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On a generalized fractional differential Cauchy problem

MESFIN ETEFA[1], GASTON M. N'GUÉRÉKATA*[2], PIERRE NGNEPIEBA[3] AND OLANIYI S. IYIOLA[4]

[1] *Department of Mathematics, NEERLab Laboratory, Bowie State University, 14000 Jericho Park Rd., Bowie, MD 20715, USA.*
[2] *University Distinguished Professor, Department of Mathematics, NEERLab Laboratory, Morgan State University, 1700 E. Cold Spring Lane, Baltimore, M.D. 21251, USA.*
[3] *College of Science and Technology, Florida A& M University, Tallahassee, FL USA.*
[4] *Department of Mathematics, Clarkson University, Potsdam, NY, USA.*

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Abstract. Qualitative results for abstract problems are very important in understanding mathematical analysis on which any application is possible. The focus of this paper is twofold: first, we investigate the existence and uniqueness of mild solutions to a generalized Cauchy problem for the nonlinear differential equation with non-local conditions in a Banach space X. This is achievable using some fixed point theorems in infinite dimensional spaces. Secondly, we study the stability results of the system in the sense of Ulam-Hyers-Rassias. Our results improve and generalize most recent related results in the literature.

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Contents

1. Introduction

During the last three decades, Fractional Calculus has become a significant research topic in mathematics on account of its wide range of applications in solving real world problems. These applications are found in different fields of studies including science, finance, mathematical biology, engineering and social sciences. Nonlocality nature of fractional order derivatives has become great tool for modeling complex phenomena for which the structures have inherent non-local properties. Examples of the applications are seen specifically in anomalous transport, and anomalous diffusion, biological modeling including pattern formation and cancer

*Corresponding author. Email address: Gaston.NGuerekata@morgan.edu

treatment, financial modeling including chaos and long memory, dynamical system control theory, random walk, viscoelasticity, and nanotechnology, see [5, 6, 8–10, 19, 21, 22]. Despite many advantages of using fractional order derivatives in modeling, rigorous theoretical results on existence and uniqueness of solutions including stability are needed to drive any meaningful analysis on the subject. This is our profound concern here.

In 2018, da C. Sousa and Olivera [11] presented for the first time, the concept of κ-Hilfer fractional derivative. This has influenced several papers recently. Our work here is based on the so-called κ-Hilfer fractional operator. The new definition is a generalization of several well-known and well-studied fractional derivatives in the literature. However, several theoretical results are lacking. Our focus in this paper is to provide some important and useful results needed to advance research on generalized fractional calculus. The main motivation of this work is based on the work done by N'Guérékata in 2009 [12], in which he proved the existence and uniqueness of mild solutions to the fractional differential equation involving nonlocal conditions:

$$D^{\alpha}u(t) = f(t, u(t)), \quad t \in I := [0, T]$$

$$u(0) + g(u) = u_0.$$

Here D^{α} represents the Caputo fractional derivative of order $0 < \alpha < 1$, $f : I \times X \to E$ is a given function with some sufficient conditions to be specified below. Furthermore, X is a (complex) Banach space with norm $\|\cdot\|$, and $I := [0, T], T > 0$. The nonlocal condition is defined as

$$g(u) = \sum_{i=1}^{n} c_i u(t_i)$$

where $c_i,\ i = 1, 2, ..., n$ are some given constants and $0 < t_1 < t_2 < ... < t_n \leq T$ a partition of I. Let's recall that the concept of "nonlocal conditions" were first introduced by K. Deng in his pioneer paper [7]. In his work, he demonstrated that using the nonlocal condition $u(0) + g(u) = u_0$ to describe for example the diffusion phenomenon of a small amount of gas in a transparent tube can provide a more efficient insight than if one consider the classical Cauchy problem $u(0) = u_0$. We observe also that since Deng's paper, such problems have attracted numerous researchers: see [4, 6, 17] and many references therein.

Recently, F. Norouzi and G. M. N'Guérékata [24] studied some existence and uniqueness results of mild solutions to the κ- Hilfer semilinear neutral fractional differential equations in a general Banach space X involving an infinite delay

$$\begin{cases} [u(t) - h(t, u_t)] = B\,u(t) + g(t, u(t), u_t),\ t \in [0, T], \quad T > 0 \\ \qquad u(t) = \psi(t), \qquad\qquad t \in (-\infty, 0] \end{cases} \tag{1.1}$$

using some classical fixed point theorems. The history function $\psi(t)$ taking values in an abstract space, and the linear operator B generates a semigroup of linear operators $(S(t)_{t\geq 0}$ which are uniformly bounded on X.

The present work is motivated by the papers mentioned above. Basically we will focus on the qualitative results (existence and stability) of mild solutions to the nonlinear abstract Cauchy problem involving nonlocal condition

$$\begin{cases} {}^{\mathbb{H}}D_{0^+}^{\alpha,\beta;\kappa}u(t) = f(t, u(t)),\ t \in I \\ I_{0^+}^{1-\sigma;\kappa}u(0) = u_0 - g(u), \end{cases} \tag{1.2}$$

Here ${}^{\mathbb{H}}D^{\alpha,\beta;\kappa}$ represents the κ-Hilfer operator, $I^{1-\sigma;\kappa}$ is the left sided $\kappa-$Riemann-Liouville fractional integral operator, α and β are the order and the type of the derivative respectively, and $\sigma = \alpha + \beta\,(1 - \alpha)$. The results are new even in the case of a finite dimensional space as mentioned below.

2. Preliminaries

Useful results involving fractional operators and their definitions are presented in this Section. We denote by $\mathcal{C} := C([a,b],E)$ the usual Banach space of all continuous functions from $[a,b]$ to E a (complex) Banach space equipped with the topology of the uniform convergence induced by the norm $\|x\|_{\mathcal{C}}. = \sup_{t\in[a,b]}\|x(t)\|$ Furthermore, we assume that $\kappa \in C^1(I,\mathbb{R})$ is a monotonically increasing function and $\kappa'(x) > 0$.

Definition 2.1. *[11] Let (a,b) be an interval such that $-\infty \le a < b \le \infty$. We assume that $\kappa(x)$ is a positive and monotonically increasing function defined on $(a,b]$ where $\kappa'(x)$ is continuous on (a,b). Then the fractional integral (left-sided) of a function w defined on on $[a,b]$ with respect to a function κ is given as*

$$I^{\alpha;\kappa}_{a^+}w(t) = \frac{1}{\Gamma(\alpha)}\int_{a^+}^{t}\kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}w(s)ds.$$

Definition 2.2. *[11] Let $\alpha > 0$, $n \in \mathbb{N}$, (a,b) be an interval such that $-\infty \le a < b \le \infty$ and let $\kappa'(x) \neq 0$. The Riemann-Liouville (left-sided) fractional derivative of a function w of order α with respect to a function κ is defined by*

$$\begin{aligned} D^{\alpha;\kappa}_{a^+}w(t) &= \left(\frac{1}{\kappa'(t)}\frac{d}{dt}\right)I^{n-\alpha;\kappa}w(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\kappa'(t)}\frac{d}{dt}\right)^n\int_a^t \kappa'(s)(\kappa(t)-\kappa(s))^{n-\alpha-1}w(s)ds. \end{aligned}$$

Definition 2.3. *[11] Let $n-1 < \alpha < n$ with $n+1,2,...,$ $I=[a,b]$ is the interval such that $-\infty \le a < b \le \infty)$, $w,\kappa \in C^n(I,\mathbb{R})$ two functions such that κ is increasing and $\kappa'(t) \neq 0$, for all $t \in I$. The left sided and right sided $\kappa-$Hilfer fractional derivative $D^{\alpha,\beta;\kappa}_{a^+}(\cdot)$ of function of order α and type $0 \le \beta \le 1$ are defined respectively by*

$$^{\mathbb{H}}D^{\alpha,\beta;\kappa}_{a^+}w(t) = I^{\beta(n-\alpha);\kappa}_{a^+}\left(\frac{1}{\kappa'(t)}\frac{d}{dt}\right)^n I^{(1-\beta)(n-\alpha);\kappa}_{a^+}w(t),$$

and

$$^{\mathbb{H}}D^{\alpha,\beta;\kappa}_{b^-}w(t) = I^{\beta(n-\alpha);\kappa}_{b^-}\left(-\frac{1}{\kappa'(t)}\frac{d}{dt}\right)^n I^{(1-\beta)(n-\alpha);\kappa}_{b^-}w(t).$$

Theorem 2.4. *[11] If $0 \le \beta \le 1$, $n-1 < \alpha < 1$ such that $\gamma = \alpha + \beta(n-\alpha)$ and $w \in C^n(I)$, then*

$$I^{\alpha;\kappa}_{a^+}\,{}^{\mathbb{H}}D^{\alpha,\beta;\kappa}_{a^+}w(t) = w(t) - \sum_{k=1}^{n}\frac{(\kappa(x)-\kappa(a))^{\gamma-k}}{\Gamma(\gamma-k+1)}w^{[n-k]}_{\kappa}I^{(1-\beta)(n-\alpha);\kappa}_{a^+}w(a),$$

and

$$I^{\alpha;\kappa}_{b^-}{}^{\mathbb{H}}D^{\alpha,\beta;\kappa}_{b^-}w(t) = w(t) - \sum_{k=1}^{n}\frac{(-1)^k(\kappa(b)-\kappa(x))^{\gamma-k}}{\Gamma(\gamma-k+1)}w^{[n-k]}_{\kappa}I^{(1-\beta)(n-\alpha);\kappa}_{b^-}w(a).$$

Theorem 2.5. *[11] Let $w \in C^1[a,b]$, $\alpha > 0$, and $0 \le \beta \le 1$, then*

$$^{\mathbb{H}}D^{\alpha,\beta;\kappa}_{a^+}I^{\alpha;\kappa}_{a^+}w(t) = w(t) \qquad \text{and} \qquad {}^{\mathbb{H}}D^{\alpha,\beta;\kappa}_{b^-}I^{\alpha;\kappa}_{b^-}w(t) = w(t).$$

3. Main Results

Let's now discuss our main results. From now on I will be the finite interval $[0,T]$. Firstly, we indicate the following assumptions which are needed in the proofs of results.

(H1): $f : \mathbb{R}\times I \to E$ is a Caratheodory function, meaning for every $u \in E$, $f(t,u)$ is strongly measurable with

respect to first variable and for every $t \in I$, $f(t,u)$ is continuous with respect to the second variable
(H2): $\|f(t,u)-f(t,v)\| \leq L_f\|u-v\|, \forall t \in \mathbb{R}, \forall u,v \in E, \gamma > 0.$
(H3): $g : \mathcal{C} \to E$ is continuous and

$$\|g(u)-g(v)\| \leq b\|u-v\|, \forall u,v \in \mathcal{C}, b > 0.$$

(H2'): There exists a function $\mu \in L^1(I)$ such that

$$\|f(t,u)-f(t,v)\| \leq \mu(t)\|u-v\|, \forall t \in I, \text{for all} u,v \in E.$$

For $0<\alpha<1$ and $0 \leq \beta \leq 1$, the nonlinear system (1.1) is then equivalent to the integral equation

$$u(t) = \frac{(\kappa(t)-\kappa(0))^{\sigma-1}}{\Gamma(\sigma)}(u_0 - g(u)) + \frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1} f(s,u(s))ds, \tag{3.1}$$

for each $t \in I$.

Proof. We do not recall the proof since it is straightforward from [11, 15]. ■

Definition 3.1. *A continuous function $x : I \to E$ is said to be a mild solution of Equations (1.1) if it can be written as*

$$u(t) = \frac{(\kappa(t)-\kappa(0))^{\sigma-1}}{\Gamma(\sigma)}(u_0 - g(u)) + \frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1} f(s,x(s))ds, \quad t \in I.$$

We now present our results.

Theorem 3.2. *Assume that assumptions **(H1-H3)** hold with*

$$b < \frac{1}{2}, \quad \frac{(\kappa(t)-\kappa(0))^{\sigma-1}}{\Gamma(\sigma)} \leq 1, \forall t \in I, \quad \textit{and} \quad \gamma < \frac{\Gamma(\alpha+1)}{2\kappa(T)^\alpha},$$

then there exists a unique mild solution to (1.1).

Proof. Consider the operator $\Omega : \mathcal{C} \to \mathcal{C}$ defined by

$$(\Omega u)(t) := \frac{(\kappa(t)-\kappa(0))^{\sigma-1}}{\Gamma(\sigma)}(u_0 - g(u)) + \frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1} f(s,u(s))ds.$$

It is obvious that this operator Ω is well-defined. From assumptions **(H1-H3)** we can define

$$M := \sup_{t\in I}||f(t,0)|| \quad \text{and} \quad P := \sup_{x\in\mathcal{C}}||g(u)||$$

and choose

$$r \geq \left(P + \|u_0\| + \frac{M\kappa(T)^\alpha}{\Gamma(\alpha+1)}\right).$$

For $B_r := \{x \in \mathcal{C} : ||u_0|| \leq r\}$, we can show that $FB_r \subset B_r$. Then if

$$\Delta_\kappa^\sigma(t,0) = \frac{(\kappa(t)-\kappa(0))^{\sigma-1}}{\Gamma(\sigma)},$$

we obtain the following estimate for $u \in B_r$

$$\|\Omega u(t)\| \leq \|\Delta_\kappa^\sigma(t,0)u_0\| + \|\Delta_\kappa^\sigma(t,0)g(u)\| + \frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}\|f(s,u(s))\|ds$$

$$\begin{aligned}
&\leq \|u_0\| + P + \frac{1}{\Gamma(\alpha)} \int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}(\|f(s,u(s)) - f(s,0)\| + \|f(s,0)\|)ds \\
&\leq \|u_0\| + P + \frac{Lr+M}{\Gamma(\alpha)} \int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1} ds \\
&= \|u_0\| + P + \left(\frac{Lr+M}{\Gamma(\alpha)}\right) \frac{(\kappa(t)-\kappa(0))^{\alpha}}{\alpha} \\
&\leq \|u_0\| + P + \left(\frac{Lr+M}{\Gamma(\alpha+1)}\right) \kappa(T)^{\alpha} \\
&\leq r,
\end{aligned}$$

with suitable choices of L_f and r.
Let $u, v \in \mathcal{C}$. Furthermore we have

$$\begin{aligned}
\|(\Omega u)(t) - (\Omega v)(t)\| &\leq \|g(u) - g(v)\| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1} \|f(s,u(s)) - f(s,v(s)\| ds \\
&\leq b\|u-v\|_{\mathcal{C}} + \frac{1}{\Gamma(\alpha)} L \|u-v\|_{\mathcal{C}} \frac{(\kappa(t)-\kappa(0))^{\alpha}}{\alpha} \\
&\leq b\|u-v\|_{\mathcal{C}} + \frac{L_f \kappa(T)^{\alpha}}{\Gamma(\alpha+1)} \|u-v\|_{\mathcal{C}} \\
&= \left(b + \frac{L_f \kappa(T)^{\alpha}}{\Gamma(\alpha+1)}\right) \|u-v\|_{\mathcal{C}} \\
&\leq \Phi \|u-v\|_{\mathcal{C}},
\end{aligned}$$

where

$$\Phi = \Phi_{b,L_f,T,\alpha} := \left(b + \frac{L_f \kappa(T)^{\alpha}}{\Gamma(\alpha+1)}\right)$$

Since $0 < \Phi < 1$ and $\|\Omega u - \Omega v\|_{\mathcal{C}} \leq \Phi \|u-v\|_{\mathcal{C}}$, Ω turns out to be a contractive mapping. We deduce that Ω has a unique fixed point, which is the mild solution of the Cauchy problem. The problem is solved. ■

Remark 3.1. *This result generalizes Theorem 2.1 in [12] in the case where $E = \mathbb{R}^n$ and $\kappa(t) = t$.*

Let's now consider the local problem associated to Equation (1.1), that is g is identically zero on J.

Theorem 3.3. *Let's suppose that ((**H1**)-(**H2**)) hold and*

$$\kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1} \leq 1.$$

Then there exists a unique mild solution to Equation (1.1).

Proof. Consider the operator Ω as in Theorem 3.2. Then we have

$$\begin{aligned}
\|(\Omega u)(t) - (\Omega v)(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1} \|f(s,u(s)) - f(s,v(s))\| ds \\
&\leq \frac{\|u-v\|_{\mathcal{C}}}{\Gamma(\alpha)} \int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1} \mu(s) ds.
\end{aligned}$$

Also we have

$$\|(\Omega u)^2(t) - (\Omega v)^2(t)\| \leq \frac{1}{\Gamma(\alpha)} \int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1} \|f(s,Fu(s)) - f(s,Fv(s))\| ds$$

$$\leq \frac{\|u-v\|_{\mathcal{C}}}{2\Gamma(\alpha)^2}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}\mu(s)\int_0^s \mu(\sigma)d\sigma ds$$
$$\leq \frac{\|u-v\|_{\mathcal{C}}}{2\Gamma(\alpha)^2}\|\mu\|^2_{L^1(I)}.$$

We obtain inductively

$$\|(\Omega u)^n(t)-(\Omega v)^n(t)\| \leq \frac{\|u-v\|_{\mathcal{C}}}{n!(\Gamma(\alpha)^n}\|\mu\|^n_{L^1(I)}.$$

If n is large enough then we can get

$$\frac{\|\mu\|^n_{L^1(I)}}{n!\Gamma(\alpha)^n} < 1$$

By a generalization of the Banach fixed point theorem, Equation (3.2) has a unique mild solution. ■

Remark 3.2. *This result is new even in the case of finite dimensional space for the fractional derivatives in the sense of Caputo or Riemann-Liouville..*

Theorem 3.4. *(Krasnoselskii). Let X be a closed convex and nonempty subset of a Banach space E. Let A_1, A_2 be two mappings such that*

(a) $A_1u + A_2v \in X$ for every $u, v \in E$;

(b) A_1 is a compact and continuous;

(c) A_2 is a contraction.

We further include the following assumption for the result that follows.

(H4): $\|f(t,u)\| \leq \mu(t), \forall (t,u) \in I \times E, \mu \in L^1(I,\mathbb{R}^+)$.

Theorem 3.5. *Suppose assumptions **(H1)**, **(H3)**, and **(H4)**. Let $b < 1$. Then Equation (1.1) has at least one mild solution on I*

Proof. Let's take r such that

$$r \geq \|u_0\| + P + \frac{\kappa(T)^\alpha \|\mu\|_{L^1}}{\Gamma(\alpha+1)}.$$

Then we define on B_r the operators A_1, A_2 by

$$(A_1u)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1} f(s,u(s))ds \tag{3.2}$$

and

$$(A_2u)(t) = \frac{(\kappa(t)-\kappa(0))^{\sigma-1}}{\Gamma(\sigma)}(u_0 - g(u)). \tag{3.3}$$

Let's prove that if $u, v \in B_r$ implies $A_1u + A_2v \in B_r$.

Indeed we have

$$\|A_1u + A_2v\| \leq \|\Delta^\sigma_\kappa(t,0)(u_0 - g(u))\| + \frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}\|f(s,u(s))\|ds$$
$$\leq \|u_0\| + P + \frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}\mu(s)ds$$

$$
\begin{aligned}
&\leq \|u_0\| + P + \frac{\|\mu\|_{L^1}}{\Gamma(\alpha)} \int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1} ds \\
&= \|u_0\| + P + \frac{\|\mu\|_{L^1}}{\Gamma(\alpha)} \left(\frac{(\kappa(t)-\kappa(0))^{\alpha}}{\alpha} \right) \\
&\leq \|u_0\| + P + \frac{\kappa(T)^{\alpha}\|\mu\|_{L^1}}{\Gamma(\alpha+1)} \\
&\leq r.
\end{aligned}
$$

In view of **(H3)**, B is a contraction mapping since $b < 1$. Continuity of u implies that $(A_1 u)(t)$ is continuous based on of **(H1)**.

Let's observe that A_1 is uniformly bounded on B_r. This is because of the inequality

$$
\|(A_1 u)(t)\| \leq \frac{\kappa(T)^{\alpha}\|\mu\|_{L^1}}{\Gamma(\alpha+1)}.
$$

In addition, we proceed to show that $(A_1 u)(t)$ is equicontinuous.

Let $t_1 \in I, \ , t_2 \in I$ and $u \in B_r$. Since u is bounded on the compact set $I \times B_r$, then $\sup_{(t,u)\in J\times B_r} \|f(t,u)\| := K < \infty$, we get

$$
\begin{aligned}
\|A_1u(t_1) - A_1u(t_2)\| &= \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_1} \kappa'(s)(\kappa(t_1)-\kappa(s))^{\alpha-1} + f(s,u(s))ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \kappa'(s)(\kappa(t_2)-\kappa(s))^{\alpha-1} f(s,u(s))ds \right\| \\
&= \frac{1}{\Gamma(\alpha)} \left\| \int_{t_2}^{t_1} \kappa'(s)(\kappa(t_1)-\kappa(s))^{\alpha-1} f(s,u(s))ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \kappa'(s)(\kappa(t_2)-\kappa(s))^{\alpha-1} f(s,u(s))ds \right\| \\
&\leq \frac{1}{\Gamma(\alpha)} \left\| \int_{t_2}^{t_1} \kappa'(s)(\kappa(t_1)-\kappa(s))^{\alpha-1} f(s,u(s))ds \right\| \\
&\quad + \frac{1}{\Gamma(\alpha)} \left(\left\| \int_0^{t_2} [\kappa'(s)(\kappa(t_2)-\kappa(s))^{\alpha-1} - \kappa'(s)(\kappa(t_1)-\kappa(s))^{\alpha-1}] f(s,u(s))ds \right\| \right) \\
&\leq \frac{K}{\Gamma(\alpha)} \left\| \frac{(\kappa(t_1)-\kappa(t_2))^{\alpha}}{\alpha} - \frac{(\kappa(t_2)-\kappa(0))^{\alpha}}{\alpha} + \frac{(\kappa(t_1)-\kappa(0))^{\alpha}}{\alpha} \right\| \\
&\leq \frac{K}{\Gamma(\alpha+1)} \|(\kappa(t_1)-\kappa(t_2))^{\alpha} - (\kappa(t_2)-\kappa(0))^{\alpha} + (\kappa(t_1)-\kappa(0))^{\alpha}\|
\end{aligned}
$$

which is independent of x and $|A_1u(t_1) - A_1u(t_2)| \to 0$ as $t_2 \to t_1$. Therefore, $A_1u(t)$ is equi-continuous. Using Arzela-Ascoli's theorem, $A_1(B_r)$ is a relatively compact subset of $\mathcal{C}$, which implies that operator A_1 is compact. We conclude the proof using Krasnoselskii's theorem. ■

4. Extension Results

For our next results, we make the following assumptions.
(H5): For every $u \in E$, there exists $c_f > 0$, such that

$$
\|f(t,u)\| \leq c_f(1 + \|u\|)
$$

and for every $u \in C(I,E)$, there exists a $c_g \in (0,1)$ such that

$$\|g(u)\| \leq c_g(1+\|u\|_{\mathcal{C}}),$$

(H6): For every $t \in I$, the set

$$\Omega = \{\kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1} f(s,u(s)):\ u \in C(J,E), 0 \leq s \leq t\}$$

is relatively compact.

Theorem 4.1. *[15] Let u, v be two integrable functions and g continuous, with domain $[a,b]$. Let $\kappa \in C^1(I)$ be an increasing function such that $\kappa'(t) \neq 0, \forall t \in I$. Assume that*
(1) u, v are nonnegative;
(2) g is nonnegative and nondecreasing.
If

$$u(t) \leq v(t) + g(t)\int_a^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1} u(\tau)d\tau,$$

then

$$u(t) \leq v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[g(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \kappa'(\tau)(\kappa(t)-\kappa(\tau))^{\alpha k-1} v(\tau)d\tau.$$

Corollary 4.1. *[15] Let v be a nondecreasing function on $[a,b]$. Under the hypothesis of the above theorem, we have*

$$u(t) \leq v(t)E_\alpha(g(t)\Gamma(\alpha)[\kappa(t)-\kappa(\tau)]^\alpha),\ \forall t \in [a,b].$$

Lemma 4.2. *For our next result, we first prove that there exists $\xi > 0$ such that $\|u(t)\| \leq \xi,\ \ \forall t \in I.$*

Proof. Using the assumptions of Theorem 3.2 and hypothesis **(H5)**, one derives the estimate

$$\begin{aligned}
\|u(t)\| &\leq \|u_0 - g(u)\| + \frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}\|f(s,x(s))\|ds, \quad t \in I. \\
&\leq \|u_0 - g(u)\| + \frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1} c_f(1+\|u(s)\|)ds \\
&\leq \|u_0\| + c_g + c_g\|u\|_{\mathcal{C}} + \frac{c_f}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}ds \\
&\quad + \frac{c_f}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}\|u(s)\|ds.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
(1-c_g)\|u\|_{\mathcal{C}} \leq\ & \|u_0\| + c_g + \frac{c_f}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}ds \\
&+ \frac{c_f}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}\|u(s)\|ds.
\end{aligned}$$

Therefore,

$$\|u(t)\| \leq \frac{\Gamma(\alpha+1)(\|u_0\|+c_g)+c_f\kappa(T)^\alpha}{(1-c_g)\Gamma(\alpha+1)} + \frac{c_f}{(1-c_g)\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}\|u(s)\|ds.$$

Using Theorem 4.2 and Corollary 4.3, we get

$$\|u(t)\| \leq \frac{\Gamma(\alpha+1)(\|u_0\|+c_g)+c_f\kappa(T)^\alpha}{(1-c_g)\Gamma(\alpha+1)} \sum_{n=0}^{\infty} \frac{\left(\frac{c_f\Gamma(\alpha)\kappa(T)^\alpha}{(1-c_g)\Gamma(\alpha)}\right)^n}{\Gamma(n\alpha+1)}$$

$$\leq \frac{\Gamma(\alpha+1)(\|u_0\|+c_g)+c_f\kappa(T)^\alpha}{(1-c_g)\Gamma(\alpha+1)}\sum_{n=0}^{\infty}\frac{(c_f\kappa(T)^\alpha)^n}{(1-c_g)^n\Gamma(n\alpha+1)}$$

Since $\sum_{n=0}^{\infty}\frac{(c_f\kappa(T)^\alpha)^n}{(1-c_g)^n\Gamma(n\alpha+1)}$ is the Mittag-Leffler function, it suffices that
$\|u(t)\| < \xi$ ■

Theorem 4.2. *Suppose **(H1)**, **(H5)**, **(H6)** hold and let*

$$c_g + \frac{c_f\kappa(T)^\alpha}{\Gamma(\alpha+1)} < 1.$$

Then the system has at least one mild solution.

Proof. The proof is given in the following steps:
Let (3) and (4) be defined on $\mathcal{C}_\xi := \{u \in C(I,E) : \|u(t)\| \leq \xi, t \in I\}$.
Step 1: We first show that $Au + Bv \in C_\xi$ for every $u, v \in C_\xi$. Using Equations (3)-(4), we have the following estimate

$$\begin{aligned}
\|(Au)(t)+(Bv)(t)\| &\leq \|u_0\| + \|g(v)\| + \frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}\|f(s,u(s))\|ds \\
&\leq \|u_0\| + c_g(1+\|v\|_{\mathcal{C}}) + \frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}c_f(1+\|u(s)\|)ds \\
&\leq \|u_0\| + c_g(1+\xi) + \frac{c_f(1+\xi)}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}ds \\
&\leq \|u_0\| + c_g(1+\xi) + \frac{c_f(1+\xi)\kappa(T)^\alpha}{\Gamma(\alpha+1)} \\
&\leq \xi.
\end{aligned}$$

which implies that $Au + Bv \in C_\xi$.
Step 2: B is a contractive operator on C_ξ.
Indeed if we take any $v_1, v_2 \in C_\xi$, then we have

$$\|Bv_1 - Bv_2\|_{\mathcal{C}} = \|g(v_1) - g(v_2)\| \leq L_g\|v_1 - v_2\|_{\mathcal{C}}.$$

Therefore B is a contraction mapping.
Step 3: A is a continuous mapping.
Using **(H1)**, if $\{u_n\}$ is a sequence in C_ξ so that $u_n \to u$ in C_ξ, then,

$$f(s,u_n(s)) \to f(s,u(s)) \quad \text{as} \quad n \to \infty.$$

For $t \in I$,we have

$$\|(Au_n)(t)-(Au)(t)\| \leq \frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}\|f(s,u_n(s)),f(s,u(s))\|ds$$

Using **(H7)**, for $t \in I$,

$$\|f(s,u_n(s)) - f(s,u(s))\| \leq L_f(\xi)\|u_n(s)-u(s)\| \leq 2\xi L_f(\xi).$$

Also, using the fact that $s \to 2\xi L_f(\xi)\kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}$ is integrable, Lebesgue's Dominated Convergence Theorem gives that

$$\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}\|f(s,u_n(s)),f(s,u(s))\|ds \to 0.$$

As $n \to \infty$, $Au_n \to Au$. Thus, A is continuous.
Step 4: A is a compact mapping.
If $\{u_n\}$ is a sequence on C_ξ, then

$$\begin{aligned}\|(Au_n)(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}\|f(s,u_n(s))\|ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}c_f(1+\|u_n(s)\|)ds \\ &\leq \frac{(1+\xi)\kappa(T)^\alpha c_f}{\Gamma(\alpha+1)}.\end{aligned}$$

Therefore, $\{u_n\}$ is uniformly bounded.
Next, we prove the equicontinuity of $\{Au_n\}$. Let $0 \leq t_1 \leq t_2 \leq T$. Then

$$\begin{aligned}&\|(Au_n)(t_1)-(Au_n)(t_2)\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \kappa'(s)(\kappa(t_1)-\kappa(s))^{\alpha-1} - \kappa'(s)(\kappa(t_2)-\kappa(s))^{\alpha-1}\|f(s,u_n(s))\|ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \kappa'(s)(\kappa(t_2)-\kappa(s))^{\alpha-1}\|f(s,u_n(s))\|ds \\ &\leq \frac{c_f}{\Gamma(\alpha)} \int_0^{t_1} \kappa'(s)(\kappa(t_1)-\kappa(s))^{\alpha-1} - \kappa'(s)(\kappa(t_2)-\kappa(s))^{\alpha-1}(1+\|u_n(s)\|)ds \\ &\quad + \frac{c_f}{\Gamma(\alpha)} \int_{t_1}^{t_2} \kappa'(s)(\kappa(t_2)-\kappa(s))^{\alpha-1}(1+\|u_n(s)\|)ds \\ &\leq \frac{c_f(1+\xi)}{\Gamma(\alpha+1)}\left((\kappa(t_1)-\kappa(t_0))^\alpha - (\kappa(t_2)-\kappa(0))^\alpha + 2(\kappa(t_2)-\kappa(t_1))^\alpha\right) \\ &\leq \frac{c_f(1+\xi)}{\Gamma(\alpha+1)}(\kappa(t_2)-\kappa(t_1))^\alpha.\end{aligned}$$

As $t_2 \to t_1$, $(\kappa(t_2)-\kappa(t_1))^\alpha \to 0$, and thus $\{Au_n\}$ is equicontinuous. In view of (H6) and theorem 3.4, $\overline{conv}K$ is a compact set. For every $t^* \in I$,

$$\begin{aligned}(Au_n)(t^*) &= \frac{1}{\Gamma(\alpha)} \int_0^t \kappa'(s)(\kappa(t^*)-\kappa(s))^{\alpha-1} f(s,u_n(s))ds \\ &= \frac{1}{\Gamma(\alpha)} \lim_{t\to\infty} \frac{t^*}{k}\left[\kappa'\left(\frac{it^*}{k}\right)\left(\kappa(t^*)-\kappa\left(\frac{it^*}{k}\right)\right)^{\alpha-1} f\left(\frac{it^*}{k}, u_n\left(\frac{it^*}{k}\right)\right)\right] \\ &= \frac{t^*}{k}\eta_n,\end{aligned}$$

where

$$\eta_n = \lim_{t\to\infty} \frac{1}{k}\left[\kappa'\left(\frac{it^*}{k}\right)\left(\kappa(t^*)-\kappa\left(\frac{it^*}{k}\right)\right)^{\alpha-1} f\left(\frac{it^*}{k}, u_n\left(\frac{it^*}{k}\right)\right)\right].$$

Since $\overline{conv}K$ is convex and compact, $\eta_n \in \overline{conv}K$. Hence for every $t^* \in I$, the set $\{Au_n\}$ is relatively compact. From Ascoli-Arzela theorem, every $\{Au_n(t)\}$ contains a uniformly convergent subsequence $\{Au_{n_k}(t)\}(k = 1,2,...)$ on I. Thus, the set $\{Au : u \in C_\xi\}$ is relatively compact. Therefore A is a completely continuous mapping. Using Krasnoselskii's Theorem, we conclude that $A+B$ possesses a fixed point on C_ξ which is the mild solution to the system (1.1). The proof is complete. ■

5. Stability Result

In this last section, we establish the stability of (1.1) with regard to Ulam-Hyers stability. We first define the following mapping $\Lambda : C(I,E) \to C(I,E)$ as follows:

$$\Lambda v(t) =^{\mathbb{H}} D_{0^+}^{\alpha,\beta;\kappa} v(t) - f(t, v(t)), t \in I.$$

Let $\epsilon > 0$ be given and $v(t) \in C(I,E)$ satisfy

$$v(t) = \frac{(\kappa(t)-\kappa(0))^{\sigma-1}}{\Gamma(\sigma)}(v_0 - g(v)) + \frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1} f(s, v(s))ds, \quad t \in J.$$

Definition 5.1. *[14] Problem (1) is said to be Ulam-Hyers stable (or stable in the sense of Ulam-Hyers) if for every $\epsilon > 0$*

$$\|\Lambda y\| \leq \epsilon,$$

and for every mild solution y of (1.1),there exists $\rho > 0$ and a mild solution $u \in C(I,E)$ to (1.1) such that

$$\|u(t) - v(t)\| \leq \rho\epsilon^*,$$

where $\epsilon^* > 0$ and depends on ϵ.

Definition 5.2. *[13] Let $m \in C(R^+, R^+)$ so that for every mild solution y of (1.1), there exists a mild solution $x \in C(I,E)$ of problem (1.1) such that*

$$\|u(t) - v(t)\| \leq m\epsilon^*, t \in I.$$

Definition 5.3. *[14] Problem (1.1) is called Ulam-Hyers-Rassias stable with respect to $\Theta \in C(I, R^+)$ if for $\epsilon > 0$,*

$$\|\Lambda v(t)\| \leq \epsilon\Theta(t), \ t \in I.$$

and there exists $\rho > 0$ and $v \in C(I,E)$ such that

$$\|u(t) - v(t)\| \leq \rho\epsilon_*\Theta(t), t \in I,$$

and $\epsilon_* > 0$ depends on ϵ

Theorem 5.4. *Assume $\|f(t,u(t))\| \leq p(t)q(\|u\|)$ where $p \in C(I, R_+)$ and $q : R_+ \to R_+$, and $\Phi < 1$. Then problem (1.1) is both Ulam-Hyers and generalized Ulam-Hyers stable.*

Proof. Let $x \in C(I,E)$ be a solution of (1.1),and let y be any solution satisfying Definition 5.1. We determine that the operators Λ and $F - Id$ (identity operator) are equivalent for all solutions $v \in C(I,E)$ of (1.1) satisfying $\Phi < 1$. Thus , using the fixed point property of operator F, we conclude that

$$\begin{aligned}
\|v(t) - u(t)\| &= \|v(t) - Fv(t) + Fv(t) - u(t)\| \\
&= \|v(t) - Fv(t) + Fv(t) - Fu(t)\| \\
&\leq \|Fv(t) - Fu(t)\| + \|Fv(t) - v(t)\| \\
&\leq \Phi\|u - v\| + \epsilon,
\end{aligned}$$

because $\Phi < 1$ and and $\epsilon > 0$,

$$\|u - v\| \leq \frac{\epsilon}{1-\Phi}.$$

Fixing $\epsilon_* = \dfrac{\epsilon}{1-\Phi}$, and $\rho = 1$ we get the Ulam-Hyers stability. Using $m\epsilon = \dfrac{\epsilon}{1-\Phi}$, we get the generalized Ulam-Hyers stability. ■

Corollary 5.1. *If definition 5.3 is satisfied for* $\Theta \in C(I, R^+)$*, and*

$$L < \frac{\Gamma(\alpha+1)(1-b)}{\kappa(T)^\alpha},$$

problem (1) is Ulam-Hyers-Rassias with respect to Θ.

Proof. Directly follows from the proof of Theorem 5.4 where

$$\|u(t) - v(t)\| \leq \epsilon_* \Theta(t), \; t \in I,$$

and

$$\epsilon_* = \frac{\epsilon}{1-\Phi}.$$

The proof is complete. ■

6. Conclusion

A generalized Cauchy problem is the central focus of this paper formulated using the generalized fractional operator called κ-Hilfer operator. The existence and uniqueness results of the abstract model are studied with nonlocal conditions using classical fixed point theorems. Furthermore, a stability result for the abstract problem is obtained in the sense of Ulam-Hyers-Rassians. The results obtained in this paper are very useful in many applications where κ-Hilfer operator is being adopted. They generalize many recent results in this field. The arguments used here can be adapted to many other problems in infinite dimensional spaces.

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