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Journal of Differential Equations 339 (2022) 555–578

Journal of Differential Equations

www.elsevier.com/locate/jde

Positivity of temperature for some non-isothermal fluid models

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Received 12 November 2021; revised 12 May 2022; accepted 24 August 2022
Available online 6 September 2022

Abstract

We establish three partial differential equation models describing the thermodynamic behavior of a fluid by combining the energetic variational approach, appropriate constitutive relations, and classical thermodynamic laws. By using an explicit algebraic approach, we show a maximum/minimum principle for some auxiliary variables involving the absolute temperature θ and density ρ under some special conditions, which then yields the positivity of the temperature. This important fact implies the thermodynamic consistency for our models.

Contents

1. Introduction 556
2. General frame to derive the non-isothermal model 558
3. Thermodynamics models 563
   3.1. Ideal gas 563
   3.2. Porous media 564
   3.3. Generalized porous media 564
4. The maximum/minimum principle for thermal ideal gas model 565

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https://doi.org/10.1016/j.jde.2022.08.025

5. The maximum/minimum principle for thermal porous media model . . . 570
6. Future directions . . . 576
Acknowledgments . . . 576
References . . . 577

1. Introduction

The study of heat transfer in fluid dynamics has recently attracted growing attention. In non-isothermal models, the temperature is a non-constant material property that flows with the fluid. This creates local changes in the density and viscosity, which then cause the fluid flow to deviate from the classically expected outcome and further influence the rate of heat transfer. This two-way coupling phenomenon is prevalent in heat exchangers, chemical reactors, atmospheric flows, and processes in which components are cooled. One example we face almost every day is the noble gas flow in fluorescent lights. If thermal effects become significant in the flow of fluid through porous media, further applications appear: solidification of binary mixtures, dehumidification, insulation, and heat pipes (see [20] for more details). Besides the applications in mechanical engineering, we also refer to [1] for an enhanced gas recovery application by non-isothermal compressible gas flow.

Due to the prevalence of thermal effects in fluid flow, the literature has seen more and more non-isothermal fluid dynamic models established and delicately analyzed. We list a few of them here. By choosing an appropriate internal energy and entropy production, Málek and Průša [28] arrive at a modern phenomenological theory (based in thermodynamics) of constitutive relations for compressible and incompressible viscous heat-conducting fluids (Navier-Stokes-Fourier), Korteweg fluids, and (in)compressible heat-conducting viscoelastic fluids (Oldroyd-B and Maxwell). The existence of weak solutions for the Navier-Stokes-Fourier system (see [14,35]) describing the evolution of a Newtonian heat-conducting fluid in a bounded domain is systematically studied in [11,29]. Generalized non-isothermal compressible and incompressible non-Newtonian fluid systems are derived in [21] using the energetic variational approach (see [16,19]) which is based on Strutt [33] and Onsager [30,31]. This approach combines the two systems derived from the least action principle and the maximum dissipation principle, and has been widely used to derive other non-isothermal models, by combining the basic thermodynamic laws. We refer to [25] for non-isothermal electrokinetics, [5] for the non-isothermal general Ericksen-Leslie system, [18] for the non-isothermal Poisson-Nernst-Planck-Fourier system, [23] for the Brinkman-Fourier system with ideal gas equilibrium, and [24] for the non-isothermal reaction-diffusion equation.

This paper is devoted exclusively to the a priori estimate of the positivity of the absolute temperature as a key postulate of thermodynamics. The crucial proof of this property for all times (starting from positive initial absolute temperature) is far from trivial in the non-isothermal setting. In [7] the invalidity of negative temperature has been proved, by demonstrating that it arises from the use of an entropy definition that is inconsistent both mathematically and thermodynamically. In the recent series works on the Navier-Stokes-Fourier system by Feireisl et al., the definition of weak solution includes the restriction on absolute temperature θ

$$\theta > 0, \quad a.e. \; in \; (0, T) \times \Omega,$$

and can actually prove the positivity of temperature if it emanates from positive initial data. See [12] for conditional regularity of very weak solutions to the Navier-Stokes-Fourier system, [10] for the existence and stability of the weak solutions to the Navier-Stokes-Fourier system and their relevance in the study of convergence of numerical schemes, and [2] for the existence of weak solutions to the stationary Navier-Stokes-Fourier system. In [34] the third author studied two hydrodynamic model problems (one incompressible and one compressible) with three-dimensional fluid flow on the torus and temperature-dependent viscosity and conductivity and established a positive lower bound for the temperature in each case. We also refer to the excellent book [8] and the monograph [9], a detailed introduction to the singular limits and scale analysis for the Navier-Stokes-Fourier system, in which the positivity of absolute temperature was proved by using a Poincaré type inequality. We have to mention that thermodynamic consistency (positivity of absolute temperature) for a class of phase change models proposed by Frémond [13] has been widely studied in the past, see [3,22,26,27,32] and references therein.

This work focuses on the transport of heat in addition to fluid flow through porous media, and gives a general framework for deriving non-isothermal models by combining the energetic variational approach and some basic thermodynamic laws. From three different given free energies, we establish three different non-isothermal models, which we call *non-isothermal ideal gas*, *non-isothermal porous media*, and *generalized non-isothermal porous media*. Moreover, in some special cases for the first two models, we find maximum/minimum principles for some auxiliary quantities related to the density and temperature (Theorems 4.1 and 5.1), by adapting an idea originally from the work [34]. This then implies the positivity of the absolute temperature. To avoid the problems connected with the boundary behavior of the fluid, we consider the problems on the torus and impose periodic boundary conditions. The proof relies heavily on the structure of the equations, and exploits genuinely nonlinear interactions in the system of equations coupling density and temperature. In the non-isothermal ideal gas model, the computations are relatively straightforward, but become significantly more intricate in the non-isothermal porous media model. The generalized non-isothermal porous media model creates yet more complicated interactions, and a similar structure does not seem evident. Extending this methodology (or finding a comprehensive approach that better incorporates the derivation) is an interesting question for future research.

We emphasize that the three models are ultimately determined by a starting choice of free energy Ψ which depends on the local state variables ρ and θ (density and temperature). This choice is not arbitrary. The first model (see Section 3.1) uses the free energy for an ideal gas, which must locally maintain that pressure be proportional to $\rho\theta$ (see (2.10), (2.11), and (3.1)). This is the simplest form that Ψ can take while maintaining physical consistency, and it leads to the simplest of our three models (3.6). The other two models (3.12) and (3.17) arise from natural variations on the ideal gas free energy, and are in a sense the “next simplest” models to consider (again maintaining physical consistency). While it is possible to consider models with a weaker coupling between ρ and θ, they would need to arise from exotic constitutive laws (which might be found in certain areas of material science). The present work aims to showcase the methodology for deriving a priori maximum principles using the nonlinear structure of the equations. We investigate models that are thermodynamically consistent and derived from fundamental principles to emphasize that the results are not “baked into” the equations.

Outline. In Section 2, we present a general derivation of models for non-isothermal fluid flow starting from a given free energy function. In Section 3, we look at three free energy functions to obtain equations for the ideal gas, porous media, and generalized porous media equations. In Section 4, we prove a priori maximum and/or minimum principles for the temperature and density of the ideal gas model, by first proving them for certain auxiliary variables adapted to the structure of the equation. In Section 5, we prove analogous (but slightly weaker) results for the generalized porous media equations. In Section 6 we give some concluding remarks and future research directions.

2. General frame to derive the non-isothermal model

This section aims to derive a system of equations involving a Darcy-type dissipative law for a suitable family of non-isothermal fluids. For more detailed physical discussions such as the choice of various energy functionals, the state variables, and also the kinematic transport relations, we will refer to the book [8] and also [23]. In this paper, for simplicity and illustration of the approach, we will derive the overall governing equations, employing some minimum ingredients and assumptions:

- The free energy of the system, which will be denoted by Ψ.
- The entropy production Δ, related to the rate of dissipation of the system.
- The kinematic transport of the state variable and the conservation of density $\rho(t,x)$ along the flow.
- The energetic variational approaches, including both the Least Action Principle and Maximum Dissipation Principle,
- Finally, the First and Second Laws of Thermodynamics.

We start from the free energy, $\Psi(\rho,\theta)$, which is a function of the density ρ and the absolute temperature θ. The (specific) entropy of the system and the (specific) internal energy can then be linked to the free energy Ψ by the standard Helmholtz relation (see formula (2.5.26) in the classical book [4])

- (specific) entropy of the system:

$$\eta(\rho,\theta) := -\partial_\theta \Psi.$$

- (specific) internal energy (the Legendre transform in θ of the free energy):

$$e(\rho,\theta) := \Psi - \partial_\theta \Psi \theta = \Psi + \eta\theta.$$

Next we are going to establish a Darcy type diffusion law by using the Least Action Principle and Maximum Dissipation Principle.

Lemma 2.1. *Assume a given energy density $\Psi(\rho,\theta)$. Assume further that the total dissipation rate of the fluid is of Darcy type, i.e., $\frac{1}{2}\rho|u|^2$, with $u(t,x)$ the velocity field of the fluid. Then the conservation of ρ and the Onsanger's principles automatically determine the equation of motion for the fluid $u(t,x)$, which corresponds to the following Darcy's law:*

$$\nabla p = -\rho u, \tag{2.1}$$

where the pressure is defined by

$$p = \Psi_\rho \rho - \rho. \tag{2.2}$$

Proof. As stated in [15,19], we start from the total energy and dissipation which are chosen as

$$E^{\text{total}} = \int_{\Omega_t^x} \Psi(\rho, \theta) dx, \quad \mathcal{D}^{\text{total}} = \frac{1}{2} \int_{\Omega_t^x} \rho u^2 dx$$

in this paper, where Ω_t^x is the deformed configuration corresponding to the reference configuration Ω_0^X; here X represents Lagrangian coordinates and x represents Eulerian coordinates. We first rewrite the energy functional in the Lagrangian coordinate system and then take the least action principle conservative force.

$$A(x(X,t)) = -\int_0^T \int_{\Omega_0^X} \Psi\left(\frac{\rho_0(X)}{J}, \theta_0(X)\right) J dX dt, \tag{2.3}$$

where $J = \det F$ and $F = \frac{\partial x}{\partial X}$ denotes the deformation gradient. Then taking the variation for any smooth compactly supported $y(X,t) = \widetilde{y}(x(X,t),t)$ with respect to x yields

$$\begin{aligned}
&\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} A(x(X,t) + \varepsilon y(X,t)) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} A(x + \varepsilon y) \\
&= -\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_0^T \int_{\Omega_0^X} \Psi\left(\frac{\rho_0(X)}{\det\frac{\partial(x+\varepsilon y)}{\partial X}}, \theta_0(X)\right)\left(\det\frac{\partial(x+\varepsilon y)}{\partial X}\right) dX dt \\
&= -\int_0^T \int_{\Omega_0^X} \left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Psi\left(\frac{\rho(X)}{\det\frac{\partial(x+\varepsilon y)}{\partial X}}, \theta_0(X)\right)\right) \cdot \left(\det\frac{\partial x}{\partial X}\right) \\
&- \Psi\left(\frac{\rho_0(X)}{\det\frac{\partial \mathrm{x}}{\partial \mathrm{X}}}, \theta_0(X)\right) \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\left(\det\frac{\partial(x+\varepsilon y)}{\partial X}\right) dX dt \\
&= -\int_0^T \int_{\Omega_0^X} \Psi_\rho\left(\frac{\rho_0(X)}{J}, \theta_0(X)\right) \cdot \left(-\frac{\rho_0(X)}{J^2}\right) \cdot J \cdot \operatorname{tr}\left(\frac{\partial X}{\partial x}\frac{\partial y}{\partial X}\right) \cdot J \\
&+ \Psi\left(\frac{\rho_0(X)}{J}, \theta_0(X)\right) \cdot J \cdot \operatorname{tr}\left(\frac{\partial X}{\partial x}\frac{\partial y}{\partial X}\right) dX dt \\
&= \int_0^T \int_{\Omega_t^x} \big(\Psi_\rho(\rho(x,t)) \cdot \rho(x,t) - \Psi(\rho(x,t))\big) \cdot (\nabla_x \cdot \widetilde{y}) dx dt \\
&= \int_0^T \int_{\Omega_t^x} -\nabla_x \big(\Psi_\rho(\rho(x,t)) \cdot \rho(x,t) - \Psi(\rho(x,t))\big) \cdot \widetilde{y} dx dt,
\end{aligned} \tag{2.4}$$

which gives the conservative force

$$\text{Force}_{\text{cons}} = -\nabla_x \left(\Psi_\rho(\rho(x,t)) \cdot \rho(x,t) - \Psi(\rho(x,t))\right) =: -\nabla p, \tag{2.5}$$

where the negative pressure gradient is by definition equal to the conservative force. On the other hand, according to the maximum dissipation principle, taking variation with respect to the velocity of the dissipation functional yields the dissipative force

$$\delta_u \mathcal{D} = \int_{\Omega_t^x} \rho u \cdot \delta u dx = \int_{\Omega_t^x} \text{Force}_{\text{diss}} \cdot \delta u dx, \tag{2.6}$$

where it is seen that $\text{Force}_{\text{diss}} = \rho u$. With these two forces in hand, we may then apply Newton's force balance law, which states that all forces, both conservative and dissipative, add up to zero ("action" equals "reaction"), thus

$$\text{Force}_{\text{cons}} = \text{Force}_{\text{diss}},$$

which yields (2.1) by combining (2.5) and (2.6). □

We also take ρ, η as new state variables and rewrite the internal energy function as

$$e_1(\rho, \eta) = e\left(\rho, \theta(\rho, \eta)\right),$$

and hence we have

Lemma 2.2. *The derivatives of the internal energy with respect to the new state variables satisfy*

$$e_{1\eta} = \theta, \quad e_{1\rho} = \Psi_\rho. \tag{2.7}$$

Proof. Direct computation yields that

$$\begin{aligned} e_{1\eta} &= \Psi_\theta \theta_\eta + \theta + \eta\theta_\eta \\ &= -\eta\theta_\eta + \theta + \eta\theta_\eta \\ &= \theta \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} e_{1\rho} &= \Psi_\rho + \Psi_\theta \theta_\rho + \eta\theta_\rho \\ &= \Psi_\rho. \quad \square \end{aligned} \tag{2.9}$$

Lemma 2.3. *By the definitions above, it holds that*

$$\partial_\theta p = \eta - \eta_\rho \rho \tag{2.10}$$

and

$$\nabla p = \rho \nabla e_{1\rho} + \eta \nabla e_{1\eta}. \tag{2.11}$$

Proof. We may get (2.10) by direct computation

$$\partial_\theta p = \Psi_{\rho\theta}\rho - \Psi_\theta = \eta - \eta_\rho \rho.$$

For (2.11), it is easy to see from the definition of pressure

$$\begin{aligned} \nabla p &= \nabla(\Psi_\rho \rho - \Psi) \\ &= (\Psi_{\rho\rho}\nabla\rho + \Psi_{\rho\theta}\nabla\theta)\rho + \Psi_\rho\nabla\rho - \Psi_\rho\nabla\rho - \Psi_\theta\nabla\theta \\ &= (\Psi_{\rho\rho}\nabla\rho + \Psi_{\rho\theta}\nabla\theta)\rho + \eta\nabla\theta. \end{aligned} \tag{2.12}$$

On the other hand, one has

$$\begin{aligned} &\rho\nabla e_{1\rho} + \eta\nabla e_{1\eta} \\ =&\rho(\Psi_{\rho\rho}\nabla\rho + \Psi_{\rho\theta}\nabla\theta) + \eta\nabla\theta, \end{aligned} \tag{2.13}$$

then (2.11) follows. □

Next we employ the following classical thermodynamic laws.

- The first one is related to the rate of change of the internal energy with dissipation and heat

$$\frac{de}{dt} = \nabla \cdot W + \nabla \cdot q. \tag{2.14}$$

  Here W denotes the amount of thermodynamic work done by the system on its surroundings and q denotes the quantity of energy supplied to the system as heat.
- The second one is related to the change of entropy

$$\partial_t \eta + \nabla \cdot (\eta u) = \nabla \cdot \left(\frac{q}{\theta}\right) + \Delta, \tag{2.15}$$

  where $\Delta \geq 0$ denotes the rate of entropy production.
- The third one is called Fourier's law

$$q = \kappa_3 \nabla\theta, \tag{2.16}$$

  where κ_3 denotes the material conductivity which may depend on ρ and θ.

Also, we make the general kinematic assumption of mass transport

$$\rho_t + \nabla \cdot (\rho u) = 0. \tag{2.17}$$

By (2.11), (2.14), (2.15), (2.16) and (2.17), one has

$$\begin{aligned}
&\frac{de_1(\rho,\eta)}{dt}\\
=&e_{1\rho}\rho_t+e_{1\eta}\eta_t\\
=&e_{1\rho}\left(-\nabla\cdot(\rho u)\right)+e_{1\eta}\left(-\nabla\cdot(\eta u)+\nabla\cdot\left(\frac{q}{\theta}\right)+\Delta\right)\\
=&-\nabla\cdot\left(e_{1\rho}\rho u+e_{1\eta}\eta u\right)+\left(\rho\nabla e_{1\rho}+\eta\nabla e_{1\eta}\right)\cdot u+\theta\nabla\cdot\left(\frac{q}{\theta}\right)+\theta\Delta\\
=&\nabla\cdot W+\nabla p\cdot u+\nabla\cdot q-\frac{q}{\theta}\cdot\nabla\theta+\theta\Delta\\
=&\nabla\cdot W-\rho u^2+\nabla\cdot q-\frac{\kappa_3|\nabla\theta|^2}{\theta}+\theta\Delta,
\end{aligned} \tag{2.18}$$

where

$$W=-\left(e_{1\rho}\rho+e_{1\eta}\eta\right)u$$

denotes the work done by the system. Set the rate of entropy production by

$$\begin{aligned}
\Delta=&\frac{1}{\theta}\left(\rho|u|^2+\frac{q\cdot\nabla\theta}{\theta}\right)\\
=&\frac{1}{\theta}\left(\rho|u|^2+\frac{\kappa_3|\nabla\theta|^2}{\theta}\right),
\end{aligned} \tag{2.19}$$

then (2.18) turns out to be (2.14).

With the above preliminaries in hand, we may establish the non-isothermal equation by (2.15). By combining (2.10) and (2.17), we have

$$\begin{aligned}
\eta_t+\nabla\cdot(\eta u)=&\eta_\theta(\theta_t+u\cdot\nabla\theta)+\eta_\rho(\rho_t+u\cdot\nabla\rho)+\eta\nabla\cdot u\\
=&\eta_\theta(\theta_t+u\cdot\nabla\theta)+\eta_\rho\left(-\rho\nabla\cdot u\right)+\eta\nabla\cdot u\\
=&\eta_\theta(\theta_t+u\cdot\nabla\theta)+\left(\eta-\eta_\rho\rho\right)\nabla\cdot u\\
=&\eta_\theta(\theta_t+u\cdot\nabla\theta)+\partial_\theta p\nabla\cdot u\\
=&\nabla\cdot j+\Delta\\
=&\nabla\cdot\left(\frac{q}{\theta}\right)+\frac{1}{\theta}\left(\rho|u|^2+\frac{q\cdot\nabla\theta}{\theta}\right),
\end{aligned} \tag{2.20}$$

which yields

$$\eta_\theta(\theta_t+u\cdot\nabla\theta)+\partial_\theta p\nabla\cdot u=\nabla\cdot\left(\frac{q}{\theta}\right)+\frac{1}{\theta}\left(\rho|u|^2+\frac{q\cdot\nabla\theta}{\theta}\right). \tag{2.21}$$

Hence, for a given free energy $\Psi(\rho,\theta)$, the non-isothermal model can be established by (2.1), (2.17), and (2.21).

3. Thermodynamics models

In this section, we introduce three specific free energies, which are related to non-isothermal ideal gas, non-isothermal porous media, and non-isothermal generalized porous media, respectively.

3.1. Ideal gas

For the ideal gas, the free energy is given by

$$\Psi(\rho,\theta) = \kappa_1\theta\rho\ln\rho - \kappa_2\rho\theta\ln\theta.$$

This gives

$$\begin{aligned} &p = \kappa_1\rho\theta,\\ &\partial_\theta p = \kappa_1\rho,\\ &\eta_\theta = \frac{\kappa_2\rho}{\theta},\\ &\rho u = -\nabla p = -\kappa_1\nabla(\rho\theta). \end{aligned} \tag{3.1}$$

Then the mass equation (2.17) becomes

$$\begin{aligned} \partial_t\rho &= -\nabla\cdot(\rho u)\\ &= \kappa_1\nabla\cdot(\nabla(\rho\theta))\\ &= \kappa_1\Delta(\rho\theta). \end{aligned} \tag{3.2}$$

Then (2.21) changes to

$$\begin{aligned} &\frac{\kappa_2\rho}{\theta}(\theta_t + u\cdot\nabla\theta) + \kappa_1\rho\nabla\cdot u\\ =&\nabla\cdot\left(\frac{\kappa_3\nabla\theta}{\theta}\right) + \frac{1}{\theta}\left(-\kappa_1\nabla(\rho\theta)\cdot u + \frac{\kappa_3|\nabla\theta|^2}{\theta}\right), \end{aligned} \tag{3.3}$$

implying

$$\kappa_2(\rho\theta)_t - \kappa_2\theta\rho_t + \kappa_2\rho u\cdot\nabla\theta + \kappa_1\nabla\cdot(\rho\theta u) = \theta\nabla\cdot\left(\frac{\kappa_3\nabla\theta}{\theta}\right) + \frac{\kappa_3|\nabla\theta|^2}{\theta}, \tag{3.4}$$

which can be simplified to

$$\kappa_2(\rho\theta)_t - \kappa_1(\kappa_1+\kappa_2)\nabla\cdot(\theta\nabla(\rho\theta)) = \nabla\cdot(\kappa_3\nabla\theta)\,. \tag{3.5}$$

Hence we get the non-isothermal model for ideal gas:

$$\begin{cases} \partial_t\rho = \kappa_1\Delta(\rho\theta),\\ \kappa_2(\rho\theta)_t - \kappa_1(\kappa_1+\kappa_2)\nabla\cdot(\theta\nabla(\rho\theta)) = \nabla\cdot(\kappa_3\nabla\theta)\,. \end{cases} \tag{3.6}$$

3.2. Porous media

For the porous media, we introduce the free energy as

$$\Psi(\rho,\theta) := \kappa_1\theta\rho^\alpha - \kappa_2\rho\theta\ln\theta, \tag{3.7}$$

with $\alpha > 1$. In this case we have

$$\begin{aligned} &p = \kappa_1(\alpha-1)\theta\rho^\alpha, \\ &\partial_\theta p = \kappa_1(\alpha-1)\rho^\alpha, \\ &\eta_\theta = \frac{\kappa_2\rho}{\theta}, \\ &\rho u = -\nabla p = -\kappa_1(\alpha-1)\nabla(\theta\rho^\alpha) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \partial_t\rho &= -\nabla\cdot(\rho u) \\ &= \kappa_1(\alpha-1)\nabla\cdot\big(\nabla(\theta\rho^\alpha)\big) \\ &= \kappa_1(\alpha-1)\Delta(\theta\rho^\alpha). \end{aligned} \tag{3.9}$$

Also, (2.21) becomes

$$\begin{aligned} &\frac{\kappa_2\rho}{\theta}(\theta_t + u\cdot\nabla\theta) + \kappa_1(\alpha-1)\theta\rho^{\alpha-1}\nabla\cdot u \\ =&\nabla\cdot\left(\frac{\kappa_3\nabla\theta}{\theta}\right) + \frac{1}{\theta}\left(-\kappa_1(\alpha-1)\nabla(\theta\rho^\alpha)\cdot u + \frac{\kappa_3|\nabla\theta|^2}{\theta}\right), \end{aligned} \tag{3.10}$$

which yields finally

$$\kappa_2(\rho\theta)_t + \kappa_2\nabla\cdot(\rho\theta u) + \kappa_1(\alpha-1)\nabla\cdot(\theta\rho^\alpha u) = \nabla\cdot(\kappa_3\nabla\theta), \tag{3.11}$$

and hence we get the non-isothermal porous media system:

$$\begin{cases} \partial_t\rho = \kappa_1(\alpha-1)\Delta\left(\theta\rho^\alpha\right), \\ \kappa_2(\rho\theta)_t - \kappa_1\kappa_2(\alpha-1)\nabla\cdot\big(\theta\nabla(\theta\rho^\alpha)\big) - \kappa_1^2(\alpha-1)^2\nabla\cdot\left(\theta\rho^{\alpha-1}\nabla(\theta\rho^\alpha)\right) \\ = \nabla\cdot(\kappa_3\nabla\theta). \end{cases} \tag{3.12}$$

3.3. Generalized porous media

We introduce the free energy for generalized porous media as

$$\Psi(\rho,\theta) = k_1\theta\rho^\alpha - k_2\rho\theta^\beta,$$

with $\alpha, \beta > 1$. Then

$$\begin{aligned} p &= k_1(\alpha-1)\theta\rho^\alpha, \\ \partial_\theta p &= k_1(\alpha-1)\rho^\alpha, \\ \eta_\theta &= k_2\beta(\beta-1)\rho\theta^{\beta-2}, \\ \rho u &= -\nabla p = -k_1(\alpha-1)\nabla(\theta\rho^\alpha), \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \partial_t \rho &= -\nabla\cdot(\rho u) \\ &= k_1(\alpha-1)\nabla\cdot\big(\nabla(\theta\rho^\alpha)\big) \\ &= k_1(\alpha-1)\Delta(\theta\rho^\alpha). \end{aligned} \tag{3.14}$$

In this case (2.21) becomes

$$\begin{aligned} &k_2\beta(\beta-1)\rho\theta^{\beta-2}(\theta_t + u\cdot\nabla\theta) + k_1(\alpha-1)\rho^\alpha\nabla\cdot u \\ =&\nabla\cdot\left(\frac{k_3\nabla\theta}{\theta}\right) + \frac{1}{\theta}\left(-k_1(\alpha-1)\nabla(\theta\rho^\alpha)\cdot u + \frac{k_3|\nabla\theta|^2}{\theta}\right), \end{aligned} \tag{3.15}$$

which yields finally

$$\begin{aligned} &k_2(\beta-1)(\rho\theta^\beta)_t - k_1k_2(\alpha-1)(\beta-1)\nabla\cdot\big(\theta^\beta\nabla(\theta\rho^\alpha)\big) \\ &-k_1^2(\alpha-1)^2\nabla\cdot\Big(\theta\rho^{\alpha-1}\nabla(\theta\rho^\alpha)\Big) = \nabla\cdot(k_3\nabla\theta), \end{aligned} \tag{3.16}$$

and hence we obtain the non-isothermal porous media system:

$$\begin{cases} \partial_t\rho = k_1(\alpha-1)\Delta\left(\theta\rho^\alpha\right), \\ k_2(\beta-1)(\rho\theta^\beta)_t - k_1k_2(\alpha-1)(\beta-1)\nabla\cdot\big(\theta^\beta\nabla(\theta\rho^\alpha)\big) \\ -k_1^2(\alpha-1)^2\nabla\cdot\Big(\theta\rho^{\alpha-1}\nabla(\theta\rho^\alpha)\Big) = \nabla\cdot(k_3\nabla\theta). \end{cases} \tag{3.17}$$

4. The maximum/minimum principle for thermal ideal gas model

In this section, we use the structure of (3.6) to establish maximum and/or minimum principles for certain auxiliary variables in the temperature and pressure. Even if one assumes a priori that a smooth solution pair (ρ,θ) exists, it is not feasible to obtain max/min principles for the two functions directly due to the complicated interdependence between ρ and θ. Indeed, maximum principles for coupled systems of partial differential equations are notoriously hard to obtain, and is one of the major obstacles in going from "scalar-valued" problems (e.g., heat, porous media, or surface quasigeostrophic equation) to "vector-valued" problems (e.g., Navier-Stokes and Euler equations). One might then search for a "state variable" $\lambda(\rho,\theta)$ that is (super- or sub-) conserved, but identifying the right variable λ is nontrivial.

Nevertheless, if the material conductivity κ_3 is proportional to $\theta\rho$, we can find two homogeneous auxiliary variables $\theta\rho^{1+\gamma_\pm}$ that, through careful cancellation in the structure of (3.6), satisfy pointwise a priori maximum or minimum principles, in the form of Theorem 4.1 below.

Rather than simply verifying the principle for the two auxiliary variables above, the proof takes a more general approach. It will look at variables $\theta\rho f(\rho)$ for a to-be-determined positive weight function f and, using the structure of (3.6), show that f must satisfy one of two possible ordinary differential equations, which naturally lead to the variables above. The method is similar to how [34] found appropriate temperature weights to close energy-type estimates for a fluid equation with thermal dissipation. The proof below is also slightly more general, as the two functions f can also be obtained implicitly when $\kappa_3 = D(\rho)\theta$, in terms of the function D. Note that the ansatz $\theta\rho f(\rho)$ for the auxiliary variables essentially covers any "state variable" of the form $\lambda(\rho,\theta) = \theta^\gamma \tilde{f}(\rho)$ (for $\gamma > 0$), so that the auxiliary variables found in the proof below are the only ones in this class to satisfy maximum or minimum principles.

Theorem 4.1. *Consider the non-isothermal ideal gas model (3.6) on $[0,T)\times\mathbb{T}^n$. Assume we have a smooth solution pair (ρ,θ) on this domain. If the material conductivity κ_3 takes the form*

$$\kappa_3 = \kappa_1 \widetilde{D}\theta\rho,$$

for $\widetilde{D} >$ a fixed constant, then we have
I. *The absolute temperature is positive on $\mathbb{T}^n \times [0,T)$.*
II. *The density is bounded from above unconditionally.*
III. *If the temperature $\theta(t,x)$ either blows up or goes to zero, then the density $\rho(t,x)$ must vanish. Precisely, we have*

$$\rho(t,x) \le \min\left\{\theta^{-c_1}, \theta^{-c_2}\right\}, \tag{4.1}$$

for some constants $c_1 > 0$, $c_2 < 0$ depending on κ_1, κ_2, and $\tilde{D}$.

Proof. We can slightly simplify system (3.6) by writing $\beta = 1 + \kappa_1/\kappa_2$ and (by abuse of notation) replacing κ_3 by $\kappa_3\kappa_2$ to obtain

$$\begin{cases} \partial_t\rho = \kappa_1\Delta(\rho\theta) \\ \partial_t(\rho\theta) = \nabla\cdot(\beta\kappa_1\theta\nabla(\rho\theta) + \kappa_3\nabla\theta) \end{cases} \tag{4.2}$$

Here κ_3 is not necessarily constant; it generally depends on θ and ρ. In the interest of making the proof more general, we will initially take $\kappa_3 = \theta D(\rho)$. Later, we make the special assumption that $D(\rho) = \kappa_1\tilde{D}\rho$.

Let $f : (0,\infty) \to (0,\infty)$ be a smooth monotone function to be determined later. We look at the quantity $f(\rho)\rho\theta$. Taking a derivative and using (4.2) yields

$$\partial_t(f\rho\theta) = \theta f'\rho\partial_t\rho + f\partial_t(\rho\theta) = \kappa_1\rho\theta f'\Delta(\rho\theta) + f\nabla\cdot(\beta\kappa_1\theta\nabla(\rho\theta) + \kappa_3\nabla\theta). \tag{4.3}$$

Let $x_0 \in \mathbb{T}^n$ be a point where $f\rho\theta$ achieves a local minimum (respectively maximum) in space. Assume that $f(\rho(x_0))\rho(x_0)\theta(x_0) > 0$. For the rest of the proof, all quantities are implicitly evaluated at x_0, though we suppress the notation. Then the gradient at this point vanishes, so that

$$(f'\rho + f)\theta\nabla\rho + f\rho\nabla\theta = 0, \tag{4.4}$$

and the quantity

$$
\begin{aligned}
L &:= \Delta(f\rho\theta) = (2f' + f''\rho)\theta|\nabla\rho|^2 + 2(f'\rho + f)\nabla\theta\cdot\nabla\rho + (f'\rho + f)\theta\Delta\rho + f\rho\Delta\theta \\
&= \frac{|\nabla\theta|^2}{\theta}\left(\frac{(f\rho)^2(2f' + f''\rho)}{(f'\rho + f)^2} - 2f\rho\right) + (f'\rho + f)\theta\Delta\rho + f\rho\Delta\theta
\end{aligned}
$$

is nonnegative (respectively nonpositive). The last equality used (4.4), which in general allows us to compare terms of the form $|\nabla\rho|^2$, $\nabla\rho\cdot\nabla\theta$, and $|\nabla\theta|^2$ to each other (recall that they are all evaluated at x_0).

Taking $\kappa_3 = \theta D(\rho)$ and expanding (4.3) (again using (4.4)) yields

$$
\begin{aligned}
\partial_t(f\rho\theta) =&\kappa_1\theta f'\rho\Delta(\rho\theta) + f\Big(\beta\kappa_1(\theta\nabla\theta\cdot\nabla\rho + \rho|\nabla\theta|^2 + \theta\Delta(\rho\theta)) \\
&+ D|\nabla\theta|^2 + D\theta\Delta\theta + D'\theta\nabla\theta\cdot\nabla\rho\Big) \\
=&\big(\kappa_1\theta f'\rho + \beta\kappa_1\theta f\big)\theta\Delta\rho + \left(\kappa_1\theta f'\rho + \beta\kappa_1\theta f + \theta f\frac{D}{\rho}\right)\rho\Delta\theta \\
&+ \left(-\frac{2\kappa_1 f f'\rho^2}{f'\rho + f} + \beta\kappa_1\left(f\rho - \frac{3f^2\rho}{f'\rho + f}\right) - \frac{f^2\rho D'}{f'\rho + f} + fD\right)|\nabla\theta|^2 \\
=&\theta(F_1\theta\Delta\rho + F_2\rho\Delta\theta) + F_3|\nabla\theta|^2.
\end{aligned}
$$

The goal then is to choose f (in terms of β, κ_2, and D) such that we may rewrite the above as

$$
\partial_t(f\rho\theta) = \theta\tilde{F}L + \tilde{G}|\nabla\theta|^2, \tag{4.5}
$$

with $\tilde{F} \geq 0$ and $\tilde{G}$ nonnegative (respectively nonpositive). This would show that $f\rho\theta$ satisfies a minimum (respectively maximum) principle for such f.

In order for (4.5) to hold, there must be some $\lambda \geq 0$ (not necessarily constant) such that

$$
\begin{aligned}
\kappa_1 f'\rho + \beta\kappa_1 f &= \lambda(f'\rho + f), \\
\kappa_1 f'\rho + \beta\kappa_1 f + f\frac{D}{\rho} &= \lambda f.
\end{aligned}
$$

Subtracting yields

$$
\lambda = -\frac{fD}{f'\rho^2},
$$

which immediately implies that

$$
f' < 0. \tag{4.6}
$$

We therefore need f to satisfy

$$
\kappa_1 f'\rho + \beta\kappa_1 f + \frac{fD}{\rho} + \frac{f^2 D}{f'\rho^2} = 0.
$$

Solving for f' formally yields

$$f' = f\frac{-\beta\kappa_1\rho - D \pm \sqrt{\beta^2\kappa_1^2\rho^2 + 2(\beta-2)\kappa_1\rho D + D^2}}{2\kappa_1\rho^2} \tag{4.7}$$

First, we remark that (4.6) is always satisfied whenever the right-hand-side of (4.7) is real-valued. The numerator is of the form $-b \pm \sqrt{b^2 - 4ac}$ where each of a, b, and c are positive (for $\rho > 0$).

Second, as long as κ_1 and κ_2 are positive (so $\beta > 1$), the discriminant of (4.7) is strictly positive for all ρ (regardless of the value of D). This is easily seen with the Schwartz inequality, and that the discriminant is bounded below by $(\beta^2 - 1)\kappa_2^2\rho^2$. Thus, (4.7) specifies two ODE's that are locally well-posed for all $\rho > 0$.

Third, it is immediate from (4.7) that $|f'| \leq C|f|(\rho^{-1} + D(\rho)\rho^{-2})$. Gronwall's inequality then guarantees that both ODE's are globally well-posed on $(0, \infty)$. That is, given $\rho_0 > 0$ and any initial datum $f_0 > 0$, there exist two unique, positive, monotone-decreasing weight functions $f_\pm$ defined on $(0, \infty)$ such that $f_\pm$ satisfies the corresponding ODE of (4.7) pointwise and $f_\pm(\rho_0) = f_0$.

Fourth, we can directly apply the Duhamel principle. Writing (4.7) as

$$f'(\rho) = f(\rho)\Gamma_\pm(\beta, \kappa_1, D, \rho),$$

we get that

$$f_\pm(\rho) = f_0 \exp\left(\int_{\rho_0}^{\rho} \Gamma_\pm(\beta, \kappa_1, D(r), r)dr\right).$$

In the special case $D(\rho) = \tilde{D}\kappa_1\rho$ (for $\tilde{D} > 0$ a fixed constant), an assumption we will make for the remainder of this proof, the functions are given explicitly as $f_\pm(\rho) = f_\pm(1)\rho^{\gamma_\pm}$ with

$$\gamma_\pm = \frac{-\beta - \tilde{D} \pm \sqrt{\beta^2 + 2(\beta-2)\tilde{D} + \tilde{D}^2}}{2}.$$

Note that $\gamma_\pm < 0$ for all values of $\tilde{D}$, β, and κ_1. Moreover, $1 + \gamma_+ > 0$ while $1 + \gamma_- < 0$.

Thus we do obtain (4.5), in the sense that

$$\partial_t(f\rho\theta) = -\theta\frac{fD}{f'\rho^2}L + \tilde{G}|\nabla\theta|^2,$$

where

$$\begin{aligned}\tilde{G} =& \frac{fD}{f'\rho^2}\left(\frac{(f\rho)^2(2f' + f''\rho)}{(f'\rho + f)^2} - 2f\rho\right) \\ &+ \left(fD + (\beta-2)\kappa_1 f\rho - (D' + \kappa_1(3\beta-2))\frac{f^2\rho}{f'\rho + f}\right).\end{aligned} \tag{4.8}$$

Remark 4.2. This is a complicated expression which can be reduced by repeated use of (4.7) until it only involves f, ρ, D, and D'. Notice that $\tilde{G}$ does not depend on θ or on the position or time variables. Its sign dictates the nature of the max/min principle satisfied by the auxiliary variable $\theta\rho f(\rho)$, and this ultimately depends only on ρ. If D is left as a generic (monotone) function of ρ, the range of possible behaviors is quite complicated, in some cases leading to "banded" structure where the auxiliary variable satisfies a maximum principle in certain interval ranges of ρ and a minimum principle on the complementary intervals (and *both* at the endpoints). For this ideal gas model, one obtains much more precise and unconditional results if one adheres to the special case $D = \tilde{D}\rho$, with more general functions D left for future work.

Since $f'\rho = \gamma_\pm f$, (4.8) reduces to the simpler expression

$$\tilde{G}_\pm = \kappa_1 f\rho \left(-\frac{(2+\gamma_\pm)\tilde{D}}{\gamma_\pm(1+\gamma_\pm)} + (\tilde{D}+\beta-2)\frac{\gamma_\pm}{1+\gamma_\pm} - \frac{2\beta}{1+\gamma_\pm} \right). \tag{4.9}$$

Recall that the minimum principle requires that $\tilde{G} \geq 0$, but the maximum principle requires that $\tilde{G} \leq 0$. However, we see that $\tilde{G}_\pm$ is in fact $C_\pm f\rho$ for some *fixed* constants $C_\pm$ that depend on the initial parameters. This guarantees that each of $f_\pm\rho\theta$ individually satisfy *either* a maximum principle *or* a minimum principle for the entire lifetime of the solution (ρ, θ).

Observing that $\gamma_+\gamma_- = \tilde{D}$, a calculation shows that

$$\begin{aligned} \tilde{G}_+ &= \frac{\kappa_1 f\rho}{1+\gamma_+}\left(-\tilde{D} - 2\gamma_- + \tilde{D}\gamma_+ + \beta\gamma_+ - 2\gamma_+ - 2\beta \right) \\ &= \frac{\kappa_1 f\rho}{1+\gamma_+}\left(\tilde{D}(1+\gamma_+) + \beta\gamma_+ \right). \end{aligned}$$

Recall that $1+\gamma_+ > 0$. We claim that $\tilde{D}(1+\gamma_+) + \beta\gamma_+$ is always negative. This is true if and only if

$$(\tilde{D}+\beta)\sqrt{(\tilde{D}+\beta)^2 - 4\tilde{D}} \leq \tilde{D}^2 + 2\beta\tilde{D} + \beta^2 - 2\tilde{D}.$$

Since both sides of the inequality are positive (recall $\beta > 1$), we may square both sides and get

$$(\tilde{D}+\beta)^4 - 4\tilde{D}(\tilde{D}+\beta)^2 \leq (\tilde{D}+\beta)^4 - 4\tilde{D}(\tilde{D}+\beta)^2 + 4\tilde{D},$$

which is always true (and therefore so is the claim). Thus, we always have $\tilde{G}_+ < 0$, guaranteeing a maximum principle for $\rho^{1+\gamma_+}\theta$.

A similar calculation shows that

$$\tilde{G}_- = \frac{\kappa_2 f\rho}{1+\gamma_-}\left(\tilde{D}(1+\gamma_-) + \beta\gamma_- \right).$$

Recall that $\gamma_- < -1$. Then $\tilde{G}_- > 0$ which gives a minimum principle for $\rho^{1+\gamma_-}\theta$.

Putting it all together, we have

$$\rho^{1+\gamma_+}\theta \leq c_1 \quad \text{and} \quad \rho^{1+\gamma_-}\theta \geq c_2.$$

The second inequality implies ***I*** (the positivity of the temperature). Further, both inequalities imply ***II***, as $\rho^{\gamma_+-\gamma_-} = \rho^{\sqrt{\beta^2+2(\beta-2)\tilde{D}+\tilde{D}^2}} \le c_1/c_2$. So the density is bounded above unconditionally.

Since ρ is bounded above, the two inequalities imply that the density must vanish if the temperature either blows up or goes to zero. This shows ***III***, as well as (4.1). □

5. The maximum/minimum principle for thermal porous media model

The non-isothermal porous media model (3.12) is more complicated than (3.6), leading to more intricate calculations for the auxiliary variables. Although they ultimately take the same form as before ($\theta\rho f(\rho)$), and there are still exactly two possibilities for f, it is no longer possible to find clean expressions for f even when κ_3 takes a simple form. The corresponding maximum and minimum principles also become more conditional, and we must resort to asymptotic analysis (i.e., large ρ and vanishing ρ limits) to determine precisely which case occurs for each of the auxiliary variables. The proof is similar to that of Theorem 4.1, but more technical and involved. For this reason, we present it here at the end, so that the proof of the previous section can be used as a reference.

Theorem 5.1. *If* (ρ, θ) *is a smooth solution pair to* (3.12) *on* $[0, T) \times \mathbb{T}^n$ *and the material conductivity* κ_3 *is given by*

$$\kappa_3 = aD\theta, \tag{5.1}$$

with constants $D > 0$ *and* $a := \alpha - 1 > 0$ *(so independent of* ρ*), then we have*

I. High density case: *There is some threshold* $\overline{\rho}$ *for which, if* $\rho(t,x) > \overline{\rho}$ *on* $\mathbb{T}^n$*, then there are constants* $c_1, c_2 > 0$ *depending on the data such that*

$$\rho^{a+1}\theta \ge c_1 \quad \textit{and} \quad \rho \exp\left(-\frac{\kappa_1 a}{\kappa_2(a+1)}\rho^a\right)\theta \ge c_2,$$

and hence

$$\theta \ge \max\left(c_1\rho^{-a-1}, c_2\rho^{-1}\exp\left(\frac{\kappa_1 a}{\kappa_2(a+1)}\rho^a\right)\right).$$

II. Low density case: *There exists a threshold* $\underline{\rho}$ *for which, if* $\rho(t,x) < \underline{\rho}$ *on* $\mathbb{T}^n$*, then the temperature* θ *is bounded from above. Moreover, there are constants* $c_1, c_2 > 0$ *depending on the data such that*

$$\theta \le c_1 \quad \textit{and} \quad \rho\theta \exp\left(\frac{D}{\kappa_2(a+1)}\rho^{-a-1}\right) \ge c_2,$$

and hence

$$\rho \exp\left(\frac{D}{\kappa_2(a+1)}\rho^{-a-1}\right) \ge \frac{c_2}{c_1}. \tag{5.2}$$

It is worth noting that, in the case of low density, the estimates hold indefinitely for certain initial data. The function on the left side of (5.2) is decreasing as ρ increases from zero, so if the constants c_1 and c_2 are appropriately chosen (i.e., the initial data is appropriately chosen), then (5.2) would fail if ρ became too large. If this threshold is less than $\underline{\rho}$, then the case of very low density also becomes *self-maintaining*; the density and temperature stay bounded above, and if temperature vanishes somewhere then so must density. Unfortunately, in the high density case, nothing prevents the temperature from becoming arbitrarily large. This then allows the density to drop, which means the lower bounds no longer apply. The case of very high density is *not self-maintaining*.

Proof. We first eliminate the κ_1 constant by rescaling. If we define

$$\tilde{\rho} := \kappa_1^{\frac{1}{a}} \rho,$$

then (3.12) (where by abuse of notation we still write ρ instead of $\tilde{\rho}$) becomes

$$\begin{cases} \partial_t \rho = a\Delta(\theta\rho^{a+1}) \\ \kappa_2 \partial_t(\rho\theta) = \nabla \cdot \Big((a\kappa_2\theta + a^2\theta\rho^a)\nabla(\theta\rho^{a+1}) \Big) + \nabla \cdot (\kappa_3 \nabla\theta) \end{cases} \tag{5.3}$$

Let $f : (0, \infty) \to (0, \infty)$ be a smooth monotone function to be determined later. We look at the quantity $f(\rho)\rho\theta$. Taking a derivative and using (5.3) yields

$$\begin{aligned} \kappa_2 \partial_t(f\rho\theta) &= \kappa_2\rho\theta f' \partial_t\rho + f\kappa_2\partial_t(\rho\theta) \\ &= \kappa_2 a\rho\theta f' \Delta(\theta\rho^{a+1}) + f\nabla \cdot \Big((a\kappa_2\theta + a^2\theta\rho^a)\nabla(\theta\rho^{a+1}) + \kappa_3\nabla\theta \Big). \end{aligned} \tag{5.4}$$

Let $x_0 \in \mathbb{T}^n$ be a point where $f\rho\theta$ achieves a local minimum (respectively maximum) in space. Assume that $\theta(x_0) > 0$ and $f(\rho(x_0))\rho(x_0) > 0$. For the rest of the proof, all quantities are implicitly evaluated at x_0, though we suppress the notation. Then the gradient at this point vanishes, so that

$$(f'\rho + f)\theta\nabla\rho + f\rho\nabla\theta = 0,$$

or

$$(f'\rho + f)\nabla\rho = -f\rho\frac{\nabla\theta}{\theta}. \tag{5.5}$$

In addition, the quantity

$$\begin{aligned} L := \Delta(f\rho\theta) &= \theta\left(f'\rho + f\right)\Delta\rho + f\rho\Delta\theta + 2\left(f'\rho + f\right)\nabla\rho\cdot\nabla\theta + \theta\left(f''\rho + 2f'\right)|\nabla\rho|^2 \\ &= \frac{|\nabla\theta|^2}{\theta}\left(\frac{f''\rho + 2f'}{(f'\rho + f)^2}f^2\rho^2 - 2f\rho\right) + \theta(f'\rho + f)\Delta\rho + f\rho\Delta\theta \end{aligned}$$

is nonnegative (respectively nonpositive). The last equality used (5.5), which in general allows us to compare terms of the form $|\nabla\rho|^2$, $\nabla\rho\cdot\nabla\theta$, and $|\nabla\theta|^2$ to each other (when they are evaluated at x_0).

Expanding (5.4) then yields

$$\begin{aligned}\kappa_2\partial_t(f\rho\theta) &= \Big(\kappa_2 a f'\rho^{a+2}\theta + (\kappa_2 a + a^2\rho^a) f\rho^{a+1}\theta + \kappa_3 f\Big)\Delta\theta \\ &+ \Big(\kappa_2 a(a+1) f'\rho^{a+1}\theta^2 + (\kappa_2 a + a^2\rho^a)(a+1) f\rho^a\theta^2\Big)\Delta\rho \\ &+ \Big(\kappa_2 a^2(a+1) f'\rho^a\theta^2 + a^3(a+1) f\rho^{2a-1}\theta^2 + (\kappa_2 a + a^2\rho^a)a(a+1) f\rho^{a-1}\theta^2\Big)|\nabla\rho|^2 \\ &+ \Big(2\kappa_2 a(a+1) f'\rho^{a+1}\theta + 3(\kappa_2 a + a^2\rho^a)(a+1) f\rho^a\theta + a^3 f\rho^{2a}\theta + f\partial_\rho\kappa_3\Big)\nabla\theta\cdot\nabla\rho \\ &+ \Big((\kappa_2 a + a^2\rho^a) f\rho^{a+1} + f\partial_\theta\kappa_3\Big)|\nabla\theta|^2.\end{aligned} \tag{5.6}$$

The ultimate goal is to evaluate (5.6) at x_0, the local minimum (respectively maximum) and obtain

$$\kappa_2\partial_t(f\rho\theta) = \tilde{F}L + \tilde{G}|\nabla\theta|^2 \tag{5.7}$$

with $\tilde{F} \geq 0$ and $\tilde{G}$ nonnegative (respectively nonpositive). This would show that $f\rho\theta$ satisfies a minimum (respectively maximum) principle for such f.

At x_0, we use (5.5) and (5.1) to turn (5.6) into

$$\begin{aligned}\kappa_2\partial_t(f\rho\theta) = a\theta &\Big(\kappa_2 f'\rho^{a+2} + (\kappa_2 + a\rho^a) f\rho^{a+1} + Df\Big)\Delta\theta \\ &+ a(a+1)\rho^a\theta^2\big(\kappa_2 f'\rho + (\kappa_2 + a\rho^a) f\big)\Delta\rho \\ &+ \frac{a^2(a+1) f^2\rho^2}{(f'\rho + f)^2}\Big(\kappa_2 f'\rho^a + af\rho^{2a-1} + (\kappa_2 + a\rho^a) f\rho^{a-1}\Big)|\nabla\theta|^2 \\ &- \frac{af\rho}{f'\rho + f}\Big(2\kappa_2(a+1) f'\rho^{a+1} + 3(\kappa_2 + a\rho^a)(a+1) f\rho^a + a^2 f\rho^{2a}\Big)|\nabla\theta|^2 \\ &+ a\Big((\kappa_2 + a\rho^a) f\rho^{a+1} + Df\Big)|\nabla\theta|^2.\end{aligned} \tag{5.8}$$

In order for (5.7) to hold, there must be some $\lambda \geq 0$ (not necessarily constant) such that

$$a\theta\Big(\kappa_2 f'\rho^{a+2} + (\kappa_2 + a\rho^a) f\rho^{a+1} + Df\Big) = \lambda f\rho, \tag{5.9}$$

$$a(a+1)\rho^a\theta^2\big(\kappa_2 f'\rho + (\kappa_2 + a\rho^a) f\big) = \lambda\theta(f'\rho + f). \tag{5.10}$$

Writing $\lambda = a\tilde{\lambda}\rho^{-1}\theta$ crucially eliminates θ from both equations above. From (5.9) we obtain

$$\tilde{\lambda} = \kappa_2\rho^{a+2}\frac{f'}{f} + (\kappa_2 + a\rho^a)\rho^{a+1} + D.$$

Plugging this into (5.10) then yields

$$\kappa_2(a+1)\rho^{a+2}f' + (a+1)(\kappa_2 + a\rho^a)\rho^{a+1}f = \left(\kappa_2\rho^{a+2}\frac{f'}{f} + (\kappa_2 + a\rho^a)\rho^{a+1} + D\right)(f'\rho + f).$$

This finally simplifies to

$$(f')^2\left(\kappa_2\rho^{a+3}\right) + ff'\left((a\rho^a - \kappa_2 a + \kappa_2)\rho^{a+2} + D\rho\right) + f^2\left(D - a(\kappa_2 + a\rho^a)\rho^{a+1}\right) = 0. \tag{5.11}$$

Using the quadratic formula yields two branches of solutions. After simplification, this becomes

$$f' = f\frac{-(a\rho^a + \kappa_2(1-a))\rho^{a+1} - D \pm \sqrt{\Delta}}{2\kappa_2\rho^{a+2}} =: f\Psi_\pm(\rho), \tag{5.12}$$

where the discriminant takes the form

$$\Delta := \left(D + \rho^{a+1}(a\rho^a - \kappa_2(a+1))\right)^2 + 4\kappa_2 a(a+1)\rho^{3a+2}.$$

Note that Δ is strictly positive for all $\rho \geq 0$. Thus (5.12) implies the existence of two solutions f_+ and f_- that both transform (5.8) into (5.7). Unfortunately, these ODE's are not explicitly solvable, and do not yield simple power laws for f.

We briefly examine now the asymptotic behavior of $f_\pm$. One can rewrite (5.12) for f_+ in the following form:

$$\begin{aligned}
&\frac{f_+'}{f_+} \\
&= \frac{(D + a\rho^{2a+1} - \kappa_2(a+1)\rho^{a+1})^2 + 4\kappa_2 a(a+1)\rho^{3a+2} - (D + a\rho^{2a+1} + \kappa_2(1-a)\rho^{a+1})^2}{2\kappa_2\rho^{a+2}(D + a\rho^{2a+1} + \kappa_2(1-a)\rho^{a+1} + \sqrt{\Delta})} \\
&= \frac{2(a^2\rho^{2a+1} + \kappa_2 a\rho^{a+1} - D)}{\rho(D + a\rho^{2a+1} + \kappa_2(1-a)\rho^{a+1} + \sqrt{\Delta})}.
\end{aligned}$$

Asymptotically, we have

$$f_+' \approx -\frac{1}{\rho}f_+ \quad \text{as} \quad \rho \to 0^+ \quad \text{and} \quad f_+' \approx \frac{a}{\rho}f_+ \quad \text{as} \quad \rho \to \infty. \tag{5.13}$$

Therefore f_+ has a positive singularity at $\rho = 0$ that grows like ρ^{-1}, decreases to a minimum value at some critical ρ_1 where $a^2\rho_1^{2a+1} + \kappa_2 a\rho_1^{a+1} = D$, then becomes increasing and grows like ρ^a.

A similar calculation for f_- shows that

$$f_-' \approx -\frac{D}{\kappa_2\rho^{a+2}}f_- \quad \text{as} \quad \rho \to 0^+ \quad \text{and} \quad f_-' \approx -\frac{a^2\rho^{a-1}}{\kappa_2(a+1)}f_- \quad \text{as} \quad \rho \to \infty. \tag{5.14}$$

Thus, f_- also has a positive singularity at $\rho = 0$ that grows like $\exp\left(\frac{D}{\kappa_2(a+1)}\rho^{-(a+1)}\right)$, stays monotone decreasing, and decays exponentially to zero with profile $\exp\left(-\frac{a}{\kappa_2(a+1)}\rho^a\right)$. The difference between f_+ and f_- comes from expanding the discriminant: $a\rho^{2a+1} + D + \sqrt{\Delta}$ has simple asymptotics, but $a\rho^{2a+1} + D - \sqrt{\Delta}$ has cancellations at both limits.

We now look at the remaining terms of (5.8) in light of (5.12). Specifically, (5.8) has now become (5.7) with

$$\tilde{F} = a\theta\left(\kappa_2\rho^a\frac{f'\rho + f}{f} + a\rho^{2a} + \frac{D}{\rho}\right),$$

and

$$\begin{aligned}
\tilde{G} = &- a\left(\kappa_2\rho^a\frac{f'\rho + f}{f} + a\rho^{2a} + \frac{D}{\rho}\right)\left(\frac{f''\rho + 2f'}{(f'\rho + f)^2}f^2\rho^2 - 2f\rho\right)\\
&+ \frac{2a^3(a+1)\rho^{2a+1}}{(f'\rho + f)^2}f^3 + \frac{\kappa_2 a(a^2-1)\rho^{a+1} - a^2(4a+3)\rho^{2a+1}}{f'\rho + f}f^2\\
&+ \left(a^2\rho^{2a+1} - \kappa_2 a(2a+1)\rho^{a+1} + aD\right)f\\
= &- a\left(\kappa_2\frac{a+1}{2}\rho^{a+1} + \frac{a}{2}\rho^{2a+1} + \frac{D \pm \sqrt{\Delta}}{2}\right)\left(\frac{\Psi_\pm\rho}{\Psi_\pm\rho + 1} + \frac{(\Psi_\pm'\rho + \Psi_\pm)\rho}{(\Psi_\pm\rho + 1)^2} - 2\right)f\\
&+ \frac{8\kappa_2^2 a^3(a+1)\rho^{4a+3}}{(\kappa_2(a+1)\rho^{a+1} - a\rho^{2a+1} - D \pm \sqrt{\Delta})^2}f\\
&+ \frac{2\kappa_2^2 a(a^2-1)\rho^{2a+2} - 2\kappa_2 a^2(4a+3)\rho^{3a+2}}{\kappa_2(a+1)\rho^{a+1} - a\rho^{2a+1} - D \pm \sqrt{\Delta}}f\\
&+ \left(a^2\rho^{2a+1} - \kappa_2 a(2a+1)\rho^{a+1} + aD\right)f
\end{aligned}$$

We now let $\rho \to \infty$ (or 0^+) for $\tilde{G}$ to obtain an asymptotic formula for that term in the limit of large (or vanishing) density. Write $\tilde{G}_\pm$ to correspond to $f_\pm$. We then obtain

$$\begin{aligned}
\tilde{G}_+ &\approx a^2\frac{a+2}{a+1}\rho^{2a+1}f + \frac{8a^3}{9(a+1)}\rho^{2a+1}f - \frac{2a^2(4a+3)}{3(a+1)}\rho^{2a+1}f + a^2\rho^{2a+1}f\\
&= \frac{a^2(2a+9)}{9(a+1)}\rho^{2a+1}f > 0 \text{ as } \rho \to \infty.
\end{aligned}$$

However, the vanishing ρ limit is made more singular due to the fact that

$$\lim_{\rho\to 0^+}\Psi_+\rho + 1 \approx \frac{\kappa_2(a+1)^2}{4D}\rho^{a+1}.$$

From this we obtain

$$\begin{aligned}\tilde{G}_+ &\approx -a\left(\frac{\kappa_2(a+1)}{2}\rho^{a+1}+D\right)\left(\frac{4D}{\kappa_2(a+1)^2}\rho^{-a-1}+\frac{8D^2}{\kappa_2^2(a+1)^2}\rho^{-2a-2}-2\right)f \\ &\quad +\frac{32D^2a^3}{\kappa_2^2(a+1)^3}\rho^{-1}f+\frac{4Da(a-1)}{a+1}f+aDf \\ &\approx -\frac{8aD^3}{\kappa_2^2(a+1)^2}\rho^{-2a-2}f<0 \text{ as } \rho\to 0^+.\end{aligned}$$

Thus, for very low values of ρ, $\tilde{G}_+$ is negative. This implies that $f_+\rho\theta$ has a *minimum principle* for minima that are above a certain threshold $\overline{\rho}$, and a *maximum principle* for maxima that are below a second threshold $\underline{\rho}$. If $\tilde{G}_+$ has only one zero, then $\overline{\rho}=\underline{\rho}$. So this corresponds to a "state change" in the material between low density and high density.

The corresponding calculations for G_- yield

$$\begin{aligned}\tilde{G}_- &\approx -a\kappa_2\frac{a+1}{2}\rho^{a+1}f+2a\kappa_2^2(a+1)\rho f+a\kappa_2(4a+3)\rho^{a+1}f+a^2\rho^{2a+1}f \\ &\approx a^2\rho^{2a+1}f \text{ as } \rho\to\infty,\end{aligned}$$

and in the low density case

$$\begin{aligned}\tilde{G}_- &\approx -\kappa_2 a(a+1)\rho^{a+1}\left(\frac{-\frac{D}{\kappa_2}\rho^{-a-1}}{1-\frac{D}{\kappa_2}\rho^{-a-1}}+\frac{(a+1)\frac{D}{\kappa_2}\rho^{-a-1}}{\left(-\frac{D}{\kappa_2}\rho^{-a-1}\right)^2}-2\right)f \\ &\quad +\frac{2a^3(a+1)\rho^{2a+1}}{\left(-\frac{D}{\kappa_2}\rho^{-a-1}\right)^2}f+\frac{\kappa_2 a(a^2-1)\rho^{a+1}}{-\frac{D}{\kappa_2}\rho^{-a-1}}f+aDf\approx aDf \text{ as } \rho\to 0^+.\end{aligned}$$

This case is of a different type. Owing to various cancellations, the last term in the formula for $\tilde{G}_-$ is the dominant one for large ρ, and the small ρ limit indicates that $\tilde{G}_-$ is always positive.

*Proof of **I***: There is some threshold $\overline{\rho}$ for which, if $\rho(t,x)>\overline{\rho}$ on $\mathbb{T}^n$, then $\tilde{G}_\pm$ is strictly positive. Thus we have a minimum principle for the quantities $f_\pm\rho\theta$. Furthermore, as long as this $\overline{\rho}$ is taken large enough, we may replace $f_\pm$ by their asymptotic profiles, so that we obtain two explicit minimum principles:

$$\rho^{a+1}\theta\geq c_1 \quad\text{and}\quad \rho\exp\left(-\frac{a}{\kappa_2(a+1)}\rho^a\right)\theta\geq c_2.$$

This implies an absolute lower bound on the temperature (if θ gets too small, ρ has to increase to keep the first quantity above its minimum, but that makes the second quantity decrease). Rigorously,

$$\theta\geq\max\left(c_1\rho^{-1-a},c_2\rho^{-1}\exp\left(\frac{a}{\kappa_2(a+1)}\rho^a\right)\right).$$

The right hand side has a positive minimum, which occurs when $c_1\rho^{-a}=c_2\exp(\rho^a/\kappa_2)$, a transcendental expression.

*Proof of **II***: Similarly, there is a second threshold $\underline{\rho}$ for which, if $\rho(t,x) < \underline{\rho}$ on $\mathbb{T}^n$, then $\tilde{G}_+ < 0$, $\tilde{G}_- > 0$, and $f_+ \approx \rho^{-1}$ while $f_- \approx \exp(D\rho^{-a-1}/(\kappa_2(a+1)))$. This implies

$$\theta \le c_1 \quad \text{and} \quad \rho\theta \exp\left(\frac{D}{\kappa_2(a+1)}\rho^{-a-1}\right) \ge c_2.$$

We get an upper bound for the temperature immediately, and (5.2) follows. □

6. Future directions

Theorems 4.1 and 5.1 show how the algebraic (and nonlinear) structure of certain consistently derived models for thermal fluids reveal hidden maximum/minimum principles. These were found through the use of auxiliary variables, yet only very specific choices for those variables (the density weight function f) could produce the necessary cancellations in the equations; the proofs in Sections 4 and 5 show how to find these precise variables for two of our three models.

Already in the second model (3.12) we see far more intricate conditions, including an apparent "local state change" depending on the sign of $\tilde{G}_+$. The third model (3.17) is yet more complicated, but a deeper treatment could similarly reveal regions in state space (i.e. ranges of ρ and θ), depending on the parameters, where the solution obeys conditional a priori bounds (and likely *disjoint* regions in state space which produce *qualitatively different* a priori bounds). The approach of this work can be further extended by considering different conductivities κ_3, though it comes with the disadvantage that we lose several explicit formulas. But perhaps the most ambitious goal for this line of research is in understanding the direct connection between the auxiliary variables and the free energy function Ψ. The above approach relied on the structure found in the system of equations *after* it was derived from the free energy, but it might be possible to *anticipate* the variables directly from the free energy and constitutive laws (perhaps even classifying which such laws lead to models with nice a priori bounds).

The results above are a priori in nature; they assume a smooth solution exists on a time interval $[0, T]$ and provide quantitative bounds on that time interval. It is then natural to pare this with an existence theory; see for instance [6], [24], and [23], which focus specifically on existence for solutions to equations derived through the energy variational approach. Solutions can generally be constructed in a weak setting through iteration schemes that rely on a linearization of the model equations. However, this often restricts the solution to the perturbative (near-equilibrium) regime, and even then only constructs solutions for a short time. The true utility of the a priori bounds above is in extending the linear theory and establishing well-posedness for these models for arbitrary times. A preliminary work in this direction is [17], which constructs weak solutions exactly for the non-isothermal ideal gas model (3.6) with a generic conductivity κ_3.

Acknowledgments

N.-A. Lai was partially supported by NSF of Zhejiang Province (LY18A010008) and NSFC (11771359), C. Liu was partially supported by NSF DMS-1759535 and DMS-1950868 and the United States - Israel Binational Science Foundation (BSF) #2024246. A. Tarfulea was partially supported by NSF DMS-2012333 and DMS-2108209.

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