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Fund Managers' Competition for Investment Flows Based on Relative Performance

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Abstract

N mutual funds compete for fund flows based on relative performance over their average returns, by choosing between an idiosyncratic and a common risky investment opportunities. The unique constant equilibrium is derived in closed form, which implies that funds generally decrease the investments in their idiosyncratic risky assets under competition, in order to lower the risk of the relative performance. It pushes all funds to herd and hurts their after-fee performance. However, the sufficiently disadvantaged funds with poor idiosyncratic investment opportunities or highly risk averse managers may take excessive risk for a better chance of attracting new investments, and their performance may improve comparing to the case without competition and benefit the investors.

Keywords Portfolio choice · Mutual funds · Fund flows · Relative performance · Equilibrium

Mathematics Subject Classification 91G10 · 91G80

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1 Introduction

Mutual funds usually pay their managers management fees which are proportional to the asset under management. This linear compensation scheme is supposed to encourage the managers to focus on the performance of the fund itself, and align the interests of investors and managers. It mitigates the moral hazard of excessive risk-taking in the fund, which may arise from the option-like performance fees widely adopted by hedge funds. On the other hand, mutual funds are also subject to profit-chasing fund flows, which are often based on the funds' performance, especially relative to their competitors—other mutual funds. Investors constantly monitor the returns of competing funds, e.g., those that mainly invest in the same asset classes. If one mutual fund outperforms others, it attracts new investments and boosts future management fees, while poor performance could lead to less new investment or even withdrawal of the current investors and magnifies the damage to the manager's compensation. While the current investors' welfare comes from the returns of the fund, the manager's welfare also includes the changes in the management fees due to fund flows. Thus, for their own interests, mutual fund managers have the tendency to hedge the risk in the competitors' investments, which enter into the fund's dynamics through the fund flows.

The competition between mutual funds and fund flows based on relative performance is well documented in the empirical literature [16, 29, 35, 46, 50]. The extant theoretical analysis mostly focuses on the competition between two funds, or in discrete-time models [5, 13, 33, 45, 51], or without fund flows (and more generally on incentives for multiple interacting agents) [2, 6, 18, 24, 31, 39, 41, 49].

We assume that each fund can invest in two investment opportunities, one idiosyncratic, representing the manager's skill, and one accessible to every fund, such as a market index, both of which are modeled by geometric Brownian motions, with general correlations. The flow of each fund is proportional to the return of the fund relative to the average of $N(\geq 2)$ funds, referred to as the industry average in the rest of the paper. The manager of each fund is assumed to have full information about other funds' investment opportunities and their portfolio choices, which is also assumed in the literature on competition between asset managers [4, 40]. It agrees with the fact that investment strategies of mutual funds are public information, and can also model the competition in a fund family managed by the same company [37].

Because fund flows are based on relative performance, the optimal strategy for each fund depends on other funds' portfolio choices, and we derive a Nash equilibrium in closed form for managers who try to maximize the expected discounted power utilities of management fees with different risk aversions. In addition to the dependence on the fund's own investment opportunities, as in the classical Merton portfolio, the equilibrium strategy also hedges against the risk in other funds' investment and thus depends on the investment opportunities that the fund does not have access to, and their correlations with the fund's own investment.

The imperfect correlations among the idiosyncratic investment opportunities force each manager to face an incomplete market. For the optimal investment of each fund, instead of solving the associated system of Hamilton–Jacobi–Bellman (HJB) equations, for which the solution and verification involve complex mathematical argument, especially if the number of funds becomes large (see, e.g., [20, 22]), we derive the dual bound of the value function and choose the risk premia which give the lowest upper bound for all admissible strategies. Then, we verify that the proposed optimal investment strategy achieves this upper bound.

Similar to [40], we search for the equilibria in which portfolios of all funds are constants, and find the unique one. It may not be the unique equilibrium if the investment strategies are allowed to be stochastic, but it is a natural choice for fund managers given the homogeneity of the power utilities and the constant investment opportunities. Furthermore, for every fund, given the constant strategies of other funds, the constant equilibrium portfolio is optimal, among all admissible, potentially stochastic strategies.

The continuous-time models in [4, 40] are closest to ours, which also consider competition between asset managers. The main difference is that instead of the comparison only at the terminal date so that the manager's utility is a function of the relative performance in these papers, we consider a competition for fund flows which happens continuously. Thus, the relative performance does not enter into the utility function, but the dynamics of the assets under management of each fund.

With fund flows based on relative performance, managers have two considerations in his/her portfolio choice, one is the total risk taking of the fund, which decides the return, and the other is the risk in the relative performance, which decides the fund flow. Our results show that in most cases the concern for the poor relative performance dominates, and managers take less risk in their idiosyncratic investment opportunity, so that the fund behaves more closely to the industry average. It indicates that competition pushes funds to herd, which agrees with the results in [43] for static models.

However, our results also show new phenomena that are not documented in the previous literature, even in the competition between two funds, which is extensively studied in, e.g., [4]. With appropriate correlation structures, if the funds' investment opportunities and the managers' risk aversions are close to each other, it could happen that every fund is further away from the average, comparing to the case without competition. In particular, if the fund is disadvantaged with poor idiosyncratic investment opportunity or the manager is of relatively high risk aversion among the group (so that he/she takes low risk without competition), then to beat the competitors and attract new investment, the fund increases the risk-taking in its idiosyncratic investment opportunity. It supports the conclusion in [4] that competition can lead to specialization, which is also discussed in [8, 9, 42, 52, 53]. It also partially agrees with the results in [4, 40] that more risk averse fund managers tend to take more risk under competition, than those with lower risk aversions. However, in addition to risk aversion, which is the only factor that plays a role in this comparison in the above papers, the Sharpe ratios of investment opportunities and their correlations also play a role in our results, which show different patterns from the previous literature depending on model parameters and is a consequence of the difference in our models.

Finally, we also analyze the fund's performance (investors' welfare) under the proposed model, which supplements the literature on the principal-agent relationship between the investors and managers, which usually focuses on the case of only one principal and one agent [1, 44]. An interesting result is that, though for most funds the after-fee performance, measured in Sharpe ratios, is lower with competition, compared to the case without fund flows, the performance of disadvantaged funds may increase in face of competition, which benefits the investors, because after all, fund flows based on relative performance push the manager to pursue superior returns over other funds.

The rest of the paper is organized as follows. Section 2 describes the model of N funds competing for fund flows and defines the equilibrium in which each manager maximizes the expected discounted power utility of management fees. Section 3 starts with the main result showing the closed-form solution to the unique constant equilibrium. Section 3.1 discusses in detail the competition between two funds, and Sect. 3.2 shows fund risk-takings, Sharpe ratios, and the herding/specialization effects in the N-fund case. In Sect. 3.3, we analyze the effect of funds with wider market access, and in Sect. 4, we extend our results to a model of stochastic investment opportunities using the forward performance criterion. All the proofs are relegated to the Appendix.

2 Model

2.1 Mutual Fund Investments and Flows

Consider a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, endowed with $N+1$ Brownian motions $W_1, W_2, \ldots, W_N$ and B, which generate the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Assume $\langle W_i, B\rangle_t = \rho_{im} t$ and $\langle W_i, W_j\rangle_t = \rho_{ij} t$, where $\rho_{im} \in (-1, 1)$ and $\rho_{ij} \in (-1, 1)$ are constants, for every $1 \leq i, j \leq N$. Denote ρ as the $N \times N$ matrix with $(\rho)_{ij} = \rho_{ij}$ and ρ_m as the N-dimensional vector with $(\rho_m)_i = \rho_{im}$.

Suppose that mutual fund i ($i = 1, \ldots, N$), in addition to a risk-free asset S_0, which earns a constant rate of return r, allocates its assets under management between two risky investment opportunities: (i) S_m, which is accessible to all investors in the market, e.g., a market index, following the dynamics

$$dS_{mt}/S_{mt} = (r + a)dt + b dB_t, \tag{1}$$

with the constants $a, b > 0$, and (ii) S_i, which only fund i has access to,[1] described by a geometric Brownian motion

$$\mathrm{d}S_{it}/S_{it} = (r + \mu_i)\mathrm{d}t + \sigma_i \mathrm{d}W_{it}, \tag{2}$$

with the constants $\mu_i, \sigma_i > 0$. Let $\lambda_i = \frac{\mu_i}{\sigma_i}$ for $1 \leq i \leq N$, $\lambda_m = \frac{a}{b}$, and the risk aversion-adjusted Sharpe ratio for each fund's idiosyncratic investment opportunity be $\lambda_{i,\gamma_i} = \frac{\lambda_i}{\gamma_i}$. Denote as π_{it} and θ_{it} the proportions of fund i's assets invested in S_i and S_m at time t. Then, R_{it}, the excess return over the risk-free rate from these investments, follows

[1] Note that even in a setting of common information in this paper, different fund managers may specialize in different investment opportunities based on their skill and preference, and for simplicity, we summarize this specialization as one idiosyncratic S_i for each fund (see the same settings in [4, 40]). The analysis in the following can adapt to the case in which each manager has access to the same N risky assets as in [3], with only notational changes.

$$\begin{aligned} \mathrm{d}R_{it} &= \pi_{it}\left(\mathrm{d}S_{it}/S_{it} - rdt\right) + \theta_{it}\left(\mathrm{d}S_{mt}/S_{mt} - r\mathrm{d}t\right) \\ &= \pi_{it}(\mu_i \mathrm{d}t + \sigma_i \mathrm{d}W_{it}) + \theta_{it}(a\mathrm{d}t + b\mathrm{d}B_t). \end{aligned}$$

where $\pi_i \in \mathcal{A}_i$ and $\theta_i \in \Theta$ are admissible, such that the above stochastic differential equation is well defined. More precisely, $\mathcal{A}_i = \{\pi_i : \mathcal{F}_t -$ progressively measurable and, $\int_0^T (|\mu_i \pi_{it}| + |\sigma_i \pi_{it}|^2)dt < \infty\}$ and $\Theta = \{\theta :$ $\mathcal{F}_t -$ progressively measurable and, $\int_0^T (|a\theta_{it}| + |b\theta_{it}|^2)dt < \infty\}$. Investors of fund i compensate the manager by management fees $\psi_i X_{it}$, where $\psi_i > 0$ is a constant, and X_{it} is fund i's value at time t.

Furthermore, assume that the N funds belong to the same peer group, e.g., because they have the same investment "style" characterized in [12], invest in the same asset class, or they belong to the same family, managed by different managers in the same firm, so that investors of each fund compares its return with the rest of the group. The current clients of fund i (or new investors) invest more into the fund, if its return is higher than the average of the group, and withdraw if it is lower. The size of the flow at time t is proportional to X_t^i, and the after-fee relative return over the industry average $(dR_{it} - \psi_i dt) - \frac{1}{N}\sum_{j=1}^N \left(dR_{jt} - \psi_j dt\right)$, and thus X_i follows

$$\begin{aligned} \mathrm{d}X_{it}/X_{it} =& r\mathrm{d}t + (\mathrm{d}R_{it} - \psi_i \mathrm{d}t) \\ & + \alpha_i \left((\mathrm{d}R_{it} - \psi_i \mathrm{d}t) - \sum_{j=1}^N \left(\mathrm{d}R_{jt} - \psi_j \mathrm{d}t\right)/N \right), \end{aligned} \tag{3}$$

where $\alpha_i > 0$ is the sensitivity of fund flows to the relative performance of fund i compared to its peers. Notice that the managerial contracts of mutual funds usually compensate managers in the above linear way. In the USA, the Investment Advisors Act requires that management fees to mutual fund managers are proportional to the assets under management [25]. Fund flows as linear functions on performance and fund size are documented in empirical studies [35, 46].

For tractability, we have abstracted away other features of the fund competition that have been shown in empirical studies. For example, in addition to relative performance, managers also tend to change investment strategies according to different regimes of market conditions [21, 38], or if they are facing employment risk out of poor performance [7, 38]. The literature also points out a convex relationship between the flow and past performance in [7, 16, 50].[2] Thus, strictly speaking, we should regard X as a proxy of the fund value, under our simplifying assumptions. In the rest of the paper, we still refer to X as the "fund value," for ease of notation.

Also, if there are linear fund flows based on absolute returns, then the coefficient of $(dR_{it} - \psi_i dt)$ in (3) becomes a constant greater than one. All the following results hold with only notational changes, and the numerical results stay qualitatively the same. Thus, we abstract away the fund flows based on absolute performance and focus on

[2] Such convexity is the lowest in the USA in cross-country comparison and is declining over time [23, 33], due to the lower participation cost to the investors, which is even more of the case in recent years.

the effect of competitions. For the similar reason, we omit the constant intercept in the linear regression of fund flows on the relative performance in (3).

From (3), in/out flows based on relative performance magnify the effect of managers' portfolio choices on their management fees. Furthermore, since the industry average enters into the dynamics of each fund, and the investments of different funds are correlated, the hedge against risks in other non-accessible investment opportunities should become part of the manager's consideration in portfolio choice, and thus every manager is facing an incomplete market.

Finally, the sum of the flows of the N funds is not necessarily zero, and there are flows in/out of the group. The funds with better performance can attract new investors apart from the current clients of the N funds, and those who lose money in the funds with inferior returns may withdraw and search for investment opportunities other than the N funds, which is consistent with the empirical evidence in [35] and the theoretical model for mutual funds tournament game over relative performance in [47]. The net flow of the whole group is

$$
\begin{aligned}
&\sum_{i=1}^{N} \alpha_i \left((\mathrm{d}R_{it} - \psi_i \mathrm{d}t) - \frac{1}{N} \sum_{j=1}^{N} \left(\mathrm{d}R_{jt} - \psi_j \mathrm{d}t \right) \right) \\
&\quad = \sum_{i=1}^{N} (\alpha_i - \bar{\alpha}) \left(\mathrm{d}R_{it} - \psi_i \mathrm{d}t \right),
\end{aligned}
$$

where $\bar{\alpha} = \sum_{i=1}^{N} \alpha_i / N$. It implies that the fund with lower flow sensitivity ($\alpha_i < \bar{\alpha}$) and a positive after-fee return has a negative contribution to the total fund flow and vice versa. The reason is that fund i's positive return pushes the industry average higher and lowers the fund flows of other funds. This effect is magnified by the larger flow sensitivity of other funds and hence a negative effect on the total fund flow. However, this would not incentivize the manager with lower flow sensitivity to pursue negative returns, because the utility function defined below still focuses on the manager's own fund and its return, instead of the group as a whole. We derive the equilibrium among managers, instead of maximizing the whole sector from a social planner's point of view.

2.2 Preferences

The manager of fund i chooses the investment strategies $(\pi_i, \theta_i) \in \mathcal{A}_i \times \Theta$ and maximizes the discounted expected power utility from management fees over the time interval $[0, T]$ (see similar settings in [31]). Since there are fund flows based on relative performance, in addition to the fund i's investment strategy (π_i, θ_i), the welfare of the manager also depends on the strategies his/her competitors are taking. Let $\pi = (\pi_1, \ldots, \pi_N)'$ and $\theta = (\theta_1, \ldots, \theta_N)'$, where the superscript $'$ (for the rest of the paper) indicates matrix transpose, and manager i's goal is $\sup_{(\pi_i, \theta_i) \in \mathcal{A}_i \times \Theta} J_i(\pi_i, \theta_i; \pi_{-i}, \theta_{-i})$, where

$$J_i(\pi_i, \theta_i; \pi_{-i}, \theta_{-i}) = \mathbb{E}\left[\int_0^T e^{-\beta_i t} \frac{(\psi_i X_{it})^{1-\gamma_i}}{1-\gamma_i} dt\right], \tag{4}$$

β_i is the manager i's subjective discount factor, and $\gamma_i > 0$ ($\neq 1$) is the coefficient of relative risk aversion. π_{-i} and θ_{-i} are the vectors representing the portfolio choices of $n-1$ managers[3], excluding manager i. Notice that the comparisons between funds are based on their return dR_i's, instead of their sizes. Thus, the sizes of other funds X_j's ($j \neq i$) do not enter into the dynamics of X_i in (3) and do not affect manager i's portfolio choice. Furthermore, since (3) and (4) imply that manager i's utility is homogeneous in the initial value X_{i0}, the latter also does not affect the optimal portfolio choice of manager i given other funds' strategies, and the same holds true for the equilibrium defined below. Thus, without loss of generality, assume that $X_{i0} = 1$ for each $1 \leq i \leq N$. We will discuss the Nash equilibrium among the N funds, which no one wants to deviate from, given the portfolio choices of others.

Definition 2.1 Let $\mathcal{A}$ and Θ^N be the Cartesian product of $\mathcal{A}_i$'s ($i = 1, \ldots, N$), and the Nth Cartesian product of Θ, respectively. $(\pi^*, \theta^*) \in \mathcal{A} \times \Theta^N$ is a Nash equilibrium if for every i, and any $(\pi_i, \theta_i) \in \mathcal{A}_i \times \Theta$,

$$J_i(\pi_i, \theta_i; \pi^*_{-i}, \theta^*_{-i}) \leq J_i(\pi^*_i, \theta^*_i; \pi^*_{-i}, \theta^*_{-i}).$$

Furthermore, (π^*, θ^*) is called a constant equilibrium if they are constants.

3 Main Results and Discussion

In this section we present the main results of this paper and discuss their implications. The following theorem shows that there exists a unique constant equilibrium. Notice that though the equilibrium (π^*, θ^*) is the unique among all constants strategies, for each $1 \leq i \leq N$, $J_i(\pi_i, \theta_i; \pi^*_{-i}, \theta^*_{-i}) \leq J_i(\pi^*_i, \theta^*_i; \pi^*_{-i}, \theta^*_{-i})$ for every $(\pi_i, \theta_i) \in \mathcal{A}_i \times \Theta$, i.e., (π^*_i, θ^*_i) is optimal among all admissible, including stochastic investment strategies, given the constant equilibrium choices of other competitors.

Theorem 3.1 *There exists a unique constant equilibrium*

$$\pi^* = A_f P_f^{-1} \gamma^{-1} \lambda_f, \tag{5}$$

$$\theta^* = A_m P_m^{-1} \left(\gamma^{-1} \eta_m + C A_f^{-1} \pi^*\right), \tag{6}$$

where λ_f and η_m are N-dimensional vectors with $(\lambda_f)_i = \lambda_i - \rho_{im}\lambda_m$ and $(\eta_m)_i = \lambda_m - \rho_{im}\lambda_i$, respectively, for $1 \leq i \leq N$. A_f, A_m and γ are diagonal matrices with the diagonal elements $(A_f)_{ii} = \frac{N}{(N+(N-1)\alpha_i)\sigma_i}$, $(A_m)_{ii} = \frac{N}{(N+(N-1)\alpha_i)b}$ and $(\gamma)_{ii} = \gamma_i$,

[3] Similarly in the rest of the paper, with positive integer n, $v \in \mathbb{R}^n$ and $D \in \mathbb{R}^{n\times n}$, let $v_{-i} \in \mathbb{R}^{n-1}$ be the vector after removing v's ith element, and $D_{-i} \in \mathbb{R}^{(n-1)\times(n-1)}$ be the matrix after removing D's ith row and ith column.

respectively, for $1 \le i \le N$. P_f, P_m *and* C *are* $N \times N$ *matrices with*

$$(P_f)_{ij} = \begin{cases} 1-\rho_{im}^2 & \text{if } i=j, \\ -c_{ij}(\rho_{ij}-\rho_{im}\rho_{jm}) & \text{if } i \neq j, \end{cases} \quad (P_m)_{ij} = \begin{cases} 1-\rho_{im}^2 & \text{if } i=j, \\ -c_{ij}(1-\rho_{im}^2) & \text{if } i \neq j. \end{cases}$$

$$(C)_{ij} = \begin{cases} 0 & \text{if } i=j, \\ c_{ij}(\rho_{jm}-\rho_{im}\rho_{ij}) & \text{if } i \neq j, \end{cases} \quad c_{ij} = \frac{\alpha_i}{N+(N-1)\alpha_j}, 1 \le i, j \le N.$$

Without fund flows ($\alpha_i = 0$), the expected utility J_i is independent of π_{-i} and θ_{-i}, and the manager essentially faces the Merton problem with two correlated risky assets, and the optimal investment strategies for fund i are constants (the verification is omitted)

$$\pi_i^M = \frac{\lambda_i - \rho_{im}\lambda_m}{\gamma_i\sigma_i(1-\rho_{im}^2)}, \quad \theta_i^M = \frac{\lambda_m - \rho_{im}\lambda_i}{\gamma_i b(1-\rho_{im}^2)}, \tag{7}$$

which only depend on the investment opportunities S_i and S_m which are accessible to fund i. With the possibility of in/out flows, since managers maximize welfare from the management fees proportional to the assets under management, they care about the total return of the fund, including the flows. The equilibrium strategies π_i^* and θ_i^* include hedging components against the risk exposure to other risky investment opportunities, and depend on their correlations and the flow sensitivities of all funds. For example, if $\lambda_i = \rho_{im} = 0$, $\pi_i^M = 0$, because S_i brings zero expected return, and cannot be used to hedge the risk in S_m. However, with competition based on relative performance, as long as S_i is not independent of other S_j's, π_i^* is not necessarily zero—S_i is worth the investment, not because of the return it provides, but the hedge it brings against the risks in other funds' investments.

In the following, we discuss how the competition affects the fund managers' equilibrium investment strategies and the investment returns for fund investors, and how they compare to the counterpart without competitions. In particular, we compare the volatility of fund investment with and without competition, denoted as σ_i^* and σ_i^M, respectively,

$$\sigma_i^* = \sqrt{\left(\pi_i^*\sigma_i\right)^2 + 2\rho_{im}\pi_i^*\theta_i^*\sigma_i b + \left(\theta_i^* b\right)^2},$$
$$\sigma_i^M = \sqrt{\left(\pi_i^M\sigma_i\right)^2 + 2\rho_{im}\pi_i^M\theta_i^M\sigma_i b + \left(\theta_i^M b\right)^2},$$

and the corresponding after-fee Sharpe ratios of the fund investment

$$\eta_i^* = \frac{-\psi_i + \pi_i^*\mu_i + \theta_i^* a}{\sqrt{\left(\pi_i^*\sigma_i\right)^2 + 2\rho_{im}\pi_i^*\theta_i^*\sigma_i b + \left(\theta_i^* b\right)^2}},$$
$$\eta_i^M = \frac{-\psi_i + \pi_i^M\mu_i + \theta_i^M a}{\sqrt{\left(\pi_i^M\sigma_i\right)^2 + 2\rho_{im}\pi_i^M\theta_i^M\sigma_i b + \left(\theta_i^M b\right)^2}}.$$

Note that since every fund invests in S_m, even with $\theta_i = 0$, fund i has exposure to the risk in S_m through fund flows. Thus, instead of θ_i, the manager actually has to choose the optimal effective investment $\zeta_i = (N + (N-1)\alpha_i)\theta_i/N - \alpha_i \sum_{j\neq i}^{N} \theta_j/N$ in S_m. However, from fund i's investors' point of view, the return on their own investments is $dR_{it}^* - \psi_i dt$ corresponding to π_i^* and θ_i^*, instead of $\frac{dX_{it}^*}{X_{it}^*}$, which includes fund flows. Thus, when we discuss the fund performance and calculate the after-fee Sharpe ratios, the calculations do not take into account fund flows.

We are also interested in how each fund's return compares to the industry average, in terms of the difference between the individual fund's after-fee return $dR_{it} - \psi_i dt$ and the industry average $\sum_{j=1}^{N}(dR_{jt} - \psi_j dt)/N$. The competition tends to move individual fund's investment in different directions: on one hand, the manager wants to deviate from the industry average, in order to outperform and attract new investments, which increases future management fees. On the other hand, the risk-averse manager also tends to mimic the competitors, which decreases the risk of outflows due to poor relative performance. The second effect of funds' competition is referred to as herding [27], and is discussed in [28, 48] for institutional investors who have reputation concerns and make investment decisions based on past performance.

Let $\bar{\theta}^* = \sum_{i=1}^{N} \theta_i^*/N$ and the average logarithmic return of the N funds in equilibrium be $\bar{R}_t^* = \sum_{i=1}^{N} R_{it}^*/N$, such that $d\bar{R}_t^* = \left(r - \sum_{i=1}^{N} \psi_i/N\right) dt + \sum_{i=1}^{N} \pi_i^*(\mu_i dt + \sigma_i dW_{it})/N + \bar{\theta}^*(adt + bdB_t)$. We use the Beta coefficient of R_i^* with respect to $\bar{R}^*$ to measure the "distance" between fund i and the industry average, denoted as Beta_i^*, and

$$\text{Beta}_i^* = \frac{N\left(q_i' \Sigma\rho\Sigma\pi^* + Nq_i'\Sigma\rho_m\bar{\theta}^* b + (\pi^*)'\Sigma\rho_m\theta_i^* b + N\theta_i^*\bar{\theta}^* b^2\right)}{(\pi^*)'\Sigma\rho\Sigma\pi^* + 2N(\pi^*)'\Sigma\rho_m\bar{\theta}^* b + N^2\left(\bar{\theta}^*\right)^2 b^2}, \tag{8}$$

where q_i is an N-dimensional vector with zero entries except that $(q_i)_i = \pi_i^*$, and Σ is an $N \times N$ diagonal matrix with $(\Sigma)_{ii} = \sigma_i$. If there is no competition, the Beta coefficient between the corresponding return R_i^M and their average $\bar{R}^M$, denoted as Beta_i^M, can be similarly calculated. Let $\bar{\theta}^M = \sum_{i=1}^{N} \theta_i^M/N$, π^M be the N-dimensional vector with $\left(\pi^M\right)_i = \pi_i^M$, and q_i^M be the N-dimensional vector with zero entries except that $\left(q_i^M\right)_i = \pi_i^M$,

$$\begin{aligned} \text{Beta}_i^M &= \frac{K_i^M}{\left(\pi^M\right)'\Sigma\rho\Sigma\pi^M + 2N\left(\pi^M\right)'\Sigma\rho_m\bar{\theta}^M b + N^2\left(\bar{\theta}^M\right)^2 b^2}, \\ K_i^M &= N\Bigg(\left(q_i^M\right)'\Sigma\rho\Sigma\pi^M + N\left(q_i^M\right)'\Sigma\rho_m\bar{\theta}^M b + \\ &\qquad \left(\pi^M\right)'\Sigma\rho_m\theta_i^M b + N\theta_i^M\bar{\theta}^M b^2\Bigg). \end{aligned} \tag{9}$$

The closer Beta_i^* (or Beta_i^M) is to 1, the more closely fund i mimics the industry average. If this is the case for most funds, then the herding effect is present.

The Beta coefficients of R_i with respect to the common investment opportunity $\frac{dS_{mt}}{S_{mt}} = adt + bdB_t$ with and without competition, denoted as Beta_{mi}^* and Beta_{mi}^M, can be computed similarly

$$\text{Beta}_{mi}^{*} = (\pi_i^* \rho_{im}\sigma_i + \theta_i^* b)/b, \quad \text{Beta}_{mi}^{M} = (\pi_i^M \rho_{im}\sigma_i + \theta_i^M b)/b.$$

They measure the "distance" between each fund's investment and S_m. The further away Beta_{mi}^* (or Beta_{mi}^M) is from 1, the more fund i specializes in its idiosyncratic investment opportunity S_i.

Finally, before we move to a detailed discussion about the case of two funds, it is worth pointing out that, in our model, the equilibrium always exists, while in [3], it depends on the model parameters, especially the risk aversion of different managers. Also, though as $N \to \infty$, $c_{ij} \to 0$, $C \to 0$, P_f and P_m converge to diagonal matrices with entries $1-\rho_{im}^2$'s, π_i^* and θ_i^* do not converge to $\frac{\pi_i^M}{1+\alpha_i}$ and $\frac{\theta_i^M}{1+\alpha_i}$ (see (7)), i.e., the limit as $N \to \infty$ and the multiplication in (5) and (6) do not commute, because the dimensions of these matrices also increase with N. In the limit, π_i^* and θ_i^* still depend on the average of model parameters of other funds, as more clearly shown in Proposition 3.5, in the case where all funds only invest in S_m.

3.1 The Case of Two Funds

We start the discussion from the case of two funds. In this case (with $j=2$ if $i=1$ and $j=1$ if $i=2$)

$$\begin{aligned}
\pi_i^* = \frac{2}{(2+\alpha_i)\sigma_i\kappa_1}\Bigg(&\frac{1}{\gamma_i}(1-\rho_{jm}^2)(\lambda_i-\rho_{im}\lambda_m)\\
&+\frac{1}{\gamma_j}\frac{\alpha_i}{2+\alpha_j}(\rho_{12}-\rho_{1m}\rho_{2m})(\lambda_j-\rho_{jm}\lambda_m)\Bigg),\\
\theta_i^* = \frac{2}{(2+\alpha_i)b\kappa_2}\Bigg(&\frac{1}{\gamma_i}(1-\rho_{jm}^2)(\lambda_m-\rho_{im}\lambda_i)\\
&+\frac{1}{\gamma_j}\frac{\alpha_i}{2+\alpha_j}(1-\rho_{im}^2)(\lambda_m-\rho_{jm}\lambda_j)\\
&+\frac{\alpha_1\alpha_2}{2(2+\alpha_j)}(1-\rho_{im}^2)(\rho_{im}-\rho_{12}\rho_{jm})\sigma_i\pi_i^*\\
&+\frac{\alpha_i}{2}(1-\rho_{jm}^2)(\rho_{jm}-\rho_{12}\rho_{im})\sigma_j\pi_j^*\Bigg),
\end{aligned}$$

where $\kappa_1 = (1-\rho_{1m}^2)(1-\rho_{2m}^2) - \frac{\alpha_1\alpha_2}{(2+\alpha_1)(2+\alpha_2)}(\rho_{12}-\rho_{1m}\rho_{2m})^2 > 0$ [4] and $\kappa_2 = (1-\rho_{1m}^2)(1-\rho_{2m}^2)\frac{4+2\alpha_1+2\alpha_2}{(2+\alpha_1)(2+\alpha_2)} > 0$. In addition to $\lambda_i, \rho_{im}, \lambda_m, \pi_i^*$ and θ_i^* also depend on the investment of the other fund, the fund's flow sensitivity and the correlations between investment opportunities, while π_i^* and θ_i^* reduce to π_i^M and θ_i^M, if $\alpha_i = 0$ for $i = 1, 2$.

Proposition 3.1 *For* $i, j = 1, 2$ *and* $j \neq i$,

[4] According to the proof of Lemma 4.3, $(1-\rho_{1m}^2)(1-\rho_{2m}^2) \geq (\rho_{12}-\rho_{1m}\rho_{2m})^2$, and thus $\kappa_1 > 0$.

(i) π_i^* *strictly increases in* λ_i. θ_i^* *strictly increases (decreases or remains a constant) in* λ_i *if* $\rho_{im} < (> \text{ or } =) \, 0$.

(ii) π_i^* *strictly increases (decreases or remains a constant) in* λ_j *if* $\rho_{12} - \rho_{1m}\rho_{2m} > (< \text{ or } =) \, 0$.

(iii) (a) *If* $\rho_{jm} = 0$, *then* θ_i^* *strictly increases (decreases or remains a constant) in* λ_j, *if* $\rho_{12}\rho_{im} < (> \text{ or } =) \, 0$.
(b) *If* $\rho_{jm} \neq 0$, *then* θ_i^* *strictly increases (decreases or remains a constant) in* λ_j, *if* $\rho_{jm} C < (> \text{ or } =) \, 0$, *where*

$$
\begin{aligned}
C = & \left(1 + \frac{\alpha_1\alpha_2}{(2+\alpha_1)(2+\alpha_2)}\right)\left(1 - \rho_{1m}^2\right)\left(1 - \rho_{2m}^2\right) \\
& - \frac{2\alpha_1\alpha_2}{(2+\alpha_1)(2+\alpha_2)}\left(\rho_{12} - \rho_{1m}\rho_{2m}\right)^2 \\
& - \left(1 - \frac{\alpha_1\alpha_2}{(2+\alpha_1)(2+\alpha_2)}\right)(1 - \rho_{jm}^2)\left(1 - \frac{\rho_{12}\rho_{im}}{\rho_{jm}}\right).
\end{aligned}
$$

Proposition 3.1 shows that each fund invests more into its idiosyncratic investment opportunity as its Sharpe λ_i becomes larger, while the change in θ_i^* depends on ρ_{im}. If $\rho_{im} > 0$, then the investment in S_m is lower because there is a larger exposure to S_m from the increase in π_i, and vice versa.

On the other hand, fund i's portfolio also changes with λ_j because the manager's compensation depends on the relative performance and thus the portfolios of fund j. A large ρ_{12} leads to an increase in π_i^*, in order to hedge the larger risk exposure to W_j from the fund flows due to the increasing π_j^*, and vice versa. However, large $\rho_{1m}\rho_{2m}$ tends to decrease π_i^*, because either $\rho_{jm} > 0$ so that θ_j^* decreases, and $\rho_{im} > 0$ so that less exposure is needed in S_i to hedge against Brownian motion B due to fund flows, or $\rho_{jm} < 0$ so that θ_j^* increases, and $\rho_{im} < 0$ so that the exposure in S_i should still decrease.

The change in θ_i^* is more delicate. If $\rho_{jm} = 0$, then according to Proposition 3.1 (i), π_j^* increases in λ_j and θ_j^* stays constant. If $\rho_{12} > 0$, then Proposition 3.1 (ii) implies that π_i^* increases in λ_j. To achieve a desired level of total risk-taking, θ_i^* tends to decrease or increase if $\rho_{im} > 0$ or $\rho_{im} < 0$, respectively. For $\rho_{12} < 0$, the sensitivity above changes to the opposite direction following similar arguments.

If $\rho_{jm} \neq 0$, θ_i^* tends to move the same way as θ_j^*, as to keep the effective exposure $\zeta_i = (2+\alpha_i)\theta_i/2 - \alpha_i\theta_j/2$ in S_m at a desired level, and thus the same sensitivity with respect to ρ_{jm} as shown in Proposition 3.1 (i). However, the manager should also consider how to hedge S_i and how S_j and S_m can substitute each other, for the best risk-return trade-off. Thus, the changes in θ_j^* also depend on the constant C as a function of the correlations and the fund flow sensitivities. Notice that $(1-\rho_{1m}^2)(1-\rho_{2m}^2) \geq (\rho_{12} - \rho_{1m}\rho_{2m})^2$. Thus, the only case in which the sign of $C\rho_{jm}$ is different from that of ρ_{jm}, is that $\rho_{12}\rho_{im}/\rho_{jm}$ is negative with a large absolute value. In other words, ρ_{jm} is close to 0, and $\rho_{12}\rho_{im}$ have a different sign than that of ρ_{jm}, which then follows the similar argument to that in the case of $\rho_{jm} = 0$.

Next, we investigate the effect of flow sensitivity α_i's, focusing on the investment in S_i's.

Assumption 3.2 $\rho_{im} = \lambda_m = 0$ for $i = 1, 2$,

In this case, $\theta_i^* = 0$ for $i = 1, 2$, and each fund invests in its own investment opportunities, which are correlated with each other—similar to the models in [4, 40]. The more general setting with investment in S_m can be analyzed exactly the same way, with all quantities calculated in closed form, though there are more complex cases to discuss.

In this setting, with $\kappa_1 = 1 - \frac{\alpha_1\alpha_2}{(2+\alpha_1)(2+\alpha_2)}\rho_{12}^2$, and $\lambda_{i,\gamma_i} = \frac{\lambda_i}{\gamma_i}$ for $i = 1, 2$,

$$\begin{aligned} \pi_1^* &= \frac{2}{(2+\alpha_1)\kappa_1\sigma_1}\left(\lambda_{1,\gamma_1} + \frac{\alpha_1}{2+\alpha_2}\rho_{12}\lambda_{2,\gamma_2}\right), \\ \pi_2^* &= \frac{2}{(2+\alpha_2)\kappa_1\sigma_2}\left(\lambda_{2,\gamma_2} + \frac{\alpha_2}{2+\alpha_1}\rho_{12}\lambda_{1,\gamma_1}\right), \end{aligned} \tag{10}$$

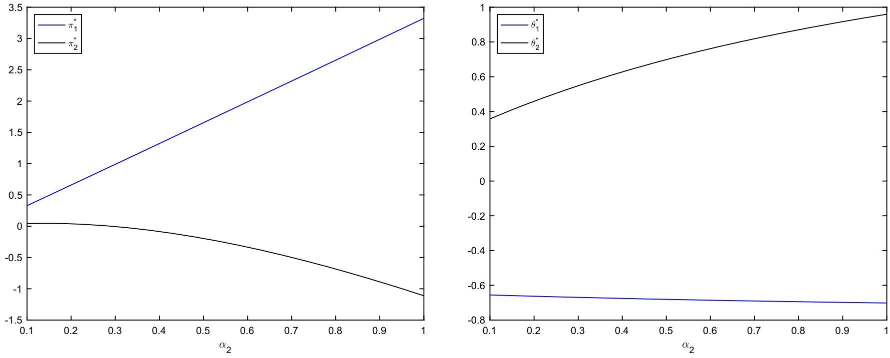
Proposition 3.2 *For $i, j = 1, 2$ and $j \neq i$,*

(i) If $2\rho_{12}\lambda_{j,\gamma_j} - \left(2 + (1-\rho_{12}^2)\alpha_j\right)\lambda_{i,\gamma_i} > (< \text{ or } =)0$, then π_i^* strictly increases (decreases or remains a constant) in α_i.
(ii) If $\rho_{12}\left(2\rho_{12}\lambda_{i,\gamma_i} - (2 + (1-\rho_{12}^2)\alpha_i)\lambda_{j,\gamma_j}\right) > (< \text{ or } =)0$, then π_i^* strictly increases (decreases or remains a constant) in α_j.

For each fund i, the fund flow magnifies the risk and return of its own investment. As α_i increases, this magnifying effect becomes larger, and manager i tends to decrease π_i^* for the desired level of risk exposure, and this tendency is larger if S_i becomes a better investment opportunity, measured by λ_{i,γ_i}. However, manager i also needs to hedge the larger risk in S_j due to fund flows. Since dX_{it} is decreasing in dR_{jt}, if $\rho_{12} > 0$, then the manager tends increase π_i and vice versa, and this tendency increases with λ_{j,γ_j}. On the other hand, α_j does not affect fund i's flow directly. But as discussed above, π_j^* changes according to the sign of $2\rho_{12}\lambda_{i,\gamma_i} - (2+(1-\rho_{12}^2)\alpha_i)\lambda_{j,\gamma_j}$. To hedge against this change in π_j^* which affects fund i's flow, π_i^* should change accordingly, depending on the sign of the correlation ρ_{12}.

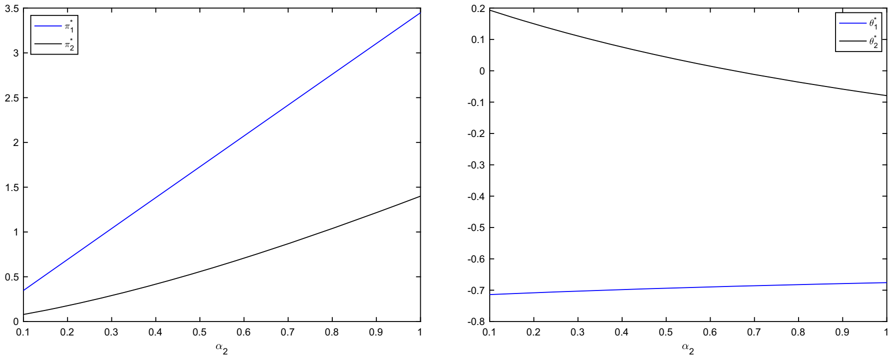
The following are numerical examples in more general settings, which allow θ_i^*'s to be non-trivial. In Fig. 1 with $\rho_{1m} = 0.3$, $\rho_{2m} = 0.5$, $\rho_{12} = -0.6$, $\alpha_1 = 0.8$, $b = 0.15$, $\sigma_1 = 0.18$, $\sigma_2 = 0.13$, $\lambda_m = 0.15$, $\lambda_1 = 1.5$, $\lambda_2 = 0.2$, $\gamma_1 = \gamma_2 = 2$, θ_2^* increases with α_2, to hedge larger risks (in absolute value) in S_2 and the fund 1's exposure to S_1. Similarly, θ_1^* moves in the opposite direction to π_1^*, while the effect is much smaller, because α_2 does not directly enter the dynamics of fund 1, and the increase in θ_2^* fulfills part of the hedging demands from the increase in π_1^*, which lowers fund 1's effective exposure to S_m, $\zeta_1 = \frac{2+\alpha_1}{2}\theta_1 - \frac{\alpha_1}{2}\theta_2$. In Fig. 2, ρ_{12} is changed to 0.6. π_2^* and θ_2^* show the opposite pattern to Fig. 1, while π_1^* and θ_1^* behave similarly, following the same intuition as above.

Next, we examine the comparison of the portfolios with and without competition under the further assumption that $\alpha_i = \alpha$ and $\psi_i = \psi$ for $i = 1, 2$. Define $\bar{\lambda} = \lambda_{2,\gamma_2}/\lambda_{1,\gamma_1}$ and without loss of generality assume $\bar{\lambda} \leq 1$. Then, the equilibrium portfolios are



(a) π_1^* and π_2^* **(b)** θ_1^* and θ_2^*

Fig. 1 Equilibrium portfolios with $\rho_{1m} = 0.3$, $\rho_{2m} = 0.5$, $\rho_{12} = -0.6$, $\alpha_1 = 0.8$, $b = 0.15$, $\sigma_1 = 0.18$, $\sigma_2 = 0.13$, $\lambda_m = 0.15$, $\lambda_1 = 1.5$, $\lambda_2 = 0.2$, $\gamma_1 = \gamma_2 = 2$, against α_2



(a) π_1^* and π_2^* with $\rho_{2m} = 0.6$ **(b)** θ_1^* and θ_2^* with $\rho_{2m} = 0.6$

Fig. 2 Equilibrium portfolios with $\rho_{1m} = 0.3$, $\rho_{2m} = 0.5$, $\rho_{12} = 0.6$, $\alpha_1 = 0.8$, $b = 0.15$, $\sigma_1 = 0.18$, $\sigma_2 = 0.13$, $\lambda_m = 0.15$, $\lambda_1 = 1.5$, $\lambda_2 = 0.2$, $\gamma_1 = \gamma_2 = 2$, against α_2

$$\pi_1^* = \frac{2\left(\lambda_{1,\gamma_1} + \frac{\alpha}{2+\alpha}\rho_{12}\lambda_{2,\gamma_2}\right)}{(2+\alpha)\sigma_1\left(1 - \left(\frac{\alpha}{2+\alpha}\right)^2\rho_{12}^2\right)}, \quad \pi_2^* = \frac{2\left(\lambda_{2,\gamma_2} + \frac{\alpha}{2+\alpha}\rho_{12}\lambda_{1,\gamma_1}\right)}{(2+\alpha)\sigma_2\left(1 - \left(\frac{\alpha}{2+\alpha}\right)^2\rho_{12}^2\right)}.$$

Proposition 3.3 *Under Assumption 3.2 and assume that* $\alpha_i = \alpha$, $\psi_i = \psi$ *for* $i = 1, 2$ *and* $\bar{\lambda} = \lambda_{2,\gamma_2}/\lambda_{1,\gamma_1} \leq 1$, *then* $\pi_1^* < \pi_1^M$, $\eta_1^* < \eta_1^M$, *and*

(i) If $\rho_{12} \geq 0$ *and* $\bar{\lambda} < \frac{2\rho_{12}}{2+\alpha(1-\rho_{12}^2)}$, *then* $\pi_2^* > \pi_2^M$ *and* $\eta_2^* > \eta_2^M$. *If* $\rho_{12} \geq 0$ *and* $\bar{\lambda} \geq \frac{2\rho_{12}}{2+\alpha(1-\rho_{12}^2)}$, *then* $\pi_2^* \leq \pi_2^M$ *and* $\eta_2^* \leq \eta_2^M$.

(ii) If $\rho_{12} < 0$, $\pi_2^* < \pi_2^M$. *If furthermore* $\bar{\lambda} < -\frac{\alpha}{2+\alpha}\rho_{12}$, $\eta_2^* > \eta_2^M$. *Otherwise* $\eta_2^* \leq \eta_2^M$.

In addition to the total risks in the fund investment, fund managers also care about the risk in the relative performance, because it affects the fund flow and thus management fees in the future. On the one hand, they want to keep investment strategy π_i^M which brings the best risk-return trade-off according to their own risk attitude. On the other hand, they may want to invest less in S_i, in order to mimic the industry average and to decrease the risk of poor performance against their competitors. Proposition 3.3 shows that for fund 1 with the larger risk aversion-adjusted Sharpe ratio λ_1/γ_1, it is relatively easier to outperform, and thus the concerns for the risks in the relative performance dominate and the manager takes less risk ($\pi_1^* \leq \pi_1^M$). The fund's performance is worse than it could have been without competition ($\eta_1^* < \eta_1^M$). The same could happen if γ_1 is small, and therefore the manager tends to take large risks without competition and has a better chance of outperforming the competitor. It is consistent with the results in [4, 40] that more risk-tolerant managers may decrease the volatility of the fund to decrease the risk in the relative performance. Though in addition to risk aversion, which is the only factor that plays a role in such comparison in the above literature, the Sharpe ratio also enters into the equation in our results.

For the relatively disadvantaged manager (with smaller λ_2 or large risk aversion γ_2), even more factors matter in such comparison and we have more cases to discuss. If $\rho_{12} \geq 0$, the portfolio choice of the competitor hedges part of the risk in the fund's own investment. Thus, if $\bar{\lambda}$ is small, i.e., the disadvantage is big, the eagerness for new investments dominates, and the manager increases the fund's risk for a better chance of winning the competition. This result partially agrees with and provides an alternative explanation for the empirical evidence in [10, 37] that the manager with relatively poorer past performance tends to increase risk-takings, though, in our model of equilibrium, the manager reacts to the disadvantaged investment opportunity by taking a larger risk at every $t \geq 0$. This may not be bad news for the clients, because the after-fee Sharpe ratio of the fund actually increases. If $\bar{\lambda}$ is sufficiently large, then the peer pressure is lighter and fund 2 invests similarly to fund 1 by decreasing the risky investment and thus lowers the performance. If $\rho_{ij} < 0$, the introduction of fund flows increases the total risks in the fund, and the concern for the fund's absolute performance leads fund 2 to decrease the investment in S_2 to hedge against the risk in S_1. π_2^*/π_2^M is increasing in $\bar{\lambda}$, and equals to $\frac{2\left(1+\frac{\alpha}{2+\alpha}\rho_{12}\right)}{(2+\alpha)\left(1-\left(\frac{\alpha}{2+\alpha}\right)^2\rho_{12}^2\right)} > 0$ at $\bar{\lambda} = 1$ and $-\infty$ at $\bar{\lambda} = 0$. Thus, similar to the case of $\rho_{12} > 0$, if $\bar{\lambda}$ is small (with a threshold different to the previous case), the big disadvantage leads fund manager 2 to take a large negative position in S_2 aiming to win the competition. If $\bar{\lambda}$ is sufficiently large, then fund 2 already has a good chance of outperforming fund 1. Therefore, the decrease in the investment in S_2 is relatively small because it increases the risk in the relative performance.

The above difference from [4, 40] is another consequence of the difference in models. The risk aversion and competitiveness parameter in these papers only enter into the utility function as powers of the fund value, so that the equilibrium portfolio decomposes into the Merton portfolio plus a hedging component. In our model, fund flow sensitivity enters into the dynamics of funds and affects the utility in a more subtle way. The simple decomposition above is not available anymore, and the manager's effective risk aversion is also affected by the fund flow sensitivity.

Proposition 3.4 *Under the assumptions in Proposition 3.3, let* $\Delta = \frac{(\alpha\rho_{12}^2+2+\alpha)^2}{(1+\alpha)^2\rho_{12}^2} - 4$. *Then,* $\left|\text{Beta}_i^* - 1\right| - \left|\text{Beta}_i^M - 1\right| \le 0$ *for both* $i = 1$ *and* 2 *if and only if one of the following holds: (i)* $\rho_{12} \ge 0$, *or (ii)* $\rho_{12} < 0$, $2\bar{\lambda} \le -\left(\frac{\alpha}{1+\alpha}\rho_{12} + \frac{2+\alpha}{1+\alpha}\frac{1}{\rho_{12}}\right) - \sqrt{\Delta}$.

Though fund 2 may behave differently according to Proposition 3.3, in most cases the competition pushes both funds' investments closer to their average. If $\rho_{12} > 0$ and $\bar{\lambda}$ is small, then π_2^M is small compared to π_1^M, and with competition, π_1^* and π_2^* move toward each other, which supports the empirical evidence of herding in [11, 14, 17, 30]. If $\bar{\lambda}$ is large, then π_1^* and π_2^* both become smaller positive numbers and thus are also closer to the average. If $\rho_{12} < 0$ and $\bar{\lambda}$ is small, π_2^* tends to be negative. Then, with a negative correlation between S_1 and S_2, the two funds actually become closer. Only in the case of $\rho_{12} < 0$ and sufficiently large $\bar{\lambda}$, i.e., fund 2 has large peer pressure, the decrease in S_2 is limited, while fund 1's risk-taking becomes much smaller. As a result, both funds are further away from their average. It partially supports [36, 54], which documents superior returns for mutual funds trading against the crowd, and [15, 19, 34], which suggests herding behavior change with market conditions, including investor sentiment.

3.2 The Equilibrium Among *N* Funds

For more than two funds, the equilibrium depends on the model parameters, especially the correlation structure, in a complex way. Explicit and simple characterization as done for the two fund case is no longer available in the most general setting. For example, it is not likely that the Beta coefficients of all funds move in the same direction as in Proposition 3.4. We first consider some special cases in which we can derive analytical results.

3.2.1 The Case of One Common Asset

Let us first consider a special case in which S_m is the only risky investment opportunity for each fund. With the investment strategy θ_i in S_m, the dynamics of X_i is

$$\begin{aligned} \frac{dX_{it}}{X_{it}} = &\left(r - \frac{N+(N-1)\alpha_i}{N}\psi_i + \frac{\alpha_i}{N}\sum_{j\neq i}^{N}\psi_j\right)dt \\ &+ \left(\frac{N+(N-1)\alpha_i}{N}\theta_{it} - \frac{\alpha_i}{N}\sum_{j\neq i}^{N}\theta_{jt}\right)(adt + bdB_t). \end{aligned} \tag{11}$$

Proposition 3.5 *If* S_m *is the only risky investment opportunity for every fund as described in (11), then there exists a unique equilibrium* $\theta^* \in \Theta^N$ *such that for each* $1 \le i \le N$,

$$\theta_i^* = \frac{\lambda_m}{b}\left(\frac{1}{1+\alpha_i}\frac{1}{\gamma_i} + \frac{\alpha_i}{1+\alpha_i}\frac{1}{\bar{\gamma}}\right), \tag{12}$$

where $1/\bar{\gamma} = \sum_{i=1}^{N} \frac{1+\bar{\alpha}}{(1+\alpha_i)\gamma_i}/N$ *and* $1/(1+\bar{\alpha}) = \sum_{i=1}^{N} \frac{1}{1+\alpha_i}/N$.

Notice that in this case, though θ_i^*'s in (12) are constants, this equilibrium is unique among all admissible strategies. Also, if there are no fund flows ($\alpha_i = 0$), the manager is facing a Merton problem with S_m being the only risky investment opportunity. The optimal strategy is $\theta_i^M = \lambda_m/(b\gamma_i)$ for each $1 \le i \le N$. Compared to θ_i^M, the manager's risk tolerance in θ_i^* shifts to a linear combination between the manager's own risk tolerance $1/\gamma_i$ and $1/\bar{\gamma}$, the average risk tolerance of all competing managers, weighted by fund flows sensitivities.

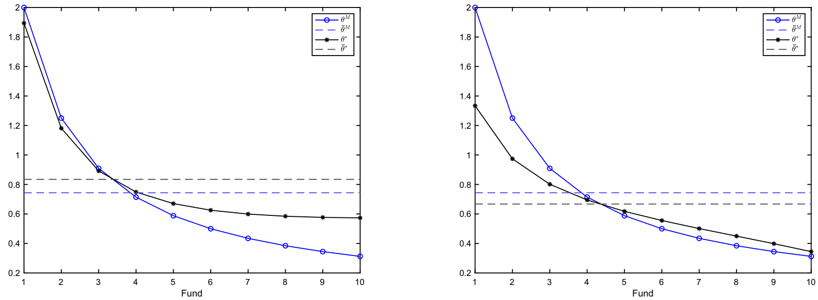
Next, we check the herding effect of competition. Since every fund invests in a common S_m, to analyze the similarities between them, it suffices to compare θ_i^*'s and θ_i^M's, and their industry average $\bar{\theta}^*$ and $\bar{\theta}^M$, respectively. The next proposition shows that relatively more risk-averse managers may take larger risks in face of competition. Also, if the fund with the more risk-averse manager has larger flow sensitivity to the relative performance, then on average the investment of the whole group becomes riskier. In the special case of constant fund flow sensitivities, θ_i^* is always closer to the industry average, than the counterpart without competition. Notice that in the case of common investment opportunities in [40], the authors derive similar equilibrium strategies with effective risk tolerance shifted according to risk aversion and competitiveness. However, our θ_i^* is always decreasing in γ_i while the monotonicity may be opposite for different combinations of risk aversion and competitiveness of the peer group in their paper.

Proposition 3.6 *Under the assumptions of Proposition 3.5, (i) If* $\gamma_i > \bar{\gamma}$*, then* $\theta_i^* > \theta_i^M$*, and vice versa.*

(ii) If $(\gamma_i - \gamma_j)(\alpha_i - \alpha_j) \ge 0$ *for every pair of* $1 \le i \le j \le N$*, then* $\bar{\theta}^* \ge \bar{\theta}^M$*, and vice versa.*

(iii) If α_i *equals a constant* $\alpha > 0$ *for every* $1 \le i \le N$*, then* $\bar{\theta}^* = \bar{\theta}^M$*, and* $\theta_i^* - \bar{\theta}^* = \frac{1}{1+\alpha}\left(\theta_i^M - \bar{\theta}^M\right)$.

The intuition for these results is that facing the same investment opportunity, the funds with more risk-averse managers tend to take less risk and thus are less likely to win the competition. The concern for relative performance pushes them to be more aggressive to keep up. On the other hand, funds with less risk-averse managers are in a better position in the competition and are thus more concerned about the risk of poor performance. They invest less in S_m to avoid possible outflows. Furthermore, if the high risk aversion is accompanied by high sensitivity α_i of fund flows, then the effect of more risk-taking for more risk-averse managers is magnified compared to the effect of less risk-taking for less risk-averse managers, and the average risk-taking of all funds with competition is higher than the counterpart without.



(a) α_i's form an arithmetic sequence from 0.1 to 1

(b) α_i's form an arithmetic sequence from 1 to 0.1

Fig. 3 θ_i^M's, θ_i^*'s and their average if managers only invest in S_m. $N = 10$, $\lambda_m = 0.15$, $b = 0.15$ and γ_i's form an arithmetic sequence from 0.5 to 3.2

Figure 3 shows θ_i^M's, θ_i^*'s and their average, with $N = 10$, $\lambda_m = 0.15$, $b = 0.15$ and γ_i's being equally spaced between 0.5 and 3.2. The left panel shows the case of increasing α_i's and $\bar{\theta}^* > \bar{\theta}^M$. In the right panel, α_i's are decreasing, and the inequality is reversed. In both graphs, similar to previous examples, θ_i^*'s are closer to $\bar{\theta}^*$, compared to the distance between θ_i^M and $\bar{\theta}^M$, with an exception of 2 out of the 10 funds. In the special case of α_i's being equal, Proposition 3.6 (iii) confirms that this comparison holds for every fund, and the competition does have herding effect on the fund investment.

3.2.2 The Case of *N* Assets

In this section, we show some analytical results of the equilibrium under parameter constraints and illustrate the more general cases with numerical examples.

Assumption 3.3 $\alpha_i = \alpha$ and $\rho_{Ni} = \eta \in [-1, 1]$ $(i \neq 1)$. $\rho_{ij} = \rho \in [0, 1]$ for all $i, j = 1, \cdots, N-1$.

First, we discuss a case with a dominant player who is not subject to performance-based fund flows while all other funds have to benchmark their performance to this player, and compare it to the case where this player also enters into the competition.

Proposition 3.7 *Under Assumptions 3.2 $(i = 1, \ldots, N)$ and 3.3, and assume that $\eta = \rho$.*

(i) If $\alpha_N = 0$, then the equilibrium strategies are $\pi_{-N}^{*0} = (A_f)_{-N}(P_f)_{-N}^{-1}\left(\gamma_{-N}^{-1}(\lambda_f)_{-N} + \frac{\alpha\rho}{N}\lambda_{1,\gamma_1} \cdot 1_{N-1}\right)$, and $\pi_N^{*0} = \lambda_{N,\gamma_N}/\sigma_N$, where A_f, P_f, γ and λ_f are defined in Theorem 3.1. The subscript $-N$ indicates deleting the last row and the last column. 1_k is a k-dimensional one vector.

(ii) *If* $\alpha_N = \alpha$*, then the equilibrium strategies are* $\pi_N^{*\alpha} = \frac{k_\rho\left(\tilde{\lambda}+\lambda_{N,\gamma_N}\right)}{\sigma_N}$ *and*

$$\pi_{-N}^{*\alpha} = (A_f)_{-N}(P_f)_{-N}^{-1}\left(\gamma_{-N}^{-1}(\lambda_f)_{-N} + \frac{\alpha\rho}{N}k_\rho\left(\tilde{\lambda} + \lambda_{N,\gamma_N}\right)\cdot 1_{N-1}\right)$$

where $k_\rho = \frac{N}{N+(N-1+\rho)\alpha}$ *and* $\tilde{\lambda} = \sum_{j=1}^{N}\frac{\alpha\rho}{N+(N-1)(1-\rho)\alpha}\lambda_{j,\gamma_j}$.

(iii) *If* $\lambda_{N,\gamma_N} \geq \varphi\sum_{j=1}^{N}\lambda_{j,\gamma_j}/N$*, where* $\varphi = \frac{\rho N^2}{(N+(N-1)(1-\rho)\alpha)(N-1+\rho)}$*, then* $\pi_i^{*0} \geq \pi_i^{*\alpha}$ *for every* $i = 1, \cdots, N$*, and vice versa.*

Notice that the parameters in the above proposition are set so that it is easy to compare equilibrium strategies in the two cases, while the mechanism as explained below follows similarly for general parameters. If fund N does not compete with others, its manager takes the Merton strategy. However, once it enters into the competition, if its own investment opportunity (summarized by λ_{N,γ_N}) is sufficiently good, then the concern for the risk in the relative performance dominates and manager N lowers the risk-taking. As a consequence, other funds also lower their risk-taking because the need for hedging fund N is smaller. On the other hand, if fund N tends to lose the competition, then the manager takes more risk in order to increase the chance of winning, and every other fund does the same in response.

As pointed out at the end of Sect. 2.1, good performance of funds with lower flow sensitivity (as fund N in Proposition 3.7 (i)) has a negative effect on the aggregate fund flows because it pushes the industry average up for those funds with larger flow sensitivities. However, since the manager's utility only relies on his/her own fund value and flows, the effect on the total cash flow of the whole sector is not part of the manager's consideration. Different α_N's decide the equilibrium mainly through the manager N's portfolio choice, which affects other funds' flow, as explained above.

As a by-product, notice that π_N^{*0} is the Merton strategy. Thus, Proposition 3.7 (iii) shows that (with the parameters therein), compared to the case without competition, managers tend to lower the risk-takings in order to hedge the risk in the relative performance. The only exception is the case of poor investment opportunity or large risk aversion (low λ_{N,γ_N}), in which the manager takes a larger risk under competition, in order to increase the chance of winning, which is consistent with the results of $N = 2$.

Next, we examine the case where S_N is negatively correlated with all other S_i's. The change in the portfolio choice follows similar patterns to the above. In a further simplified setting, we show that every fund becomes closer to the industry average under competition, though the investment strategy can change in different directions compared to the Merton strategy.

Proposition 3.8 *Under Assumptions 3.2 and 3.3, and assume that* $\eta = -\rho$*, with* κ_ρ *and* φ *defined in Proposition 3.7,*

(i) *The unique constant equilibrium is* $\pi_i^* = k_\rho\left((-1)^{\delta_N(i)}\hat{\lambda} + \lambda_{i,\gamma_i}\right)/\sigma_i$*, where* δ_N *is the indicator function of* N *and* $\hat{\lambda} = \sum_{j=1}^{N}\frac{\alpha\rho(-1)^{\delta_N(j)}}{N+(N-1)(1-\rho)\alpha}\lambda_{j,\gamma_j}$.

(ii) $\pi_i^M \geq \bar{\pi}_i^*$ *if and only if* $\lambda_{i,\gamma_i} \geq (-1)^{\delta_N(i)}\varphi\sum_{j=1}^{N}(-1)^{\delta_N(j)}\lambda_{j,\gamma_j}/N$.

(iii) Denote as $\bar{\lambda} = \frac{\lambda_N/\gamma_N}{\lambda/\bar{\gamma}_{-N}}$ *and* $\bar{\gamma}_{-N} = \frac{N-1}{\sum_{j=1}^{N-1} 1/\gamma_j}$*. Further assume*[5] *that* $\lambda_i = \lambda, \sigma_i = \sigma$ *for* $i = 1, \ldots, N-1$, $\rho = 1$, *and*[6] $\bar{\lambda} \neq N-1$. *Then,* $\left|\text{Beta}_i^* - 1\right| < \left|\text{Beta}_i^M - 1\right|$ *for every* $1 \leq i \neq N$.

Next, we check the effect of competition by numerical experiments in more general settings. The results, which are generally consistent with the conclusions for the case of $N = 2$ and the above analytical results given simplifying parameters, show that the competition tends to push funds to decrease risk-takings in their idiosyncratic investment opportunities, in order to decrease the risk in the relative performance. This usually leads to worse performance in terms of Sharpe ratios. However, managers with big disadvantages tend to take larger idiosyncratic risk in order to beat the average. In terms of herding effect, while the fund flows generally push funds to become closer to the industry average, if no funds are severely disadvantaged (in other words, the competition is severe), then the competition may push all competitors to move away from the industry average.

For $N = 5$, Fig. 4 plots the funds' portfolios with (π_i^*'s and θ_i^*'s) and without competition (π_i^M's and θ_i^M's), the corresponding Sharpe ratios η_i^*'s and η_i^M's, and volatilities σ_i^* and σ_i^M, with $\sigma_i = 0.2$, $\psi_i = 0.02$, $\alpha = 0.5$, $\gamma_i = 2$, $\rho_{im} = 0.1$ for every $1 \leq i \leq 5$, λ_i's forming an arithmetic sequence from 0.1 to 0.5, $\rho_{ij} = 0.2$ ($1 \leq i \neq j \leq 5$), $\lambda_m = 0.15$, $b = 0.15$, $r = 0.05$. Compared to the case where the managers do not have to care about relative performance, all the funds facing the competition have lower after-fee Sharpe ratios. The main reason is similar to the case of $N = 2$ that the managers are concerned about the risk of underperformance. Thus, they take less risk in the idiosyncratic opportunity S_i and more in the common investment opportunity S_m. This change is larger for funds with better investment opportunities. It lowers the expected return of the fund. On the other hand, since the decrease in π_i^* from π_i^M is much larger than the increase in θ_i^* from θ_i^M, the total risk-taking of the fund is also smaller, and thus the Sharpe ratio decreases, but not as much.

Figure 5 illustrates the case where $\lambda_i = 0.3$ and ρ_{im}'s form an arithmetic sequence from -0.2 to 0.6. Similar to the previous case, all the funds take less risk in their idiosyncratic investment opportunities in face of competition. If S_i is more positively correlated with S_m, fund i invests more in S_m, even changing from negative to positive amount in some cases. Only for fund 1 with $\rho_{1m} < 0$, $\theta_1^* < \theta_1^M$, because the need for hedging the risk in S_1 is reduced. Similar to the previous case, the total risk of the

[5] Though it is a simplified setting in which the first $N-1$ assets are perfectly correlated and the last one is perfectly negatively correlated with other assets, the manager's portfolio choice problem is not trivial. For fund $i \in \{1, \ldots, N-1\}$, the fund flow due to the last fund brings positive exposure to their own idiosyncratic risk. Yet to hedge such a risk manager i may not want to lower the risky investment by too much, because it may hurt the absolute return of the fund. Also, this setting does not create arbitrage opportunities, because each fund only has access to one investment opportunity.

[6] If $\bar{\lambda} = N-1$, then the industry average both with and without competition is riskless and Beta coefficients are not well defined.

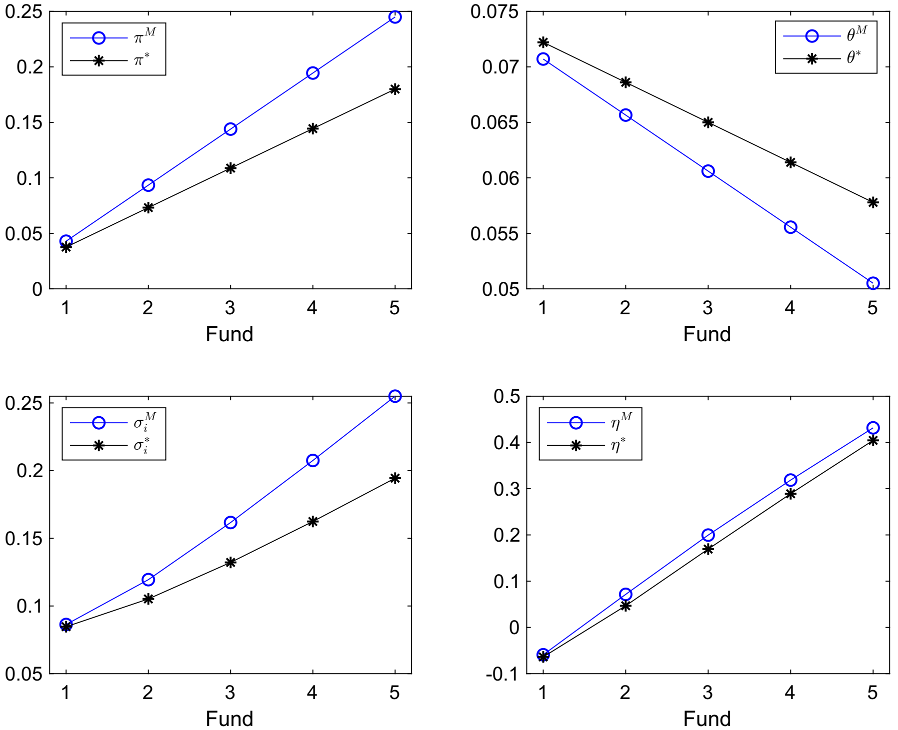


Fig. 4 Funds' portfolios, volatility and Sharpe ratios, with $\sigma_i = 0.2, \psi_i = 0.02, \alpha = 0.5, \gamma_i = 2, \rho_{im} = 0.1$ for every $1 \le i \le 5$, λ_i's form an arithmetic sequence from 0.1 to 0.5, $\rho_{ij} = 0.2$ $(1 \le i \ne j \le 5)$, $\lambda_m = 0.15$, $b = 0.15$, $r = 0.05$

funds and the Sharpe ratios decrease. Notice that in this case, λ_i/γ_i is a constant across 5 funds, and the result agrees with Proposition 3.3 in the change of Sharpe ratios.

Figure 6 illustrates the case where λ_i's form an arithmetic sequence from 0.1 to 1.3, γ_i's form an arithmetic sequence from 0.5 to 4, $\rho_{im} = 0$ for every $1 \le i \le 5$, $\rho_{i5} = -0.2$ $(i \ne 5)$, $\rho_{ij} = 0.2$ $(1 \le i \ne j \le 5)$, and other parameters are the same as in the previous cases. It shows some features that we do not see in the case of $N = 2$, due to the complex dependence on the correlation structure. λ_i's have larger differences than in the previous cases, but λ_i/γ_i actually become closer than in Fig. 4. With negative correlations between some S_i's, while other funds behave similarly as in 4, the most disadvantaged manager (of fund 1) takes the larger risk in the fund's idiosyncratic investment opportunity, and less in S_m, in order to have a better chance of beating the competitors and attracting new investments. As a result, the total risk of fund 1's investments is larger compared to the case without competition.

Next, we use the Beta coefficients with respect to the industry average and S_m to measure the herding and specialization effect, respectively, of the competition. In particular, if $|\text{Beta}_i^* - 1| < |\text{Beta}_i^M - 1|$, then fund i is closer to the industry average, and if $|\text{Beta}_{mi}^* - 1| > |\text{Beta}_{mi}^M - 1|$, then fund i tends to specialize more in its idiosyncratic investment opportunity rather than S_m.

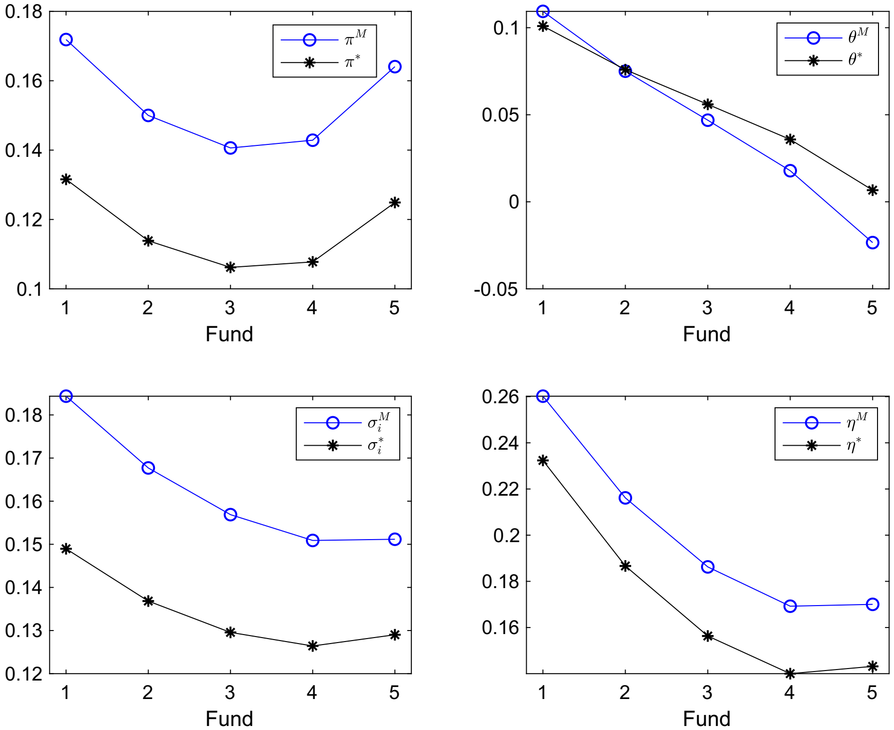


Fig. 5 Funds' portfolios, volatility and Sharpe ratios, with $\sigma_i = 0.2$, $\psi_i = 0.02$, $\alpha = 0.5$, $\gamma_i = 2$, $\lambda_i = 0.3$ for every $1 \le i \le 5$, $\rho_{ij} = 0.2$ $(1 \le i \ne j \le 5)$, ρ_{im}'s form an arithmetic sequence from -0.2 to 0.6, $\lambda_m = 0.15$, $b = 0.15$, $r = 0.05$

Figure 7 illustrates the case with the same parameters as for Fig. 4. From Fig. 4, to hedge the risk from relative performance, the decrease in π_i^* from π_i^M is more than the increase in θ_i^* from θ_i^M. Since $\rho_{im} > 0$ for every $1 \le i \le 5$, the combined effect is that each fund's investment is further away from S_m (Beta_{mi}^* is further away from 1 than Beta_{mi}^M), and at the same time closer to the industry average. In Fig. 8, the model parameters are the same as for Fig. 5. As shown in Fig. 5, each fund decreases its investment in the idiosyncratic investment opportunity, and most of them increase the investment in S_m, except fund 1, because S_1 is negatively correlated with S_m. Thus, though the Beta coefficients with respect to R_m do not change much with and without competition, Beta_i^*'s are always closer to 1 than Beta_i^M's. Both the above results show that in general, the competition pushes mutual funds to herd. Also, though θ_i^* in most cases are greater than θ_i^M, because of the positive correlations between S_i's and S_m, each fund is further away from the common investment opportunity.

In Fig. 9, the model parameters are the same as for Fig. 6, and the funds' behaviors change drastically. While funds 2–5 decrease their investment in their idiosyncratic risk, the disadvantaged fund 1 takes an excessive risk in S_1. It shifts the industry average so large that $|\text{Beta}_i^* - 1| > |\text{Beta}_i^M - 1|$ for every $1 \le i \le 5$, which means that competition actually increases the risk in relative performance for every fund, even

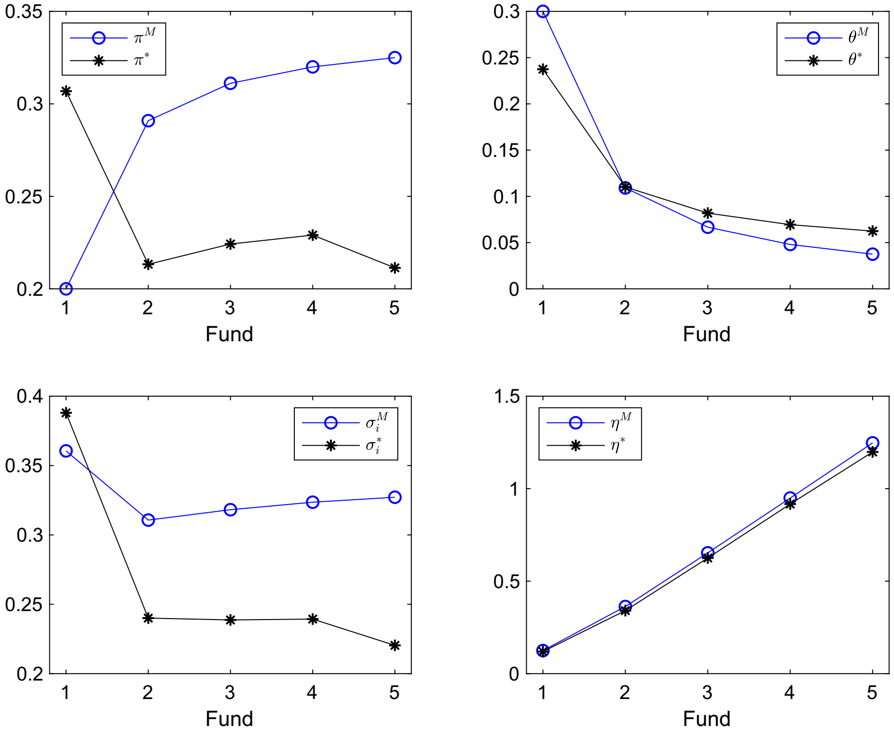


Fig. 6 Funds' portfolios, volatility and Sharpe ratios, with $N = 5$, $\sigma_i = 0.2$, $\psi_i = 0.02$, $\alpha = 0.5$, $\rho_{im} = 0$ for every $1 \leq i \leq 5$, λ_i's form an arithmetic sequence from 0.1 to 1.3, γ_i's form an arithmetic sequence from 0.5 to 4, $\rho_{i5} = -0.2$ $(i \neq 5)$, $\rho_{ij} = 0.2$ $(1 \leq i \neq j \leq 5)$, $\lambda_m = 0.15$, $b = 0.15$, $r = 0.05$

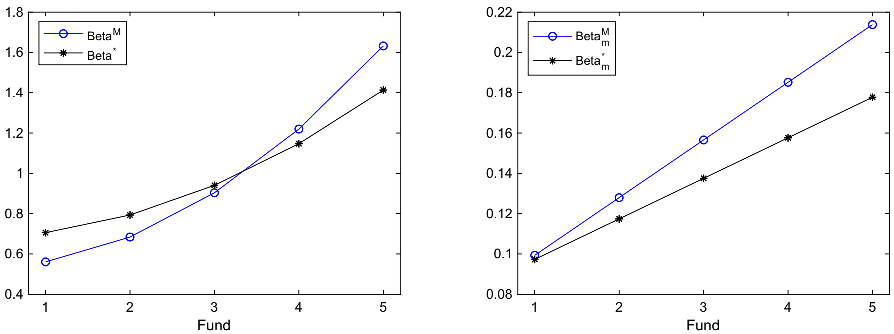


Fig. 7 Funds' Beta coefficients with and without competition, with $N = 5$, $\gamma_i = 2$, $\alpha_i = 0.5$, $\sigma_i = 0.2$ and $\rho_{im} = 0.2$, $1 \leq i \leq 5$, λ_i's form an arithmetic sequence from 0.1 to 0.5, and $\rho_{ij} = 0.1$ for $1 \leq i \neq j \leq 5$

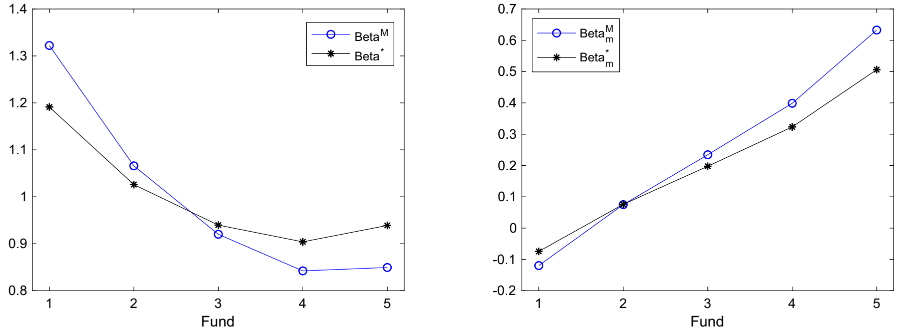


Fig. 8 Funds' Beta coefficients with and without competition, with $N = 5$, $\lambda_i = 0.3$, $\gamma_i = 2$, $\alpha_i = 0.5$, $\sigma_i = 0.2$, $1 \leq i \leq 5$, ρ_{im}'s form an arithmetic sequence from -0.2 to 0.6, and $\rho_{ij} = 0.2$ for $1 \leq i \neq j \leq 5$. $\lambda_m = 0.15$, $b = 0.15$

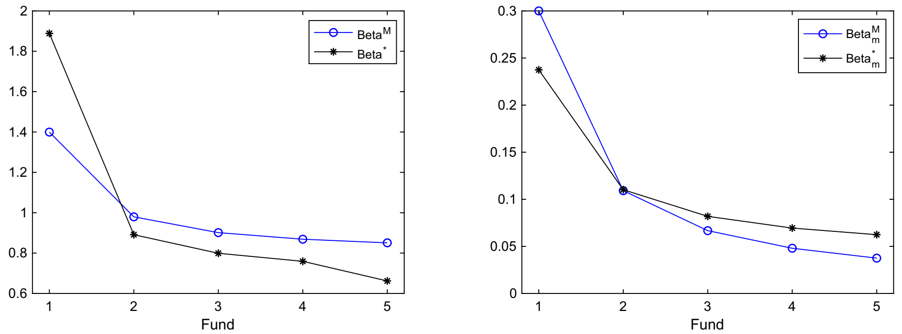


Fig. 9 Funds' Beta coefficients with and without competition, with $N = 5$, $\sigma_i = 0.2$, $\psi_i = 0.02$, $\alpha = 0.5$, $\rho_{im} = 0$ for every $1 \leq i \leq 5$, λ_i's form an arithmetic sequence from 0.1 to 1.3, γ_i form an arithmetic sequence from 0.5 to 4, $\rho_{i5} = -0.2$ $(i \neq 5)$, $\rho_{ij} = 0.2$ $(1 \leq i \neq j \leq 5)$, $\lambda_m = 0.15$, $b = 0.15$, $r = 0.05$

though most of them choose the optimal portfolios to avoid this. In this case, funds 2–5 take more exposure in S_m and only fund 1 specializes more in its idiosyncratic risk. Notice that in this case, the risk aversion-adjusted Sharpe ratios for all funds are very close to each other. Thus, this result offers some new insights into the effect of the managers' competition, which is not shown in the previous literature focusing on the case of two funds. Instead of herding, severe competition may push all funds to move away from their competitors. It is caused by the large risk exposure the disadvantaged manager adopts, in order to survive the competition. However, from the systemic point of view, it is not a big concern, because from Fig. 6, most funds take less risk under competition, and the whole group takes a more diversified portfolio.

3.3 Wider Market Access

In the previous discussion, each fund has its own investment opportunities that are not perfectly correlated, which is similar to the case of asset specialization in [2].

Next, we discuss the case where one or more manager has access to more investment opportunities, e.g., by hiring experts on new asset classes.

If there exists a common (and sufficiently large) set of assets that every fund has access to, as in the case of diversification in [2, 4], then deriving the equilibrium becomes simpler than the previous case, because every manager faces a complete market. The constant equilibrium can be calculated also in closed form and is actually the unique equilibrium among all possible strategies, similar to the case of Proposition 3.5.

The market completeness offers the manager a better deal and the need for speculation and hedging can be spread among different assets, depending on their Sharpe ratios and correlations. The asset allocation is still driven by the same two main considerations: pursuing superior absolute returns and hedging against the risk in the relative performance.

If $N = 2$ and both fund has access to S_1 and S_2, and Assumption 3.2 holds so that there is no investment in S_m, then the equilibrium portfolio for manager i in both assets is

$$\pi_{i1}^* = \frac{4 + 2\alpha_1 + 2\alpha_2}{(2 + \alpha_1)(2 + \alpha_2)} \frac{\lambda_1 - \rho_{12}\lambda_2}{(1 - \rho_{12}^2)\sigma_1\gamma_i}, \pi_{i2}^* = \frac{4 + 2\alpha_1 + 2\alpha_2}{(2 + \alpha_1)(2 + \alpha_2)} \frac{\lambda_2 - \rho_{12}\lambda_1}{(1 - \rho_{12}^2)\sigma_2\gamma_i}.$$

Thus, each manager essentially takes the Merton strategy, with a decreased risk exposure ($4 + 2\alpha_1 + 2\alpha_2 < (2 + \alpha_1)(2 + \alpha_2)$), and their Beta$_i^*$'s are the same as in the case without competition, which is different from the specialization case in Propositions 3.3 and 3.4. Next, we compare equilibrium strategies in the current setting of diversification with those in the case of specialization. We omit the proof which only involves algebraic calculations.

Proposition 3.9 *For π_1^* and π_2^* in (10), $\pi_i^* \geq \pi_{ii}^*$ if and only if ($j \in \{1, 2\}$ and $j \neq i$)*

$$\frac{2 + \alpha_j}{\kappa_1} \left(1 + \rho_{12} \frac{\alpha_i \lambda_j \gamma_i}{(2 + \alpha_j)\lambda_i \gamma_j}\right) \geq \frac{2 + \alpha_1 + \alpha_2}{1 - \rho_{12}^2} \left(1 - \frac{\rho_{12}\lambda_j}{\lambda_i}\right).$$

Whether manager i increases or decreases the investment in S_i once fund i also has access to S_j mainly depends on the correlation between the two assets. Notice that $\kappa_1 = 1 - \frac{\alpha_1\alpha_2\rho_{12}^2}{(2+\alpha_1)(2+\alpha_2)} > 1 - \rho_{12}^2$, thus $(2 + \alpha_j)/\kappa_1 < (2 + \alpha_1 + \alpha_2)/(1 - \rho_{12}^2)$. If $\rho_{12} < 0$, then $\pi_{ii}^* > \pi_i^*$ - manager i invests more in S_i because S_2 can be used as a hedging tool for the larger risk in S_1, which achieves a better risk-return trade-off. If $\rho_{12} > 0$, then with access to S_j, manager i may not need as much risk exposure in S_i as in the specialization case. This does not always happen and depends on whether S_j is indeed a good substitute for S_i (e.g., with large λ_j compared to λ_i).

Next, we consider the case where fund 1 can only invest in asset 1, and fund 2 has access to both asset 1 and 2. The excess return of the latter is $dR_{2t} = \pi_{2t}(\mu_2 dt + \sigma_2 dW_{2t}) + \pi_{3t}(\mu_1 dt + \sigma_1 dW_{1t})$, where π_{3t} is fund 2's proportional investment in S_1. Their accounts follow

$$\begin{aligned}\frac{dX_{1t}}{X_{1t}} &= \left(r - \frac{2+\alpha_1}{2}\psi_1 + \frac{\alpha_1}{2}\psi_2\right)dt + \frac{2+\alpha_1}{2}\pi_1(\mu_1 dt + \sigma_1 dW_{1t}) \\ &\quad - \frac{\alpha_1}{2}\left(\pi_2(\mu_2 dt + \sigma_2 dW_{2t}) + \pi_3(\mu_1 dt + \sigma_1 dW_{1t})\right), \\ \frac{dX_{2t}}{X_{2t}} &= \left(r - \frac{2+\alpha_2}{2}\psi_2 + \frac{\alpha_2}{2}\psi_1\right)dt - \frac{\alpha_2}{2}\pi_1(\mu_1 dt + \sigma_1 dW_{1t}) \\ &\quad + \frac{2+\alpha_2}{2}\left(\pi_2(\mu_2 dt + \sigma_2 dW_{2t}) + \pi_3(\mu_1 dt + \sigma_1 dW_{1t})\right).\end{aligned}$$

The equilibrium portfolios in this setting (following similar calculations to those for Theorem 3.1) are

$$\begin{aligned}\hat{\pi}_1^* &= \frac{2+\alpha_2}{(2+\alpha_1+\alpha_2)\,\sigma_1}\left[\lambda_{1,\gamma_1} + \frac{\alpha_1}{2+\alpha_2}\lambda_{1,\gamma_2}\right], \\ \hat{\pi}_2^* &= \frac{2}{(2+\alpha_2)\sigma_2}\frac{\lambda_{2,\gamma_2} - \rho_{12}\lambda_{1,\gamma_2}}{1-\rho_{12}^2}, \\ \hat{\pi}_3^* &= \frac{2}{(2+\alpha_2)\sigma_1}\frac{\lambda_{1,\gamma_2} - \rho_{12}\lambda_{2,\gamma_2}}{1-\rho_{12}^2} + \frac{\alpha_2}{(2+\alpha_1+\alpha_2)\sigma_1}\left[\lambda_{1,\gamma_1} + \frac{\alpha_1}{2+\alpha_2}\lambda_{1,\gamma_2}\right],\end{aligned}$$

They immediately show the advantage fund 2 of obtaining wider market access, thus facing a complete market. Fund 2 can focus on the risk-return trade-off of the absolute return and not worry about the risk in the relative performance. Whatever π_1 is, manage 2 can change π_3 accordingly to completely eliminate the effect of π_1 on the dynamics of X_2 due to fund flows. Thus, essentially, π_3 is of the form $\pi_3 = A + \frac{\alpha_2}{2+\alpha_2}\pi_1$. Then, manager 2 only has to make sure that $\frac{2+\alpha_2}{2}\pi_2$ and $\frac{2+\alpha_2}{2}A$ equal the Merton strategy, and that is where $\hat{\pi}_2^*$ and the first term of $\hat{\pi}_3^*$, which we refer to as Merton component$_2$ and Merton component$_3$, come from.

For fund 1, the risk exposure to dW_1 is $\frac{2+\alpha_1}{2}\pi_1 - \frac{\alpha_1}{2}\pi_3 = \frac{2+\alpha_1+\alpha_2}{2+\alpha_2}\pi_1 - \frac{\alpha_1}{2}$Merton component$_3$, and the exposure of to dW_1 is $-\frac{\alpha_1}{2}\hat{\pi}_2^* = -\frac{\alpha_1}{2}$Merton component$_2$. Since $dW_2 = \rho_{12}dW_1 + \sqrt{1-\rho^2}dB_t$, where B is a Brownian motion independent of W_1, and manager 1 can do nothing about, the manager essentially faces a Merton problem with an exposure of $\frac{2+\alpha_1+\alpha_2}{2+\alpha_2}\pi_1 - \frac{\alpha_1\lambda_{1,\gamma_2}}{2+\alpha_2}$ to W_1, and thus $\hat{\pi}_1^*$, which happens to be independent of ρ_{12} in this case. Direct algebraic calculations show the following:

Proposition 3.10 *For π_i^*'s in (10), $\pi_1^* \geq \hat{\pi}_1^*$ if and only if $D \geq 0$, and $\pi_2^* \geq \hat{\pi}_2^*$ if and only if $\rho_{12}D \leq 0$, where*

$$\begin{aligned}D = \rho_{12}&\left(1 - \frac{\alpha_1\alpha_2}{(2+\alpha_1)(2+\alpha_2)}\right)\frac{\lambda_2}{\lambda_1} - \\ &\left(1 - \frac{\alpha_1\alpha_2}{(2+\alpha_1)(2+\alpha_2)}\rho_{12}^2 + \frac{\alpha_2}{2+\alpha_1}(1-\rho_{12}^2)\frac{\gamma_2}{\gamma_1}\right).\end{aligned}$$

If $\rho_{12} = 0$, then manager 2 cannot hedge the risk in S_1 by the investment in S_2; thus, the optimal choice of π_2 does not change with access to S_1. Notice that $\frac{\alpha_1\alpha_2}{(2+\alpha_1)(2+\alpha_2)} < 1$

and $1 - \frac{\alpha_1\alpha_2}{(2+\alpha_1)(2+\alpha_2)}\rho_{12}^2 + \frac{\alpha_2}{2+\alpha_1}(1-\rho_{12}^2)\frac{\gamma_2}{\gamma_1} > 0$. Thus, if $\rho_{12} < 0$, then $D < 0$ and $\hat{\pi}_2^* > \pi_2^*$, because the risk exposure to S_2 can be hedged by investment in S_1, and the optimal π_2 becomes larger to achieve a better risk-return trade-off, and becomes a Merton strategy. On the other hand, if $\rho_{12} > 0$ and λ_2/λ_1 is small so that $D < 0$, then the Merton strategy is smaller than π_2^* because part of the desired risk exposure is fulfilled by the investment in S_1. If λ_2/λ_1 is sufficiently large so that $D > 0$, then the Merton strategy is larger because, without access to S_1, the concerns for the risk in the relative return make manager 2 conservative and cannot take full advantage of the good investment opportunity in S_2.

On the other hand, the above discussion shows that in addition to fully eliminating the risk from fund 1, fund 2 essentially takes the Merton portfolio, and the combined effect of $\hat{\pi}_2^*$ and $\hat{\pi}_3^*$ is a fixed negative exposure $\frac{\alpha_1\lambda_{1,\gamma_2}}{2+\alpha_2}$ to W_1. Thus, fund 1 optimally increases π_1 to hedge this risk, because π_1^* is relatively smaller in the case of specialization, either due to the positive exposure to W_1 from fund flows ($\rho_{12} < 0$) or relatively small λ_2 (see Proposition 3.1(ii)). The only exception is the case of $\rho_{12} > 0$ and large λ_2/λ_1 so that $D > 0$. In this case π_1^* is larger (see Proposition 3.1) than $\hat{\pi}_1^*$, which is needed to hedge a large negative exposure to W_1.

4 Forward Relative Performance Criteria

In this section, we investigate the Nash equilibrium under the forward performance criterion introduced in [2]. It allows us to relax the assumption of the common planning horizon and generalize our results to the setting of stochastic investment opportunities. Since all the definitions follow closely to those in [2] and the calculations are quite similar to previous sections, we omit all the proofs in this section.

Assume that the investment opportunities are described by (1) and (2), with r, b, a, μ_i's and σ_i's, ρ_{im}'s and ρ_{ij}'s being $\mathcal{F}_t$-adapted processes. All the Sharpe ratios λ_m and λ_i's are bounded. The admissible sets of trading strategies are $\mathcal{A}_i$'s and Θ can be defined the same way. Instead of a given utility function, the managers' preference is characterized by the forward performance criterion in the following:

Definition 4.1 ([2][Definition 1 and Definition 2])

(i) Let $\mathcal{U}$ be the set of random functions $u(t,x)$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}_+$, such that for each $t \geq 0$ and P-a.s., the mapping $x \to u(t,x)$ is strictly concave and strictly increasing, and $u(t,x) \in C^{1,4}$.

(ii) Given $(\pi_{-i}, \theta_{-i}) \in \bigotimes_{j\neq i} A_j \times \Theta^{N-1}$, an $\mathcal{F}_t$-adapted process $(V_i(t, x_i; \pi_{-i}, \theta_{-i}))_{t\geq 0}$ is a best forward relative performance criteria for fund manager i if the following conditions hold:

   (a) For each $t \geq 0$, $V_i(t, x_i; \pi_{-i}, \theta_{-i}) \in \mathcal{U}$ a.s.
   (b) For each $(\pi_i, \theta_i) \in \mathcal{A}_i \times \Theta$, $V_i(t, X_{it}; \pi_{-i}, \theta_{-i})$ is a (local) supermartingale, where X_i follows (3) with (π_i, θ_i) plugged in.
   (c) There exists $(\hat{\pi}_i, \hat{\theta}_i) \in \mathcal{A} \times \Theta$ such that $V_i(t, \hat{X}_{it}; \pi_{-i}, \theta_{-i})$ is a (local) martingale, where $\hat{X}_i$ follows (3) with $(\hat{\pi}_i, \hat{\theta}_i)$ plugged in.

For tractability, we look for the locally riskless performance process V_i such that $dV_i(t, x_i; \pi_{-i}, \theta_{-i}) = \nu_i(t, x_i; \pi_{-i}, \theta_{-i})dt$, for some $\mathcal{F}_t$-adapted process $\nu_i(t, x_i; \pi_{-i}, \theta_{-i})$, as in [2]. To compare with the results in the previous section, we focus on the CRRA type of V_i.

Lemma 4.1 *Given* $(\pi_{-i}, \theta_{-i}) \in \bigotimes_{j \neq i} A_j \times \Theta^{N-1}$ *and denote as*

$$\eta_i = h_i' w_i h_i - 2\gamma_i \left(r - \tilde{\psi}_i - \pi_{-i}' C_i \lambda_{-i} + h_i' w_i^{-1} w_{-i} w_i C_i \pi_{-i} \right) + \gamma_i^2 \pi_{-i}' C_i \left(\rho_{-i} - w_{-i}' w_i^{-1} w_{-i} \right) C_i \pi_{-i},$$

where $\tilde{\psi}_i, h_i, w_i, w_{-i}, C_i, \lambda_{-i}$ *are defined in the proof of Theorem 3.1 and Lemma 4.2, and* ρ_{-i} *is the* $(N-1) \times (N-1)$ *correlation matrix of* W_j*'s* $(j \neq i)$*. Then, the process*

$$V_i(t, x_i; \pi_{-i}, \theta_{-i}) = \frac{x_i^{1-\gamma_i}}{1-\gamma_i} e^{-\int_0^t \frac{1-\gamma_i}{\gamma_i} \eta_i ds}$$

is a locally riskless best-response forward criteria for manager i *and the optimal policy is*

$$\pi_{it}^* = \frac{N}{(N + (N-1)\alpha_i)\sigma_{it}} \left(\frac{\lambda_{it} - \rho_{imt}\lambda_{mt}}{(1 - \rho_{imt}^2)\gamma_{it}} + \frac{\alpha_i}{N} \sum_{j \neq i}^{N} \pi_{jt}\sigma_{jt} \frac{\rho_{ijt} - \rho_{imt}\rho_{jmt}}{1 - \rho_{imt}^2} \right),$$

$$\theta_{it}^* = \frac{N}{(N + (N-1)\alpha_i)b_t} \left(\frac{\lambda_{mt} - \rho_{imt}\lambda_{it}}{(1 - \rho_{imt}^2)\gamma_{it}} + \frac{\alpha_i}{N} \sum_{j \neq i}^{N} \left(\pi_{jt}\sigma_{jt} \frac{\rho_{jmt} - \rho_{imt}\rho_{ijt}}{1 - \rho_{imt}^2} + \theta_{jt} b_t \right) \right).$$

We can then define the Nash equilibrium among all the funds based on the above relative forward performance criteria.

Definition 4.2 A forward Nash equilibrium is an N-dimensional vector consisting of $\mathcal{F}_t$-adapted process $\left(V_i(t, x_i; \pi_{-i}^*, \theta_{-i}^*), (\pi_i^*, \theta_i^*)\right)$ with the following properties: for any $i = 1, \cdots, N$,

(i) $(\pi_i^*, \theta_i^*) \in \mathcal{A}_i \times \Theta$.
(ii) $V_i(t, X_i; \pi_{-i}^*, \theta_{-i}^*) \in \mathcal{U}$ a.s.
(iii) For every $(\pi_i, \theta_i) \in \mathcal{A}_i \times \Theta$, $V_i(t, X_i; \pi_{-i}^*, \theta_{-i}^*)$ is a (local) supermartingale and $V_i(t, X_i^*; \pi_{-i}^*, \theta_{-i}^*)$ is a (local) martingale, where X_i, X_i^* follow (3) with (π_i, θ_i) and (π_i^*, θ_i^*) plugged in, respectively.

With closed-form best-response strategy derived in Lemma 4.1, we can easily calculate the forward Nash equilibrium. The following proposition shows that the equilibrium in Theorem 3.1 can be easily generalized to an equilibrium under the forward performance criterion. Thus, our discussions about the equilibrium strategies apply to the more general setting with stochastic market parameters and without the assumption of the common planning horizon across all funds.

Proposition 4.1 *With π^* and θ^* in (5) and (6) (after plugging the stochastic model parameters), $\left(V_i(t, x_i; \pi^*_{-i}, \theta^*_{-i}), (\pi^*_i, \theta^*_i)\right)_{1\le i\le N}$ is a forward Nash equilibrium.*

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

The proof of Theorem 3.1 The first step is to find the optimal portfolio choice of the fund i, given the investment strategies π_{-i} and θ_{-i} of other funds. Since we focus on constant equilibria, assume that π_{-i} and θ_{-i} are constants. Then, with $\tilde{X}_{it} = \exp\left(-\left(r - \tilde{\psi}_i\right) t\right) X_{it}$, where $\tilde{\psi}_i = \left(1 + \frac{N-1}{N}\alpha_i\right) \psi_i - \frac{\alpha_i}{N} \sum_{j\neq i}^N \psi_j$, Fubini's theorem implies that

$$\mathbb{E}\left[\int_0^T e^{-\beta_i t} \frac{(\psi_i X_{it})^{1-\gamma_i}}{1-\gamma_i} dt\right] = \int_0^T e^{\left(-\beta_i + r - \tilde{\psi}_i\right)t} \psi_i^{1-\gamma_i} \frac{\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right]}{1-\gamma_i} dt.$$

For any $(\pi_i, \theta_i) \in \mathcal{A}_i \times \Theta$ $(1 \le i \le N)$, let $\tilde{\pi}_{it} = \left(1 + \frac{N-1}{N}\alpha_i\right) \sigma_i \pi_{it}$, $\tilde{\theta}_{it} = \left(1 + \frac{N-1}{N}\alpha_i\right) b\theta_{it}$, so that $\pi = (\pi_1, \cdots, \pi_N) = A_f \tilde{\pi}$ and $\theta = A_m \tilde{\theta}$, where $\tilde{\pi}$ and $\tilde{\theta}$ are N-dimensional vectors with $(\tilde{\pi})_i = \tilde{\pi}_i$ and $(\tilde{\theta})_i = \tilde{\theta}_i$. Thus,

$$\frac{d\tilde{X}_{it}}{\tilde{X}_{it}} = \tilde{\pi}_{it}(\lambda_i dt + dW_{it}) + \left(\tilde{\theta}_{it} - \sum_{j\neq i}^N c_{ij}\tilde{\theta}_j\right)(\lambda_m dt + dB_t) - \sum_{j\neq i}^N c_{ij}\left(\tilde{\pi}_j(\lambda_j dt + dW_{jt})\right).$$

With $\phi_{it} = \begin{bmatrix} \tilde{\pi}_{it} \\ \tilde{\theta}_{it} - \sum_{j\neq i}^N c_{ij}\tilde{\theta}_j \end{bmatrix}$, $h_i = \begin{bmatrix} \lambda_i \\ \lambda_m \end{bmatrix}$, $F_{it} = \begin{bmatrix} W_{it} \\ B_t \end{bmatrix}$, $\lambda_{-i} = \left[\cdots \lambda_{i-1}\ \lambda_{i+1} \cdots\right]'$, and $W_{-it} = \left[\cdots W_{(i-1)t}\ W_{(i+1)t} \cdots\right]'$, the dynamics of $\tilde{X}$ is

$$\frac{d\tilde{X}_{it}}{\tilde{X}_{it}} = \phi'_{it}(h_i dt + dF_{it}) - \tilde{\pi}'_{-i} C_i(\lambda_{-i} dt + dW_{-it}),$$

where C_i is an $(N-1)$-dimensional matrix with diagonal entries c_{ij} for $1 \leq j \leq N$ and $j \neq i$.

Lemma 4.2 shows that $\hat{\phi}_i = \frac{1}{\gamma_i} w_i^{-1} h_i + w_i^{-1} w_{-i} C_i \pi_{-i}$ maximizes $\frac{\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right]}{1-\gamma_i}$. Since it is a constant strategy independent of t, it also maximizes the discounted expected utility from management fees for each manager i

$$\int_0^T e^{\left(-\beta_i + r - \tilde{\psi}_i\right)t} \psi_i^{1-\gamma_i} \frac{\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right]}{1-\gamma_i} dt.$$

$\hat{\phi}_i = \frac{1}{\gamma_i} w_i^{-1} h_i + w_i^{-1} w_{-i} C_i \tilde{\pi}_{-i}$ for $1 \leq i \leq N$ are $2N$ equations of constants $\tilde{\pi} = (\tilde{\pi}_1, \cdots, \tilde{\pi}_N)$ and $\tilde{\theta} = (\tilde{\theta}_1, \cdots, \tilde{\theta}_N)$:

$$P_f \tilde{\pi} = \gamma^{-1} \lambda_f, \quad P_m \tilde{\theta} = \gamma^{-1} \eta_m + C\tilde{\pi},$$

of which the solution corresponds to the equilibrium strategies of the N funds. Since Lemma 4.3 shows that P_f and P_m are invertible, there exists a unique solution $\tilde{\pi} = P_f^{-1} \gamma^{-1} \lambda_f$, $\tilde{\theta} = P_m^{-1} \left(\gamma^{-1} \eta_m + C\tilde{\pi}\right)$. Therefore, $\pi^* = A_f P_f^{-1} \gamma^{-1} \lambda_f$, and $\theta^* = A_m P_m^{-1} \left(\gamma^{-1} \eta_m + C A_f^{-1} \pi^*\right)$. □

Lemma 4.2 *Given constant* π_{-i} *and* θ_{-i}, $\underset{\phi_i:(\pi_i,\theta_i)\in\mathcal{A}_i\times\Theta}{\arg\max} \frac{\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right]}{1-\gamma_i} = \hat{\phi}_i = \frac{1}{\gamma_i} w_i^{-1} h_i + w_i^{-1} w_{-i} C_i \pi_{-i}$, *for every* $0 \leq t \leq T$, *where* $w_i = \begin{bmatrix} 1 & \rho_{im} \\ \rho_{im} & 1 \end{bmatrix}$, $w_{-i} = \begin{bmatrix} (\rho_i)'_{-i} \\ (\rho_m)'_{-i} \end{bmatrix}$, *and* ρ_i *is the* N*-dimensional vector with* $(\rho_i)_j = \rho_{ij}$.

Proof We prove the case of $0 < \gamma_i < 1$ and focus on $\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right]$ because $1 - \gamma_i > 0$. The case of $\gamma > 1$ follows similarly. Define a stochastic process ξ such that $\xi_0 = 1$ and

$$-\frac{d\xi_t}{\xi_t} = \left(M'_i w_{-i} + M'_{-i} \rho_{-i} - \lambda'_{-i}\right) C_i \pi_{-i} dt + M'_i dF_{it} + M'_{-i} dW_{-it},$$

where M_i and M_{-i} are two constant vectors to be determined later, which satisfy $w_i M_i + w_{-i} M_{-i} = h_i$. Then,

$$\begin{aligned} \frac{d\xi_t \tilde{X}_{it}}{\xi_t \tilde{X}_{it}} &= -\left(M'_i w'_i + M'_{-i} w'_{-i} - h'_i\right) \phi_{it} dt + (\phi'_{it} - M'_i) dF_{it} \\ &\quad - (\pi'_{-i} C_i + M'_{-i}) dW_{-it} \\ &= (\phi'_{it} - M'_i) dF_{it} - (\pi'_{-i} C_i + M'_{-i}) dW_{-it}. \end{aligned}$$

Thus, $\xi_t \hat{X}_{it}$ is a nonnegative local martingale and hence a supermartingale. Therefore (ignoring the positive $1-\gamma_i$), by Hölder's inequality and noticing that $\tilde{X}_{i0} = X_{i0} = 1$,

$$\begin{aligned}\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right] &\le \mathbb{E}\left[\xi_t \tilde{X}_{it}\right]^{1-\gamma_i} \mathbb{E}\left[\xi_t^{\frac{\gamma_i-1}{\gamma_i}}\right]^{\gamma_i} \le \mathbb{E}\left[\xi_t^{\frac{\gamma_i-1}{\gamma_i}}\right]^{\gamma_i} \\ &= \exp\left((1-\gamma_i)\left(\left(M_i' w_{-i} + M_{-i}'\rho_{-i} - \lambda_{-i}'\right) C_i \pi_{-i} + \frac{1}{2\gamma_i} M_i' w_i M_i \right.\right. \\ &\quad \left.\left. + \frac{1}{2\gamma_i} M_{-i}' \rho_{-i} M_{-i} + \frac{1}{\gamma_i} M_i' w_{-i} M_{-i}\right) t\right),\end{aligned}$$

which is an upper bound for $\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right]$ corresponding to any $(\pi_i, \theta_i) \in \mathcal{A}_i \times \Theta$.

Next we search for the minimum among all such upper bounds corresponding to different choices of M_i and M_{-i}, by considering the following constrained minimization problem:

$$\begin{aligned}&\min_{\{M_i, M_{-i}\}} \frac{1}{2\gamma_i} M_i' w_i M_i + \frac{1}{2\gamma_i} M_{-i}' \rho_{-i} M_{-i} + \frac{1}{\gamma_i} M_i' w_{-i} M_{-i} + \\ &\qquad \left(M_i' w_{-i} + M_{-i}' \rho_{-i}\right) C_i \pi_{-i}, \\ &\text{subject to: } w_i M_i + w_{-i} M_{-i} = h_i.\end{aligned}$$

The corresponding Lagrangian function, with Lagrange multiplier l, is

$$\begin{aligned}\mathcal{L} = &\frac{1}{2\gamma_i}\left(M_i' w_i M_i + M_{-i}' \rho_{-i} M_{-i} + 2 M_i' w_{-i} M_{-i}\right) + \\ &\left(M_i' w_{-i} + M_{-i}' \rho_{-i}\right) C_i \pi_{-i} + l' \left(h_i - w_i M_i - w_{-i} M_{-i}\right).\end{aligned}$$

The first-order conditions for M_i, M_{-i} and l are

$$M_i = \gamma_i l - w_i^{-1} w_{-i} M_{-i} - \gamma_i w_i^{-1} w_{-i} C_i \pi_{-i}, \tag{13}$$

$$0 = \frac{1}{\gamma_i} \rho_{-i} M_{-i} + \frac{1}{\gamma_i} w_{-i}' M_i + \rho_{-i} C_i \pi_{-i} - w_{-i}' l,$$

$$0 = h_i - w_i M_i - w_{-i} M_{-i}. \tag{14}$$

Plugging (13) into (14) implies that

$$0 = \left(\rho_{-i} - w_{-i}' w_i^{-1} w_{-i}\right)\left(M_{-i} + \gamma_i C_i \pi_{-i}\right).$$

Instead of discussing the uniqueness of solutions to the above equation, we pick out one of them $M_{-i} = -\gamma_i C_i \pi_{-i}$, $M_i = \gamma_i \hat{\phi}_i$, $l = \hat{\phi}_i$, and verify that the candidate strategy $\hat{\phi}_i$ can achieve the upper bound corresponding to M_{-i} and M_i, which verifies that $\hat{\phi}_i$ is indeed the maximizer of $\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right]$.

The upper bound corresponding to $M_{-i} = -\gamma_i C_i \pi_{-i}$ and $M_i = \gamma_i \hat{\phi}_i$ is

$$\exp\left((1-\gamma_i)\left(\left(M_i' w_{-i} + M_{-i}'\rho_{-i} - \lambda_{-i}'\right) C_i \pi_{-i} + \frac{1}{2\gamma_i} M_i' w_i M_i \right.\right.$$
$$\left.\left. + \ \frac{1}{2\gamma_i} M_{-i}'\rho_{-i} M_{-i} + \frac{1}{\gamma_i} M_i' w_{-i} M_{-i}\right) t\right)$$
$$= \exp\left((1-\gamma_i)\left(\left(\gamma_i \hat{\phi}_i' w_{-i} - \gamma_i \pi_{-i}' C_i \rho_{-i} - \lambda_{-i}'\right) C_i \pi_{-i}\right.\right.$$
$$\left.\left. + \ \frac{\gamma_i}{2}\left(\hat{\phi}_i' w_i \hat{\phi}_i + \pi_{-i}' C_i \rho_{-i} C_i \pi_{-i} - 2\hat{\phi}_i' w_{-i} C_i \pi_{-i}\right)\right) t\right)$$
$$= \exp\left(\left(-(1-\gamma_i)\lambda_{-i}' C_i \pi_{-i} + \frac{(1-\gamma_i)\gamma_i}{2}\left(\hat{\phi}_i' w_i \hat{\phi}_i - \pi_{-i}' C_i \rho_{-i} C_i \pi_{-i}\right)\right) t\right). \tag{15}$$

On the other hand, for $\tilde{X}_i$ corresponding to $\hat{\phi}_i$,

$$\tilde{X}_{it} = \exp\left(\hat{\phi}_i'(h_i t + F_{it}) - \pi_{-i}' C_i(\lambda_{-i} t + W_{-it})\right.$$
$$\left. + \left(-\frac{1}{2}\hat{\phi}_i' w_i \hat{\phi}_i - \frac{1}{2}\pi_{-i}' C_i \rho_{-i} C_i \pi_{-i} + \hat{\phi}_i' w_{-i} C_i \pi_{-i}\right) t\right).$$

Thus,

$$\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right]$$
$$= \exp\left((1-\gamma_i)\left(\hat{\phi}_i' h_i - \pi_{-i}' C_i \lambda_{-i} - \frac{1}{2}\hat{\phi}_i' w_i \hat{\phi}_i - \frac{1}{2}\pi_{-i}' C_i \rho_{-i} C_i \pi_{-i} + \right.\right.$$
$$\hat{\phi}_i' w_{-i} C_i \pi_{-i} + \frac{(1-\gamma_i)}{2}\hat{\phi}_i' w_i \hat{\phi}_i' + \frac{(1-\gamma_i)}{2}\pi_{-i}' C_i \rho_{-i} C_i \pi_{-i}$$
$$\left.\left. -(1-\gamma_i)\hat{\phi}_i' w_{-i} C_i \pi_{-i}\right) t\right)$$
$$= \exp\left(\left(-(1-\gamma_i)\pi_{-i}' C_i \lambda_{-i} + \frac{\gamma_i(1-\gamma_i)}{2}\left(\hat{\phi}_i' w_i \hat{\phi}_i' - \pi_{-i}' C_i \rho_{-i} C_i \pi_{-i}\right)\right) t\right),$$

which coincides with the upper bound in (15). □

Lemma 4.3 *P_f and P_m are invertible.*

Proof P_f and P_m can be rewritten as $P_f = A_1 P_{diag} P_1 P_{diag} A_2$, and $P_m = P_{diag}^2 A_1 P_2 A_2$, where A_1, A_2 and $\mathbf{P}_{diag}$ are $N \times N$ diagonal matrices with $(A_1)_{ii} = \frac{1}{N+(N-1)\alpha_i}$, $(A_2)_{ii} = \alpha_i$ and $(P_{diag})_{ii} = \sqrt{1-\rho_{im}^2}$, and P_1 and P_2 are $N \times N$ matrices with

$$(P_1)_{ij} = \begin{cases} \frac{1}{c_{ii}} & \text{if } i = j, \\ -\frac{\rho_{ij} - \rho_{im}\rho_{jm}}{\sqrt{1-\rho_{im}^2}\sqrt{1-\rho_{jm}^2}} & \text{if } i \neq j, \end{cases} \quad (P_2)_{ij} = \begin{cases} \frac{1}{c_{ii}} & \text{if } i = j, \\ -1 & \text{if } i \neq j. \end{cases}$$

On the other hand, for $i \neq j$, Brownian motions W_i and W_j can be written as

$$W_{it} = \rho_{im} B_t + \sqrt{1-\rho_{im}^2} Z_{it}, \quad W_{jt} = \rho_{jm} B_t + \sqrt{1-\rho_{jm}^2} Z_{jt},$$

where Z_i, Z_j are Brownian motions independent of B. Suppose that $\langle Z_i, Z_j \rangle_t = \rho_{ij}^z t$, and then $\rho_{ij} = \rho_{im}\rho_{jm} + \sqrt{(1-\rho_{im}^2)(1-\rho_{jm}^2)}\rho_{ij}^z$, which implies that $\frac{(\rho_{ij}-\rho_{im}\rho_{jm})^2}{(1-\rho_{im}^2)(1-\rho_{jm}^2)} = (\rho_{ij}^z)^2 \leq 1$. Since $c_{ii} = \frac{\alpha_i}{N+(N-1)\alpha_i} < \frac{1}{N-1}$, both P_1 and P_2 are strictly diagonally dominated matrices, and hence invertible. Therefore, P_f and P_m are invertible, because the diagonal matrices A_1, A_2 and P_{diag} are also invertible. □

The proof of Proposition 3.1 (i) The claim follows from the fact that $\frac{\partial \pi_i^*}{\partial \lambda_i} = \frac{2(1-\rho_{jm}^2)}{(2+\alpha_i)\sigma_i\gamma_i\kappa_1}$ and $\kappa_1 > 0$.

(ii) The claim follow from $\frac{\partial \pi_i^*}{\partial \lambda_j} = \frac{2\alpha_i(\rho_{12}-\rho_{1m}\rho_{2m})}{(2+\alpha_i)(2+\alpha_j)\sigma_i\gamma_j\kappa_1}$.

(iii) The claims are direct results of the following derivatives and the fact that $(1-\rho_{1m}^2)(1-\rho_{2m}^2) \geq (\rho_{12}-\rho_{1m}\rho_{2m})^2$ from the proof of Lemma 4.3.

$$\begin{aligned}
\frac{\partial \theta_i}{\partial \lambda_i} &= -\frac{2(1-\rho_{jm}^2)\rho_{im}}{(2+\alpha_i)b\kappa_2\gamma_i\kappa_1}\left[\left(1+\frac{\alpha_1\alpha_2}{(2+\alpha_1)(2+\alpha_2)}\right)(1-\rho_{1m}^2)(1-\rho_{2m}^2)\right.\\
&\quad \left. -\frac{2\alpha_1\alpha_2}{(2+\alpha_1)(2+\alpha_2)}(\rho_{12}-\rho_{1m}\rho_{2m})^2\right],\\
\frac{\partial \theta_i}{\partial \lambda_j} &= -\frac{2\alpha_i(1-\rho_{im}^2)}{(2+\alpha_1)(2+\alpha_2)b\kappa_2\gamma_j\kappa_1}\left[\left(1+\frac{\alpha_1\alpha_2}{(2+\alpha_1)(2+\alpha_2)}\right)\cdot\right.\\
&\quad \left(1-\rho_{1m}^2\right)\left(1-\rho_{2m}^2\right)\rho_{jm} - \frac{2\alpha_1\alpha_2\rho_{jm}}{(2+\alpha_1)(2+\alpha_2)}(\rho_{12}-\rho_{1m}\rho_{2m})^2\\
&\quad \left. -\left(1-\frac{\alpha_1\alpha_2}{(2+\alpha_1)(2+\alpha_2)}\right)(1-\rho_{jm}^2)(\rho_{jm}-\rho_{12}\rho_{im})\right].
\end{aligned}$$

□

The proof of Proposition 3.2 The claims follow from the derivatives of π_i^* with respect to α_i and α_j.

$$\begin{aligned}
\frac{\partial \pi_i^*}{\partial \alpha_i} &= \frac{2}{(2+\alpha_i)\sigma_i\kappa_1}\left(\frac{\rho_{12}}{2+\alpha_j}\left(1-\frac{\alpha_i}{2+\alpha_i}\frac{1-\frac{\alpha_j}{2+\alpha_j}\rho_{12}^2}{\kappa_1}\right)\lambda_{j,\gamma_j}\right.\\
&\quad \left. -\frac{1-\frac{\alpha_j}{2+\alpha_j}\rho_{12}^2}{(2+\alpha_i)\kappa_1}\lambda_{i,\gamma_i}\right)\\
&= \frac{2}{(2+\alpha_i)^2(2+\alpha_j)\sigma_i\kappa_1^2}\left(2\rho_{12}\lambda_{j,\gamma_j} - \left(2+(1-\rho_{12}^2)\alpha_j\right)\lambda_{i,\gamma_i}\right),
\end{aligned}$$

$$\begin{aligned}\frac{\partial \pi_i^*}{\partial \alpha_j} &= \frac{2}{(2+\alpha_i)\sigma_i \kappa_1}\left(\frac{2\alpha_i \rho_{12}^2}{(2+\alpha_i)(2+\alpha_j)^2\kappa_1}\lambda_{i,\gamma_i}\right.\\ &\qquad \left. - \frac{\alpha_i \rho_{12}}{(2+\alpha_j)^2}\left(1-\frac{2\alpha_i\rho_{12}^2}{(2+\alpha_i)(2+\alpha_j)\kappa_1}\right)\lambda_{j,\gamma_j}\right)\\ &= \frac{2\alpha_i\rho_{12}}{(2+\alpha_i)^2(2+\alpha_j)^2\sigma_i\kappa_1^2}\left(2\rho_{12}\lambda_{i,\gamma_i} - (2+(1-\rho_{12}^2)\alpha_i)\lambda_{j,\gamma_j}\right). \qquad \square\end{aligned}$$

The proof of Proposition 3.3 Following (7) and (10), with $\rho_{12} \in (-1, 1)$ and $\bar{\lambda} \leq 1$,

$$\begin{aligned}\frac{\pi_1^*}{\pi_1^M} &= \frac{2\left(1+\frac{\alpha}{2+\alpha}\rho_{12}\bar{\lambda}\right)}{(2+\alpha)\left(1-\left(\frac{\alpha}{2+\alpha}\right)^2\rho_{12}^2\right)} = \frac{2(2+\alpha)\left(1+\frac{\alpha}{2+\alpha}\rho_{12}\bar{\lambda}\right)}{4\alpha+\alpha^2(1-\rho_{12}^2)+4}\\ &\leq \frac{4+4\alpha}{\alpha^2(1-\rho_{12}^2)+4\alpha+4} < 1.\\ \eta_1^* - \eta_1^M &= \lambda_1 - \frac{\psi(2+\alpha)\left(1-\left(\frac{\alpha}{2+\alpha}\right)^2\rho_{12}^2\right)}{2\lambda_{1,\gamma_1}\left(1+\frac{\alpha}{2+\alpha}\rho_{12}\bar{\lambda}\right)} - \left(\lambda_1 - \frac{\psi}{\lambda_{1,\gamma_1}}\right)\\ &= -\frac{\alpha\psi\left((2+\alpha(1-\rho_{12}^2)) - 2\rho_{12}\bar{\lambda}\right)}{2(2+\alpha)\lambda_{1,\gamma_1}\left(1+\frac{\alpha}{2+\alpha}\rho_{12}\bar{\lambda}\right)}\\ &< -\frac{\alpha\psi\left(2-2\bar{\lambda}\right)}{2(2+\alpha)\lambda_{1,\gamma_1}\left(1+\frac{\alpha}{2+\alpha}\rho_{12}\bar{\lambda}\right)} \leq 0.\end{aligned}$$

On the other hand,

$$\frac{\pi_2^*}{\pi_2^M} - 1 = \frac{2\left(\frac{\alpha}{2+\alpha}\rho_{12}+\bar{\lambda}\right)}{\bar{\lambda}(2+\alpha)\left(1-\left(\frac{\alpha}{2+\alpha}\right)^2\rho_{12}^2\right)} - 1 = \frac{\frac{2\alpha\rho_{12}}{2+\alpha}+\alpha\left(\frac{\alpha\rho_{12}^2}{2+\alpha}-1\right)\bar{\lambda}}{\bar{\lambda}(2+\alpha)\left(1-\left(\frac{\alpha}{2+\alpha}\right)^2\rho_{12}^2\right)}.$$

Thus, $\pi_2^* \leq \pi_2^M$ is equivalent to $\frac{2\alpha\rho_{12}}{2+\alpha} + \alpha\left(\frac{\alpha\rho_{12}^2}{2+\alpha}-1\right)\bar{\lambda} \leq 0$, which, since $\frac{\alpha\rho_{12}^2}{2+\alpha} < 1$, always holds for $\rho_{12} < 0$, and is equivalent to $\bar{\lambda} \geq \frac{2\rho_{12}}{2+\alpha(1-\rho_{12}^2)}$ if $\rho_{12} \geq 0$. Finally,

$$\eta_2^* - \eta_2^M = -\frac{\psi\alpha\left((2+\alpha(1-\rho_{12}^2))\,\bar{\lambda} - 2\rho_{12}\right)}{2(2+\alpha)\lambda_{2,\gamma_2}\left(\bar{\lambda}+\frac{\alpha\rho_{12}}{2+\alpha}\right)}.$$

Thus, if $\rho_{12} \geq 0, \bar{\lambda}+\frac{\alpha\rho_{12}}{2+\alpha} > 0$, and $\eta_2^* \leq \eta_2^M$ if and only if $\left(2+\alpha(1-\rho_{12}^2)\right)\bar{\lambda}-2\rho_{12} \geq 0$, or equivalently $\bar{\lambda} \geq \frac{2\rho_{12}}{2+\alpha(1-\rho_{12}^2)}$. If $\rho_{12} < 0$, $\left(2+\alpha(1-\rho_{12}^2)\right)\bar{\lambda}-2\rho_{12} > 0$, and $\eta_2^* \leq \eta_2^M$ if and only if $\bar{\lambda} \geq -\frac{\alpha\rho_{12}}{2+\alpha}$. □

The proof of Proposition 3.4 Following (8) and (9),

$$
\begin{aligned}
\mathrm{Beta}_1^* - 1 &= \frac{\left(1-\left(\frac{\alpha}{2+\alpha}\right)^2\rho_{12}^2\right)(1-\bar{\lambda}^2)}{K_1^*} \geq 0, \\
K_1^* &= \left(1+\frac{\alpha}{2+\alpha}\rho_{12}\bar{\lambda}\right)^2 + 2\rho_{12}\left(1+\frac{\alpha}{2+\alpha}\rho_{12}\bar{\lambda}\right)\left(\frac{\alpha}{2+\alpha}\rho_{12}\bar{\lambda}\right) \\
&\quad + \left(\frac{\alpha}{2+\alpha}\rho_{12}+\bar{\lambda}\right)^2, \\
\mathrm{Beta}_1^M - 1 &= \frac{1-\bar{\lambda}^2}{1+2\rho_{12}\bar{\lambda}^2+\bar{\lambda}^2} \geq 0.
\end{aligned}
$$

Therefore, if $\bar{\lambda} = 1$, $\mathrm{Beta}_1^* = \mathrm{Beta}_1^M = 1$. Otherwise, since $\bar{\lambda} < 1$, the sign of $|\mathrm{Beta}_1^* - 1| - |\mathrm{Beta}_1^M - 1| = \mathrm{Beta}_1^* - \mathrm{Beta}_1^M$ is the same as that of

$$
\begin{aligned}
&\left(1-\left(\frac{\alpha}{2+\alpha}\right)^2\rho_{12}^2\right)\left(1+2\rho_{12}\bar{\lambda}+\bar{\lambda}^2\right) - \left(1+\frac{\alpha}{2+\alpha}\rho_{12}\bar{\lambda}\right)^2 \\
&\quad - 2\rho_{12}\left(1+\frac{\alpha}{2+\alpha}\rho_{12}\bar{\lambda}\right)\left(\frac{\alpha}{2+\alpha}\rho_{12}+\bar{\lambda}\right) - \left(\frac{\alpha}{2+\alpha}\rho_{12}+\bar{\lambda}\right)^2 \\
&\quad = -\frac{4\alpha(1+\alpha)}{(2+\alpha)^2}\rho_{12}^2\left(1+\left(\frac{2+\alpha}{1+\alpha}\frac{1}{\rho_{12}}+\frac{\alpha}{1+\alpha}\rho_{12}\right)\bar{\lambda}+\bar{\lambda}^2\right). \qquad (16)
\end{aligned}
$$

If $\rho_{12} \geq 0$, $1+\left(\frac{2+\alpha}{1+\alpha}\frac{1}{\rho_{12}}+\frac{\alpha}{1+\alpha}\rho_{12}\right)\bar{\lambda}+\bar{\lambda}^2 \geq 0$, and hence $|\mathrm{Beta}_1^*-1|-|\mathrm{Beta}_1^M-1| \leq 0$. If $\rho_{12} < 0$, $|\mathrm{Beta}_1^* - 1| - |\mathrm{Beta}_1^M - 1| \leq 0$ is equivalent to

$$
1+\left(\frac{2+\alpha}{1+\alpha}\frac{1}{\rho_{12}}+\frac{\alpha}{1+\alpha}\rho_{12}\right)\bar{\lambda}+\bar{\lambda}^2 \geq 0. \qquad (17)
$$

Since $\frac{\alpha}{1+\alpha}\rho_{12}+\frac{2+\alpha}{1+\alpha}\frac{1}{\rho_{12}}$ is negative and is decreasing in $\rho_{12} \in [-1, 0)$, the maximum value at $\rho_{12} = -1$ is -2, and $\Delta \geq 0$. Since $\lambda \leq 1$, the two roots of the left hand side of (17) are

$$
\frac{-\left(\frac{\alpha}{1+\alpha}\rho_{12}+\frac{2+\alpha}{1+\alpha}\frac{1}{\rho_{12}}\right)+\sqrt{\Delta}}{2} \geq 1 \text{ and } 0 \leq \frac{-\left(\frac{\alpha}{1+\alpha}\rho_{12}+\frac{2+\alpha}{1+\alpha}\frac{1}{\rho_{12}}\right)-\sqrt{\Delta}}{2}.
$$

Thus, $|\mathrm{Beta}_1^* - 1| - |\mathrm{Beta}_1^M - 1| \leq 0$ if $\bar{\lambda} \leq \frac{-\left(\frac{\alpha}{1+\alpha}\rho_{12}+\frac{2+\alpha}{1+\alpha}\frac{1}{\rho_{12}}\right)-\sqrt{\Delta}}{2}$, and otherwise the inequality is reversed.

On the other hand, algebraic calculations show that $\text{Beta}_2^* - 1 = -\left(\text{Beta}_1^* - 1\right) \leq 0$ and $\text{Beta}_2^M - 1 = -\left(\text{Beta}_1^M - 1\right) \leq 0$. Therefore, the sign of $|\text{Beta}_2^* - 1| - |\text{Beta}_2^M - 1| = -\text{Beta}_2^* + \text{Beta}_2^M$ is the same as (16), and equivalent conditions for Fund 1 still hold. □

The proof of Proposition 3.5 Let $\tilde{X}_{it} = \exp\left(-\left(r - \tilde{\psi}_i\right) t\right) X_{it}$. Then, with $\zeta_{it} = \frac{N+(N-1)\alpha_i}{N}\theta_{it} - \frac{\alpha_i}{N}\sum_{j\neq i}^{N}\theta_{jt}$, $\tilde{X}_i$ follows

$$\frac{d\tilde{X}_{it}}{\tilde{X}_{it}} = \zeta_{it}(adt + bdB_t), \quad \tilde{X}_{i0} = X_{i0} = 1.$$

We first calculate the optimal ζ_i (or equivalently the optimal θ_i) given θ_j's ($j \neq i$) of other funds. With $d\xi_t/\xi_t = -\lambda_m dB_t$ and $\xi_0 = 1$, $d(\xi_t \tilde{X}_{it}) = \xi_t \tilde{X}_{it}\,(\zeta_{it} b - \lambda_m)\, dB_t$, which is a nonnegative local martingale, and thus a supermartingale. Then, for $0 < \gamma_i < 1$ (the case of $\gamma_i > 1$ follows similarly), by Hölder's inequality, for any $\theta = (\theta_1, \ldots, \theta_N) \in \Theta^N$,

$$\mathbb{E}\left[\frac{\tilde{X}_{it}^{1-\gamma_i}}{1-\gamma_i}\right] = \frac{\mathbb{E}\left[\left(\xi_t \tilde{X}_{it}\right)^{1-\gamma_i} \xi_t^{\gamma_i - 1}\right]}{1-\gamma_i}$$
$$\leq \frac{\mathbb{E}\left[\xi_t \hat{X}_{it}\right]^{1-\gamma_i} \mathbb{E}\left[\xi_t^{-\frac{1-\gamma_i}{\gamma_i}}\right]^{\gamma_i}}{1-\gamma_i} \leq \frac{\exp\left(\frac{1-\gamma_i}{2\gamma_i}\lambda_m^2 t\right)}{1-\gamma_i},$$

which gives an upper bound of $\mathbb{E}\left[\frac{\tilde{X}_{it}^{1-\gamma_i}}{1-\gamma_i}\right]$. On the other hand, with $\theta_{it} = \frac{N\left(\frac{\lambda_m}{\gamma_i b} + \frac{\alpha_i}{N}\sum_{j\neq 1}^{N}\theta_{jt}\right)}{N+(N-1)\alpha_i}$, and thus $\zeta_{it} = \frac{\lambda_m}{\gamma_i b}$, $\mathbb{E}\left[\frac{1}{1-\gamma_i}\tilde{X}_{it}^{1-\gamma_i}\right] = \frac{\exp\left(\frac{1-\gamma_i}{2\gamma_i}\lambda_m^2 t\right)}{1-\gamma_i}$, which indicates that ζ_i is the maximizer of $\mathbb{E}\left[\frac{\tilde{X}_{it}^{1-\gamma_i}}{1-\gamma_i}\right]$. Since $\zeta_{it} = \frac{\lambda_m}{\gamma_i b}$ is a constant strategy, independent of t, it also maximizes manager i's expected utility

$$\int_0^T e^{\left(-\beta_i + r - \tilde{\psi}_i\right)t} \psi_i^{1-\gamma_i} \frac{\mathbb{E}\left[\tilde{X}_{it}^{1-\gamma_i}\right]}{1-\gamma_i} dt,$$

and the optimal strategy given θ_j's ($j \neq i$) is $\theta_{it} = \frac{N\left(\frac{\lambda_m}{\gamma_i b} + \frac{\alpha_i}{N}\sum_{j\neq i}^{N}\theta_{jt}\right)}{N+(N-1)\alpha_i}$.

To find the equilibrium, it suffices to solve the system of N equations, each representing the optimal strategy given the portfolio of other funds: $P_I \theta_t = \frac{\lambda_m}{b}\gamma^{-1} e$, where $\theta_t = (\theta_{1t}, \cdots, \theta_{Nt})'$, e is the N-dimensional vector with all entries equal to 1, P_I the $N \times N$ matrix with $(P_I)_{i,j} = \begin{cases} \frac{N+(N-1)\alpha_i}{N} & \text{if } i = j \\ -\frac{\alpha_i}{N} & \text{if } i \neq j, \end{cases}$ and γ is defined in Theorem 3.1. Since P_I is strictly diagonally dominated and thus invertible, there exists a unique solution $\theta^* = \frac{\lambda_m}{b} P_I^{-1} \gamma^{-1} e$ for every $0 \le t \le T$. Furthermore, since $P_I = -\frac{1}{N} A_2(D + ee')$, where A_2 is the diagonal matrix defined in the proof of Lemma 4.3, D is an $N \times N$ diagonal matrix with $(D)_{ii} = -\frac{1}{c_{ii}} - 1$, $1 \le i \le N$, by Sherman–Morrison–Woodbury formula (See equation 2.1.4 in [26]),

$$(P_I^{-1})_{i,j} = \begin{cases} \frac{1}{1+\alpha_i} + \frac{1+\bar{\alpha}}{N} \frac{\alpha_i}{(1+\alpha_i)^2} & , i = j \\ \frac{1+\bar{\alpha}}{N} \frac{\alpha_i}{(1+\alpha_i)(1+\alpha_j)} & , i \ne j \end{cases},$$

and this solution reduces to $\theta_i^* = \frac{\lambda_m}{b} \left(\frac{1}{1+\alpha_i} \frac{1}{\gamma_i} + \frac{\alpha_i}{1+\alpha_i} \frac{1}{\bar{\gamma}} \right)$ for $1 \le i \le N$. □

The proof of Proposition 3.6 (i) Both θ_i^* and θ_i^M are proportion to $\frac{\lambda_m}{b}$, while the coefficient for θ_i^M is $\frac{1}{\gamma_i}$ and that for θ_i^* is a convex combination between $\frac{1}{\gamma_i}$ and $\frac{1}{\bar{\gamma}}$. The claim follows by the comparing the two coefficients.

(ii) Since $\frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{1+\alpha_i} \frac{1}{\gamma_i} + \frac{\alpha_i}{1+\alpha_i} \frac{1}{\bar{\gamma}} \right) = \frac{1}{\bar{\gamma}}$, $\bar{\theta}^* = \frac{\lambda_m}{b\bar{\gamma}}$, and

$$\begin{aligned} \bar{\theta}^* - \bar{\theta}^M &= \frac{\lambda_m}{b} \left(\frac{1}{\bar{\gamma}} - \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\gamma_i} \right) = \frac{\lambda_m}{b} \left(\frac{1+\bar{\alpha}}{N} \sum_{i=1}^{N} \frac{1}{(1+\alpha_i)\gamma_i} - \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\gamma_i} \right) \\ &= \frac{\lambda_m}{b} \frac{1}{N} \sum_{i=1}^{N} \left(\frac{1+\bar{\alpha}}{1+\alpha_i} - 1 \right) \frac{1}{\gamma_i}. \end{aligned}$$

Since $(\gamma_i - \gamma_j)(\alpha_i - \alpha_j) \ge 0$ for every pair of i and j, $\left(\frac{1+\bar{\alpha}}{1+\alpha_i} - 1 \right)$'s and $\frac{1}{\gamma_i}$'s are similarly ordered, and from Tchebychef's inequality [32, 2.17.1], the above is greater than or equal to

$$\frac{\lambda_m}{b} \frac{1}{N} \sum_{i=1}^{N} \left(\frac{1+\bar{\alpha}}{1+\alpha_i} - 1 \right) \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\gamma_i} = 0,$$

and the inequality is reversed if $(\gamma_i - \gamma_j)(\alpha_i - \alpha_j) \le 0$ for every pair of i and j.

(iii) $\bar{\theta}^* = \bar{\theta}^M$ follows from (ii). Furthermore, since $\frac{1}{\bar{\gamma}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\gamma_i}$,

$$\begin{aligned} \theta_i^* - \bar{\theta}^* &= \frac{\lambda_m}{b} \left(\frac{1}{1+\alpha} \frac{1}{\gamma_i} + \sum_{j=1}^{N} \left(\frac{\alpha}{N(1+\alpha)} \frac{1}{\gamma_j} \right) - \frac{1}{N} \sum_{j=1} \frac{1}{\gamma_j} \right) \\ &= \frac{\lambda_m}{b} \left(\frac{1}{1+\alpha} \frac{1}{\gamma_i} - \frac{1}{1+\alpha} \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\gamma_j} \right) \\ &= \frac{1}{1+\alpha} \frac{\lambda_m}{b} \left(\frac{1}{\gamma_i} - \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\gamma_j} \right) = \frac{1}{1+\alpha} \left(\theta_i^M - \bar{\theta}^M \right). \end{aligned}$$

□

The proof of Proposition 3.7 (i) and (ii) follow the same calculation as for Theorem 3.1. For (iii), notice that $\pi_N^{*\alpha} - \pi_N^{*0} = (k_\rho(\tilde{\lambda} + \lambda_{N,\gamma_N}) - \lambda_{N,\gamma_N})/\sigma_1$. Thus, the comparison between π^{*0} and $\pi^{*\alpha}$ is reduced to

$$\begin{aligned}
&k_\rho(\tilde{\lambda} + \lambda_{N,\gamma_N}) - \lambda_{N,\gamma_N} \\
&\quad = k_\rho \left(\sum_{j=1}^{N} \frac{\alpha\rho}{N + (N-1)(1-\rho)\alpha} \lambda_{j,\gamma_j} + \left(1 - \frac{N + (N-1+\rho)\alpha}{N}\right) \lambda_{N,\gamma_N} \right) \\
&\quad = \frac{\alpha(N-1+\rho)k_\rho}{N} \left(\frac{\rho N^2 \frac{1}{N} \sum_{j=1}^{N} \lambda_{j,\gamma_j}}{((1+(1-\rho)\alpha)N - (1-\rho)\alpha)\,(N-(1-\rho))} - \lambda_{N,\gamma_N} \right) \\
&\quad = \frac{\alpha(N-1+\rho)k_\rho}{N} \left(\varphi \frac{1}{N} \sum_{j=1}^{N} \lambda_{j,\gamma_j} - \lambda_{N,\gamma_N} \right).
\end{aligned}$$

Thus, if $\lambda_{N,\gamma_N} \geq \varphi \sum_{j=1}^{N} \lambda_{j,\gamma_j}/N$, then $\lambda_{N,\gamma_N} \geq k_\rho(\tilde{\lambda} + \lambda_{1,\gamma_1})$, and $\pi_i^{*0} \geq \pi_i^{*\alpha}$ for every i. □

The proof of Proposition 3.8 (i), (ii) is a direct application of Theorem 3.1 and follows similar arguments to those for Proposition 3.7 (iii).

(iii) Given π_i^*'s, $\text{Beta}_N^* = \frac{(N+\alpha)\bar{\lambda} - (N-1)\alpha}{(1+\alpha)(\bar{\lambda} - (N-1))}$, $\text{Beta}_N^M = \frac{N\bar{\lambda}}{\bar{\lambda} - (N-1)}$. Thus,

$$\begin{aligned}
\left|\text{Beta}_N^M - 1\right| - \left|\text{Beta}_N^* - 1\right| &= \frac{(1+\alpha)(N-1)(\bar{\lambda}+1) - (N-1)\left(\bar{\lambda}+1\right)}{(1+\alpha)|\bar{\lambda} - (N-1)|} \\
&= \frac{\alpha(N-1)(\bar{\lambda}+1)}{(1+\alpha)|\bar{\lambda} - (N-1)|} > 0.
\end{aligned}$$

For fund $i \neq N$, $\text{Beta}_i^* = \frac{1}{1+\alpha} - \frac{N}{(1+\alpha)(\bar{\lambda} - (N-1))} \frac{\bar{\gamma}_{-N}}{\gamma_i}$ and $\text{Beta}_i^M = -\frac{N}{\bar{\lambda} - (N-1)} \frac{\bar{\gamma}_{-N}}{\gamma_i}$. If $\bar{\lambda} > N-1$, both Beta's are less than 1, and $\left|\text{Beta}_i^M - 1\right| \geq \left|\text{Beta}_i^* - 1\right|$ because

$$\left|\text{Beta}_i^M - 1\right| - \left|\text{Beta}_i^* - 1\right| = \frac{\alpha}{1+\alpha} \left(1 + \frac{N}{\bar{\lambda} - (N-1)} \frac{\bar{\gamma}_{-N}}{\gamma_i}\right) \geq 0. \tag{18}$$

If $\bar{\lambda} < N-1$ and $\frac{\bar{\gamma}_{-N}}{\gamma_i} \geq \max(\alpha, 1) \frac{N-1-\bar{\lambda}}{N}$, both Betas are greater than or equal to 1, and

$$\left|\text{Beta}_i^M - 1\right| - \left|\text{Beta}_i^* - 1\right| = -\frac{\alpha}{1+\alpha} \left(1 + \frac{N}{\bar{\lambda} - (N-1)} \frac{\bar{\gamma}_{-N}}{\gamma_i}\right) \geq 0.$$

Other cases of $\bar{\gamma}_{-N}/\gamma_i$ follow by similar arguments. □

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