REVISTA DE LA UNIÓN MATEMÁTICA ARGENTINA
Vol. 64, No. 2, 2022, Pages 271–280
Published online: November 28, 2022
https://doi.org/10.33044/revuma.3065

INEQUIVALENT REPRESENTATIONS OF THE DUAL SPACE

TEPPER L. GILL, DOUGLAS MUPASIRI, AND ERDAL GÜL

ABSTRACT. We show that there exist inequivalent representations of the dual space of $\mathbb{C}[0,1]$ and of $L_p[\mathbb{R}^n]$ for $p \in [1,\infty)$. We also show how these inequivalent representations reveal new and important results for both the operator and the geometric structure of these spaces. For example, if $\mathcal{A}$ is a proper closed subspace of $\mathbb{C}[0,1]$, there always exists a closed subspace $\mathcal{A}^{\perp}$ (with the same definition as for $L_2[0,1]$) such that $\mathcal{A} \cap \mathcal{A}^{\perp} = \{0\}$ and $\mathcal{A} \oplus \mathcal{A}^{\perp} = \mathbb{C}[0,1]$. Thus, the geometry of $\mathbb{C}[0,1]$ can be viewed from a completely new perspective. At the operator level, we prove that every bounded linear operator A on $\mathbb{C}[0,1]$ has a uniquely defined adjoint A^* defined on $\mathbb{C}[0,1]$, and both can be extended to bounded linear operators on $L_2[0,1]$. This leads to a polar decomposition and a spectral theorem for operators on the space. The same results also apply to $L_p[\mathbb{R}^n]$. Another unexpected result is a proof of the Baire one approximation property (every closed densely defined linear operator on $\mathbb{C}[0,1]$ is the limit of a sequence of bounded linear operators). A fundamental implication of this paper is that the use of inequivalent representations of the dual space is a powerful new tool for functional analysis.

INTRODUCTION

The dual concept for a function space began with concrete examples in the early 1900s. Riesz and others introduced the idea of linear functionals without a structure for them. Riesz proved his representation theorem for $L_2[0,1]$ in 1910 and by 1918 had already proven special cases of the Hahn–Banach theorem for $L_p[0,1]$, ℓ_p and $BV[0,1]$ (see [15, 16, 17, 18]). Following Riesz, Helly defined a (general) normed sequence space and its dual (see [8, 9, 10, 11]). Hahn was the first to define a general (real) normed space in 1922, and in 1923 Banach and Wiener gave their (independent) definitions (see [7, 1, 20]). It is unfortunate that all these ideas were developed before any consideration of the finite-dimensional case. Unlike the times of Riesz, we now know that every finite-dimensional (real) Banach space V

2020 *Mathematics Subject Classification.* Primary 46B20; Secondary 46B10, 46C99.
Key words and phrases. Duality, adjoint operator, approximation property, Baire one approximation.
The work of T. L. Gill and D. Mupasiri was supported by the National Science Foundation under grant DMS-1440140, National Security Agency under grant H98230-20-1-0015, and the Sloan Foundation under grant G-2020-12602 while the authors participated in a program hosted by the Mathematical Sciences Research Institute in Berkeley, California, during the summer of 2020.

of dimension n is isomorphic to the Hilbert space $\mathbb{R}^n$ and all norms are equivalent. Thus, it is doubtful that operator theory or the geometry of Banach spaces would have developed as they did if the finite-dimensional case had been studied first.

Perspective. If the dimension of a Banach space $\mathcal{B}$ is infinite, all spaces are no longer isomorphic, so the natural question is: "what properties are preserved between the finite and infinite dimensional cases?" At that time, this question was not considered, and methods of study for Hilbert and non-Hilbert Banach spaces began to diverge from the beginning. For example, on a Hilbert space $\mathcal{H}$, the definition of orthogonality is unique, but on a Banach space $\mathcal{B}$ there are at least four different definitions (see [3]). Following Dunford and Schwartz [3], two closed subspaces U and V are said to be *complementary* if $U \cap V = \{0\}$ and $U \oplus V = \mathcal{B}$. In $\mathcal{H}$ the concepts of orthogonality and complementarity of closed subspaces are very close, but in $\mathcal{B}$ they are not. For example, in $\mathcal{H}$, if U is a closed subspace, $U^{\perp} = \{x \in \mathcal{H} : (x, y) = 0 \, \forall y \in U\}$ and $U \oplus U^{\perp} = \mathcal{H}$. For $\mathcal{B}$, $U^{\perp} = \{x^* \in \mathcal{B}^* : x^*(y) = 0 \, \forall y \in U\}$ and $U^{\perp}$ is not a subspace of $\mathcal{B}$. Thus, we already see that the complemented subspace problem is fundamentally different for $\mathcal{H}$ and $\mathcal{B}$. We see the same kind of difference in the definition of the adjoint for a linear operator.

In this paper, we want to reinvestigate this question relative to the spaces $\mathcal{B} = \mathbb{C}[0,1]$ and $\mathcal{B} = L_p[\mathbb{R}^n]$, for $1 \leq p < \infty$.

0.1. **Preliminaries.** The following is due to Lax (see [13, Theorem I]).

Theorem 0.1 (Lax's Theorem). *Let $\mathcal{B}$ be a separable Banach space that is continuously and densely embedded in a Hilbert space $\mathcal{H}$, and let T be a bounded linear operator on $\mathcal{B}$ that is symmetric with respect to the inner product of $\mathcal{H}$ (i.e., $(Tu, v)_{\mathcal{H}} = (u, Tv)_{\mathcal{H}}$ for all $u, v \in \mathcal{B}$). Then T is bounded with respect to the $\mathcal{H}$ norm, and*

$$\|T^*T\|_{\mathcal{H}} = \|T\|_{\mathcal{H}}^2 \leqslant k \, \|T\|_{\mathcal{B}}^2$$

for some positive constant k.

1. The case $\mathcal{B} = \mathbb{C}[0,1]$

In this section, we investigate the relationship between $\mathbb{C}[0,1]$ and $L_2[0,1]$, where $\mathbb{C}[0,1]$ is the Banach space of continuous functions on $[0,1]$ and $L_2[0,1]$ is the Hilbert space of functions f such that $\int_0^1 |f(x)|^2 \, dx < \infty$, with the Lebesgue measure on $[0,1]$ and inner product $(f,g)_2 = \int_0^1 f(x)\bar{g}(x) \, dx$. In this case, $\mathbb{C}[0,1] \subset L_2[0,1]$ as a continuous dense embedding (i.e., $\|f\|_2 \leqslant \|f\|_{\mathbb{C}}$, $f \in \mathbb{C}[0,1]$). It is well known that every bounded linear functional on $\mathbb{C}[0,1]$ has a representation of the form

$$\langle f, \alpha \rangle_{\mathbb{C}^*} = \int_0^1 f(x) \, d\alpha(x), \quad \text{where } \alpha(x) \in \mathbb{C}^*[0,1] = NBV[0,1],$$

the functions of normalized bounded variation on $[0,1]$ (i.e., $\alpha(0) = 0$). However, every bounded linear functional on $L_2[0,1]$, when restricted to $\mathbb{C}[0,1]$, is a bounded linear functional on $\mathbb{C}[0,1]$. Thus, for each $u \in \mathbb{C}[0,1]$, there is a function $u^* = \alpha_u \in NBV[0,1]$ and a constant $c_u > 0$, depending on u, such that

$$\langle f, u^* \rangle_{C^*} = \int_0^1 f(x)\, d\alpha_u(x) = c_u \int_0^1 f(x)\bar{u}(x)\, dx = c_u (f,u)_2.$$

If we define

$$\mathbb{C}_2^*[0,1] = \{u^* \in \mathbb{C}^*[0,1] : u^* = c_u(\,\cdot\,,u)_2,\ u \in \mathbb{C}[0,1]\},$$

then $\mathbb{C}_2^*$ is a new representational subspace of $NBV[0,1]$. Furthermore,

$$\|u\|_2^2 = c_u^{-1} \int_0^1 u(x)\, d\alpha_u(x) = \int_0^1 |u(x)|^2\, dx \quad \text{and}$$

$$\|u\|_{\mathbb{C}}^2 = \int_0^1 u(x)\, d\alpha_u(x) = c_u(u,u)_2 = \langle u\,, u^* \rangle_{\mathbb{C}^*} = \|u^*\|_{\mathbb{C}^*}^2,$$

so that $(\,\cdot\,, c_u u)_2 = u^*$ is a duality mapping for u. From the two equations above, we see that $c_u = \|u\|_{\mathbb{C}}^2 / \|u\|_2^2$. This leads to the following result:

Theorem 1.1. *The space* $\mathbb{C}_2^*[0,1] \subset L_2^*[0,1]$ *is a conjugate isometric isomorphic copy of* $\mathbb{C}[0,1]$.

Definition 1.2. We call $\mathbb{C}_2^*[0,1]$ the *Zachary representation* of $\mathbb{C}[0,1]$ in $NBV[0,1]$, and let $u_z^* = (\,\cdot\,, c_u u)_2$ for each $u \in \mathbb{C}[0,1]$.

1.1. **The adjoint.** The adjoint for an operator on a Banach space is not the same as on a Hilbert space. Our inner product representation offers a new perspective on the whole question of an adjoint. Let $\mathbf{J}_2 : L_2[0,1] \to L_2^*[0,1]$ be the standard conjugate isomorphism, and let $\mathbf{J}_{\mathbb{C}}$ be the restriction of $\mathbf{J}_2$ to $\mathbb{C}[0,1]$ so that $\mathbf{J}_{\mathbb{C}} : \mathbb{C}[0,1] \to L_2^*[0,1]$.

Define $\mathbb{C}_h^* = \{u_h = (\,\cdot\,,u)_2 : u \in \mathbb{C}[0,1]\}$ so that $\mathbf{J}_{\mathbb{C}}(u) = u_h$. Let $\mathcal{C}[\mathbb{C}[0,1]]$ be the set of closed densely defined linear operators on $\mathbb{C}[0,1]$.

Theorem 1.3. *If* $A \in \mathcal{C}[\mathbb{C}[0,1]]$, *then there is a unique operator* $A^* \in \mathcal{C}[\mathbb{C}[0,1]]$ *that satisfies the following:*

(1) $(aA)^* = \bar{a}A^*$,
(2) $A^{**} = A$,
(3) $(A^* + B^*) = A^* + B^*$,
(4) $(AB)^* = B^*A^*$ *on* $D(A^*) \cap D(B^*)$, *and*
(5) *if* A *is bounded* ($A \in \mathcal{L}[\mathbb{C}[0,1]]$), *then* $\|A^*A\|_{\mathcal{B}} \leq M\, \|A\|_{\mathcal{B}}^2$ *(for some constant* M*) and* A *has a bounded extension to* $L_2[0,1]$.

Proof. If $A \in \mathcal{C}[\mathbb{C}[0,1]]$, then the dual operator $A' : NBV[0,1] \to NBV[0,1]$. As a mapping on $NBV[0,1]$, A' is closed and weak* densely defined. However, since $\mathbb{C}[0,1]$ is dense in $L_2[0,1]$, $\mathbb{C}_h^*$ is strongly dense in $L_2^*[0,1]$. It follows that $A'\mathbf{J}_{\mathbb{C}}$, mapping $\mathbb{C}_h^* \subset L_2^*[0,1] \to L_2^*[0,1]$, is a closed (strongly) densely defined linear operator. Thus, $\mathbf{J}_{\mathbb{C}}^{-1} A' \mathbf{J}_{\mathbb{C}} : \mathbb{C}[0,1] \to \mathbb{C}[0,1]$ is a closed and densely defined linear operator. We define $A^* = [\mathbf{J}_{\mathbb{C}}^{-1} A' \mathbf{J}_{\mathbb{C}}] \in \mathcal{C}[\mathbb{C}[0,1]]$. If A is bounded, A^* is defined on all of $\mathbb{C}[0,1]$. According to the closed graph theorem, A^* is bounded. The proofs of (1)–(3) are straightforward. To prove (4), let $u \in D(A^*) \cap D(B^*)$; then,

$$\begin{aligned}(BA)^* u &= [\mathbf{J}_{\mathbb{C}}^{-1}(BA)'\mathbf{J}_{\mathbb{C}}]u = [\mathbf{J}_{\mathbb{C}}^{-1}A'B'\mathbf{J}_{\mathbb{C}}]u \\ &= \left[\mathbf{J}_{\mathbb{C}}^{-1}A'\mathbf{J}_{\mathbb{C}}\right]\left[\mathbf{J}_{\mathbb{C}}^{-1}B'\mathbf{J}_{\mathbb{C}}\right]u = A^*B^*u.\end{aligned} \tag{1.1}$$

If we replace B by A^* in equation (1.1), noting that $A^{**} = A$, we also find that $(A^*A)^* = A^*A$.

The proof of the first part of (5) follows from

$$\|A^*A\|_{\mathbb{C}} \leqslant \|A^*\|_{\mathbb{C}}\|A\|_{\mathbb{C}} \leqslant \|\mathbf{J}_{\mathbb{C}}\|_{\mathbb{C}^*}\left\|\mathbf{J}_{\mathbb{C}}^{-1}\right\|_{\mathbb{C}}\|A'\|_{\mathbb{C}^*}\|A\|_{\mathbb{C}} = M\|A\|_{\mathbb{C}}^2$$

for some constant M. A proof of the second part is a special case of Theorem 0.1. From (4), $S = A^*A$ is self-adjoint; thus, from Theorem 0.1, S has a bounded extension to $L_2[0,1]$ and

$$\|A\|_2^2 = \|A^*A\|_2 \leqslant k\|A^*A\|_{\mathbb{C}} \leqslant kM\|A\|_{\mathbb{C}}^2.$$

Therefore, A has a bounded extension $\bar{A}$ to $L_2[0,1]$ so that $\mathcal{L}[\mathbb{C}[0,1]]$ is continuously embedded into $\mathcal{L}[L_2[0,1]]$, the bounded linear operators on $L_2[0,1]$. □

The last result also shows that $\mathcal{L}[\mathbb{C}[0,1]]$ is a *algebra.

Theorem 1.4 (Polar Representation)**.** *If $A \in \mathcal{C}[\mathbb{C}[0,1]]$, then there exists a partial isometry U and a self-adjoint operator T, $T = T^*$, with $D(T) = D(A)$ and $A = UT$.*

Proof. Let $\bar{A}$ be the (closed densely defined) extension of A to $L_2[0,1]$. On $L_2[0,1]$, $\bar{T}^2 = \bar{A}^*\bar{A}$ is self-adjoint, and there exists a unique partial isometry $\bar{U}$, with $\bar{A} = \bar{U}\bar{T}$. Thus, the restriction to $\mathbb{C}[0,1]$ provides us $A = UT$, and U is a partial isometry on $\mathbb{C}[0,1]$. (It is easy to check that $A^*A = T^2$.) □

Theorem 1.5 (Spectral Representation)**.** *Let $A \in \mathcal{C}[\mathbb{C}[0,1]]$ be a self-adjoint linear operator. There exists an operator-valued spectral measure E_x defined for each $x \in \mathbb{R}$, and for each $u \in D(A)$,*

$$Au = \int_{-\infty}^{\infty} x\, dE_x(u).$$

The next result easily follows from examination of the previous proofs.

Theorem 1.6. *Let $\mathcal{B}$ be any Banach space that is a continuous dense embedding in $L_2[0,1]$; then, all the results of this section hold for $\mathcal{B}$.*

Since $L_p[0,1] \subset L_2[0,1]$, $p > 2$, as a continuous dense embedding, we conclude that Theorems 1.1, 1.3, and 1.4 hold for all $L_p[0,1]$, $p > 2$.

If $u \in L_p[0,1]$, $2 < p < \infty$, then the standard duality mapping is

$$u^* = \|u\|_p^{2-p}|u(x)|^{p-2}u(x) \in L_q[0,1], \quad \tfrac{1}{p} + \tfrac{1}{q} = 1. \tag{1.2}$$

Furthermore,

$$\langle u, u^* \rangle = \|u\|_p^{2-p}\int_0^1 |u(x)|^p\, d\lambda_n(x) = \|u\|_p^2 = \|u^*\|_q^2.$$

Applying our earlier observation to $L_p[0,1]$, $p > 2$, we find that, for each $u \in L_p[0,1]$, there is an $\alpha_u \in NBV[0,1]$, a constant $c_{pu} > 0$, and a unique $u^* \in L_q[0,1]$, $\frac{1}{p} + \frac{1}{q} = 1$, such that (see (1.2))

$$\langle f, u^* \rangle_p = \|u\|_p^{2-p} \int_0^1 f(x)|u(x)|^{p-2} u(x)\, dx = c_{pu}^{-1} \int_0^1 f(x)\, d\alpha_u(x)$$

for all $f \in \mathbb{C}[0,1]$. It follows that there are an infinite number of possible inequivalent representations for the linear functionals on $\mathbb{C}[0,1]$.

1.2. **Complemented subspaces.**

Definition 1.7. We say that two subspaces of $\mathbb{C}[0,1]$, U and V, are *orthogonal* ($U \perp V$ in $\mathcal{B}$) if $u_z^*(v) = 0$ for all $u \in U$ and $v \in V$.

Lemma 1.8. *$U \perp V$ in $\mathbb{C}[0,1]$ if and only if $V \perp U$ in $\mathbb{C}[0,1]$.*

We now have the following:

Theorem 1.9. *Let $\mathbb{A}$ be a proper closed subspace of $\mathbb{C}[0,1]$ and let $\mathcal{H}_1$ be the closure of $L_2[0,1] \cap \mathbb{A}$ in the $L_2[0,1]$ topology, so that $L_2[0,1] = \mathcal{H}_1^\perp \oplus \mathcal{H}_1$. If $\mathbb{A}^\perp = \mathcal{H}_1^\perp \cap \mathbb{C}[0,1]$, then $\mathbb{C}[0,1] = \mathbb{A} \oplus \mathbb{A}^\perp$.*

Proof. It is clear that $\mathcal{H}_1^\perp \cap \mathcal{H}_1 = \{0\}$ implies that $\mathbb{A} \cap \mathbb{A}^\perp = \{0\}$. Furthermore, as $L_2[0,1] = \mathcal{H}_1^\perp \oplus \mathcal{H}_1$, we see that $\mathbb{C}[0,1] = \mathbb{A} \oplus \mathbb{A}^\perp$, provided that $\mathbb{A}^\perp$ is closed in $\mathbb{C}[0,1]$. To prove this, suppose that x is a limit point of $\mathbb{A}^\perp$ in the sup norm, which is not in $\mathbb{A}^\perp$. Since the embedding is continuous, x is a limit point in the L_2 norm. It follows that $x \in \mathcal{H}_1^\perp$ so it must be in $\mathbb{A}^\perp$, which is a contradiction, so $\mathbb{A}^\perp$ is closed and $\mathbb{C}[0,1] = \mathbb{A} \oplus \mathbb{A}^\perp$. Thus, $\mathbb{A}$ is complemented in $\mathbb{C}[0,1]$. □

Remark 1.10. It is well known that there exist closed subspaces of separable Banach spaces, which are not complemented. However, the meaning of "complemented" is different for the two cases. The change for Banach spaces was based on the implicit assumption that it was impossible to have (essentially) the same definition as for a Hilbert space.

2. The space $L_p[\mathbb{R}^n]$

In this section, we want to consider the space $L_p[\mathbb{R}^n]$ for $1 \le p < \infty$, with $n \in \mathbb{N}$. It is well known that these spaces have a common dense core, but it is not well known that they are densely (and continuously) contained in a separable Hilbert space. We begin this section with the construction of the Kuelbs–Steadman space, $KS^2[\mathbb{R}^n]$. This space contains the class of non-absolutely integrable functions, all the $L_p[\mathbb{R}^n]$ and the class of test functions $\mathcal{D}(\mathbb{R}^n)$ as continuous dense embeddings. The space was first used to provide a rigorous foundation for the Feynman path integral formulation of quantum mechanics (see [5]). We provide proofs of important results to make the section self-contained.

Let $\mathbb{Q}^n$ be the set $\{\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n\}$ such that x_l is rational for each l. Since this is a countable dense set in $\mathbb{R}^n$, we can arrange it as $\mathbb{Q}^n = \{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \dots\}$. For each i and j, let $\mathbf{B}_i(\mathbf{x}^j)$ be the closed cube centered at $\mathbf{x}^j$, with sides parallel to the coordinate axes and edge $e_i = \frac{1}{2^{i-1}\sqrt{n}}$, $i \in \mathbb{N}$. Now choose the natural order that maps $\mathbb{N} \times \mathbb{N}$ bijectively to $\mathbb{N}$:

$$\{(1,1),\ (2,1),\ (1,2),\ (1,3),\ (2,2),\ (3,1),\ (3,2),\ (2,3),\ \ldots\}.$$

Let $\{\mathbf{B}_k,\ k \in \mathbb{N}\}$ be the resulting set of (all) closed cubes $\{\mathbf{B}_i(\mathbf{x}^j) : (i,j) \in \mathbb{N} \times \mathbb{N}\}$ centered at a point in $\mathbb{Q}^n$, and let $\mathcal{E}_k(\mathbf{x})$ be the characteristic function of $\mathbf{B}_k$ so that $\mathcal{E}_k(\mathbf{x})$ is in $L^p[\mathbb{R}^n]$ for $1 \le p \le \infty$. Define $F_k(\,\cdot\,)$ on $L^p[\mathbb{R}^n]$ for $1 \le p \le \infty$, by

$$F_k(f) = \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\, d\lambda_n(\mathbf{x}).$$

It is clear that $F_k(\,\cdot\,)$ is a bounded linear functional on $L^p[\mathbb{R}^n]$ for each k, $\|F_k\| \le 1$, and, if $F_k(f) = 0$ for all k, then $f = 0$, so that $\{F_k\}$ is fundamental on $L^p[\mathbb{R}^n]$ for $1 \le p \le \infty$. Set $t_k = 2^{-k}$ so that $\sum_{k=1}^{\infty} t_k = 1$ and define an inner product $(\,\cdot\,,\,\cdot\,)$ on $L^1[\mathbb{R}^n]$ by

$$(f,g) = \sum_{k=1}^{\infty} t_k F_k(f) \bar{F}_k(g).$$

We call the completion of $L^1[\mathbb{R}^n]$, with the above inner product, the *Kuelbs–Steadman space*, $KS^2[\mathbb{R}^n]$. To see that this space contains non-absolutely integrable functions, suppose that f is non-absolutely integrable, say a Henstock–Kurzweil integral ([4]); then,

$$\begin{aligned}\|f\|_{KS^2}^2 &= \sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\, d\lambda_n(\mathbf{x}) \right|^2 \\ &\leqslant \sup_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\, d\lambda_n(\mathbf{x}) \right|^2 < \infty.\end{aligned}$$

Theorem 2.1. *The space $KS^2[\mathbb{R}^n]$ contains $L^p[\mathbb{R}^n]$ (for each p, $1 \leqslant p \leqslant \infty$) as a continuous dense compact embedding.*

Proof. By construction, $KS^2[\mathbb{R}^n]$ contains $L^1[\mathbb{R}^n]$ densely, so we need only show that $KS^2[\mathbb{R}^n] \supset L^q[\mathbb{R}^n]$ for $q \neq 1$. If $f \in L^q[\mathbb{R}^n]$ and $q < \infty$, we have

$$\begin{aligned}\|f\|_{KS^2} &= \left(\sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\, d\lambda_n(\mathbf{x}) \right|^{\frac{2q}{q}} \right)^{1/2} \\ &\leqslant \left(\sum_{k=1}^{\infty} t_k \left(\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) \left| f(\mathbf{x}) \right|^q \, d\lambda_n(\mathbf{x}) \right)^{\frac{2}{q}} \right)^{1/2} \\ &\leqslant \sup_k \left(\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) \left| f(\mathbf{x}) \right|^q \, d\lambda_n(\mathbf{x}) \right)^{\frac{1}{q}} \leqslant \|f\|_q \,.\end{aligned}$$

Hence, $f \in KS^2[\mathbb{R}^n]$. For $q = \infty$, first note that $\mathrm{vol}(\mathbf{B}_k)^2 \leq \left(\frac{1}{2\sqrt{n}}\right)^{2n}$, so we have

$$\|f\|_{KS^2} = \left(\sum_{k=1}^{\infty} t_k \left|\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\, d\lambda_n(\mathbf{x})\right|^2\right)^{1/2}$$
$$\leqslant \left(\left(\sum_{k=1}^{\infty} t_k (\mathrm{vol}(\mathbf{B}_k))^2\right) (\operatorname{ess\,sup} |f|)^2\right)^{1/2} \leqslant \left(\frac{1}{2\sqrt{n}}\right)^n \|f\|_\infty .$$

Thus $f \in KS^2[\mathbb{R}^n]$, and $L^\infty[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$.

To see that the embedding is compact, suppose that $f_n \to f$ weakly. Since $\mathcal{E}_k \in L_p[\mathbb{R}^n]$ for $p \in [1, \infty]$, we see that $F_k(f_n) \to F_k(f)$ for all $k \in \mathbb{N}$. It follows that $\|f_n - f\|_{KS_p} \to 0$ for all p, so that the embedding is compact. □

We can also define and construct the class of $KS^p[\mathbb{R}^n]$ spaces for $p \neq 2,\ 1 \leq p \leq \infty$. First, for $p \neq 2$, set

$$\|f\|_{KS^p} = \begin{cases} \left(\sum_{k=1}^\infty t_k \left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\, d\lambda_n(\mathbf{x})\right|^p\right)^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \sup_{k \geqslant 1} \left|\int_{\mathbb{R}^n_I} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\, d\lambda_n(\mathbf{x})\right| & \text{if } p = \infty. \end{cases}$$

It is easy to see that $\|\cdot\|_{KS^p}$ defines a norm on L^p. If KS^p is the completion of L^p with respect to this norm, we have:

Theorem 2.2. *For each q, $1 \leqslant q \leqslant \infty$, $KS^p[\mathbb{R}^n] \supset L^q[\mathbb{R}^n]$ as a dense continuous embedding.*

Since we don't need or use this result, the proof is omitted (see [4]).

Lemma 2.3. *The space $KS^\infty \subset KS^2$, as a continuous embedding.*

Proof. First we note that $f \in KS^\infty$ implies that $\left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\, d\lambda_n(\mathbf{x})\right|$ is uniformly bounded for all k. It follows that $\left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\, d\lambda_n(\mathbf{x})\right|^2$ is uniformly bounded. It is now clear from the definition of KS^∞ that

$$\|f\|_{KS^2} = \left(\sum_{k=1}^{\infty} t_k \left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x})\, d\lambda_n(\mathbf{x})\right|^2\right)^{1/2} \leqslant \|f\|_{KS^\infty} < \infty,$$

and therefore $f \in KS^2$. □

In light of the importance of the theory of distributions and the general belief that test functions cannot be included in a Banach space structure, the following result is instructive.

Theorem 2.4. *The space of test functions $\mathcal{D}(\mathbb{R}^n) \subset KS^2[\mathbb{R}^n]$ as a continuous dense embedding.*

Proof. The proof is easy. Since $KS^\infty(\mathbb{R}^n)$ is continuously embedded in $KS^2(\mathbb{R}^n)$, it suffices to prove the result for $KS^\infty(\mathbb{R}^n)$. Suppose that $\phi_j \to \phi$ in $\mathcal{D}[\mathbb{R}^n]$, so there exists a compact set $K \subset \mathbb{R}^n$ containing the support of $\phi_j - \phi$, and $D^\alpha \phi_j$

converges to $D^\alpha\phi$ uniformly on K for every multi-index α. Let $L = \{l \in \mathbb{N} :$ the support of $\mathcal{E}_l$, $\operatorname{supp}(\mathcal{E}_l) \subset K\}$; then

$$\begin{aligned}\lim_{j\to\infty} \|D^\alpha\phi - D^\alpha\phi_j\|_{KS} &= \lim_{j\to\infty} \sup_{l\in L} \left| \int_{\mathbb{R}^n} [D^\alpha\phi(x) - D^\alpha\phi_j(x)]\, \mathcal{E}_l(x)\, d\lambda_n(x) \right| \\ &\le \sup_{l\in L} \operatorname{vol}(\mathbf{B}_l) \lim_{j\to\infty} \sup_{x\in K} |D^\alpha\phi(x) - D^\alpha\phi_j(x)| \\ &\leqslant \lim_{j\to\infty} \sup_{x\in K} |D^\alpha\phi(x) - D^\alpha\phi_j\,(x)| = 0. \end{aligned}$$ □

Remark 2.5. Since $\mathcal{D}(\mathbb{R}^n)$ is a dense topological vector subspace of $KS^2[\mathbb{R}^n]$, by the Hahn–Banach theorem, each continuous linear functional T on $\mathcal{D}(\mathbb{R}^n)$ has a continuous extension to $KS^2[\mathbb{R}^n]$. However, from the Riesz representation theorem, every continuous linear functional on $KS^2[\mathbb{R}^n]$ is of the form $T(f) = (f, g)_{KS^2}$ for some unique $g \in KS^2[\mathbb{R}^n]$, so that $\mathcal{D}'(\mathbb{R}^n) \subset KS^2[\mathbb{R}^n]$. This is the property that suggested the use of $KS^2[\mathbb{R}^n]$ for the Feynman path integral representation of quantum mechanics [5].

We close this section with the following:

Theorem 2.6. *If $\mathcal{B} = L_p[\mathbb{R}^n]$ and $\mathcal{H} = KS_2[\mathbb{R}^n]$, then all of the theorems proven for the couple ($\mathbb{C}[0,1]$, $L_2[0,1]$) also hold for the couple ($\mathcal{B}$, $\mathcal{H}$).*

3. The Baire one approximation problem

Let $\mathcal{B}$ be a given separable Banach space and let $\hat{I}_u$ be the bounded linear functional on $\mathcal{B}^*$ given by $\hat{I}_u(v^*) = v^*(u)$, so that $\hat{I}_u \in \mathcal{B}^{**}$. This defines a mapping $\hat{I}$ of $\mathcal{B}$ into $\mathcal{B}^{**}$, which we call the *canonical evaluation map.*

Definition 3.1. A Banach space $\mathcal{B}$ is said to be:

(1) *reflexive* if $\hat{I} : \mathcal{B} \to \mathcal{B}^{**}$ is surjective;
(2) *quasi-reflexive* if $\dim[\mathcal{B}^{**}/\mathcal{B}] < \infty$;
(3) *nonquasi-reflexive* if $\dim[\mathcal{B}^{**}/\mathcal{B}] = \infty$.

In this section, $\mathcal{B}$ is $\mathbb{C}[0,1]$ or $L_p[\mathbb{R}^n]$, with $1 \le p < \infty$ and $n \in \mathbb{N}$.

Definition 3.2. If $A \in \mathcal{C}[\mathcal{B}]$ and there exists a sequence $\{A_n\}$ of bounded linear operators such that $A_n u \to Au$ for $u \in D(A)$, we say that A is *of Baire class one.*

The theorem by Vinokurov et al. [19] shows that, for every nonquasi-reflexive Banach space $\mathcal{B}$ (for example, $\mathbb{C}[0,1]$ or $L_1[\mathbb{R}^n]$, $n \in \mathbb{N}$), there is at least one closed densely defined linear operator A, which is not of Baire class one. This means, in particular, that there does not exist a sequence of bounded linear operators $A_n \in \mathcal{L}[\mathcal{B}]$ such that, for $g \in D(A)$, $A_n g \to Ag$ as $n \to \infty$. We have the following contradiction, showing that their conclusion implicitly assumes a unique representation for the dual functionals on $\mathcal{B}$.

Our proof exploits the existence of A^* and the polar decomposition of $A \in \mathcal{C}[\mathcal{B}]$, $A = U|A^*A|^{1/2}$. Let $T = |A^*A|^{1/2}$ and $\bar{T} = |AA^*|^{1/2}$. The Hille–Yosida Theorem ensures that both T and $\bar{T} = |AA^*|^{1/2}$ generate strongly continuous contraction semigroups (see Pazy [14]). Furthermore, a result of Kato shows that $AT = \bar{T}A$;

the resolvent of T, $R(\lambda, T)$ exists for every $\lambda > 0$; and $AR(\lambda, T) = R(\lambda, \bar{T})A$ (see [12]). The following results can be found in Pazy:

Theorem 3.3. *If $S(t)$ is the contraction semigroup generated by T and $v \in D(A) = D(T)$, then*

(1) *for each $\lambda > 0$, $R(\lambda, T) = \int_0^\infty e^{-\lambda t} S(t)\, dt$;*
(2) *for each $\lambda > 0$, $(\lambda I - T)R(\lambda, T)v = R(\lambda, T)(\lambda I - T)v = v$;*
(3) $\left\| A \left(I - \frac{1}{\lambda} T \right)^{-1} v \right\| = \left\| \left(I - \frac{1}{\lambda} \bar{T} \right)^{-1} Av \right\| \leq \frac{1}{\lambda} \|Av\|$.

Remark 3.4. We note that $\left(I - \frac{1}{\lambda} T \right)^{-1} = \lambda R(\lambda, T)$. Let $R_n = nR(n, T)$.

Lemma 3.5. *If $\mathcal{B} = \mathbb{C}[0,1]$ or $L_1[\mathbb{R}^n]$ and $A \in \mathcal{C}[\mathcal{B}]$, then there exists a sequence of operators $A_n \in \mathcal{L}[\mathcal{B}]$ such that $A_n \to A$ on $D(A)$.*

Proof. In either case, if $A \in \mathcal{C}[\mathcal{B}]$, let $T = |A^* A|^{1/2}$ and $R_n = (I + n^{-1}T)$. Then $(R_n v, v)_{\mathcal{H}} > 0$ for all $0 \neq v \in D(A)$. Thus, the range of R_n is $\mathcal{B}$ and $A_n = AR_n^{-1} \in \mathcal{L}[\mathcal{B}]$. It is clear that $A_n \to A$ on $D(A)$, so that A is of Baire class one. □

Conclusion

In this paper, we have shown that the existence of inequivalent representations of the dual space for a particular class of separable Banach spaces can lead to totally unexpected insight into the geometry of the space and the structure of the operators on the space. Our results show that these new representations offer a powerful tool for functional analysis.

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.* **3** (1922), 133–181. MR 3949898.

[2] S. Banach, *Théorie des opérations linéaires*, Monografje Matematyczne, 1, Warsaw, 1932. https://eudml.org/doc/271931.

[3] N. Dunford and J. T. Schwartz, *Linear Operators. Part I*, reprint of the 1958 original, Wiley Classics Library, John Wiley & Sons, New York, 1988. MR 1009162.

[4] T. L. Gill, Banach spaces for the Schwartz distributions, *Real Anal. Exchange* **43** (2018), no. 1, 1–36. MR 3816428.

[5] T. L. Gill and W. W. Zachary, *Functional Analysis and the Feynman Operator Calculus*, Springer, Cham, 2016. MR 3468941.

[6] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, Graduate Studies in Mathematics, 4, American Mathematical Society, Providence, RI, 1994. MR 1288751.

[7] H. Hahn, Über Folgen linearer Operationen, *Monatsh. Math. Phys.* **32** (1922), no. 1, 3–88. MR 1549169.

[8] E. Helly, Über lineare Funktionaloperationen, *Wien. Ber.* **121** (1912), 265–297.

[9] E. Helly, Über Systeme linearer Gleichungen mit unendlich vielen Unbekannten, *Monatsh. Math. Phys.* **31** (1921), no. 1, 60–91. MR 1549097.

[10] E. Helly, Über Mengen konvexer Körper mit gemeinschaftlichen Punkten, *Deutsche Math.-Ver.* **32** (1923), 175–176. https://eudml.org/doc/145659.

[11] E. Helly, Über Systeme von abgeschlossenen Mengen mit gemeinschaftlichen Punkten, *Monatsh. Math. Phys.* **37** (1930), no. 1, 281–302. MR 1549795.

[12] T. Kato, *Perturbation Theory for Linear Operators*, second edition, Grundlehren der Mathematischen Wissenschaften, Band 132, Springer-Verlag, Berlin, 1976. MR 0407617.

[13] P. D. Lax, Symmetrizable linear transformations, *Comm. Pure Appl. Math.* **7** (1954), 633–647. MR 0068116.

[14] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, 44, Springer-Verlag, New York, 1983. MR 0710486.

[15] F. Riesz, Sur certains systèmes d'équations fonctionnelles et l'approximation des fonctions continues, *C. R. Acad. Sci. Paris* **150** (1910), 674–677.

[16] F. Riesz, Sur certains systèmes singuliers d'équations intégrales, *Ann. Sci. École Norm. Sup. (3)* **28** (1911), 33–62. MR 1509135.

[17] F. Riesz, *Les systèmes d'équations linéaires à une infinité d'inconnues*, Gauthier-Villars, Paris, 1913.

[18] F. Riesz, Über lineare Funktionalgleichungen, *Acta Math.* **41** (1916), no. 1, 71–98. MR 1555146.

[19] V. A. Vinokurov, Ju. Petunīn, and A. N. Pličko, Measurability and regularizability of mappings that are inverses of continuous linear operators (in Russian), *Mat. Zametki* **26** (1979), no. 4, 583–591, 654. MR 0552720. Translated in *Math. Notes* **26** (1979), 781–785.

[20] N. Wiener, Note on a paper of M. Banach, *Fund. Math.* **4** (1923), 136–143.

Tepper L. Gill✉
Department of EECS and Mathematics, Howard University, Washington DC 20059, USA
tgill@howard.edu

Douglas Mupassiri
Department of Mathematics, University of Northern Iowa, Cedar Falls, Iowa, USA
douglas.mupasiri@uni.edu

Erdal Gül
Department of Mathematics, Faculty of Arts and Science, Yildiz Technical University,
34210 Davutpasa, İstanbul, Turkey
gul@yildiz.edu.tr

Received: October 6, 2021
Accepted: December 7, 2021