Brief Announcement: Distributed Lightweight Spanner Construction for Unit Ball Graphs in Doubling Metrics

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ABSTRACT

Resolving an open question from 2006 [4], we prove the existence of light-weight bounded-degree $(1+\epsilon)$-spanners for unit ball graphs in the metrics of bounded doubling dimension, and we design a simple $O(\log^* n)$-round distributed algorithm in the LOCAL model for finding such spanners using only 2-hop neighborhood information. We further study the problem in the two dimensional Euclidean plane and we propose a construction with similar properties that has a low-intersection property as well. Lastly, we provide experimental results that confirm the performance of our algorithms.

CCS CONCEPTS

• **Theory of computation** → **Distributed algorithms**.

KEYWORDS

Unit ball graphs, doubling metrics, lightness, topology control

ACM Reference Format:
David Eppstein and Hadi Khodabandeh. 2022. Brief Announcement: Distributed Lightweight Spanner Construction for Unit Ball Graphs in Doubling Metrics. In *Proceedings of the 34th ACM Symposium on Parallelism in Algorithms and Architectures (SPAA '22), July 11–14, 2022, Philadelphia, PA, USA.* ACM, New York, NY, USA, 3 pages. https://doi.org/10.1145/3490148.3538553

1 INTRODUCTION

Given a collection of points V in a metric space with doubling dimension d, the weighted *unit ball graph (UBG)* on V is defined as a weighted graph $G(V, E)$ where two points $u, v \in V$ are connected if and only if their metric distance $\|uv\| \leq 1$. The weight of the edge uv of the UBG is $\|uv\|$ if the edge exists. Unit ball graphs in the Euclidean plane are called unit disk graphs (UDGs) and are frequently used to model ad-hoc wireless communication networks, where every node in the network has an effective communication range R, and two nodes are able to communicate if they are within a distance R of each other.

The necessity of a connected and energy-efficient topology for high-level routing protocols led researchers to develop many spanning algorithms for ad-hoc networks and in particular, UDGs. But the decentralized nature of ad-hoc networks demands that these algorithms be local instead of centralized. In these applications, it is important that the resulting topology is connected, has a low weight, and has a bounded degree, implying also that the number of edges is linear in the number of vertices.

Spanner construction has been the topic of many studies in both centralized and distributed settings [1, 3, 5, 6, 9, 11]. The best known result for the case of unit ball graphs in the distributed setting belongs to Damian, Pandit, and Pemmaraju [4], where they design a distributed algorithm for UBGs lying in d*-dimensional Euclidean space*. Their algorithm runs in $O(\log^* n)$ rounds of communication, where n is the number of vertices, and produces a $(1 + \epsilon)$-spanner of the UBG, that has constant bounds on its maximum degree and lightness, with the lightness being defined as the weight of the spanner divided by the weight of the minimum spanning tree on the same point set. They used the so-called *leapfrog* property to prove the constant lightness bound on the weight of the spanner, which does not hold for doubling spaces in general. Instead, they showed that the lightness of their spanner in doubling spaces would be bounded by $O(\log \Delta)$, where Δ is the ratio of the length of the longest edge to the length of the shortest edge in the UBG.

Apart from being a generalization of the Euclidean space, the importance of the spaces of bounded doubling dimension comes from the fact that a small perturbation in the pairwise distances does not affect the doubling dimension of the point set by much, while it can change their Euclidean dimension significantly, or the resulting distances might not even be embeddable in Euclidean metrics at all [2].

Since the work of [4] in 2006, it has remained open whether UBGs in doubling spaces possess lightweight bounded-degree $(1 + \epsilon)$-spanners and whether they can be found efficiently in a distributed model of computation. In this paper, we resolve this long-standing open question and we present centralized and distributed algorithms for finding such spanners.

We resolve this question by proving the existence of light-weight bounded-degree $(1 + \epsilon)$-spanners of unit ball graphs in the spaces of bounded doubling dimension. We provide a centralized as well as a distributed construction for building such spanners, both of which have constant bounds on the maximum degree and the lightness. Furthermore, our distributed spanner can be built locally in $O(\log^* n)$ rounds of communication, where n is the number of vertices. In the full version of the paper [7], we modify this construction for the two dimensional Euclidean plane in order to have a linear number of edge intersections in total, implying a constant average number of edge intersection per node. We also provide experimental results on random point sets in the two dimensional Euclidean plane that confirm the efficiency of our distributed construction.

SPAA '22, July 11–14, 2022, Philadelphia, PA, USA

ACM ISBN 978-1-4503-9146-7/22/07.
https://doi.org/10.1145/3490148.3538553

2 CENTRALIZED CONSTRUCTION

In this section we propose our centralized construction for a light-weight bounded-degree $(1+\epsilon)$-spanner for unit ball graphs in a metric of bounded doubling dimension. Later in section 3 we use this centralized construction to design a distributed algorithm that delivers the same features.

It is worth mentioning that the greedy spanner would be a $(1+\epsilon)$-spanner of the UBG if the algorithm stops after visiting the pairs of distance at most 1, and it even has a weight of $O(1)\omega(MST)$, but as we mentioned earlier, there are metrics with doubling dimension 1 in which its degree may be unbounded.

To construct a lightweight bounded-degree $(1+\epsilon)$-spanner of the unit ball graph, we start with the spanner of [10], called Approximate-Greedy, which is returns a spanner of the complete graph. It is proven in [12] that Approximate-Greedy has the desired properties, i.e. bounded-degree and lightness, for complete weighted graphs in Euclidean metrics, but as stated in [8], the proof only relies on the triangle inequality and packing argument which both work for doubling metrics as well. Therefore, we may safely assume that Approximate-Greedy finds a light-weight bounded-degree $(1+\epsilon)$-spanner of the complete weighted graph defined on the point set. The main issue is that the edges of length more than 1 are not allowed in a spanner of the unit ball graph on the same point set. Therefore, a replacement procedure is needed to substitute these edge with edges of length at most 1. Peleg and Roditty [13] introduced a refinement process which removes the edges of length larger than 1 from the spanner and replaces them with three smaller edges to make the output a subgraph of the UBG. The main issue with their approach is that it can lead to vertices having unbounded degrees in the spanner, therefore missing an important feature. Here, we introduce our own refinement process that not only replaces edges of larger than 1 with smaller edges and makes the spanner a subgraph of the unit ball graph, but also guarantees a constant bounded on the degrees of the resulting spanner.

Algorithm 1 A centralized spanner construction.

Input. A unit ball graph $G(V,E)$ in a metric with doubling dimension d.
Output. A light-weight bounded-degree $(1+\epsilon)$-spanner of G.

procedure Centralized-Spanner(G, ϵ)
  $\epsilon' \leftarrow \epsilon/36$
  $S \leftarrow$ Approximate-Greedy(V, ϵ')
  R $\leftarrow \varnothing$
  for $e = (u, v)$ in S **do**
    if $|e| > 1$ **then**
      Remove e from S
    if $|e| \in (1, 1+\epsilon']$ **then**
      if $\exists(x,y) \in E$ that $\|ux\| \le \epsilon'$ and $\|vy\| \le \epsilon'$ **then**
        if $\nexists(x',y') \in R$ that $\|ux'\| \le 2\epsilon'$ and $\|vy'\| \le 2\epsilon'$ **then**
          $S \leftarrow S \cup \{(x,y)\}$
          $R \leftarrow R \cup \{(x,y)\}$
  return S

We prove that the output S of Algorithm 1 is a light-weight bounded-degree $(1+\epsilon)$-spanner of the unit ball graph G.

Theorem 2.1 (Centralized Spanner). *Given a weighted unit ball graph G in a metric of bounded doubling dimension and a constant $\epsilon > 0$, the spanner returned by Centralized-Spanner(G,ϵ) is a $(1+\epsilon)$-spanner of G and has constant bounds on its lightness and maximum degree. These constant bounds only depend on ϵ and the doubling dimension.*

3 DISTRIBUTED CONSTRUCTION

In this section we propose our distributed construction for finding a $(1+\epsilon)$-spanner of a unit ball graph using only 2-hop neighborhood information. The spanner returned by our algorithm has constant bounds on its maximum degree and its lightness. This is the first light-weight distributed construction for unit ball graphs in doubling metrics, to the best of our knowledge.

We propose Algorithm 2 which has a preprocessing step of finding a maximal independent set I of G. This can be done using the distributed algorithm of [14] in $O(\log^* n)$ rounds. We refer to this algorithm by Maximal-Independent. Then the Local-Greedy subroutine is run on every vertex $w \in I$ to find a $(1+\epsilon)$-spanner S_w of the 2-hop neighborhood of w, denoted by $\mathcal{N}^2(w)$. At the final step, every $w \in I$ sends its local spanner edges to the corresponding endpoints of every edge. Symmetrically, every vertex listens for the edges sent by the vertices in I and once a message is received, it stores the edges in its local storage. In other words, the final spanner is the union of all these local spanners. We use the centralized algorithm of section 2 for every local neighborhood $\mathcal{N}^2(w)$ to guarantee the bounds that we need.

Algorithm 2 The localized greedy algorithm.

Input. A unit ball graph $G(V,E)$ in a metric with doubling dimension d and an $\epsilon > 0$.
Output. A light-weight bounded-degree $(1+\epsilon)$-spanner of G.

procedure Distributed-Spanner(G, ϵ)
  Find a maximal independent set I of G using [14]
  Run Local-Greedy on the vertices of G
function Local-Greedy(vertex w)
  Retrieve $\mathcal{N}^2(w)$, the 2-hop neighborhood information of w
  if w is in I **then**
    $\mathcal{S}_w \leftarrow$ Centralized-Spanner($\mathcal{N}^2(w)$, ϵ)
    for $e = (u,v)$ in $\mathcal{S}_w$ **do**
      Send e to u and v
  Listen to incoming edges and store them

We show that the spanner returned by Algorithm 2 possesses the desired properties. The round complexity follows from the round complexity of [14], and the stretch-factor and the degree bounds follow from the corresponding bounds on the centralized spanner, with some adjustments that are explained in the full version of the paper [7]. The weight bound however requires more attention.

In order to bound the lightness of the output, we assume that $\epsilon \le 1$ and we make a few comparisons. First, for any $w \in I$ we compare the weight of $\mathcal{S}_w$ to the weight of the minimum spanning tree on $\mathcal{N}^2(w)$. Then we compare the weight of the minimum spanning tree on $\mathcal{N}^2(w)$ to the weight of the minimum Steiner tree on $\mathcal{N}^3(w)$, where the required vertices are $\mathcal{N}^2(w)$ and 3-hop vertices are optional. Finally, we compare the weight of this minimum

Steiner tree to the weight of the induced subgraph of CENTRALIZED-SPANNER(G, ϵ) on the subset of vertices $\mathcal{N}^3(w)$, which later implies that the overall weight of $\mathcal{S}_w$s is bounded by a constant factor of the weight of the minimum spanning tree on G.

COROLLARY 3.1. $\omega(\mathcal{S}_w) = O(1)\omega(MST(\mathcal{N}^2(w)))$

Next we compare $\omega(MST(\mathcal{N}^2(w)))$ to the weight of the minimum Steiner tree of $\mathcal{N}^3(w)$ on the required vertices $\mathcal{N}^2(w)$.

LEMMA 3.2. *Define $\mathcal{T}$ to be the optimal Steiner tree on the set of vertices $\mathcal{N}^3(w)$, where only vertices in $\mathcal{N}^2(w)$ are required and the rest of them are optional. Then*

$$\omega(MST(\mathcal{N}^2(w))) \leq 2\omega(\mathcal{T})$$

We then compare the weight of $\mathcal{T}$ to the weight of induced subgraph of CENTRALIZED-SPANNER(G, ϵ) on the subset of vertices $\mathcal{N}^3(w)$. The main observation here is that when $\epsilon \leq 1$ the induced subgraph of the centralized spanner on $\mathcal{N}^3(w)$ would be a feasible solution to the minimum Steiner tree problem on $\mathcal{N}^3(w)$, with the required vertices being the vertices in $\mathcal{N}^2(w)$. This will imply that the weight of the induced subgraph is at least equal to the weight of the minimum Steiner tree.

LEMMA 3.3. *Let $\mathcal{S}^*$ be the output of CENTRALIZED-SPANNER(G, ϵ) and let $\mathcal{S}^*_w$ be the induced subgraph of $\mathcal{S}^*$ on $\mathcal{N}^3(w)$. Then*

$$\omega(\mathcal{T}) \leq \omega(\mathcal{S}^*_w)$$

PROOF. We prove that for $\epsilon \leq 1$, $\mathcal{S}^*_w$ forms a forest that connects all the vertices in $\mathcal{N}^2(w)$ in a single component. So $\mathcal{S}^*_w$ is a feasible solution to the minimum Steiner tree problem on the set of vertices $\mathcal{N}^3(w)$ with required vertices being $\mathcal{N}^2(w)$. Thus $\omega(\mathcal{T}) \leq \omega(\mathcal{S}^*_w)$.

Now we just need to prove that the vertices in $\mathcal{N}^2(w)$ are connected in $\mathcal{S}^*_w$. Let u be an i-hop neighbor of w and v be an $i+1$-hop neighbor of w for some $w \in I$ and $i = 0, 1$. Assume that $(u, v) \in E$. It is enough to prove that u and v are connected in $\mathcal{S}^*_w$. In order to do so, we observe that there is a path of length at most $(1+\epsilon)\|uv\|$ between u and v in $\mathcal{S}^*$. We show that this path is contained in $\mathcal{N}^3(w)$ and we complete the proof in this way, because $\omega(\mathcal{S}^*_w)$ is nothing but the induced subgraph of $\mathcal{S}^*$ on $\mathcal{N}^3(w)$.

Assume, on the contrary, that there is a vertex $x \notin \mathcal{N}^3(w)$ on the $(1+\epsilon)$-path between u and v. This means that x is not a 1-hop neighbor of any of u and v, because otherwise x would have been in $\mathcal{N}^3(w)$. So $\|ux\| > 1$ and $\|vx\| > 1$. Thus the length of the path would be at least $\|ux\| + \|xv\| > 2 \geq (1+\epsilon) \geq (1+\epsilon)\|uv\|$ which is a contradiction. □

PROPOSITION 3.4. *The spanner returned by DISTRIBUTED-SPANNER has a weight of $O(1)\omega(MST)$.*

PROOF. By Corollary 3.1, Lemma 3.2, and Lemma 3.3,

$$\omega(\mathcal{S}_w) = O(1)\omega(\mathcal{S}^*_w)$$

Summing up together these inequalities for $w \in I$,

$$\omega(\text{output}) = O(1) \sum_{w \in I} \omega(\mathcal{S}^*_w)$$

But we recall that every vertex, and hence every edge of $\mathcal{S}^*$, is repeated $O(1)$ times in the summation above, so

$$\omega(\text{output}) = O(1)\omega(\mathcal{S}^*) = O(1)\omega(MST(G))$$

□

Therefore we have all the ingredients to prove Theorem 3.5.

THEOREM 3.5 (DISTRIBUTED SPANNER). *Given a weighted unit ball graph G with n vertices in a metric of bounded doubling dimension and a constant $\epsilon > 0$, the algorithm DISTRIBUTED-SPANNER(G,ϵ) runs in $O(\log^* n)$ rounds of communication and returns a $(1+\epsilon)$-spanner of G that has constant bounds on its lightness and maximum degree. These constant bounds only depend on ϵ and the doubling dimension.*

4 CONCLUSION

In this paper we resolve an open question from 2006 and we prove the existence of light-weight bounded-degree $(1+\epsilon)$-spanners for unit ball graphs in the spaces of bounded doubling dimension. Moreover, we provide a centralized construction and a distributed construction that finds a spanner with these properties in $O(\log^* n)$ rounds of communication. In the full version of the paper [7], we adjust these algorithms for the case of unit disk graphs in the two dimensional Euclidean plane, and we present the first centralized and distributed constructions for a light-weight bounded-degree $(1+\epsilon)$-spanner that also has a linear number of edge intersections in total. We also propose our experimental results on random point sets in the two dimensional Euclidean plane, to ensure that our theoretical bounds are also supported by enough empirical evidence. Our results show that our construction performs efficiently with respect to the maximum degree, size, and total weight.

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