

Frequency Response of Transmission Lines with Unevenly Distributed Properties with Application to Railway Safety Monitoring*

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Abstract—This paper proposes a method to efficiently compute the voltage and current along a transmission line which can be “damaged”; that is its electrical properties can be unevenly distributed. The method approximates a transmission line by a self-similar circuit network and leverages our previous work regarding the frequency response for that class of networks. The main motivation arises from research for railway track circuit systems where transmission line models are often employed. Determining deviations from baseline properties of the railway circuit is important for health monitoring of the system and furthermore, changes in circuit properties due to a train occupying a segment of the track also is of great interest as a means to ensure safety. Thus, in addition to monitoring the integrity of the railway circuit, our approach also could provide a means for safe operation in that it can be used to detect segments of the rail system that are occupied by trains.

I. INTRODUCTION

Transmission line theory, credited to Oliver Heaviside, determines the voltage and current along a transmission line with respect to both spatial and temporal parameters. Various transmission line effects, such as can become significant, especially when dealing with high frequency signals [1]. Transmission line theory is widely applied in electrical engineering. For antenna systems, it is used to determine the matching networks balancing a load with its source [2]. It builds the foundation for physical models of power line communications channel representing signal propagation effects [3]. It is also employed to analyze the cylindrical body model for studying interaction of a human with electromagnetic fields [4].

The main motivation of this work is the implementation of transmission line theory in railway track circuits which automatically detects whether a sector of track is occupied [5]–[8]. One drawback of applying the classical transmission line theory to this problem is its assumption of uniformly distributed electrical properties, which is unlikely for track circuits. For instance, humidity in both soil and ballast bed can impact those properties as indicated by [9]. In addition, some sudden external influences, like lightning, can damage a track circuit system which may lead to a catastrophic safety monitoring failures where two trains are present within the same track segment.

*The support of the US National Science Foundation under Grant No. CMMI 1826079 is gratefully acknowledged.

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As a result, in this work, we propose a method to quickly compute the voltage and current along a transmission line when its electrical properties are unevenly distributed with example applications to simulating a damaged track circuit system, as well as simulating a train passing along an intact track circuit. The contribution of this paper can be further employed more generally in the health monitoring research area for track circuits.

The proposed method approximates a transmission line by an electrical network with many subsections, and the electrical properties are lumped in each subsection. The electrical properties at one subsection can be different from the others, which imitates a transmission line with unevenly distributed properties. The transmission line model and its approximated counterpart, the circuit network model, are shown in Figures 1 and 2, and are similar to the networks constructed in [10], [11].

Leveraging a frequency-domain network modeling algorithm proposed in our previous work [12], we can obtain the impedance V_g/I_g and the voltage gain V_{out}/V_g at each subsection g . Then, given one voltage observation inside the network, e.g. v_{out} , we can obtain the voltage and current at every node connecting two adjacent subsections. Note that the network modeling algorithm presented in our previous work [12] can be applied to any self-similar one-dimensional networks. Therefore, the network model is not limited to Figure 2, which is selected here merely due to its consistency with the transmission line model in Figure 1. Readers with different circuit network models can still follow the same procedure proposed in this paper as well as the modeling algorithm in [12]. This modeling approach yields a frequency domain model where damage (or the fact that a rail segment is occupied) is represented as a multiplicative disturbance, which is particularly convenient for robust control analyses.

The rest of this paper is organized as follows. Section II briefly reviews transmission line theory. Section III describes our method to approximate voltage and current along a transmission line through the circuit network model. To validate the correctness of our approximation result, we compare it to the transmission line theory in Section IV when the electrical attributes are distributed evenly. In Section V, we illustrate our method’s capability of evaluating voltage and current along an unevenly distributed transmission line. That capability is illustrated by two examples. The first example is computing voltage and current along a railway track circuit when a ballast degradation occurs. The second one is assessing the current as a train is passing through an intact track circuit. Finally, Section VI concludes this paper.

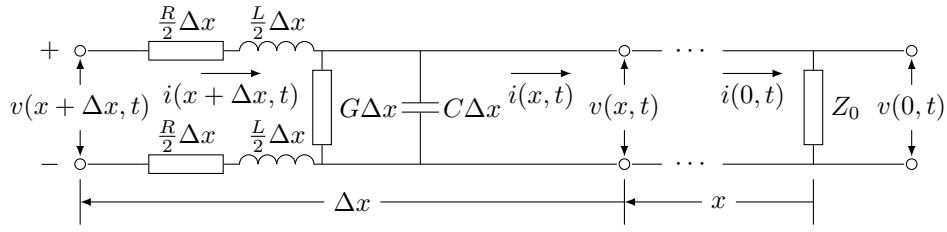


Fig. 1. Model for transmission line theory. The left hand side is the input/transmitter end. The right hand side is the output/receiver end, where $x = 0$

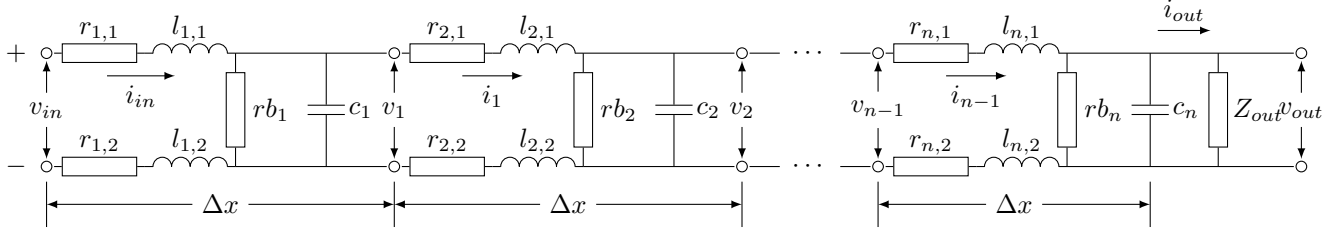


Fig. 2. Circuit network model with n subsections to approximate a transmission line

R	Series resistance	Ω/m
L	Series inductance	H/m
G	Shunt conductance	S/m
C	Shunt capacitance	F/m

TABLE I

NOTATIONS OF THE ELECTRICAL ATTRIBUTES USED IN THE TRANSMISSION LINE MODEL

II. TRANSMISSION LINE THEORY

In this section, we briefly review transmission line theory for a model shown in Figure 1. The goal is to obtain the spatial and temporal distribution of voltage and current, *i.e.* $v(x, t)$ and $i(x, t)$, given the boundary conditions at $x = 0$ and the values of the electrical attributes listed in Table I.

For an evenly distributed transmission line, those electrical attributes are constant. Therefore, by using Kirchhoff's circuit laws, we have the following within an infinitesimal distance Δx .

$$v(x + \Delta x, t) - R\Delta x i(x + \Delta x, t) - L\Delta x \frac{\partial i(x + \Delta x, t)}{\partial t} - v(x, t) = 0,$$

and

$$i(x + \Delta x, t) - G\Delta x v(x, t) - C\Delta x \frac{\partial v(x, t)}{\partial t} - i(x, t) = 0.$$

Taking $\Delta x \rightarrow 0$, we obtain the following partial differential equations.

$$\begin{aligned} \frac{\partial v(x, t)}{\partial x} &= Ri(x, t) + L \frac{\partial i(x, t)}{\partial t}, \\ \frac{\partial i(x, t)}{\partial x} &= Gv(x, t) + C \frac{\partial v(x, t)}{\partial t}. \end{aligned}$$

Using separation of variables, we assume $v(x, t) = Re\{v(x)e^{j(\omega t + \phi)}\}$ and $i(x, t) = Re\{i(x)e^{j(\omega t + \phi)}\}$, which

result in two decoupled second-order ordinary differential equations.

$$\frac{d^2 v}{dx^2} = \gamma^2 v, \text{ and } \frac{d^2 i}{dx^2} = \gamma^2 i,$$

where $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$. Using the boundary conditions at $x = 0$, the final result is

$$v(x, t) = Re\{v(x)e^{j(\omega t + \phi)}\}, \quad (1)$$

$$i(x, t) = Re\{i(x)e^{j(\omega t + \phi)}\}, \quad (2)$$

where

$$\begin{aligned} v(x) &= \frac{v(0)}{1 + \mu}(e^{\gamma x} + \mu e^{-\gamma x}), \\ i(x) &= \frac{i(0)}{1 - \mu}(e^{\gamma x} - \mu e^{-\gamma x}), \\ \mu &= \frac{Z_0 - Z_c}{Z_0 + Z_c}, \text{ and } Z_c = \sqrt{\frac{R + j\omega L}{G + j\omega C}}. \end{aligned}$$

Note that for an unevenly distributed transmission line, those electrical attributes may vary with the distance x , which makes such simple solutions difficult or impossible.

III. CIRCUIT NETWORK MODEL

In this section, we propose our procedure to quickly approximate the voltage and current along a transmission line which can have unevenly distributed physical parameters. Our method divides a long transmission line into n subsections with equal length to form a circuit network where electrical attributes lump into each subsection as shown in Figure 2. The goal is to compute the voltage and current at every node connecting two adjacent subsections, *i.e.* v_g and i_g in Figure 2. Those would be discrete approximations of the continuous results, $v(x, t)$ and $i(x, t)$ in Equations (1) and (2), given by transmission line theory.

It is clear that the electrical components in Figure 2 can be either same or different, which incidentally does not affect

the capability of our proposed method. When all components are the same, that is

$$\begin{aligned} r_{1,1} &= r_{1,2} = r_{2,1} = r_{2,2} = \dots = r, \\ l_{1,1} &= l_{1,2} = l_{2,1} = l_{2,2} = \dots = l, \\ rb_1 &= rb_2 = \dots = rb, \\ c_1 &= c_2 = \dots = c, \end{aligned}$$

we call the network “undamaged,” and those constants r , l , rb , and c are the undamaged constants. Otherwise, the network is damaged, in which case we use a pair of two lists, (l, e) , to describe a specific damage case, where l is the list of damaged components, and e is the corresponding list of damage amounts. For example, the damage case $(l, e) = ([rb_1, c_2], [0.1, 2])$ means $rb_1 = 0.1rb$, and $c_2 = 2c$, while all the other components are undamaged.

Our previous work proposed algorithms to compute frequency response and transfer functions for one-dimensional self-similar networks. For the specific application in this paper, we can use a recursive algorithm from [12] to obtain the impedance $Z_g = V_g/I_g$ and voltage gain $H_g = V_{out}/V_g$ at each node in Figure 2.

That algorithm is listed in Algorithm 1. The returned values Z and H are the impedance and the voltage gain at the transmitter end. The input argument nG is the number of subsections of the circuit network, and w is the angular frequency at which the computation conducts, which is same as the ω in Equations (1) and (2). In addition, $zOut$ is the impedance at the receiver end. The undamaged constants r , l , rb , c , are grouped into the input argument $undCst$.

Algorithm 1 Computing the impedance Z and voltage gain H at the angular frequency w for a circuit network with nG number of generations given its damage case (l, e) , the undamaged constants $undCst$, and the impedance at the output end $zOut$

```

1: function  $[Z, H] = fR(l, e, undCst, zOut, w, nG)$ 
2:  $s = j * w$ ;
3:  $[l1, e1, lS, eS] = partition(l, e)$ ;
4:  $g1Cst = getG1Cst(l1, e1, undCst)$ ;
5: if  $nG == 0$  then
6:    $[Z, H] = G0(zOut, s)$ ;
7: else
8:    $nG = nG - 1$ ;
9:    $[ZS, HS] = fR(lS, eS, undCst, zOut, w, nG)$ ;
10:   $Z = Zr(g1Cst, ZS, s)$ ;
11:   $H = Hr(g1Cst, Z, HS, s)$ ;
12:  save  $(Z, H)$ ;
13: end if

```

In Algorithm 1, the `partition()` function splits the damage case (l, e) for the entire network into two parts, where $(l1, e1)$ is the damage case at the first generation, and (lS, eS) is the damage case with respect to the subnetwork after the first generation. Then, based on $(l1, e1)$, the `getG1Cst()` function computes the values of the constants at the first generation, that is the values of $r_{1,1}$, $r_{1,2}$, $l_{1,1}$, $l_{1,2}$,

rb_1 and c_1 . This is done simply by multiplying the default value by the damage amount for each element included in the damage list for that generation. If the network in question has zero generations, the returned impedance $Z = zOut$, and the returned voltage gain $H = 1$. Otherwise, the input argument nG is reduced by one, and a recursive call is made to obtain the impedance ZS and voltage gain HS for the subnetwork given the relevant damage case (lS, eS) . Those two quantities are used to compute the final result, the impedance Z and voltage gain H for the entire network. By using series and parallel connection rules of idealized electrical components, from Figure 4, we can obtain that

$$\begin{aligned} Z(s) &= r_{1,1} + r_{1,2} + l_{1,1}s + l_{1,2}s + \frac{1}{\frac{1}{rb_1} + c_1s + \frac{1}{Z_s(s)}}, \\ H(s) &= H_s(s) \left(1 - \frac{r_{1,1} + r_{1,2} + l_{1,1}s + l_{1,2}s}{Z(s)} \right), \end{aligned} \quad (3)$$

which are computed by the `Zr()` and `Hr()` functions. Finally, the resultant Z and H are saved externally. Due to the recursive nature of Algorithm 1, by doing so, all Z_g and H_g can be saved externally for any node g between two neighboring subsections. Then, since V_{out} is known, which serves as the boundary condition $v(0, t)$ in Section II, we can evaluate V_g and I_g at every node too.

The algorithm can be applied to any self-similar one dimensional network. In the case of a segmented transmission line, the interpretation of the algorithm is fairly straight forward. Eqns. 3 uses series and parallel rules to compute the impedance of the n th component, which is then simply placed in series with the part of the network already computed. This is continued recursively until all the elements of the network are included.

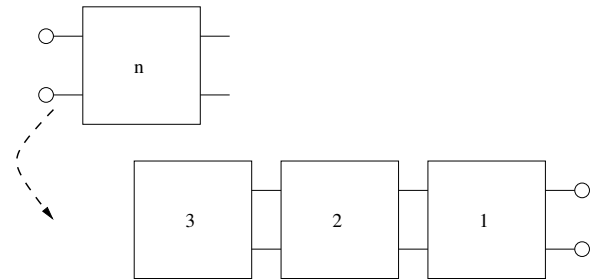


Fig. 3. Schematic of algorithm implementation on track circuit.

Note that Algorithm 1 is not limited to the circuit network shown in Figure 2. For other similar networks, the structure of Algorithm 1 stays the same with necessary modifications on the detailed computations inside some subordinate functions. A more comprehensive explanation of Algorithm 1 can be found in our previous work [12]. It is also worth noting that examples provided in that reference are relatively simple, but the example showcased in this paper is much more complicated and realistic.

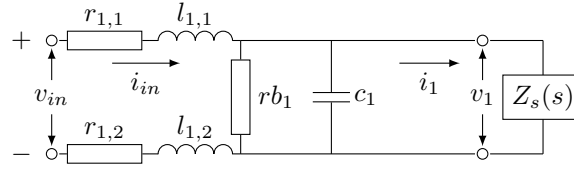


Fig. 4. When coding the recursive Algorithm 1, we only focus on the first generation, because the impedance of the subnetwork $Z_s(s) = V_1/I_1$ and the voltage gain of the subnetwork $H_s(s) = V_{out}/V_1$ are computed by the recursive call in Algorithm 1. ($H_s(s)$ is not shown here.)

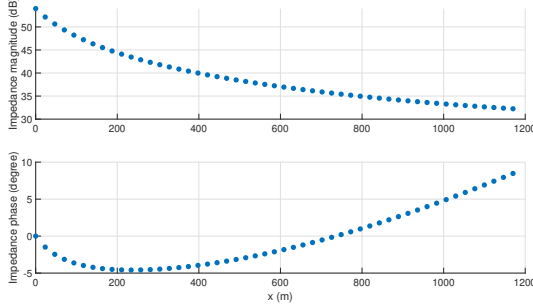


Fig. 5. Impedance Z at each node connecting two adjacent subsections obtained by the circuit network model

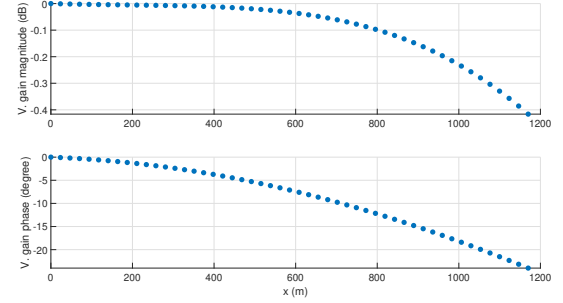


Fig. 6. Voltage gain H at each node connecting two adjacent subsections obtained by the circuit network model

IV. EVENLY DISTRIBUTED ELECTRICAL ATTRIBUTES

To prove the correctness of our calculation in Section III, we compare our discrete approximations to the continuous results given by the transmission line theory when electrical attributes are evenly distributed.

The boundary conditions at the receiver end are assumed to be ($Z_0 = 500\Omega$)

$$\begin{aligned} v(0, t) &= \text{Re}\{110e^{j4600\pi t}\}V, \\ i(0, t) &= \text{Re}\{0.22e^{j4600\pi t}\}A. \end{aligned} \quad (4)$$

The above voltage and frequency are taken from [6]. The length of the entire transmission line is set to be 1170m, which is from [7]. The constants for electrical attributes are from [9] at the frequency 2300Hz, where $R = 2.5m\Omega/m$, $L = 1.8\mu H/m$, $G = 20\mu S/m$, $C = 0.2nF/m$. By knowing the above quantities, we can compute the voltage $v(x, t)$ and current $i(x, t)$ given by the transmission line theory according to Equations (1) and (2).

On the other hand, for our circuit network model, if we use n subsections, the distance of each one is $\Delta x = 1170/n$ meters. Therefore, the undamaged constants are $r = 2.5\Delta x/2 m\Omega$, $l = 1.8\Delta x/2 \mu H$, $rb = 1/(20 \times 10^{-6}\Delta x) \Omega$, $c = 0.2\Delta x nF$, which are grouped into the `undCst` to call Algorithm 1. In addition, both `l` and `e` are empty lists indicating the intact case, `zOut` = 500, `w` = 4600 π , and `nG` = n . Then, Algorithm 1 saves the impedance and voltage gain at each node between two adjacent subsections, which are shown in Figures 5 and 6 for the undamaged network with fifty generations. After that, given the same boundary condition $v(0, t)$ in Equation (4), we can obtain the voltage and current at those nodes.

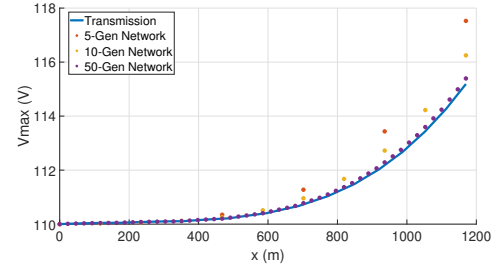


Fig. 7. Maximum voltage $\max_t(|v(x, t)|)$ versus the distance x . The blue curve is given by transmission line theory. The dots are given by our circuit network model

The comparison between the transmission line theory's result and our circuit network's result are shown in Figures 7 and 8, where $\max_t(|v(x, t)|)$ and $\max_t(|i(x, t)|)$ are plotted versus the distance x . For the circuit network, we test three networks with 5, 10, and 50 generations. From Figure 7, we see that the V_{\max} given by the circuit model converges to the one obtained by using transmission line theory as the number of generations increases. From Figure 8, we observe that the I_{\max} given by both methods are nearly identical. In addition, to show that both magnitudes and phases are correct, Figure 9 compares $v(1170, t)$ at the receiver end given by both methods. Hence, we can confirm that our method provides a reasonable approximation of transmission line model.

V. UNEVENLY DISTRIBUTED ELECTRICAL ATTRIBUTES

In this section, we illustrate our method's capability of computing voltage and current along an unevenly distributed transmission line with the applications to railway track circuit networks. All constants used in this section are same as those

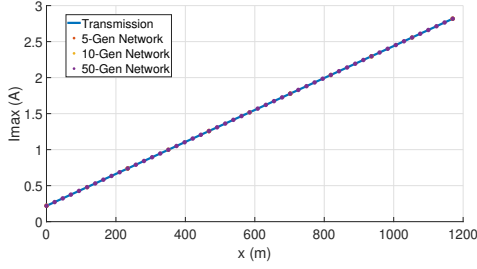


Fig. 8. Maximum current $\max_t(|i(x, t)|)$ versus the distance x . The blue curve is given by transmission line theory. The dots are given by our circuit network model

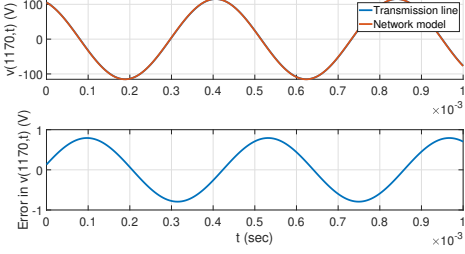


Fig. 9. The upper figure plots $v(1170, t)$ obtained by two methods. The blue curve is from transmission line theory, while the red curve is from our network model with fifty generations, where two curves almost overlap each other. The lower figure shows the error between those two

in Section IV. Besides, we fix the number of subsections at 117 here, so each subsection takes $10m$. In addition, we assume V_{max} at the transmitter end is fixed at the one in Figure 7, i.e. $V_{max} = 115V$ at $x = 1170m$.

The first example of unevenly distribution we showcase here is ballast degradation, which means unusual current leakage between the rails through the ballast [13]. In this damage case, some shunt resistance rb_g become lower and shunt capacitance c_g become higher. Here, we suppose a ballast degradation happens between $x = 100m$ and $x = 1000m$, that is between subsection 18 and subsection 107. Note that the order of subsections is opposite to the direction of x . Hence, the list of damaged components is $\mathbf{l} = [rb_{18} \cdots rb_{107} \ c_{18} \cdots c_{107}]$. The assumed list of damage amounts \mathbf{e} is plotted in Figure 10. When the above damage case (\mathbf{l}, \mathbf{e}) is inputted to Algorithm 1, the resultant

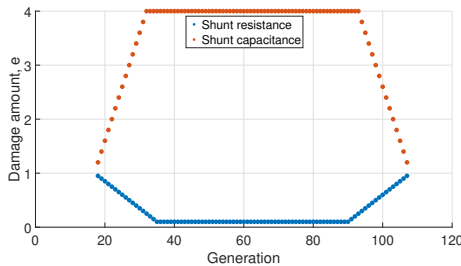


Fig. 10. The damage amounts \mathbf{e} are plotted versus generations where the blue dots are for the shunt resistance from rb_{18} to rb_{107} , and the red dots are for the shunt capacitance from c_{18} to c_{107}

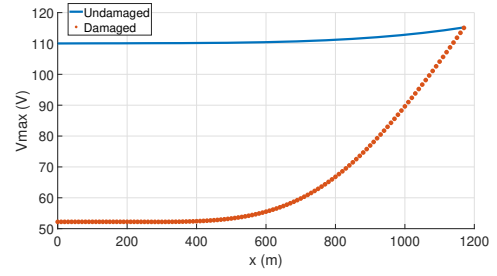


Fig. 11. The distribution of $\max_t(|v(x, t)|)$ at each node between two adjacent subsections obtained by the circuit network model when the ballast degradation occurs. The undamaged curve is obtained by transmission line theory which is same as that in Figure 7

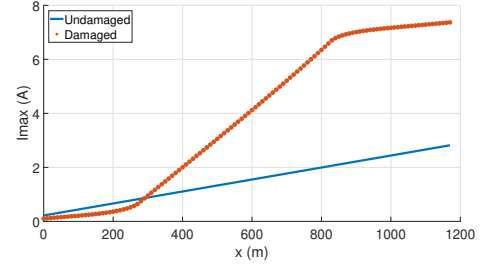


Fig. 12. The distribution of $\max_t(|i(x, t)|)$ at each node between two adjacent subsections obtained by the circuit network model when the ballast degradation occurs. The undamaged curve is obtained by transmission line theory which is same as that in Figure 8

voltage and current distribution along the track obtained by our circuit network model for this type of ballast degradation are shown in Figures 11 and 12. Note that the other two damage cases mentioned in [13], insulation imperfections and rail conductance impairments, can also be simulated by our circuit network model.

The second example of uneven distribution is when a train is moving along an intact track, which can be equivalently viewed as a number of damage cases. We assume that a 190-meter train is moving from the receiver end to the transmitter end at a constant speed $100m/s$, that is it passes one subsection every $0.1sec$. Also assume that the train has one wheel base every 10 meters, so there are 20 wheel bases in total. Each wheel base acts as an additional shunt resistor across two rails with resistance $r_w = 102.0408\Omega$. The undamaged value of the shunt resistance in this case is $rb = 1/(20 \times 10^{-6} \times 10) = 5000\Omega$. When the shunt resistance rb is connected to the wheel base's resistance r_w in parallel, the equivalent resistance is 100Ω . In other words, when one wheel base is within one subsection, we can consider that as if that corresponding shunt resistance is damaged by a factor of 0.02. Therefore, this example of a train moving along an intact track can be regarded as a time series of damage cases, where the correspondence between the time instance and the damage case is listed in Table II. At each time instance in Table II, we use the corresponding damage case in the second column to call Algorithm 1. Then, we can evaluate the current at the receiver end, $\max_t(|i(0, t)|)$ as the train is passing through this sector of track, which

Time (sec)	Damage case (\mathbf{l}, \mathbf{e})
0.1	$([rb_{117}], [0.02])$
0.2	$([rb_{117}, rb_{116}], [0.02, 0.02])$
\vdots	\vdots
2	$([rb_{117}, \dots, rb_{98}], [0.02, \dots, 0.02])$
2.1	$([rb_{116}, \dots, rb_{97}], [0.02, \dots, 0.02])$
\vdots	\vdots
11.7	$([rb_{20}, \dots, rb_1], [0.02, \dots, 0.02])$
11.8	$([rb_{19}, \dots, rb_1], [0.02, \dots, 0.02])$
\vdots	\vdots
13.6	$([rb_1], [0.02])$

TABLE II

THE EQUIVALENT DAMAGE CASE AT EACH TIME INSTANCE FOR THE TRAIN PASSING EXAMPLE

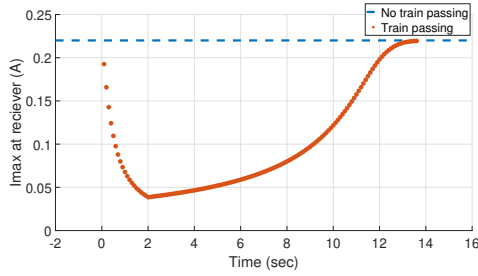


Fig. 13. The variation in the current at the receiver end, $\max_t |i(0, t)|$, as the train passing through an intact track obtained by our modeling algorithm

is plotted in Figure 13. Note that Figure 13 shares similar characteristics with real measurements presented in [13].

Every damage case only takes around one second to compute with the setup of the 117-generation network in this section. The computation is run on MATLAB R2019b with a single CPU of Intel Core i7-4510U Processor.

VI. CONCLUSIONS

This paper presents a method to rapidly and accurately compute voltage and current along a transmission line, especially when its electrical properties are unevenly distributed. The proposed method divides a transmission line into many subsections to form a circuit network, where electrical attributes are lumped at each generation. Thus, that model offers opportunities to imitate an unevenly distributed transmission line by a damaged network. Thanks to a modular recursive algorithm designed to compute frequency response for a general class of one-dimensional self-similar networks, we can thereby quantify the voltage and current at each node inside that network. The correctness of our method is proved by comparing our result to the one given by the transmission line theory when the electrical attributes are evenly distributed. In addition, we illustrate our method's ability to simulate an unevenly distributed transmission line

by providing two real application examples of railway track circuit systems. One of them concerns a damage situation when a ballast degradation occurs. The other quantifies the current's variation as a train is passing through. The proposed method of quickly simulating a railway track circuit under various conditions can be applicable to the relevant health monitoring research area. For example, for some damage detection methods with supervised learning tools, it can offer a baseline case of how the circuit would respond with respect to different types of situations.

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