

# Breakdown of hydrodynamics below four dimensions in a fracton fluid

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We present the nonlinear fluctuating hydrodynamics which governs the late time behavior of a chaotic many-body system with simultaneous charge/mass, dipole/center of mass, and momentum conservation. This hydrodynamic effective theory is unstable below four spatial dimensions: dipole-conserving fluids at rest are unstable to fluctuations, which drive the system to a novel dynamical universality class. In one spatial dimension, our construction is reminiscent of the well-established renormalization group flow of the stochastic Navier-Stokes equations; however, the fixed point we find possesses subdiffusive scaling, rather than the superdiffusive scaling of the Kardar-Parisi-Zhang universality class. We numerically simulate many-body classical dynamics in one- and two-dimensional models with dipole and momentum conservation, and find evidence for the predicted breakdown of hydrodynamics. Our theory provides a controlled example of how kinematic constraints lead to a rich landscape of dynamical universality classes in high dimensional models.

*Introduction.*— One of the oldest and most applicable theories in physics is hydrodynamics. While hydrodynamics was first understood as a phenomenological set of equations that govern liquids and gases [1], over the past century we have instead recognized that hydrodynamics is best understood as the universal effective field theory that governs thermalization in a chaotic many-body system [2–4]. Due to this universality, the same theories of hydrodynamics can describe diverse phases of classical or quantum matter, including ultracold atoms [5, 6], quark-gluon plasma [7], and electrons and phonons in high-purity solids [8–11].

Novel phases of matter arise when the microscopic degrees of freedom are *fractons* — excitations which are individually immobile, and can only move in tandem [12–30] (these are different from a previous use of the word “fracton” in the context of high-energy physics [31]). As a simple example, we can consider a phase of matter in which the global charge (or mass) is conserved, together with the global dipole moment (or center of mass). In this case, a single particle cannot move without violating the dipole conservation law. If such a phase of matter, realized on a lattice, can thermalize [32–34], it is described by a novel hydrodynamics in which Fick’s law of charge diffusion is replaced by slower subdiffusion [35–40]. The emergence of subdiffusion is not special to peculiar microscopic details of particular lattice models; it is guaranteed by the symmetries of the dynamics. This robustness of hydrodynamics to microscopic peculiarities makes it an experimentally ideal probe for constrained dynamics [41].

Here, we study such a dipole-conserving theory which is also translation-invariant. In this case, charge, dipole and momentum are all conserved quantities. We show that these fluids exhibit a highly unusual hydrodynamics, with magnon-like propagations with subdiffusive decay

rates. More importantly, as we now explain, below four spatial dimensions these fluids are violently unstable to thermal fluctuations. Hydrodynamics, meant as a set of deterministic conservation laws, will thus not exist in any experimentally-realizable spatial dimension.

This differs from what happens in fluids with conventional conserved quantities. Such systems are well described by deterministic hydrodynamic equations in three spatial dimensions  $d = 3$ , with marginally irrelevant fluctuations in  $d = 2$ . In  $d = 1$ , fluctuations become overwhelmingly large and the dynamics on long time/length scales is governed by the Kardar-Parisi-Zhang (KPZ) universality class [42, 43]. The KPZ fixed point is remarkably generic: it also arises in the study of growing surfaces [42], quantum Hall edge states [44], and dynamics of integrable systems [45–51]. Among the many non-equilibrium universality classes that have been discovered in statistical mechanics, ranging from flocking [52] in active matter, to driven-dissipative condensates [53], to fluctuating smectic liquid crystals [54], the KPZ fixed point is unique in that it describes the instability of an ordinary undriven fluid at rest without any spontaneous symmetry breaking.

In this paper, we discover a new dynamical universality class which we call the dipole-momentum conserving (DMC) fixed point. Although both KPZ and DMC fixed points can arise out of an instability of a fluid at rest, the **DMC** fixed point is distinct. Microscopically, the dynamics is *kinematically constrained* due to the conservation of the total dipole moment. As a consequence, this leads to a dramatic slowdown of the macroscopic dynamics. In particular, transport in DMC is *subdiffusive*, while KPZ has superdiffusive scaling. Another distinction from ordinary fluids is that, as a consequence of the modified scaling, fluctuations are relevant below four dimensions, whereas in ordinary fluids fluctuations are relevant below two dimensions. This is the first example known to us where the hydrodynamics of conserved quantities in a thermalizing system, without symmetry breaking, is un-

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stable to relevant fluctuations in dimensions greater than one.

Just as hydrodynamics is robust against microscopic details, so too is the universality class of non-equilibrium dynamics that emerges out of a hydrodynamic instability. The universality class of a dipole-conserving fluid can be realized in any medium with exact or emergent dipole and momentum conservation. While we are not aware of a current experimental platform exhibiting these conservation laws, we outline possible strategies to building them in what follows. Independently of experiment, it is possible to realize these conservation laws in classical or quantum dynamics which can be simulated numerically. We have numerically simulated chaotic classical dynamics with dipole and momentum conservation in one and two dimensions, and find evidence for the DMC fixed point. Our work establishes an unexpected and profound connection between non-equilibrium statistical mechanics and fracton phases of matter.

*Microscopic Models.*—Given the seemingly abstract nature of how fluids might simultaneously have both dipole and momentum conservation, before diving into the theoretical framework of fluid dynamics, it is instructive to first describe a microscopic model which would lie in such a universality class. For simplicity in the discussion which follows, we focus on one dimensional systems. Let us begin by considering  $N$  particles with momenta  $p_i$  and displacements  $x_i$  ( $i = 1, \dots, N$ ), coupled together by the following Hamiltonian:

$$H = \sum_{n=1}^{N-1} \frac{(p_n - p_{n+1})^2}{2} + V(x_n - x_{n+1}). \quad (1)$$

Here  $V(x) = V_2 x^2 + V_3 x^3 + \dots$  is a generic polynomial. This is qualitatively similar to a simple model of one-dimensional solids with anharmonicity, except for the kinetic energy, which depends on only the difference of momenta. Somewhat similar models have arisen in the “dipole fermion” picture of fractional quantum Hall states [55, 56].

This curious kinetic energy is all that we need to have an emergent dipole conservation. Using the Poisson brackets  $\{x_n, p_m\} = \delta_{nm}$ , we find that the dipole moment  $D$ , “charge”  $Q$  and momentum  $P$ , given by  $Q = \sum_{n=1}^N 1$ ,  $D = \sum_{n=1}^N x_n$  and  $P = \sum_{n=1}^N p_n$  obey the classical multipole algebra [57]

$$\{D, P\} = Q, \quad (2)$$

with all the other Poisson brackets vanishing. Since  $Q$ ,  $D$  and  $P$  commute with the Hamiltonian, we have conservation of charge, dipole and momentum all at once, in a spatially local theory. Note that energy is also conserved if the dynamics is Hamiltonian. But it is straightforward to modify the dynamics to no longer conserve energy.

If we assume that the dynamics is close to equilibrium (i.e. displacements are small), then we can safely set (without loss of generality)  $V(x) = \frac{1}{2}x^2$ . A

simple calculation reveals that the normal modes of this quadratic and integrable system are of the form  $(x_n, p_n) \propto e^{ikn - i\omega_k t}$ , and that when  $k \ll 1$ ,  $\omega_k \approx \pm k^2$ . As promised in the introduction, we have uncovered a magnon-like dissipationless dispersion relation, which we will show later is universal and follows entirely from simultaneous dipole and momentum conservation, even when we include higher order terms (which destroy integrability) in  $V(x)$ .

As we have found a magnon-like dispersion relation, it is tempting to push the analogy between dipole conserving fluids, and isotropic ferromagnets, a little further. Consider the isotropic Heisenberg ferromagnet

$$H = - \sum_{n=1}^{N-1} \mathbf{S}_n \cdot \mathbf{S}_{n+1}, \quad (3)$$

where  $\mathbf{S}_n = (S_n^x, S_n^y, S_n^z)$  and  $\{S_n^i, S_m^j\} = \epsilon^{ijk} S_n^k \delta_{nm}$ . Imposing the constraint  $S_n^x S_n^x + S_n^y S_n^y + S_n^z S_n^z = 1$ , and perturbing around the minimal energy configuration  $S_n^z = 1$ , we observe that if we identify charge  $Q$ , dipole  $D$  and momentum  $P$  with the total  $z$ ,  $x$  and  $y$  components of spin, the spin algebra is equivalent to (2). Moreover, Taylor expanding  $H$  to leading order in small  $S_n^{x,y}$ , we observe that up to a global constant,  $H$  is approximately given by (1) with  $V(x) = \frac{1}{2}x^2$ .

Remarkably, we see that the isotropic ferromagnet has an *approximate* dipole and momentum conservation close to equilibrium. However, nonlinearities in the ferromagnet do not preserve  $S_n^z = 1$ , and so the nonlinear theories differ [58, 59]. Despite this nonlinear discrepancy, we note that nonlinearities are known to be strongly relevant below six dimensions in the Heisenberg ferromagnet [60]. In the exact dipole-conserving theory, we will show that the upper critical dimension is four. It would be interesting to investigate whether a ferromagnet can be modified by realizable interactions to better stabilize the dipole-conserving hydrodynamics.

*Hydrodynamics.*—We now use canonical arguments, based on the second law of thermodynamics, to derive the hydrodynamics of conserved momentum, charge, and dipole. The fundamental assumption of hydrodynamics is that the late time physics is governed locally by the independent quantities of the system, which we write as

$$P^i = \int d^d x \pi^i, \quad Q = \int d^d x n \quad (4)$$

where  $\pi^i$  and  $n$  are the momentum and charge density, respectively.  $ij$  indices in what follows run over spatial dimensions  $i = 1, \dots, d$ , and repeated indices are summed over. The dynamics of the densities  $n$  and  $\pi^i$  is given by the local conservation laws:

$$\partial_t \pi^i + \partial_j T^{ji} = 0, \quad \partial_t n + \partial_i J^i = 0, \quad (5)$$

where  $T^{ij}$  and  $J^i$  are stress and charge flux, and are assumed to be local expressions of  $\pi^i, n$ . Crucially, we also need to demand

$$J^i = \partial_j J^{ji}, \quad (6)$$

which comes from dipole conservation:

$$\partial_t \int d^d x x^i n = - \int d^d x x^i \partial_j J^j = \int d^d x J^i, \quad (7)$$

where the right-hand side vanishes only if  $J^i$  satisfies (6). We will also demand that  $J^{ij}$ , which can be interpreted as a dipole flux tensor, be local in the densities.

We have not included the dipole density as a separate degree of freedom. Indeed, let us decompose the dipole charge as:  $D^i = D_0^i + S^i$ , where  $D_0^i = \int d^d x x^i n$  is the “orbital” component and  $S^i$  a remainder, corresponding to a density of microscopic dipoles. In general, we only expect the sum  $D^i$  to be conserved, and not each component separately. Therefore, the  $S^i$  charge will typically relax into the local density  $x^i n$ , and dipole density is not a separate hydrodynamic degree of freedom [61]. This is analogous to the reason why a fluid with angular momentum conservation does not have a new hydrodynamic mode associated to angular momentum density.

Upon specifying the explicit dependence of  $T^{ij}$  and  $J^i$  on  $\pi^i$  and  $n$ , (5) and (6) will completely specify the time evolution of  $\pi^i$  and  $n$ . To find such explicit dependence, we shall write down the most general expressions of  $T^{ij}$  and  $J^i$  in terms of  $\pi^i$  and  $n$  following a derivative expansion, and then impose that the dynamics be consistent with the local second law of thermodynamics. This amounts at finding a vector  $J_S^i$  such that

$$\partial_t s + \partial_i J_S^i \geq 0 \quad (8)$$

when evaluated on solutions to hydrodynamics, where  $s$  is the thermodynamic entropy density. This basic constraint will uniquely determine the concrete expressions of  $T^{ij}$ ,  $J^i$  in terms of  $\pi^i$ ,  $n$ , order by order in derivatives, up to phenomenological coefficients that are determined by the specific underlying system.

Let us first review what happens in the absence of momentum conservation [35]. We solve for the dynamics of  $n$  alone. Eq. (6) and the fluctuation-dissipation theorem demand that  $J_{ij} = D_1(\rho) \partial_i \partial_j n + D_2(\rho) \delta_{ij} \partial^2 n$ . This starkly contrasts with a conventional diffusive system, where without dipole conservation,  $J_i = -D \partial_i n$ . The continuity equation implies that (within linear response)  $\partial_t n = -(D_1 + D_2)(\partial^2)^2 n$ : density obeys a *subdiffusion equation* with dynamical critical exponent  $z = 4$  relating the scaling of time and space. At leading order  $J_S^i \propto J^i$ .

The thermodynamics and hydrodynamics of dipole/momentum-conserving systems are special: contrary to all cases known to us, the homogeneous part of momentum density decouples from the dynamics. Indeed, we begin by first assuming that the entropy density is a function of momentum and charge densities  $s = s(\pi^i, n)$ , as in conventional thermodynamics. We now show that this breaks (6). Recall the thermodynamic relation

$$T ds = -V^i d\pi^i - \mu dn, \quad (9)$$

where  $V^i$  and  $\mu$  are the velocity and chemical potential of the system. We recall that we are assuming absence of

energy conservation, so we will take  $T$  to be a constant set by noise in this discussion, and will study hydrodynamics with energy conservation in a technical companion paper [62]. Combining this thermodynamic relation with the fact that, in non-dissipative hydrodynamics, entropy is locally conserved, we arrive at

$$T(\partial_t s + \partial_i J_S^i) = -V^i(\partial_t \pi^i + \partial_j T^{ji}) - \mu(\partial_t n + \partial_i J^i). \quad (10)$$

The most general expressions for the currents are  $J_S^i = s_1 V^i$ ,  $T^{ij} = p \delta^{ij} + h_1 \pi^j V^i$ ,  $J^i = h_2 V^i$ , where  $s_1, p, h_1, h_2$  are functions of  $\pi^i$  and  $n$ . Plugging these expressions in (10) gives  $s_1 = s - p$ ,  $p = Ts + \mu n + V^i \pi^i$ ,  $h_1 V^i = \pi^i$  and  $h_2 = n$ , which are just the standard constitutive relations of the hydrodynamics of a charged fluid: indeed, we have not used anywhere the fact that we are dealing with a dipole-conserving fluid. In particular, these results together with (6) lead to the relation

$$n V^i = \partial_j J^{ji}, \quad (11)$$

which would naively imply that  $J^{ij}$  is non-local in the hydrodynamic variables, thus violating our basic assumptions.

The only way for (11) to be consistent with locality, therefore, is to demand that the velocity  $V^i$  is itself a total derivative (divided by  $n$ ). Since  $V^i$  is defined as the chemical potential of  $\pi^i$ , such requirement is only possible if the entropy density has the following dependence on  $\pi^i$ :

$$s = s(\partial_i v_j, n), \quad v_i \equiv \frac{\pi^i}{n}. \quad (12)$$

Again demanding (10), we find

$$T^{ij} = p \delta^{ij} + V^i \pi^j - \psi_{ik} \partial_j v_k, \quad (13a)$$

$$J^{ij} = \psi_{ij}, \quad (13b)$$

where the velocity and the thermodynamic pressure are

$$V^i = \frac{1}{n} \partial_j \psi_{ji}, \quad p = Ts - nT \frac{\partial s}{\partial n}, \quad (14)$$

and we have defined the quantity

$$\psi_{ij} \equiv T \frac{\partial s}{\partial (\partial_i v_j)} \Big|_n. \quad (15)$$

Unlike ordinary fluids, the velocity is a higher-derivative expression of momentum density. This is the only way to reconcile (11) with locality. In a rotationally invariant theory, we find that  $T_{ij}$  is symmetric up to total derivatives, consistent with conservation of angular momentum. An explicit derivation of these facts is provided in the Supplementary Material (SM).

Note that the entropy density  $s$  as well as equations of motion are invariant under the shift

$$\pi^i \rightarrow \pi^i + n c^i, \quad T^{ij} \rightarrow T^{ij} + J^j c^i, \quad (16)$$

where  $c^i$  is a constant vector. This invariance is a manifestation of the dipole algebra (2). Indeed, (2) implies,

using locality:  $\{D^i, \pi^j\} = n\delta^{ij}$ , which is equivalent to the symmetry (16). In fact, using (16) as the only input, one immediately infers (12), which in turn imply (13a)-(15) and in particular (6). This symmetry-based approach confirms that the hydrodynamics (13a)-(15) is valid for arbitrary strongly-coupled systems and thus universal.

It is straightforward to derive first order dissipative corrections to hydrodynamics. We do this calculation in the SM, and now summarize the results. We find that  $J^{ij} = J_{(0)}^{ij} + J_{(1)}^{ij}$  and  $T^{ij} = T_{(0)}^{ij} + T_{(1)}^{ij}$ , where  $J_{(0)}^{ij}$  and  $T_{(0)}^{ij}$  correspond to the ideal hydrodynamic results derived above, and

$$-T_{(1)}^{ij} = \eta^{ijkl} \partial_k V_l + \alpha^{ijkl} \partial_k \partial_l \mu \quad (17a)$$

$$J_{(1)}^{ij} = \kappa^{ijkl} \partial_k V_l + C^{ijkl} \partial_k \partial_l \mu. \quad (17b)$$

The tensor structures are detailed in the SM, and are similar to shear and bulk viscosities of an isotropic fluid.

To complete our hydrodynamic description, we finally add the effect of thermal fluctuations. Generalizing the standard fluctuation-dissipation theorem [1], we add noise to the currents:  $T^{ij} \rightarrow T^{ij} + \tau^{ij}$  and  $J^{ij} \rightarrow J^{ij} + \xi^{ij}$ , where the variance is determined by the dissipative coefficients of (17):  $\langle \tau^{ij}(t, x) \tau^{kl}(0, 0) \rangle \propto T \delta(t) \delta^{(d)}(x)$ , and similarly for the two-point function of  $\xi^{ij}$  and its product with  $\tau^{ij}$ . More details can be found in the SM.

In the SM, we derive the propagating hydrodynamic modes of this theory: namely, we look for solutions to the hydrodynamic equations in which  $n, v_i \propto e^{ikx - i\omega t}$ . We find a magnon-like “sound” mode with dispersion relation [60, 63]  $\omega = \pm ck^2 - i\gamma k^4$ , and  $d - 1$  subdiffusive modes for transverse momentum with dispersion relation  $\omega = -i\gamma' k^4$ . Explicit expressions for  $c, \gamma, \gamma'$  are not illuminating and are provided in the SM. Note that the qualitative structure of these quasinormal modes matches those of an ordinary fluid, except that each power of wave number  $k$  is doubled.

*Instability of Hydrodynamics.*— In fact, the true dispersion relations differ from those we found from linear response above. Relevant nonlinearities couple to thermal fluctuations and lead to anomalous scaling, severely affecting the long-time behavior of general dipole-conserving hydrodynamics. For simplicity, we shall present the explicit nonlinearities only in one dimension; the higher dimensional counterpart is qualitatively similar and can be found in the SM. We consider perturbations of an equilibrium fluid at rest where  $v_x$  and  $\bar{n} = n - n_0$  are regarded as small:

$$\partial_t v_x + \frac{1}{\chi} \partial_x \bar{n} + \lambda \bar{n} \partial_x \bar{n} + \lambda' \bar{n}^2 \partial_x \bar{n} + \tilde{\Gamma} \partial_x^4 v_x + \frac{1}{n_0} \partial_x \tau = 0, \quad (18a)$$

$$\partial_t \bar{n} - a T \partial_x^3 v_x + \frac{C}{\chi} \partial_x^4 \bar{n} + \partial_x^2 \xi = 0. \quad (18b)$$

where  $\tilde{\Gamma} = \frac{aT}{n_0} \Gamma$ , and we dropped dissipative/nonlinear terms which are not important for what follows. We in-

cluded stochastic fluctuations in the equations. The values of constants  $\lambda, \lambda', \chi, a, \tilde{\Gamma}$ , and  $C$  do not depend on  $v_x$  or  $\bar{n}$ .

At mean-field level, the dissipative scaling is  $\omega \sim k^4$ , while the noise scales as  $\tau, \eta \sim k^{\frac{d+4}{2}}$ . Starting from the linearized theory and assuming that  $\chi, \tilde{\Gamma}$  etc. are scale ( $k$ ) independent, we find  $\pi^i \sim v^i \sim k^{\frac{d-2}{2}}$  and  $n \sim k^{\frac{d}{2}}$ . As per the usual renormalization group analysis, the nonlinearity  $\lambda$  must scale as  $k^{\frac{d-4}{2}}$ , making it *relevant* when  $d < 4$ ;  $\lambda'$  scales as  $k^{d-2}$  and is relevant when  $d < 2$ . As a consequence, we can anticipate anomalous dissipative scaling: the magnon-like sound mode will have dispersion relation  $\omega \sim k^2 - ik^z$ , with  $z < 4$ . We crudely estimate  $z$  by assuming that, even after fluctuations are accounted for, the scaling of the densities does not renormalize. Assuming  $\lambda \neq 0$  and balancing the time derivative with nonlinearities  $\partial_t \pi^i \sim \nabla(\bar{n})^2, \nabla(\nabla \pi)^2$  leads to  $k^{z+\frac{d-2}{2}} \sim k^{d+1}$ , or  $z \sim d/2 + 2$ . In one dimension, this gives  $z \sim 2.5$ , and in two dimensions  $z \sim 3$ .

*Numerical Simulations.*— Having predicted both the exotic dissipative hydrodynamics of a dipole-conserving theory, together with its breakdown due to thermal fluctuations, we now describe two models which we have used to numerically test our predictions.

**Model A:** Our first model starts with Hamiltonian (1), with

$$V(x) = \frac{1}{2} x^2 + k_3 x^3 + k_4 x^4. \quad (19)$$

Note that Model A has energy conservation, and strictly speaking our hydrodynamic derivation above does not. Energy conservation changes the universality class and critical exponents of hydrodynamics [62], and so strictly speaking, model A does not lie in the universality class predicted above. However, we empirically find that the coupling with energy density is negligible within numerical accuracy, and model A is effectively described by the hydrodynamics above. In the SM, we study a stochastic version of model A with noise and dissipation, which is not energy conserving and is predicted to lie in our universality class. Further simulations, discussion of the role of energy conservation, and extensions of this model to higher dimensions, are also described there.

**Model B:** Our second model corresponds to

$$H = \sum_{i=1}^N (-\cos p_i - F x_i) + V(x), \quad (20)$$

with  $V(x)$  given in (19). Note that this model does *not* have explicit dipole conservation, nor momentum conservation. However, analogous to the emergence of dipole conservation out of energy conservation in systems placed in strong tilt fields [35, 41, 64], we predict the emergence of dipole *and* momentum conserving hydrodynamics in this model. Indeed, in the SM, we explicitly show that the linearized equations in model B exhibit fast Bloch oscillations superimposed on top of magnon-like hydrody-



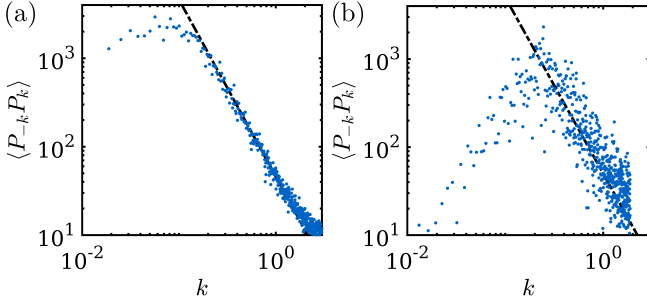


FIG. 1. Distribution function of  $\langle p_{-k}(t)p_k(t) \rangle$  correlations at late times for (a) model A and (b) model B. The dashed line is the theoretical  $k^{-2}$  prediction.

namic modes. Model B is inspired by one possible experimental realization of dipole-conserving hydrodynamics, in which ultracold fermionic atoms are placed in a tilted optical lattice [41]. The  $\cos p_i$  kinetic energy arises from the finite bandwidth of a lattice model, and the  $Fx_i$  force field comes from the tilt. In order for this realization to be appropriate, it is important for umklapp scattering to be suppressed and for momentum to be approximately conserved.

We now present large scale simulations for each model. A “thermodynamic” check for dipole-conserving hydrodynamics is to look at the equilibrium fluctuations of the momentum density, which are proportional to the momentum susceptibility: using (11) and (14),

$$\chi_{PP} = \frac{\pi_i}{V_i} \propto k^{-2}. \quad (21)$$

As  $k \rightarrow 0$ , we predict a clear divergence in the equal time correlation functions  $\langle p_k(t)p_{-k}(t) \rangle$ , with  $\langle \dots \rangle$  an average over times and/or initial conditions; here  $p_k$  denotes the discrete Fourier transform of  $p_i$ . Figure 1 demonstrates this divergence is present in both models.

Next, we study the thermalization time scale of each model. Choosing random initial conditions, higher-momenta modes will thermalize first, so that  $\langle p_{-k}(t)p_k(t) \rangle$  will develop a maximum at some momentum  $k_*$ . For a dispersion relation with  $\text{Im}\omega \propto k^z$ , the thermalization time of a mode with momentum  $k$  scales as  $k^{-z}$ , which will yield

$$k_*(t) \propto t^{-1/z}; \quad (22)$$

note that  $z = 4$  in linearized hydrodynamics. Figure 2 shows numerical simulations in both models, which demonstrate that  $z \approx 4$  when  $k_3 = 0$ , but that  $z \approx 2.5$  when  $k_3 \neq 0$ . Crucially, we observe both (i) a large scale deviation from  $z = 4$ , which is compatible with our crude estimates for  $z$  at the non-equilibrium fixed point, and (ii) much weaker instability (in fact, we did not numerically detect one unambiguously) when  $k_3 = 0$ , which is a consequence of  $\lambda \approx 0$  in this model in (18). Moreover, our data exhibits the strongest scaling collapse when  $z \approx 2.5$  (when  $k_3 \neq 0$ ). This constitutes strong numerical evidence for the existence of the same non-equilibrium fixed

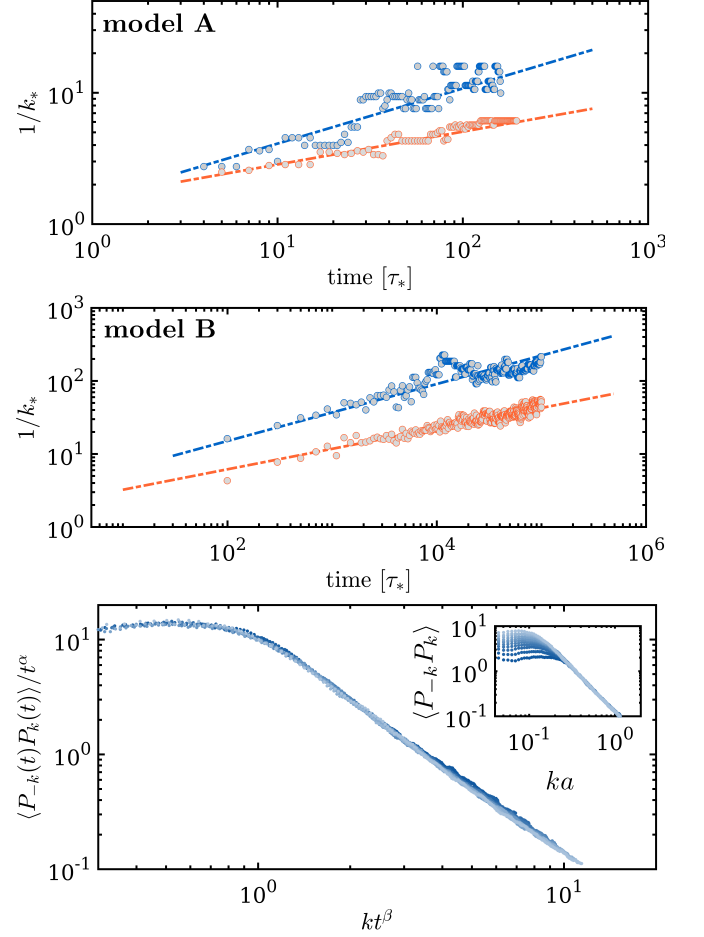


FIG. 2. Temporal dependence of  $1/k_*$  showing anomalous scaling in model A (top) and model B (middle). Shown with blue (orange) is the case when  $k_3 \neq 0$  ( $k_3 = 0$ ). We estimate  $1/z = 0.40 \pm 0.03$  when  $k_3 \neq 0$  and  $1/z = 0.26 \pm 0.01$  when  $k_3 = 0$  in model A;  $1/z = 0.41 \pm 0.02$  when  $k_3 \neq 0$  and  $1/z = 0.27 \pm 0.02$  when  $k_3 = 0$  in model B. In model A,  $k_3 = 3 \cdot 10^3$  when nonzero,  $k_4 = 2 \cdot 10^3$ ,  $N = 10^3$ , averaging over 50 different initial conditions. In model B,  $F = 50$ ,  $k_3 = 3 \cdot 10^3$  when nonzero,  $k_4 = 1.8 \cdot 10^6$ ,  $N = 10^4$ , with one random realization shown. The characteristic timescale  $\tau_*$  is  $\tau_* = 2000/\varepsilon_*$  in the upper panel, whereas  $\tau_* = 1/\varepsilon_*$  in the lower panel, with  $\varepsilon_*$  the energy density. **Bottom:** Scaling collapse of the equal-time momentum correlation function in model A for  $1/z \approx 0.4$ . Shown in the inset is the raw equal-time momentum correlation function without rescaling.

point, arising both in Models A and B. Remarkably, the observed value of  $z$  is extremely close to our simplistic estimate of 2.5.

In Model A, we have also studied the fate of the magnon-like sound mode by studying the Fourier transform of unequal time correlation functions  $\langle p_{-k}(0)p_k(t) \rangle$ . Figure 3 shows that this correlation function is sharply peaked near  $\omega = ck^2$ , with a decay rate consistent with  $z \approx 2.5$ . This suggests that the real part of the dispersion relation remains quadratic at the dipole-conserving

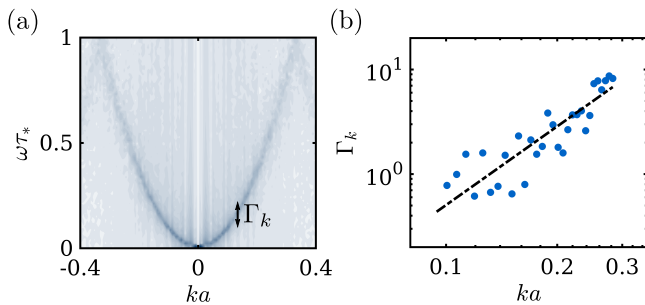


FIG. 3. (a) Absolute value of the temporal Fourier transform of  $\langle P_{-k}(0)P_k(t) \rangle$  showing the quadratic dispersion of a propagating mode in model A. (b) Linewidth of the quadratic excitations in panel (a) as a function of momentum. Shown with dashed-dotted lines is the  $k^{2.5}$  fit.

fixed point, and is consistent with our assumption that the densities do not pick up anomalous exponents at this new fixed point.

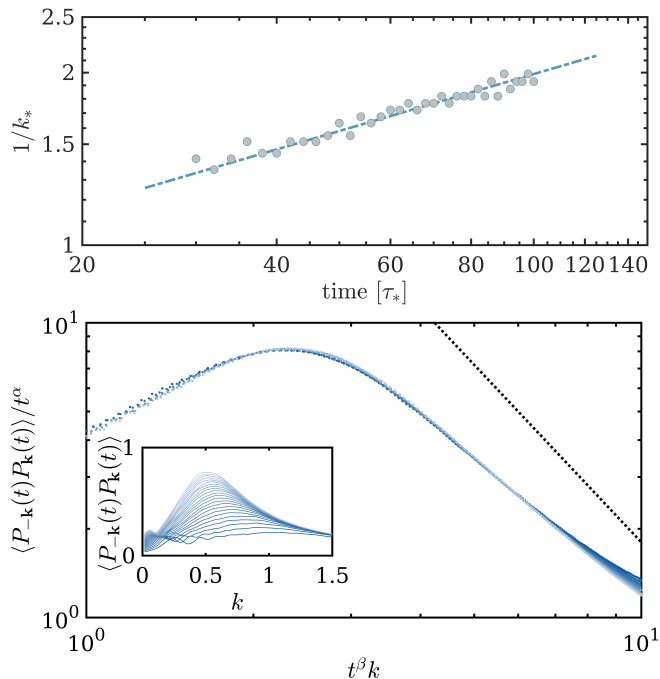


FIG. 4. Temporal dependence of  $1/k_*$  showing anomalous scaling in the two-dimensional version of model A (**top**) and scaling collapse of the equal-time total momentum correlation function (**bottom**). Shown in the inset is the raw equal-time momentum correlation function without rescaling. The dashed line is the theoretical  $k^{-2}$  prediction. We estimate  $1/z = 0.33 \pm 0.03$ . The characteristic timescale  $\tau_*$  is  $\tau_* = 100/\varepsilon_*$ , with  $\varepsilon_*$  the energy density. Simulations occurred on a  $400 \times 400$  grid with  $k_3 = 3$ ,  $k_4 = 4$ , averaged over 100 realizations.

We have also simulated Model A in  $d = 2$ . Figure 4 shows a scaling collapse consistent with  $z \sim 3$ , again very close to the estimated exponent predicted in the

previous section, and clearly distinct from the Gaussian prediction ( $z = 4$ ). This is the first instance known to us where hydrodynamics is strongly unstable to fluctuations in dimension  $d > 1$ . Finally, Figure 4 also indicates consistency with the divergent behavior of  $\chi_{PP}$  predicted by the thermodynamic relation (21).

We remark that the numerical detection of anomalous transport scaling can be sometimes quite subtle. For example, it has long been known that energy transport in one-dimensional “standard” (non-fractonic) hydrodynamics is anomalous [65]; however, this statement has been subject to relatively recent debates, where certain models were observed to display ordinary diffusion [66, 67]. While the general consensus is now that energy transport in these systems is always anomalous [68], these works instruct us that unambiguous determinations of non-equilibrium critical exponents may be quite non-trivial. The relatively weak mixing of energy density with momentum and charge densities in our simulations is consistent with the fact that our model A appears to access the DMC fixed point at numerically accessible time scales, even without any noise.

*Outlook.*—We have discovered a new universality class which is undriven, yet out-of-equilibrium. Our construction was inspired by the physics of fractons, which were originally devised [12] to protect quantum information against thermalization, but have since revealed deep connections between quantum information, condensed matter physics, quantum field theory, and (due to this work) non-equilibrium statistical mechanics.

Our discovery of hydrodynamics with both momentum and dipole conservation may also provide a key path towards understanding whether it is possible to couple fracton matter to gravity (a spacetime metric) – and if so, how [69]: the natural language for describing momentum-conserving hydrodynamics as an effective theory involves coupling to background metrics [2–4]. Moreover, performing a renormalization group analysis in fracton systems is still a technical and conceptual challenge [70–72]. This paper provides a rare example where such an analysis can be performed, and may shed light into field theories of other fractonic phases of matter.

Although we have focused on the hydrodynamics of a dipole-conserving fluid here, we anticipate infinitely many additional non-equilibrium universality classes arising from the consideration of higher multipole conservation laws [35], subsystem symmetries [35, 73, 74], or explicit/spontaneous symmetry breaking. We look forward to the systematic classification of fracton-inspired non-equilibrium universality classes, and hope for their ultimate discovery in experiment.

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*Note Added.*—While preparing this work, the preprint [75], which discusses non-dissipative response in a fluid

with conserved dipole and momentum, was posted.

*Author Contributions.*—PG and AL developed the theoretical framework and classical models. JFRN and JG performed the numerical simulations and statistical analysis of the data, and prepared the figures. PG and AL wrote the manuscript with input from all authors.

*Conflicts of Interest.*—The authors declare no conflict of interest.

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# Supplemental Material for Breakdown of hydrodynamics below four dimensions in a fracton fluid

## Appendix A: Zeroth order hydrodynamics

In this appendix, we derive the zeroth order hydrodynamics. Our starting point is (10). First collecting the terms with time derivatives, we find

$$\begin{aligned} T\partial_t s &= -V^i \partial_t \pi^i - \mu \partial_t n \\ T \left( \frac{\partial s}{\partial n} \partial_t n + \frac{\partial s}{\partial(\partial_k v^l)} \partial_t \partial_k v_l \right) &= -V^i \partial_t (n v^i) - \mu \partial_t n \\ \left( T \frac{\partial s}{\partial n} + V^i v^i + \mu \right) \partial_t n &+ \psi_{kl} \partial_t \partial_k v_l + n V^i \partial_t v^i = 0. \end{aligned} \quad (\text{A1})$$

Since the first term must vanish, we conclude that the chemical potential is given by

$$\mu = -T \frac{\partial s}{\partial n} - V^i v^i. \quad (\text{A2})$$

For the second term, we integrate by parts to find

$$\partial_k (\psi_{kl} \partial_t v_l) + (n V^i - \partial_k \psi_{kl}) \partial_t v^i = 0. \quad (\text{A3})$$

Setting the second term to zero implies the first equation in (14).

The remaining divergence term can be absorbed into the entropy current, and indeed we now turn to the evaluation of the entropy current by studying the spatial derivative terms in (10):

$$\begin{aligned} T \partial_i J_S^i + \partial_k (\psi_{kl} \partial_t v_l) &= -V^i \partial_j T^{ji} - \mu \partial_i J^i \\ &= -V^i \partial_j T^{ji} + \left( T \frac{\partial s}{\partial n} + V^k v^k \right) \partial_i (n V^i + \tilde{J}^i). \end{aligned} \quad (\text{A4})$$

In the second line above, we have plugged in the ansatz  $J^i = n V^i + \tilde{J}^i$ . Ultimately we will show that  $\tilde{J}^i = 0$  at ideal fluid level, so in the equations that follow we will drop this term to avoid clutter. With a bit of hindsight, we claim that

$$J_S^i = n \frac{\partial s}{\partial n} V^i - \psi_{ij} \partial_t \psi_j. \quad (\text{A5})$$

The latter term clearly cancels out a term in the first line of (A4); as for the former, observe that if we plug (A5) into (A4), we find

$$\begin{aligned} -V^i \partial_j T^{ji} + V^k v^k \partial_i (n V^i) &= T n V^i \partial_i \frac{\partial s}{\partial n} \\ &= T V^i \partial_i \left( n \frac{\partial s}{\partial n} \right) - T V^i \frac{\partial s}{\partial n} \partial_i n \\ &= T V^i \partial_i \left( n \frac{\partial s}{\partial n} - s \right) + T V^i \frac{\partial s}{\partial \partial_k v^l} \partial_i \partial_k v^l \end{aligned}$$

$$\begin{aligned} &= T V^i \partial_i \left( n \frac{\partial s}{\partial n} - s \right) + V^i \partial_k (\psi_{kl} \partial_i v^l) \\ &\quad - n V^i V^l \partial_i v^l. \end{aligned} \quad (\text{A6})$$

This equation is satisfied if

$$T^{ji} = T \left( s - n \frac{\partial s}{\partial n} \right) \delta^{ji} + n V^j v^i - \psi_{jk} \partial_i v_k. \quad (\text{A7})$$

Let us confirm that this ideal stress tensor  $T^{ij}$  is symmetric in the indices  $i, j$  up to total derivatives. To this aim, consider a constant antisymmetric matrix  $\Omega_{ij}$  and write

$$\begin{aligned} &\int d^d x T^{ij} \Omega_{ij} \\ &= -T \int d^d x \left( \frac{\partial s}{\partial(\partial_k v_i)} \partial_k v_j + \frac{\partial s}{\partial(\partial_i v_k)} \partial_j v_k \right) \Omega_{ij} \\ &= -T \int d^d x \frac{\partial s}{\partial(\partial_i v_j)} \delta(\partial_i v_j) \end{aligned} \quad (\text{A8})$$

where  $\delta(\partial_i v_j) = (\Omega_{jk} \partial_i v_k + \Omega_{ik} \partial_k v_j)$  is the variation of  $\partial_i v_j$  with respect to the infinitesimal rotation  $x^i \rightarrow x^i + \Omega^{ij} x^j$ . Since the entropy density  $s$  is invariant under rotations, the above expression vanishes. Hence,  $T_{ij}$  is symmetric, up to total derivative contributions.

## Appendix B: Dissipative hydrodynamics

Introduce higher-derivative corrections to the currents:  $T^{ij} = T_{(0)}^{ij} + T_{(1)}^{ij}$ ,  $J^{ij} = J_{(0)}^{ij} + J_{(1)}^{ij}$ , and  $J^i = J_{S(0)}^i + J_{S(1)}^i$  where the quantities with subscript (1) denote higher-derivative corrections. Using the conservation laws (5) together with thermodynamic relations (14) and (A2), expand

$$\begin{aligned} T \partial_t s &= T \frac{\partial s}{\partial n} \partial_t n + \psi_{ij} \partial_t \partial_i v_j \\ &= \mu \partial_i \partial_j J^{ji} + V^j \partial_i T^{ij} + \partial_i (\psi_{ij} \partial_t v_j) \\ &= \partial_i (\mu \partial_j J^{ji} - J^{ij} \partial_j \mu + V^j T^{ij} \psi_{ij} \partial_t v_j) \\ &\quad + J^{ij} \partial_i \partial_j \mu - T^{ij} \partial_i V_j. \end{aligned} \quad (\text{B1})$$

Eq. (8) then gives

$$\begin{aligned} 0 &\leq \partial_i (J_{S(1)}^i + \mu \partial_j J_{(1)}^{ji} + V^j T_{(1)}^{ij} \psi_{ij} (\partial_t v_j)_{(1)} - J_{(1)}^{ij} \partial_j \mu) \\ &\quad + J_{(1)}^{ij} \partial_i \partial_j \mu - T_{(1)}^{ij} \partial_i V_j, \end{aligned} \quad (\text{B2})$$

where the leading derivative terms cancel out due to (10), and where  $(\partial_t v_j)_{(1)}$  denotes the dissipative contribution to  $\partial_t v_j$ . The most general expressions for the dissipative stress tensor and charge current are

$$-T_{(1)}^{ij} = \eta_1^{ijkl} \partial_k \partial_l \partial_p v_q + \alpha^{ijkl} \partial_k \partial_l \mu \quad (\text{B3a})$$

$$J_{(1)}^{ij} = \kappa_1^{ijkql} \partial_k \partial_l \partial_p v_q + C^{ijkl} \partial_k \partial_l \mu, \quad (\text{B3b})$$

where  $\eta_1^{ijkql}, \dots$  are regarded as constant tensors, and where we only kept terms that are linear in  $v_i$  and  $\mu$  as these are sufficient to capture the anomalous scalings: any nonlinear dissipative term will be irrelevant. The tensor  $\eta_1^{ijkql}$  has the general decomposition

$$\eta_1^{ij(kqp)l} = b_1 \delta^{ij} \delta^{(kp} \delta^{q)l} + b_2 \delta^{i(k} \delta^{j|p} \delta^{q)l} + b_3 \delta^{jl} \delta^{i(k} \delta^{p)q} + b_4 \delta^{il} \delta^{j(k} \delta^{p)q}, \quad (\text{B4})$$

where  $A^{(k|r_1 r_2 \dots |p q)}$  denotes total symmetrization with respect to the indices  $k, p, q$ . From (B8a) and (C2) below, we also have

$$\begin{aligned} \eta^{ij(k|r|a^{p|r|q)l} = & \left( \zeta - \frac{2}{d} \eta \right) \left( a_1 + 2 \frac{d-1}{d} a_2 \right) \delta^{ij} \delta^{(kp} \delta^{q)l} \\ & + 2\eta \left( a_1 + \frac{d-2}{d} a_2 - a_3 \right) \delta^{i(k} \delta^{j|p} \delta^{q)l} \\ & + (\eta + \bar{\eta})(a_2 + a_3) \delta^{i(k} \delta^{p)q} \delta^{lj} \\ & + (\eta - \bar{\eta})(a_2 + a_3) \delta^{il} \delta^{j(k} \delta^{p)q}. \end{aligned} \quad (\text{B5})$$

Eqs. (B4) and (B5) have the same tensor structure. From this and (C3) we conclude that, by a suitable choice of coefficients, the first term in (B3a) can be written as

$$\eta_1^{ijklpq} \partial_k \partial_l \partial_p v_q = \eta^{ijkl} \partial_k V_l \quad (\text{B6})$$

A similar conclusion applies to the first term in (B3b). This allows us to write the currents in terms of  $V^i$ :

$$-T_{(1)}^{ij} = \eta^{ijkl} \partial_k V_l + \alpha^{ijkl} \partial_k \partial_l \mu \quad (\text{B7a})$$

$$J_{(1)}^{ij} = \kappa^{ijkl} \partial_k V_l + C^{ijkl} \partial_k \partial_l \mu, \quad (\text{B7b})$$

where

$$\eta^{ijkl} = \zeta \delta^{ij} \delta^{kl} + 2\eta \delta^{i<k} \delta^{l>j} + 2\bar{\eta} \delta^{i[k} \delta^{l]j} \quad (\text{B8a})$$

$$\kappa^{ijkl} = \kappa_1 \delta^{ij} \delta^{kl} + 2\kappa_2 \delta^{i<k} \delta^{l>j} + 2\kappa_3 \delta^{i[k} \delta^{l]j} \quad (\text{B8b})$$

$$\alpha^{ijkl} = \alpha_1 \delta^{ij} \delta^{kl} + 2\alpha_2 \delta^{i<k} \delta^{j>l} \quad (\text{B8c})$$

$$C^{ijkl} = C_1 \delta^{ij} \delta^{kl} + 2C_2 \delta^{i<k} \delta^{j>l}, \quad (\text{B8d})$$

where  $A^{<ij>} = \frac{1}{2}(A^{ij} + A^{ji} - \frac{1}{d} \delta^{ij} A^{kk})$  and  $A^{[ij]} = \frac{1}{2}(A^{ij} - A^{ji})$  denote traceless symmetrization and anti-symmetrization with respect to indices  $i, j$ , respectively. The coefficients in (B7) can be regarded as a matrix acting on the vector  $(\partial_i V_j, \partial_i \partial_j \mu)$ . Choosing  $J_{S(1)}^i$  so that the total derivative on the right-hand side of (B2) vanishes then implies positive semi-definiteness of such transport matrix. Moreover, Onsager's principle implies that this matrix should be symmetric. The last term in (B8a) would contribute to entropy production through rigidly rotating the fluid as  $dS \sim \bar{\eta}(\partial_i V_j)^2$ ; this cannot happen for rotationally invariant fluids, and we shall set it to zero. Similarly, by coupling the system to a suitable

background higher-rank gauge field, one can infer that  $\kappa_3$  also vanishes. We shall explicitly derive these constraints in [62]. Putting all the constraints together, we then find

$$\begin{aligned} \zeta, \eta, C_1, C_2 &\geq 0, & \alpha_1 &= \kappa_1, & \alpha_2 &= \kappa_2 \\ \alpha_1^2 &\leq \zeta C_1, & \alpha_2^2 &\leq \eta C_2. \end{aligned} \quad (\text{B9})$$

Finally, using the fluctuation-dissipation theorem gives the following two-point functions for the noise introduced in the main text:

$$\langle \tau^{ij}(t, x) \tau^{kl}(0, 0) \rangle = 2T \eta^{ijkl} \delta(t) \delta^{(d)}(x) \quad (\text{B10a})$$

$$\langle \xi^{ij}(t, x) \xi^{kl}(0, 0) \rangle = 2T C^{ijkl} \delta(t) \delta^{(d)}(x) \quad (\text{B10b})$$

$$\langle \tau^{ij}(t, x) \xi^{kl}(0, 0) \rangle = T(\alpha^{ijkl} + \kappa^{klij}) \delta(t) \delta^{(d)}(x) \quad (\text{B10c})$$

### Appendix C: Linearized hydrodynamics

We now analyze the dispersion relations of the linearized hydrodynamics around a homogeneous background charge density  $n_0$ , ignoring the effects of fluctuations. Taking  $n = n_0 + \bar{n}$ , where  $\bar{n}$  and  $v_i$  are regarded as small, expand

$$s = -\frac{1}{2} a^{ijkl} \partial_i v_j \partial_k v_l - \frac{\mu_0}{T} \bar{n} - \frac{1}{2\chi T} \bar{n}^2, \quad (\text{C1})$$

where the tensor  $a^{ijkl} = a^{klij}$  has the following general decomposition:

$$a^{ijkl} = a_1 \delta^{ij} \delta^{kl} + 2a_2 \delta^{i<k} \delta^{l>j} + 2a_3 \delta^{i[k} \delta^{l]j} \quad (\text{C2})$$

In (C1),  $\mu_0 = \mu(n_0)$  is the chemical potential evaluated on the background charge density, and  $\chi = \frac{\partial n}{\partial \mu}$  is charge susceptibility. Thermodynamic stability requires that  $a_1, a_2, a_3, \chi \geq 0$ . The thermodynamic relations in Eqs. (14), (A2) then give

$$\begin{aligned} V^i &= -\frac{T}{n_0^2} a^{ijkl} \partial_j \partial_k \pi_l, & \delta\mu &= \chi^{-1} \bar{n} \\ p &= \mu_0 n_0 + \chi^{-1} n_0 \bar{n} \end{aligned} \quad (\text{C3})$$

where  $\delta\mu = \mu - \mu_0$ , and the ideal part of the currents read

$$T_{(0)}^{ij} = (\mu_0 n_0 + n_0 \chi^{-1} \bar{n}) \delta^{ij}, \quad (\text{C4a})$$

$$J_{(0)}^{ij} = -\frac{T}{n_0} a^{ijkl} \partial_k \pi_l, \quad (\text{C4b})$$

while the dissipative contributions  $T_{(1)}^{ij}$  and  $J_{(1)}^{ij}$  are precisely given by (17). Plugging these in the conservation equations (5) yields

$$\begin{aligned} \partial_t \pi^i + \frac{n_0}{\chi} \partial_i \bar{n} + \frac{aT}{n_0^2} \Gamma_1 \partial^4 \pi_i + \frac{aT}{n_0^2} \Gamma_2 \partial^2 \partial_i \partial_j \pi_j \\ - \frac{A}{\chi} \partial_i \partial^2 \bar{n} = 0 \end{aligned} \quad (\text{C5a})$$

$$\partial_t \bar{n} - \frac{aT}{n_0} \partial^2 \partial_i \pi_i + \frac{C}{\chi} \partial^4 \bar{n} - \frac{aT}{n_0^2} K \partial^4 \partial_i \pi_i = 0 \quad (\text{C5b})$$

where

$$a = a_1 + 2 \frac{d-1}{d} a_2, \quad (\text{C6a})$$

$$C = C_1 + 2 \frac{d-1}{d} C_2, \quad (\text{C6b})$$

$$K = \kappa_1 + 2 \frac{d-1}{d} \kappa_2, \quad (\text{C6c})$$

$$\Gamma_1 = \frac{a_2 + a_3}{a} \eta, \quad (\text{C6d})$$

$$\Gamma_2 = \zeta + \left( 2 \frac{d-1}{d} a - \frac{1}{d^2} a_2 - a_3 \right) \frac{\eta}{a}, \quad (\text{C6e})$$

$$A = \alpha_1 + 2 \frac{d-1}{d} \alpha_2. \quad (\text{C6f})$$

These equations lead to the following dispersion relations:

$$\omega = \pm \sqrt{\frac{aTn_0}{\chi}} k^2 - \frac{i}{2} \left( \frac{C}{\chi} + \frac{aT}{n_0^2} (\Gamma_1 + \Gamma_2) \right) k^4 \quad (\text{C7a})$$

$$\omega = -i \frac{aT}{n_0^2} \Gamma_1 k^4 \quad (\text{C7b})$$

where the two modes in the first line arise from the coupling between  $\bar{n}$  and the longitudinal component of momentum  $\partial_i \pi_i$ , while the modes in the second line have multiplicity  $d-1$  and are associated to the transverse components of momentum  $\partial^{-2}(\pi_i - \partial_i \partial_j \pi_j)$ .

#### Appendix D: Equations of motion with relevant nonlinearities

Here we give explicit expressions for various quantities, valid beyond linear response. The input is the entropy density

$$s = -\frac{1}{2} a^{ijkl} \partial_i v_j \partial_k v_l + \bar{s}(n), \quad (\text{D1})$$

whose form is sufficient to capture all the relevant nonlinearities, and  $\bar{s}(n)$  is a generic function of  $n$ . Using (13a)-(15) we have

$$\begin{aligned} \psi_{ij} &= -T a^{ijkl} \partial_k v_l, \quad n V^i = -T a^{ijkl} \partial_j \partial_k v_l \\ p &= -\frac{T}{2} a^{ijkl} \partial_i v_j \partial_k v_l + \bar{p}(n) \end{aligned} \quad (\text{D2})$$

where  $\bar{p} = T(\bar{s} - n \partial_n \bar{s})$ . The currents are

$$\begin{aligned} T^{ij} &= \left( \bar{p} - \frac{T}{2} a^{klpq} \partial_k v_l \partial_p v_q \right) \delta^{ij} - T a^{pikl} \partial_p \partial_k v_l v^j \\ &\quad + T a^{ikpl} \partial_p v_l \partial_j v_k - \eta^{ijkl} \partial_k V_l - \alpha^{ijkl} \partial_k \partial_l \mu + \tau^{ij} \\ J^{ij} &= -T a^{ijkl} \partial_k v_l + C^{ijkl} \partial_k \partial_l \mu + \alpha^{ijkl} \partial_k V_l + \xi^{ij}, \end{aligned} \quad (\text{D3})$$

where we already accounted for the constraints discussed around (B9). It will be convenient to express the conservation equations in terms of  $v_i = \frac{\pi_i}{n}$ , for which we have

$$\partial_t v^i + \frac{1}{n} (\partial_j T^{ji} - v^i \partial_k \partial_j J^{jk}) = 0. \quad (\text{D4})$$

Plugging in the above expressions and substituting (C2) for  $a^{ijkl}$ , we find

$$\begin{aligned} \partial_t v_i + \frac{1}{n} \partial_i \bar{p} + \frac{2T}{n} (a - a_2 - a_3) \partial_j \partial_k v_k \partial_{[i} v_{j]} \\ + \frac{2T}{n} (a_2 + a_3) \partial_j^2 v_k \partial_{[i} v_{k]} + \frac{aT}{n^2} \Gamma_1 \partial^4 v_i \\ + \frac{aT}{n^2} \Gamma_2 \partial^2 \partial_i \partial_j v_j - \frac{A}{n\chi} \partial_i \partial^2 n + \frac{1}{n} \partial_j \tau^{ji} = 0 \end{aligned} \quad (\text{D5a})$$

$$\partial_t n - aT \partial^2 \partial_i v_i + \frac{C}{\chi} \partial^4 n - \frac{aT}{n} A \partial^4 \partial_i v_i + \partial_i \partial_j \xi^{ij} = 0 \quad (\text{D5b})$$

where we neglected nonlinear dissipative terms, since they are irrelevant.

Expanding

$$\bar{s}(n) = -\frac{\mu_0}{T} \bar{n} - \frac{1}{2\chi T} \bar{n}^2 - \frac{\lambda}{6T} \bar{n}^3 - \frac{\lambda_1}{12T} \bar{n}^4, \quad (\text{D6})$$

we find that the contribution from  $\bar{p}$  in (D5a) is

$$\frac{1}{n} \partial_i \bar{p} = \frac{1}{\chi} \partial_i \bar{n} + \lambda \bar{n} \partial_i \bar{n} + \lambda' \bar{n}^2 \partial_i \bar{n}, \quad (\text{D7})$$

where  $\lambda' = \lambda_1 - \frac{1}{\chi n_0^2}$ .

The evaluation of the dynamical exponent  $z$  is not amenable to perturbation theory. One can verify this by evaluating the Kubo formula for frequency-dependent transport; for example for the viscosity

$$\begin{aligned} \eta(\omega) &\sim \int dt e^{-i\omega t} \int d^d x \langle T^{xx}(t, \vec{x}) T^{xx}(0, 0) \rangle \\ &\sim \int dt e^{-i\omega t} \int d^d x \langle (\bar{n}(t, \vec{x}))^2 (\bar{n}(0, 0))^2 \rangle \\ &\sim \int d^d k dt e^{-i\omega t} e^{-2\gamma k^4 t} \sim \omega^{\frac{d-4}{4}}, \end{aligned} \quad (\text{D8})$$

where we only considered the nonlinearity  $T^{ij} \sim \bar{n}^2 \delta^{ij}$  and assumed factorization of the 4-point function of  $\bar{n}$ . For simplicity we assumed absence of propagating modes, so that  $\langle n(t, k) n(0, -k) \rangle \sim e^{-\gamma k^4 t}$ . We then see that the correction to the mean-field viscosity is divergent as  $\omega \rightarrow 0$ , making perturbation theory ill-defined.



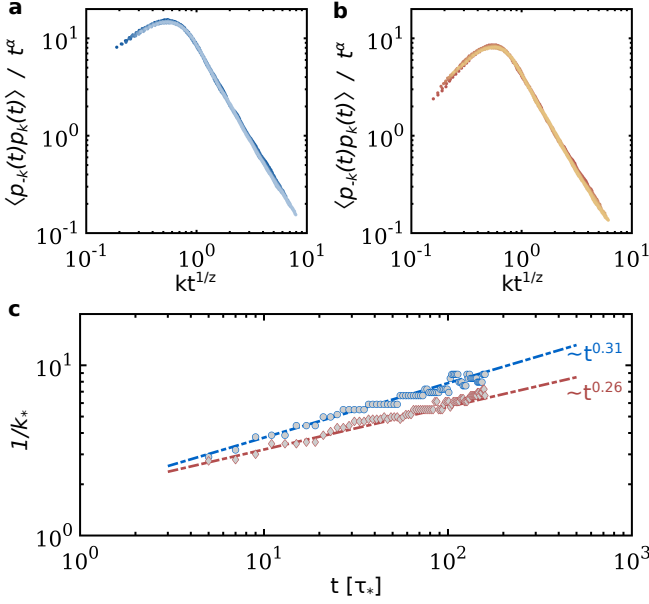


FIG. S2. Scaling of momentum correlation for the stochastic model A. Shown in panels (a-b) are the self-similar collapse of the correlations for (a)  $\nu = 0.1$ , and (b)  $\nu = 1$ . Shown in panel (c) is the position of the peak of the distribution as a function of time for (a) and (b). The linear fitting yields  $1/z = 0.31 \pm 0.02$  for  $\nu = 0.1$ , and  $1/z = 0.26 \pm 0.02$  for  $\nu = 1$ .

### Appendix E: Stochastic Model A

We now overview a stochastic version of model A, given by the equations:

$$\partial_t p_n - V'(x_{n-1} - x_n) + V'(x_n - x_{n+1}) + \nu(2\partial_t x_n - \partial_t x_{n-1} - \partial_t x_{n+1}) + \xi_n - \xi_{n-1} = 0 \quad (\text{E1a})$$

$$\partial_t x_n - (2p_n - p_{n+1} - p_{n-1}) - \nu(2\partial_t p_n - \partial_t p_{n-1} - \partial_t p_{n+1}) + \zeta_n - \zeta_{n-1} = 0 \quad (\text{E1b})$$

where  $\nu$  is a friction coefficient, and the noise correlations are chosen so that the fluctuation-dissipation theorem is satisfied:

$$\langle \xi_n(t) \xi_m(0) \rangle = \langle \zeta_n(t) \zeta_m(0) \rangle = \nu T \delta_{mn} \delta(t). \quad (\text{E2})$$

Setting  $\nu = 0$ , one recovers the Hamiltonian model A. The model above can serve to verify that the scaling exponent  $z$  remains smaller than 4 even after energy conservation is broken, which happens, at sufficiently long times, as soon as  $\nu$  is turned on. Figure S2 shows that, for  $\nu$  not too large the scaling exponent is still anomalous, thus corroborating the fact that  $z < 4$  independently of the existence of a diffusive energy conservation law. We emphasize that in the true thermodynamic limit, theories with any amount of noise are expected to flow to the same DMC fixed point – the fact that our estimate of  $z$  changes with  $\nu$  is a consequence of finite size effects.

### Appendix F: Model A in higher dimensions

Here we describe a generalization of model A in  $d$  dimensions. This is a rather straightforward extension of the one-dimensional case of (1):

$$H = \sum_r \left\{ \sum_{\mu\nu} \frac{(p_\mu(r) - p_\mu(r + e_\nu))^2}{2} + V \left( \sum_\mu (x_\mu(r) - x_\mu(r + e_\mu)) \right) \right\}, \quad (\text{F1})$$

with the Poisson brackets

$$\{p_\mu(r), x_\nu(r')\} = \delta_{\mu,\nu} \delta_{r,r'}. \quad (\text{F2})$$

Here,  $r = (r_1, \dots, r_d)$ , with  $r_1, \dots, r_d = 1, \dots, N - 1$  denotes the lattice site.  $p_\mu$  with  $\mu = 1, \dots, d$  denotes the  $d$ -dimensional particle momentum, and similarly for  $x_\mu$ .  $e_\mu$  represents a unit vector in each direction  $\mu$ . One can easily see that in one-dimension this Hamiltonian reduces to (1). The potential is taken to have the same form as in (19).

Note that this model has a large number of non-dynamical degrees of freedom corresponding to the components of  $p_\mu$  and  $x_\mu$  which cannot be written as gradients of a scalar function on the lattice. We focus on the “longitudinal sector” only when studying correlation functions. It is possible to couple this transverse sector to the longitudinal one, if desired, by multiplying the kinetic terms by a function of  $\sum_\mu (x_\mu(r) - x_\mu(r + e_\mu))$  or by letting the potential depending on the “transverse increments” of  $x_\mu$ :  $V(x_\mu(r) - x_\mu(r + e_\nu))$ .

### Appendix G: Statistical analysis of numerical data for model A in $d = 2$

To find the anomalous scaling exponent  $z$  that best fits the numerical data, we minimize a function that quantifies the dispersion of the collapsed datapoints, as we describe next. We denote by  $k_n$  the values of the radial component of the wavevectors, and by  $t_m$  the timesteps of the simulation, where  $m, n = 0, 1, 2, \dots$ . Fixing the value of  $(\alpha, \beta)$ , we define the auxiliary variables

$$y_{n,m} = t_m^{-\alpha} \langle P_{-k_n} P_{k_n} \rangle(t_m), \quad x_{n,m} = t_m^\beta k_n, \quad (\text{G1})$$

with  $\beta = 1/z$ . For each value of  $t = t_m$ , these functions define an implicit function  $y_m(x)$  which we can approximate by linearly interpolating the datapoints  $(x_{n,m}, y_{n,m})$ . As such, the dispersion function can be defined as

$$\sigma(\alpha, \beta) = \sum_x \sum_{n,m \in I_x} \sum_{n',m' \in I_x} (y_{n,m} - y_{n',m'})^2, \quad (\text{G2})$$

where  $I_x$  are bins centered at discretized positions  $x$  with an appropriately chosen width. The value of  $\sigma(\alpha, \beta)$  for the  $d = 2$  version of model A is plotted in Figure S3, where in particular we observe that the minimum dispersion is obtained for  $\beta = 0.33 \pm 0.03$ .

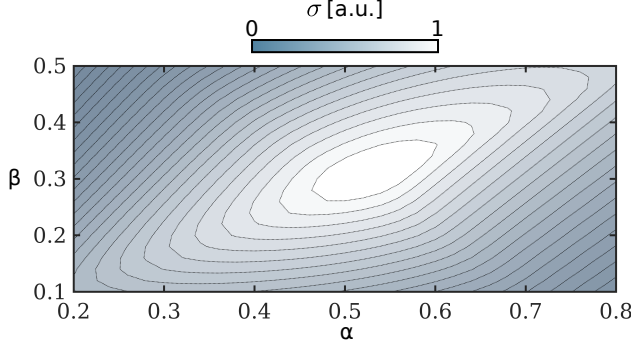


FIG. S3. Contour plot of the dispersion function  $\sigma$  defined in Eq. (G2) as a function of the parameters  $(\alpha, \beta)$ . Lighter colors indicate smaller values.

### Appendix H: Linearized dynamics in model B

Here we show that the linearized equations in model B exhibit fast Bloch oscillations superimposed on top of magnon-like hydrodynamic modes. Consider the Hamiltonian of model B with  $V(x) = \frac{1}{2}x^2$ :

$$H = \sum_{i=1}^N (-\cos p_i - Fx_i) + \frac{1}{2}(x_i - x_{i-1})^2. \quad (\text{H1})$$

We observe that the nonlinear equations of motion:

$$\begin{aligned} \partial_t x_i &= \sin p_i \\ \partial_t p_i &= F + 2x_i - x_{i+1} - x_{i-1}, \end{aligned} \quad (\text{H2})$$

admit the following exact solution corresponding to collective Bloch oscillations:

$$x_i^0(t) = \frac{1 - \cos(Ft)}{F}, \quad (\text{H3a})$$

$$p_i^0(t) = Ft. \quad (\text{H3b})$$

Now consider perturbations around this solution:  $\delta x_i = x_i - x_i^0$  and  $\delta p_i = p_i - p_i^0$ . The linearized equations of motion for the perturbations are

$$\begin{aligned} \partial_t \delta x_k &= \cos Ft \delta p_k, \\ \partial_t \delta p_k &= (2 - 2 \cos k) \delta x_k, \end{aligned} \quad (\text{H4})$$

where  $\delta x_k$  and  $\delta p_k$  are the discrete Fourier transform of  $\delta x_i$  and  $\delta p_i$ . When  $k \ll 1$  (the long wavelength limit), (H4) implies

$$\partial_t^2 \delta p_k = k^2 \cos Ft \delta p_k, \quad (\text{H5})$$

which is the well-studied Mathieu equation.

The general form of the Mathieu equation is an ordinary differential equation with real coefficients:

$$u'' + (a - 2q \cos 2\tau)u = 0. \quad (\text{H6})$$

In our case,  $a = 0$ ,  $\tau = Ft/2$ ,  $q = 2k^2/F^2 \ll 1$ . By Floquet's theorem, (H6) has solutions of the form  $e^{i\nu\tau}\Phi(\tau)$  and  $e^{-i\nu\tau}\Phi(-\tau)$ , where  $\Phi(\tau)$  is periodic with period  $\pi$ . Assuming  $\nu$  is not an integer,  $e^{i\nu\tau}\Phi(\tau)$  and  $e^{-i\nu\tau}\Phi(-\tau)$  are linearly independent, so we can expand  $u(\tau)$  as

$$u(\tau) = A_1 e^{i\nu\tau} \sum_{n=-\infty}^{\infty} c_{2n} e^{2ni\tau} + A_2 e^{-i\nu\tau} \sum_{n=-\infty}^{\infty} c_{2n} e^{-2ni\tau}. \quad (\text{H7})$$

Substituting (H7) in (H6) gives a recurrence relationship for  $c_{2n}$ :

$$\gamma_n(\nu) c_{2n-2} + c_{2n} + \gamma_n(\nu) c_{2n+2} = 0 \quad (\text{H8})$$

where

$$\gamma_n(\nu) = \frac{q}{(2n - \nu)^2 - a} = \frac{2k^2}{F^2(2n - \nu)^2}. \quad (\text{H9})$$

The equations (H8) can be written as a matrix equation  $M_{mn} c_n = 0$ , with repeated indices summed over, and with tridiagonal matrix  $M_{mn} = \delta_{mn} + \delta_{m,n-1} \gamma_m + \delta_{m,n+1} \gamma_m$ . The exponent  $\nu$  is fixed by the condition  $\det(M) = 0$ .

Assuming  $\nu \ll 1$ , we have the relations  $\gamma_0(\nu) \gg \gamma_n(\nu)$  and  $\gamma_n(\nu) \ll 1$  for any  $n \neq 0$ . So  $M$  is now a tridiagonal matrix whose diagonal elements are 1 and all off-diagonal elements other than  $M_{0,1}$  and  $M_{0,-1}$  are perturbatively small. As a result, we can estimate the value of  $\nu$  by only considering the matrix elements  $M_{mn}$  for  $m, n = -1, 0, 1$ :

$$\det(M) \approx \det \begin{pmatrix} 1 & \frac{q}{\nu^2} & 0 \\ \frac{q}{\nu^2} & 1 & \frac{q}{\nu^2} \\ 0 & \frac{q}{\nu^2} & 1 \end{pmatrix} = 1 - \frac{q^2}{2\nu^2} = 0 \quad (\text{H10})$$

We get  $\nu = \frac{q}{\sqrt{2}} = \frac{\sqrt{2}k^2}{F^2} \ll 1$ . Since now  $\frac{q}{\nu^2} = \frac{2}{q} \gg 1$ , we have  $|c_0| \gg |c_{\pm 2}| \gg |c_{\pm 4}|$  for any  $n > 1$ , so approximately  $c_2 = c_{-2} \approx -\frac{q}{4} c_0$ .

Assuming the initial condition  $\delta p_k(0) = 0$ , we have  $A_1 = -A_2$ , so

$$\begin{aligned} \delta p_k(t) &\approx (A_1 e^{i\nu\tau} + A_2 e^{-i\nu\tau}) (c_0 + c_{\pm 2} e^{2i\tau} + c_{\pm 2} e^{-2i\tau}) \\ &= A \sin \nu\tau \left( 1 - \frac{q}{2} \cos 2\tau \right) \\ &= A \sin \frac{k^2 t}{\sqrt{2}F} \left( 1 - \frac{k^2}{F^2} \cos Ft \right) \end{aligned} \quad (\text{H11})$$

We can also find  $\delta x_k$ :

$$\delta x_k(t) \approx \frac{A}{\sqrt{2}F} \cos \frac{k^2 t}{\sqrt{2}F} + \frac{A}{F} \sin Ft \sin \frac{k^2 t}{\sqrt{2}F}. \quad (\text{H12})$$

We observe that a fraction of both  $\delta p_k$  and  $\delta x_k$  oscillate at the frequency  $k^2$ , confirming our claim that this model has hydrodynamic magnon modes propagating on top of fast Bloch oscillations.