H})$. Consider a k -partition $V(H) = V_1 \sqcup \dots \sqcup V_k$ that minimizes $\sum_{i=1}^k e(V_i)$, which is the minimum number of edges to delete from H to obtain a k -colorable subgraph. By Lemma 3.7, we have

$$\begin{aligned} 2 \sum_{i=1}^k e(V_i) &= \sum_{i=1}^k e(V_i, V_i) \geq \sum_{i=1}^k \left(\frac{d}{N} |V_i|^2 - \lambda |V_i| \right) = \frac{d}{N} \sum_{i=1}^k |V_i|^2 - \lambda N \stackrel{(*)}{\geq} \frac{dN}{k} - \lambda N \stackrel{(**)}{\geq} \frac{dN}{2k} \\ &\geq \Omega_r(N^2/k^{r+1}). \end{aligned}$$

Here $(*)$ follows by convexity of $f(x) = x^2$ and $(**)$ follows from $\lambda \leq c_r \sqrt{d} \leq \frac{d}{2k}$, which in turn follows from $N \geq (2c_r k)^{2r+1}$ and $d \geq \frac{1}{8}N^{2/(2r+1)}$. Hence, $h(n, k, \mathcal{H}) \geq t^2 h(H, k) = \Omega_r(n^2/k^{r+1})$, which completes the proof. \blacksquare

We can strengthen Theorem 1.4 when H is a triangle. Guo and Warnke [33] show the existence of triangle-free graphs with discrepancy like random graphs by using a semi-random variant of the triangle-free process (studied in [10]).

Theorem 3.13 (Theorem 4, [33]). *There exist $\beta_0, D_0 > 0$ such that for all $\gamma, \delta \in (0, 1], \beta \in (0, \beta_0)$ and $C \geq D_0/(\delta^2 \sqrt{B} \gamma)$, the following holds for all $n \geq n_0(\gamma, \delta, \beta, C)$ with $\rho := \sqrt{\beta \log n/n}$: for any n -vertex graph G' , there exists a triangle-free subgraph $G \subset G'$ on the same vertex set such that*

$$e_G(A, B) = (1 \pm \delta) \rho e_{G'}(A, B)$$

for all vertex-sets $A, B \subset V(G')$ with $|A| = |B| = \lceil C \sqrt{n \log n} \rceil$ and $e'_{G'}(A, B) \geq \gamma |A| |B|$.

This yields an improved lower bound on $h(n, k, C_3)$.

Proposition 3.14. *There exists an absolute constant c such that for all $n \geq ck^2 \log k$, there exists a triangle-free graph Γ on n vertices with $h(\Gamma, k) = \Omega(n^2/k^2)$.*

Proof. We apply Theorem 3.13 with $G' = K_N$, $\gamma = 1$, and $\delta = 1/2$. This yields a triangle-free graph $G \subset K_N$ such that for $A \subset V$ with $|A| = \lceil C\sqrt{N \log N} \rceil$, we have that

$$e_G(A) \geq \frac{1}{2} \sqrt{\beta \log n/N} \cdot |A|^2,$$

i.e. G resembles a random graph with edge probability $\Theta(\sqrt{\log N/N})$ and $\alpha(G) \leq 2\lceil C\sqrt{N \log N} \rceil$. We choose $N = ck^2 \log k$ for sufficiently large absolute constant c . Any k -partition of G as above has relatively dense parts. Precisely, we have at least $\frac{1}{4} \sqrt{\beta \log N/N} \cdot \frac{N^2}{k}$ edges internal to any k -partition. Then, taking a balanced blowup of G (in which each part has either $\lfloor n/N \rfloor$ or $\lceil n/N \rceil$ vertices) gives by Lemma 3.9 a triangle-free graph Γ on n vertices with $h(\Gamma, k) = \Omega(n^2/k^2)$. ■

Note that Proposition 3.14 is essentially best possible. This follows from the construction of Kim [35], refined by Fiz Pontiveros, Griffiths and Morris [41] of triangle-free graphs on at most $(4 + o(1))k^2 \log k$ vertices with chromatic number at most k .

4. APPLICATIONS TO OTHER FORBIDDEN SUBGRAPHS

So far, we have only proved bounds on $h(n, k, H)$ when H is an odd cycle or clique. In this section, we obtain bounds for a broader class of graphs H . In particular, we prove that if H' is a subgraph of a fixed graph H , and H has a homomorphism to H' , then $h(n, k, H)$ is within $o(n^2)$ of $h(n, k, H')$.

4.1. Graph Homomorphisms. To obtain these results, we use the graph removal lemma, which first appeared in [6, 31]. It extends the triangle removal lemma of Ruzsa and Szemerédi (see the survey [15] for details).

Theorem 4.1 (Graph removal lemma). *For any graph H on h vertices and any $\varepsilon > 0$, there exists $\delta > 0$ such that any graph on n vertices that contains at most δn^h copies of H can be made H -free by removing at most εn^2 edges.*

Recall that a homomorphism from a graph H to a graph H' is a (not necessarily injective) map $\rho: V(H) \rightarrow V(H')$ that maps edges of H to edges of H' . We also use the following lemma of Erdős.

Lemma 4.2 ([21]). *For $\delta > 0$, $r \geq 2$, $t \geq 1$, and sufficiently large n , every r -uniform hypergraph Γ on n vertices with at least δn^r edges contains a complete r -partite, r -uniform subhypergraph with parts of order t .*

Theorem 4.3. *Suppose \mathcal{H} and \mathcal{H}' are fixed finite families of graphs such that for each $H' \in \mathcal{H}'$, there is some $H \in \mathcal{H}$ such that H has a homomorphism to H' . If k is a fixed positive integer, then*

$$h(n, k, \mathcal{H}) \leq h(n, k, \mathcal{H}') + o(n^2).$$

Proof. Let H' be a graph in \mathcal{H}' and H be some graph in \mathcal{H} for which H' has a homomorphism to H . Let r denote the number of vertices of H' , and t denote the number of vertices of H . Label the vertices of H' as $\{1, \dots, r\}$. Fix any $\varepsilon > 0$ and let $\delta > 0$ be as in the graph removal lemma for H' . Let G be an \mathcal{H} -free graph on n vertices. Consider the r -uniform r -partite hypergraph X with parts V_1, \dots, V_r with each V_i a copy of $V(G)$, and $(v_1, v_2, \dots, v_r) \in V_1 \times V_2 \times \dots \times V_r$ is an edge of X if there is a copy of H' with v_i a copy of i for $i \in \{1, \dots, r\}$.

If there are at least δn^r copies of H' in G , then X contains at least $\delta n^r = \delta t^{-r} |V(X)|^r$ edges. As we may assume n is sufficiently large, Lemma 4.2 implies that X contains a copy of the complete r -partite r -uniform hypergraph with parts of order t . As H has a homomorphism to H' , we can then find a copy of H with vertices among the vertices of the copy of the complete r -partite r -uniform hypergraph with parts of order t , contradicting that G is \mathcal{H} -free.

So we may suppose G has less than δn^r copies of H' . By the graph removal lemma applied to H' , we can remove εn^2 edges from G to make it H' -free. We can do this edge removal for each $H' \in \mathcal{H}'$ to make the graph \mathcal{H}' -free. We can then remove an additional $h(n, k, \mathcal{H}')$ edges to make it k -partite. We have thus obtained the desired upper bound on $h(n, k, \mathcal{H})$. ■

We have the following immediate corollary by taking \mathcal{H} and \mathcal{H}' to each consist of a single graph.

Corollary 4.4. *If H and H' are fixed graphs for which H has a homomorphism to H' and k is a fixed positive integer, then $h(n, k, H) \leq h(n, k, H') + o(n^2)$.*

In the special case that H is not bipartite, H and k are fixed, and H' is a subgraph of H , we get that these parameters are asymptotically equal, as in Theorem 1.6.

Proof of Theorem 1.6. The upper bound follows from Corollary 4.4. As H' is a subgraph of H , then any H' -free graph is also H -free, and hence $h(n, k, H) \leq h(n, k, H')$. ■

Remark 4.5. Note that Theorem 1.6 immediately implies a weaker version of Theorem 1.3, namely that for $\mathcal{H}' = \{C_3, C_5, \dots, C_{2r-1}, C_{2r+1}\}$,

$$h(n, k, C_{2r+1}) = h(n, k, \mathcal{H}') + o_r(n^2).$$

As another example, let $K_{2,2,2}$ be the complete tripartite graph on six vertices with two vertices in each part. From Theorems 4.3 and 1.6, we have

$$h(n, k, K_3) \leq h(n, k, K_{2,2,2}) \leq h(n, k, K_3) + o(n^2).$$

Consequently, although we don't know for any fixed $k \geq 2$ the asymptotic value of $h(n, k, K_3)$ or $h(n, k, K_{2,2,2})$, we know that they are asymptotically the same.

4.2. Forbidding a wheel. Recall that the *wheel* W_l is the graph of $l + 1$ vertices consisting of an l -cycle and an additional vertex adjacent to all of the vertices of the l -cycle. The above results on graph homomorphisms allow us to bound how many edges we need to delete to make a W_l -free graph on n vertices k -colorable. We first get an asymptotic answer for even wheels W_l (when l is even). Since the wheel W_l has a triangle, and with l even is 3-colorable, we have the following corollary of Corollary 1.6 and Theorems 1.1 and 1.4.

Proposition 4.6. *Fix integers $r \geq 2, k \geq 1$. Then, $h(n, k, K_3) \leq h(n, k, W_{2r}) \leq h(n, k, K_3) + o(n^2)$. In particular, for n sufficiently large, we have $h(n, k, W_{2r}) = \Theta(n^2/k^2)$.*

We suspect $h(n, k, W_l)$ depends significantly on the parity of l , and this is related to odd wheels not being 3-colorable. The following proposition makes an initial observation for odd wheels.

Proposition 4.7. *There are positive constants $c_1, c_2 > 0$ such that the following holds. For fixed positive integers r, k and every sufficiently large positive integer n , we have*

$$c_1 \frac{n^2}{k^2} \leq h(n, k, W_{2r+1}) \leq c_2 \frac{n^2}{k^{3/2}}.$$

Proof. As K_3 is a subgraph of W_{2r+1} , which in turn is a subgraph of K_{2r+2} , we have $h(n, k, W_{2r+1}) \geq h(n, k, K_3) \geq c_1 \frac{n^2}{k^2}$ for a positive constant c_1 , where the last bound is from Theorem 1.4 when $r = 1$. For the upper bound, we observe that W_{2r+1} is 4-colorable as C_{2r+1} is 3-colorable, and hence W_{2r+1} has a homomorphism to K_4 . It follows from Theorems 4.3 and 1.1 and n is sufficiently larger that $h(n, k, W_{2r+1}) \leq h(n, k, K_4) + o(n^2) \leq c_2 \frac{n^2}{k^{3/2}}$. ■

Note that the lower and upper bounds are rather far apart, and the above result does not give an indication of which of the two bounds $h(n, k, W_{2r+1})$ is closer to. We obtain Theorem 1.5 by a careful analysis that combines the methods used to give upper bounds on $h(n, k, K_r)$ and $h(n, k, C_{2r+1})$. This gives a much better upper bound on $h(n, k, W_{2r+1})$ which we conjecture to be tight up to a constant factor depending only on r .

Proof of Theorem 1.5. Let ℓ be the largest integer at most k which is twice a perfect $(r+1)$ -power. We have $k \leq 2^{r+2}\ell$. Let $s = (\ell/2)^{1/(r+1)}$ and $t = \ell/(2s)$, so s and t are integers and $\ell = 2st$. Let $H = W_{2r+1}$ and v be the vertex of the wheel W_{2r+1} of degree $2r+1$, so $H_v = C_{2r+1}$. By Lemma 2.13 and Theorem 1.3, we have

$$\begin{aligned} h(n, k, W_{2r+1}) &\leq \frac{2n^2}{est^2} + t \cdot h\left(\frac{2n}{t}, s, C_{2r+1}\right) \\ &\leq \frac{2n^2}{est^2} + t \cdot \beta_r \cdot \frac{(2n/t)^2}{s^{r+1}} \\ &= \left(\frac{2}{e} + 4\beta_r\right) \frac{n^2}{(\ell/2)^{2-1/(r+1)}} \\ &\leq 2^{2r+6} \left(\frac{2}{e} + 4\beta_r\right) \frac{n^2}{k^{2-1/(r+1)}}, \end{aligned}$$

where β_r only depends on r . Letting $c_r = 2^{2r+6} \left(\frac{2}{e} + 4\beta_r\right)$ completes the proof. \blacksquare

5. FROM MAX- k -CUTS TO MAX- l -CUTS

Note that $h(G, k)$ and $\text{Max-}k\text{-Cut}(G)$ are related through the identity

$$h(G, k) = e(G) - \text{Max-}k\text{-Cut}(G).$$

Consequently, the results proved in the previous section on $h(n, k, C_{2r+1})$ yield bounds on $\text{Max-Cut}(G)$ for C_{2r+1} -free graphs G , and on $\text{Max-}l\text{-Cut}(G)$ for $l > 2$. We can also relate $\text{Max-}l\text{-Cut}(G)$ to $h(G, k)$ for $l < k$.

Definition 5.1. For $G = (V, E)$, let

$$d_l(G) = \frac{\text{Max-}l\text{-Cut}(G)}{|E|},$$

be the fraction of edges of G that can cross an optimal l -cut of G .

In particular, for a graph G on m edges, we have that $d_2(G)m = \text{Max-Cut}(G)$. We can bound $d_l(G)$ in terms of $d_k(G)$ for $k \geq l$.

Proposition 5.2. For $G = (V, E)$ and positive integers $l \leq k$, $d_l(G) \geq d_l(K_k)d_k(G)$.

Proof. For $G = (V, E)$ with $|V| = n$, $|E| = m$, fix a k -partition $V = V_1 \sqcup \dots \sqcup V_k$ with $\text{Max-}k\text{-Cut}(G)$ edges between parts. Choose a random, equitable partition of the set $\{1, \dots, k\}$ into l parts S_1, \dots, S_l (so each part has size either $\lfloor k/l \rfloor$ or $\lceil k/l \rceil$). Let $W_i = \bigcup_{j \in S_i} V_j$ for $i = 1, \dots, l$. Then $V = W_1 \sqcup \dots \sqcup W_l$ is an l -partition of V .

We count the expected fraction of edges internal to the l -cut $V = W_1 \sqcup \dots \sqcup W_l$. Any edge $e \in E$ internal to some V_j will remain internal in the l -cut, and a fraction $1 - d_k(G)$ of the edges are of this form. All other edges $e \in E$ have endpoints in V_{j_1}, V_{j_2} for $j_1 \neq j_2$. The probability that e is internal in the l -cut is the probability that $V_{j_1}, V_{j_2} \in W_i$ for some single part of the l -partition, which is $1 - d_l(K_k)$. Thus, the expected fraction of edges internal to the l -cut W_1, \dots, W_l is

$$(1 - d_l(K_k))d_k(G) + (1 - d_k(G)) = 1 - d_l(K_k)d_k(G).$$

This gives the desired bound. \blacksquare

By considering a uniformly random partition of a graph G , we have the bound

$$\text{Max-}l\text{-Cut}(G) \geq \frac{l-1}{l} \cdot e(G).$$

Definition 5.3. The *surplus* of the Max- l -Cut of a graph G is given by

$$\pi(l, G) := \text{Max-}l\text{-Cut}(G) - \left(1 - \frac{1}{l}\right) e(G).$$

The *surplus* of a graph G is the *surplus* of the Max-2-Cut of G .

The surplus measures how much larger the Max- l -Cut is above the random bound. We are often interested in how large can we show the surplus is for graphs with certain properties. We recall a standard method for giving a lower bound on the Max- l -Cut(G) in terms of the Max- l -Cut of smaller induced subgraphs of G .

Lemma 5.4. *Given $G = (V, E)$ and a vertex k -partition $V = V_1 \sqcup \dots \sqcup V_k$, then*

$$\pi(l, G) \geq \sum_{i=1}^k \pi(l, G[V_i])$$

Proof. For each i , fix an l -partition $V_i = W_{i1} \sqcup \dots \sqcup W_{il}$ such that the number of edges between different W_{ij} is Max- l -Cut($G[V_i]$). From this, we construct an l -cut of $V = U_1 \sqcup \dots \sqcup U_l$. For each i , fix a random permutation $\sigma \in S_l$ and assign W_{ij} to $U_{\sigma(j)}$. Note that the Max- l -Cut(G) is at least the expected size of this random l -cut. In this process, all edges not contained in $G[V_i]$ for some i have endpoints randomly assigned and thus cross the resulting cut with probability $1 - 1/l$. Therefore,

$$\text{Max-}l\text{-Cut}(G) \geq \left(1 - \frac{1}{l}\right) \left(e(G) - \sum_{i=1}^k e(G[V_i])\right) + \sum_{i=1}^k \text{Max-}l\text{-Cut}(G[V_i]).$$

Since Max- l -Cut($G[V_i]$) = $(1 - 1/l)e(G[V_i]) + \pi(l, G[V_i])$, the above inequality implies the desired result. \blacksquare

6. MAX-CUT IN GRAPHS WITH A FORBIDDEN ODD CYCLE

We leverage the previous results to prove the following lower bound on Max-Cut for C_{2r+1} -free graphs. This bound will be helpful to apply to graphs which are reasonably dense.

Lemma 6.1. *For any positive integer r , there exists $c = c(r) > 0$ such that the following holds. If G is a C_{2r+1} -free graph with n vertices and m edges, then*

$$\text{Max-Cut}(G) \geq \frac{m}{2} + \Omega_r((m/n^2)^{1/r} m).$$

Proof. Let k be the smallest even integer that is at least $(2c_r n^2/m)^{1/r}$, where c_r is the implicit constant in Theorem 1.3. In particular, $c_r n^2/k^{r+1} \leq m/(2k)$. Let m_0 be the number of edges we delete from G to get a k -partite graph, so $d_k(G)m = m - m_0$. By Theorem 1.3, we have $m_0 \leq c_r n^2/k^{r+1}$. Note as k is even,

$$d_2(K_k) = \frac{(k/2)^2}{\binom{k}{2}} = \frac{k}{2(k-1)} = \frac{1}{2} \left(1 + \frac{1}{k-1}\right),$$

which is realized by assigning $k/2$ vertices to each of 2 parts arbitrarily. By Proposition 5.2, we obtain that

$$\begin{aligned} \text{Max-Cut}(G) &= d_2(G)m \geq d_2(K_k)d_k(G)m = d_2(K_k)d_2(K_k)(m - m_0) \\ &\geq \frac{1}{2} \left(1 + \frac{1}{k-1}\right) \left(m - c_r \frac{n^2}{k^{r+1}}\right) \geq \frac{1}{2} \left(1 + \frac{1}{k-1}\right) \left(m - \frac{m}{2k}\right) = \frac{m}{2} + \frac{m}{4(k-1)}. \end{aligned}$$

This last inequality gives the desired bound. \blacksquare

We will be able to show another lower bound on $\text{Max-Cut}(G)$ for graphs $G = (V, E)$, using the following result of Alon, Krivelevich, and Sudakov.

Lemma 6.2 (Lemma 3.3 [9]). *There is an absolute positive constant ϵ such that for every positive constant M , there is a $\alpha = \alpha(M) > 0$ with the following property. If $G = (V, E)$ is a graph with m edges such that the induced subgraph on any set of $N \geq M$ vertices all of which have a common neighbor contains at most $\epsilon N^{3/2}$ edges, then*

$$\text{Max-Cut}(G) \geq \frac{m}{2} + \alpha \sum_{v \in V} \sqrt{d(v)}.$$

In particular, this condition holds for graphs with a small forbidden cycle.

Corollary 6.3. *For every positive integer k there is $\alpha = \alpha(k) > 0$ such that if $G = (V, E)$ is a C_k -free graph, then*

$$\text{Max-Cut}(G) \geq \frac{m}{2} + \alpha \sum_{v \in V} \sqrt{d(v)}.$$

Proof. Let $M = M(k) = \epsilon^{-2}k^2$, where $\epsilon > 0$ is the absolute constant from Lemma 6.2. Let $G' = (V', E')$ be an induced subgraph of G with $|V'| = N \geq M$ and $V' \subset N(v)$ for some $v \in V$. It suffices to show that $|E'| \leq \epsilon N^{3/2}$. Since G is C_k -free, G' is P_{k-1} -free. Hence,

$$|E'| \leq \text{ex}(N, P_{k-1}) < Nk/2 < \epsilon N^{3/2},$$

where the middle inequality is due to Erdős and Gallai [27]. ■

We leverage these results to give an improved upper bound on the Max-Cut of a C_{2r+1} -free graph on m edges.

Proof of Theorem 1.7. Let $k = 2r + 1$ with r a positive integer and observe that $(k + 5)/(k + 7) = 1 - 1/(r + 4)$. Let $G = (V, E)$ be a C_{2r+1} -free graph with m edges. Let $U \subset V$ be the set of vertices of degree at least $m^{2/(r+4)}$. As the sum of the degrees of vertices in G is $2m$, we have $|U| \leq (2m)/m^{2/(r+4)} = 2m^{1-2/(r+4)}$.

If the induced subgraph $G[U]$ contains at least $m/2$ edges, then applying Lemma 6.1 to $G[U]$, we obtain that the surplus of $G[U]$ is $\Omega_r((m/|U|^2)^{1/r}m)$, which is at least $\Omega_r(m^{1-1/(r+4)})$. Lemma 5.4 implies that the surplus of a graph is at least the surplus of any induced subgraph. Hence, the surplus of G is also $\Omega_r(m^{1-1/(r+4)})$.

Otherwise, the induced subgraph $G[U]$ has less than $m/2$ edges, and so $\sum_{v \in V \setminus U} d(v) \geq m/2$. By Corollary 6.3, the surplus of G is, for some positive constant c_r , at least

$$c_r \sum_{v \in V} \sqrt{d(v)} \geq c_r \sum_{v \in V \setminus U} \sqrt{d(v)} \geq c_r \frac{m/2}{m^{2/(r+4)}} \sqrt{m^{2/(r+4)}} = \Omega_r(m^{1-1/(r+4)}),$$

where the last inequality uses concavity of the function $f(x) = x^{1/2}$ and that $d(v) < m^{2/(r+4)}$ for all $v \in V \setminus U$. ■

7. CONCLUDING REMARKS

Theorems 1.3 and 1.4 together determine $h(n, k, C_{2r+1})$ up to a constant factor depending only on r for sufficiently large n . The lower bound in Theorem 1.4 relies on the construction of Alon of a relatively dense pseudorandom C_{2r+1} -free graph as in Theorem 3.12. A corresponding pseudorandom graph construction of K_r -free graphs does not exist apart from the case $r = 3$ (as $K_3 = C_3$). However, it is conjectured that such a graph exists. If so, the proof would carry over and we would obtain the following conjecture, which would show that Theorem 1.1 is tight up to a constant function of r for sufficiently large n .

Conjecture 7.1. *If $r \geq 3$ and $n \gg k$, then $h(n, k, K_r) = \Omega_r(n^2/k^{(r-1)/(r-2)})$.*

It is known that other interesting results would follow from knowing the existence of such pseudorandom K_r -free graphs, including giving nearly tight bounds for off-diagonal Ramsey numbers (see [34, 39]).

It would also be interesting to get better bounds on the Max-Cut of H -free graphs for other forbidden subgraphs H . The methods of Sections 5 and 6 can be used to obtain improved bounds for some other H , like odd wheels.

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